Dade's Conjecture

in the Finite Special Unitary Groups

BY

KATHERINE BIRD
Bachelor of Science in Mathematics, University of Illinois at Chicago, 2000
Master of Science in Mathematics, University of Illinois at Chicago, 2004

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2012

Chicago, Illinois

Defense Committee:

Bhama Srinivasan, Chair and Advisor
Paul Fong
Ramin Takloo-Bighash
Brooke Shipley
Stephen Doty, Loyola University Chicago

To Lynn.

ACKNOWLEDGMENTS

I thank my advisor Bhama Srinivasan for her wisdom and patience in guiding me through my years at UIC, in particular through the process of completing my thesis. I thank Paul Fong for his careful reading of my thesis and his advice. I extend my deepest thanks to all the members of my defense committee.

I thank the NSF for the Vigre fellowship which supported the first three years of my graduate studies. Finally, I thank the entire UIC mathematics department and its staff for the uncountably many tangible and intangible means of support given to me.

KAB

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

SUMMARY

We prove Dade's ordinary conjecture (DOC) for the finite Special Unitary groups in the defining characteristic. We base our approach on the following existing work. Olsson and Uno proved DOC for the finite General Linear groups. Sukizaki then modified their approach in order to show that DOC holds for the finite Special Linear groups. The key aspect of his approach is to reformulate the alternating sum for $\mathrm{SL}_n(q)$ in terms of an alternating sum for $\mathrm{GL}_n(q)$ via Clifford theory. Chao Ku modified Olsson and Uno's work to show DOC for the finite Unitary groups in his Caltech Ph.D. thesis. In this thesis, we take Sukizaki's method and apply it to Ku's work. Some modifications are necessary as the structure of the finite unitary groups is more complicated than that of the finite general linear groups. In particular the cancellations, in the alternating sum in the statement of DOC, in the unitary case are very different from the cancellations that occur in the general linear case.

The combinatorial details involved in counting characters of parabolic subgroups are fairly involved. Hence we devote a chapter by way of example to the manageable case when $n = 4$. This is the smallest case which is just large enough to be non trivial with respect to the cancellation in the sum.

CHAPTER 1

INTRODUCTION

Let G be a finite group. An ordinary representation of G is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

where V is a finite dimensional vector space over a field K of characteristic zero. Then V is the G-module, or KG-module, afforded by ρ. If we fix a basis for V, then $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}_n(K)$. Then the degree of the representation ρ is n. We define the character χ of ρ by $\chi(g) = \mathrm{Tr}(\rho(g))$. In general, though we may consider K to be the field of complex numbers, in practice we may take K to be a sufficiently large extension of $\mathbb{Q}$. A character χ is irreducible if the associated vector space V afforded by ρ has no proper nonzero KG-submodules. We will denote the ordinary irreducible characters of G by $\mathrm{Irr}(G)$. The set $\mathrm{Irr}(G)$ forms an orthonormal basis for the K-space of class functions on G.

Throughout this thesis we assume that p is a fixed prime number. A modular representation of G is defined similarly, the key difference being that the module afforded is a vector space over a field of characteristic p. A modular representation of G is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(W)$$

where W is a vector space over a field k of characteristic p. Then W is the G-module, or kG-module, afforded by ρ. Restricting our attention to the elements of G with order prime to p, we may define a so called Brauer character, a complex valued class function on elements with order prime to p.

The philosophy of modular representations goes back to Brauer and in part relates "global information" to "p-local information". By global we mean the group G, and by p-local we mean subgroups related to p-subgroups of G, for example normalizers of p-subgroups. Information can refer to a variety of things including numbers of characters or character values.

Modular representations of G give rise to a partition of $\mathrm{Irr}(G)$ into blocks. Dade's Conjecture was first presented in a series of papers entitled *Counting Characters in Blocks*. The interplay between global and local blocks is informative in both directions. The conjecture involves an alternating sum involving characters of p-local subgroups. In order to state Dade's fairly elaborate conjecture we must first assemble some concepts.

1 Modular Representations

Let K be an algebraic number field, a splitting field for G and its subgroups. Let $\mathcal{O}$ be the ring of algebraic integers in K. Let $\mathcal{P}$ be a prime ideal in $\mathcal{O}$ containing p. Let $\mathcal{R}$ be the ring of $\mathcal{P}$-integral elements of K, i.e. the localization of $\mathcal{O}$ at $\mathcal{P}$. Let $\pi\mathcal{R}$ be the unique maximal ideal in $\mathcal{R}$ with respect to a valuation associated with the prime ideal $\mathcal{P}$. We define k to be the residue field $\mathcal{R}/\pi\mathcal{R}$ which is isomorphic to $\mathcal{O}/\mathcal{P}$. Then k has characteristic p. The triple

$(K, \mathcal{R}, k)$ is called a p-modular system. We may look at the following group algebras: KG, $\mathcal{R}G$, and kG.

We use $\mathcal{R}G$ in order to pass from ordinary to p-modular representations of G. As an $\mathcal{R}$-module, the $\mathcal{R}$-algebra $\mathcal{R}G$ is free and of finite rank. We have

$$KG = K \otimes_{\mathcal{R}} \mathcal{R}G \text{ and}$$

$$kG = k \otimes_{\mathcal{R}} \mathcal{R}G = {}^{\mathcal{R}G}/_{\pi \mathcal{R}G}\,.$$

A finitely generated $\mathcal{R}$-free $\mathcal{R}G$-module is called an $\mathcal{R}G$-lattice. If V is a KG-module then there exists an $\mathcal{R}G$-lattice M with

$$K \otimes_{\mathcal{R}} M \cong V.$$

Note that this is not unique as it depends on our choice of basis for V. Set

$$\overline{M} = k \otimes_{\mathcal{R}} M = M/\pi M \text{ as } kG\text{-module.}$$

Then $\overline{M}$ is a modular representation, or module, of G. The composition factors of $\overline{M}$ are unique up to isomorphism and do not depend on the choice of M.

The K-algebra KG is semi-simple and thus completely reducible. However kG is not semi-simple if $p \mid |G|$. Rather it can only be written as a sum of indecomposable two sided ideals.

These indecomposable subalgebras of kG are called the p-blocks of kG. A decomposition of kG into blocks

$$kG = B_1 \oplus B_2 \oplus \cdots \oplus B_s$$

corresponds to a decomposition of the identity $1 = e_1 + e_2 + \cdots + e_s$ where the e_i are orthogonal primitive central idempotents in kG. This is given by $B_i = e_i kG$. The class sums of elements in G form a basis for both $Z(\mathcal{R}G)$ and $Z(kG)$. Hence reducing mod $\pi\mathcal{R}$ is a surjective map $Z(\mathcal{R}G) \longrightarrow Z(kG)$. We may lift the e_i to orthogonal primitive central idempotents f_i in $\mathcal{R}G$. Then the decomposition $1 = f_1 + f_2 + \cdots + f_s$ in $\mathcal{R}G$ corresponds to a decomposition of $\mathcal{R}G$ into two sided ideals also called the p-blocks of $\mathcal{R}G$.

$$\mathcal{R}G = \hat{B}_1 \oplus \hat{B}_2 \oplus \cdots \oplus \hat{B}_s$$

where $\hat{B}_i = f_i \mathcal{R}G$ and $B_i = k \otimes_{\mathcal{R}} \hat{B}_i$.

If V is an irreducible KG-modules then $f_i V = V$ for a unique f_i and $f_j V = 0$ for all $j \neq i$. We say that V belongs to the block $B = f_i KG$. If V affords the character χ, we also say that χ is in B. This gives rise to a partition of the set of ordinary characters of G into blocks. We may informally think of a p-block B of G as simultaneously being all of the following related objects:

- an indecomposable two-sided ideal kG-module $e_i kG$ for primitive idempotent $e_i \in Z(kG)$
- an indecomposable two-sided ideal $\mathcal{R}G$-module$f_i \mathcal{R}G$ for primitive idempotent $f_i \in Z(\mathcal{R}G)$ where f_i is the lift of e_i.

- the set of irreducible KG-modules V for which $f_iV = V$
- the set of ordinary characters of the KG-modules V as above

The two-sided ideal summands of kG (or $\mathcal{R}G$) are the same as the direct summands of kG (or $\mathcal{R}G$) as a $k(G \times G)$ (or $\mathcal{R}(G \times G)$) -module where the action of $k(G \times G)$ (or $\mathcal{R}(G \times G)$) on kG (or $\mathcal{R}G$) is given by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.

Let B be a p-block of G. Then B has associated to it a p-group D called a defect group of B and a non-negative integer called the defect of B. The subgroup D is a minimal subgroup of G such that every B-module is a direct summand of an induced module from D ((19), p. 122). If D is such a defect group and $|D| = p^d$ then B has defect d. We define the defect of a character χ to be the maximum power of p dividing $\frac{|G|}{\chi(1)}$. Clearly, the defect of a character is inversely related to the power of the p-part of its degree. If B has defect d, then B contains a character of defect d and the defect of all other characters in B is less than or equal to d. We have two extremes. Write $|G| = p^e m$ where $p \nmid m$. If B contains a linear character then the defect of B is e. In this case we say B has full defect. For example the block containing the trivial module K, equivalently containing the trivial character, is called the principal block and has full defect. If B contains a character of degree divisible by p^e, then B has zero defect. It turns out that a block B of defect zero contains exactly one character ((7), Proposition 56.31).

Brauer's First Main Theorem states that if D is a p-subgroup of G then there exists a bijection between blocks B of G with defect group D and blocks b of $N_G(D)$ with defect group D. Let $H \leq G$ satisfy $DC_G(D) \leq H \leq N_G(D)$. Let B be a block of G and b be a block of

H. We say b induces to B and write $b^G = B$ if b, as a $k(H \times H)$-module, is a summand in the restriction $B_{H\times H}$ of the $k(G \times G)$-module B to $H \times H$ and that B is the only block for which this holds ((1), p.101). For H as above, b^G is always defined.

2 Statement of Dade's Ordinary Conjecture

Let G be a finite group and p a prime. Given a chain of p-subgroups $C : U_0 < U_1 < \cdots < U_l$ in G we define the length of C, $|C| = l$. We say that C is radical if $U_0 = O_p(G)$, the maximal normal p-subgroup of G and $U_i = O_p(\cap_{j=0}^{i} N_G(U_j))$ for $1 \leq i \leq l$. Let $N_G(C)$ denote $\cap_{j=0}^{l} N_G(U_j)$. Observe that if two chains C_1 and C_2 are conjugate to one another, then $N_G(C_1) \cong N_G(C_2)$. If b is a p-block of $N_G(C)$, then $b^G = B$ is defined. Let

$$\mathrm{Irr}(N_G(C), B, d) = \left\{\psi \in \mathrm{Irr}(N_G(C)) \mid \psi \in b \text{ where } b^G = B \text{ and } \psi \text{ has defect } d\right\}.$$

We will set $k(N_G(C), B, d) = |\mathrm{Irr}(N_G(C), B, d)|$.

Conjecture 1.2.1 Dade's Ordinary Conjecture (DOC) *((8), Conjecture 6.3). Let G be a finite group with $O_p(G) = 1$ so that all radical chains in G begin with the trivial group $U_0 = 1$. Let B be a block of G of nonzero defect. Then the following holds:*

$$\sum{}' (-1)^{|C|} k(N_G(C), B, d) = 0, \qquad \forall d \geq 0$$

where $|C| = l$ is the length of C and $\sum'$ indicates the sum over a set of representatives of conjugacy classes of radical chains in G.

3 Refinement of DOC for Certain Finite Reductive Groups

DOC reduces nicely for certain finite reductive groups in the defining characteristic. Let G be a finite reductive group of characteristic p. Then G is the group of fixed points of a Frobenius endomorphism of a connected reductive algebraic group. We consider the p-blocks of G. Let I be an index set for the distinguished generators of the Weyl group W of G. Let B be a Borel subgroup of G. In this thesis P_J will denote the parabolic subgroup $BW_{I\setminus J}B$. For example, $P_I = B$ is the Borel subgroup of G (rather than $P_\emptyset$). It is also useful to think of parabolics indexed in the following way: If $\{P_j \mid j \in I\}$ is a complete set of maximal parabolic subgroups in G, then

$$P_J = \bigcap_{j \in J} P_j.$$

Let $C : U_0 < U_1 < \cdots < U_l$ be a radical chain of p-subgroups in G. Then $U_0 = O_p(G) = 1$. Moreover, $U_1 = O_p(N_G(U_1))$ and hence, by ((4), Corollary), U_1 must be the unipotent radical U_J of a parabolic subgroup P_J of G with $N_G(U_J) = P_J$. We have the familiar Levi decomposition $P_J = L_J U_J$. It is obvious that $U_1 \subseteq B$. Notice that $P_J/U_J \cong L_J$ is itself a finite group of Lie type with Borel subgroup isomorphic to $B \cap L_J$. The quotient U_2/U_J is isomorphic to a p-group of $B \cap L_J$ and hence is isomorphic to a unipotent radical of a parabolic subgroup of L_J. Since $U_2 = O_p(P_J \cap N_G(U_2)) = O_p(N_{P_J}(U_2))$, we must have $U_2 = U_{J'}$ where $J' \supset J$. Hence C is a chain of unipotent radicals and $N_G(C)$ is equal to $N_G(U_l)$ the normalizer of the last term so that $N_G(C) = P_J$ for suitable J depending only on the last term of the chain C.

It turns out that there is considerable cancellation amongst the G-conjugacy classes of chains of unipotent radicals for G. The collection of all such chains C which terminate with a fixed U_J and thus have $N_G(C) = P_J$ cancels almost entirely, due to the alternating parity of the involved chains. One uncancelled chain remains of maximal length J. By a standard argument ((14), p.58),

$$\sum_{C}(-1)^{|C|}k(N_G(C), B, d) = \sum_{J\subseteq I}(-1)^{|J|}k(P_J, B, d)$$

where the sum on the left is taken over a set of representatives of G-conjugacy classes of chains of unipotent radicals.

The possible defect of p-blocks is well known for finite groups of Lie type, otherwise known as finite reductive groups, of characteristic p ((13)). The only possibilities are blocks of zero defect and blocks of full defect. In Humphreys' concluding remarks he notes that the number of blocks of zero defect is equal to the index of the derived subgroup G' in G and that the number of block of full defect is equal to the order of the center of G.

Let us now restrict our attention to $G = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{U}_n(q)$, or $\mathrm{SU}_n(q)$, where q is a power of p. In fact we have a bijection from the set of p-blocks of P_J to the set of p-blocks of G of full defect. Indeed, the center of G is a torus and hence has order prime to p. Hence $O_{p'}(Z(G)) = Z(G)$ and $O_p(Z(G)) = 1$. Let $Z(G) = Z$. The group Z centralizes U_J so $U_JZ \subseteq U_JC_G(U_J)$ certainly holds. It happens that $U_JC_G(U_J) \subseteq U_JZ$ holds for these four families of groups. Thus by ((16), Lemma 2.1), ψ and ψ' lie in the same block b of P_J if and only if their restrictions to Z have the same constituent. In other words, P_J has $|Z|$ blocks and a block b of P_J is determined by a unique character $\rho \in \mathrm{Irr}(Z)$. The induced block $b^G = B$ is defined. B has full defect and is determined by the same ρ. The proof for $\mathrm{U}_n(q)$ is analogous to the proof for $\mathrm{GL}_n(q)$ in ((16), Lemma 2.1). If $\psi \in \mathrm{Irr}(P_J)$ restricted to Z contains ρ we will say that ψ lies over ρ.

Write $|G| = p^e m$. Each parabolic subgroup P_J contains U_I the unipotent radical of the Borel subgroup of G. This is a Sylow p-subgroup of G. Hence $|P_J|$ is divisible by p^e for every $J \subseteq I$. As noted above the defect of a character is inversely related to the power of p dividing its degree. If the p-part of $\psi(1)$ is p^a for $\psi \in \mathrm{Irr}(P_J)$, then the defect of ψ is $e - a$. Hence it is equivalent to count characters by their so called p-height rather than their defect.

Definition 1.3.1 *We define the p-height of ψ to be d if $p^d \| \psi(1)$. Similarly, we define the q-height ψ to be d if $q^d \| \psi(1)$.*

Remark: This definition is not entirely standard. In the literature p-height is generally defined with reference to the defect of the block containing the character. For example Brauer's definition of height in his Height Conjecture is more standard. However if ψ is in a block of full defect, then the p-height as it is usually defined is equal to the maximal power of p dividing $\psi(1)$ and hence our definition coincides with the standard.

Let ρ be an irreducible character of the center of G and define

$$k_d(P_J, \rho) = \left| \left\{ \psi \in \mathrm{Irr}(P_J) \mid \psi \text{ lies over } \rho \text{ and } p^d \| \psi(1) \right\} \right|.$$

Then DOC is equivalent to the following:

Conjecture 1.3.2 *Let $q = p^a$. Let $G = \mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{U}_n(q)$, or $\mathrm{SU}_n(q)$, with parabolic subgroups P_J indexed by subsets $J \subseteq I$. Let Z be the center of G. Let $|G| = p^e m$ where $p \nmid m$. Then $\forall \rho \in \mathrm{Irr}(Z)$*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho) = 0 \ \forall d \ , \ 0 \leq d < e.$$

If G is either of $\mathrm{GL}_n(q)$ or $\mathrm{U}_n(q)$, then the p part of the degree of characters in $\mathrm{Irr}(G)$ are powers of q. Indeed, this is well known for $\mathrm{GL}_n(q)$ and follows for $\mathrm{U}_n(q)$ by replacing q in the $\mathrm{GL}_n(q)$-character theory by $-q$ in the $\mathrm{U}_n(q)$-theory, by Ennola's conjecture, now proved. For $G = \mathrm{GL}_n(q)$, or $\mathrm{U}_n(q)$, let $S = \mathrm{SL}_n(q)$, or $\mathrm{SU}_n(q)$ respectively. The group S is the kernel of the determinant map on G. Moreover the quotient G/S is cyclic of order $q-1$ or $q+1$ respectively, in either case prime to p. Take any $\psi \in \mathrm{Irr}(S)$, then there exists $\chi \in \mathrm{Irr}(G)$ such that ψ is a constituent of the restriction of χ to S. By Frobenius reciprocity, we may choose any irreducible χ appearing in the induction of ψ from S to G. Then, by 2.4.1 and 2.4.4 in the next chapter,

$$\chi|_S = \psi_1 + \psi_2 + \cdots + \psi_r \text{ where the } \psi_i \in \mathrm{Irr}(S) \text{ are } G\text{-conjugates of } \psi$$

and r divides $|G/S|$. Thus

$$\chi(1) = \psi_1(1) + \psi_2(1) + \cdots + \psi_r(1).$$

Since r is prime to p, it follows from Clifford theory that the p-height of ψ is equal to the p-height of χ and hence is also a power of q. Suppose $\chi \in \mathrm{Irr}(G)$, or $\mathrm{Irr}(S)$ has p-height d. Then d is certainly divisible by a so χ has q-height d/a. It turns out that from Olsson and Uno's construction for $\mathrm{GL}_n(q^2)$ and Ku's construction for $\mathrm{U}_n(q)$ the characters of parabolic subgroups of G also have degrees with p part equal to a power of q. As we will see in chapter 4, parabolic subgroups of S are in fact the kernel of the determinant map restricted to parabolic subgroups of G. Thus, by the same reasoning as above, they also have degrees with p part equal to a power of q. Hence in statement 1.3.2 of DOC, for d not divisible by a the left hand side of the sum is empty and so vacuously true. This allows us to simplify our notation by counting characters via q-height rather than p-height. Henceforth and for the rest of this thesis we redefine the subscript d so that it indicates q-height so for example $\mathrm{Irr}_d(P_J, \rho)$ will denote irreducible characters of P_J lying over ρ with q-height d.

