

**An investigation of the forced Navier-Stokes equations
in two and three dimensions.**

by

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To my wife,

Laura Ann Kavlie, M.M.S., PA-C,

without whom I would never have made it this far.

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SUMMARY

This dissertation is devoted to expanding the classical theory of the forced Navier-Stokes equations. First, we study the regularity of solutions to the two dimensional Navier-Stokes equations with a singular or “fractal” forcing term. The classical theory tells us that the two dimensional Navier-Stokes equations gain two derivatives on a sufficiently smooth force. Following these classical methods we extend this result to spaces with negative fractional derivatives. However, these methods break down at a critical value. In this case, we show that one can still gain two derivatives locally in time.

Next, we investigate the long-term behavior of both the two dimensional and three dimensional Navier-Stokes equations with a time-dependent force. When the force is independent of time, it is known that the long-term behavior of the Navier-Stokes equations is encapsulated within a set called the global attractor. The global attractor has a nice characterization, even in the three dimensional case, where we still do not know if there exists unique solutions. We present a framework for studying the existence of an analogous object, the pullback attractor, when the force depends on time. We study the existence and structure of these pullback attractors as well as the relationship between the pullback attractor and other existing notions of attractors.

Finally, we apply our framework to the two dimensional and three dimensional Navier-Stokes equations with an appropriate time-dependent force. We also study the effect that the size of the force has on the size of the pullback attractor. Finally, we show that if the force

SUMMARY (Continued)

is sufficiently small and periodic, there must exist a unique, smooth, periodic solution to the three dimensional Navier-Stokes equations.

CHAPTER 1

INTRODUCTION

1.1 Basic Definitions, Notation, and Classical Theory

The Navier-Stokes equations model the evolution of the velocity of a viscous, incompressible fluid. They are given by

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f \\ \nabla \cdot u = 0. \end{cases} \quad (1.1)$$

where u is the fluid velocity, ν is the kinematic viscosity, p is the kinematic (internal) pressure, and f is the (external) force acting on the fluid. The first equation is derived from Newton's second law $F = ma$. It enforces conservation of momentum for the fluid. For this reason, we will call this equation the momentum equation. The second equation is the conservation of mass for the fluid. It is also known within the literature as the incompressibility condition or the divergence-free condition. We will consider these equations on the n -dimensional torus, \mathbb{T}^n where $n = 2, 3$. We will also assume that for some $s \in \mathbb{R}$ and any $t \in \mathbb{R}$ that

$$\int_{\mathbb{T}^n} u(x, s) dx = \int_{\mathbb{T}^n} f(x, t) dx = 0.$$

That way, $\int_{\mathbb{T}^n} u(x, t) dx = 0$ for all $t \geq s$ as a formal calculation shows.

These equations have been the source of intensive study in recent years, particularly in the three dimensional case. The first major breakthrough began with the seminal work of Jean

Leray (1) who used functional analysis techniques to build solutions to these equations in a weak or distributional sense. His first observation is that the pressure term is related to the nonlinear term through the incompressibility condition. Indeed, taking the divergence of the momentum equation, a formal calculation gives us that

$$-\Delta p = \nabla \cdot (u \cdot \nabla)u - \nabla \cdot f.$$

Thus, if one knows the velocity, then the pressure can be recovered by solving Laplace's equation. In this way, it becomes convenient to remove the pressure term from the equations by projecting onto divergence-free vector fields. This is an extension of the Helmholtz decomposition which says that for any vector field $F \in C^2$,

$$F = -\nabla\phi + \nabla \times G.$$

That is, F can be decomposed into the curl of some vector field G (which is divergence-free) and a gradient of some scalar function ϕ .

Applying the Leray projection P_σ onto divergence-free vector fields to Equation 1.1 gives us the projected equation

$$u_t + B(u, u) + \nu Au = f. \tag{1.2}$$

For notational simplicity, we will assume that $P_\sigma f = f$. Here,

$$B(u, u) := P_\sigma(u \cdot \nabla)u$$

$$Au := P_\sigma(-\Delta)u.$$

This equation determines the evolution of the vector field u within a space of functions, an infinite-dimensional space.

The Stokes operator A is an unbounded, self-adjoint, positive definite operator with discrete eigenvalues $0 < \mu_1 \leq \mu_2 \leq \dots$ (we avoid the classical use of λ_p as that notation has a special designation which will be introduced in the next section). With corresponding eigenfunctions w_j , it is known that for $u = \sum_j a_j w_j$,

$$Au = \sum_j \mu_j a_j w_j.$$

Thus, we may define the fractional Stokes operator as

$$A^\alpha u := \sum_j \mu_j^\alpha a_j w_j$$

Recall that the Sobolev space $H^\alpha(\mathbb{T}^2)$ for $\alpha \in \mathbb{R}$ is a Hilbert space with the norm

$$\|u\|_{H^\alpha(\mathbb{T}^2)} := \left(\sum_{n \in \mathbb{Z}^2} |n|^{2\alpha} |\hat{u}(n)|^2 \right)^{1/2}$$

where $\hat{u}(n)$ is the n th Fourier coefficient of u . Using the operator A , an equivalent definition of the $H^\alpha(\mathbb{T}^2)$ norm is given by

$$\|u\|_{H^\alpha(\mathbb{T}^2)} = \|A^{\alpha/2}u\|_{L^2(\mathbb{T}^2)}.$$

Remark 1.1.1. The constant μ_1 is known as the Poincaré constant. It is well-known that if $\int_{\mathbb{T}^n} u dx = 0$ (the 0th Fourier coefficient is 0), then

$$\|u\|_{H^\alpha(\mathbb{T}^n)}^2 \leq \mu_1 \|u\|_{H^{\alpha+1}(\mathbb{T}^n)}^2,$$

known as the Poincaré inequality.

In this setting, the following theorem was proven:

Theorem 1.1.2 (Leray, Hopf). *For each $u_0 \in L^2(\mathbb{T}^n)$ with $\nabla \cdot u_0 = 0$ in a distributional sense and for any $f \in L^2_{loc}(\mathbb{R}, H^{-1}(\mathbb{T}^n))$, there exists a weak solution $u \in L^\infty_{loc}([s, \infty), L^2(\mathbb{T}^n)) \cap L^2_{loc}([s, \infty), H^1(\mathbb{T}^n))$ of Equation 1.2 on $[s, \infty)$ so that $u(s) = u_0$. Moreover, for each $t \geq t_0$, t_0 a.e. in $[s, \infty)$ we have the following energy inequality:*

$$\|u(t)\|_{L^2(\mathbb{T}^n)}^2 + 2\nu \int_{t_0}^t \|u(r)\|_{H^1(\mathbb{T}^n)}^2 dr \leq \|u(t_0)\|_{L^2(\mathbb{T}^n)}^2 + 2 \int_{t_0}^t \langle f(r), u(r) \rangle_{H^{-1}(\mathbb{T}^n)} dr. \quad (1.3)$$

We call (weak) solutions satisfying Equation 1.3 Leray-Hopf (weak) solutions.

Definition 1.1.3. A weak solution to Equation 1.1 satisfying Equation 1.3 will be called a Leray-Hopf (weak) solution.

In fact, the method of building these weak solutions allows one to prove the existence of solutions which are continuous at the starting time s . For instance, one could use the method of Galerkin approximations to prove their existence. We call these solutions Leray solutions.

Definition 1.1.4. A Leray-Hopf weak solution $u \in L_{loc}^\infty([s, \infty), L^2(\mathbb{T}^n)) \cap L_{loc}^2([s, \infty), H^1(\mathbb{T}^n))$ to [Equation 1.2](#) on $[s, \infty)$ satisfying the energy inequality

$$\|u(t)\|_{L^2(\mathbb{T}^n)}^2 + 2\nu \int_s^t \|u(r)\|_{H^1(\mathbb{T}^n)}^2 dr \leq \|u(s)\|_{L^2(\mathbb{T}^n)}^2 + 2 \int_s^t \langle f(r), u(r) \rangle_{H^{-1}(\mathbb{T}^n)} dr$$

for each $t \geq s$ is called a Leray solution.

In two dimensions, we can say more.

Theorem 1.1.5. *For each $u_0 \in L^2(\mathbb{T}^2)$ satisfying $\nabla \cdot u_0 = 0$ in a distributional sense and for $f \in L_{loc}^2(\mathbb{R}, H^{-1}(\mathbb{T}^2))$, there exists a unique solution $u \in C_{loc}([s, \infty), L^2(\mathbb{T}^2))$ of [Equation 1.2](#) on $[s, \infty)$ so that $u(s) = u_0$. Moreover, for each $t \geq t_0$, in $[s, \infty)$ we have the following energy equality:*

$$\|u(t)\|_{L^2(\mathbb{T}^n)}^2 + 2\nu \int_{t_0}^t \|u(r)\|_{H^1(\mathbb{T}^n)}^2 dr = \|u(t_0)\|_{L^2(\mathbb{T}^n)}^2 + 2 \int_{t_0}^t \langle f(r), u(r) \rangle_{H^{-1}(\mathbb{T}^n)} dr.$$

On the other hand, the analogue in three dimensions has eluded mathematicians. In fact, no matter how smooth your initial conditions and your forcing term are, we still do not know that there exist unique solutions to the Navier-Stokes equations [\(2\)](#).

For more information on the classical theory of Navier-Stokes, see the books (3), (4), or (5), among others.

Remark 1.1.6. For the remainder of this paper, we will unambiguously omit \mathbb{T}^n from our norms as it is implied. The dimension n will be clear from context. Moreover, we will use the standard convention

$$\|u\|_p := \|u\|_{L^p}$$

Remark 1.1.7. For a Hilbert space H , we denote by $\langle \cdot, \cdot \rangle_H$ the associated inner product for H . We will use this same notation for the functional action between a Banach space X and its dual X^* . That is $\langle \cdot, \cdot \rangle_X : X^* \times X \rightarrow F$, where $F = \mathbb{R}$ or \mathbb{C} . In addition we denote by X_w the associated topological space endowed with the weak topology.

Remark 1.1.8. It is customary to project the spaces H^α when using the projected equations. We will not make this distinction to more easily track the underlying space in which we are working. The fact that the evolution occurs in a projected space is coincidentally implied. Moreover, we will always assume that our initial conditions u_0 satisfy $\nabla \cdot u_0 = 0$ in a distributional sense.

Remark 1.1.9. The above theorems have analogous statements without the use of the projected equations.

1.2 Dynamics of the Navier-Stokes Equations

A dynamical system is a rule which determines the evolution of points in an underlying space, called the phase space. In the study of dynamical systems, a common question is “what is the asymptotic or long-term behavior of the system?” Equation 1.2 defines a dynamical

system on an infinite-dimensional space of functions. Typically, we use the phase space L^2 (or the associated projected space of divergence-free vector fields as mentioned in Remark 1.1.8). If the force f does not depend on time, we call this dynamical system autonomous. In this case, the evolution only depends on the starting point, not the starting time. If the force $f(t)$ does depend on time, we call this dynamical system nonautonomous. For a nonautonomous system, the evolution depends on both the starting position and the starting time.

Due to Theorem 1.1.5, we may define a solution operator in the two-dimensional case which carries points in the phase space to other points in the phase space in a well-defined manner. In the autonomous case, we have a solution semigroup

$$S(t)u_0 := u(t) \quad \text{where} \quad u(0) = u_0;$$

in the nonautonomous case, we have a solution process

$$P(t, s)u_0 := u(t) \quad \text{where} \quad u(s) := u_0.$$

Remark 1.2.1. Another common framework for studying nonautonomous dynamical is using cocycles. Cocycles give a clear advantage when studying the evolution in a more complicated scenarios, such as the study of the Navier-Stokes equations on manifolds or the study of random dynamical systems. However, for purpose of presenting an introduction to the theory, we summarize the theory using processes.

Studying the asymptotic behavior of these solutions amounts to studying the limit

$$\lim_{t \rightarrow \infty} S(t)u_0$$

in the autonomous case. This gives rise to the theory of global attractors. In the nonautonomous case, we have two options. We may study either of the following “natural” limits:

$$\begin{aligned} \lim_{t \rightarrow \infty} P(t, s)u_0 \\ \lim_{s \rightarrow -\infty} P(t, s)u_0. \end{aligned}$$

The former limit gives rise to the theory of forward attractors, whereas the latter limit gives rise to the theory of pullback attractors.

In three dimensions, where uniqueness of solutions remains unsolved, no such well-defined solution operators on the phase space are known to exist. Even so, the study of the asymptotic nature of these equations can still be explored, as we will make rigorous later.

1.2.1 Autonomous Dynamical Systems

Let (X, d) , a complete metric space, be our phase space. A global attractor \mathcal{A} is the minimal closed subset of X which uniformly attracts all bounded subsets of X . That is, \mathcal{A} is the minimal closed set so that for any bounded set $B \subset X$,

$$\lim_{t \rightarrow \infty} d(S(t)B, \mathcal{A}) = 0$$

where d is the Hausdorff semi-distance between the two sets. This means that the global attractor \mathcal{A} contains all of the asymptotic information of the dynamical system.

Remark 1.2.2. In the classical theory of global attractors, the global attractor is the unique, compact, invariant set ($S(t)\mathcal{A} = \mathcal{A}$) which uniformly attracts all bounded subsets of the phase space. The two definitions are equivalent under some minimal assumptions. However, the general setting in which we are interested requires weakening the compactness criterion.

There are several ways to prove the existence of a global attractor for your dynamical system. In the classical theory, you will first prove that your dynamical system has a compact absorbing set K . Then, the omega-limit set of K is the global attractor.

Definition 1.2.3. A point x is in the omega limit set of the set K provided there exists a sequence $t_n \rightarrow \infty$ and $x_n \in S(t_n)K$ with $x_n \rightarrow x$.

A more general method is to first prove that your dynamical system is asymptotically compact (which is implied by the former method).

Definition 1.2.4. A dynamical system $S(t)$ on X is asymptotically compact provided every sequence $t_n \rightarrow \infty$ and $x_n \in S(t_n)X$ has a convergent subsequence.

The existence of a global attractor for the 2D Navier-Stokes equations was first proven by Foias and Prodi (6) and later by Ladyzhenskaya (7) using the language of dynamical systems. For more information on this theory, see the books (5) and (8).

As was previously pointed out, the existence of a semigroup of solution operators is not known for the 3D Navier-Stokes equations. Even so, Foias and Temam proved the existence of

a weak global attractor for Leray-Hopf weak solutions for the 3D Navier-Stokes equations (9). Specifically, they showed that

$$\{u(0) : u \text{ is a complete, bounded, weak solution to Equation 1.2}\} \quad (1.4)$$

uniformly attracts all bounded subsets of L^2 . Yet, they did so without a dynamical framework. Even so, several frameworks now exist for studying the asymptotic dynamics of a system without uniqueness of solution. For a comparison between two canonical frameworks, see Caraballo et al (10). In their paper, they compare the framework of multivalued semiflows used by Melnik and Valero (11) to the framework of generalized semiflows developed by Ball (12).

A semiflow, or set-valued semigroup is a set-valued function with some inclusion properties that account for the lack of uniqueness. That is,

$$S(t)B := \{u(t) : u(0) \in B\}.$$

This set-valued function tracks the evolution of one set into another, defining a dynamical system on the power set of X . One can then define and prove the existence of a global attractor in this setting.

The approach used by Ball involves defining trajectories in the phase space. In a dynamical system, a trajectory is the set of points “swept out” by the evolution of a point. That is,

$$\{S(t)b : t \geq 0\}$$

for any $b \in X$. Ball's construction allowed for more than one trajectory starting from a single starting value.

For multivalued semiflows or multivalued semigroups, one often loses individual trajectories. Therefore, recovering results such as Equation 1.4 become problematic. On the other hand, Ball's construction requires certain assumptions about trajectories including the ability to concatenate them which are still not known for the Leray-Hopf weak solutions to the 3D Navier-Stokes equations.

In their paper (13), Cheskidov and Foias introduced the concept of an evolutionary system allowing them to construct a framework based on known results for the Leray-Hopf weak solutions to the autonomous 3D NSEs. Moreover, they proved the existence of the global attractor in this setting. Cheskidov later added the idea of a trajectory attractor for an evolutionary system (14) an idea which grew from the nonautonomous theory. We will discuss uniform attractors further in the next section.

1.2.2 Nonautonomous Dynamical Systems

As stated earlier, evolution in a nonautonomous dynamical system not only depends on the starting position in the phase space, it also depends on the starting time. Thus, always assuming the starting time of 0 loses generality. Most techniques for studying these systems have to do with reducing the nonautonomous system to an autonomous system. Then, the classical theory will allow one to prove the existence of an attractor for the resultant autonomous system. The first such example is that of Chepyzhov and Vishik's use of time symbols (15).

Chepyzhov and Vishik introduced the idea of a time symbol encapsulating the time-dependent portion of the nonautonomous system. For example, if the Navier-Stokes equations have a time-dependent forcing term $f(t)$, this time-dependent force is the “time symbol” for the equation. They then observed that switching from one trajectory to another involved shifting the time symbol. That is, they introduced a one-parameter family of processes P_σ with $\sigma \in \Sigma$. They then assumed that

$$P_\sigma(t + s, \tau + s) = P_{T(s)\sigma}(t, \tau)$$

for $T(s)$ a translation operator acting on the complete metric space Σ in a one-to-one fashion ($t \geq \tau \in \mathbb{R}$ and $s \geq 0$). For example, consider

$$P_\sigma(t, s)u_0 := u(t)$$

where $u(s) := u_0$ is a solution to the Navier-Stokes equation

$$u_t + \nu Au + B(u, u) = f(\cdot + \sigma)$$

with $\sigma \in \mathbb{R}$.

This allowed them to define an action on the product space $X \times \Sigma$ which reduces the nonautonomous system to an autonomous system. Moreover, if they are able to assume that Σ is compact, or able to be compactified in a reasonable way, then a compact absorbing set in

the phase space would automatically mean the existence of a global attractor for the resulting autonomous dynamical system. This attractor is called the uniform attractor.

In their paper, Cheskidov and Lu introduced the concept of a nonautonomous evolutionary system, adding the concept of symbols. They then applied their framework to the 3D NSEs and certain reaction-diffusion equations in (16) and (17), respectively, proving the existence of uniform attractors in both occasions.

Another common framework is that of trajectory attractors. This framework stems from the observation that if $u(\cdot)$ is a solution to

$$u_t(\cdot) + \nu Au(\cdot) + B(u(\cdot), u(\cdot)) = f(\cdot),$$

then $u(\cdot + h)$ for any $h \geq 0$ is a solution to

$$u_t(\cdot) + \nu Au(\cdot) + B(u(\cdot), u(\cdot)) = f(\cdot + h),$$

With the very same time symbols as before, we can define an autonomous dynamical system on the space of trajectories

$$\{u(\cdot) : u \text{ is a solution to Equation 1.2 with force } f(\cdot + h)\}$$

Again, we take the closure of the symbol space to compactify the space. Then, we can investigate the existence of an attractor in the space of trajectories. This attractor, however, exists in the

space of trajectories, not in the physical space. Thus, it is natural to project our attractor into the physical space by evaluating at some fixed time. This gives a deep connection between the theory of uniform attractors and trajectory attractors. For more information on trajectory attractors for a wide range of equations, including the 2D and 3D Navier-Stokes equations, see the book (15).

The uniform attractor, under sufficient conditions is fibered over the symbol space into “kernel sections,” consisting of complete, bounded solutions evaluated at some fixed time. These kernel sections, however, do not have classical attraction properties. The attraction is in a pullback sense, letting the initial time go to minus infinity (18). This leads us to our next technique for dealing with nonautonomous systems, pullback attractors.

Studying the asymptotic pullback dynamics of a nonautonomous system amounts to studying the following limit

$$\lim_{s \rightarrow -\infty} P(t, s)u_0,$$

where t is fixed. Studying the existence of attractors for this system necessitates the use of a one-parameter family of sets $\mathcal{A}(t)$ where t is determined by the “stop time.” The pullback attractor $\mathcal{A}(t)$ is the minimal family of closed sets which uniformly attracts all bounded subsets of the phase space in a pullback sets. That is, for any bounded set $B \subset X$,

$$\lim_{s \rightarrow -\infty} d(P(t, s)B, \mathcal{A}(t)) \rightarrow 0$$

The concept of a pullback attractor originated in the work of Crauel, Flandoli, Kloeden, and Schmalfuss (19), (20) who were studying random dynamical systems. For more information on pullback attractors, see the books (21), (22), and (23).

Remark 1.2.5. As in the autonomous case, the classical definition of a the pullback attractor $\mathcal{A}(t)$ is the unique invariant family $(P(t, s)\mathcal{A}(s) = \mathcal{A}(t))$ of compact sets which uniformly attracts all bounded subsets of the phase space. As before, under some minimal assumptions, these two definitions are equivalent. But, in a more general setting, the requirement that the pullback attractor be compact is too restrictive.

It is worthwhile to note that in the autonomous case, $P(t, s) := S(t - s)$. Thus, sending $s \rightarrow -\infty$ is equivalent to sending $t - s \rightarrow \infty$ starting from $t - s = 0$. Thus, the concept of a pullback attractor accurately extends the autonomous theory. In fact, several concepts extend from the autonomous theory to this setting. In particular, to prove the existence of a pullback attractor, you can prove the existence of a compact pullback absorbing set $K(t)$. Then, the pullback attractor is given by

$$\mathcal{A}(t) = \overline{\bigcup_{\substack{B \subset X \\ B \text{ bounded}}} \omega(B, t)}$$

where $\omega(B, t)$ is the pullback omega-limit set of B .

Definition 1.2.6. The point x is in the pullback omega-limit set of B , $\omega(B, t)$ if and only if there exists sequences $s_n \rightarrow -\infty$, $s_n \leq t$ and $x_n \in P(t, s_n)B$ so that $x_n \rightarrow x$.

The second, more general way to prove the existence of a pullback attractor is to prove that your nonautonomous dynamical system is pullback asymptotically compact.

Definition 1.2.7. A nonautonomous dynamical system $P(t, s)$ on X is pullback asymptotically compact if for each $s_n \rightarrow -\infty$, $s_n \leq t$ and $x_n \in P(t, s_n)X$ contains a convergent subsequence.

Remark 1.2.8. As in the case of autonomous dynamical systems without uniqueness, much work is being done to study the theory of set-valued processes (24), (25). As in the autonomous case, we define

$$P(t, s)B := \{u(t) : u(s) \in B\}.$$

This shares the inability to track the location of individual trajectories with generalized semi-flows as previously discussed.

1.3 Dissertation Organization

1.3.1 Chapter 2

In Chapter 2, we study the autonomous two-dimensional Navier-Stokes equations with a singular forcing term $f \in H^\alpha$ for $\alpha \in [-1, 0)$. The $\alpha = 0$ case has been thoroughly studied in the classical literature (3), (5), (4). In this setting, when $f \in L^2$, then (Equation 1.1) admits unique solutions $u \in C([0, T], H^1) \cap L^\infty([0, T], H^2)$ for all times $T \geq 0$. On the other hand, for many linear parabolic equations, like the heat equation, we find that $u \in C([0, T], H^{\alpha+2})$ as long as $u(0) \in H^{\alpha+2}$ and $f \in H^\alpha$. In colloquial terms, the heat equation always gains two derivatives on the force, even with a singular force, such as $f \in H^{-1}$.

This gap between the Navier-Stokes equations, and linear parabolic equations such as the heat equation was studied by Constantin and Seregin for the Navier-Stokes equations (26) and later for the Fokker-Plank equations (27). Their analysis involves using the modulus of

continuity in physical space. They find that with a forcing term $f \in W^{-1,q}$ with $q > 4$, the solution u remains Hölder continuous with exponent $1 - 4/q$. That is, in an L^∞ sense, the function gains $2 - \epsilon$ derivatives, for any $\epsilon \geq 0$ (with “smoother forces” required to reach a two derivative gain). To further bridge this gap, our analysis uses the technique of Littlewood-Paley decompositions in Fourier space to show that u remains in H^1 locally in time, a gain of two derivatives on the force $f \in H^{-1} = W^{-1,2}$. Specifically, we prove the following theorem, Theorem 2.3.1:

Theorem 1.3.1. *Let u be the unique solution to Equation 1.1 with $u(0) := u_0 \in H^1$. Then, there exists $T := T(u_0, f)$ so that $u \in L^\infty([0, t_0], H^1)$ for all $0 < t_0 < T$.*

Intervals where $u \in H^1$ are known as intervals of regularity for the Navier-Stokes equations. Using Theorem 2.3.1, one can prove a Leray characterization for the 2D Navier-Stokes equations with force $f \in H^{-1}$. That is, $[0, \infty) = \cup_j [a_j, b_j)$ with $u(t) \in H^1$ for all $t \in [a_j, b_j)$. This result is well-known for the 3D Navier-Stokes equations with a force $f \in L^2$.