4 Some Results for Dade's Conjecture and Implications

We summarize the cases for which some version of Dade's conjecture has been shown, including the result of this thesis. References for this section are ((12), Section 5) and on the web at ((17)).

1. Classical Groups:

$\mathrm{GL}_n(q)$	ord., $p \mid q$	Olssen, Uno
$\mathrm{U}_n(q)$	ord., $p \mid q$	Ku
$\mathrm{GL}_n(q)$, $\mathrm{U}_n(q)$	invar., $p \nmid q$	An
$Sp_{2n}(q)$, $SO_m^{\pm}(q)$	ord., $p \nmid q$, p, q odd	An
L_2	final	Dade
L_3	final, $p \mid q$	Dade
L_n	ord., $p \mid q$	Sukizaki
$Sp_4(2^n)$	final, $p = 2$	An, Himstedt, Huang
$SU_4(2^{2n})$		
$Sp_4(q)$	invar., $p \mid q$, p odd	An, Himstedt, Huang, Yamada
$\mathrm{SU}_n(q)$	ord., $p \mid q$	Bird

2. Sporadic Simple Groups:

M_{11}, M_{12}, J_1, J_2	final	Dade
M_{22}	final	Huang
M_{23}, M_{24}	final	Schwartz, An, Conder
J_3	final	Kotlica
McL	final	Murray, Entz, Pahlings
Ru	final	Dade, An, O'Brien
He	final	An
HS	final	Hassan, Horváth
Co_1	final	An, O'Brien
Co_2	final	An, O'Brien
Co_3	final	An
Suz	final	Himstedt
$O'N$	final	An, O'Brien, Uno, Yoshiara
Th	final	Uno
Ly	final	Sawabe, Uno
HN	final	An, O'Brien
Fi_{23}	final	An, O'Brien
Fi_{22}	invar.	An, O'Brien
J_4		An, O'Brien, Wilson
B	p odd	An, Wilson
Fi'_{24}		An, Cannon, O'Brien, Unger

3. Exceptional Groups:

${}^2B_2(2^{2n+1})$	final	Dade
${}^2G_2(3^{2n+1})$	final	$p \neq 3$ An, $p = 3$ Eaton
$G_2(q)$	final, $2, 3 \mid q$, $p \nmid q$ $q \neq 3, 4$	An
$G_2(q)$	final, $p \mid q (p \geq 5)$, $q = 3, 4$	Huang
${}^3D_4(q)$	final, $p \nmid q$	An
${}^3D_4(q)$	final, $p \mid q$ ($p = 2$ or odd)	An, Himstedt, Huang
${}^2F_4(2^{2n+1})$	ord., $p \neq 2$	An
${}^2F_4(2^{2n+1})$	final, $p = 2$	Himstedt, Huang
${}^2F_4(2)'$	final	An

4. Other cases:

S_n	ord., $p \neq 2$	Olssen, Uno
S_n	ord., $p = 2$	An
A_n, abelian defect	ord.	Fong, Harris
Cyclic defect group	final	Dade
Tame block	invar.	Uno
Abelian defect unipotent blocks	ord.	Broué, Malle, Michel
Abelian defect principal blocks	ord., $p = 2$	Fong, Harris
Abelian defect some cases	ord.	Piug, Usami
p-solvable	proj.	Robinson
$O_p(G)$ cyclic, $O_p(G)/G$ p-Sylow TI	proj.	Eaton
Nilpotent blocks		

The sequence of most interest with respect to this thesis is the following: Olsson and Uno proved DOC for $\mathrm{GL}_n(q)$ in the defining characteristic (16). Sukizaki proved it for $\mathrm{SL}_n(q)$ also in the defining characteristic (22). Chao Ku verified DOC in his doctoral thesis for $\mathrm{U}_n(q)$.

Assuming that Dade's Ordinary Conjecture is true for all finite groups implies a number of other conjectures. In this sense DOC encodes a variety of information. DOC implies Alperin's Weight Conjecture which counts Brauer characters. DOC implies the Alperin-McKay Conjecture which is a refinement of the McKay Conjecture. DOC also implies one direction of Brauer's Height Conjecture which involves abelian defect groups.

CHAPTER 2

PRELIMINARIES

We begin these preliminaries with our definition of the finite unitary and special unitary groups. Throughout this thesis q is a fixed power of the prime p. Let $K = \overline{F_q}$, and $\widetilde{G} = \mathrm{GL}_n(K)$. Define the matrix

$$M = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

Define the following Frobenius map F on $\widetilde{G}$ by:

$$F(a_{i,j}) = M(a_{j,i}^q)^{-1}M^{-1}.$$

The group of fixed points $\widetilde{G}^F$ is the finite unitary group $\mathrm{U}_n(q)$, i.e.

$$\mathrm{U}_n(q) = \{(a_{i,j}) \mid M = (a_{i,j})M(a_{j,i}^q)\}.$$

Clearly $\mathrm{U}_n(q) \leq \mathrm{GL}_n(q^2)$. The advantage of this definition is that F fixes the subgroup of upper triangular matrices in $\mathrm{GL}_n(q^2)$. We can define the special unitary groups in two equivalent ways. On the one hand, the group of fixed points of $\mathrm{SL}_n(K)$ under F is $\mathrm{SU}_n(q)$. On the other hand,

$$\mathrm{SU}_n(q) = \{A \in \mathrm{U}_n(q) \mid \det(A) = 1\}.$$

The Weyl group W of $\mathrm{U}_n(q)$ is of type B_m, where $n = 2m$, or $2m+1$, and is isomorphic to the wreath product $C_2 \wr S_m$. The symmetric group on m elements is generated by reflections indexed by $\{1, 2, \ldots, m-1\}$ and the cyclic group of order 2 is generated by the reflection indexed by $\{m\}$. With this identification, the distinguished generators of W may be indexed by $I = \{1, 2, \ldots, m\}$ denoted by $[m]$.

1 Some Notation

Throughout this thesis we will make use of the following notation. Let q be a fixed power of prime p. We consider the finite field F_{q^2} and its group of units $F_{q^2}^*$. For divisors h of $q^2 - 1$, let $\mathbb{C}_h$ denote the cyclic subgroup of order h in $F_{q^2}^*$. So in particular $\mathbb{C}_{q+1}$ denotes the cyclic subgroup of order $q+1$ in $\mathbb{C}_{q^2-1}$.

2 On radical p-chains

In order to reformulate DOC for the finite special unitary groups we will need the following proposition due to Sukizaki.

Proposition 2.2.1 *((22), Proposition 2.1) Let G be a finite group and let H be a subgroup of G. If H contains all p-subgroups of G and satisfies $O_p(G) = O_p(H)$, then any radical p-chain of H is a radical p-chain of G.*

3 Certain functions on partitions

In this section we are following the development of Olsson and Uno ((16)), Sukizaki ((22)), and Ku ((15)). To that end we discuss partitions. Further we define two important functions α and β on pairs (μ, a) where μ is a partition and a is a field element. The function α was defined in ((16), p.363). The function β was introduced as a unitary version of α and was defined in ((15), p.16). These functions are involved in expressing the q-height of characters. Further, it turns out that they are also involved in the splitting of characters upon restriction to certain subgroups. We assert some combinatorial facts about the behavior of these functions.

Let $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$, where $a_1 > a_2 \cdots > a_r > 0$. We define $|\mu| = \sum_{i=1}^{r} l_i a_i = n$. Let the number of distinct parts of the partition be $\delta(\mu) = r$ and the length of the partition $l(\mu) = \sum_{i=1}^{r} l_i$. We define $\gamma(\mu) = \gcd(a_1, a_2, \ldots, a_r)$ and $\lambda(\mu) = \gcd(l_1, l_2, \ldots, l_r)$.

Given $\mu_1 \vdash n_1$ and $\mu_2 \vdash n_2$ we can define $2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2$. In order to define this new partition write $\mu_i = (1^{m_{i1}}, 2^{m_{i2}}, \ldots, n_i^{m_{in_i}})$, so that for nonzero m_{it}, the integer t appears in μ_i with multiplicity m_{it}. Then define $2\mu_1 \cup \mu_2 = (1^{2m_{11}+m_{21}}, 2^{2m_{12}+m_{22}}, \ldots, n^{2m_{1n}+m_{2n}})$.

Definition 2.3.1 *Let $a \in \mathbb{C}_{q^2-1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. We define a function α by*

$$\alpha(\mu, a) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q^2-1})^r \mid (-1)^n {x_1}^{a_1} x_2^{a_2} \ldots x_r^{a_r} = a\}|.$$

Lemma 2.3.2 *With μ defined as above and $a \in \mathbb{C}_{q^2-1}$ we have*

$$\alpha(\mu, a) = (q^2 - 1)^{r-1}\alpha(\gamma, a)$$

where $\gamma = \gamma(\mu)$ and

$$\alpha(\gamma, a) = \begin{cases} gcd(q^2-1, \gamma), & \text{if } a \in \mathbb{C}_{(q^2-1)/gcd(q^2-1,\gamma)}; \\ 0, & \text{otherwise.} \end{cases}$$

See ((16), p.363) for the proof.

Definition 2.3.3 *Let $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. We define a function β*

$$\beta(\mu, b) = |\{(x_1, x_2, \ldots, x_r) \in (\mathbb{C}_{q+1})^r \mid (-1)^n {x_1}^{a_1} x_2^{a_2} \ldots x_r^{a_r} = b\}|.$$

Lemma 2.3.4 *With μ defined as above and $b \in \mathbb{C}_{q+1}$ we have*

$$\beta(\mu, b) = (q+1)^{r-1}\beta(\gamma, b)$$

where $\gamma = \gamma(\mu)$ and

$$\beta(\gamma, b) = \begin{cases} gcd(q+1, \gamma), & \text{if } b \in \mathbb{C}_{(q+1)/gcd(q+1,\gamma)}; \\ 0, & \text{otherwise.} \end{cases}$$

The proof is similar to the proof of Lemma 2.3.2.

In practice we will be restricting our attention to elements b in $\mathbb{C}_{q+1}$ and hence have need of the following modification of our α function which was defined in ((15), p.16).

Definition 2.3.5 *For $b \in \mathbb{C}_{q+1}$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$ we define*

$$\overline{\beta}(\mu, b) = \sum_{\substack{a \in \mathbb{C}_{q^2-1} \\ a^{q-1}=b}} \alpha(\mu, a).$$

Some important technical facts from ((15), (16), and (22)) regarding α, β, and $\overline{\beta}$ are summarized in the following lemmas.

Lemma 2.3.6 *If $(k) \vdash k$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$, then*

$$\sum_{\substack{b_1, b_2 \in \mathbb{C}_{q+1} \\ b_1 b_2 = b}} \beta((k), b_1)\beta(\mu, b_2) = \beta(\lambda, b)$$

where $\lambda = ((a_1 + k)^{l_1}, (a_2 + k)^{l_2}, \ldots, (a_r + k)^{l_r}, k^x) \vdash (n + (l(\mu) + x)k)$.

Lemma 2.3.7 *If $\mu_i \vdash n_i$, for $i = 1, 2$, and $\mu = 2\mu_1 \cup \mu_2 \vdash n = 2n_1 + n_2$, then*

$$\sum_{\substack{b_1, b_2 \in \mathbb{C}_{q+1} \\ b_1 b_2 = b}} \overline{\beta}(\mu_1, b_1)\beta(\mu_2, b_2) = (q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1, \mu_2)}\beta(\mu, b)$$

where $c(\mu_1, \mu_2)$ is the number of distinct entries that μ_1 and μ_2 have in common.

Lemma 2.3.8 *If $\mu \vdash n$, then*

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2}} q^{2(l(\mu_1)-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = q^{l(\mu)-\delta(\mu)}.$$

Notice that this sum is taken over all pairs of partitions (μ_1, μ_2) such that $\mu = 2\mu_1 \cup \mu_2$.

These results are proved in ((15)). We mention that the last 2.3.8 is proved by associating to the pair (μ_1, μ_2) a matrix and its shadow which are defined as follows:

Definition 2.3.9 *For two partitions $\mu_i = (t^{m_{it}})$, $i = 1, 2$, with $\mu = (2\mu_1 \cup \mu_2) \vdash n$ we define the 2 by n matrix*

$$A(\mu_1, \mu_2) = A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix}$$

Given such an A we define the shadow of A to be the 2 by n matrix (c_{ij}) where the ij-entry

$$c_{ij} = \begin{cases} 1, & \text{if } m_{ij} \text{ is nonzero;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus taking the sum over pairs (μ_1, μ_2) in Lemma 2.3.8 is equivalent to taking the sum over possible matrices $A(\mu_1, \mu_2)$. We also remark that with this definition of the shadow, the number of entries that μ_1 and μ_2 have in common $c(\mu_1, \mu_2) = \sum_{t=1}^{n} c_{1t}c_{2t}$.

Suppose that we have a pair (μ_1, μ_2) with $\mu = 2\mu_1 \cup \mu_2$ a partition of n and all nonzero multiplicities of μ_i, $i = 1, 2$, are divisible by a fixed integer j, i.e. $j \mid \gcd(\lambda(\mu_1), \lambda(\mu_2))$. First of all it is clear that $j|\lambda(\mu)$. Write $\mu = (t^{m_t})$ so that $2m_{1t}+m_{2t} = m_t$. Observe that $tm_t \leq n$ must hold. In particular $(n/j)m_{n/j} \leq n$ implies that $m_{n/j}$ is the last possibly nonzero exponent in μ. In other words the matrix $A(\mu_1, \mu_2)$ must have zero entries to the right of the n/j-column. Furthermore, A may be decomposed:

$$A = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \end{pmatrix} = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \end{pmatrix} = jB.$$

If we remove the zero columns to the right of the n/j-column of B, we obtain $A(\kappa_1, \kappa_2)$, the 2 by n/j matrix associated to $\kappa_1 = (t^{k_{1t}})$ and $\kappa_2 = (t^{k_{2t}})$ for $1 \leq t \leq n/j$. If $\mu = (t^{2m_{1t}+m_{2t}})$, then $\kappa = (t^{2k_{1t}+k_{2t}}) = 2\kappa_1 \cup \kappa_2$, a partition of n/j. We have the following equalities:

$$l(\mu_1)/j = l(\kappa_1); \quad \delta(\mu_1) = \delta(\kappa_1);$$

$$c(\mu_1, \mu_2) = c(\kappa_1, \kappa_2); \quad l(\mu)/j = l(\kappa); \quad \delta(\mu) = \delta(\kappa).$$

Thus for a fixed partition $\mu \vdash n$

$$\sum_{\substack{(\mu_1,\mu_2) \\ \mu=2\mu_1\cup\mu_2 \\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}$$

$$= \sum_{\substack{(\kappa_1,\kappa_2) \\ \kappa=2\kappa_1\cup\kappa_2}} q^{2(l(\kappa_1)-\delta(\kappa_1))}(q-1)^{\delta(\kappa_1)}(q+1)^{c(\kappa_1,\kappa_2)}$$

Hence we have the following important corollary to 2.3.8:

Corollary 2.3.10 *If $\mu \vdash n$, then*

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)} = q^{l(\mu)/j-\delta(\mu)}.$$

4 Applications of the Clifford Theory

We will make abundant use of Clifford Theory. A reference for this section is ((7), Chapter 11). We summarize here the results that we need. In this section we assume that G is a finite group with a normal subgroup H. For $\psi \in \mathrm{Irr}(H)$ and $g \in G$ we define the character ${}^g\psi$ by ${}^g\psi(h) = \psi(g^{-1}hg)$ for all $h \in H$. Let $T_G(\psi)$ denote the stabilizer of ψ in G so that

$$T_G(\psi) = \{g \in G \mid {}^g\psi = \psi\}.$$

We define a subset of characters in $\mathrm{Irr}(G)$

$$\mathrm{Irr}(G,\psi) = \{\chi \in \mathrm{Irr}(G) \mid (\chi|_H, \psi)_H \neq 0\}.$$

If $\chi \in \mathrm{Irr}(G,\psi)$ we will say that χ corresponds to ψ.

We define a subset of characters in $\mathrm{Irr}(H)$

$$\mathrm{Irr}(H,\chi) = \{\psi \in \mathrm{Irr}(H) \mid (\chi|_H, \psi)_H \neq 0\}.$$

This definition is equivalent to saying that the χ appear in the induced character of ψ to G. Thus we will say that ψ corresponds to χ if $\psi \in \mathrm{Irr}(H, \chi)$.

Theorem 2.4.1 *((7), Proposition 11.4) Let $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G, \psi)$. Then*

$$\chi|_H = e \left(\sum_{x \in G/T_G(\psi)} {}^x\psi \right)$$

where e is a positive integer.

Theorem 2.4.2 *((7), Theorem 11.5) Let $\psi \in \mathrm{Irr}(H)$ and suppose that $\psi = \tilde{\psi}|_H$ for some character $\tilde{\psi}$ of $T_G(\psi)$, that is, suppose that ψ can be extended to a character $\tilde{\psi}$ of $T_G(\psi)$. Write $T = T_G(\psi)$ then*

$$\mathrm{Irr}(T, \psi) = \{ \ \theta\tilde{\psi} \mid \theta \in \mathrm{Irr}(T/H) \ \}, \quad \textit{and}$$

$$\mathrm{Irr}(G, \psi) = \{ \ (\theta\tilde{\psi})^G \mid \theta \in \mathrm{Irr}(T/H) \ \}.$$

Here we regard θ as a character of T.

We have the following corollary which is a simple consequence of the transitivity of character induction.

Corollary 2.4.3 *If $\psi \in Irr(H)$ and P is any subgroup of G that contains $T_G(\psi)$ then*

$$|\mathrm{Irr}(P, \psi)| = |\mathrm{Irr}(G, \psi)|.$$

Moreover, there is a 1-1 correspondence between characters in each of these sets given by

$$\phi = (\theta\tilde{\psi})^P \leftrightarrow (\theta\tilde{\psi})^G = \chi.$$

If $\phi \in \mathrm{Irr}(P, \psi)$ corresponds to $\chi \in \mathrm{Irr}(G, \psi)$ and $\chi(1)$ has p-part equal to p^d then the p-part of $\phi(1)$ is $p^{d-d'}$ where $|P\backslash G|$ has p-part $p^{d'}$.

Lemma 2.4.4 *((23), Lemma 2.5) If G/H is cyclic then the following hold:*

1. *Characters of G restricted to H are multiplicity free. In other words $e = 1$ in 2.4.1.*
2. *Two characters of G either restrict to the same character of H or have disjoint irreducible components.*
3. *If $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G, \psi)$ then $|\mathrm{Irr}(H, \chi)| = |G|/(|H||\mathrm{Irr}(G, \psi)|)$.*

We have another important consequence of Clifford Theory concerning when an extension of a character exists when G is a semi-direct product.

Lemma 2.4.5 *((22), Theorem 2.5) Let G be a finite group with $G = P \ltimes M$.*

1. *If $\tau \in \mathrm{Irr}(M)$ is linear, then τ extends to an irreducible character $\tilde{\tau}$ of $T = T_G(\tau)$. Moreover*

$$\mathrm{Irr}(G, \tau) = \{(\theta\tilde{\tau})^G \mid \theta \in \mathrm{Irr}(T/M)\}.$$

2. *Let H be a normal subgroup of G containing M and suppose that G/H is cyclic. If $\theta \in \mathrm{Irr}(T/M)$, then*

$$|\mathrm{Irr}(H, (\theta\tilde{\tau})^G)| = |G : TH||\mathrm{Irr}(T_H(\tau), \theta)|.$$

We will be interested in the existence of extensions of non-linear characters of certain normal subgroups. The following result of Dade's on the extendibility of characters of normal extraspecial p-subgroups is certainly relevant.

Lemma 2.4.6 *((9)) Let E be an extra special p-group and $G = H \ltimes E$ with $Z(E) \leq Z(G)$. Assume that for each normal p'-subgroup K of H, the commutator subgroup $[K, E] = 1$. If $\psi \in \mathrm{Irr}(E)$ is non-linear, then ψ is extendible to G.*

5 On a Product of Groups

We will be examining the splitting of characters of direct products upon restriction to certain normal subgroups. We note that if $G = G_1 \times G_2$, then an irreducible character of G is of the form $\chi_1\chi_2$ where $\chi_i \in \mathrm{Irr}(G_i)$. We will have need of the following result.

Lemma 2.5.1 *((22)) Let $G = G_1 \times G_2$ where the group homomorphism $\phi_i : G_i \to F^*_{q^2}$ has image $\mathbb{C}_{h_i}$ for $i = 1, 2$. Set*

$$H = \{(g_1, g_2) \in G \mid \phi_1(g_1)\phi_2(g_2) = 1\}.$$

If χ_i has m_i irreducible constituents upon restriction to $\ker \phi_i$, then $\chi = \chi_1\chi_2$ restricted to H has m irreducible constituents, where

$$m = \frac{\gcd\left(m_1(q^2-1)/h_1, m_2(q^2-1)/h_2\right)}{\gcd\left((q^2-1)/h_1, (q^2-1)/h_2\right)}$$

.

6 Restriction of Characters to the Kernel of the Determinant Map

In this section we present results for $\mathrm{GL}_n(q^2)$ and $\mathrm{U}_n(q)$. Sukizaki's result is proved using G.I. Lehrer's work which uses an earlier parametrization of the characters of $\mathrm{GL}_n(q)$. We will use the more modern approach of Deligne-Lusztig theory. However, our goal remains the same in that we construct sequences of polynomials corresponding to characters and use them to count the number of characters.

Definition 2.6.1 *Let $G = \mathrm{GL}_n(q^2)$ or $\mathrm{U}_n(q)$. Given a homomorphism $\phi : G \to (F_{q^2})^*$ and $\rho \in \mathrm{Irr}(Z(G))$, we define the following:*

1. *Let $\mathrm{Irr}_d(G, \rho, \phi, j)$ be the set of irreducible ordinary characters χ of G with q-height d and lying over ρ such that the restriction of χ to the kernel of the map ϕ has j irreducible components.*

2. *Let $k_d(G, \rho, \phi, j)$ denote the number of irreducible ordinary characters χ of G with q-height d and lying over ρ such that the restriction of χ to the kernel of the map ϕ has j' irreducible components, where j divides j', i.e.*

$$k_d(G, \rho, \phi, j) = \sum_{\substack{j' \\ j|j'}} |\mathrm{Irr}_d(G, \rho, \phi, j')|.$$

We will be considering the determinant map on $\mathrm{U}_n(q)$ and certain subgroups. For a matrix element A, $\det(A)$ denotes the usual matrix determinant. We will consider subgroups of $\mathrm{U}_n(q)$

whose elements are block matrices. If A is a block matrix with block matrices $A_1, A_2, \ldots, A_s$ down its diagonal then $\det(A) = \det(A_1)\det(A_2)\cdots\det(A_s)$. Moreover, if certain of the A_i are repeated then $\det(A)$ may involve powers of $\det(A_i)$. Define $\det^h(A) = (\det(A))^h$. We apply definition 2.6.1 below with $\phi = \det$.

We now fix an isomorphism between $(\overline{F}_q)^*$ and $\mathrm{Irr}((\overline{F}_q)^*)$ and consider it fixed for the rest of this thesis. In practice, we are primarily interested in the subgroup $F^*_{q^2}$. The group $Z = Z(\mathrm{GL}_n(q^2)) \cong F^*_{q^2} = \mathbb{C}_{q^2-1}$. Further, assume that the induced isomorphism of $\mathbb{C}_{q^2-1}$ with $\mathrm{Irr}(\mathbb{C}_{q^2-1})$ is given by the following. Let ε generate $\mathbb{C}_{q^2-1}$. Define the isomorphism via

$$\varepsilon \mapsto \rho_\varepsilon \text{ where } \rho_\varepsilon(\varepsilon) = e^{(2\pi i)/(q^2-1)}.$$

Under this isomorphism, $\rho \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$ corresponds to $a_\rho \in \mathbb{C}_{q^2-1}$. Equivalently $a \in \mathbb{C}_{q^2-1}$ corresponds to $\rho_a \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$. This induces an isomorphism of $\mathrm{Irr}(Z(\mathrm{U}_n(q)))$ with $\mathbb{C}_{q+1}$.

The following integer valued function on partitions of n is involved in the q-height of characters for $\mathrm{GL}_n(q^2)$ and $\mathrm{U}_n(q)$.

Definition 2.6.2 *We define $n'(\mu)$:*

$$n'(\mu) = \sum_{i=1}^{r} l_i \binom{a_i}{2}.$$

Our first proposition is a slight reformulation of Sukizaki's result in (22). This is needed as our ground field is F_{q^2} rather than F_q. For $\mu \vdash n$, recall $\lambda(\mu)$, $l(\mu)$, and $\delta(\mu)$ defined on page 16.

Proposition 2.6.3 *((22), Lemma 4.1) Let* $\rho \in \mathrm{Irr}(\mathbb{C}_{q^2-1})$. *Then*

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j \mid \gcd(q^2-1, \lambda(\mu))}} q^{2(l(\mu)/j-\delta(\mu))}\alpha(\mu, a_\rho).$$

For $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ we define

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det^{1-q}, j) = \sum_{\substack{\rho' \in \mathrm{Irr}(\mathbb{C}_{q^2-1}) \\ \rho'|_{\mathbb{C}_{q+1}} = \rho}} k_{2d}(\mathrm{GL}_n(q^2), \rho', det^{1-q}, j).$$

Then since $(q^2-1)/(q-1) = q+1$, by Sukizaki's equation following equation 3-5 in (22) we have a disjoint union

$$\mathrm{Irr}_{2d}(\mathrm{GL}_n(q^2), \rho', \det^{1-q}, j) = \bigsqcup_{\substack{j' \\ j=\gcd(q+1,j')}} \mathrm{Irr}_{2d}(\mathrm{GL}_n(q^2), \rho', \det, j').$$

This together with our definition of $\overline{\beta}$ from earlier in this chapter implies the following corollary:

Corollary 2.6.4 *Let $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$. Then*

$$k_{2d}(\mathrm{GL}_n(q^2), \rho, \det{}^{1-q}, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} q^{2(l(\mu)/j-\delta(\mu))}\overline{\beta}(\mu, a_\rho).$$

We present the case for $\mathrm{U}_n(q)$ now for completeness. We will prove this in the next chapter.

Proposition 2.6.5 *Let $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$. Then*

$$k_d(\mathrm{U}_n(q), \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu, a_\rho)$$

Remark: Let χ be an irreducible character of $\mathrm{GL}_n(q^2)$. If $\chi\mid_{\ker\det}$ has j irreducible constituents, then j divides $\gcd(q^2-1, n)$. ((20), Theorem 4.7). If $\chi\mid_{\ker\det^{1-q}}$ has j irreducible constituents, then j divides $\gcd(q+1, n)$. Now let χ be an irreducible character of $\mathrm{U}_n(q)$. If $\chi\mid_{\ker\det}$ has j irreducible constituents, then j divides $\gcd(q+1, n)$. An identical theorem for the unitary case may be obtained making the following simple modifications to the proof of ((20), Theorem 4.7): Change Definition 4.6 by defining

$$M(d) = \{A \in \mathrm{U}_n(q) \mid \det A = \xi^{dk}\ , k = 1, \ldots, (q+1)/d\}$$

where $d = \gcd(n, q+1)$ and $\xi = \varepsilon^{1-q}$, a generator of the subgroup $\mathbb{C}_{q+1}$ in $F^*_{q^2}$. Then in Lemma 4.6 and Theorem 4.7 replace $\mathrm{GL}_n(q)$ with $\mathrm{U}_n(q)$ and $\mathrm{SL}_n(q)$ with $\mathrm{SU}_n(q)$.

CHAPTER 3

CHARACTERS OF $\mathbf{U_N(Q)}$ RESTRICTED TO $\mathbf{SU_N(Q)}$

In this chapter we prove Proposition 2.6.5. We start by parameterizing irreducible characters χ of $\mathrm{U}_n(q)$ via pairs (s, λ), and construct a unique sequence of polynomials $(h_1(x), h_2(x), \ldots)$ corresponding to χ. The subgroup $\mathrm{SU}_n(q)$ is normal in $\mathrm{U}_n(q)$ with cyclic quotient isomorphic to $\mathbb{C}_{q+1}$ which acts naturally on $\mathrm{Irr}(\mathrm{SU}_n(q))$ via $\mathrm{U}_n(q)$-conjugation. In the last chapter, we fixed an isomorphism $\mathbb{C}_{q^2-1} \cong \mathrm{Irr}(\mathbb{C}_{q^2-1})$ and hence we have an isomorphism $\mathbb{C}_{q+1} \simeq \mathrm{Irr}(\mathbb{C}_{q+1})$. The group $\mathrm{Irr}(\mathbb{C}_{q+1})$ acts on $\mathrm{Irr}(\mathrm{U}_n(q))$. Indeed, if $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ then we have a corresponding linear character of $\mathrm{U}_n(q)$ also denoted by ρ. Then $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ acts on $\mathrm{Irr}(\mathrm{U}_n(q))$ by

$$\chi \mapsto \rho \otimes \chi \text{ abbreviated by } \rho\chi.$$

Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$. By Clifford Theory, χ restricted to $\mathrm{SU}_n(q)$ is multiplicity free. If

$$\chi|_{\mathrm{SU}_n(q)} = \psi_1 + \psi_2 + \cdots \psi_j \qquad \text{where } \psi_i \in \mathrm{Irr}(\mathrm{SU}_n(q))$$

then the ψ_i are $\mathrm{U}_n(q)$-conjugates of one another.

The following lemma uses the well-known fact on characters of finite groups that if G is a finite group and $H \leq G$, and ϑ, η are characters of H, G respectively, then

$$\eta \mathrm{Ind}_H^G(\vartheta) = \mathrm{Ind}_H^G(\eta|_H \vartheta).$$

Lemma 3.0.6 *Let χ, $\chi' \in \mathrm{Irr}(\mathrm{U}_n(q))$. Then χ, χ' have the same restriction to $\mathrm{SU}_n(q)$ if and only if $\chi' = \rho\chi$ for some $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$.*

Proof: Let $\chi' = \rho\chi$. Since $\mathrm{SU}_n(q)$ is in the commutator subgroup of $\mathrm{U}_n(q)$, ρ is trivial on $\mathrm{SU}_n(q)$ and hence χ, χ' have the same restriction to $\mathrm{SU}_n(q)$.

Suppose ψ is a common constituent of χ, χ' restricted to $\mathrm{SU}_n(q)$. Let T be the stabilizer of ψ in $\mathrm{SU}_n(q)$. Then ψ extends to $\tilde{\psi} \in \mathrm{Irr}(T)$, and we have

$$\chi = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi}\phi), \qquad \chi' = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi}\phi')$$

where ϕ, ϕ' are lifts to T of characters of $T/\mathrm{SU}_n(q)$, denoted ϕ_1, ${\phi'}_1$. Then ϕ_1, ${\phi'}_1$ can be extended to characters ξ_1, ${\xi'}_1$ of $\mathrm{U}_n(q)/\mathrm{SU}_n(q)$, which can be lifted to characters ξ, ξ' of $\mathrm{U}_n(q)$. Then we have

$$\chi = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi})\xi, \qquad \chi' = \mathrm{Ind}_T^{\mathrm{U}_n(q)}(\tilde{\psi})\xi'.$$

Thus χ, χ' differ by a linear character. Since every linear character of $\mathrm{U}_n(q)$ is of the form ρ_z for some $z \in \mathbb{C}_{q+1}$ we have $\chi' = \rho\chi$.

Let $\mathcal{E} = \{\psi_1, \psi_2, \ldots, \psi_j\}$ and let $\mathcal{F} = \{\chi_1, \chi_2, \ldots, \chi_r\}$ where the χ_i are the constituents of the induced character $\mathrm{Ind}_{\mathrm{SU}_n(q)}^{\mathrm{U}_n(q)}(\psi)$ for any $\psi \in \mathcal{E}$. Then $\mathcal{E}$ is a $\mathbb{C}_{q+1}$-stable subset of $\mathrm{Irr}(\mathrm{SU}_n(q))$ and $\mathcal{F}$ is a $\mathrm{Irr}(\mathbb{C}_{q+1})$-stable subset of $\mathrm{Irr}(\mathrm{U}_n(q))$. Hence $r = (q+1)/j$ and our original character χ is stabilized by $\rho_z \in \mathrm{Irr}(\mathbb{C}_{q+1})$ where $z \in \mathbb{C}_{q+1}$ is a primitive j-th root of unity, i.e.

$$\chi = \rho_z \chi.$$

This forces certain conditions on the coefficients in the polynomials in $(h_1(x), h_2(x), \ldots)$ corresponding to χ and allows us to count how many χ of a fixed q-height are fixed by a j-th root of unity.

1 Pairs (s, λ)

Let $K = \overline{F}_q$. Consider the algebraic group $\widetilde{G} = \mathrm{GL}_n(K)$ with Frobenius endomorphism defined in the last chapter, $F : \widetilde{G} \to \widetilde{G}$ by $F((a_{ij})) = M(a_{ji}^q)^{-1}M^{-1}$. Then let $G = \widetilde{G}^F = \mathrm{U}_n(q)$.

A reference for the following is ((11), section 1). A subgroup L of G is Levi if $L = \widetilde{L}^F$ for some F-stable Levi subgroup $\widetilde{L}$ of a parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$. For a Levi subgroup L of G, let R_L^G be the additive operator from $X(L)$ to $X(G)$ defined in the Deligne-Lusztig theory, where $X(L)$ and $X(G)$ are the character rings of representations of L and G over $\overline{\mathbb{Q}}_l$, an algebraic closure of the l-adic field $\overline{\mathbb{Q}}_l$ $(l \neq p)$. Recall, in the previous chapter we fixed an isomorphism between $(\overline{F}_q)^*$ and $\mathrm{Irr}((\overline{F}_q)^*)$. Providing a coherent choice of roots of unity (via monomorphisms of multiplicative groups) has been made, this leads to an isomorphism

$$Z(L) \cong \mathrm{Irr}(Z(L)) = \mathrm{Hom}(Z(L), \overline{\mathbb{Q}}_l)$$

as in ((5), Section 8.2). Recall, ρ_s is the linear character of L corresponding to $s \in Z(L)$. We have a Jordan decomposition of characters of G. Namely the set of ordinary irreducible characters of G is in one-to-one correspondence with the set of pairs (s, λ). In our case, this means

$$\chi \leftrightarrow (s, \lambda)$$

where s is a representative of a semi-simple conjugacy class of G and λ is a unipotent character of $L = C_G(s)$, i.e. λ appears as a constituent of $R_T^L(1)$ for some maximal torus T of L.

Let $\epsilon_L = (-1)^d$ where d is the dimension of a maximal F_{q^2} split torus of L. Then

$$\chi = \epsilon_G \epsilon_L R_L^G(\rho_s \lambda) \text{ by ((11), p.116).}$$

Proposition 3.1.1 *For $\rho_z \in \mathrm{Irr}(Z(G))$, and $\chi \in \mathrm{Irr}(G)$*

$$\chi \leftrightarrow (s, \lambda) \Leftrightarrow \rho_z \chi \leftrightarrow (zs, \lambda).$$

where (s, λ) is the Jordan decomposition of χ.

Proof: Let $L = C_G(s) = C_G(sz)$. Then

$$\chi = \epsilon_G \epsilon_L R_L^G(\rho_s \lambda).$$

Moreover

$$\begin{aligned} \rho_z \chi &= \epsilon_G \epsilon_L R_L^G(\rho_z(\rho_s \lambda)), \text{ by ((5), Proposition 8.20)} \\ &= \epsilon_G \epsilon_L R_L^G((\rho_z \rho_s)\lambda), \end{aligned}$$

which corresponds to the pair (zs, λ).

Remark: The above discussion also holds for the finite group $\mathrm{GL}_n(q) = \widetilde{G}^{F'}$ where the Frobenius map is given by $F'((a_{i,j})) = (a_{i,j}^q)$.

2 Sequences of polynomials

In order to count efficiently we make use of polynomial sequences. We construct certain sequences which correspond to the irreducible characters of $\mathrm{U}_n(q)$. This identification arises naturally out of Deligne-Lusztig Theory. These polynomials encode information about both s and λ. This procedure is known in the case of $\mathrm{GL}_n(q)$ where if χ corresponds to the pair (s, λ) then χ corresponds to a sequence $(h_1(x), h_2(x), \ldots)$. The $h_i(x)$ are products of powers of irreducible polynomials over F_q which are elementary divisors of s; the powers of these divisors come from λ. In precisely the same spirit, irreducible characters of $\mathrm{U}_n(q)$ can be identified with sequences $(h_1(x), h_2(x), \ldots)$ where the $h_i(x)$ are products of powers of polynomials over F_{q^2}, appropriate for $\mathrm{U}_n(q)$, which are elementary divisors of s. We proceed with this identification.