To complete our study of singular forces, we explore the use of classical techniques in the intermediate spaces where $f \in H^\alpha$ with $\alpha \in (-1, 0)$. We show that classical techniques work to give a global gain of two derivatives. That is, we prove the following theorem, Theorem 2.2.3:

Theorem 1.3.2. *Let $f \in H^\alpha$ and $u(0) := u_0 \in H^{\alpha+2}$ for some $\alpha \in (-1, 0)$. Then, there exists a solution $u(t)$ to Equation 1.1 so that $u \in L^\infty([0, \infty), H^{\alpha+2})$.*

Both arguments require the use of Littlewood-Paley theory. Thus, we include a short introduction to this theory at the beginning of Chapter 2.

1.3.2 Chapter 3

Chapter 3 begins our study of pullback attractors. To begin, we present a framework for studying nonautonomous dynamical systems without uniqueness called a generalized evolutionary system. This framework requires the use of two metrics on your phase space known as the weak metric and the strong metric. This construction mirrors the two metrics on a bounded subset of a separable Banach space induced by the norm topology and weak topology, respectively. By construction, we can now examine attraction in either a weak sense, using the weak metric, or in a strong sense, using the strong metric. This framework differs from that of (13) and (14) by removing the ability to “shift trajectories.”

After presenting the basic definitions, we prove the following theorem, Theorem 3.1.11:

Theorem 1.3.3. *Every generalized evolutionary system possesses a weak pullback attractor $\mathcal{A}_w(t)$. Moreover, if the strong pullback attractor $\mathcal{A}_s(t)$ exists, then $\overline{\mathcal{A}_s(t)}^w = \mathcal{A}_w(t)$.*

We then provide several examples to build the reader’s intuition.

To finish this chapter, we add the assumption of pullback asymptotic compactness. With this added assumption, we prove the following theorem, Theorem 3.3.3:

Theorem 1.3.4. *If a generalized evolutionary system is pullback asymptotically compact, then $\mathcal{A}_w(t)$ is a strongly compact strong pullback attractor.*

That is, the weak pullback attractor already proven to exist must also be the strong pullback attractor.

1.3.3 Chapter 4

We begin Chapter 4 by adding additional assumptions to our generalized evolutionary system. These are denoted A1, A2, and A3. We also introduce several notions of invariance. We will show that if the generalized evolutionary system satisfies A1, then Corollary 4.1.4, Theorem 4.1.6, and Theorem 4.1.8 give us the following result:

Theorem 1.3.5. *Let \mathcal{E} be a generalized evolutionary system satisfying A1. Then, the weak pullback attractor is the maximal pullback quasi-invariant and maximal pullback invariant subset of X . In particular,*

$$\mathcal{A}_w(t) = \{u(t) : u \text{ is a complete, bounded trajectory}\}$$

Second, \mathcal{E} satisfies the weak tracking property. In addition, if the strong pullback attractor exists, $\mathcal{A}_s(t) = \mathcal{A}_w(t)$.

If we include the assumption that our generalized evolutionary system is pullback asymptotically compact, this theorem is strengthened to the following via Theorem 3.3.3, Theorem 4.1.7, and Corollary 4.1.9:

Theorem 1.3.6. *Let \mathcal{E} be a pullback asymptotically compact generalized evolutionary system satisfying A1. Then, the strong pullback attractor is the maximal pullback quasi-invariant and maximal pullback invariant subset of X . In particular,*

$$\mathcal{A}_s(t) = \{u(t) : u \text{ is a complete, bounded trajectory}\}$$

Second, \mathcal{E} satisfies the strong tracking property. In addition, we know that $\mathcal{A}_s(t) = \mathcal{A}_w(t)$, the weak pullback attractor.

Pullback asymptotic compactness has proven to be quite powerful. Thus, in the next section, we study some minimal conditions in which one may prove that their generalized evolutionary system is pullback asymptotically compact. To this end, we add the assumptions **A2** and **A3** to our existing assumption **A1**. We then prove the following theorem, Theorem 4.2.2:

Theorem 1.3.7. *Let \mathcal{E} be a generalized evolutionary system satisfying **A1**, **A2**, and **A3**. Assume, also, that complete, bounded trajectories are strongly continuous. Then, \mathcal{E} is pullback asymptotically compact.*

The assumption that complete, bounded trajectories are strongly continuous for the 3D Navier-Stokes equations remains an open question. As these trajectories hold all of the asymptotic information for the Navier-Stokes equations, this is related to the Prodi Conjecture which asserts that solutions to the 3D Navier-Stokes equations are asymptotically regular.

The rest of Chapter 4 is devoted to studying the relationship between the theory of pullback attractors, the theory of global attractors (13), (14), and the theory of uniform attractors (16), (17). In both the autonomous case and nonautonomous case, we begin by reviewing the previous definitions and major theorems. In the autonomous case, we find the following theorem, Theorem 4.3.10:

Theorem 1.3.8. *Let \mathcal{E} be an evolutionary system. Then, the weak global attractor \mathcal{A}_w and the weak pullback attractor $\mathcal{A}_w(t)$ exist, and $\mathcal{A}_w = \mathcal{A}_w(t)$ for each $t \in \mathbb{R}$. Moreover, the strong global attractor \mathcal{A}_s exists if and only if the strong pullback attractor $\mathcal{A}_s(t)$ exists, and $\mathcal{A}_s = \mathcal{A}_s(t)$.*

This theorem shows that the theory of pullback attractors for nonautonomous dynamical systems without uniqueness accurately extends the classical theory of autonomous dynamical systems without uniqueness.

The nonautonomous case proves to be much more interesting. For a fixed symbol σ , we find that the nonautonomous evolutionary system \mathcal{E}_σ induces a generalized evolutionary system. We then find the following inclusion in Corollary 4.3.20

Theorem 1.3.9. *Let \mathcal{E}_σ be a nonautonomous evolutionary system. Then, the weak uniform attractor \mathcal{A}_w^Σ exists. Similarly, for each $\sigma \in \Sigma$, the induced generalized evolutionary system, there exists a weak pullback attractor $\mathcal{A}_w^\sigma(t)$. Moreover,*

$$\overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(t_0)}^w \subseteq \mathcal{A}_w^\Sigma$$

for any fixed $t_0 \in \mathbb{R}$.

The reverse inclusion is untrue. We construct a counterexample using a modified heat equation with time-dependent viscosity. It is unknown at this time what minimal assumptions are required for the uniform attractor to be the closed union of pullback attractors. On the other hand, this is evidence that the uniform attractor may be “too big” when studying the asymptotic nature of nonautonomous dynamical systems.

1.3.4 Chapter 5

In Chapter 5, we apply our framework to the Navier-Stokes equations. We work with the projected equations, Equation 1.2 and a translationally-bounded force. We begin by deriving the phase space in either dimension by showing that there exists an absorbing ball X for Leray solutions to either the 2D or 3D Navier-Stokes equations. We then define the generalized evolutionary system \mathcal{E} as

$$\begin{aligned}\mathcal{E}([s, \infty)) &:= \{u : u \text{ is a Leray-Hopf solution of Equation 1.2 on } [s, \infty) \\ &\quad \text{and } u(t) \in X \text{ for } t \in [T, \infty)\}, \\ \mathcal{E}((-\infty, \infty)) &:= \{u : u \text{ is a Leray-Hopf solution of Equation 1.2 on } (-\infty, \infty) \\ &\quad \text{and } u(t) \in X \text{ for } t \in (-\infty, \infty)\}.\end{aligned}$$

With a little more work, we show that the Navier-Stokes equations satisfy the extra assumptions A1, A2, and A3. This gives us the following theorem, Theorem 5.2.3:

Theorem 1.3.10. *The weak pullback attractor for the generalized evolutionary system \mathcal{E} of the 2D or 3D Navier-Stokes equations, $\mathcal{A}_w(t)$, is the maximal pullback quasi-invariant and maximal pullback invariant subset of X . Also,*

$$\begin{aligned}\mathcal{A}_w(t) &= \{u(t) : u \text{ is a complete, bounded, Leray-Hopf solution} \\ &\quad \text{to the Navier-Stokes equations}\}.\end{aligned}$$

This result generalizes the autonomous result first proven by Foias and Temam (Equation 1.4).

In two dimensions, we can say more as found in Theorem 5.2.4:

Theorem 1.3.11. *The generalized evolutionary system \mathcal{E} for the 2D Navier-Stokes equations is pullback asymptotically compact. Thus, \mathcal{E} has a strongly compact, strong pullback attractor $\mathcal{A}_s(t)$ given by*

$$\mathcal{A}_s(t) = \{u(t) : u \text{ is a complete, bounded, Leray-Hopf solution} \\ \text{to the 2D Navier-Stokes equations}\}.$$

In three dimensions, we have a similar result in Theorem 5.2.5.

Theorem 1.3.12. *Assume that the generalized evolutionary system \mathcal{E} for the 3D Navier-Stokes equations satisfies the property that complete, bounded, Leray-Hopf solutions are strongly continuous. Then, \mathcal{E} is pullback asymptotically compact. In this case, \mathcal{E} has a strongly compact, strong pullback attractor $\mathcal{A}_s(t)$ given by*

$$\mathcal{A}_s(t) = \{u(t) : u \text{ is a complete, bounded, Leray-Hopf solution} \\ \text{to the 3D Navier-Stokes equations}\}.$$

The extra assumption is necessary because it is not known that complete, bounded, Leray-Hopf solutions to the 3D Navier-Stokes equations are strongly continuous.

1.3.5 Chapter 6

A natural question in the study of attractors for dissipative partial differential equations is what conditions on the force necessitate a trivial attractor. That is, under what conditions on the force do we find that the attractor $\mathcal{A} = \{z\}$, a single point. This is closely related to the question of dimensionality of the attractor. For the Navier-Stokes equations, it has long been known that they possess a compact global attractor in two dimensions (6). The dimension of this global attractor is controlled by the Grashof number $G = \frac{\|f\|_2^2}{\nu^2 \lambda_1}$ (28), (29). In particular, when the Grashof number is small enough, the attractor is trivial. For a proof of this fact, see the book (15), although the argument used goes back to (30). That is, $\mathcal{A} = \{z\}$ where z is the unique stationary solution to the Stokes system. An analogous result was proven by Chepyzhov and Vishik using trajectory attractors in three dimensions where the Grashof number is given by $G = \frac{\|f\|_2^2}{\nu^2 \lambda_1^{3/4}}$ (15). This result can easily be extended to the theory of weak attractors as developed in (9), (13), (14).

Now, in the book by Carvalho, Langa, and Robinson (21), they produce a theorem giving sufficient conditions under which the pullback attractor $\mathcal{A}(t)$ for the 2D Navier-Stokes equations is a single point. They find that if some form of the Grashof number is small enough, then the pullback attractor is degenerate. That is, if

$$G(t) := \frac{1}{\nu^2 \lambda_1} \left(\limsup_{s \rightarrow -\infty} \frac{1}{t-s} \int_s^t \|f(r)\|_2^2 dr \right)^{1/2}$$

is small enough, then the pullback attractor $\mathcal{A}(t)$ is trivial. We present an analogous result for the 3D Navier-Stokes equations assuming a translationally bounded force in $L^2_{loc}(\mathbb{R}, L^2)$. We show that if a form of the Grashof number is small enough, then the weak pullback attractor $\mathcal{A}_w(t) = \{v(t)\}$ for a complete, bounded solution v .

To begin, we expand the definition of translationally-boundedness in Chapter 6. The earlier definition of translationally-boundedness depends on the size of the interval you integrate over. When the size of the interval is exactly $(\nu\mu_1)^{-1}$, we can express our results in terms of a form of the Grashof number. We first show that if the force is small enough, then complete, bounded Leray-Hopf solutions $u(t)$ to the 3D Navier-Stokes equations are strong. By this, we mean that $u \in L^\infty_{loc}(\mathbb{R}, H^1)$. Next, we prove that if the force is assumed to be small enough, then the pullback attractor is, in fact, trivial. That is, we prove the following theorem, Theorem 6.2.6:

Theorem 1.3.13. *Let f be translationally bounded in $L^2_{loc}(\mathbb{R}, L^2)$. Assume that the translationally-bounded norm of f is sufficiently small, then the weak pullback attractor for Equation 1.2 is a single point,*

$$\mathcal{A}_w(t) = \{v(t)\}$$

for some complete, bounded, strong solution to Equation 1.2.

The argument is a modification of Serrin's argument (31) for the uniqueness of Leray-Hopf weak solutions on an interval of regularity.

Finally, we apply this result to the case where the force is small and periodic. In this scenario, when the force is sufficiently small, we show that there exists a unique, periodic, strong solution to the 3D Navier-Stokes equations in Theorem [6.2.3](#).

CHAPTER 2

REGULARITY FOR THE 2D NAVIER-STOKES EQUATIONS WITH A SINGULAR FORCE

2.1 Littlewood-Paley Decomposition

In this section, we briefly describe the Littlewood-Paley decomposition and the Littlewood-Paley theorem which are integral to the following arguments. This describes how to relate Sobolev norms in physical space via a particular breakdown in Fourier space. For more information on this theory, see, for example, the book by Chemin (32), among others.

Choose a nonnegative radial function $\chi \in C_0^\infty(\mathbb{R}^2)$ so that

$$\chi(\xi) = \begin{cases} 1, & |\xi| \leq \frac{1}{2} \\ 0, & |\xi| > 1. \end{cases}$$

Let $\phi(\xi) := \chi(\lambda_1^{-1}\xi) - \chi(\xi)$. For each $q \geq 0$, we let $\phi_q(\xi) := \phi(\lambda_q^{-1}\xi)$. For technical reasons, let $\phi_{-1}(\xi) := \chi(\xi)$.

Given a tempered distribution vector field u on \mathbb{T}^2 and $q \geq 1$, an integer, the q th Littlewood-Paley projection of u is given by

$$u_q(x) := \Delta_q u(x) := \sum_{k \in \mathbb{Z}^2} \hat{u}(k) \phi_q(k) e^{ik \cdot x}$$

where $\hat{u}(k)$ is the k th Fourier coefficient of u . Note that, by the Littlewood-Paley theorem,

$$\|u\|_{H^s} \sim \left(\sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2}$$

for each $u \in H^s$ and $s \in \mathbb{R}$.

2.2 The H^α Case for $\alpha \in (-1, 0)$

In this section, we study the regularity of solutions for the 2D Navier-Stokes equations with force $f \in H^\alpha$ for $\alpha \in (-1, 0)$. In this scenario, we can use classical methods including energy methods and analyticity methods to prove that the solution u gains two derivatives on the force, globally.

2.2.1 Estimating the Nonlinear Term

Lemma 2.2.1. *Let $u \in H^1 \cap H^{1-\beta}$ for $\beta \in (0, 1)$. Then,*

$$\|B(u, u)\|_{H^{-\beta}} \leq C \|u\|_{H^{1-\beta}} \|u\|_{H^1}.$$

Note that when $\beta = 1$, the estimate becomes

$$\|B(u, u)\|_{H^{-1}} \leq C \|u\|_2 \|u\|_{H^1}.$$

This is the classical estimate which is obtained using Hölder's inequality and the Ladyzhenskaya inequality. On the other hand, when $\beta = 0$, we may use interpolation to say that

$$\|B(u, u)\|_2 \leq C\|u\|_{H^1}\|u\|_{H^1} \leq C\|u\|_2^{1/2}\|u\|_{H^2}^{1/2}\|u\|_{H^1}.$$

This is the classical estimate which is obtained using Hölder's inequality followed by Agmon's inequality. Thus, our estimate accurately generalizes the classical estimates.

Proof. Let $v \in H^\beta$. We must estimate the integral

$$\int_{\mathbb{T}^2} u \cdot \nabla u \cdot v dx.$$

To do so, we will use Bony's paraproduct. Separating each term into its Littlewood-Paley pieces, we apply the necessary cancellations to find that

$$\begin{aligned} \left| \int_{\mathbb{T}^2} u \cdot \nabla u \cdot v dx \right| &\leq \sum_{\substack{|p-q| \leq 2 \\ r < q+1}} \int_{\mathbb{T}^2} |u_p \cdot \nabla u_q \cdot v_r| dx \\ &\quad + \sum_{\substack{|p-r| \leq 2 \\ q < r+1}} \int_{\mathbb{T}^2} |u_p \cdot \nabla u_q \cdot v_r| dx \\ &\quad + \sum_{\substack{|q-r| \leq 2 \\ p < r+1}} \int_{\mathbb{T}^2} |u_p \cdot \nabla u_q \cdot v_r| dx \\ &=: I + II + III. \end{aligned}$$

To estimate I , we use Hölder's inequality followed by Bernstein's inequality to find that

$$I \leq \sum_{\substack{|p-q| \leq 2 \\ r < p+1}} \|u_p\|_2 \|\nabla u_q\|_2 \|v_r\|_\infty \leq C \|u\|_{H^1} \sum_{r < p+1} \|u_p\|_2 \lambda_r \|v_r\|_2.$$

Splitting the derivative λ_r and applying Cauchy-Schwarz inequality gives us that

$$I \leq C \|u\|_{H^1} \underbrace{\left(\sum_{r < p+1} \lambda_r^{2-2\beta-2\epsilon} \lambda_p^{2\epsilon} \|u_p\|_2^2 \right)^{1/2}}_{=: I_A} \underbrace{\left(\sum_{r < p+1} \lambda_r^{2\beta+2\epsilon} \lambda_p^{-2\epsilon} \|v_r\|_2^2 \right)^{1/2}}_{=: I_B}$$

where $\epsilon \ll 1$ is chosen so that $\beta + \epsilon < 1$.

For the first sum, I_A^2 , we see that

$$\begin{aligned} I_A^2 &= \sum_{p=-1}^{\infty} \sum_{r=-1}^p \lambda_r^{2-2\beta-2\epsilon} \lambda_p^{2\epsilon} \|u_p\|_2^2 \\ &\leq C \sum_{p=-1}^{\infty} \lambda_p^{2-2\beta} \|u_p\|_2^2 \\ &\leq C \|u\|_{H^{1-\beta}}^2. \end{aligned}$$

For the second sum, I_B^2 , we must switch the order of summation as show below:

$$\begin{aligned}
I_B^2 &= \sum_{p=-1}^{\infty} \sum_{r=-1}^p \lambda_r^{2\beta+2\epsilon} \lambda_p^{-2\epsilon} \|v_r\|_2^2 \\
&= \sum_{r=-1}^{\infty} \sum_{p=r}^{\infty} \lambda_r^{2\beta+2\epsilon} \lambda_p^{-2\epsilon} \|v_r\|_2^2 \\
&\leq C \sum_{r=-1}^{\infty} \lambda_r^{2\beta} \|v_r\|_2^2 \\
&\leq C \|v\|_{H^\beta}^2.
\end{aligned}$$

To estimate II , we proceed as with I using Hölder's inequality followed by Bernstein's inequality to give us that

$$II \leq \sum_{\substack{|p-r|\leq 2 \\ q < r+1}} \|u_p\|_2 \|\nabla u_q\|_\infty \|v_r\|_2 \leq C \|u\|_{H^1} \sum_{q < r+1} \lambda_q \|u_q\|_2 \|v_r\|_2.$$

Similarly with I , we split the derivative λ_q and apply Cauchy-Schwarz to get

$$II \leq C \|u\|_{H^1} \underbrace{\left(\sum_{q < r+1} \lambda_q^{2-2\beta+2\epsilon} \lambda_r^{-2\epsilon} \|u_q\|_2^2 \right)^{1/2}}_{=: II_A} \underbrace{\left(\sum_{q < r+1} \lambda_q^{2\beta-2\epsilon} \lambda_p^{2\epsilon} \|v_r\|_2^2 \right)^{1/2}}_{=: II_B}$$

Switching the order of summation in II_A and proceeding as in I , we find that

$$II_A^2 \leq C \|u\|_{H^{1-\beta}}^2$$

$$II_B^2 \leq C \|v\|_{H^\beta}^2.$$

Finally, to estimate III , we again use Hölder's inequality followed by Bernstein's inequality to get

$$III \leq \sum_{\substack{|q-r| \leq 2 \\ p < r+1}} \|u_p\|_\infty \|\nabla u_q\|_2 \|v_r\|_2 \leq C \|u\|_{H^1} \sum_{p < r+1} \lambda_p \|u_p\|_2 \|v_r\|_2.$$

The rest of the estimates for III proceed exactly as in the case for II . \square

2.2.2 Gaining One Derivative

In this section, as well as the following section, we make *a priori* estimates. The calculations are done on the level of Galerkin approximations. One can then pass to the limit to obtain the stated bounds for the actual solutions.

Theorem 2.2.2. *Let $f \in H^\alpha$ and $u(0) := u_0 \in H^{\alpha+1}$ for some $\alpha \in (-1, 0)$. Then, there exists a solution $u(t)$ to Equation 1.2 so that $u \in L^\infty([0, \infty), H^{\alpha+1})$.*

Proof. Taking the inner product of Equation 1.2 with $A^{\alpha+1}u$ and integrating in space, we find that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{H^{\alpha+1}}^2 + \nu \|u\|_{H^{\alpha+2}}^2 &\leq \int_{\mathbb{T}^2} |B(u, u) \cdot A^{\alpha+1}u| \, dx + \int_{\mathbb{T}^2} |f \cdot A^{\alpha+1}u| \, dx \\ &\leq \|B(u, u)\|_{H^\alpha} \|u\|_{H^{\alpha+2}} + \|f\|_{H^\alpha} \|u\|_{H^{\alpha+2}}. \end{aligned}$$

Using Lemma 2.2.1 and Young's inequality, we find that

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{\alpha+1}}^2 + \nu \|u\|_{H^{\alpha+2}}^2 \leq C \|u\|_{H^1} \|u\|_{H^{\alpha+1}} \|u\|_{H^{\alpha+2}} + \|f\|_\alpha \|u\|_{H^{\alpha+2}} \quad (2.1)$$

$$\frac{d}{dt} \|u\|_{H^{\alpha+1}}^2 + \nu \|u\|_{H^{\alpha+2}}^2 \leq \frac{C}{\nu} \|u\|_{H^1}^2 \|u\|_{H^{\alpha+1}}^2 + \frac{2}{\nu} \|f\|_\alpha^2. \quad (2.2)$$

Dropping the $H^{\alpha+2}$ term, and using Gronwall, we have that for $t \geq t_0 \geq 0$,

$$\|u(t)\|_{H^{\alpha+1}}^2 \leq \left(\|u(t_0)\|_{H^{\alpha+1}}^2 + \frac{2}{\nu} \|f\|_{H^\alpha}^2 (t - t_0) \right) \exp \left(\frac{C}{\nu} \int_{t_0}^t \|u(s)\|_{H^1}^2 ds \right).$$

After using the embedding $H^1 \subset H^{\alpha+1}$, this becomes

$$\|u(t)\|_{H^{\alpha+1}}^2 \leq \left(\|u(t_0)\|_{H^1}^2 + \frac{2}{\nu} \|f\|_{H^\alpha}^2 (t - t_0) \right) \exp \left(\frac{C}{\nu} \int_{t_0}^t \|u(s)\|_{H^1}^2 ds \right).$$

By the energy inequality [Equation 1.3](#), we know that

$$\nu \int_{t_0}^t \|u(s)\|_{H^1}^2 ds \leq \|u_0\|_2^2 + \frac{1}{\nu} \|f\|_{H^{-1}}^2 (t - t_0).$$

Therefore, for $0 < \epsilon \leq t$, we have that

$$|\{t_0 \in [t - \epsilon, t] : \|u(t_0)\|_{H^1} \geq M\}| \leq \frac{1}{M^2} \left(\frac{1}{\nu} \|u_0\|_2^2 + \frac{\epsilon}{\nu^2} \|f\|_{H^{-1}}^2 \right)$$

Letting

$$M := \sqrt{\frac{2}{\epsilon} \left(\frac{1}{\nu} \|u_0\|_2^2 + \frac{\epsilon}{\nu^2} \|f\|_{H^{-1}}^2 \right)},$$

we find that

$$|\{t_0 \in [t - \epsilon, t] : \|u(t_0)\|_{H^1} \geq M\}| \leq \frac{\epsilon}{2}.$$

Therefore, there exists $t_0 \in [t - \epsilon, t]$ so that

$$\|u(t_0)\|_{H^1}^2 \leq \frac{2}{\epsilon} \left(\frac{1}{\nu} \|u_0\|_2^2 + \frac{\epsilon}{\nu^2} \|f\|_{H^{-1}}^2 \right).$$

So, the above argument shows us that for $t \geq \epsilon$,

$$\begin{aligned} \|u(t)\|_{H^{\alpha+1}}^2 &\leq \left(\frac{2}{\epsilon\nu} \|u_0\|_2^2 + \frac{2}{\nu^2} \|f\|_{H^{-1}}^2 + \frac{2\epsilon}{\nu} \|f\|_{H^\alpha}^2 \right) e^{C_0} \\ &\leq \left(\frac{2}{\epsilon\nu} \|u_0\|_{H^{\alpha+1}}^2 + \frac{2}{\nu^2} \|f\|_{H^\alpha}^2 + \frac{2\epsilon}{\nu} \|f\|_{H^\alpha}^2 \right) e^{C_0} \end{aligned}$$

for

$$C_0 := C \|u_0\|_{H^{\alpha+1}}^2 + \frac{C\epsilon}{\nu} \|f\|_{H^\alpha}^2.$$

This gives boundedness of u in $H^{\alpha+1}$ for all $t \geq \epsilon$ for any fixed $\epsilon > 0$. To show the boundedness of u in $H^{\alpha+1}$ for small t , we go back to equation [Equation 2.1](#). Using interpolation, we have that

$$\|u\|_{H^1} \leq C \|u\|_2^{\frac{1}{\alpha+2}} \|u\|_{H^{\alpha+2}}^{\frac{\alpha+1}{\alpha+2}}.$$

Thus, after using interpolation and Young's inequality on [Equation 2.1](#), we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|_{H^{\alpha+1}}^2 + \nu \|u\|_{H^{\alpha+2}}^2 \leq \frac{C}{\nu^{2\alpha+3}} \|u\|_2^2 \|u\|_{H^{\alpha+1}}^{2\alpha+4} + \frac{1}{\nu} \|f\|_\alpha^2$$

Dropping the $H^{\alpha+2}$ term, we may use nonlinear Gronwall to say that the $H^{\alpha+1}$ norm remains bounded for small time.

Combining this short-term bound with the previous long-term bound gives us that $u \in L^\infty([0, \infty), H^{\alpha+1})$, as required.

□

2.2.3 Gaining Two Derivatives

We combine the result of the previous section with an analyticity argument to show the uniform gain of two derivatives. That is, we prove the following theorem:

Theorem 2.2.3. *Let $f \in H^\alpha$ and $u(0) := u_0 \in H^{\alpha+2}$ for some $\alpha \in (-1, 0)$. Then, there exists a solution $u(t)$ to [Equation 1.2](#) so that $u \in L^\infty([0, \infty), H^{\alpha+2})$.*

To use analyticity arguments, we first need to complexify the spaces H^α as well as the Navier-Stokes equations themselves. First, the complexified space $H_\mathbb{C}^\alpha$ is given by

$$H_\mathbb{C}^\alpha := \{u = u_1 + iu_2 : u_1, u_2 \in H^\alpha\}$$

with the inner product defined via linearity as

$$\langle u_1 + iu_2, v_1 + iv_2 \rangle_{H_\mathbb{C}^\alpha} := \langle u_1, v_1 \rangle_{H^\alpha} + \langle u_2, v_2 \rangle_{H^\alpha} + i(\langle u_2, v_1 \rangle_{H^\alpha} - \langle u_1, v_2 \rangle_{H^\alpha}).$$

We will let the time $t := se^{i\theta}$. It is known, in this setting, that there exist unique, analytic solutions to the Galerkin system for complex time t in some neighborhood of the origin. Moreover, the restriction of these solutions to the real line agree with the usual Galerkin approximations.

Proof. To begin, we multiply the complexified Navier-Stokes equations by $e^{i\theta}$, take the inner product with $A^{\alpha+1}u$, and take the real part. This gives us

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \|u(se^{i\theta})\|_{H^{\alpha+1}}^2 + \nu \cos(\theta) \|u(se^{i\theta})\|_{H^{\alpha+2}}^2 \\ = \text{Real} \left[e^{i\theta} (\langle B(u, u), A^{\alpha+1}u \rangle_{H_{\mathbb{C}}^{-1}} + \langle f, A^{\alpha+1}u \rangle_{H_{\mathbb{C}}^{-1}}) \right]. \end{aligned}$$

Estimating the right-hand side, we first see, as in the real case, that

$$\left| \langle f, A^{\alpha+1}u \rangle_{H_{\mathbb{C}}^{-1}} \right| \leq \|f\|_{H^{\alpha}} \|u\|_{H^{\alpha+2}} \leq \frac{1}{\nu \cos(\theta)} \|f\|_{H^{\alpha}}^2 + \frac{\nu \cos(\theta)}{4} \|u\|_{H^{\alpha+2}}^2$$

Next, we use Lemma 2.2.1 to see that

$$\begin{aligned} \left| \langle B(u, u), A^{\alpha+1}u \rangle_{H_{\mathbb{C}}^{-1}} \right| &\leq \|B(u, u)\|_{H^{\alpha}} \|u\|_{H^{\alpha+2}} \\ &\leq C \|u\|_{H^{\alpha+1}}^2 \|u\|_{H^{\alpha+2}}^{\frac{2\alpha+3}{\alpha+2}} \\ &\leq \frac{C}{(\nu \cos(\theta))^{2\alpha+3}} \|u\|_{H^{\alpha+1}}^{2\alpha+6} + \frac{\nu \cos(\theta)}{4} \|u\|_{H^{\alpha+2}}^2. \end{aligned}$$

Thus, we obtain the Riccati-type inequality

$$\frac{d}{dt} \|u\|_{H^{\alpha+1}}^2 + \nu \cos(\theta) \|u\|_{H^{\alpha+2}}^2 \leq \frac{2}{\nu \cos(\theta)} \|f\|_{H^{\alpha}}^2 + \frac{C}{(\nu \cos(\theta))^{2\alpha+3}} \|u\|_{H^{\alpha+1}}^{2\alpha+6}.$$

This inequality shows us that for some time $\|u(t)\|_{H^{\alpha+1}} \leq M$ for some fixed $M > 0$ and $t \leq T := T(\|u_0\|_{H^{\alpha+1}}, \nu, f, \theta)$. Therefore, the solutions to the complexified Navier-Stokes equations extend to analytic solutions in a neighborhood D of the origin given by

$$D := \{t = se^{i\theta} : 0 < s < T, |\theta| < \pi/2\}.$$

Note that D is symmetric across the real axis, by construction. Also, note that within D , $\|u(t)\|_{H^{\alpha+1}} < M$.

Fix a compact set $K \subset D$. By Cauchy's formula with γ a circle in K of radius $r < d(K, \partial D)$, we have that

$$\frac{du}{dt}(t) = \frac{1}{2\pi i} \int_{\gamma} \frac{u(z)}{(z-t)^2} dz$$

for all $t \in K$. Taking the $H^{\alpha+1}$ norm of this equation, we get that

$$\|u_t(t)\|_{H^{\alpha+1}} \leq \frac{M}{r}.$$

Within K , we find that

$$\begin{aligned} \nu \|Au\|_{H^\alpha} &\leq \|u_t\|_{H^\alpha} + \|B(u, u)\|_{H^\alpha} + \|f\|_{H^\alpha} \\ &\leq \mu_1 \|u_t\|_{H^{\alpha+1}} + C \|u\|_{H^1} \|u\|_{H^{\alpha+1}} + \|f\|_{H^\alpha} \\ &\leq \mu_1 \|u_t\|_{H^{\alpha+1}} + C \|u\|_2^{\frac{1}{\alpha+2}} \|u\|_{H^{\alpha+1}} \|u\|_{H^{\alpha+2}}^{\frac{\alpha+1}{\alpha+2}} + \|f\|_{H^\alpha} \\ &\leq \mu_1 \|u_t\|_{H^{\alpha+1}} + \frac{C}{\nu^{\alpha+1}} \|u\|_2 \|u\|_{H^{\alpha+1}}^{\alpha+2} + \frac{\nu}{2} \|u\|_{H^{\alpha+2}} + \|f\|_{H^\alpha} \end{aligned} \tag{2.3}$$

where we used the Poincaré inequality along with Lemma 2.2.1 in the second line; we used interpolation in the second line; and we used Young's inequality in the final line.

Moving the $H^{\alpha+2}$ terms to the same side of the equation and noting that $\|u_t\|_{H^{\alpha+1}}$ is bounded by analyticity, $\|u\|_2$ is bounded by the energy inequality (Equation 1.3), and $\|u\|_{H^{\alpha+1}}$ is bounded by Theorem 2.2.2, we now know that $\|u\|_{H^{\alpha+2}}$ is bounded in the compact set K .

Using the uniform boundedness of $\|u\|_{H^{\alpha+1}}$ obtained in Theorem 2.2.2, we can rerun this argument with the same bounds at each starting point $t_0 \in [0, \infty)$. Thus, $\|u\|_{H^{\alpha+2}}$ is uniformly bounded in a complex neighborhood of the real axis. In particular, $\|u(t)\|_{H^{\alpha+2}} < C < \infty$ for each $t \in [0, \infty)$. \square

2.3 The H^{-1} Case

This section is devoted to proving the following theorem:

Theorem 2.3.1. *Let u be the unique solution to Equation 1.1 with $u(0) := u_0 \in H^1$. Then, there exists $T := T(u_0, f)$ so that $u \in L^\infty([0, t_0], H^1)$ for all $0 < t_0 < T$.*

The reason analyticity methods from the previous section fails are as follows: Note that when $\alpha = -1$, the inequality in Equation 2.3 becomes

$$\nu \|Au\|_{H^{-1}} \leq \|u_t\|_{H^{-1}} + C\|u\|_{H^1}\|u\|_{H^2} + \|f\|_{H^{-1}}$$

since interpolating between the H^1 norm between L^2 and $H^{\alpha+2}$ fails. We are now unable to use Young's inequality to split the H^1 and H^2 norms to move the $\|u\|_{H^2}$ terms to the left-hand side. Thus, we must use another tactic.

Again, note that in this section, we use the unprojected Navier-Stokes equations so that the use of harmonic analysis techniques are more easily followed.

Proof. Multiply the first equation in [Equation 1.1](#) with $(u_q)_q := \Delta_q (\Delta_q u)$ and integrate in space. This becomes

$$\frac{1}{2} \frac{d}{dt} \|u_q(t)\|_2^2 + \nu \lambda_q^2 \|u_q(t)\|_2^2 = \int_{\mathbb{T}^2} (u \cdot \nabla u) \cdot (u_q)_q dx + \int_{\mathbb{T}^2} f_q \cdot u_q dx.$$

Apply Cauchy-Swartz and Young's inequality to the second term on the right-hand side. This gives

$$\frac{1}{2} \frac{d}{dt} \|u_q(t)\|_2^2 + \nu \lambda_q^2 \|u_q(t)\|_2^2 \leq \int_{\mathbb{T}^2} u \cdot \nabla u \cdot (u_q)_q dx + \frac{\nu \lambda_q^2}{4} \|u_q(t)\|_2^2 + \frac{1}{\nu \lambda_q^2} \|f_q\|_2^2.$$

For the first term on the right-hand side, use Hölder's inequality.

$$\int_{\mathbb{T}^2} u \cdot \nabla u \cdot (u_q)_q \leq C \|u\|_r \|u\|_{H^1} \|u_q\|_\rho \quad \text{where} \quad \frac{1}{r} + \frac{1}{\rho} = \frac{1}{2}.$$

Assume that $2 < r < \infty$. Then, Applying the Sobolev and Bernstein inequalities give us that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_q(t)\|_2^2 + \frac{3}{4} \nu \lambda_q^2 \|u_q(t)\|_2^2 &\leq C \|u(t)\|_r \|u(t)\|_{H^1} \|u_q(t)\|_\rho + \frac{1}{\nu \lambda_q^2} \|f_q\|_2^2 \\ &\leq C \|u(t)\|_{H^1}^2 \lambda_q^{(\rho-2)/2\rho} \|u_q(t)\|_2 + \frac{1}{\nu \lambda_q^2} \|f_q\|_2^2 \end{aligned}$$

For simplicity of notation, let $p \in (0, 1)$ be given by $p := (\rho - 2)/2\rho$. Then, using Young's inequality, we find that

$$\frac{d}{dt} \|u_q(t)\|_2^2 + \nu \lambda_q^2 \|u_q(t)\|_2^2 \leq \frac{C}{\nu \lambda_q^{2-2p}} \|u(t)\|_{H^1}^4 + \frac{2}{\nu \lambda_q^2} \|f_q\|_2^2.$$

Next, we apply Duhamel's principle. This gives

$$\begin{aligned} \|u_q(t)\|_2^2 &\leq e^{-\nu \lambda_q^2 t} \|u_q(0)\|_2^2 + \frac{2}{\nu^2} \lambda_q^{-4} \|f_q\|_2^2 \left[1 - e^{-\nu \lambda_q^2 t}\right] \\ &\quad + \frac{C}{\nu} \int_0^t e^{\nu \lambda_q^2 (s-t)} \lambda_q^{2p-2} \|u(s)\|_{H^1}^4 ds \end{aligned} \tag{2.4}$$

Multiply this through by λ_q^2 and sum in q . We find that

$$\|u(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \frac{2}{\nu^2} \|f\|_{H^{-1}}^2 + \frac{C}{\nu} \int_0^t \sum_q e^{\nu \lambda_q^2 (s-t)} \lambda_q^{2p} \|u(s)\|_{H^1}^4 ds. \tag{2.5}$$

Remark 2.3.2. It is worthwhile to note that [Equation 2.4](#), obtained via integrating the force term, can also be obtained using a nonautonomous force. To obtain this, we need $f \in L_{loc}^\infty([0, \infty), H^{-1})$ with a “dominating function in Fourier space.” We mean that there exists a $g \in H^{-1}$ with

$$\|f_q(t)\|_2 \leq \|g_q\|_2$$

for all $t \geq 0$ and $q \geq Q$ for some finite integer $Q \geq -1$.

Taking a closer look at the final integral, consider the sum

$$\sum_q e^{\nu\lambda_q^2(s-t)} \lambda_q^{2p}. \quad (2.6)$$

Fix a constant $\gamma > 0$ to be determined later. Then, let $Q_0 > 1$ be chosen so that $\ln \lambda_q \leq (\lambda_q)^\gamma$ for all $q \geq Q_0$. Then, define

$$Q(s) := \min \left\{ q \geq Q_0 : e^{\nu\lambda_q^2(s-t)} \leq \lambda_q^{-2p-1} \right\}$$

$$\Lambda(s) := \lambda_{Q(s)}.$$

We estimate the integral as follows

$$\int_0^t \sum_q e^{\nu\lambda_q^2(s-t)} \lambda_q^{2p} \|u(s)\|_{H^1}^4 ds \leq \underbrace{\int_{[0,t] \cap 1_{Q(s) \leq Q_0}}}_{:=I} + \underbrace{\int_{[0,t] \cap 1_{Q(s) > Q_0}}}_{:=II}$$

where 1_E is the characteristic function of the set E .

For I , we have that

$$\begin{aligned} I &\leq \int_0^t \|u(s)\|_{H^1}^4 \left(\sum_{q \leq Q_0} e^{\nu\lambda_q^2(s-t)} \lambda_q^{2p} + \sum_{q > Q_0} \lambda_q^{-2} \right) ds \\ &\leq \int_0^t \|u(s)\|_{H^1}^4 (Q_0 \lambda_{Q_0}^{2p} + 1) ds \\ &\leq C \int_0^t \|u(s)\|_{H^1}^4 ds. \end{aligned}$$

For II , we see that for a fixed s , Equation 2.6 can be estimated by

$$\sum_{q \leq Q(s)} e^{\nu \lambda_q^2(s-t)} \lambda_q^{2p} + \sum_{q > Q(s)} \lambda_q^{-1} \leq Q(s) \Lambda(s)^{2p} + 1.$$

By the definition of Λ , $2^{-1}\Lambda$ satisfies

$$2^{2p+1} \Lambda^{-2p-1} \leq e^{\nu 2^{-2} \Lambda^2(s-t)} \iff \Lambda^2 \leq \frac{8(p+1/2)}{\nu(t-s)} \ln \Lambda.$$

But, $\ln \Lambda \leq \Lambda^\gamma$ by definition. Thus,

$$\Lambda^{2-\gamma} \leq \frac{8(p+1/2)}{\nu(t-s)}.$$

for $\gamma < 2$.

Proceeding in much the same way as we did with I , we see that

$$\begin{aligned} II &\leq \int_0^t \|u(s)\|_{H^1}^4 \left(\sum_{q \leq Q(s)} \Lambda(s)^{2p} + \sum_{q > Q(s)} \lambda_q^{-2} \right) ds \\ &\leq C \int_0^t \|u(s)\|_{H^1}^4 (Q(s) \Lambda(s)^{2p} + 1) ds \\ &\leq C \int_0^t \|u(s)\|_{H^1}^4 (\ln \Lambda(s) \Lambda(s)^{2p} + 1) ds \\ &\leq C \int_0^t \left(\frac{1}{(\nu(t-s))^{(2p+\gamma)/(2-\gamma)}} + 1 \right) \|u(s)\|_{H^1}^4 ds. \end{aligned}$$

Putting these estimates together with [Equation 2.5](#), we have that

$$\begin{aligned} \|u(t)\|_{H^\alpha}^2 &\leq \|u_0\|_{H^\alpha}^2 + \frac{2}{\nu^2} \|f\|_{H^{\alpha-2}}^2 \\ &\quad + \frac{C}{\nu} \int_0^t \left(\frac{1}{(\nu(t-s))^{(2p+\gamma)/(2-\gamma)}} + 1 \right) \|u(s)\|_{H^1}^4 ds. \end{aligned}$$

Next, let $p := 1/4$ and $\gamma := 1/2$. Then, $(2p + \gamma)/(2 - \gamma) = 2/3$. Note that this choice of γ means that $Q_0 = 2$. This gives us that

$$\begin{aligned} \|u(t)\|_{H^1}^2 &\leq \|u_0\|_{H^1}^2 + \frac{2}{\nu^2} \|f\|_{H^{-1}}^2 \\ &\quad + \frac{C}{\nu} \int_0^t \left((\nu(t-s))^{-2/3} + 1 \right) \|u(s)\|_{H^1}^4 ds. \end{aligned}$$

An application of nonlinear Gronwall leads to the desired result. □

CHAPTER 3

PULLBACK ATTRACTORS

3.1 Generalized Evolutionary System

3.1.1 Preliminaries

We start with the setup as it first appeared in (13). So, let $(X, d_s(\cdot, \cdot))$ be a metric space with a metric d_s known as the strong metric on X . Let d_w be another metric on X satisfying the following conditions:

1. X is d_w compact.
2. If $d_s(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$ for some $u_n, v_n \in X$ then $d_w(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$.

As justified by property (2), we will call d_w the weak metric on X . Denote by \overline{A}^\bullet the closure of the set $A \subseteq X$ in the topology generated by d_\bullet . Note that any strongly compact set (d_s -compact) is also weakly compact (d_w -compact), and any weakly closed set (d_w -closed) is also strongly closed (d_s -closed).

Let $C([a, b]; X_\bullet)$, where $\bullet = s$ or w , be the space of d_\bullet -continuous X -valued functions on $[a, b]$ endowed with the metric

$$d_{C([a, b], X_\bullet)}(u, v) := \sup_{t \in [a, b]} d_\bullet(u(t), v(t)).$$

Let also $C([a, \infty), X_\bullet)$ be the space of all d_\bullet -continuous X -valued functions on $[a, \infty)$ endowed with the metric

$$d_{C([a, \infty), X_\bullet)}(u, v) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\sup\{d_\bullet(u(t), v(t)) : a \leq t \leq a + n\}}{1 + \sup\{d_\bullet(u(t), v(t)) : a \leq t \leq a + n\}}.$$

Let

$$\mathcal{T} := \{I \subset \mathbb{R} : I = [T, \infty) \text{ for some } T \in \mathbb{R}\} \cup \{(-\infty, \infty)\},$$

and for each $I \in \mathcal{T}$, let $\mathcal{F}(I)$ denote the set of all X -valued functions on I .

Definition 3.1.1. A map \mathcal{E} that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subset \mathcal{F}(I)$ will be called a generalized evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}([\tau, \infty)) \neq \emptyset$ for each $\tau \in \mathbb{R}$.
2. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subseteq \mathcal{E}(I_2)$ for each $I_1, I_2 \in \mathcal{T}$ with $I_2 \subseteq I_1$.
3. $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[T, \infty)} \in \mathcal{E}([T, \infty)) \ \forall T \in \mathbb{R}\}.$

We will refer to $\mathcal{E}(I)$ as the set of all trajectories on the time interval I . Trajectories in $\mathcal{E}((-\infty, \infty))$ are called complete. Next, for each $t \geq s \in \mathbb{R}$ and $A \subseteq X$, we define the map

$$P(t, s) : \mathcal{P}(X) \rightarrow \mathcal{P}(X),$$

$$P(t, s)A := \{u(t) : u(s) \in A, u \in \mathcal{E}([s, \infty))\}.$$

We get, for each $t \geq s \geq r \in \mathbb{R}$ and $A \subseteq X$

$$P(t, r)A \subset P(t, s)P(s, r)A.$$

We will also study generalized evolutionary systems endowed with the following properties:

A1 : $\mathcal{E}([s, \infty))$ is compact in $C([s, \infty), X_w)$ for each $s \in \mathbb{R}$.

A2 : (Energy Inequality) Let X be a set in some Banach space H satisfying the Radon-Riesz Property (see below) with norm $\|\cdot\|_H$ so that $d_s(x, y) = \|x - y\|_H$ for each $x, y \in X$, and assume that d_w induces the weak topology on X . Assume that for each $\epsilon > 0$ and each $s \in \mathbb{R}$ there is a $\delta := \delta(\epsilon, s)$ so that for every $u \in \mathcal{E}([s, \infty))$ and $t > s \in \mathbb{R}$

$$\|u(t)\|_H \leq \|u(t_0)\|_H + \epsilon$$

for t_0 a.e. in $(t - \delta, t)$.

A3 : (Strong Convergence a.e.) Let $u, u_n \in \mathcal{E}([s, \infty))$ be so that $u_n \rightarrow u$ in $C([s, t], X_w)$ for some $s \leq t \in \mathbb{R}$. Then, $u_n(t_0) \xrightarrow{d_s} u(t_0)$ for a.e. $t_0 \in [s, t]$.

Remark 3.1.2. A Banach space H with norm $\|\cdot\|_H$ satisfies the Radon-Riesz property if $x_n \rightarrow x$ in norm if and only if $x_n \rightarrow x$ weakly and

$$\lim_{n \rightarrow \infty} \|x_n\|_H = \|x\|_H.$$

Often, X will be a closed, bounded subset of a separable, reflexive Banach space. By the Troyanski Renorming Theorem (33), we can assume that our norm makes H a locally uniformly convex space, at which point the Radon-Riesz property is satisfied.

To see how this relates back to the classical setting, let H be a separable, reflexive Banach space, which we call the phase space. Let $S(\cdot, \cdot)$ be a process on H . That is, for each $t \geq s$, we have that $S(t, s) : H \rightarrow H$ with the following properties:

$$S(t, s) = S(t, r)S(r, s)$$

$$S(t, t) = Id_H$$

for any $t \geq r \geq s$. A trajectory on H is a mapping $u : [s, \infty) \rightarrow H$ so that $u(t) = S(t, s)u(s)$ for each $t \geq s$. A set $X \subseteq H$ will be called absorbing if, for each $s \in \mathbb{R}$ and $B \subseteq H$ bounded, there is $t_0 := t_0(B, s)$ so that for $t \geq t_0$,

$$S(t, s)B \subseteq X.$$

If there exists a closed absorbing ball X , then we call the process S dissipative.