It is well known that the conjugacy class of an element in $\mathrm{GL}_n(q^2)$ may be described by the elementary divisors of the rational canonical form. These divisors are powers of monic irreducible polynomials in $F_{q^2}[x]$ with non-zero roots. View $\mathrm{U}_n(q)$ as a subgroup of $\mathrm{GL}_n(q^2)$. Let $g \in \mathrm{GL}_n(q^2)$ have $\mathrm{GL}_n(q^2)$-conjugacy class $[g]$. The intersection $[g] \cap \mathrm{U}_n(q)$ is either a $\mathrm{U}_n(q)$-conjugacy class or is empty ((11), p.111). Let f be a monic irreducible polynomial in $F_{q^2}[x]$ of degree d with nonzero roots $\{\omega\}$. We define $\tilde{f}$ to be the polynomial in $F_{q^2}[x]$ with roots $\{\omega^{-q}\}$. Let $m_{f^i}(g)$ denote the multiplicity of f^i as an elementary divisor of g. Then $[g] \cap \mathrm{U}_n(q)$ is nonempty precisely when $m_{f^i}(g) = m_{\tilde{f}^i}(g)$ holds $\forall f$ and $\forall i$. Hence the conjugacy class of an element in $\mathrm{U}_n(q)$ is given by the elementary divisors of its rational canonical form and these divisors are powers of polynomials in a subset $\mathfrak{F}$ of $F_{q^2}[x]$.

Definition 3.2.1 *Let* $\mathfrak{F}_1 = \{f | f \neq x$ *is monic, irreducible and* $f = \tilde{f}\}$ *and let* $\mathfrak{F}_2 = \{f\tilde{f} | f \neq x$ *is monic, irreducible and* $f \neq \tilde{f}\}$. *Let* $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$.

Notice that for every polynomial $f \in \mathfrak{F}$, $f = \tilde{f}$. Members of $\mathfrak{F}_1$ have odd degree and members of $\mathfrak{F}_2$ have even degree. The latter fact is obvious. The former can be observed by noting that since f is irreducible the roots are the Galois conjugates of ω. Suppose that $d = 2k$. If $f = \tilde{f}$, then $\{\omega, \omega^{-q}, \ldots, \omega^{(-q)^{d-1}}\} = \{\omega^{-q}, \omega^{q^2}, \ldots, \omega^{(-q)^d}\}$. Hence $\omega = \omega^{(-q)^n} = \omega^{q^d}$ so $\omega \in F_{q^d}$. But F_{q^d} is an extension of F_{q^2} of degree k, hence f is reducible, a contradiction.

An element $g \in \mathrm{U}_n(q)$ is semi-simple if and only if $m_{f^i}(g) = 0$ for all $i > 0$. Given a semi-simple $s \in \mathrm{U}_n(q)$, we want to describe its centralizer $C_{\mathrm{U}_n(q)}(s)$. Let s have primary decomposition $s = \prod_{f \in \mathfrak{F}} s_f$ where s_f is the primary component corresponding to elementary divisor f. Let s have minimal polynomial $\min(x) = \prod_{f \in \mathfrak{F}} f$ and characteristic polynomial $ch(x) = \prod_{f \in \mathfrak{F}} f^{m_f(s)}$. Then s has rational canonical form

$$s = \bigoplus_{\mathfrak{F}} m_f(s)c(f)$$

where $c(f)$ denotes the $d_f \times d_f$ companion matrix of the polynomial f with degree d_f and for nonzero multiplicity $m_f(s)$, $m_f(s)c(f)$ denotes the $m_f(s)d_f \times m_f(s)d_f$ matrix with $m_f(s)$ copies of $c(f)$.

Proposition 3.2.2 *((11), Proposition 1A) Let s have primary decomposition $s = \prod_{f \in \mathfrak{F}} s_f$ and rational canonical form*

$$s = \bigoplus_{f \in \mathfrak{F}} m_f(s)c(f).$$

The structure of the centralizer of s is given by

$$C_{\mathrm{U}_n(q)}(s) = \prod_{f \in \mathfrak{F}} C(s_f), \text{ where}$$

1. *If $f \in \mathfrak{F}_1$, then $C(s_f) = \mathrm{U}_{m_f(s_f)}(F_f)$, where $|F_f : F_{q^2}| = \deg(f)$.*
2. *If $f \in \mathfrak{F}_2$, then $C(s_f) = \mathrm{GL}_{m_f(s_f)}(F_f)$, where $|F_f : F_{q^2}| = \frac{1}{2}\deg(f)$.*

Hence the centralizer of an element is a product of general linear and unitary groups. A unipotent character of such a product is a product of unipotent characters. Moreover, the unipotent characters of both the general linear and unitary groups are indexed by partitions of the dimension of the underlying vector space. In particular, the unipotent characters of $\mathrm{U}_n(q^m)$ and $\mathrm{GL}_n(q^m)$ are given by partitions of n, for any exponent m.

Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ correspond to the pair (s, λ). Since λ is a unipotent character of $C_{\mathrm{U}_n(q)}(s)$ it is a product of unipotent characters of the $C(s_f)$ which are general linear or unitary groups each of which corresponds to a partition $\mu_f \vdash m_f(s)$. Let $\mathfrak{P}$ denote the set of all partitions including the empty partition. We define the map

$$\begin{aligned} \Lambda : \mathfrak{F} &\longrightarrow \mathfrak{P} \\ f &\mapsto \mu_f. \end{aligned}$$

Notice that $\sum_{f \in \mathfrak{F}} d_f m_f(s) = n$.

Our construction is summarized in the following often quoted proposition which originates with Green's important paper on general linear characters, and has been modified for $\mathrm{U}_n(q)$ by several authors. Here we use the notation of Ku ((15)). For a partition μ recall the definitions of $|\mu|$ on page 16 and $n'(\mu)$ on page 26 in the previous chapter.

Proposition 3.2.3 *((15), Proposition 4.2.2 and Lemma 4.2.3) Let $\mathfrak{P}$ denote the set of all partitions of all integers $n > 0$, together with the empty partition. The irreducible characters of* $\mathrm{U}_n(q)$ *are in one-to-one correspondence with maps Λ from $\mathfrak{F}$ to $\mathfrak{P}$ which satisfy the following:*

$$\sum_{f \in \mathfrak{F}} |\Lambda(f)| d_f = n.$$

If $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ corresponds to such a map Λ, then the following hold:

1. *The q-height of χ is $\sum_{f \in \mathfrak{F}} d_f n'(\Lambda(f)')$ where $\Lambda(f)'$ is the conjugate partition of $\Lambda(f)$.*
2. *The character χ lies over $\rho \in \mathrm{Irr}(Z(\mathrm{U}_n(q)))$ where a_ρ is the product of the roots of $\prod_{f \in \mathfrak{F}} f^{|\Lambda(f)|}$.*

Let $\chi \in \mathrm{Irr}(G)$ be associated to the pair (s, λ) which is in turn associated to the map $\Lambda : \mathfrak{F} \to \mathfrak{P}$. For each $f \in \mathfrak{F}$, write the conjugate partition $\Lambda(f)' = (t^{m_{f,t}})$. Using these exponents, we may now define for χ a unique sequence of polynomials $(h_1(x), h_2(x), \ldots)$ by letting

$$h_i(x) = \prod_{f \in \mathfrak{F}} f^{m_{f,i}}.$$

We will be concerned with examining classes of irreducible characters which share certain properties. We want to group characters by their q-height and also by their splitting upon restriction to certain subgroups. To that end we make the following definition which will be of utmost importance in this endeavor.

Definition 3.2.4 *If χ determines the sequence $(h_1(x), h_2(x), \ldots)$, we will say that χ is of μ-type where $\mu = (t^{\deg(h_t(x))}) \vdash n$.*

3 Proof of Proposition 2.6.5

In this section we verify Proposition 2.6.5. Recall $k_d(\mathrm{U}_n(q), \rho, \det, j)$ is the number of $\chi \in \mathrm{Irr}(\mathrm{U}_n(q))$ of q-height d lying over ρ such that $\chi|_{\mathrm{SU}_n(q)}$ has j' irreducible constituents where $j|j'$. Let $\rho \in \mathrm{Irr}(\mathbb{C}_{q+1})$ and $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_r^{l_r}) \vdash n$. Let $\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)$ denote the irreducible characters of $\mathrm{U}_n(q)$ of μ-type lying over ρ. Let $\chi \in \mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)$ correspond to (s, λ) and $(h_1, h_2, \ldots)$. Suppose ρ_z is the linear character of $\mathrm{U}_n(q)$ corresponding to $z \in \mathbb{C}_{q+1}$, a primitive j-th root of unity and that

$$\chi = \rho_z \chi.$$

By 3.1.1 $\rho_z \chi$ corresponds to (zs, λ). If h_i in $(h_1, h_2, \ldots)$ has roots $\{\omega\}$ then (zs, λ) corresponds to $(g_1, g_2, \ldots)$ where g_i has roots $\{z\omega\}$. Then we have

$$(h_1, h_2, \ldots) = (g_1, g_2, \ldots).$$

Since χ is of μ-type, $h_{a_i}(x)$ is a polynomial of degree l_i and hence $g_{a_i}(x)$ also has degree l_i. Let

$$\{\omega_{i,k} | 1 \leq k \leq l_i\} \text{ denote the roots of } h_{a_i}(x).$$

Then

$$h_{a_i}(x) = \prod_{k=1}^{l_i}(x - \omega_{i,k}) \qquad\qquad g_{a_i}(x) = \prod_{k=1}^{l_i}(x - z\omega_{i,k})$$

$$= x^{l_i} + \cdots + b_{i,1}x + b_{i,0} \qquad\qquad = x^{l_i} + \cdots + z^{l_i-1}b_{i,1}x + z^{l_i}b_{i,0}.$$

Recall $\lambda(\mu)$ was defined on page 16 and is equal to $\gcd(l_1, l_2, \ldots, l_r)$. Our first observation is that $b_{i,0}$ is nonzero. Hence $z^{l_i} = 1$ for each $i = 1, 2, \ldots, r$ thus j divides $\lambda(\mu)$. Secondly, we must have $b_{i,j}x^k = z^{l_i-k}b_{i,k}x^k$. If $l_i - k$ is not divisible by j, i.e. j doesn't divide k, the coefficient $b_{i,k} = 0$. This reduces the possible number of nonzero coefficients.

If χ lies over ρ then $(-1)^n\prod_{i=1}^r(b_{i,0})^{a_i} = a_\rho$ by construction. The $b_{i,k}$ are symmetric functions of the roots. Simplifying notation for a moment, since $h(x) = x^m + \ldots + b_1x + b_0$ is a product of polynomials in $\mathfrak{F}$, the coefficients satisfy $b_{m-i} = b_0b_i^q$. If m is even $b_{m/2}^{1-q} = b_0$. Hence we have $(l_i/j - 1)/2$ degrees of freedom in the nonconstant coefficients, i.e. $q^{2(l_i/j-1)/2}$ choices for the $b_{i,k}$ and thus

$$|\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)| = q^{l(\mu)/j-\delta(\mu)}\beta(\mu, a_\rho).$$

The left hand side of the sum in Proposition 2.6.5 can now be evaluated.

$$\begin{aligned}
k_d(\mathrm{U}_n(q), \rho, \det, j) &= \sum_{\substack{j' \\ j|j'}} |\mathrm{Irr}_d(\mathrm{U}_n(q), \rho, \det, j')| \\
&= \sum_{\substack{j' \\ j|j'}} \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ j' |\gcd(q+1,\lambda(\mu))}} |\mathrm{Irr}(\mathrm{U}_n(q), \mu, \rho)| \\
&= \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu, a_\rho).
\end{aligned}$$

CHAPTER 4

THE FINITE SPECIAL UNITARY GROUPS

1 A Reduction of DOC for the Finite Special Unitary Groups

In this chapter we make use of Clifford theory in the manner of Sukizaki ((22)) to reformulate DOC for the Special Unitary group $\mathrm{SU}_n(q)$ in terms of $\mathrm{U}_n(q)$. In this section we will distinguish between subgroups of $\mathrm{U}_n(q)$ and $\mathrm{SU}_n(q)$ with superscripts as indicated. Recall $I = [m]$ is our index set for the distinguished generators of the Weyl group for $\mathrm{U}_n(q)$. We have $n = 2m$ or $2m+1$. In keeping with established notation, let $K = \overline{F}_q$. Let $\widetilde{B}$ be the Borel subgroup of upper triangular matrices of the linear algebraic group $\widetilde{G} = \mathrm{GL}_n(K)$. A Frobenius endomorphism on $\widetilde{G}$ was defined by

$$F(a_{ij}) = M(a_{ji}^q)^{-1}M^{-1},$$

and the unitary group $\mathrm{U}_n(q) = \widetilde{G}^F$. The special unitary group is defined

$$\mathrm{SU}_n(q) = \{g \in \mathrm{U}_n(q) \mid \det g = 1\}.$$

Note that except for the cases $n = 2$ and $q \leq 3$, the derived subgroup $\widetilde{G}' = \mathrm{SL}_n(K)$ so $\mathrm{SU}_n(q) = \widetilde{G}'^F$. The group of fixed points of $\widetilde{B}$ under F is a Borel subgroup for $\mathrm{U}_n(q)$. Let B^{U} be this subgroup. Notice that B^{U} is upper triangular. We will fix a Borel subgroup $B^{\mathrm{SU}} = B^{\mathrm{U}} \cap \mathrm{SU}_n(q)$ for $\mathrm{SU}_n(q)$. Notice that B^{SU} is the group of fixed points of the Frobenius restricted to $\mathrm{SL}_n(K)$ and is also upper triangular. We have corresponding Levi decompositions $B^{\mathrm{U}} = T \ltimes U$ and $B^{\mathrm{SU}} = S \ltimes U$ where $S = T \cap \mathrm{SU}_n(q)$. By standard parabolic subgroups we mean subgroups containing B^{U} or B^{SU}. For $J \subseteq I$ let P_J^{U} or P_J^{SU} be the standard parabolic group of $\mathrm{U}_n(q)$ or $\mathrm{SU}_n(q)$ respectively corresponding to J. For fixed $J \subset I$, we have $P_J^{\mathrm{U}} = N_{\mathrm{U}_n(q)}(U_J)$ and $P_J^{\mathrm{SU}} = N_{\mathrm{SU}_n(q)}(U_J)$, both containing the same upper triangular unipotent radical, i.e. $O_p(P_J^{\mathrm{U}}) = O_p(P_J^{\mathrm{SU}}) = U_J$

The group $\mathrm{SU}_n(q)$ contains every p-subgroup of $\mathrm{U}_n(q)$ and $O_p(\mathrm{SU}_n(q)) = O_p(\mathrm{U}_n(q)) = 1$. Thus any radical p-chain of $\mathrm{SU}_n(q)$ is a radical p-chain of $\mathrm{U}_n(q)$ by Proposition 2.2.1. Conversely, let

$$C : U_0 < U_1 < \cdots < U_l$$

be a radical p-chain of $\mathrm{U}_n(q)$. The U_i are unipotent radicals of parabolic subgroups of $\mathrm{U}_n(q)$. Hence each U_i is conjugate to a standard unipotent radical U_{J_i}, i.e. for each i there exists $g_i \in \mathrm{U}_n(q)$ such that

$$U_i = g_i U_{J_i} {g_i}^{-1}.$$

For all $x \in F^*_{q^2}$, the matrix

$$\overline{x} = \begin{pmatrix} x & 0 & \dots & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ & & & 1 & 0 \\ 0 & & \dots & 0 & x^{-q} \end{pmatrix} \text{ stabilizes all standard } U_{J_i}.$$

In particular, this holds for $\overline{x_i}$ such that $x_i^{1-q} = (\det(g_i))^{-1}$. Moreover, $\forall g_i \in \mathrm{U}_n(q)$ there exists such an $x_i \in F^*_{q^2}$. Thus

$$U_i = g_i(\overline{x_i} U_{J_i} \overline{x_i}^{-1}) g_i^{-1} = (g_i\overline{x_i}) U_{J_i} (g_i\overline{x_i})^{-1}$$

where $g_i\overline{x_i} \in \mathrm{SU}_n(q)$. Thus C is $\mathrm{SU}_n(q)$-conjugate to a radical p-chain of $\mathrm{SU}_n(q)$.

For an irreducible character ρ of Z^{SU}, the center of $\mathrm{SU}_n(q)$, recall $k_d(P_J^{\mathrm{SU}}, \rho) = |\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho)|$ is the number of irreducible characters of P_J^{SU} lying over ρ (i.e. in the p-block corresponding to ρ) with q-height d. The p-part of $|\mathrm{SU}_n(q)|$ is equal to $q^{\binom{n}{2}}$. As we saw in the introduction DOC can be written:

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J^{\mathrm{SU}}, \rho) = 0, \qquad \text{for all } \rho \text{ in } \mathrm{Irr}(Z^{\mathrm{SU}}) \text{ and nonnegative integers } d < \tbinom{n}{2}. \quad (4.1)$$

We now reformulate this statement using Clifford Theory. Let det be the determinant map on $\mathrm{U}_n(q)$. Then $\det(\mathrm{U}_n(q)) = \mathbb{C}_{q+1}$ and $\ker\det = \mathrm{SU}_n(q)$. Moreover, restricting the determinant map to parabolic subgroups P_J^{U} we have $\det(P_J^{\mathrm{U}}) = \mathbb{C}_{q+1}$ and $\ker\det|_{P_J^{\mathrm{U}}} = P_J^{\mathrm{SU}}$. The group P_J^{SU} is normal in P_J^{U} and hence P_J^{U} acts on the set $\mathrm{Irr}(P_J^{\mathrm{SU}})$ in the natural way. For $g \in P_J^{\mathrm{U}}$ and $\phi \in \mathrm{Irr}(P_J^{\mathrm{SU}})$, $g \cdot \phi = {}^g\phi$ where ${}^g\phi(x) = \phi(gxg^{-1})$ as defined in Section 4 of Chapter 2. The quotient group is cyclic

$$P_J^{\mathrm{U}}/P_J^{\mathrm{SU}} \cong \mathbb{C}_{q+1}.$$

.

Let $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$ denote the irreducible characters $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}})$ such that ϕ lies over ρ, has q-height d, and the P_J^{U} orbit of ϕ contains j characters. Then the following implies Equation 4.1:

For integers $0 \leq d < \binom{n}{2}$, $1 \leq j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$

$$\sum_{J \subseteq I} (-1)^{|J|} |\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)| = 0. \tag{4.2}$$

Rather than counting characters of P_J^{SU} we count characters of P_J^{U}. Let $\chi \in \mathrm{Irr}(P_J^{\mathrm{U}})$. By Clifford Theory, χ restricted to P_J^{SU} is multiplicity free. The restrictions of two irreducible characters χ and χ' of P_J^{U} to P_J^{SU} have the same irreducible constituents or are disjoint (Lemma 2.4.4). If $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$, then the P_J^{U}-orbit of ϕ contains j characters. For $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$, let $\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)$ denote the subset of $\mathrm{Irr}_d(P_J^{\mathrm{U}})$ consisting of characters such that their restrictions to $\ker\det$ belong to $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$.