If S is dissipative, and we can ensure that it is dissipative arbitrarily far in the past (that is, for each $s \in \mathbb{R}$, there is a trajectory $u : [s, \infty) \rightarrow X$), then studying the asymptotic pullback dynamics of S on H amounts to studying the asymptotic pullback dynamics of S on X . That is, using the definition of the pullback attractor Definition 3.1.4, one can show that if X has a pullback attractor $\mathcal{A}(t)$ (by restricting S to X), then $\mathcal{A}(t)$ is a pullback attractor for H . Note

that since H is a separable reflexive Banach space, both the strong and weak topologies on X are metrizable. We define a generalized evolutionary system on X by

$$\mathcal{E}([s, \infty)) := \{u(\cdot) : u(t) = S(t, s)u(s), u(t) \in X \ \forall t \geq s\}.$$

In particular, this also gives us the following characterization for each $t \gg s \in \mathbb{R}$ and $A \subseteq X$

$$P(t, s)A = S(t, s)A.$$

As we will see later, by Theorem 3.1.10 and Theorem 3.1.11 that the weak pullback attractor exists for \mathcal{E} and

$$\mathcal{A}_w(t) = \Omega_w(X, t) = \bigcap_{s \leq t} \overline{\bigcup_{r \leq s} S(t, r)X}^w.$$

Moreover, if we know that A1 holds (that $\mathcal{E}([s, \infty))$ is compact in $C([s, \infty), X_w)$ for each $s \in \mathbb{R}$), then we get, using Theorem 4.1.8 that

$$\mathcal{A}_w(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

Finally, if we also have that A2 and A3 also hold and complete trajectories are strongly continuous $\mathcal{E}((-\infty, \infty)) \subseteq C((-\infty, \infty), X)$, then by Corollary 4.2.3, \mathcal{E} possesses a strongly compact, strong pullback attractor $\mathcal{A}_s(t)$. In fact, by Corollary 4.1.3,

$$\mathcal{A}_s(t) = \mathcal{A}_w(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

3.1.2 Pullback Attracting Sets, Ω -limits, and Pullback Attractors

Let \mathcal{E} be a fixed generalized evolutionary system on a metric space X . For $A \subseteq X$ and $r > 0$, denote $B_\bullet(A, r) := \{x \in X : d_\bullet(x, A) < r\}$, where

$$d_\bullet(x, A) := \inf_{a \in A} d_\bullet(x, a), \quad \bullet = s, w.$$

A family of sets $A(t) \subseteq X$, $t \in \mathbb{R}$ (uniformly) pullback attracts a set $B \subseteq X$ in the d_\bullet -metric ($\bullet = s, w$) if for any $\epsilon > 0$, there exists an $s_0 := s_0(B, \epsilon, t) < t \in \mathbb{R}$ so that for $s \leq s_0$,

$$P(t, s)B \subseteq B_\bullet(A(t), \epsilon).$$

Definition 3.1.3. A family of sets $A(t) \subseteq X$ for $t \in \mathbb{R}$ are d_\bullet -pullback attracting ($\bullet = s, w$) if they pullback attract X in the d_\bullet -metric.

Definition 3.1.4. A family of sets $\mathcal{A}_\bullet(t) \subseteq X$ is the d_\bullet -pullback attractor of X if for each t , $\mathcal{A}_\bullet(t)$ is d_\bullet -closed, d_\bullet -pullback attracting and $\mathcal{A}_\bullet(t)$ is minimal with respect to these properties.

Next, we define the concept of the pullback Ω_\bullet -limit.

Definition 3.1.5. For each $A \subseteq X$ and $t \in \mathbb{R}$, we define the pullback Ω -limit ($\bullet = s, w$) of A as

$$\Omega_\bullet(A, t) := \bigcap_{s \leq t} \overline{\bigcup_{r \leq s} P(t, r)A}^\bullet.$$

Equivalently, we have that $x \in \Omega_\bullet(A, t)$ if there exist sequences $s_n \rightarrow -\infty$, $s_n \leq t$, $x_n \in P(t, s_n)A$ so that $x_n \rightarrow x$ in the d_\bullet -metric. We now present some basic properties of Ω_\bullet .

Lemma 3.1.6. *Let $A \subseteq X$ and $t \in \mathbb{R}$. Then,*

1. $\Omega_{\bullet}(A, t)$ is d_{\bullet} -closed ($\bullet = s, w$).
2. $\Omega_s(A, t) \subseteq \Omega_w(A, t)$.
3. *If $\Omega_w(A, t)$ is strongly compact and uniformly, strongly, pullback attracts A , then $\Omega_s(A, t) = \Omega_w(A, t)$.*

Proof. Part 1 is obvious from the definition. For part 2, let $x \in \Omega_s(A, t)$. Then, there exists sequences $s_n \leq t$, $s_n \rightarrow -\infty$ and $x_n \in P(t, s_n)A$ with $x_n \xrightarrow{d_s} x$. But then, $x_n \xrightarrow{d_w} x$ and $x \in \Omega_w(A, t)$.

Now, suppose that $\Omega_w(A, t)$ is strongly compact and d_s -pullback attracts A . Let $x \in \Omega_w(A, t)$. Then, by definition, there are sequences $s_n \leq t$, $s_n \rightarrow -\infty$ and $x_n \in P(t, s_n)A$ with $x_n \xrightarrow{d_w} x$. Since $\Omega_w(A, t)$ d_s -pullback attracts A , there exists a sequence $y_n \in \Omega_w(A, t)$ with $d_s(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. Because $d_s(x_n, y_n) \rightarrow 0$, $d_w(x_n, y_n) \rightarrow 0$. Since $\Omega_w(A, t)$ is compact, there is some subsequence $y_{n_k} \xrightarrow{d_s} y$ for some $y \in \Omega_w(A, t)$. But then, $x_{n_k} \xrightarrow{d_s} y$ which means that $x_{n_k} \xrightarrow{d_w} y$. Thus, $y = x$ which means that $x_{n_k} \xrightarrow{d_s} x$. That is, $x \in \Omega_s(A, t)$. \square

Lemma 3.1.7. *Let $A(t)$ be a family of d_{\bullet} -closed, d_{\bullet} -pullback attracting sets ($\bullet = s, w$). Then $\Omega_{\bullet}(X, t) \subseteq A(t)$.*

Proof. Let $x \in \Omega_{\bullet}(X, t)$. Then, there exist sequences $s_n \leq t$, $s_n \rightarrow -\infty$, and $x_n \in P(t, s_n)X$ with $x_n \xrightarrow{d_{\bullet}} x$. Since $A(t)$ is d_{\bullet} -pullback attracting, there exists $a_n \in A(t)$ with $d_{\bullet}(x_n, a_n) \rightarrow 0$ as $n \rightarrow \infty$. But, $x_n \xrightarrow{d_{\bullet}} x$ which gives us that $a_n \xrightarrow{d_{\bullet}} x$. Since $A(t)$ is d_{\bullet} -closed, $x \in A(t)$. \square

Now, we are ready to show that if the d_{\bullet} -pullback attractor exists, then it is unique.

Theorem 3.1.8. *If the pullback attractor $\mathcal{A}_\bullet(t)$ exists ($\bullet = s, w$), then*

$$\mathcal{A}_\bullet(t) = \Omega_\bullet(X, t).$$

Proof. By the above lemmas, $\Omega_\bullet(X, t) \subseteq \mathcal{A}_\bullet(t)$. Now, let $x \in \mathcal{A}_\bullet(t) \setminus \Omega_\bullet(X, t)$. Then, there exists $\epsilon > 0$ and $s_0 \leq t$ so that for $s \leq s_0$,

$$P(t, s)X \cap B_\bullet(x, \epsilon) = \emptyset. \quad (3.1)$$

Otherwise, for each n and $t - n \leq 0$ there exists $s_n \leq t - n$ with

$$x_n \in P(t, s_n)X \cap B_\bullet(x, 1/n) \neq \emptyset.$$

But, then $x_n \xrightarrow{d_\bullet} x$, and $x \in \Omega_\bullet(X, t)$. This is a contradiction. Thus, Equation 3.1 holds. In this case, $\mathcal{A}_\bullet(t) \setminus B_\bullet(x, \epsilon)$ is a strict subset of $\mathcal{A}_\bullet(t)$ which is d_\bullet -closed d_\bullet -pullback attracting. This contradicts the definition of $\mathcal{A}_\bullet(t)$. \square

An immediate consequence of Theorem 3.1.8 and Lemma 3.1.7 is the following:

Corollary 3.1.9. *The pullback attractor $\mathcal{A}_\bullet(t)$ exists if and only if $\Omega_\bullet(X, t)$ is a d_\bullet -pullback attracting set.*

Next, we study the structure of $\Omega_w(A, t)$ for some $A \subseteq X$ and $t \in \mathbb{R}$.

Theorem 3.1.10. *Let $A \subseteq X$ be such that for each $t \in \mathbb{R}$ and $r \leq t$, there is some $u \in \mathcal{E}([r, \infty))$ with $u(t) \in A$. Then, $\Omega_w(A, t)$ is a nonempty, weakly compact set. Moreover, $\Omega_w(A, t)$ weakly pullback attracts A .*

Proof. Due to the assumptions on A , we have that $P(t, r)A \neq \emptyset$. Also, due to the fact that X is weakly compact, we have that

$$W(s) := \overline{\bigcup_{r \leq s} P(t, r)A}^w$$

is nonempty and weakly compact for each $s \leq t$. Moreover, for $s_0 \leq s_1 \leq t$, $W(s_0) \subset W(s_1)$.

Thus, by Cantor's intersection theorem,

$$\Omega_w(A, t) = \bigcap_{s \leq t} W(s)$$

is a nonempty weakly compact set.

To see that $\Omega_w(A, t)$ weakly pullback attracts A , suppose for contradiction that it doesn't.

Then, there exists some $\epsilon > 0$ and a sequence $s_n \rightarrow -\infty$, $s_n \leq t$ with

$$P(t, s_n)A \cap B_w(\Omega_w(A, t), \epsilon)^c \neq \emptyset.$$

Therefore,

$$K_n := \overline{\bigcup_{r \leq s_n} P(t, r)A}^w \cap B_w(\Omega_w(A, t), \epsilon)^c \neq \emptyset.$$

Passing to a subsequence if necessary and reindexing, we can assume that the s_n 's are monotonically decreasing. Thus, we get a decreasing sequence of nonempty weakly compact sets. Again, by Cantor's intersection theorem, we have that $x \in \cap_n K_n \neq \emptyset$. That is,

$$x \in \bigcap_{s_n \leq t} \overline{\bigcup_{r \leq s_n} P(t, r)A}^w = \Omega_w(A, t)$$

This contradicts the definition of the K_n 's. □

Using the above results, we have the following:

Theorem 3.1.11. *Every generalized evolutionary system possesses a weak pullback attractor $\mathcal{A}_w(t)$. Moreover, if the strong pullback attractor $\mathcal{A}_s(t)$ exists, then $\overline{\mathcal{A}_s(t)}^w = \mathcal{A}_w(t)$.*

Proof. Due to Theorem 3.1.10, $\Omega_w(X, t)$ is a non-empty weakly closed, weakly pullback attracting set. Therefore, by Theorems 3.1.9 and 3.1.8, $\mathcal{A}_w(t) = \Omega_w(X, t)$ is the weak pullback attractor.

Now, suppose the strong pullback attractor $\mathcal{A}_s(t)$ exists. Then, by Theorem 3.1.8, $\mathcal{A}_s(t) = \Omega_s(X, t)$. Then, since $\mathcal{A}_s(t)$ strongly pullback attracts X , we have that $\mathcal{A}_s(t)$ must weakly pullback attract X . If not, there is some $\epsilon > 0$ and a sequence $s_n \rightarrow -\infty$ with

$$x_n \in P(t, s_n)X \cap B_w(\mathcal{A}_s(t), \epsilon)^c \neq \emptyset.$$

Since $\mathcal{A}_s(t)$ is strongly pullback attracting, there is a sequence $a_n \in \mathcal{A}_s(t)$ with $d_s(x_n, a_n) \rightarrow 0$.

But, then $d_w(x_n, a_n) \rightarrow 0$. Then, for n sufficiently large, $x_n \in B_w(\mathcal{A}_s(t), \epsilon)$ which contradicts

the choice of x_n . In addition, $\overline{\mathcal{A}_s(t)}^w$ is weakly closed and weakly pullback attracting. thus, by Lemma 3.1.7, $\mathcal{A}_w(t) = \Omega_w(X, t) \subseteq \overline{\mathcal{A}_s(t)}^w$. Finally, by Lemma 3.1.6,

$$\overline{\mathcal{A}_s(t)}^w = \overline{\Omega_s(X, t)}^w \subseteq \Omega_w(X, t) = \mathcal{A}_w(t).$$

□

3.2 Examples

3.2.1 A Single Trajectory

For our first example, let our generalized evolutionary system on an arbitrary phase space X consist of a single trajectory $u \in \mathcal{C}((-\infty, \infty))$ and all of its restrictions, $\mathcal{C}([s, \infty)) = \{u|_{[s, \infty)}\}$.

Then, we have that

$$P(t, s)X = \{u(t)\}$$

is a single point. Therefore, we have that the strong and weak pullback attractors both exist.

Moreover,

$$\mathcal{A}_s(t) = \mathcal{A}_w(t) = \{u(t)\}.$$

3.2.2 An Abstract Example on $\ell^2(\mathbb{Z})$

For our second example, let X be the unit ball in $\ell^2(\mathbb{Z})$. Let the strong metric on $\ell^2(\mathbb{Z})$ be the metric induced by the norm on $\ell^2(\mathbb{Z})$. That is, given by

$$d_s(x, y) = \|x - y\|_{\ell^2(\mathbb{Z})} = \sqrt{\sum_{n \in \mathbb{Z}} (x_n - y_n)^2}.$$

In a similar fashion, X with the weak topology is metrizable using the weak metric

$$d_w(x, y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

Now, consider the following complete trajectory on X : Let $n \in \mathbb{Z}$. Then, let

$$u(t) := \frac{(1 + n - t)e_n + (t - n)e_{n+1}}{\|(1 + n - t)e_n + (t - n)e_{n+1}\|_{\ell^2(\mathbb{Z})}}$$

for $t \in [n, n + 1]$ and $\{e_n\}$ the standard basis vectors in $\ell^2(\infty)$. That is, we interpolate between the basis vectors and normalize onto the boundary of the unit ball. Next, let $\mathcal{E}((-\infty, \infty)) := \{u(\cdot + r) : r \in \mathbb{R}\}$. That is, the above complete trajectory and all of its shifts. As in the previous example, we complete our definition of a generalized evolutionary system by letting $\mathcal{E}([s, \infty)) := \{u|_{[s, \infty)} : u \in \mathcal{E}((-\infty, \infty))\}$. As we will see in Section 4.3, this turns \mathcal{E} into an autonomous evolutionary system. Therefore, we find that

$$P(t, s)X = \{u(r) : r \in \mathbb{R}\}.$$

This is strongly closed but not weakly closed. Thus, we find that

$$\mathcal{A}_s(t) = \{u(r) : r \in \mathbb{R}\} \quad \text{and} \quad \mathcal{A}_w(t) = \overline{\mathcal{A}_s(t)}^w = \{u(r) : r \in \mathbb{R}\} \cup \{0\}$$

for each $t \in \mathbb{R}$. In particular, the weak and strong pullback attractors are not equal.

3.2.3 The Heat Equation

For our next example, let X be the unit ball in $L^2(\mathbb{R})$. The strong metric on X is the metric induced by the norm on $L^2(\mathbb{R})$. That is, for any $f, g \in L^2(\mathbb{R})$,

$$d_s(f, g) := \|f - g\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x) - g(x)|^2 dx \right)^{1/2}.$$

For the weak metric, we first choose any countable dense subset ϕ_n for $L^2(\mathbb{R})$ for $n \in \mathbb{N}$. For example, one could use wavelets as an orthonormal basis, as is explained in (34). Then, the weak metric on X is given by

$$d_w(f, g) := \sum_{k \in \mathbb{N}} \frac{1}{2^k} \frac{|\langle f, \phi_k \rangle_{L^2(\mathbb{R})} - \langle g, \phi_k \rangle_{L^2(\mathbb{R})}|}{1 + |\langle f, \phi_k \rangle_{L^2(\mathbb{R})} - \langle g, \phi_k \rangle_{L^2(\mathbb{R})}|}.$$

Now, consider the heat equation on X . That is, for some starting time $s \in \mathbb{R}$,

$$\begin{cases} u_t = u_{xx} \\ u(s) = f(x) \end{cases} \quad (3.2)$$

for some $f \in X$. Then, using the Fourier transform,

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{ix \cdot \xi} dx,$$

we find that a solution to Equation 3.2 is given by

$$\hat{u}(\xi, t) = e^{\xi^2(s-t)} \hat{f}(\xi). \quad (3.3)$$

Note that by Plancherel's theorem, we may work exclusively in Fourier space. Define a generalized evolutionary system on X via

$$\mathcal{E}([s, \infty)) := \{u : u \text{ is a solution to Equation 3.2}\}.$$

We will see that the weak pullback attractor $\mathcal{A}_w(t)$ is given by the single point $\{0\}$ for each t .

On the other hand, the strong pullback attractor $\mathcal{A}_s(t)$ does not exist.

To see this, we first note that for fixed $t \in \mathbb{R}$, $\|u(t)\|_{L^2(\mathbb{R})} \rightarrow 0$ as $s \rightarrow -\infty$. This gives us the candidate weak and strong pullback attractor $\{0\}$. In the weak metric, this is the pullback attractor. On the other hand, by Theorem 3.1.11, if the strong pullback attractor exists, it must be the case that $\mathcal{A}_w(t) = \overline{\mathcal{A}_s(t)}^w = \{0\}$. So, the only possibility for the strong pullback attractor is $\mathcal{A}_s(t) = \{0\}$. However, we fail to have uniform convergence in the strong metric. By definition, if $\{0\}$ was the strong pullback attractor, then, for any $\epsilon > 0$, there is an $s_0 \leq t$ with

$$P(t, s)X \subseteq B_s(\{0\}, \epsilon)$$

for each $s \leq s_0$. So, let $\epsilon := 1/2$ and let $s_0 \leq t$ be given. Then, consider $f \in L^2(\mathbb{R})$ with $\|f\|_{L^2(\mathbb{R})} = 1$ and $\text{supp}(\hat{f}) \subseteq \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$ for some $j \in \mathbb{Z}$ to be determined later.

Then, we see that

$$\begin{aligned}
\|u(\xi, t)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} e^{2\xi^2(s_0-t)} \hat{f}(\xi)^2 d\xi \\
&= \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \exp(2\xi^2(s_0-t)) \hat{f}(\xi)^2 d\xi \\
&\geq \exp(2 \cdot 2^{2j-2}(s_0-t)) \int_{2^{j-1} \leq |\xi| \leq 2^{j+1}} \hat{f}(\xi)^2 d\xi \\
&= \exp(2^{2j-1}(s_0-t)) \|f\|_{L^2(\mathbb{R})}^2 \\
&= \exp(2^{2j-1}(s_0-t)).
\end{aligned}$$

Therefore, we find that

$$\|u(\xi, t)\|_{L^2(\mathbb{R})} \geq \exp(2^{2j-2}(s_0-t)).$$

This is greater than or equal to $\epsilon := 1/2$ provided that

$$j \leq \frac{1}{2} \left(\log_2 \left(\frac{\ln(2)}{t-s_0} \right) + 2 \right).$$

Therefore, the convergence to 0 is not uniform, and no strong pullback attractor exists.

3.2.4 A Phase Space that is not Weakly Compact

Finally, a simple example showing the importance of the compactness of X in the weak topology. Let $X := \mathbb{R}$ with the weak and strong metrics both given by $d_s(x, y) := d_w(x, y) := |x - y|$ for any $x, y \in \mathbb{R}$. For each $s \in \mathbb{R}$, define

$$\mathcal{E}([s, \infty)) := \{u(t) := t - s\}.$$

Then, we have that for some $t \in \mathbb{R}$ and some $s \leq t$

$$P(t, s)X = \{u(t) : u(s) \in X, u \in \mathcal{E}([s, \infty))\} = \{t - s\}.$$

But, as $s \rightarrow -\infty$, the limit does not exist. Thus, the weak and strong pullback attractors do not exist.

3.3 Existence of a Strong Pullback Attractor

Definition 3.3.1. A generalized evolutionary system is pullback asymptotically compact if for any $t \in \mathbb{R}$, $s_n \rightarrow -\infty$ with $s_n \leq t$, and any $x_n \in P(t, s_n)X$, the sequence $\{x_n\}$ is relatively strongly compact.

Theorem 3.3.2. *Let \mathcal{E} be pullback asymptotically compact. Let $A \subseteq X$ be so that for each $t \in \mathbb{R}$ and $r \leq t$, there is some $u \in \mathcal{E}([r, \infty))$ with $u(t) \in A$. Then, $\Omega_s(A, t)$ is a nonempty strongly compact set which strongly pullback attracts A . Moreover, $\Omega_s(A, t) = \Omega_w(A, t)$.*

Proof. By Theorem 3.1.8, our assumptions on A imply that $\Omega_w(A, t) \neq \emptyset$. We will see that $\Omega_w(A, t)$ strongly pullback attracts A . Suppose it does not. Then, there is some $\epsilon > 0$ and a sequence $s_n \rightarrow -\infty$ with

$$x_n \in P(t, s_n)A \cap B_s(\Omega_w(A, t), \epsilon)^c \neq \emptyset.$$

Since X is pullback asymptotically compact, this sequence has a convergent subsequence. After passing to a subsequence and dropping a subindex, we have that $x_n \xrightarrow{d_s} x$. But then, $x_n \xrightarrow{d_w} x$. Therefore, by the equivalent definition of $\Omega_w(A, t)$, $x \in \Omega_w(A, t)$. However, for large enough n , we must then have $x_n \in B_s(\Omega_w(A, t), \epsilon)^c$ which contradicts our choice of x_n .

By Lemma 3.1.6, $\Omega_s(A, t) \subseteq \Omega_w(A, t)$. For the other inclusion, let $x \in \Omega_w(A, t)$. By the equivalent definition of $\Omega_w(A, t)$, there are sequences $s_n \rightarrow -\infty$ with $s_n \leq t$ and $x_n \in P(t, s_n)A$ so that $x_n \xrightarrow{d_w} x$. By pullback asymptotic compactness, there is a subsequence $\{x_{n_k}\}$ with $x_{n_k} \xrightarrow{d_s} y$ for some $y \in X$. But then, $x_{n_k} \xrightarrow{d_w} y$ which gives us that $x = y$ and thus, $x_n \xrightarrow{d_s} x$. That is, $x \in \Omega_s(A, t)$ and $\Omega_s(A, t) = \Omega_w(A, t)$.

Finally, we establish the strong compactness of $\Omega_s(A, t)$. So, let $\{x_n\}$ be any sequence in $\Omega_s(A, t)$. By the equivalent definition of $\Omega_s(A, t)$, there is a corresponding sequence $\{s_k^n\}$ for each x_n with $s_k^n \rightarrow -\infty$, $s_k^n \leq t$, and $x_k^n \in P(t, s_k^n)A$ so that $x_k^n \xrightarrow{d_s} x_n$. Letting y_n, x_n be the diagonals of these families, we have that

$$d_s(y_n, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By pullback asymptotic compactness, $\{y_n\}$ is relatively strongly compact. Hence, $\{x_n\}$ is also relatively strongly compact. Since $\Omega_s(A, t)$ is closed, the limit of this subsequence lies in $\Omega_s(A, t)$ giving us that $\Omega_s(A, t)$ is compact. \square

Using this result, we have the following existence result for strong pullback attractors.

Theorem 3.3.3. *If a generalized evolutionary system \mathcal{E} is pullback asymptotically compact, then $\mathcal{A}_w(t)$ is a strongly compact strong pullback attractor.*

Proof. By Theorem 3.3.2, $\Omega_s(X, t)$ is strongly compact strong pullback attracting set with $\Omega_s(X, t) = \Omega_w(X, t) = \mathcal{A}_w(t)$, the weak pullback attractor. Therefore, by Theorem 3.1.8 and Corollary 3.1.9, the strong pullback attractor $\mathcal{A}_s(t)$ exists and $\mathcal{A}_s(t) = \Omega_s(X, t) = \mathcal{A}_w(t)$. \square

CHAPTER 4

THE STRUCTURE OF PULLBACK ATTRACTORS

4.1 Invariance and Tracking Properties

Now, we assume that \mathcal{E} satisfies **A1**. That is,

A1: $\mathcal{E}([s, \infty))$ is compact in $C([s, \infty), X_w)$ for each $s \in \mathbb{R}$.

Moreover, we introduce the following variation of the mapping P : for $A \subseteq X$ and $s \leq t \in \mathbb{R}$

$$\tilde{P}(t, s)A := \{u(t) : u(s) \in A, u \in \mathcal{E}((-\infty, \infty))\}.$$

Definition 4.1.1. We say that a family of sets $\mathcal{B}(t) \subseteq X$ is pullback semi-invariant if for each $s \leq t \in \mathbb{R}$,

$$\tilde{P}(t, s)\mathcal{B}(s) \subseteq \mathcal{B}(t).$$

We say that $\mathcal{B}(t)$ is pullback invariant if for $s \leq t \in \mathbb{R}$,

$$\tilde{P}(t, s)\mathcal{B}(s) = \mathcal{B}(t).$$

We say that $\mathcal{B}(t)$ is pullback quasi-invariant if for each $b \in \mathcal{B}(t)$, there is some complete trajectory $u \in \mathcal{E}((-\infty, \infty))$ with $u(t) = b$ and $u(s) \in \mathcal{B}(s)$ for each $s \leq t$.

Note that if $\mathcal{B}(t)$ is pullback quasi-invariant, then for each $s \leq t$,

$$\mathcal{B}(t) \subseteq \tilde{P}(t, s)\mathcal{B}(s) \subseteq P(t, s)\mathcal{B}(s). \quad (4.1)$$

Therefore, if $\mathcal{B}(t)$ is pullback quasi-invariant and pullback semi-invariant, then $\mathcal{B}(t)$ is pullback invariant.

Theorem 4.1.2. *Let \mathcal{E} be a generalized evolutionary system satisfying [A1](#). Then, $\Omega_w(A, t)$ is pullback quasi-invariant for each $A \subseteq X$.*

Proof. Let $x \in \Omega_w(A, t)$. Then, there are sequences $s_n \rightarrow -\infty$ with $s_n \leq t$ and $x_n \in P(t, s_n)A$ so that $x_n \xrightarrow{d_w} x$. Note that by passing to a subsequence, we can assume without loss of generality that s_n is a monotonically decreasing sequence. Since $x_n \in P(t, s_n)A$, there is some $u_n \in \mathcal{E}([s_n, \infty))$ with $x_n = u_n(t)$, $u_n(s_n) \in A$. Using [A1](#), $\mathcal{E}([s_n, \infty))$ is compact in $C([s_n, \infty), X_w)$. Moreover, by the definition of \mathcal{E} ,

$$\{u|_{[s_1, \infty)} : u \in \mathcal{E}([s_n, \infty))\} \subseteq \mathcal{E}([s_1, \infty)).$$

Thus, using compactness on $\{u_n|_{[s_1, \infty)}\}$, we can pass to a subsequence and drop a subindex obtaining $u^1 \in \mathcal{E}([s_1, \infty))$ so that

$$u_n|_{[s_1, \infty)} \rightarrow u^1 \text{ in } C([s_1, \infty), X_w).$$

Repeating the argument above with our subsequence, we can find another subsequence which, after dropping another subindex, gives us some $u^2 \in \mathcal{E}([s_2, \infty))$ with

$$u_n|_{[s_2, \infty)} \rightarrow u^2 \text{ in } C([s_2, \infty), X_w).$$

Note that, by construction, $u^2|_{[s_1, \infty)} = u^1$. Continuing, inductively, we get $u^k \in \mathcal{E}([s_k, \infty))$ with

$$u_n|_{[s_k, \infty)} \rightarrow u^k \text{ in } C([s_k, \infty), X_w).$$

and $u^k|_{[s_{k-1}, \infty)} = u^{k-1}$. A standard diagonalization process gives us some subsequence of u_n and $u \in \mathcal{E}((-\infty, \infty))$ so that $u|_{[-T, \infty)} \in \mathcal{E}([-T, \infty))$ and $u_n \rightarrow u$ in $C([-T, \infty), X_w)$ for any $T > 0$. That is, $u \in \mathcal{E}((-\infty, \infty))$, by definition.

Note that $u(t) = x$, by construction. Now, let $s \leq t$. Then, $u_n(s) \xrightarrow{d_w} u(s)$. By definition, since $u_n(s_n) \in A$, we then get that $u_n(s) \in P(s, s_n)A$ for $s_n \leq s$ (n sufficiently large). Hence, $u(s) \in \Omega_w(A, s)$ and $\Omega_w(A, t)$ is pullback quasi-invariant. \square

This characterization of $\Omega_w(A, t)$ gives us the following important consequences.

Corollary 4.1.3. *Let \mathcal{E} be a generalized evolutionary system satisfying [A1](#). Let $A \subseteq X$ be such that $\Omega_w(A, t) \subseteq A$ for each $t \in \mathbb{R}$. Then, $\Omega_w(A, t) = \Omega_s(A, t)$.*

Proof. By Theorem [4.1.2](#), we have that $\Omega_w(A, t)$ is pullback quasi-invariant. Thus, by [Equation 4.1](#), we have that $\Omega_w(A, t) \subseteq P(t, s)\Omega_w(A, s)$ for each $s \leq t$. By assumption, $\Omega_w(A, s) \subseteq A$,

thus $\Omega_w(A, t) \subseteq P(t, s)A$ for each $s \leq t$. Therefore, $\Omega_w(A, t) \subseteq \Omega_s(A, t)$. On the other hand, by Lemma 3.1.6, $\Omega_s(A, t) \subseteq \Omega_w(A, t)$. That is, $\Omega_w(A, t) = \Omega_s(A, t)$. \square

The following result is a direct result of Corollary 4.1.3 and Theorem 3.1.8.

Corollary 4.1.4. *Let \mathcal{E} be a generalized evolutionary system satisfying A1. Then, if the strong pullback attractor $\mathcal{A}_s(t)$ exists, $\mathcal{A}_s(t) = \mathcal{A}_w(t)$, the weak pullback attractor.*

In fact, we get a new characterization of pullback invariance for a weakly closed set A .

Theorem 4.1.5. *Let \mathcal{E} be a generalized evolutionary system satisfying A1. Then, for a family of weakly closed subsets $\mathcal{B}(t) \subseteq X$, $\mathcal{B}(t)$ is pullback invariant if and only if $\mathcal{B}(t)$ is pullback semi-invariant and pullback quasi-invariant.*

Proof. If $\mathcal{B}(t)$ is pullback semi-invariant and pullback quasi-invariant, then by the definition of pullback semi-invariant and Equation 4.1, $\mathcal{B}(t)$ is pullback invariant.

For the other direction, assume $\mathcal{B}(t)$ is pullback invariant. Then, we have that $\mathcal{B}(t)$ is clearly pullback semi-invariant. To see pullback quasi-invariance, let $b \in \mathcal{B}(t)$. Then, by pullback invariance, we can construct a monotonically decreasing sequence $s_n \rightarrow -\infty$ and find $u_n \in \mathcal{E}([s_n, \infty))$ with $u_n(s_n) \in \mathcal{B}(s_n)$ and $u_n(t) = b$. As in Theorem 4.1.2, we can find a subsequence which, after dropping a subindex, is so that $u_n \rightarrow u$ for some $u \in \mathcal{E}((-\infty, \infty))$ in the sense of $C([-T, \infty), X_w)$ for each $T > 0$. Moreover, $u(t) = b$ and $u(s) \in \mathcal{B}(s)$ for each $s \leq t$ since each $\mathcal{B}(s)$ is weakly closed. Therefore, $\mathcal{B}(t)$ is pullback quasi-invariant. \square

Let $\mathcal{J}(t)$ be a family of subsets of X given by

$$\mathcal{J}(t) := \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

Then, $\mathcal{J}(t)$ is both pullback semi-invariant and pullback quasi-invariant. Moreover, $\mathcal{J}(t)$ contains every pullback quasi-invariant and every pullback invariant set. Thus, by Theorem 4.1.2,

$$\Omega_w(A, t) \subseteq \mathcal{J}(t)$$

for each $t \in \mathbb{R}$ and each $A \subseteq X$.

Now, we will show that $\Omega_w(A, t)$ contains all the asymptotic behavior (as the initial time goes to $-\infty$) of every trajectory starting in A , provided A1 holds.

Theorem 4.1.6 (Weak pullback tracking property). *Let \mathcal{E} be a generalized evolutionary system satisfying A1, and let $A \subseteq X$. Then, for each $\epsilon > 0$ and each $t \in \mathbb{R}$, there is some $s_0 := s_0(\epsilon, t) \leq t$ so that for $s' < s_0$ and $u \in \mathcal{E}([s', \infty))$ with $u(s') \in A$ satisfies*

$$d_{C([s', \infty), X_w)}(u, v) < \epsilon$$

for some $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \Omega_w(A, s)$ for each $s \leq t$.

Proof. For contradiction, suppose not. Then, there exists $\epsilon > 0$ and sequences $s_n \leq t$ with $s_n \rightarrow -\infty$, $u_n \in \mathcal{E}([s_n, \infty))$ with the property that $u_n(s_n) \in A$ and

$$d_{C([s_n, \infty), X_w)}(u_n, v) \geq \epsilon \quad (4.2)$$

for each n and each $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \Omega_w(A, s)$ for $s \leq t$. As in the proof of Theorem 4.1.2, we find a $u \in \mathcal{E}((-\infty, \infty))$ and a subsequence which after dropping the subindex can be written as u_n with $u_n \rightarrow u$ in $C([-T, \infty), X_w)$ for each $T > 0$. In particular, $u(s) \in \Omega_w(A, s)$ for each $s \leq t$. In particular, for large enough n , $d_{C([s_n, \infty), X_w)}(u_n, u) < \epsilon$ which contradicts Equation 4.2. \square

Theorem 4.1.7 (Strong pullback tracking property). *Let \mathcal{E} be a pullback asymptotically compact generalized evolutionary system satisfying A1 and let $A \subseteq X$. Then, for each $\epsilon > 0$, $t \in \mathbb{R}$, and $T > 0$, there is some $s_0 := s_0(\epsilon, t, T) \leq t$ so that for $s' < s_0$ and each $u \in \mathcal{E}([s', \infty))$ with $u(s') \in A$, we have*

$$d_s(u(\hat{s}), v(\hat{s})) < \epsilon$$

for each $\hat{s} \in [s', s' + T]$ and some $v \in \mathcal{E}((-\infty, \infty))$ so that $v(s) \in \Omega_s(A, s)$ for each $s \leq t$.

Proof. Again, suppose not. Then, there is some $\epsilon > 0$, $T > 0$, and sequences $s_n \leq t$ with $s_n \rightarrow -\infty$, $u_n \in \mathcal{E}([s_n, \infty))$ so that $u_n(s_n) \in A$ and

$$\sup_{\hat{s} \in [s_n, s_n + T]} d_s(u_n(\hat{s}), v(\hat{s})) \geq \epsilon \quad (4.3)$$

for each n and each $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \Omega_s(A, s)$ for each $s \leq t$.

By Theorem 4.1.6, there exists a sequence $v_n \in \mathcal{E}((-\infty, \infty))$ with $v_n(s) \in \Omega_w(A, s)$ for $s \leq t$ so that

$$\lim_{n \rightarrow \infty} \sup_{\hat{s} \in [s_n, s_n + T]} d_w(u_n(\hat{s}), v_n(\hat{s})) = 0. \quad (4.4)$$

Using the pullback asymptotic compactness of \mathcal{E} we get that $\Omega_s(A, t') = \Omega_w(A, t')$ for all $t' \in \mathbb{R}$.

Then, by Equation 4.3, there is a sequence $\hat{s}_n \in [s_n, s_n + T]$ so that

$$d_s(u_n(\hat{s}_n), v_n(\hat{s}_n)) \geq \epsilon/2. \quad (4.5)$$

Again, using the pullback asymptotic compactness of \mathcal{E} , the sequences $\{u_n(\hat{s}_n)\}$ and $\{v_n(\hat{s}_n)\}$ have convergent subsequences. So, passing to a subsequence and dropping a subindex, we have that $u_n(\hat{s}_n) \xrightarrow{d_s} x$, $v_n(\hat{s}_n) \xrightarrow{d_s} y$ for some $x, y \in X$. By Equation 4.4, $x = y$ contradicting Equation 4.5. \square

Next, we use the above tracking properties to show that the weak pullback attractor $\mathcal{A}_w(t) = \mathcal{I}(t)$ if the generalized evolutionary system \mathcal{E} satisfies A1.

Theorem 4.1.8. *Let \mathcal{E} be a generalized evolutionary system satisfying A1. Then, the weak pullback attractor $\mathcal{A}_w(t) = \mathcal{I}(t)$, and $\mathcal{A}_w(t)$ is the maximal pullback quasi-invariant and maximal pullback invariant subset of X . Moreover, for each $\epsilon > 0$ and $t \in \mathbb{R}$ there is some $s_0 := s_0(\epsilon, t) \leq t$ so that for $s' < s_0$ and every trajectory $u \in \mathcal{E}([s', \infty))$ has*

$$d_{C([s', \infty), X_w)}(u, v) < \epsilon$$

for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.

Proof. Since $\mathcal{A}_w(t) = \Omega_w(X, t)$ and $\Omega_w(X, t)$ is pullback quasi-invariant by Theorem 4.1.2, we have by Equation 4.1 that $\mathcal{A}_w(t) \subseteq \mathcal{J}(t)$. For the other inclusion, let $u(t) \in \mathcal{J}(t)$. Suppose $u(t) \notin \mathcal{A}_w(t)$. Then, since $\mathcal{A}_w(t)$ is weakly closed, there is some $\epsilon > 0$ and $B_w(u(t), \epsilon)$ so that $\mathcal{A}_w(t) \cap B_w(u(t), \epsilon) = \emptyset$. By the weak pullback tracking property on $\mathcal{A}_w(t) = \Omega_w(X, t)$, there is some s' so that for any $\hat{u} \in \mathcal{E}([s', \infty))$,

$$d_{C([s', \infty), X_w)}(\hat{u}, v) < \epsilon$$

for some $v \in \mathcal{E}((-\infty, \infty))$ with $v(s) \in \Omega_w(X, s)$ for each $s \leq t$. In particular, $u \in \mathcal{E}([s', \infty))$ for each $s' < t$. Thus, there is some $v \in \mathcal{E}((-\infty, \infty))$ so that $v(t) \in \Omega_w(X, t)$ and

$$d_w(u(t), v(t)) \leq d_{C([s', \infty), X_w)}(u, v) < \epsilon.$$

This contradicts that $\mathcal{A}_w(t) \cap B_w(u(t), \epsilon) = \emptyset$. The rest of the theorem follows from Theorem 4.1.6. \square

Putting together this result as well as Theorem 3.3.3 and Theorem 4.1.7, we have the following corollary.

Corollary 4.1.9. *Let \mathcal{E} be a pullback asymptotically compact generalized evolutionary system satisfying A1. Then, the strong pullback attractor $\mathcal{A}_s(t) = \mathcal{J}(t)$ and $\mathcal{A}_s(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set. Moreover, for each $\epsilon > 0$, $t \in \mathbb{R}$,*

and $T > 0$, there is some $s_0 := s_0(\epsilon, t, T) \leq t$ so that for $s' < s_0$, every trajectory $u \in \mathcal{E}([s', \infty))$ satisfies

$$d_s(u(s), v(s)) < \epsilon$$

for each $s \in [s', s' + T]$ and some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.

4.2 Energy Inequality

In this section, we assume that our generalized evolutionary system satisfies properties **A2** and **A3**. That is, for **A2**, we let X be a set in some Banach space H satisfying the Radon-Riesz Property with norm $\|\cdot\|_H$ so that $d_s(x, y) = \|x - y\|_H$ for each $x, y \in X$, and assume that d_w induces the weak topology on X . Assume that for each $\epsilon > 0$ and each $s \in \mathbb{R}$ there is a $\delta := \delta(\epsilon, s)$ so that for every $u \in \mathcal{E}([s, \infty))$ and $t > s \in \mathbb{R}$ that

$$\|u(t)\|_H \leq \|u(t_0)\|_H + \epsilon$$

for t_0 a.e. in $(t - \delta, t)$. For **A3** we assume that if $u, u_n \in \mathcal{E}([s, \infty))$ with $u_n \rightarrow u$ in $C([s, t], X_w)$ for some $s \leq t \in \mathbb{R}$, then, $u_n(t_0) \xrightarrow{d_s} u(t_0)$ for a.e. $t_0 \in [s, t]$.

Theorem 4.2.1. *Let \mathcal{E} be a generalized evolutionary system satisfying **A2** and **A3**. Let $u_n \in \mathcal{E}([s, \infty))$ be so that $u_n \rightarrow u$ in $C([s, t], X_w)$ for some $u \in \mathcal{E}([s, \infty))$. If $u(t)$ is strongly continuous at some $t^* \in (s, t)$, then $u_n(t^*) \xrightarrow{d_s} u(t^*)$.*

Proof. By [A3](#), there is a set $E \subset [s, t]$ of measure zero so that $u_n(t_0) \xrightarrow{d_s} u(t_0)$ on $[s, t] \setminus E$. Let $\epsilon > 0$. By the energy inequality [A2](#) and the strong continuity of $u(t)$, there is some $t_0 \in [s, t^*) \setminus E$ so that

$$\|u_n(t^*)\|_H \leq \|u_n(t_0)\|_H + \epsilon/2, \quad \|u(t_0)\|_H \leq \|u(t^*)\|_H + \epsilon/2,$$

for each n . Taking the upper limit, we then have that

$$\limsup_{n \rightarrow \infty} \|u_n(t^*)\|_H \leq \limsup_{n \rightarrow \infty} \|u_n(t_0)\|_H + \epsilon/2 = \|u(t_0)\|_H + \epsilon/2 \leq \|u(t^*)\|_H + \epsilon.$$

Letting $\epsilon \rightarrow 0$, we have that

$$\limsup_{n \rightarrow \infty} \|u_n(t^*)\|_H \leq \|u(t^*)\|_H.$$

Since $u_n(t^*) \xrightarrow{d_w} u(t^*)$, by assumption, we know that $\liminf_{n \rightarrow \infty} \|u_n(t^*)\|_H \geq \|u(t^*)\|_H$. Thus, $\lim_{n \rightarrow \infty} \|u_n(t^*)\|_H = \|u(t^*)\|_H$, and, using the Radon-Riesz property, we have that $u_n(t^*) \xrightarrow{d_s} u(t^*)$. \square

Theorem 4.2.2. *Let \mathcal{E} be a generalized evolutionary system satisfying [A1](#), [A2](#), and [A3](#). If $\mathcal{E}((-\infty, \infty)) \subseteq C((-\infty, \infty), X_s)$, then \mathcal{E} is pullback asymptotically compact.*

Proof. Let $s_n \rightarrow -\infty$, $s_n \leq t$ for some $t \in \mathbb{R}$ and $x_n \in P(t, s_n)X$. Since X is weakly compact, we can pass to a subsequence and drop a subindex to assume that $x_n \xrightarrow{d_w} x$ for some $x \in X$.

Next, since $x_n \in P(t, s_n)X$, there is some $u_n \in \mathcal{E}([s_n, \infty))$ with $u_n(t) = x_n$ for each n . Using [A1](#) and the usual diagonalization process, we can pass to a subsequence and drop a subindex

to find that $u_n \rightarrow u$ in $C((-\infty, \infty), X_w)$ for some $u \in \mathcal{E}((-\infty, \infty)) \subseteq C((-\infty, \infty), X_s)$. Since u is strongly continuous at t , Theorem 4.2.1 implies that $u_n(t) = x_n \xrightarrow{d_s} x = u(t)$. Therefore, \mathcal{E} is pullback asymptotically compact. \square

Together with Theorem 3.3.3, we have the following:

Corollary 4.2.3. *let \mathcal{E} be a generalized evolutionary system satisfying A1, A2, and A3. If every complete trajectory is strongly continuous, then \mathcal{E} possesses a strongly compact, strong pullback attractor $\mathcal{A}_s(t)$.*

In fact, following the proofs of Theorem 3.3.2 and Theorem 4.2.2, we have the following generalization.

Theorem 4.2.4. *Let \mathcal{E} be a generalized evolutionary system satisfying A1, A2, and A3. Let $A \subseteq X$ be such that for each $s \in \mathbb{R}$ there exists some $u \in \mathcal{E}([s, \infty))$ with $u(t) \in A$. Assume that u is strongly continuous at t for each $u \in \mathcal{E}((-\infty, \infty))$ with $u(t) \in \Omega_w(A, t)$. Then, $\Omega_w(A, t)$ is a nonempty, strongly compact set that strongly pullback attracts A . Moreover, $\Omega_s(A, t) = \Omega_w(A, t)$.*

4.3 Pullback Attractors for Evolutionary Systems

4.3.1 Autonomous Case

We will begin with the definitions and major results for autonomous evolutionary systems as given in (14), (13). Note that X has the same structure as it had in Section 3.1. That is, X is endowed with two metrics d_s known as the strong metric and d_w known as the weak metric so that X is d_w -compact and every d_s -convergent sequence is also d_w -convergent.

Definition 4.3.1. (14) A map \mathcal{E} that associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}(I) \subseteq \mathcal{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}([0, \infty)) \neq \emptyset$.
2. $\mathcal{E}(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}(I)\}$ for all $s \in \mathbb{R}$.
3. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}(I_1)\} \subseteq \mathcal{E}(I_2)$ for all pairs $I_1, I_2 \in \mathcal{T}$, so that $I_2 \subseteq I_1$.
4. $\mathcal{E}((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[T, \infty)} \in \mathcal{E}([T, \infty)) \forall T \in \mathbb{R}\}$.

As with a generalized evolutionary system, $\mathcal{E}(I)$ is referred to as the set of all trajectories on the time interval I . Trajectories in $\mathcal{E}((-\infty, \infty))$ are known as complete. Let $\mathcal{P}(X)$ be the set of all subsets of X . For each $t \geq 0$, the map

$$R(t) : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is defined by

$$R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0, \infty))\} \text{ for } A \subseteq X.$$

By the definitions of \mathcal{E} and $R(t)$, R has the following property for each $A \subseteq X$, $t, s \geq 0$,

$$R(t + s)A \subseteq R(t)R(s)A.$$

Definition 4.3.2. (14) A set $\mathcal{A}_\bullet \subseteq X$ is a d_\bullet -global attractor ($\bullet = s$ or w) if \mathcal{A}_\bullet is a minimal set which is

1. d_\bullet -closed.
2. d_\bullet -attracting: for any $B \subseteq X$ and any $\epsilon > 0$, there is a $t_0 := t_0(B, \epsilon)$ so that

$$R(t)B \subseteq B_\bullet(\mathcal{A}_\bullet, \epsilon) := \{u : \inf_{x \in \mathcal{A}_\bullet} d_\bullet(u, x) < \epsilon\} \text{ for all } t \geq t_0.$$

Definition 4.3.3. (14) The ω_\bullet -limit ($\bullet = s$ or w) of a set $A \subseteq X$ is

$$\omega_\bullet(A) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R(t)A}^\bullet.$$

Equivalently, $x \in \omega_\bullet(A)$ if there exist sequences $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $x_n \in R(t_n)A$, such that $x_n \xrightarrow{d_\bullet} x$ as $n \rightarrow \infty$.

To extend the notion of invariance from a semiflow to an evolutionary system, the following mapping is used for $A \subseteq X$ and $t \in \mathbb{R}$:

$$\tilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}((-\infty, \infty))\}.$$

Definition 4.3.4. (14) A set $A \subseteq X$ is positively invariant if for each $t \geq 0$,

$$\tilde{R}(t)A \subseteq A.$$

We say that A is invariant if for each $t \geq 0$,

$$\tilde{R}(t)A = A.$$

A is quasi-invariant if for every $a \in A$, there exists a complete trajectory $u \in \mathcal{E}((-\infty, \infty))$ with $u(0) = a$ and $u(t) \in A$ for all $t \in \mathbb{R}$.