A character $\phi \in \mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$ extends to $\widetilde{\phi} \in \mathrm{Irr}(T_{P_J^{\mathrm{U}}}(\phi))$. The induced character $(\widetilde{\phi}\theta)^{P_J^{\mathrm{U}}}$ is irreducible where θ is the lift to $T_{P_J^{\mathrm{U}}}(\phi)$ of an irreducible character of $T_{P_J^{\mathrm{U}}}(\phi)/P_J^{\mathrm{SU}}$. Then

$$\left|T_{P_J^{\mathrm{U}}}(\phi)/P_J^{\mathrm{SU}}\right| = \frac{q+1}{j} \text{ since } \left|P_J^{\mathrm{U}}/T_{P_J^{\mathrm{U}}}(\phi)\right| = j.$$

Let k_J be the number of P_J^{U}-orbit representatives in $\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)$. Then

$$\left|\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)\right| = j \cdot k_J$$

$$\left|\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)\right| = \tfrac{q+1}{j} \cdot k_J.$$

Then

$$\sum_{J \subseteq I} (-1)^{|J|} |\mathrm{Irr}_d(P_J^{\mathrm{SU}}, \rho, j)| = \sum_{J \subseteq I} (-1)^{|J|} j \cdot k_J = j \cdot \sum_{J \subseteq I} (-1)^{|J|} k_J = 0$$

holds if and only if $\sum_{J \subseteq I} (-1)^{|J|} k_J = 0$ if and only if

$$0 = \tfrac{q+1}{j} \cdot \sum_{J \subseteq I} (-1)^{|J|} k_J = \sum_{J \subseteq I} (-1)^{|J|} \tfrac{q+1}{j} \cdot k_J = \sum_{J \subseteq I} (-1)^{|J|} |\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)|.$$

Hence the following equation is equivalent to Equation 4.2:

For integers $0 \leq d < \binom{n}{2}$, $1 \leq j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$:

$$\sum_{J \subseteq I} (-1)^{|J|} |\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)| = 0. \tag{4.3}$$

We will need the case where $d = \binom{n}{2}$. Recall from Chapter 2, the isomorphism of $(F_{q^2})^*$ and $\mathrm{Irr}((F_{q^2})^*)$ so that $a_\rho \in (F_{q^2})^*$ corresponds to $\rho \in \mathrm{Irr}((F_{q^2})^*)$ under this isomorphism.

This induces an isomorphism of the cyclic subgroups Z^{U} and Z^{SU} with $\mathrm{Irr}(Z^{\mathrm{U}})$ and $\mathrm{Irr}(Z^{\mathrm{SU}})$ respectively. Also recall from Chapter 2 the definition (2.3.3) of $\beta(\mu, a_\rho)$.

For $\rho' \in \mathrm{Irr}(Z^{\mathrm{U}})$ let $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho')$ be the set of characters $\chi \in \mathrm{Irr}(P_J^{\mathrm{U}})$ such that χ lies over ρ'. For $J \neq \emptyset$, $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho')$ consists of a unique p-block corresponding to ρ'. However if $J = \emptyset$ then $P_J^{\mathrm{U}} = \mathrm{U}_n(q)$ and $\mathrm{Irr}(\mathrm{U}_n(q), \rho')$ consists of two p-blocks one of zero defect the other of full defect. Observe $|\mathrm{Irr}_{\binom{n}{2}}(\mathrm{U}_n(q), \rho')|$ is one, the number of irreducible characters of $\mathrm{U}_n(q)$ of full q-height lying over ρ'. For $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$, let $\mathrm{Irr}(P_J^{\mathrm{U}}, \rho)$ denote the set of irreducible characters of P_J^{U} that lie over $\rho' \in \mathrm{Irr}(Z^{\mathrm{U}})$ where ρ' lies over ρ. Then for $\rho \in \mathrm{Irr}(Z^{\mathrm{SU}})$ we have the following disjoint union

$$\mathrm{Irr}(P_J^{\mathrm{U}}, \rho) = \bigsqcup_{\substack{\rho' \in \mathrm{Irr}(Z^{\mathrm{U}}) \\ \rho'|_{Z^{\mathrm{SU}}} = \rho}} \mathrm{Irr}(P_J^{\mathrm{U}}, \rho').$$

We focus now on irreducible characters in $\mathrm{Irr}(Z^{\mathrm{U}})$ and so switch the roles of ρ and ρ'. The following implies Equation 4.3:

For integers $0 \leq d$, $1 \leq j$ and any $\rho \in \mathrm{Irr}(Z^{\mathrm{U}})$,

$$\sum_{J \subseteq I} (-1)^{|J|} |\mathrm{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j)| = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and j=1;} \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Given $\rho \in \operatorname{Irr}(Z^{\mathrm{U}})$, let $k_d(P_J^{\mathrm{U}}, \rho, \det, j)$ denote the number of irreducible characters $\chi \in \operatorname{Irr}(P_J^{\mathrm{U}})$ such that χ lies over ρ, has q-height d and $\chi|_{\ker \det}$ has j' irreducible constituents where j divides j'. Observe that

$$\ker(\det) = P_J^{\mathrm{U}} \cap \mathrm{SU}_n(q) = P_J^{\mathrm{SU}} \text{ as mentioned.}$$

Then

$$k_d(P_J^{\mathrm{U}}, \rho, \det, j) = \sum_{j|j'} |\operatorname{Irr}_d(P_J^{\mathrm{U}}, \rho, \det, j')|.$$

We may now drop the superscript notation and restrict our attention to irreducible characters of parabolic subgroups of $\mathrm{U}_n(q)$.

Taking into account that Equation 4.4 implies Equation 4.3 which implies Equation 4.2 which implies Equation 4.1, we will have proved DOC for $\mathrm{SU}_n(q)$ if we prove the following theorem, which is the main result in this thesis.

Theorem 4.1.1 (Main) *Let $Z = Z(\mathrm{U}_n(q))$ and $\{P_J | J \subseteq I\}$ the set of standard parabolic subgroups in $\mathrm{U}_n(q)$. For any $\rho \in \operatorname{Irr}(Z)$, any positive integer j, and all nonnegative integers d we have*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j{=}1; \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove Theorem 4.1.1 we first break the left hand side into two sub-sums, the second of which will reduce quite spectacularly. Let us differentiate between characters χ counted by $k_d(P_J, \rho, \det, j)$ for which $\ker \chi$ contains U_J or not.

Definition 4.1.2 *Let $k_d^0(P_J, U_J, \rho, \det, j)$ be the number of characters counted by $k_d(P_J, \rho, \det, j)$ which contain U_J in their kernel and let $k_d^1(P_J, U_J, \rho, \det, j)$ count those characters which do not contain U_J in their kernel.*

Then

$$\begin{aligned}\sum_{J\subseteq I}(-1)^{|J|}k_d(P_J, \rho, \det, j) &= \sum_{J\subseteq I}(-1)^{|J|}(k_d^0(P_J, U_J, \rho, \det, j) + k_d^1(P_J, U_J, \rho, \det, j)) \\ &= \sum_{J\subseteq I}(-1)^{|J|}k_d^0(P_J, U_J, \rho, \det, j) + \sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J, U_J, \rho, \det, j)\end{aligned} \tag{4.5}$$

We will show the following:

Proposition 4.1.3 *For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d*

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^0(P_J, U_J, \rho, \det, j) = \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu, a_\rho) \tag{4.6a}$$

$$\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J, U_J, \rho, \det, j) = -\sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu, a_\rho). \tag{4.6b}$$

Clearly Proposition 4.1.3 implies Theorem 4.1.1.

CHAPTER 5

AUXILIARIES FOR THE PROOF OF THE MAIN THEOREM

This chapter is dedicated to proving Equation 4.6a, the first half of Proposition 4.1.3. Our first observation is that if χ is an irreducible character of P_J containing U_J in its kernel then we may consider χ as an irreducible character of the Levi subgroup L_J of P_J. We must be careful in applying the determinant map.

Let $J = \{j_1, j_2, \ldots, j_s\}$. Then L_J can be written as the following direct product:

$$L_J = \mathrm{GL}_{n_1}(q^2) \times \mathrm{GL}_{n_2}(q^2) \times \cdots \times \mathrm{GL}_{n_s}(q^2) \times \mathrm{U}_{n-2j_s}(q)$$

where $n_1 = j_1$, $n_i = j_i - j_{i-1}$ for $2 \leq i \leq s$. Then since $P_J = L_J U_J$ for $x \in P_J$, $x = lu$ where $l \in L_J$ and $u \in U_J$, so the determinant $\det(x) = \det(lu) = \det(l)\det(u) = \det(l)$ since u is unipotent.

Recall our definition

$$\mathrm{U}_n(q) = \{(a_{i,j}) \in \mathrm{GL}_n(q^2) \mid \ M = (a_{i,j})M(a_{j,i}^q)\}$$

where M is the $n \times n$ matrix with ones down the reverse diagonal.

With this definition the fixed Borel subgroup of $\mathrm{U}_n(q)$ is upper triangular. Thus for $x \in P_J$ we may write x as a block matrix:

$$x = \begin{pmatrix} A_1 & * & \cdots & & & & \cdots & * & * \\ 0 & A_2 & \cdots & & & & \cdots & * & * \\ \vdots & \vdots & \ddots & & & & & \vdots & \vdots \\ & & & A_s & & & & & \\ & & & & B & & & & \\ & & & & & \widetilde{A_s} & & & \\ \vdots & \vdots & & & & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & & & & \cdots & \widetilde{A_2} & * \\ 0 & 0 & \cdots & & & & \cdots & 0 & \widetilde{A_1} \end{pmatrix}$$

where $A_k \in \mathrm{GL}_{n_k}(q^2)$, $B \in \mathrm{U}_{n_{s+1}}(q)$, and if $A_k = (a_{i,j})$, then $\widetilde{A_k} = M(a^q_{j,i})^{-1}M^{-1}$.

The determinant of x as an element in P_J which is embedded in $\mathrm{U}_n(q)$ may be defined in terms of the determinant map on the component factors of the Levi subgroup L_J in P_J. We have $\det(x) = (\det(A_1)\det(A_2)\ldots\det(A_s))^{1-q}\det(B)$ since $\det(\widetilde{A_k}) = \det(A_k)^{-q}$.

Thus $k^0_d(P_J, U_J, \rho, \det, j) = k_d(L_J, \rho, \det|_{L_J}, j)$ where the determinant map $\det|_{L_J}$ is as indicated. Hence we are proving the equivalent statement:

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(L_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu, a_\rho). \tag{5.1}$$

We make the following observation. Suppose $L_J = G_1 \times G_2$ where $G_1 \cong \mathrm{GL}_{n_1}(q^2)$ and $G_2 \cong \mathrm{U}_{n_2}(q)$ with $2n_1+n_2 = n$. As a subgroup embedded in $\mathrm{U}_n(q)$ the determinant map on elements in L_J is defined in terms of the determinant map on the factors G_1 and G_2. For $x = g_1g_2$ in L_J with $g_i \in G_i$ $\det(x) = \det(g_1)^{1-q}\det(g_2)$. Recall $\det(g_1)^{1-q}$ is denoted $\det^{1-q}(g_1)$ Then $\ker(\det) = \{(g_1, g_2) \mid \det^{1-q}(g_1)\det(g_2) = 1\}$, $\det^{1-q}(G_1) = \mathbb{C}_{q+1}$, and $\det(G_2) = \mathbb{C}_{q+1}$.

If χ_i lies over $\rho_i \in \mathrm{Irr}(\mathbb{C}_{q+1})$, then $\chi = \chi_1\chi_2$ lies over $\rho = \rho_1\rho_2$ since for $z \in Z(\mathrm{U}_n(q))$ $\chi(z) = \chi_1(1)\chi_2(1)\rho_1(z)\rho_2(z)$. Hence 2.5.1 implies that

$$k_d(L_J, \rho, \det, j) = \sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} k_{2d_1}(G_1, \rho_1, \det^{1-q}, j)k_{d_2}(G_2, \rho_2, \det, j). \tag{5.2}$$

We proceed by induction on n or equivalently by induction on m, where $n = 2m$ or $2m + 1$.

1 Small Case

Let $n = 1$ so that $m = 0$ and I is empty. Then we have but one levi subgroup, $\mathrm{U}_1(q)$ itself which is equal to its center. The determinant map is just the identity and hence $\ker\det$ is trivial so that the left hand side of Equation 5.1 is

$$k_d(\mathrm{U}_1(q), \rho, \det, j) = \begin{cases} 1, & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Certainly this is equal to the right hand side of Equation 5.1 since we have but one partition of 1 and $\beta((1), a_\rho) = 1$

Let $m = 1$ so that $n = 2$ or $n = 3$. In either case we have but two levi subgroups $\mathrm{U}_n(q)$ and the borel levi subgroup L_I. First suppose that $n = 2$. Then $L_I = \mathrm{GL}_1(q^2)$ and we may apply 2.6.5 and 2.6.4 directly. The left hand side of Equation 5.1 is

$$k_d(\mathrm{U}_2(q), \rho, \det, j) - k_d(\mathrm{GL}_1(q^2), \rho, \det^{1-q}, j) = \begin{cases} \beta((2), a_\rho) - 0, & \text{if } d = 1 \text{ and } j = 1; \\ \beta((1^2), a_\rho) - 0, & \text{if } d = 0 \text{ and } j = 2; \\ q\beta(1^2, a_\rho) - \overline{\beta}(1, a_\rho), & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This is equal to the right hand side of Equation 5.1.

We continue to assume that $m = 1$. Now suppose that $n = 3$. Then $L_I = \mathrm{GL}_1(q^2) \times \mathrm{U}_1(q)$ so that Equation 5.2 implies

$$k_d(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q), \rho, \det, j) = \sum_{\substack{2d_1 + d_2 = d \\ \rho_1\rho_2 = \rho}} k_{2d_1}(\mathrm{GL}_1(q^2), \rho_1, \det^{1-q}, j) k_{d_2}(\mathrm{U}_1(q), \rho_2, \det, j)$$

which is nonzero only for $d = 0$. Then the left hand side of Equation 5.1 is

$$k_d(\mathrm{U}_3(q), \rho, \det, j) - k_d(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q), \rho, \det, j)$$
$$= \begin{cases} \beta((3), a_\rho) - 0, & \text{if } d = 3 \text{ and } j = 1; \\ \beta((2,1), a_\rho) - 0, & \text{if } d = 1 \text{ and } j = 1; \\ \beta(1^3, a_\rho) - 0, & \text{if } d = 0 \text{ and } j = 3; \\ q^2\beta(1^3, a_\rho) - (q-1)(q+1), & \text{if } d = 0 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}$$

This is equal to the right hand side of (Equation 5.1).

2 Inductive Case

We assume that Equation 5.1 holds for all dimensions strictly less than n. Our first observation is that for fixed J with minimal element $j_1 = k$ we may write $L_J = \mathrm{GL}_k(q^2) \times L_{J'}$ where $L_{J'}$ is a levi subgroup in $U_{n-2k}(q)$ and $J' = \{j_i - k \mid 2 \leq i \leq s\}$. Note $|J'| = |J| - 1$. We will use superscripts to indicate the dimension of the ambient group when necessary. So for example $L_{J'}$ will be written as $L_{J'}^{n-2k}$. For such a J we have

$$k_d(\mathrm{GL}_k(q^2) \times L_{J'}^{n-2k}, \rho, \det, j) = \sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \det^{1-q}, j) k_{d_2}(L_{J'}^{n-2k}, \rho_2, \det, j).$$

We remark that $k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \det^{1-q}, j) = 0$ for k not divisible by j. We could eliminate from our sum all J whose smallest members are not all multiples of j. This isn't necessary though since the contribution is just zero. In fact we may discard all Js not contained in $\{j, 2j, 3j, \ldots\} \subseteq I$ but again this isn't necessary for our induction.

Definition 5.2.1 *Fix $k \geq 1$ and let $\mathbf{J_k}$ be the collection of all $J \subseteq I$ with minimal member k.*

We have

$$\begin{aligned}
\sum_{J \in \mathbf{J_k}} (-1)^{|J|} k_d(L_J, \rho, \det, j) &= \sum_{J' \subseteq I^{m-k}} (-1)^{|J|} k_d(\mathrm{GL}_k(q^2) \times L_{J'}^{n-2k}, \rho, \det, j) \\
&= \sum_{J' \subseteq I^{m-k}} \sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} (-1)^{|J'|+1} k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \det^{1-q}, j) k_{d_2}(L_{J'}^{n-2k}, \rho_2, \det, j) \\
&= -\sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} \left(k_{2d_1}(\mathrm{GL}_k(q^2), \rho_1, \overset{1-q}{\det}, j) \sum_{J \subseteq I^{m-k}} (-1)^{|J|} k_{d_2}(L_J^{n-2k}, \rho_2, \det, j) \right) \\
&= -\sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} \left(\sum_{\substack{\mu_1 \vdash k \\ n'(\mu_1)=d_1 \\ j \mid \gcd(q+1, \lambda(\mu_1))}} q^{2(l(\mu_1)/j - \delta(\mu_1))} \overline{\beta}(\mu_1, a_{\rho_1}) \times \sum_{\substack{\mu_2 \vdash (n-2k) \\ n'(\mu_2)=d_2 \\ j \mid \gcd(q+1, \lambda(\mu_2))}} \beta(\mu_2, a_{\rho_2}) \right) \\
&= -\sum_{\substack{2d_1+d_2=d \\ \rho_1\rho_2=\rho}} \sum_{\substack{(\mu_1,\mu_2) \\ \mu_1 \vdash k \\ \mu_2 \vdash (n-2k) \\ n'(\mu_1)=d_1 \\ n'(\mu_2)=d_2 \\ j \mid \gcd(q+1, \lambda(\mu_1), \lambda(\mu_2))}} q^{2(l(\mu_1)/j - \delta(\mu_1))} \overline{\beta}(\mu_1, a_{\rho_1}) \beta(\mu_2, a_{\rho_2})
\end{aligned} \tag{5.3}$$

by our inductive assumption, for indeed $n - 2k$ is strictly less than n.