Definition 4.3.5. (14) The evolutionary system \mathcal{E} is asymptotically compact if for any $t_k \rightarrow \infty$ and any $x_k \in R(t_k)X$, the sequence $\{x_k\}$ is relatively strongly compact.

Here are other assumptions that are imposed on \mathcal{E} .

B1 : $\mathcal{E}([0, \infty))$ is a compact set in $C([0, \infty), X_w)$.

B2 : Assume that X is a set in some Banach space H satisfying the Radon-Riesz property with the norm denoted by $\|\cdot\|_H$, so that $d_s(x, y) = \|x - y\|_H$ for $x, y \in X$ and d_w induces the weak topology on X . Assume also that for any $\epsilon > 0$, there exists $\delta := \delta(\epsilon)$, such that for every $u \in \mathcal{E}([0, \infty))$ and $t > 0$,

$$\|u(t)\|_H \leq \|u(t_0)\|_H + \epsilon,$$

for t_0 a.e. in $(t - \delta, t)$.

B3 : Let $u, u_n \in \mathcal{E}([0, \infty))$, be so that $u_n \rightarrow u$ in $C([0, T], X_w)$ for some $T > 0$. Then, $u_n(t) \rightarrow u(t)$ strongly a.e. in $[0, T]$.

Theorem 4.3.6. (14) Let \mathcal{E} be an evolutionary system. Then,

1. If the d_\bullet -global attractor \mathcal{A}_\bullet exists, then $\mathcal{A}_\bullet = \omega_\bullet(X)$.
2. The weak global attractor \mathcal{A}_w exists.

Furthermore, if \mathcal{E} satisfies B1, then

3. $\mathcal{A}_w = \omega_w(X) = \omega_s(X) = \{u_0 : u_0 = u(0) \text{ for some } u \in \mathcal{E}((-\infty, \infty))\}$.
4. \mathcal{A}_w is the maximal invariant and maximal quasi-invariant set.
5. (Weak uniform tracking property) For any $\epsilon > 0$, there exists a $t_0 := t_0(\epsilon)$, so that for any $t > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies $d_{C([t, \infty), X_w)}(u, v) < \epsilon$, for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.

Theorem 4.3.7. (14) Let \mathcal{E} be an asymptotically compact evolutionary system. Then,

1. The strong global attractor \mathcal{A}_s exists, it is strongly compact, and $\mathcal{A}_s = \mathcal{A}_w$.

Furthermore, if \mathcal{E} satisfies B1, then

2. (Strong uniform tracking property) for any $\epsilon > 0$ and $T > 0$, there exists $t_0 := t_0(\mathcal{E}, T)$, so that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}([0, \infty))$ satisfies $d_s(u(t), v(t)) < \epsilon$, for all $t \in [t^*, t^* + T]$, for some complete trajectory $v \in \mathcal{E}((-\infty, \infty))$.

Theorem 4.3.8. (14) Let \mathcal{E} be an evolutionary system satisfying B1, B2, and B3 and so that every complete trajectory is strongly continuous. Then, \mathcal{E} is asymptotically compact.

Now, we consider the existence of the pullback attractor in the context of an evolutionary system. Note that every evolutionary system is also a generalized evolutionary system. In this case, we have the following relationship for the set functions P and R : Let $s \leq t \in \mathbb{R}$, and let $A \subseteq X$, then

$$P(t, s)A = P(t - s, 0)A = R(t - s)A.$$

The following results can also be easily verified:

1. A set A is d_\bullet -attracting if and only if the family of sets $A(t) := A$ for all $t \in \mathbb{R}$ is d_\bullet -pullback attracting.
2. \mathcal{E} is asymptotically compact if and only if \mathcal{E} is pullback asymptotically compact.
3. \mathcal{E} satisfies **B1** if and only if \mathcal{E} satisfies **A1**.
4. \mathcal{E} satisfies **B2** if and only if \mathcal{E} satisfies **A2**.
5. \mathcal{E} satisfies **B3** if and only if \mathcal{E} satisfies **A3**.

Moreover, for $B \subseteq X$ and $B(t) := B$ for all $t \in \mathbb{R}$, we have that the following invariance relations:

6. B is positively invariant if and only if $B(t)$ is pullback semi-invariant.
7. B is invariant if and only if $B(t)$ is pullback invariant.
8. If B is quasi-invariant, then $B(t)$ is pullback quasi-invariant.

This gives us the following characterization of the ω_\bullet -limit and Ω_\bullet -limit sets.

Theorem 4.3.9. *Let \mathcal{E} be an evolutionary system. Let $t \in \mathbb{R}$ and $A \subseteq X$ then,*

$$\Omega_\bullet(A, t) = \omega_\bullet(A) \quad (\bullet = s, w).$$

Proof. Let $x \in \Omega_\bullet(A, t)$. Then, there exist sequences $s_n \rightarrow -\infty$, $s_n \leq t$, and $x_n \in P(t, s_n)A$ so that $x_n \xrightarrow{d_\bullet} x$. Then, $(t - s_n) \rightarrow \infty$ for $(t - s_n) \geq 0$, and $x_n \in P(t, s_n)A = P(t - s_n, 0)A = R(t - s_n)A$. That is, $x \in \omega_\bullet(A)$.

Now, let $x \in \omega_\bullet(A)$. Then, there exist sequences $t_n \rightarrow \infty$, $t_n \geq 0$, and $x_n \in R(t_n)A$ so that $x_n \xrightarrow{d_\bullet} x$. But, $t_n = t - (t - t_n)$. Therefore,

$$x_n \in R(t_n)A = R(t - (t - t_n))A = P(t - (t - t_n), 0)A = P(t, t - t_n)A$$

for $t - t_n \leq t$ and $t - t_n \rightarrow -\infty$. That is, $x \in \Omega_\bullet(A, t)$. \square

Using this result as well as Theorem 4.3.6, Theorem 3.1.8, and Theorem 3.1.11, we have the following corollary.

Theorem 4.3.10. *Let \mathcal{E} be an evolutionary system. Then, the weak global attractor \mathcal{A}_w and the weak pullback attractor $\mathcal{A}_w(t)$ exist, and $\mathcal{A}_w = \mathcal{A}_w(t)$ for each $t \in \mathbb{R}$. Moreover, the strong global attractor \mathcal{A}_s exists if and only if the strong pullback attractor $\mathcal{A}_s(t)$ exists, and $\mathcal{A}_s = \mathcal{A}_s(t)$.*

Proof. Using Theorem 4.3.6, we know that the weak global attractor \mathcal{A}_w exists, and $\mathcal{A}_w = \omega_w(X)$. By Theorem 3.1.8 and Theorem 3.1.11, the weak pullback attractor $\mathcal{A}_w(t)$ exists and $\mathcal{A}_w(t) = \Omega_w(X, t)$. By Theorem 4.3.9, we have that

$$\mathcal{A}_w = \omega_w(X) = \Omega_w(X, t) = \mathcal{A}_w(t).$$

Now, suppose the strong global attractor \mathcal{A}_s exists. Then, as in the above section, we have that

$$\mathcal{A}_s = \omega_s(X) = \Omega_s(X, t).$$

But, then $\Omega_w(X, t)$ is d_s -attracting which means that it is d_s -pullback attracting. Therefore, the strong pullback attractor $\mathcal{A}_s(t)$ exists, and $\mathcal{A}_s(t) = \mathcal{A}_s$. An analogous argument shows that if the strong pullback attractor $\mathcal{A}_s(t)$ exists, then the strong global attractor \mathcal{A}_s exists and $\mathcal{A}_s = \mathcal{A}_s(t)$. \square

Furthermore, if \mathcal{E} is asymptotically compact, then \mathcal{E} is pullback asymptotically compact, and the strong global attractor \mathcal{A}_s and strong pullback attractor $\mathcal{A}_s(t)$ both exist, and $\mathcal{A}_s(t) = \mathcal{A}_s$.

4.3.2 Nonautonomous Case

The modern theory of uniform attractors (using a symbol space across which attraction is uniform) was first introduced by Chepyzhov and Vishik. They applied this framework to the 2D Navier-Stokes equations with an appropriate forcing term. For more information on this theory, see (15). They also proved the existence of uniform attractors for the 3D Navier-Stokes equations using the framework of trajectory attraction (35). A similar result appeared earlier in a paper by Sell using the restriction of semiflows to an invariant set (36). Using the framework of evolutionary systems, Cheskidov and Lu added a structure theorem and tracking properties of the uniform attractor (17). We follow the closely-related framework in (17) to compare the structure of the weak uniform attractor to the weak pullback attractor.

Following the theory of (15) the concept of symbols is introduced. So, let Σ be a parameter set and $\{T(s)\}_{s \geq 0}$ be a family of operators acting on Σ satisfying $T(s)\Sigma = \Sigma$, for each $s \geq 0$. Any element $\sigma \in \Sigma$ will be called a (time) symbol and Σ will be called the (time) symbol space. For instance, in many applications $\{T(s)\}$ is the translation semigroup and Σ is the translation

family of time dependent items of the system being considered or its closure in some appropriate topological space.

Definition 4.3.11. (17) A family of maps \mathcal{E}_σ , $\sigma \in \Sigma$ which for each $\sigma \in \Sigma$ associates to each $I \in \mathcal{T}$ a subset $\mathcal{E}_\sigma(I) \subseteq \mathcal{F}(I)$ will be called a nonautonomous evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}_\sigma([\tau, \infty)) \neq \emptyset$ for each $\tau \in \mathbb{R}$.
2. $\mathcal{E}_\sigma(I + s) = \{u(\cdot) : u(\cdot + s) \in \mathcal{E}_{T(s)\sigma}(I)\}$ for each $s \geq 0$.
3. $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathcal{E}_\sigma(I_1)\} \subseteq \mathcal{E}_\sigma(I_2)$ for each $I_1, I_2 \in \mathcal{T}$, $I_2 \subseteq I_1$.
4. $\mathcal{E}_\sigma((-\infty, \infty)) = \{u(\cdot) : u(\cdot)|_{[\tau, \infty)} \in \mathcal{E}_\sigma([\tau, \infty)) \forall \tau \in \mathbb{R}\}$.

Analogous to our previous definitions, $\mathcal{E}_\sigma(I)$ is called the set of all trajectories with respect to the symbol σ on the time interval I . Trajectories in $\mathcal{E}_\sigma((-\infty, \infty))$ are called complete with respect to σ . Note that if we fix any symbol $\sigma \in \Sigma$ in a nonautonomous evolutionary system \mathcal{E}_σ , then we obtain a generalized evolutionary system. On the other hand, if we let $\Sigma := \mathbb{R}$ with $T(s)t := t + s$, the translation semigroup, as well as

$$\mathcal{E}_t([T, \infty)) := \{u(\cdot) : u(\cdot - t) \in \mathcal{E}([T + t, \infty))\},$$

we obtain a nonautonomous evolutionary system from a given generalized evolutionary system.

For every $t \geq \tau$, $\tau \in \mathbb{R}$, $\sigma \in \Sigma$, the map

$$R_\sigma(t, \tau) : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$$

is defined by

$$R_\sigma(t, \tau)A := \{u(t) : u(\tau) \in A, u \in \mathcal{E}_\sigma([\tau, \infty))\} \text{ for } A \subseteq X. \quad (4.6)$$

By the assumptions on \mathcal{E}_σ for each $\sigma \in \Sigma$, it is found that

$$R_\sigma(t, \tau)A \subseteq R_\sigma(t, s)R_\sigma(s, \tau)A$$

for each $A \subseteq X$, $t \geq s \geq \tau \in \mathbb{R}$. Using the following Lemma, one can reduce a nonautonomous evolutionary system to an evolutionary system.

Lemma 4.3.12. *(17) Let $\tau_0 \in \mathbb{R}$ be fixed. Then, for any $\tau \in \mathbb{R}$ and $\sigma \in \Sigma$, there exists at least one $\sigma' \in \Sigma$ so that*

$$\mathcal{E}_\sigma([\tau, \infty)) = \{u(\cdot) : u(\cdot + \tau - \tau_0) \in \mathcal{E}_{\sigma'}([\tau, \infty))\}.$$

Thus, it is found that for $A \subseteq X$, $\tau \in \mathbb{R}$ and $t \geq 0$,

$$\bigcup_{\sigma \in \Sigma} R_\sigma(t, 0)A = \bigcup_{\sigma \in \Sigma} R_\sigma(t + \tau, \tau)A.$$

Moreover, defining

$$\mathcal{E}_\Sigma(I) := \bigcup_{\sigma \in \Sigma} \mathcal{E}_\sigma(I)$$

for $I \in \mathcal{T}$, then \mathcal{E}_Σ defines an autonomous evolutionary system. Also, for $A \subseteq X$, $t \geq 0$, the map $R_\Sigma(t) : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is given by

$$R_\Sigma(t)A := \bigcup_{\sigma \in \Sigma} R_\sigma(t, 0)A.$$

Let $\omega_\bullet^\Sigma(A)$ be the corresponding omega-limit set for \mathcal{E}_Σ . That is,

$$\omega_\bullet^\Sigma(A) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R_\Sigma(t)}^\bullet = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \bigcup_{\sigma \in \Sigma} R_\sigma(t, 0)}^\bullet.$$

Definition 4.3.13. (17) For the autonomous evolutionary system \mathcal{E}_Σ , we denote its \mathbf{d}_\bullet -global attractor (if it exists) by $\mathcal{A}_\bullet^\Sigma$. We call $\mathcal{A}_\bullet^\Sigma$ the \mathbf{d}_\bullet -uniform attractor for \mathcal{E}_Σ .

The following results for \mathcal{E}_Σ are then attained using Theorem 4.3.6.

Theorem 4.3.14. (17) Let \mathcal{E}_Σ be a nonautonomous evolutionary system. Then, if the \mathbf{d}_\bullet -uniform attractor exists, then $\mathcal{A}_\bullet^\Sigma = \omega_\bullet^\Sigma(X)$. Also, the weak uniform attractor \mathcal{A}_w^Σ exists.

Here are additional assumptions imposed on \mathcal{E}_Σ .

C1 : $\mathcal{E}_\Sigma([0, \infty))$ is precompact in $C([0, \infty), X_w)$.

C2 : Assume that X is a set in some Banach space H satisfying the Radon-Riesz property

with the norm denoted by $\|\cdot\|_H$, such that $\mathbf{d}_s(x, y) := \|x - y\|_H$ for all $x, y \in X$ and \mathbf{d}_w

induces the weak topology on X . Assume also that for any $\epsilon > 0$, there exists $\delta := \delta(\epsilon)$, so that for each $u \in \mathcal{E}_\Sigma([0, \infty))$ and $t > 0$,

$$\|u(t)\|_H \leq \|u(t_0)\|_H + \epsilon,$$

for t_0 a.e. in $(t - \delta, t)$.

C3 : Let $u_k \in \mathcal{E}_\Sigma([0, \infty))$ be so that u_k is $d_{C([0, T], X_w)}$ -Cauchy sequence in $C([0, T], X_w)$ for some $T > 0$. Then, $u_k(t)$ is d_s -Cauchy for a.e. $t \in [0, T]$.

Next, the closure of the evolutionary system \mathcal{E}_Σ is introduced. This is given by $\bar{\mathcal{E}}$ defined as follows:

$$\bar{\mathcal{E}}([\tau, \infty)) := \overline{\mathcal{E}_\Sigma([\tau, \infty))}^{C([\tau, \infty), X_w)}$$

for each $\tau \in \mathbb{R}$. This is an evolutionary system. Let $\bar{\omega}_\bullet(A)$ and $\bar{\mathcal{A}}_\bullet$ be the corresponding omega-limit set and global attractor for $\bar{\mathcal{E}}$, respectively. Then, $\bar{\mathcal{E}}$ has the following properties:

Lemma 4.3.15. *(17) If \mathcal{E}_Σ satisfies **C1**, then $\bar{\mathcal{E}}$ satisfies **B1**. Moreover, if \mathcal{E}_Σ satisfies **C2** and **C3**, then $\bar{\mathcal{E}}$ satisfies **B2** and **B3**.*

Theorem 4.3.16. *(17) Assume that \mathcal{E}_Σ satisfies **C1**. Then, the weak uniform attractor exists by Theorem 4.3.14. Also,*

$$1. \mathcal{A}_w^\Sigma = \omega_w^\Sigma(X) = \bar{\omega}_w(X) = \bar{\omega}_s(X) = \bar{\mathcal{A}}_w = \{u_0 \in X : u_0 = u(0) \text{ for some } u \in \bar{\mathcal{E}}((-\infty, \infty))\}.$$

$$2. \mathcal{A}_w^\Sigma \text{ is the maximal invariant and maximal quasi-invariant set with respect to } \bar{\mathcal{E}}.$$

3. (Weak uniform tracking property) For any $\epsilon > 0$, there exists a $t_0 := t_0(\epsilon, T)$ so that for any $t^* > t_0$, every trajectory $u \in \mathcal{E}_\Sigma([0, \infty))$ satisfies $d_{C([t^*, \infty), X_w)}(u, v) < \epsilon$ for some complete trajectory $v \in \bar{\mathcal{E}}((-\infty, \infty))$.

If \mathcal{E}_Σ is an asymptotically compact evolutionary system (not necessarily satisfying **C1**), then

4. The strong uniform attractor \mathcal{A}_s^Σ exists, is strongly compact, and $\mathcal{A}_s^\Sigma = \mathcal{A}_w^\Sigma$.

Furthermore, if \mathcal{E}_Σ is asymptotically compact and satisfies **C1**, then

5. (Strong uniform tracking property) For any $\epsilon > 0$ and $T > 0$, there exists $t_0 := t_0(\epsilon, T)$ so that for $t^* > t_0$, every trajectory $u \in \mathcal{E}_\Sigma([0, \infty))$ satisfies $d_s(u(t), v(t)) < \epsilon$ for each $t \in [t^*, t^* + T]$, for some complete trajectory $v \in \bar{\mathcal{E}}((-\infty, \infty))$.

Theorem 4.3.17. (17) Let \mathcal{E}_Σ be an evolutionary system satisfying **C1**, **C2**, and **C3**, and assume that $\bar{\mathcal{E}}((-\infty, \infty)) \subseteq C((-\infty, \infty), X_s)$. Then, \mathcal{E}_Σ is asymptotically compact.

Let \mathcal{E} be a nonautonomous evolutionary system with symbol space Σ and shift operators $T(s) : \Sigma \rightarrow \Sigma$ for each $s \geq 0$. Then, we have that $P_\sigma(t, s) = R_\sigma(t, s)$ for all $t \geq s$. We can use property (2) in Definition 4.3.11, to obtain the following identity for any $\sigma \in \Sigma$, $t \geq r \in \mathbb{R}$, $s > 0$ and $A \subseteq X$,

$$R_\sigma(t + s, r + s)A = R_{T(s)\sigma}(t, r)A. \quad (4.7)$$

Using this fact, we have can say that

$$\Omega_\bullet^\sigma(A, t + s) = \Omega_\bullet^{T(s)\sigma}(A, t)$$

for any $\sigma \in \Sigma$, $t \in \mathbb{R}$, and $A \subseteq X$. Thus, we get that

$$\bigcup_{\sigma \in \Sigma} \bigcup_{t \in \mathbb{R}} \Omega_{\bullet}^{\sigma}(A, t) = \bigcup_{\sigma \in \Sigma} \Omega_{\bullet}^{\sigma}(A, t_0).$$

for any fixed $t_0 \in \mathbb{R}$. Now, we have can draw the following relationship between $\bigcup_{\sigma} \Omega_{\bullet}^{\sigma}(A, t_0)$ and the uniform attractor $\mathcal{A}_{\bullet}^{\Sigma}$ (if it exists).

Theorem 4.3.18. *Let \mathcal{E}_{σ} be a nonautonomous evolutionary system. Then, if the d_{\bullet} -uniform attractor $\mathcal{A}_{\bullet}^{\Sigma}$ exists, we have that*

$$\overline{\bigcup_{\sigma \in \Sigma} \Omega_{\bullet}^{\sigma}(X, t_0)}^{\bullet} \subseteq \mathcal{A}_{\bullet}^{\Sigma}$$

for any fixed $t_0 \in \mathbb{R}$.

Remark 4.3.19. Note that $x \in \omega_{\bullet}^{\Sigma}(A)$ if and only if there exist sequences $\sigma_n \in \Sigma$, $t_n \geq 0$ with $t_n \rightarrow \infty$, and $x_n \in R_{\sigma_n}(t_n, 0)A$ with $x_n \xrightarrow{d_{\bullet}} x$. Similarly, if $x \in \overline{\bigcup_{\sigma} \Omega_{\bullet}^{\sigma}(A, t_0)}^{\bullet}$ then there exist sequences $\sigma_n \in \Sigma$, $s_n \in \mathbb{R}$, $s_n \leq t_0$ with $s_n \rightarrow -\infty$, and $x_n \in R_{\sigma_n}(t_0, s_n)A$ so that $x_n \xrightarrow{d_{\bullet}} x$.

Proof. Let $x \in \bigcup_{\sigma} \Omega_{\bullet}^{\sigma}(X, t_0)$. Then, there exist sequences $\sigma_n \in \Sigma$, $s_n \rightarrow -\infty$ with $s_n \leq t_0$ and $x_n \in R_{\sigma_n}(t_0, s_n)X$ with $x_n \xrightarrow{d_{\bullet}} x$. Without loss of generality, we can pass to a subsequence and assume that $s_n \leq 0$ for each n . Using the fact that $R_{\sigma_n}(t_0, s_n)X = R_{\sigma'_n}(t - s_n, 0)X$ for any σ'_n so that $T(-s_n)\sigma'_n = \sigma_n$, we see that $x \in \omega_{\bullet}^{\Sigma}(X)$ by Remark 4.3.19. Thus,

$$\overline{\bigcup_{\sigma \in \Sigma} \Omega_{\bullet}^{\sigma}(X, t_0)}^{\bullet} \subseteq \omega_{\bullet}^{\Sigma}(X) = \mathcal{A}_{\bullet}^{\Sigma}$$

by Theorem 4.3.14. □

Using this result, Theorem 4.3.16, Theorem 3.1.9, and Theorem 3.1.11, we get the following corollary.

Corollary 4.3.20. *Let \mathcal{E}_σ be a nonautonomous evolutionary system. Then, the weak uniform attractor \mathcal{A}_w^Σ exists. Similarly, for each $\sigma \in \Sigma$, the induced generalized evolutionary system, there exists a weak pullback attractor $\mathcal{A}_w^\sigma(t)$. Moreover,*

$$\overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(t_0)}^w \subseteq \mathcal{A}_w^\Sigma$$

for any fixed $t_0 \in \mathbb{R}$.

Combining Theorem 4.3.18 with Theorem 4.3.17, the second half of Theorem 4.3.16, Theorem 3.3.3, and the following Lemma 4.3.21, we get a similar embedding of the union of the strong pullback attractors within the strong uniform attractor. But first, we need to know that the asymptotic compactness of \mathcal{E}_Σ guarantees the pullback asymptotic compactness of each \mathcal{E}_σ . This is given in the following lemma.

Lemma 4.3.21. *Let \mathcal{E}_Σ , the induced autonomous evolutionary system from the nonautonomous evolutionary system \mathcal{E}_σ be asymptotically compact. Then, for each fixed $\sigma \in \Sigma$, the induced generalized evolutionary system \mathcal{E}_σ is pullback asymptotically compact.*

Proof. Let $\sigma \in \Sigma$ be fixed. Let $s_n \leq t$ be so that $s_n \rightarrow -\infty$. Also, let $x_n \in R_\sigma(t, s_n)X$. Then, using (Equation 4.7), we get that

$$x_n \in \bigcup_{\sigma \in \Sigma} R_\sigma(t, s_n)X = \bigcup_{\sigma \in \Sigma} R_\sigma(t - s_n, 0)X = R_\Sigma(t - s_n)X$$

for $t - s_n \rightarrow \infty$. Thus, x_n has a convergent subsequence by the asymptotic compactness of \mathcal{E}_Σ □

Theorem 4.3.22. *Let \mathcal{E}_Σ , the induced autonomous evolutionary system from the nonautonomous evolutionary system \mathcal{E}_σ , be asymptotically compact or let \mathcal{E}_Σ satisfy C1, C2, and C3 with complete trajectories strongly continuous. Then, the weak uniform attractor \mathcal{A}_w^Σ is the strongly compact strong uniform attractor \mathcal{A}_s^Σ . Also, for each fixed $\sigma \in \Sigma$, the weak pullback attractor $\mathcal{A}_w^\sigma(t)$ is a strongly compact strong pullback attractor $\mathcal{A}_s^\sigma(t)$. Moreover,*

$$\overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_s^\sigma(t_0)}^s \subseteq \overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_s^\sigma(t_0)}^w = \overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(t_0)}^w \subseteq \mathcal{A}_w^\Sigma = \mathcal{A}_s^\Sigma$$

for any fixed $t_0 \in \mathbb{R}$.