Summing over all possible values for k and switching the order of summation we have

$$\begin{aligned}
&\sum_{k=1}^{m}\sum_{J\in\mathbf{J_k}}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&\quad=-\sum_{k=1}^{m}\sum_{\substack{2d_1+d_2=d\\ \rho_1\rho_2=\rho}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu_1\vdash k\\ \mu_2\vdash(n-2k)\\ n'(\mu_1)=d_1\\ n'(\mu_2)=d_2\\ j|\gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})\\
&\quad=-\sum_{\substack{\mu\vdash n\\ n'(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ |\mu_1|\neq 0\\ j|\gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}}\sum_{\substack{\rho_1,\rho_2\\ \rho_1\rho_2=\rho}} q^{2(l(\mu_1)/j-\delta(\mu_1))}\overline{\beta}(\mu_1,a_{\rho_1})\beta(\mu_2,a_{\rho_2})\\
&\quad=-\sum_{\substack{\mu\vdash n\\ n'(\mu)=d}}\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ |\mu_1|\neq 0\\ j|gcd(q+1,\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}\beta(\mu,a_\rho).
\end{aligned}\tag{5.4}$$

Now we remind the reader that from 2.3.10 we have the following for each $\mu\vdash n$

$$\sum_{\substack{(\mu_1,\mu_2)\\ \mu=2\mu_1\cup\mu_2\\ j|\gcd(\lambda(\mu_1),\lambda(\mu_2))}} q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}=q^{l(\mu)/j-\delta(\mu)}.\tag{5.5}$$

Notice that this sum includes the pair $(\mu_1,\mu_2)=(\emptyset,\mu)$ and that for this particular pair

$$q^{2(l(\mu_1)/j-\delta(\mu_1))}(q-1)^{\delta(\mu_1)}(q+1)^{c(\mu_1,\mu_2)}=1.$$

We are now ready to prove Equation 5.1:

$$
\begin{aligned}
\sum_{J\subseteq I}(-1)^{|J|}k_d(L_J,\rho,\det,j) &= k_d(U_n(q),\rho,\det,j)+\sum_{\emptyset\neq J\subseteq I}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&= k_d(\mathrm{U}_n(q),\rho,\det,j)+\sum_{k=1}^{m}\sum_{J\in\mathbf{J_k}}(-1)^{|J|}k_d(L_J,\rho,\det,j)\\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu,a_\rho)\\
&\qquad -\left(\sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}} q^{l(\mu)/j-\delta(\mu)}\beta(\mu,a_\rho)-\sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}}\beta(\mu,a_\rho)\right)\\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ j|\gcd(q+1,\lambda(\mu))}}\beta(\mu,a_\rho).
\end{aligned}
\tag{5.6}
$$

And we are done.

CHAPTER 6

MODULES FOR PARABOLIC SUBGROUPS

Recall the set up in Chapter 5. Let $J \subset I = [m]$, an index set for the distinguished generators of the Weyl group of type B_m. Let P_J be the standard parabolic subgroup of $\mathrm{U}_n(q)$ corresponding to J. Then $P_J = \bigcap_{j \in J} P_j$ where P_j is the maximal parabolic subgroup corresponding to j. We have the usual Levi decomposition of $P_J = L_J U_J$ where L_J is a levi subgroup and U_J is the unipotent radical of P_J. Thus far we have seen a reduction of DOC in the finite special unitary case to Proposition 4.1.3 which involves two sub-sums of certain characters of parabolic subgroups in $\mathrm{U}_n(q)$. Having proved Equation 4.6a we now devote the rest of this thesis to proving Equation 4.6b.

The first step in the process of finding a formula for $k_d^1(P_J, U_J, \rho, \det, j)$ is to examine orbits in submodules afforded by quotient groups in the normal series for U_J discussed below. We are faced with the task of describing certain irreducible characters of parabolic subgroups P_J in $\mathrm{U}_n(q)$ which do not contain U_J in their kernels. In general this is not an easy task as $\mathrm{Irr}(P_J)$ is not known. We avoid this difficulty by examining P_J orbits of irreducible U_J characters.

In order to distinguish between parabolic subgroups in general linear and in unitary groups of varying dimensions, we make the following definition.

Definition 6.0.2 *1. Let P_J^{+n} denote a parabolic subgroup in $\mathrm{GL}_n(q^2)$, where $J \subseteq [n-1]$.*

2. Let P_J^n denote a parabolic subgroup in $\mathrm{U}_n(q)$, where $n = 2m$, or $2m+1$ and $J \subseteq [m]$.

Write $J = \{j_1, j_2, \ldots, j_s\}$ in increasing order.

Definition 6.0.3 *The following subsets of J are defined*

1. $J(\geq j_i) = \{j_i, j_{i+1}, \ldots, j_s\}$ *and* $J(> j_i) = \{j_{i+1}, \ldots, j_s\}$
2. $J(\leq j_i) = \{j_1, j_2, \ldots, j_i\}$ *and* $J(< j_i) = \{j_1, j_2, \ldots, j_{i-1}\}$.

The unipotent radical U_J has the following normal series:

$$U_J = U_{J(\geq j_1)} > U_{J(\geq j_2)} > \cdots > U_{J(\geq j_s)} = U_{j_s} \geq Z(U_{j_s}) > 1 \tag{6.1}$$

The quotient groups in this series are elementary abelian. The set of all irreducible characters of P_J divides into subsets of characters in the following way. Let χ be an irreducible character of P_J then there exits a group N in the above series such that the kernel of χ contains N but does not contain the previous subgroups in the series. Since $N \trianglelefteq P_J$ and $N \subseteq \ker(\chi)$ for $\chi \in \mathrm{Irr}(P_J)$, we may consider χ as an irreducible character of the quotient group P_J/N. If N' is the previous subgroup in the normal series, then N'/N is elementary abelian and hence may be regarded as a P_J/N-module over F_p. There are four types of characters originally found by KU which we now briefly describe. What distinguishes these four types is the action induced by conjugation by P_J on the abelian quotients in Equation 6.1.

1. *Levi Characters:* Suppose $\chi \in \mathrm{Irr}(P_J)$ and U_J is contained in $\ker(\chi)$. Then we may consider χ a character of the quotient group P_J/U_J which is of course isomorphic to L_J. Such a character χ is trivial on U_J and has already been accounted for in Equation 4.6a.

2. *General Linear Characters:* Suppose $\chi \in \mathrm{Irr}(P_J)$ and for fixed i satisfying $1 \leq i < s$ we have $U_{J(\geq j_{i+1})} \subseteq \ker(\chi)$ but $U_{J(\geq j_i)} \nsubseteq \ker(\chi)$. We may consider χ as a character of $P_J/U_{J(\geq j_{i+1})}$. Then

$$P_J/U_{J(\geq j_{i+1})} \cong P^{+j_{i+1}}_{J(\leq j_i)} \times L_{J'} \quad ((15), 7.1.2.2)$$

where $J' = \{j - j_{i+1} | j \in J(> j_{i+1})\}$ and $L_{J'}$ is a levi subgroup in $\mathrm{U}_{n-2j_{i+1}}(q)$. Let $V(j_i, j_{i+1})$ denote the quotient group $U_{J(\geq j_i)}/U_{J(\geq j_{i+1})}$.

$$P^{+j_{i+1}}_{J(\leq j_i)} \cong \left(P^{+j_i}_{J(<j_i)} \times \mathrm{GL}_{j_{i+1}-j_i}(q)\right) \ltimes V(j_i, j_{i+1}).$$

We call $V(j_i, j_{i+1})$ a general linear module to indicate that a general linear group is acting on the module. The character χ restricted to $V(j_i, j_{i+1})$ is nontrivial. Hence $\chi = \chi_1\chi_2$ where χ_1 corresponds to some irreducible character of $V(j_i, j_{i+1})$ and χ_2 is an irreducible character of the factor $L_{J'}$.

3. *Unitary Linear Characters:* Now we suppose that $\chi \in \mathrm{Irr}(P_J)$ with $Z(U_{j_s}) \subseteq \ker(\chi)$ but $U_{j_s} \nsubseteq \ker(\chi)$. We may consider χ as a character of

$$P_J/Z(U_{j_s}) \cong \left(P^{+j_s}_{J(<j_s)} \times \mathrm{U}_{n-2j_s}(q)\right) \ltimes (U_{j_s}/Z(U_{j_s})) \quad ((15), 7.1.2.3).$$

Then χ corresponds to an irreducible character of the quotient $U_{j_s}/Z(U_{j_s})$, a unitary linear module to indicate that a unitary group is acting on the module.

4. *Central Characters:* Finally we make take $\chi \in \mathrm{Irr}(P_J)$ with only the trivial subgroup contained in $\ker(\chi)$ so that $Z(U_{j_s}) \nsubseteq \ker(\chi)$. In this case χ corresponds to a non trivial character of $Z(U_{j_s})$.

As mentioned these four types are outlined by Ku in (15). The main difference in this thesis is the introduction of two extra parameters in the manner of Sukizaki's approach (22) to the special linear case. Amazingly, it turns out that parabolic characters as parameterized by Ku are very well behaved with regard to their splitting upon restriction to the kernel of the determinant map. This is very convenient and not an obvious fact. One important result of this fact is that Ku's parametrization of the character q-heights is sufficient.

Conjugation by P_J on the abelian quotients in Equation 6.1 gives rise to the aforementioned internal modules with non-trivial action by a group H_J in a quotient group P_J/N of P_J, where N appears in the series Equation 6.1. Hence we need to examine what occurs at the level of H_J.

1 Parabolic Actions

In this section we study the modules which arise in the following situation. Fix positive integers n_1 and n_2. Let V_1 be the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 the dual of the natural module for $\mathrm{GL}_{n_2}(q^2)$. Fix a basis for V_1, n_1-dimensional column vectors $\{e_1, e_2, \ldots, e_{n_1}\}$ where e_i has a 1 in the i-position and zeros elsewhere. Fix a basis for V_2, n_2-dimensional row vectors $\{e^1, e^2, \ldots, e^{n_2}\}$ where e^j has a 1 in the j-position and zeros elsewhere. Set $V = V_1 \otimes V_2 \cong M_{n_1,n_2}(q^2)$. A basis for V is given by $\{E_{i,j}\}$ where $E_{i,j} = e_i \otimes e^j$ the $n_1 \times n_2$-matrix with (i,j)-entry 1 and zeros elsewhere. Then $\mathrm{GL}_{n_1}(q^2) \times \mathrm{GL}_{n_2}(q^2)$ acts on V in the natural way via left multiplication by $\mathrm{GL}_{n_1}(q^2)$ and right multiplication by inverses in $\mathrm{GL}_{n_2}(q^2)$. Let G_i be a subgroup of $\mathrm{GL}_{n_i}(q^2)$. Then $G = G_1 \times G_2$ acts on V and hence V is a module for G. In subsequent sections we will be considering the following cases:

1. The group $G_1 = P_J^{+n_1}$ and $G_2 = \mathrm{GL}_{n_2}(q^2)$,
2. the group $G_1 = P_J^{+n_1}$ and $G_2 = \mathrm{U}_{n_2}(q)$, and
3. the group $G_1 = P_J^{+n_1}$ and G_2 is an isomorphic copy of G_1. In this case, as we will see, the module we consider is a subgroup of V isomorphic to $M_{n_1,n_2}(q)$. We will discuss this in the central module section.

The vector space V is an elementary abelian group. We have the following G-isomorphisms of abelian groups ((15), 6.1.2):

$$\mathrm{Irr}(V) = \mathrm{Hom}(V, \mathbb{C}^*) \cong \mathrm{Hom}(V, \mathbb{C}_p) \cong \mathrm{Hom}_{F_p}(V, F_p) \cong \mathrm{Hom}_{F_{q^2}}(V_1 \otimes V_2, F_{q^2}) \cong \mathrm{Hom}_{F_{q^2}}(V_1, V_2^*)$$

where now V_2^* is the restriction to G_2 of the natural module for $\mathrm{GL}_{n_2}(q^2)$.

The first isomorphism is clear since complex characters of V take values in $\mathbb{C}_p$. The second isomorphism is also clear since the multiplicative group $\mathbb{C}_p$ can be identified with the additive group F_p. The last isomorphism is also clear by adjoint associativity of the tensor product. The penultimate isomorphism is less clear. The field F_{q^2} is a finite extension of F_p. Let θ generate

F_{q^2} so that $\{1, \theta, \theta^2, \ldots\}$ is a basis for F_{q^2} as a vector space over F_p. Note that $\theta^0 = 1$. Define the projection

$$\pi : F_{q^2} \longrightarrow F_p \ , \qquad \sum_i a_i \theta^i \mapsto a_0.$$

We fixed a basis $\{E_{i,j}\}$ for V over F_{q^2} above. Let U be the F_p-span of $\{E_{i,j}\}$. Then

$$V = F_{q^2} \otimes_{F_p} U$$

so that $\{\theta^k \otimes E_{i,j}\}$ is an F_p-basis of V. The following is a well-defined isomorphism

$$\mathrm{Hom}_{F_{q^2}}(V, F_{q^2}) \longrightarrow \mathrm{Hom}_{F_p}(V, F_p) \ , \qquad f \mapsto \pi \circ f.$$

Hence we have $\mathrm{Hom}_{F_p}(V, F_p) \cong \mathrm{Hom}_{F_{q^2}}(V_1 \otimes V_2, F_{q^2})$.

The action of G on V gives rise to a parallel action of G on $\mathrm{Irr}(V)$ and hence on $\mathrm{Hom}_{F_{q^2}}(V_1, V_2^*)$. Since we fixed a basis for V_2, the dual V_2^* has basis $\{(e^1)^*, (e^2)^*, \ldots, (e^{n_2})^*\}$ where $(e^j)^*$ is the n_2-dimension row vector with 1 in the j-position and zeros elsewhere. Take $\tau \in \mathrm{Irr}(V)$. Then τ corresponds to $f \in \mathrm{Hom}_{F_{q^2}}(V_1, V_2^*)$ and

$$V_1 / \ker(f) \cong f(V_1) \cong (V_2 / \mathrm{Ann}(f))^* \, .$$

where

$$\ker(f) = \{v \in V_1 \mid f(v)(w) = 0 \ \forall w \in V_2\} \text{ is the kernel of } f \text{, and}$$

$$\mathrm{Ann}(f) = \{w \in V_2 \mid f(v)(w) = 0 \ \forall v \in V_1\} \text{ is the annihilator of } f(V_1) \text{ in } V_2.$$

The codimension of both of these subspaces is the same. If r is the codimension of $\ker(f)$ and $\mathrm{Ann}(f)$ we will say that f has rank r. In this way if $\tau \in \mathrm{Irr}(V)$ corresponds to $f \in \mathrm{Hom}_{F_{q^2}}(V_1, V_2^*)$ then τ is said to have rank r. We remark that via the isomorphism between $\mathrm{Irr}(V)$ and V, if τ corresponds to the matrix v then the rank of τ is the row rank of v, which is invariant under the action of G.

We are interested in describing representatives of G-orbits in $\mathrm{Irr}(V)$ and their stabilizers in G. Given $\tau \in \mathrm{Irr}(V)$, let $\ker(\tau)$ and $\mathrm{Ann}(\tau)$ denote $\ker(f)$ and $\mathrm{Ann}(f)$, respectively, where τ corresponds to the element $f \in \mathrm{Hom}_{F_{q^2}}(V_1, V_2^*)$. Write $\overline{V}_1 = V_1/\ker(\tau)$ and $\overline{V}_2 = V_2/\mathrm{Ann}(\tau)$. The dimension of both quotient spaces is r. Let

$$C_{G_i}(\overline{V}_i) = \{\ g \in G_i \mid g \cdot \overline{v} = \overline{v} \ \forall \overline{v} \in \overline{V}_i\ \}$$

where as noted above $g \cdot \overline{v}$ indicates left or right multiplication depending on i equal to 1 or 2, respectively. As stated by Ku ((15), p.67) we have

$$C_{G_1}(\overline{V}_1) \times C_{G_2}(\overline{V}_2) \leq T_G(\tau) \leq T_{G_1}(\ker(\tau)) \times T_{G_2}(\mathrm{Ann}(\tau)).$$

Before describing these orbits and stabilizers we pause to consider some cancellation which occurs in the alternating sum of DOC.

2 Some Useful Cancellation

An alternating sum may reduce to a smaller alternating sum in an advantageous way. One may approach this from a combinatorial perspective in which case, like terms that appear with opposite parity cancel one another. We will make use of this approach later in the thesis. One may also approach the reduction of an alternating sum from a topological perspective. We discuss this now and apply the results in order to make a first reduction of Equation 4.6b.

Consider the Burnside ring $b(G)$ of a finite group G the free abelian group on equivalence classes $[G/H]$, where $[G/H]$ is equal to $[G/K]$ if and only if H and K are conjugate subgroups of G. A typical element of $b(G)$ is of the form

$$a_1[G/H_1] + a_2[G/H_2] + \cdots + a_N[G/H_N]$$

where $a_i \in \mathbb{Z}$ and the H_i are representatives of conjugacy classes of subgroups in G. Multiplication in $b(G)$ is given by

$$[G/H] \cdot [G/K] = [G/(H \cap K)].$$

Let G act on a finite poset P ordered by inclusion. The simplicial complex $\mathcal{O}(P)$ consists of chains,

$$c : x_0 < x_1 < \cdots < x_k \qquad x_i \in P.$$

where we require strict inclusion. The chain c as above has length $k+1$. The chains of length $k+1$ form the k-simplices of $\mathcal{O}(P)$. By convention the -1-simplex is 1 the trivial G-set. Let $\Delta(P) = \{1 < x_0 < x_1 < \cdots < x_k \mid x_i \in P\}$. Every chain in $\Delta(P)$ in (including the trivial chain) begins with the trivial G-set 1. For $c : 1 < x_0 < x_1 < \cdots < x_k \in \Delta(P)$ define the absolute value $|c| = k+1$. The reduced Lefshetz element of P in G is an element of $b(G)$ and is defined

$$\Lambda_G(P) = \sum_{c \in \Delta(P)/G} (-1)^{|c|} [G/G_c]$$

where c runs over a set of G-orbit representatives in $\Delta(P)$ and G_c is the stabilizer $T_G(c)$ of c in G.

We may assign topological concepts to posets in the following sense. When we say a poset P has a certain property we mean that its associated simplicial complex has the property. See ((18)) for a discussion of this.

Let G act on posets P and Q. A poset map $f : P \longrightarrow Q$ is a map that preserves ordering, i.e. $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$. The poset map f gives rise to a simplicial map $f_{\mathcal{O}} : \mathcal{O}(P) \longrightarrow \mathcal{O}(Q)$. This map is G-equivariant if $gf_{\mathcal{O}}(c) = f_{\mathcal{O}}(gc)$ for all $c \in \mathcal{O}(P)$. Two simplicial maps $f_{\mathcal{O}}$, $g_{\mathcal{O}} : \mathcal{O}(P) \longrightarrow \mathcal{O}(Q)$ are homotopic if there exists a continuous map $H : \mathcal{O}(P) \times [0,1] \longrightarrow \mathcal{O}(Q)$

such that $H(c,0) = f_{\mathcal{O}}(c)$ and $H(c,1) = g_{\mathcal{O}}(c)$ for all $c \in \mathcal{O}(P)$. If $f,\, g : P \longrightarrow Q$ are poset maps such that $f(x) \leq g(x)$, for all $x \in P$, then $f_{\mathcal{O}}$ and $g_{\mathcal{O}}$ are homotopic.