The reverse inclusion in Corollary 4.3.20 is untrue as the following example shows. So, consider the following nonautonomous heat equation acting on $L^2(\mathbb{R}^n)$:

$$\begin{cases} u_t = e^{-t} \Delta u \\ u(s) = u_s. \end{cases}$$

This system is solved in Fourier space (see Example 3.2.3) as

$$\hat{u}(t) = \exp \left[|\xi|^2 (e^{-t} - e^{-s}) \right] \hat{u}_s.$$

This forms a nonautonomous evolutionary system on X , the closed unit ball in $L^2(\mathbb{R}^n)$, using

$$\mathcal{E}_\sigma([s, \infty)) := \left\{ \hat{u}(t) = \exp \left[|\xi|^2 e^{-\sigma} (e^{-t} - e^{-s}) \right] \hat{u}_s : \hat{u}_s \in L^2 \right\}.$$

with time symbol $\sigma(t) := e^{-t}$ and with $T(h)e^{-t} := e^{-(t+h)}$ the associated family of operators.

Then, for any fixed symbol σ , we find that

$$\mathcal{A}_w^\sigma(0) := \{0\}.$$

On the other hand, we find that for any $s \in \mathbb{R}$,

$$\bar{\mathcal{E}}([s, \infty)) := \overline{\mathcal{E}_\Sigma([s, \infty))}^{C([s, \infty), X_w)}$$

contains all constant functions $\hat{u}(t) \equiv \hat{u}_s$. Therefore, the uniform attractor is the entire closed unit ball in $L^2(\mathbb{R}^n)$. That is,

$$\mathcal{A}_w^\Sigma = X.$$

Thus,

$$\overline{\bigcup_{\sigma \in \Sigma} \mathcal{A}_w^\sigma(0)}^w = \{0\} \neq X = \mathcal{A}_w^\Sigma.$$

CHAPTER 5

PULLBACK ATTRACTORS FOR THE NAVIER-STOKES EQUATIONS

In this chapter, we will be studying the 2D and 3D Navier-Stokes equations simultaneously. Many of the below arguments are valid in either dimension. Thus, if the dimension is not noted, assume that the arguments valid in either dimension.

5.1 Deriving the Phase Space X

Assume f is translationally bounded in $L^2_{\text{loc}}(\mathbb{R}, H^{-1})$. That is,

$$\|f\|_{L^2_b}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|_{H^{-1}}^2 ds < \infty.$$

We will show that there exists a bounded set $X \subset L^2$ which captures all of the asymptotic dynamics of Leray solutions with a translationally bounded force f . We will be more precise with what this means in a moment. But first, we need a preliminary definition and a lemma. The lemma's proof can be found in (15).

Definition 5.1.1. A function $g(s)$ is almost everywhere equal to a monotonic non-increasing function on $[a, b]$ if $g(t) \leq g(\tau)$ for any $t, \tau \in [a, b] \setminus Q$ with $\tau \leq t$ and the measure of Q is zero.

Lemma 5.1.2. *Let $g(s) \in L_1([a, b])$. Then, the function $g(s)$ is almost everywhere equal to a monotone non-increasing function on $[a, b]$ if and only if, for any $\phi \in C_0^\infty((a, b))$ with $\phi(s) \geq 0$, one has*

$$\int_a^b g(s)\phi'(r)dr \geq 0.$$

So, let u be a Leray solution to Equation 1.2 with f translationally bounded and starting time s . That is, a point where the energy inequality Equation 1.3 is satisfied. Then, applying Young's inequality followed by the Poincaré inequality, we find that

$$\|u(t)\|_2^2 + \nu\mu_1 \int_s^t \|u(r)\|_2^2 dr \leq \|u(s)\|_2^2 + \frac{1}{\nu} \int_s^t \|f(r)\|_{H^{-1}}^2 dr. \quad (5.1)$$

Let $\phi \in C_0^\infty((s, \tau))$ for $\tau \geq t$. Then, the above inequality is equivalent to the following distributional inequality

$$-\int_s^\tau \|u(r)\|_2^2 \phi'(r) dr + \nu\mu_1 \int_s^\tau \|u(r)\|_2^2 \phi(r) dr \leq \frac{1}{\nu} \int_s^\tau \|f(r)\|_{H^{-1}}^2 \phi(r) dr. \quad (5.2)$$

Replacing ϕ with $e^{\nu\mu_1 r} \phi(r) \in C_0^\infty((s, \tau))$, we have that

$$\begin{aligned} -\int_s^\tau \|u(r)\|_2^2 e^{\nu\mu_1 r} \phi'(r) dr &\leq \frac{1}{\nu} \int_s^\tau \|f(r)\|_{H^{-1}}^2 e^{\nu\mu_1 r} \phi(r) dr \\ &= \frac{1}{\nu} \int_s^\tau \frac{d}{dr} \left(\int_0^r \|f(\rho)\|_{H^{-1}}^2 e^{\nu\mu_1 \rho} d\rho \right) \phi(r) dr \\ &= -\frac{1}{\nu} \int_s^\tau \left(\int_0^r \|f(\rho)\|_{H^{-1}}^2 e^{\nu\mu_1 \rho} d\rho \right) \phi'(r) dr. \end{aligned}$$

Rearranging, and using Lemma 5.1.2, we get that

$$\|u(t)\|_2^2 e^{\nu\mu_1 t} - \|u(s)\|_2^2 e^{\nu\mu_1 s} \leq \frac{1}{\nu} \int_s^t \|f(r)\|_{H^{-1}}^2 e^{\nu\mu_1 r} dr. \quad (5.3)$$

It remains to estimate the right-hand side of Equation 5.3. As in (37), we have that

$$\begin{aligned} \int_s^t \|f(r)\|_{H^{-1}}^2 e^{\nu\mu_1 r} dr &\leq \int_{t-1}^t \|f(r)\|_{H^{-1}}^2 e^{\nu\mu_1 r} dr + \int_{t-2}^{t-1} \|f(r)\|_{H^{-1}}^2 e^{\nu\mu_1 r} dr + \dots \\ &\leq e^{\nu\mu_1 t} (1 + e^{-\nu\mu_1} + e^{-2\nu\mu_1}) \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(r)\|_{H^{-1}}^2 dr \\ &\leq \frac{e^{\nu\mu_1 t}}{1 - e^{-\nu\mu_1}} \|f\|_{L_b^2}^2. \end{aligned}$$

Thus, we arrive at the following inequality

$$\|u(t)\|_2^2 \leq \|u(s)\|_2^2 e^{\nu\mu_1(s-t)} + \frac{1}{\nu} \frac{\|f\|_{L_b^2}^2}{1 - e^{-\nu\mu_1}}.$$

Let

$$R^2 := \frac{2\|f\|_{L_b^2}^2}{\nu(1 - e^{-\nu\mu_1})}.$$

Then, X given by

$$X := \{u \in L^2 : \|u\|_2 \leq R\}$$

is a closed absorbing ball in L^2 for Leray solutions. Moreover, X is weakly compact and contains all of the asymptotic dynamics of Leray solutions by the above argument. Define the strong and weak distances on X , respectively, by

$$d_s(u, v) := \|u - v\|_2 \quad \text{and} \quad d_w(u, v) := \sum_{k \in \mathbb{Z}^3} \frac{1}{2^{|k|}} \frac{|\hat{u}_k - \hat{v}_k|}{1 + |\hat{u}_k - \hat{v}_k|}$$

for $u, v \in L^2$ where \hat{u}_k and \hat{v}_k are the Fourier coefficients of u and v , respectively. Note that the above weak metric d_w induces the weak topology L_w^2 on X . Next, we define our generalized evolution system on X by

$$\mathcal{E}([s, \infty)) := \{u : u \text{ is a Leray-Hopf solution of Equation 1.2 on } [s, \infty)$$

$$\text{and } u(t) \in X \text{ for } t \in [T, \infty)\},$$

$$\mathcal{E}((-\infty, \infty)) := \{u : u \text{ is a Leray-Hopf solution of Equation 1.2 on } (-\infty, \infty)$$

$$\text{and } u(t) \in X \text{ for } t \in (-\infty, \infty)\}.$$

Then, \mathcal{E} satisfies the necessary properties in Definition 3.1.1 and forms a generalized evolutionary system on X . We must use Leray-Hopf solutions as our generalized evolutionary system since the restriction of a Leray solution may not be a Leray solution, but it is always a Leray-Hopf solution.

In three dimensions, note that an absorbing ball does not exist for the Leray-Hopf weak solutions. An absorbing ball is a bounded set $X \subset L^2$ so that for any $B \subset L^2$ bounded and

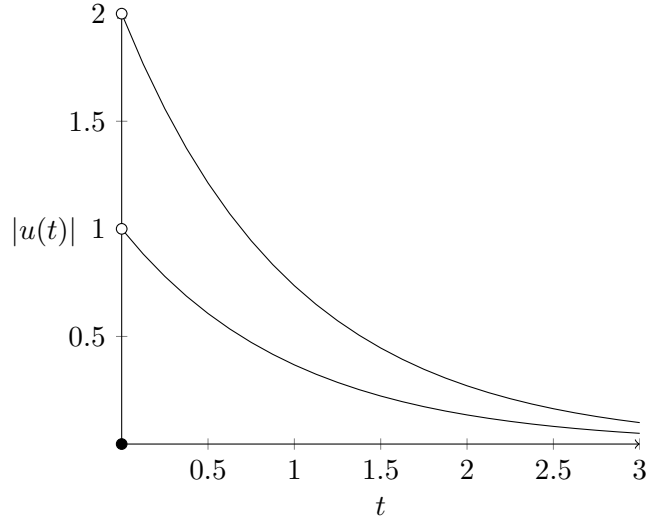


Figure 1. Possible Leray-Hopf weak solutions for the Navier-Stokes equations with zero forcing.

any $s_0 \in \mathbb{R}$, there is some $\sigma := \sigma(B) \geq s_0$ so that if $u(s_0) \in B$, then $u(s) \in X$ for $s \geq \sigma$. This requires uniformity in B . However, in three dimensions, Leray-Hopf solutions may have “jumps” at the starting point which can be as large as you like. Thus, even if you were to consider the bounded set $B := \{0\}$ for the Leray-Hopf solutions with $s_0 := 0$ in the autonomous case, you may not have such a structure as [Figure 1](#) illustrates.

5.2 The Structure of Pullback Attractors for the Navier-Stokes Equations

By Theorem [3.1.8](#), \mathcal{E} has a weak pullback attractor. Next, we will show that \mathcal{E} satisfies [A1](#), [A2](#), and [A3](#). We start with a preliminary lemma.

Lemma 5.2.1. *Let u_n be a sequence of Leray-Hopf weak solutions of Equation 1.1 on $[s, \infty)$, so that $u_n(t) \in X$ for all $t \geq s$ for some $s \in \mathbb{R}$. Then, there exists a subsequence n_j so that u_{n_j} converges to some u in $C([t_1, t_2], L_w^2)$. That is,*

$$\langle u_{n_j}, v \rangle_{L^2} \rightarrow \langle u, v \rangle_{L^2}$$

uniformly on $[t_1, t_2]$ as $n_j \rightarrow \infty$ for all $v \in L^2$.

Proof. The major arguments in this lemma are classical. For more information, see (3), (4), and (5), among others. Note that several of the arguments hinge on the dimension. We will take special care to point out when differences arise.

First, using Equation 1.3 as well as the definition of \mathcal{E} , we have that u_n is uniformly bounded in $L^\infty([t_1, t_2], L^2)$ and in $L^2([t_1, t_2], H^1)$. Thus, we use Alaoglu compactness theorem to find subsequences (which we will keep reindexing as u_n) which converge to some u weak-* in $L^\infty([t_1, t_2], L^2)$ and weakly in $L^2([t_1, t_2], H^1)$.

Next, using the fact that $A : H^1 \rightarrow H^{-1}$ is continuous using the assignment

$$\langle Av, \phi \rangle_{H^{-1}} := \langle A^{1/2}v, A^{1/2}\phi \rangle_{L^1} = \langle v, \phi \rangle_{H^1},$$

we get that Au_n is uniformly bounded in $L^2([t_1, t_2], H^{-1})$. Thus, we can extract a subsequence and relabel as u_n so that Au_n converges to Au weakly in $L^2([t_1, t_2], H^{-1})$.

A classical estimate (or Lemma 2.2.1) gives us that

$$\|B(u, u)\|_{H^{-1}} \leq C\|u\|_2\|u\|_{H^1} \quad \text{in two dimensions}$$

$$\|B(u, u)\|_{H^{-1}} \leq C\|u\|_2^{1/2}\|u\|_{H^1}^{3/2} \quad \text{in three dimensions.}$$

for any $u \in H^1$. Thus, using the fact that u_n is uniformly bounded in both $L^\infty([t_1, t_2], L^2)$ and $L^2([t_1, t_2], H^1)$, we see that $B(u_n, u_n)$ is uniformly bounded in $L^2([t_1, t_2], H^{-1})$ in two dimensions. In three dimensions, we see that $B(u_n, u_n)$ is uniformly bounded in $L^{4/3}([t_1, t_2], H^{-1})$. First, in two dimensions,

$$\begin{aligned} \|B(u_n, u_n)\|_{L^2([t_1, t_2], H^{-1})}^2 &\leq C \int_{t_1}^{t_2} \|u_n(r)\|_2^2 \|u_n(r)\|_{H^1}^2 dr \\ &\leq C \|u_n\|_{L^\infty([t_1, t_2], L^2)}^2 \|u_n\|_{L^2([t_1, t_2], H^1)}^2. \end{aligned}$$

On the other hand, in three dimensions,

$$\begin{aligned} \|B(u_n, u_n)\|_{L^{4/3}([t_1, t_2], H^{-1})}^{4/3} &\leq C \int_{t_1}^{t_2} \|u_n(r)\|_2^{2/3} \|u_n(r)\|_{H^1}^2 dr \\ &\leq C \|u_n\|_{L^\infty([t_1, t_2], L^2)}^{2/3} \|u_n\|_{L^2([t_1, t_2], H^1)}^2. \end{aligned}$$

Since Au_n , $B(u_n, u_n)$, and f are uniformly bounded sequences in $L^2([t_1, t_2], H^{-1})$ in two dimensions, we use Equation 1.2 to say that so is $\frac{d}{dt}u_n$. In three dimensions, we find that $\frac{d}{dt}u_n$ is uniformly bounded in $L^{4/3}([t_1, t_2], H^{-1})$.

So, we have that $\frac{d}{dt}u_n, B(u_n, u_n)$ converges weakly in $L^2([t_1, t_2], H^{-1})$ in two dimensions. In three dimensions, $\frac{d}{dt}u_n, B(u_n, u_n)$ converges weakly in $L^{4/3}([t_1, t_2], H^{-1})$. Moreover, a standard compactness argument gives us that then u_n converges strongly to u in $L^2([t_1, t_2], L^2)$ in either dimension. Using the strong convergence of u_n as well as the uniform bound of u_n in $L^\infty([t_1, t_2], L^2)$, a well known result shows that $B(u_n, u_n)$ converges weakly to $B(u, u)$ in $L^2([t_1, t_2], H^{-1})$ in two dimensions. In three dimensions, $B(u_n, u_n)$ converges weakly to $B(u, u)$ in $L^{4/3}([t_1, t_2], H^{-1})$.

Passing to the limit gives us that

$$\frac{d}{dt}u + \nu Au + B(u, u) = f$$

in H^{-1} . Now, take the inner product with $v \in H^1$ and integrate from t to $t + h$ where $t_1 \leq t < t + h \leq t_2$. In either dimension, we get that

$$\langle u(t+h) - u(t), v \rangle_{L^2} = -\nu \int_t^{t+h} \langle u(r), v \rangle_{H^1} dr - \int_t^{t+h} \langle B(u(r), u(r)), v \rangle_{H^{-1}} dr + \int_t^{t+h} \langle f(r), v \rangle_{H^{-1}} dr.$$

Taking the absolute value and using Cauchy-Swartz followed by Hölder's inequality, we find that

$$\begin{aligned}
|\langle u(t+h) - u(t), v \rangle_{L^2}| &\leq \nu \left(\int_t^{t+h} \|u(r)\|_{H^1}^2 dr \right)^{1/2} \left(\int_t^{t+h} \|v\|_{H^1}^2 dr \right)^{1/2} \\
&\quad + \left(\int_t^{t+h} \|B(u(r), u(r))\|_{H^{-1}}^2 dr \right)^{1/2} \left(\int_t^{t+h} \|v\|_{H^1}^2 dr \right)^{1/2} \\
&\quad + \left(\int_t^{t+h} \|f(r)\|_{H^{-1}}^2 dr \right)^{1/2} \left(\int_t^{t+h} \|v\|_{H^1}^2 dr \right)^{1/2}
\end{aligned}$$

in two dimensions. A similar process in three dimensions gives us that

$$\begin{aligned}
|\langle u(t+h) - u(t), v \rangle_{L^2}| &\leq \nu \left(\int_t^{t+h} \|u(r)\|_{H^1}^2 dr \right)^{1/2} \left(\int_t^{t+h} \|v\|_{H^1}^2 dr \right)^{1/2} \\
&\quad + \left(\int_t^{t+h} \|B(u(r), u(r))\|_{H^{-1}}^{4/3} dr \right)^{3/4} \left(\int_t^{t+h} \|v\|_{H^1}^4 dr \right)^{1/4} \\
&\quad + \left(\int_t^{t+h} \|f(r)\|_{H^{-1}}^2 dr \right)^{1/2} \left(\int_t^{t+h} \|v\|_{H^1}^2 dr \right)^{1/2}.
\end{aligned}$$

Using the above uniform bounds then gives us that

$$\lim_{h \rightarrow 0} \langle u(t+h) - u(t), v \rangle_{L^2} = 0$$

in either dimension. Since H^1 is dense in L^2 , $u \in C([t_2, t_1], L_w^2)$, and we are done. \square

Theorem 5.2.2. *The generalized evolutionary system for the 2D or 3D Navier-Stokes equations*

\mathcal{E} satisfies [A1](#), [A2](#), and [A3](#).

Proof. Let u_n be a sequence in $\mathcal{E}([s, \infty))$ for some $s \in \mathbb{R}$. Then, repeatedly using Lemma 5.2.1, there is a subsequence which we reindex as u_n that converges to some $u^1 \in C([s, s+1], L_w^2)$. A further subsequence converges to some $u^2 \in C([s, s+2], L_w^2)$ with $u^1(t) = u^2(t)$ on $[s, s+1]$. Continuing this diagonalization process, we get that there is some subsequence u_{n_j} converging to $u \in C([s, \infty), L_w^2)$. Note that the convergence in Lemma 5.2.1 gives us that the energy inequality

$$\|u_n(t)\|_2^2 + 2\nu \int_{t_0}^t \|u_n(r)\|_{H^1}^2 dr \leq \|u_n(t_0)\|_2^2 + 2 \int_{t_0}^t \langle f(r), u_n(r) \rangle_{H^{-1}} dr$$

converges as well to

$$\|u(t)\|_2^2 + 2\nu \int_{t_0}^t \|u(r)\|_{H^1}^2 dr \leq \|u(t_0)\|_2^2 + 2 \int_{t_0}^t \langle f(r), u(r) \rangle_{H^{-1}} dr$$

for $t \geq t_0$ and t_0 a.e. in $[s, \infty)$. That is, $u \in \mathcal{E}([s, \infty))$, and A1 is proven.

Next, for A2, let $u \in \mathcal{E}([s, \infty))$ for some $s \in \mathbb{R}$. Let $\epsilon > 0$. Then, using the Leray-Hopf energy inequality Equation 1.3, we get that

$$\|u(t)\|_2^2 \leq \|u(t_0)\|_2^2 + \frac{1}{\nu} \int_{t_0}^t \|f(r)\|_{H^{-1}}^2 dr$$

from Equation 5.1 for all $s \leq t_0 \leq t$, t_0 a.e. in $[s, \infty)$. Since $f \in L^2_{loc}(\mathbb{R}, H^{-1})$, there is a $\delta := \delta(\epsilon, s) > 0$ so that for t_0 a.e. in $(t - \delta, t)$, we have that

$$\|u(t)\|_2^2 \leq \|u(t_0)\|_2^2 + \epsilon,$$

and A2 follows.

For A3, we again see a slight difference depending on the dimension. To begin, let $u_n \in \mathcal{E}[s, \infty)$ for some $s \in \mathbb{R}$ and let $u_n \rightarrow u \in \mathcal{E}([s, \infty))$ in $C([s, t], L^2_w)$. Then, as we saw in Lemma 5.2.1, u_n is uniformly bounded in $L^2([s, t], H^1)$ for any $T \geq s$. In two dimensions, $\frac{d}{dt}u_n$ is uniformly bounded in $L^2([s, t], H^{-1})$. In three dimensions, $\frac{d}{dt}u_n$ is uniformly bounded in $L^{4/3}([s, t], H^{-1})$. In either scenario, we find that $u_n \rightarrow u$ strongly in $L^2([s, t], L^2)$. In particular, we have

$$\int_s^t \|u_n(r) - u(r)\|_2^2 dr \rightarrow 0$$

as $n \rightarrow \infty$. Thus, $\|u_n(t_0)\|_2 \rightarrow \|u(t_0)\|_2$ a.e. on $[s, t]$. □

Therefore, using A1 we have the following results for \mathcal{E} .

Theorem 5.2.3. *The weak pullback attractor for the generalized evolutionary system \mathcal{E} of the 2D or 3D Navier-Stokes equations, $\mathcal{A}_w(t)$, is the maximal pullback quasi-invariant and maximal pullback invariant subset of X . Also,*

$$\mathcal{A}_w(t) = \mathcal{I}(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

Moreover, \mathcal{E} satisfies the weak pullback tracking property and if the strong pullback attractor $\mathcal{A}_s(t)$ exists, $\mathcal{A}_w(t) = \mathcal{A}_s(t)$.

Using this, we now have that \mathcal{E} is pullback asymptotically compact, assuming that complete trajectories are strongly continuous. This is known to be the case in two dimensions as was stated in Theorem 1.1.5. In three dimensions, it is not currently known. This gives us the following two theorems which are a direct consequence of Corollary 4.1.9, Theorem 4.1.7, and Theorem 4.2.2:

Theorem 5.2.4. *The generalized evolutionary system \mathcal{E} for the 2D Navier-Stokes equations is pullback asymptotically compact. Therefore, \mathcal{E} has strongly compact, strong pullback attractor $\mathcal{A}_s(t)$. Also, the strong and weak pullback attractors coincide giving us that*

$$\mathcal{A}_s(t) = \mathcal{A}_w(t) = \mathcal{J}(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

That is, $\mathcal{A}_s(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set. Finally, \mathcal{E} has the strong pullback attracting property.

Theorem 5.2.5. *Suppose the generalized evolutionary system \mathcal{E} for the 3D Navier-Stokes equations has the property that $\mathcal{E}((-\infty, \infty)) \subseteq C((-\infty, \infty), X_s)$. Then, \mathcal{E} has strongly compact, strong pullback attractor $\mathcal{A}_s(t)$. Also, the strong and weak pullback attractors coincide giving us that*

$$\mathcal{A}_s(t) = \mathcal{A}_w(t) = \mathcal{J}(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

That is, $\mathcal{A}_s(t)$ is the maximal pullback invariant and maximal pullback quasi-invariant set.

Finally, \mathcal{E} has the strong pullback attracting property.

CHAPTER 6

TRIVIAL PULLBACK ATTRACTORS

In this section, we deal exclusively with the 3D Navier-Stokes equations.