The spaces $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are said to be G-homotopy equivalent if there exist G-equivariant maps $f_{\mathcal{O}} : \mathcal{O}(P) \to \mathcal{O}(Q)$ and $g_{\mathcal{O}} : \mathcal{O}(Q) \to \mathcal{O}(P)$ such that $f_{\mathcal{O}} \circ g_{\mathcal{O}}$, $id_{\mathcal{O}(P)}$ are G-equivariant homotopic and $g_{\mathcal{O}} \circ f_{\mathcal{O}}$, $id_{\mathcal{O}(Q)}$ are G-equivariant homotopic ((25), p.352). We may say that the posets P and Q are G-homotopy equivalent, by which we mean that $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent. Given our interest in the reduced Lefshetz element we may also say that that $\Delta(P)$ and $\Delta(Q)$ are G-homotopy equivalent by which we mean again that $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent.

It is well known that, if $\mathcal{O}(P)$ and $\mathcal{O}(Q)$ are G-homotopy equivalent, then $\Lambda_G(P) = \Lambda_G(Q)$.

Let $S(G)$ denote the set of subgroups of G. Let $f : S(G) \longrightarrow \mathbb{Z}$ be a G-stable map, i.e. constant on conjugate subgroups of G. Set $f([G/G_c]) = f(G_c)$ and extend linearly to $b(G)$. We may apply f to $\Lambda_G(P)$,

$$f(\Lambda_G(P)) = \sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c).$$

This is an integer. Moreover if P and Q are G-homotopy equivalent then $f(\Lambda_G(P)) = f(\Lambda_G(Q))$, i.e.

$$\sum_{c \in \Delta(P)/G} (-1)^{|c|} f(G_c) = \sum_{c \in \Delta(Q)/G} (-1)^{|c|} f(G_c).$$

Given a G-poset P, we may make use of a contractible sub-poset in order to find a smaller complex which is G-homotopy equivalent to $\mathcal{O}(P)$. A poset Q is G-contractible if Q is itself

G-homotopy equivalent to a point. Let Q contain a single element x. Then $\Delta(Q)$ consists of two chains

$$c_1 : 1 \text{ with } |c_1| = 0 \text{ and } c_2 : 1 < x \text{ with } |c_2| = 1,$$

which are both necessarily stabilized by G. Observe that $\Lambda_G(Q) = [G/G] - [G/G] = 0$ and thus for any G-stable map f, $f(\Lambda_G(Q)) = 0$

A reference for more material on G-homotopy is ((21), Chapters 3 and 4).

An important example of homotopy equivalence: Let V be a vector space over a finite field, and let P be the poset of non-trivial proper subspaces of V ordered by inclusion. Let w be a non-trivial proper subspace of V. Let Q be the sub-poset of P whose elements are subspaces which are not complements in V to w. Then $\Delta(P)$ is the set of chains of subspaces of V beginning with the zero subspace and $\Delta(Q)$ is the subset of chains which do not contain a complement in V to w. Let $G = T_{\mathrm{GL}(V)}(w)$. The group G certainly acts on P and Q. The vector space w is in Q and is fixed by G. It turns out that Q is G-contractible ((24), Corollary 1.9). Thus $\Lambda_G(Q) = 0$. Let $\Delta(P, w)$ denote the sub-set of $\Delta(P)$ consisting of chains which contain complements to w. Then $\Delta(P) = \Delta(Q) \bigsqcup \Delta(P, w)$ ((15), Lemma 5.2.13). Since Q is G-contractible, $\Lambda_G(P) = \Lambda_G(P, w)$.

We are now ready to apply this discussion to our situation. For the rest of this section let $G = \mathrm{GL}_{n_1}(q^2) \times G_2$ where G_2 is a subgroup of $\mathrm{GL}_{n_2}(q^2)$. For now, we will be considering G_2

to be either $\mathrm{GL}_{n_2}(q^2)$ or $\mathrm{U}_{n_2}(q)$. Let $H = G \ltimes V$ where $V = V_1 \otimes V_2$ where V_1 is the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{n_1}(q^2)$, as above. For each $J \subseteq [n_1 - 1]$, we let G_J denote $P_J^{+n_1} \times G_2$ and H_J denote $G_J \ltimes V$.

Definition 6.2.1 *Let* $\mathrm{Irr}(V, r)$ *denote the subset of characters in* $\mathrm{Irr}(V)$ *of rank* r*, where we recall that the rank of* τ *is the codimension of* $\ker(\tau)$ *in* V_1*, viewing* τ *as a homomorphism from* V_1 *to* V_2^**.*

Set $X = \mathrm{Irr}(V, r)$. Let $Y = \{y \leq V_1 \mid \mathrm{codim}(y) = r\}$ and for $y \in Y$ let $X(y) = \{\tau \in X \mid \ker(\tau) = y\}$. We extend the action of $\mathrm{GL}_{n_1}(q^2)$ on V_1 to H by declaring that $G_2 \rtimes V$ act trivially on V_1. In this way H acts on X and Y but also on the poset $P = P(V_1)$ of subspaces of V_1 ordered by inclusion. Then $\Delta(P)$ is the set of flags in P together with the empty flag and so H also acts on $\Delta(P) \times X$. A set of orbit representatives for the action of $\mathrm{GL}(V_1)$ and hence H on $\Delta(P)$ is given by $\{c_J \mid J \subseteq [n_1 - 1]\}$, where if $J = \{j_1, j_2, \ldots, j_s\}$ then c_J is the flag

$$0 < V_{j_1} < V_{j_2} < \cdots < V_{j_s}$$

where V_{j_i} is the subspace of V_1 spanned by the vectors $\{e_1, e_2, \ldots, e_{j_i}\}$. We also note that $P_J^{+n_1}$ is the stabilizer of the flag c_J.

The proposition below is a modification of ((15), Proposition 6.2.1), where we obtain a first reduction for equation Equation 4.6b. Ku's proof makes use of some fairly elaborate abstractions involving alternating sums. We present a more direct proof with the necessary adjustments which include extra parameters in the definition of an H-stable function which lead to the desired reduction. The significance of this result is that for fixed r we need only describe orbit representatives for the subset $X(w) = \{\tau \mid \ker \tau = w\}$ in $\mathrm{Irr}(V, r)$, where w is a complement in V_1 of the r-dimensional subspace stabilized by $P_r^{+n_1}$. In fact, since $T_{H_J}(\tau) \leq T_{H_J}(w) \leq H_J$ and $T_{H_J}(w) = T_{G_J}(w) \ltimes V$ by Corollary 2.4.3 it is sufficient to find representatives for $T_{G_J}(w)$-orbits in $X(w)$. What is more we only need the stabilizers in H_J where $J \cup \{n_1\}$ contains the element r.

We will be considering the group $H_J = G_J \ltimes V$ as embedded in $\mathrm{U}_n(q)$.

Definition 6.2.2 *Define the map on G_J,*

$$\mathcal{D} : P_J^{+n_1} \times G_2 \longrightarrow F_{q^2}^* , \qquad (x, y) \mapsto \det(x)^{i(1-q)} \det(y)^k,$$

for some positive integer i where the exponent of $\det(y)$ depends on G_2 in the following way:

$$k = \begin{cases} 1 - q, & \text{if } G_2 = \mathrm{GL}_{n_2}(q^2); \\ 1, & \text{if } G_2 = \mathrm{U}_{n_2}(q). \end{cases}$$

Extend $\mathcal{D}$ to H_J by letting $\mathcal{D}(v) = 1$ for all $v \in V$. Observe that $\ker(\mathcal{D})$ is normal in H_J, contains V, and the quotient $H_J/\ker(\mathcal{D})$ is cyclic. The map $\mathcal{D}$ is constructed to be the restriction to H_J of the determinant map on $\mathrm{U}_n(q)$. The value of the integer i depends on the embedding of H_J in $\mathrm{U}_n(q)$.

Definition 6.2.3 *For a subset $X \subseteq \mathrm{Irr}(V)$, let $k_d(H_J, X, \rho, \mathcal{D}, j)$ denote the number of irreducible characters χ of H_J of q-height d, lying over ρ, and corresponding to $\tau \in X$ such that χ restricted to the kernel of the map $\mathcal{D}$ is a sum of j' irreducible characters where j divides j'.*

Proposition 6.2.4 *Let $Z \leq Z(G)$ and $\rho \in Irr(Z)$. Fix $1 \leq r \leq \min(n_1, n_2)$. Let $X = \mathrm{Irr}(V, r)$ and Let $Y = \{y \leq V_1 \mid \mathrm{codim}(y) = r\}$. Let w be a complement in V_1 to the r-dimensional subspace stabilized by $P_r^{+n_1}$, in its action on the natural module for $\mathrm{GL}_{n_1}(q^2)$. Let $X(w) = \{\tau \in X \mid \ker(\tau) = w\}$. Let $\mathcal{D}$ be the map on H_J defined above. For $d \geq 0$ the following holds*

1. *If $r = n_1$, then $Y = \{0\}$ and $X = X(0)$.*
2. *If $r < n_1$, then*

$$\sum_{J \subseteq [n_1 - 1]} (-1)^{|J|} k_d(H_J, X, \rho, \mathcal{D}, j) = \sum_{\substack{J \subseteq [n_1 - 1] \\ r \in J}} (-1)^{|J|} k_d(H_J, X(w), \rho, \mathcal{D}, j).$$

Proof: The first part is immediate since the trivial subspace is the only subspace of V_1 of codimension n_1. The proof of the second part is based on the exposition on homotopy equivalences which begins this section.

Let $r < n_1$. We define a function

$$f : \Delta(P) \times Y \longrightarrow \mathbb{Z}\ , \quad (c_J, y) \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j).$$

This is a well defined H-stable function on $\Delta(P) \times Y$, i.e. constant on the H-orbits. Indeed, the action of H on $\Delta(P)$ preserves chain type and thus a set of H-orbit representatives is given by

$$\{\ c_J \mid \in J \subseteq [n_1 - 1]\}.$$

Moreover, $k_d(H_J, \tau, \rho, \mathcal{D}, j) = k_d(H_J, {}^h\tau, \rho, \mathcal{D}, j)$ for any $h \in H_J$. If $y' = gy$ for $g \in P_J^{+n_1}$, then

$$\begin{aligned}
k_d(H_J, X(y'), \rho, \mathcal{D}, j) &= \sum_{\ker \tau = y'} k_d(H_J, \tau, \rho, \mathcal{D}, j) \\
&= \sum_{\ker \tau = gy} k_d(H_J, \tau, \rho, \mathcal{D}, j) \\
&= \sum_{g^{-1} \ker \tau = y} k_d(H_J, {}^{g^{-1}}\tau, \rho, \mathcal{D}, j) \\
&= \sum_{\ker \tau = y} k_d(H_J, \tau, \rho, \mathcal{D}, j) \\
&= k_d(H_J, X(y), \rho, \mathcal{D}, j).
\end{aligned}$$

The function f induces the following projection functions. For fixed $c_J \in \Delta(P)$

$$f_{c_J} : Y \longrightarrow \mathbb{Z}\ , \quad y \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(c_J) = H_J$-stable on Y; for fixed $y \in Y$

$$f_y : \Delta(P) \longrightarrow \mathbb{Z}\ , \quad c_J \mapsto k_d(H_J, X(y), \rho, \mathcal{D}, j)$$

is $T_H(y)$-stable on $\Delta(P)$.

For fixed $\tau \in X(y)$, $T_{H_J}(\tau) \leq T_{H_J}(y) \leq H_J$. Thus by Corollary 2.4.3

$$\begin{aligned} k_d(H_J, X(y), \rho, \mathcal{D}, j) &= \sum_{\tau \in X(y)/H_J} k_d(H_J, \tau, \rho, \mathcal{D}, j) \\ &= \sum_{\tau \in X(y)/T_{H_J}(y)} k_{d-d'}(T_{H_J}(y), \tau, \rho, \mathcal{D}, j') \\ &= k_{d-d'}(T_{H_J}(y), X(y), \rho, \mathcal{D}, j') \end{aligned}$$

where d' is the exponent of q in $|T_{H_J}(y)\backslash H_J|$ and j' is the least positive integer such that

$$j | j' \cdot |T_{H_J}(y) \ker(\mathcal{D}) \backslash H_J|.$$

In all three cases which we will consider $j' = j$ will hold. Indeed, if $G_2 = \mathrm{GL}_{n_2}(q^2)$, or $\mathrm{U}_{n_2}(q)$ then this is certainly true since $n_2 \neq 0$ so

$$T_{H_J}(y) = \left(T_{P_J^{+n_1}} \times G_2\right) \ltimes V,$$

hence $\mathcal{D}(T_{H_J}(y)) = \mathcal{D}(H_J)$. We will be considering the central case where G_2 is an isomorphic copy of $P_J^{+n_1}$. In this instance, the fact that $r < n_1$ is sufficient to imply that $j' = j$.

Let d_0 be the exponent of q in $|H|$. Let w be a subspace of V_1 as defined in the statement of the proposition. Define the following function on subgroups of $T_H(w)$

$$
\begin{aligned}
g : S(T_H(w)) &\longrightarrow \mathbb{Z} \\
K &\mapsto \begin{cases} k_{d-(d_0-d(K))}(K, X(w), \rho, \mathcal{D}, j), & \text{if } V \leq K; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

where $d(K)$ is the exponent of q in $|K|$. This is a $T_H(w)$-stable function. For $c_J \in \Delta(P)$, $T_H(c_J) = H_J$. Since the q-height of H is equal to the q-height of H_J,

$$d_0 - d(K) = |T_{H_J}(w) \backslash H_J|.$$

Hence $g(H_J) = f_w(c_J)$. Applying g to the reduced Lefshetz element we have

$$g(\Lambda_{T_H(w)}(P)) = g(\Lambda_{T_H(w)}(P, w)).$$

We put the above together

$$\begin{aligned}
&\sum_{J\subseteq[n_1-1]}(-1)^{|J|}k_d(H_J,X,\rho,\mathcal{D},j)\\
&\qquad=\sum_{c_J\in\Delta(P)/H}(-1)^{|J|}\sum_{y\in Y/H_J}k_d(H_J,X(y),\rho,\mathcal{D},j)\\
&\qquad=\sum_{y\in Y/H}\sum_{c_J\in\Delta(P)/T_H(y)}(-1)^{|J|}k_d(H_J,X(y),\rho,\mathcal{D},j)\\
&\qquad=\sum_{c_J\in\Delta(P)/T_H(y)}(-1)^{|J|}k_d(H_J,X(w),\rho,\mathcal{D},j),\ \text{since there is a unique } H\text{-orbit in } Y\\
&\qquad=\sum_{c_J\in\Delta(P)/T_H(w)}(-1)^{|J|}k_{d-d'}(T_{H_J}(w),X(w),\rho,\mathcal{D},j)\\
&\qquad=\sum_{c_J\in\Delta(P,w)/T_H(w)}(-1)^{|J|}k_{d-d'}(T_{H_J}(w),X(w),\rho,\mathcal{D},j)\\
&\qquad=\sum_{c_J\in\Delta(P,w)/T_H(w)}(-1)^{|J|}k_d(H_J,X(w),\rho,\mathcal{D},j)\\
&\qquad=\sum_{\substack{J\subseteq[n_1-1]\\ r\in J}}(-1)^{|J|}k_d(H_J,X(w),\rho,\mathcal{D},j),
\end{aligned}$$

and we are done.

3 General Linear Modules

In this section we first summarize Ku's results and then examine a map which is related to the usual determinant map. In this section G_1 is $P_J^{+n_1}$, a parabolic subgroup of $\mathrm{GL}_{n_1}(q^2)$ for fixed $J \subset [n_1-1]$. Let G_2 be $\mathrm{GL}_{n_2}(q^2)$. The module V_1 is the restriction to $P_J^{+n_1}$ of the natural module for $\mathrm{GL}_{n_1}(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{n_2}(q^2)$. Recall, in this context $G_J = P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2)$. Fix $r \in J$ or $r = n_1$. If $r < n_1$ then let $w \leq V_1$ be a complement to the r-dimensional subspace stabilized by $P_r^{+n_1}$. If $r = n_1$ then let $w \leq V_1$ be the trivial subspace. Let $\tau \in X(w)$ so that τ has rank r and $\ker \tau = w$. We can use r to divide the set J into two subsets:

$$\{j \in J \mid j < r\} \quad \text{and} \quad \{j \in J \mid r < j\}.$$

Let us set $J_2 = \{j \mid j \in J \text{ and } j < r\}$ and $J_1 = \{j - r \mid j \in J \text{ and } r < j\}$. Notice that we have the containment $J_2 \subseteq [r-1]$ and $J_1 \subseteq [n_1 - r]$.

Proposition 6.3.1 *((15), Lemma 6.2.2) The group* $T_{G_J}(w) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r} \times \mathrm{GL}_{n_2}(q^2)$ *and is transitive on* $X(w)$*. The stabilizer of* τ *in* G_J *is*

$$T_{G_J}(\tau) = \begin{cases} P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2}, & \text{if } r = n_2; \\ P_{J_1}^{+(n_1-r)} \times P_{J_2 \cup \{r\}}^{+n_2}, & \text{if } r < n_2. \end{cases} \tag{6.2}$$

3.1 The determinant map in the general linear module context

We would like to determine the determinant map on

$$G_J = P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2),$$

as embedded in a larger unitary group. At this stage we consider the map

$$\begin{aligned} \mathcal{D} : P_J^{+n_1} \times \mathrm{GL}_{n_2}(q^2) &\longrightarrow F_{q^2}^* \\ (x, y) &\mapsto (\det(x)^i \det(y))^{1-q}, \end{aligned}$$

for some positive integer i. As we will see $\mathcal{D}$ is constructed to be the restriction to G_J of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of G_J in $\mathrm{U}_n(q)$. For the rest of this subsection let $K = \ker(\mathcal{D})$

Our first observation is that the image of G_J under this map is all of $\mathbb{C}_{(q+1)}$ since $n_2 \neq 0$. We need to examine the map $\mathcal{D}$ restricted to the subgroup $T_{G_J}(\tau)$ and calculate $|T_{G_J}(\tau)K \backslash G_J|$. The cases depend on r, n_1, and n_2. In order to calculate $|T_{G_J}(\tau)K \backslash G_J|$ we need to find the image of $T_{G_J}(\tau)$ under $\mathcal{D}$. If $\mathcal{D}(T_{G_J}(\tau)) = \mathbb{C}_{(q+1)/h}$ then $\mathcal{D}(T_{G_J}(\tau)K) = \mathbb{C}_{(q+1)/h}$ so that

$$|T_{G_J}(\tau)K \backslash G_J| = |\mathbb{C}_{(q+1)/h}| \backslash |\mathbb{C}_{(q+1)}| = h.$$

We examine the four cases:

1. If $r = n_1 = n_2$ then $T_{G_J}(\tau) = P_J^{+r}$. If $(x, y) \in G_J$ stabilizes τ then $x = y = B$ and

$$\begin{aligned} \mathcal{D} : \quad P_J^{+r} &\longrightarrow F_{q^2}^* \\ B &\mapsto (\det(B)^{i+1})^{1-q}. \end{aligned}$$

Also notice that $\mathcal{D}(P_J^{+r}) = \mathbb{C}_{(q+1)/\gcd(q+1,i+1)}$.