6.1 Trivial Pullback Attractors for the 3D Navier-Stokes Equations

To begin, we will expand our definition of translationally boundedness from Chapter 5. To that end, Fix $\tau > 0$. Assume f is translationally bounded in $L^2_{\text{loc}}(\mathbb{R}, H^{-1})$. That is,

$$\|f\|_{L^2_b(\tau)}^2 := \sup_{t \in \mathbb{R}} \frac{1}{\tau} \int_t^{t+\tau} \|f(r)\|_{H^{-1}}^2 dr < \infty.$$

In the original definition of translationally boundedness, we had chosen $\tau := 1$.

First, note that $\|f\|_{H^{-1}}$ and $\|f\|_{L^2_b(\tau)}$ have the same dimensions. Next, note that the choice of τ is not particularly important. In fact, for any $\tau, \rho > 0$, we have that the norms $\|\cdot\|_{L^2_b(\tau)}$ and $\|\cdot\|_{L^2_b(\rho)}$ are equivalent.

Lemma 6.1.1. *Let $\tau, \rho > 0$ be given. Assume, without loss of generality that $\tau \leq \rho$. Then, for any translationally bounded $f \in L^2_{\text{loc}}(\mathbb{R}, H^{-1})$,*

$$\frac{\tau}{\rho} \|f\|_{L^2_b(\tau)}^2 \leq \|f\|_{L^2_b(\rho)}^2 \leq \frac{N\tau}{\rho} \|f\|_{L^2_b(\tau)}^2,$$

where N is any integer so that $N\tau \geq \rho$.

The proof is elementary and is thus omitted. Therefore, we may use whatever $\tau > 0$ we like in our calculations. Later, we will choose $\tau := (\nu\mu_1)^{-1}$.

Using a similar calculation to Chapter 5, there exists an absorbing ball for Leray solutions of the projected Navier-Stokes equations, Equation 1.2. That is, from the energy inequality and the fact that g is translationally bounded, one can derive the following inequality:

$$\|u(t)\|_2^2 \leq \|u(s)\|_2^2 e^{\nu\mu_1(s-t)} + \frac{\tau \|f\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\mu_1\tau})}.$$

Letting

$$R^2 := \frac{2\tau \|f\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\mu_1\tau})}, \quad (6.1)$$

we define

$$X := \{u \in L^2 : \|u\|_2 \leq R\}$$

as a closed absorbing ball in L^2 . We may then define a generalized evolutionary system on X exactly as it appears in Chapter 5.

By Theorem 5.2.3, the weak pullback attractor $\mathcal{A}_w(t)$, is nonempty, and is given by

$$\mathcal{A}_w(t) = \{u(t) : u \in \mathcal{E}((-\infty, \infty))\}.$$

In particular, there exists a complete bounded (in the sense of L^2) weak solution to the 3D Navier-Stokes equations. In the next section, we will present an argument demonstrating that

when the force is small enough, the weak pullback attractor consists of only one such solution.

In this case, we have that $\mathcal{A}_w(t) = \{u(t)\}$, is trivial.

6.2 Degenerate Pullback Attractors

6.2.1 A Criterion for Strong Solutions

In our goal of proving the existence of that the pullback attractor consists of a single point, we will begin by showing that if the force is small enough, then a complete bounded solution guaranteed by Theorem 5.2.3 is, in fact, a strong solution.

Definition 6.2.1. A weak solution u to Equation 1.2 will be called strong if $u \in L_{loc}^\infty(\mathbb{R}, H^1)$.

Let v be a complete bounded solution to Equation 1.2 as discussed in the previous section. In particular, v satisfies the inequality Equation 1.3. Using the Cauchy-Schwarz inequality followed by Young's inequality, we find that

$$\|v(t)\|_2^2 + \nu \int_s^t \|v(r)\|_{H^1}^2 dr \leq \|v(s)\|_2^2 + \frac{1}{\nu} \int_s^t \|f(r)\|_{H^{-1}}^2 dr. \quad (6.2)$$

Using the radius of the absorbing ball given in Equation 6.1 and dropping the first term on the left-hand side, we find that

$$\nu \int_s^t \|v(r)\|_{H^1}^2 dr \leq \frac{2\tau \|f\|_{L_b^2(\tau)}^2}{\nu(1 - e^{-\nu\mu_1\tau})} + \frac{1}{\nu} \int_s^t \|f(r)\|_{H^{-1}}^2 dr.$$

Thus, we find that for any $s \in \mathbb{R}$

$$\int_s^{s+\tau} \|v(r)\|_{H^1}^2 dr \leq \frac{\tau \|f\|_{L_b^2(\tau)}^2 (3 - e^{-\nu\mu_1\tau})}{\nu^2(1 - e^{-\nu\mu_1\tau})}. \quad (6.3)$$

Hence, we find that for any $M \geq 0$,

$$|\{x \in [s, s + \tau] : \|v(x)\|_{H^1} \geq M\}| \leq \frac{1}{M^2} \frac{\tau \|f\|_{L_b^2(\tau)}^2 (3 - e^{-\nu\mu_1\tau})}{\nu^2(1 - e^{-\nu\mu_1\tau})}.$$

Letting $M := \left(\frac{2\|f\|_{L_b^2(\tau)}^2 (3 - e^{-\nu\mu_1\tau})}{\nu^2(1 - e^{-\nu\mu_1\tau})} \right)^{-1/2}$, we have that

$$|\{x \in [s, s + \tau] : \|v(x)\|_{H^1} \geq M\}| \leq \frac{\tau}{2}.$$

We encapsulate the above remarks into the following lemma.

Lemma 6.2.2. *Let v be any complete, bounded solution to [Equation 1.2](#) with f translationally bounded in $L_{loc}^2(\mathbb{R}, H^{-1})$ whose existence is guaranteed by [Theorem 5.2.3](#). Then, for any $t \in \mathbb{R}$, there exists a point $s \in [t, t + \tau]$ so that*

$$\|v(s)\|_{H^1}^2 \leq \frac{2\|f\|_{L_b^2(\tau)}^2 (3 - e^{-\nu\mu_1\tau})}{\nu^2(1 - e^{-\nu\mu_1\tau})} < \infty.$$

Now, we add the assumption that f is translationally bounded in $L^2_{loc}(\mathbb{R}, L^2)$ which will be assumed for the remainder of the paper. That is, we assume that

$$\|f\|_{L^2_0(\tau)}^2 := \sup_{t \in \mathbb{R}} \frac{1}{\tau} \int_t^{t+\tau} \|f(r)\|_2^2 dr < \infty.$$

Note that using the Poincaré inequality, we have that

$$\|f\|_{L^2_b(\tau)}^2 \leq \mu_1^{-1} \|f\|_{L^2_0(\tau)}^2.$$

We will show that if $\|f\|_{L^2_0(\tau)}$ is sufficiently small, then $v \in L^\infty(\mathbb{R}, H^1)$.

To do this, let $t_0 \in \mathbb{R}$ be arbitrary. Then, consider the interval $[t_0 - \tau, t_0]$. By Lemma 6.2.2, there exists a point $s \in [t_0 - \tau, t_0]$ so that

$$\|v(s)\|_{H^1}^2 \leq \frac{2\|f\|_{L^2_b(\tau)}^2(3 - e^{-\nu\mu_1\tau})}{\nu^2(1 - e^{-\nu\mu_1\tau})} < \infty.$$

Thus, by Leray's characterization (1), there is an $\epsilon > 0$ so that v is a strong solution on $[s, s + \epsilon)$.

We investigate the length of this interval.

Starting with Equation 1.2, we take the inner product with Av giving us that

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^1}^2 + \nu \|v\|_{H^2}^2 \leq |\langle B(v, v), Av \rangle_{H^{-1}}| + |\langle f, Av \rangle_{L^2}|. \quad (6.4)$$

Classical estimates give us that

$$|\langle B(v, v), Av \rangle_{H^{-1}}| \leq c_0 \|v\|_{H^1}^{3/2} \|v\|_{H^2}^{3/2} \quad (6.5)$$

$$|\langle f, Av \rangle_{L^2}| \leq \|f\|_2 \|v\|_{H^2}. \quad (6.6)$$

Next, we apply Young's inequality on each of these terms to get that

$$\begin{aligned} |\langle B(v, v), Av \rangle_{H^{-1}}| &\leq \frac{\nu}{4} \|v\|_{H^2}^2 + \frac{c_0}{\nu^3} \|v\|_{H^1}^6 \\ |\langle f, Av \rangle_{L^2}| &\leq \frac{1}{\nu} \|f\|_2^2 + \frac{\nu}{4} \|v\|_{H^2}^2. \end{aligned}$$

Using these estimates as well as the Poincaré inequality, [Equation 6.4](#) reduces to

$$\frac{d}{dt} \|v\|_{H^1}^2 + \nu \mu_1 \|v\|_{H^1}^2 \leq \frac{2}{\nu} \|f\|_2^2 + \frac{c_0}{\nu^3} \|v\|_{H^1}^6. \quad (6.7)$$

Now, assume that

$$\|f\|_{L_0^2(\tau)}^2 \leq \frac{c_0^{-1/2} \nu^4 \mu_1^{3/2}}{2c_1 + 4\nu \mu_1 \tau}$$

where $c_1 := \frac{2(3-e^{-\nu \mu_1 \tau})}{1-e^{-\nu \mu_1 \tau}}$. Then, we will show that $\|v(t_0)\|_{H^1}^2 \leq c_0^{-1/2} \nu^2 \mu_1^{1/2}$. The following is a modification of the argument given in [\(3\)](#). For completeness, we present the argument in its entirety.

First, note that the criterion on $\|f\|_{L_0^2(\tau)}$ guarantees that

$$\begin{aligned} \|v(s)\|_2^2 + \frac{2}{\nu} \int_s^{s+\tau} \|f(r)\|_2^2 dr &\leq \frac{c_1}{\nu^2} \|f\|_{L_b^2(\tau)}^2 + \frac{2\tau}{\nu} \|f\|_{L_0^2(\tau)}^2 \\ &\leq \frac{c_1}{\nu^2 \mu_1} \|f\|_{L_0^2(\tau)}^2 + \frac{2\tau}{\nu} \|f\|_{L_0^2(\tau)}^2 \\ &\leq \frac{c_0^{-1/2} \nu^2 \mu_1^{1/2}}{2}. \end{aligned}$$

Then, certainly $\|v(s)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2}$. Let

$$T := \sup\{t \in [s, s + \tau] : \|v(t)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2}\}.$$

Since v is a strong solution at s we get that $T > s$. Assume that $T < s + \tau$. Using $\|v(T_0)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2}$ for each $T_0 \leq T$, we find that

$$\nu \mu_1 \|v(T_0)\|_{H^1}^2 - \frac{c_0}{\nu^3} \|v(T_0)\|_{H^1}^6 = \nu \mu_1 \|v(T_0)\|_{H^1}^2 \left(1 - \frac{c_0}{\nu^4 \mu_1} \|v(T_0)\|_{H^1}^4\right) \geq 0.$$

Thus, we integrate [Equation 6.7](#) from s to T and get that

$$\begin{aligned} \|v(T)\|_{H^1}^2 &\leq \|v(s)\|_{H^1}^2 + \frac{2}{\nu} \int_s^T \|f(r)\|_2^2 dr \\ &\leq \|v(s)\|_{H^1}^2 + \frac{2}{\nu} \int_s^{s+\tau} \|f(r)\|_2^2 dr \\ &\leq \frac{c_0^{-1/2} \nu^2 \mu_1^{1/2}}{2}. \end{aligned}$$

Thus, we must have that $T = s + \tau$. In particular, this is true of $t_0 \in [s, s + \tau]$. Since $t_0 \in \mathbb{R}$ was arbitrary, we have that

$$\|v(s)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2} \quad (6.8)$$

for all $s \in \mathbb{R}$. This completes the proof of the following theorem.

Theorem 6.2.3. *Suppose f is translationally bounded in $L_{loc}^2(\mathbb{R}, L^2)$ so that*

$$\|f\|_{L_0^2(\tau)}^2 \leq \frac{c_0^{-1/2} \nu^4 \mu_1^{3/2}}{2c_1 + 4\nu \mu_1 \tau}$$

for $c_1 := \frac{2(3-e^{-\nu \mu_1 \tau})}{1-e^{-\nu \mu_1 \tau}}$ and c_0 the constant given in [Equation 6.5](#). Then, there exists a complete, bounded, strong solution to [Equation 1.2](#) so that $v \in L^\infty(\mathbb{R}, H^1)$. In particular, $\|v(s)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2}$ for all $s \in \mathbb{R}$.

Now, let $\tau := (\nu \mu_1)^{-1}$. For simplicity, we let

$$\|f\|_{L_b^2} := \|f\|_{L_b^2((\nu \mu_1)^{-1})}$$

$$\|f\|_{L_0^2} =: \|f\|_{L_0^2((\nu \mu_1)^{-1})}.$$

Then, we can express [Theorem 6.2.3](#) in terms of the non-dimensional 3D Grashof number

$$G := \frac{\|f\|_{L_0^2}}{\nu^2 \mu_1^{3/4}}.$$

Corollary 6.2.4. *Suppose f is translationally bounded in $L^2_{loc}(\mathbb{R}, L^2)$ so that*

$$G^2 = \frac{\|f\|_{L^2_0}^2}{\nu^4 \mu_1^{3/2}} \leq \frac{c_0^{-1/2}}{2c_1 + 4}$$

for $c_1 := \frac{2(3-e^{-1})}{1-e^{-1}}$ and c_0 the constant given in [Equation 6.5](#). Then, there exists a complete, bounded, strong solution to [Equation 1.2](#) so that $v \in L^\infty(\mathbb{R}, H^1)$. In particular, $\|v(s)\|_{H^1}^2 < c_0^{-1/2} \nu^2 \mu_1^{1/2}$ for all $s \in \mathbb{R}$.

Remark 6.2.5. It is also worthwhile to note that the above argument proves the strongness of all complete trajectories in our generalized evolutionary system \mathcal{E} . In fact, it proves that if $u \in \mathcal{E}([s, \infty))$, then for $t > s + \tau$, u is strong.

6.2.2 A Serrin-type Argument

In [\(31\)](#), Serrin presents an argument for the uniqueness of weak solutions in an interval of regularity (where a strong solution exists). Using a modification of the argument as it is presented in [\(4\)](#), we obtain the required argument for the existence of degenerate pullback attractors.

Let v be a complete, bounded strong solution to [Equation 1.2](#) on $(-\infty, \infty)$ guaranteed by [Theorem 6.2.3](#). Let u be another Leray-Hopf weak solution to [Equation 1.2](#) on $[T, \infty)$ and let $w := u - v$. Then, u and v satisfy

$$\|u(t)\|_2^2 + 2\nu \int_s^t \|u(r)\|_{H^1}^2 dr \leq \|u_0\|_2^2 + 2 \int_s^t \langle f(r), u(r) \rangle_{L^2} dr, \quad (6.9)$$

$$\|v(t)\|_2^2 + 2\nu \int_s^t \|v(r)\|_{H^1}^2 dr = \|v_0\|_2^2 + 2 \int_s^t \langle f(r), v(r) \rangle_{L^2} dr, \quad (6.10)$$

respectively for each $t \geq s \geq T$ since v is a strong solution. Also, as seen in Temam's book (4)

$$\begin{aligned} \langle u(t), v(t) \rangle_{L^2} + 2\nu \int_s^t \langle u(r), v(r) \rangle_{H^1} dr &= \langle u(s), v(s) \rangle_{L^2} \\ &+ \int_s^t \langle f(r), u(r) + v(r) \rangle_{L^2} dr \\ &- \int_s^t \langle B(w(r), w(r)), v(r) \rangle_{H^{-1}} dr. \end{aligned} \quad (6.11)$$

Adding Equation 6.9 to Equation 6.10 and then subtracting twice Equation 6.11, we get that

$$\|w(t)\|_2^2 + 2\nu \int_s^t \|w(r)\|_{H^1}^2 dr \leq \|w(s)\|_2^2 + 2 \int_s^t \langle B(w(r), w(r)), v(r) \rangle_{H^{-1}} dr. \quad (6.12)$$

We estimate the nonlinear term using classical estimates. That is, we find that

$$\begin{aligned} |\langle B(w, w), v \rangle_{H^{-1}}| &\leq C \|w\|_2^{1/4} \|w\|_{H^1}^{7/4} \|v\|_2^{1/4} \|v\|_{H^1}^{3/4} \\ &\leq \frac{\nu}{2} \|w\|_{H^1}^2 + \frac{C}{\nu^7} \|v\|_2^2 \|v\|_{H^1}^6 \|w\|_2^2 \end{aligned}$$

after applying the Young's inequality. Since $v \in L^\infty(\mathbb{R}, H^1) \cap L^\infty(\mathbb{R}, L^2)$, we use Equation 6.1 and Equation 6.8 to estimate Equation 6.12 by

$$\|w(t)\|_2^2 - \|w(s)\|_2^2 \leq \nu \mu_1 \int_s^t \left(C \frac{\tau \|f\|_{L_0^2(\tau)}^2}{\nu^3 \mu_1^{1/2}} - 1 \right) \|w(r)\|_2^2 dr. \quad (6.13)$$

Assuming that $\|f\|_{L_0^2(\tau)}^2$ is sufficiently small, we can ensure that $C\tau\|f\|_{L_0^2(\tau)}^2 < \nu^3\mu_1^{1/2}$ giving us that

$$\|w(t)\|_2^2 - \|w(s)\|_2^2 \leq -M \int_s^t \|w(r)\|_2^2 dr \quad (6.14)$$

for $M := \nu\mu_1 \left(1 - C \frac{\tau\|f\|_{L_0^2(\tau)}^2}{\nu^3\mu_1^{1/2}}\right) > 0$. Thus, after applying Gronwall's inequality, we have that

$$\|w(t)\|_2^2 \leq \|w(s)\|_2^2 e^{M(s-t)}.$$

In particular, for t fixed and $s \rightarrow -\infty$, $\|w(t)\|_2 \rightarrow 0$. This completes the proof of the following theorem.

Theorem 6.2.6. *Let f be translationally bounded in $L_{loc}^2(\mathbb{R}, L^2)$. Assume that $\|f\|_{L_0^2(\tau)}$ is sufficiently small, then the weak pullback attractor for [Equation 1.2](#) is a single point,*

$$\mathcal{A}_w(t) = \{v(t)\}$$

for some complete, bounded, strong solution to [Equation 1.2](#).

Again, if we let $\tau := (\nu\mu_1)^{-1}$, then [Equation 6.13](#) simplifies to

$$\|w(t)\|_2^2 - \|w(s)\|_2^2 \leq \nu\mu_1 \int_s^t (CG^2 - 1) \|w(r)\|_2^2 dr. \quad (6.15)$$

So, we can restate [6.2.6](#) once again in terms of the 3D Grashof constant.

Corollary 6.2.7. *Let f be translationally bounded in $L^2_{loc}(\mathbb{R}, L^2)$. Assume that the Grashof number G given by*

$$G = \frac{\|f\|_{L^2_b}}{\nu^2 \mu_1^{3/4}}$$

is sufficiently small, then the weak pullback attractor for Equation 1.2 is a single point,

$$\mathcal{A}_w(t) = \{v(t)\}$$

for some complete, bounded, strong solution to Equation 1.2.

6.2.3 Periodic Force

The existence of a unique periodic solution to the 3D Navier-Stokes equations is a remarkable consequence of this Theorem. To begin, let the force f in Equation 1.2 be periodic in $L^2_{loc}(\mathbb{R}, L^2)$ with period ρ . A straightforward argument shows that f is translationally bounded. Thus, by Theorem 6.2.6, if f is sufficiently small, there exists a unique, strong solution w to Equation 1.2. We will show that w is, in fact, periodic.

Theorem 6.2.8. *Let f be periodic in $L^2_{loc}(\mathbb{R}, L^2)$ with period ρ . Assume that f is sufficiently small. Then, there exists a unique, periodic, strong solution w to Equation 1.2. In particular, w has period ρ .*

Proof. Due to Theorem 6.2.6, we only must show that w has period ρ . To this end, note that w satisfies the equation

$$\frac{d}{dt}w(t) + \nu Aw(t) + B(w(t), w(t)) = f(t). \quad (6.16)$$

Then, of course, w satisfies

$$\frac{d}{dt}w(t + \rho) + \nu Aw(t + \rho) + B(w(t + \rho), w(t + \rho)) = f(t + \rho).$$

But, $f(t + \rho) = f(t)$. So, $w(\cdot + \rho)$ also satisfies Equation 6.16. By uniqueness, $w(t + \rho) = w(t)$. \square

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VITA

Education

PhD in Mathemematics, The University of Illinois at Chicago, Chicago, IL (2015)

MS in Mathematics, The University of Wisconsin - Milwaukee, Milwaukee, WI (2011)

BA in Mathematics and Computer Science, Calvin College, Grand Rapids, MI (2010)

Publications

1. Alexey Cheskidov and Landon Kavlie. Gaining two derivatives on a singular force in the 2D Navier-Stokes equations. In preparation.
2. Gung-Min Gie, Christopher Henderson, Gautam Iyer, Landon Kavlie, and Jared Whitehead. Stability of Oseen's vortex in an extended Navier-Stokes system. Submitted.
3. Alexey Cheskidov and Landon Kavlie. Degenerate pullback attractors for the 3D Navier-Stokes equations. [*arXiv e-prints* 1403.6200](#). Accepted for publication in the *Journal of Mathematical Fluid Mechanics*.
4. Alexey Cheskidov and Landon Kavlie. Pullback attractors for generalized evolutionary systems. [*arXiv e-prints* 1310.4917](#). Accepted for publication in *Discrete and Continuous Dynamical Systems Series B*.
5. Michael Bolt, Timothy Ferdinands, and Landon Kavlie. The most general planar transformations that map parabolas into parabolas. *Involve*, 2(1):79-88, 2009.

6. Timothy Ferdinands and Landon Kavlie. A Beckman-Quarles type theorem for Laguerre transformations in the dual plane. *Rose-Hulman Undergraduate Math Journal*, 10(1), 2009.

Invited Talks

1. Fluid Flow, Turbulence, and Chaos. Wilkes University (2015).
2. The structure of pullback attractors for the 3D Navier-Stokes equations. UIC Applied Math Seminar (2014).
3. On the existence of pullback attractors for the 3D Navier-Stokes equations. SIAM conference on the Analysis of PDEs (2013).
4. On the existence of pullback attractors for the 3D Navier-Stokes equations. AMS special session on PDEs of Fluid Mechanics (2013).
5. A Beckman-Quarles type theorem for Laguerre transformations in the dual plane. Poster session at the 2009 joint meetings of the MAA and AMS. Washington, DC (2009).
6. A Beckman-Quarles type theorem for Laguerre transformations in the dual plane. Calvin College mathematics and statistics colloquium (2008).

Teaching Experience

Fall 2014:	Precalculus - Teaching Assistant
Summer 2014:	Calculus III - Teaching Assistant
Spring 2014:	Applied Linear Algebra - Teaching Assistant

Spring 2014:	Calculus III - Teaching Assistant
Fall 2013:	Differential Equations - Teaching Assistant
Summer 2013:	Differential Equations - Teaching Assistant
Spring 2013:	Calculus II - Teaching Assistant
Fall 2012:	Introduction to Advanced Math - Teaching Assistant
Summer 2012:	Intermediate Algebra - Lecturer
Spring 2012:	Calculus II - Teaching Assistant
Fall 2011:	Intermediate Algebra - Teaching Assistant

Honors and Awards

Spring 2015:	Research Assistant. University of Illinois at Chicago.
2013-2014:	Victor Twersky award for applied mathematics. University of Illinois at Chicago.
Spring 2014:	AMS travel grant.
Fall 2013:	AMS travel grant.
Summer 2013:	Selected to attend the mathematics research community on regularity problems for nonlinear partial differential equations modeling fluids and complex fluids.
2012-2013:	Outstanding teaching assistant of the year. University of Illinois at Chicago.

2011-2014: Teaching assistant. University of Illinois at Chicago.

2010-2011: Research fellowship. University of Wisconsin - Milwaukee.

Service

2015: Co-organizer for the Chicago Area SIAM Student Conference.

2014-2015: President of the UIC SIAM student chapter.

2013-2014: Vice-president of the UIC SIAM student chapter.

2014: Co-organizer for the Chicago Area SIAM Student Conference.

2014: Volunteer grader for the City of Chicago Math League.

2013: Co-organizer of the graduate analysis seminar.