2. If $r = n_1$ but $r < n_2$ then $T_{G_J}(\tau) = P_{J\cup\{r\}}^{+n_2} = \left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2)$. If $(x, y) \in G_J$ stabilizes τ then

$$x = B \text{ and } y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.$$

Since $\mathcal{D}$ is trivial on the normal factor $M_{r,n_2-r}(q^2)$ we have

$$\begin{array}{rccc} \mathcal{D}: & \left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) & \longrightarrow & F_{q^2}^* \\ & (B, C) & \mapsto & (\det(B)^{i+1}\det(C))^{1-q}. \end{array}$$

Observe that $\mathcal{D}(\left(P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2)) = \mathbb{C}_{(q+1)}$ since $n_2 - r \neq 0$.

3. If $r < n_1$ but $r = n_2$, then $T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r}$. If $(x, y) \in G_J$ stabilizes τ, then

$$x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \text{ and } y = B.$$

We have

$$\begin{array}{rccc} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r} & \longrightarrow & F_{q^2}^* \\ & (A, B) & \mapsto & (\det(A)^i(\det(B)^{i+1})^{1-q}. \end{array}$$

Take any element α of $F_{q^2}^*$. Then the diagonal matrix $A = (\alpha^{-1}, 1, 1, \cdots, 1)$ is in $P_{J_1}^{+(n_1-r)}$ and the diagonal matrix $B = (\alpha, 1, 1, \cdots, 1)$ is in $P_{J_2}^{+r}$. Then

$$\mathcal{D}(A, B) = (\alpha^{-i}\alpha^{i+1})^{1-q} = \alpha^{1-q}$$

and thus $\mathcal{D}(P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+r})$ is all of $\mathbb{C}_{(q+1)}$.

4. Finally we suppose that $r < n_1$ and $r < n_2$. We have

$$\begin{aligned} T_{G_J}(\tau) &= P_{J_1}^{+(n_1-r)} \times P_{J_2}^{+n_2} \\ &= P_{J_1}^{+(n_1-r)} \times \left(P_{J_2}^{+r} \times \mathrm{GL}_{n_2-r}(q^2)\right) \ltimes M_{r,n_2-r}(q^2). \end{aligned}$$

If $(x, y) \in G_J$ stabilizes τ, then

$$x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \text{ and } y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.$$

As above, $\mathcal{D}$ is trivial on the normal factor $M_{r,n_2-r}(q^2)$ and

$$\begin{array}{rlcl} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_J^{+r} \times \mathrm{GL}_{n_2-r}(q^2) & \longrightarrow & F_{q^2}^* \\ & (A, B, C) & \mapsto & (\det(A)^i \det(B)^{i+1} \det(C))^{1-q}. \end{array}$$

In this case we also have $\mathcal{D}(T_{G_J}(\tau))$ equal to all of $\mathbb{C}_{(q+1)}$ since $n_2 - r \neq 0$.

Summarizing these results, we get the following proposition which is a step in computing the number of irreducible characters of P_J (in an appropriate p-block, of appropriate q-height) which split as desired upon restriction to $P_J \cap \mathrm{SU}_n(q)$:

Proposition 6.3.2

$$|T_{G_J}(\tau)K\backslash G_J| = \begin{cases} \gcd(q+1, i+1), & \textit{if } r = n_1 = n_2; \\ 1, & \textit{otherwise.} \end{cases}$$

4 Unitary Modules

In this section we first summarize Ku's results and then examine a map which is related to the usual determinant map. We make use of the following notations which are due to Ku: $S^u(V, J, r)$, $S^{su}(V, J, r)$, and $S^{nu}(V, J, r)$. In this section G_1 is $P_J^{+n_1}$ a parabolic subgroup of $\mathrm{GL}_{n_1}(q^2)$ for fixed $J \subset [n_1 - 1]$ as above. In the previous section we had $G_2 = \mathrm{GL}_{n_2}(q^2)$ which is transitive on subspaces of the same dimension in the dual space V_2^*. Now we consider $G_2 = \mathrm{U}_{n_2}(q)$. Recall, in this context $G_J = P_J^{+n_1} \times \mathrm{U}_{n_2}(q)$. The module V_1 is still the restriction to $P_J^{+n_1}$ of the natural module for $\mathrm{GL}_{n_1}(F_{q^2})$. However, now V_2 is the dual of the natural module for $\mathrm{U}_{n_2}(q)$ and thus has a unitary structure. In this case, G_2 is not transitive on subspaces of the same dimension of V_2^*.

Recall if U is a unitary vector space, a subspace W is totally isotropic if $< v, w >= 0$ for all vectors v, w in W, where $<, >$ indicates the hermitian form on U. A totally isotropic subspace W is degenerate since its radical $rad(W) = W \cap W^{\perp} = W$. A chain of totally isotropic subspaces will be called a singular chain. A chain of subspaces which are not all totally isotropic will be called a non-singular chain. A subspace W is non-degenerate if $rad(W) = 0$ in which case $V = W \oplus W^{\perp}$.

Let W be a non-degenerate subspace of V_2^* of dimension r. Then

$$T_{\mathrm{U}_{n_2}(q)}(W) = \mathrm{U}_{n_2-r}(q) \times \mathrm{U}_r(q).$$

Now consider W a totally isotropic subspace. A basis for V_2^* has already been fixed. To simplify notation, denote $(e^j)^*$ by e_j so that the basis is $\{e_1, e_2, \ldots, e_{n_2}\}$. Further suppose that with respect to the inner product on V_2^* we have

$$< e_i, e_j >= \begin{cases} 1, & \text{if } i + j = n_2 + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let W be the totally isotropic subspace equal to $< e_1, e_2, \ldots, e_r >$. Note that r must be less than or equal to $[n_2/2]$. Then the stabilizer is a maximal parabolic subgroup

$$T_{\mathrm{U}_{n_2}(q)}(W) = P_r^{n_2}.$$

Just as in the previous section we fix r where either $r \in J$ or $r = n_1$. If $r = n_1$ then let $w \leq V_1$ be the trivial subspace. If $r < n_1$ then let $w \leq V_1$ be a complement to the r-dimensional subspace stabilized by $P_r^{+n_1}$. Let $\tau \in X(w)$ so that τ has rank r and $\ker \tau = w$. Ku parameterizes the $T_{G_J}(w)$ orbits on $X(w)$ with chains of subspaces in V_2^*. These are so-called normal flags of fixed type depending on J and reflect the unitary structure in V_2^* which we define now.

Definition 6.4.1 *A normal flag in V_2^* is a chain of subspaces*

$$c : 0 < V_1 < V_2 < \cdots < V_s$$

satisfying the following. There exists $0 = i_0 < i_1 < \cdots i_k \leq s$, $k \geq 0$, such that for all $0 \leq j \leq k$

1. *V_{i_j} is either a non-degenerate subspace in V_2^* or the zero subspace and*
2. *for any $i_j < i < i_{j+1}$ we have $V_i = V_{i_j} \oplus rad(V_i)$, where we assume $i_{k+1} = s+1$ and $V_{s+1} = V_2^*$.*

Take linear maps $f,\ g : V_1 \to V_2^*$ with kernel w. The stabilizer $T_{G_J}(w) = P_{J_1}^{+n_1-r} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q)$. Under its action on $X(w)$, f and g are in the same orbit if and only if $f(c_J)$ and $g(c_J)$ are isomorphic as flags in V_2^*. Observe that $f(c_J)$ and $g(c_J)$ are both flags of type $\{J(< r) \cup \{r\}\} \backslash \{n_2\}$ because we can choose a basis so that $w =< e_{r+1}, \ldots, e_{n_1} >$ and $f(w) = g(w) = 0$ the trivial subspace. Let $\mathcal{P}(V_2^*)$ be the poset of subspaces in V_2^* ordered by inclusion.

If $r = n_1$ then $X(w) = X$ and $T_{G_J}(w) = G_J$. The G_J-orbits in X are in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$-orbits on chains of type $\{J \cup \{r\}\} \backslash \{n_2\}$ in $\mathcal{P}(V_2^*)$. If τ corresponds to chain c of such type then

$$T_{G_J}(\tau) = T_{\mathrm{U}_{n_2}(q)}(c).$$

See ((15),Lemma 6.3.1).

If $r < n_1$ then w is non-trivial and

$$T_{G_J}(w) = P_{J_1}^{+n_1-r} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q)$$

where $J_1 = \{j - r \mid j \in J(> r)\}$. The group $P_{J_1}^{+n_1-r}$ acts trivially on the quotient space $\overline{V}_1 = V_1/w$ and hence acts trivially on $X(w)$ which is isomorphic to $\mathrm{Irr}(\overline{V}_1 \otimes V_2, r)$. By the above discussion for the case $r = n_1$ the $(P_{J(<r)}^{+r} \times \mathrm{U}_{n_2}(q))$-orbits in $\mathrm{Irr}(\overline{V}_1 \otimes V_2, r)$ are in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$-orbits on chains of type $\{J(< r) \cup \{r\}\}\backslash\{n_2\}$ in $\mathcal{P}(V_2^*)$.

Definition 6.4.2 *Let $S^u(V, J, r)$ denote the set of $T_{G_J}(w)$-orbits in $X(w)$ labeled by normal flags of type $\{J(< r) \cup \{r\}\}\backslash\{n_2\}$ in $\mathcal{P}(V_2^*)$.*

The set $S^u(V, J, r)$ is defined for $r \in J \cup \{r\}$ and is in 1-1 correspondence with the $\mathrm{U}_{n_2}(q)$-orbits on normal chains of type $\{J(< r) \cup \{r\}\}\backslash\{n_2\}$. Let us examine the possible flags of such type in $\mathcal{P}(V_2^*)$. Let c be such a flag. Since V_2^* has a unitary structure, subspaces fall into two categories. They are either totally isotropic or not. The action of $\mathrm{U}_{n_2}(q)$ on V_2^* preserves this structure.

Definition 6.4.3 *If c is a flag of totally isotropic subspaces we will say that c is singular. If c contains a non-degenerate subspace we will say that c is a non-singular flag. Moreover we define the non-singular rank of c to be the dimension of the minimal non-degenerate subspace.*

If $\tau \in X(w)$ has rank r and corresponds to a non-singular flag c with non-singular rank r', then τ itself is said to have non-singular rank r'. Moreover $r' \in J \cup \{r\}$ since r' is the dimension of a subspace in the flag c corresponding to τ.

Suppose c is a flag of type $\{J(< r) \cup \{r\}\} \backslash \{n_2\}$ of non-singular rank r' with $1 \leq r' \leq r$. We have r', $r \in J \cup \{r\}$. Let $\widetilde{J} = \{k_1, k_2, \ldots, k_s\}$ be the type of c. The element r' divides $\widetilde{J}$ into two subsets

$$\widetilde{J}_1 = \{\ k \mid k \in \widetilde{J}, k < r'\} \quad \text{and} \quad \widetilde{J}_2 = \{\ k \mid k \in \widetilde{J}, r' < k\}.$$

Write

$$c : 0 < V_{k_1} < \cdots < V_{r'} < \cdots < V_{k_s}.$$

We assign to c a pair (c_1, c_2) of shorter flags in the following way. The subspace $V_{r'}$ is the minimal non-degenerate subspace. We define

$$c_1 : 0 < V_{k_1} < \cdots < V_{\max(\widetilde{J}_1)}$$

$$c_2 : 0 < V_{\min(\widetilde{J}_2)} \cap V_{r'}^{\perp} < \cdots < V_{k_s} \cap V_{r'}^{\perp}$$

The flag c_1 is a singular flag in the unitary space of dimension r'. The flag c_2 is a flag in the unitary space of dimension $n_2 - r'$.

Definition 6.4.4 *We define the following subsets of $S^u(V, J, r)$:*

1. *Let $S^{su}(V, J, r)$ be the subset of $S^u(V, J, r)$ labeled by singular flags.*

2. Let $S^{nu}(V,J,r)$ be the subset of $S^u(V,J,r)$ labeled by non-singular flags.

3. Finally, let $S^{nu}_{r'}(V,J,r)$ be the subset of $S^{nu}(V,J,r)$ with non-singular rank r'.

Observe that the set $S^{su}(V,J,r)$ is non-empty if and only if $r \leq n_2/2$, in which case it consists of a single member. Notice also that $S^{nu}_{r'}(V,J,r)$ is nonempty if and only if $r' \in J(< r)$ and $J(< r') \subseteq [n_2/2]$. We have the following structure of stabilizers of characters in these sets given by Ku.

Proposition 6.4.5 *((15), Remark 6.3.12)*

1. For $\tau \in S^{su}(V,J,r)$ we have $T_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times P^{n_2}_{\{J(<r)\cup\{r\}\}\setminus\{n_2\}}$

2. For $\tau \in S^{nu}_{r'}(V,J,r)$ we have $T_{G_J}(\tau) = P^{+(n_1-r)}_{J_1} \times P^{r'}_{J(<r')} \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$

where $J_1 = \{j - r \mid j \in J(> r)\}$ and c_2 is obtained as above from c which corresponds to τ.

4.1 The determinant map in the unitary linear module context

We would like to determine the determinant map on

$$G_J = P^{+n_1}_J \times \mathrm{U}_{n_2}(q),$$

as embedded in a larger unitary group. At this stage we consider the map

$$\begin{array}{rccc} \mathcal{D}: & P^{+n_1}_J \times \mathrm{U}_{n_2}(q) & \longrightarrow & F^*_{q^2} \\ & (x,y) & \mapsto & \det(x)^{i(1-q)}\det(y). \end{array}$$

for some positive integer i. The map $\mathcal{D}$ is constructed to be the restriction to G_J of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of G_J in $\mathrm{U}_n(q)$. For the rest of this subsection let $K = \ker(\mathcal{D})$

Observe the image of G_J under this map is all of $\mathbb{C}_{(q+1)}$ since $n_2 \neq 0$. We need to examine the map $\mathcal{D}$ restricted to the subgroup $T_{G_J}(\tau)$ and calculate $|T_{G_J}(\tau)K\backslash G_J|$.

Let $\tau \in S^{su}(V, J, r)$ so that

$$\begin{aligned} T_{G_J}(\tau) &= P_{J_1}^{+(n_1-r)} \times P_{J(<r)\cup\{r\}\backslash\{n_2\}}^{n_2} \\ &= P_{J_1}^{+(n_1-r)} \times \left(\left(P_{J(<r)}^{+r} \times \mathrm{U}_{n_2-2r}(q)\right) \ltimes U_r^{n_2}\right) \end{aligned}$$

where $U_r^{n_2}$ is the unipotent radical in the maximal parabolic subgroup $P_r^{n_2}$ in $\mathrm{U}_{n_2}(q)$. If $(x, y) \in G_J$ stabilizes τ then

$$x = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \text{ and } y = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}.$$

The map $\mathcal{D}$ restricted to $T_{G_J}(\tau)$ is trivial on the normal factor $U_r^{n_2}$ and

$$\begin{array}{rclcl} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_{J(<r)}^{+r} \times \mathrm{U}_{n_2-2r}(q) & \longrightarrow & F_{q^2}^* \\ & (A, B, C) & \mapsto & (\det(A)^i \det(B)^{i+1})^{1-q} \det(C) \end{array}$$

where it is understood that if any of these dimensions are zero, we assume $\det(A) = 1$ for A in any group of dimension zero. The image of $\mathcal{D}$ on $T_{G_J}(\tau)$ depends on r, n_1, and n_2. There are three cases:

1. If $r = n_1 = n_2/2$, then the image of $T_{G_J}(\tau)$ under $\mathcal{D}$ is $\mathbb{C}_{(q+1)/\gcd(q+1,i+1)}$.

2. If $r < n_2/2$, then $\mathcal{D}(T_{G_J}(\tau)) = \mathbb{C}_{(q+1)}$.

3. If $r < n_1$ but $r = n_2/2$, then for any $\alpha \in F_{q^2}^*$, the diagonal matrix $A = (\alpha^{-1}, 1, \ldots, 1)$ is in $P_{J_1}^{+(n_1-r)}$ and the diagonal matrix $B = (\alpha, 1, \ldots, 1)$ is in $P_{J(<r)}^{+r}$. We have $\mathcal{D}(AB) = (\alpha^{-i}\alpha^{i+1})^{1-q} = \alpha^{1-q}$ and thus $\mathcal{D}(T_{G_J}(\tau))$ is all of $\mathbb{C}_{(q+1)}$.

Summarizing these results, we get the following proposition which is another step in computing the number of irreducible characters of P_J (in an appropriate p-block, of appropriate q-height) which split as desired upon restriction to $P_J \cap \mathrm{SU}_n(q)$:

Proposition 6.4.6 *For $\tau \in S^{su}(V, J, r)$*

$$|T_{G_J}(\tau)K \backslash G_J| = \begin{cases} \gcd(q+1, i+1), & \text{if } r = n_1 = n_2/2; \\ 1, & \text{otherwise.} \end{cases}$$

Now take $\tau \in S^{nu}(V, J, r)$ with non-singular rank $r' \neq 0$. We have

$$T_{G_J}(\tau) = P_{J_1}^{+(n_1-r)} \times P_{J(<r')}^{r'} \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$$

where τ corresponds to c which corresponds to the pair (c_1, c_2) as above so that c_2 is a flag in the unitary space of dimension $n_2 - r'$. If $(x, y) \in G_J$ stabilizes τ then

$$x = \begin{pmatrix} B & 0 & 0 \\ 0 & C' & 0 \\ 0 & 0 & A \end{pmatrix} \text{ and } y = \begin{pmatrix} B & 0 & 0 \\ 0 & C' & * \\ 0 & 0 & C'' \end{pmatrix}.$$

Let

$$C = \begin{pmatrix} C' & * \\ 0 & C'' \end{pmatrix}.$$

Then

$$\begin{array}{rlcl} \mathcal{D}: & P_{J_1}^{+(n_1-r)} \times P_J^{r'}(< r') \times T_{\mathrm{U}_{n_2-r'}(q)}(c_2) & \longrightarrow & F_{q^2}^* \\ & (A, B, C) & \mapsto & \det(A)^{i(1-q)} \det(B)^{2i+1} \mathcal{D}'(C). \end{array}$$

where $\mathcal{D}'(C)$ is the determinant of C as an element embedded in $\mathrm{U}_n(q)$. In other words $\mathcal{D}'(C)$ depends on the embedding of C'. We take note of the restriction of $\mathcal{D}$ to the factor $P_{J(<r')}^{r'}$ in the stabilizer $T_{G_J}(\tau)$. For $B \in P_{J(<r')}^{r'}$, $\mathcal{D}(B) = \det(B)^{2i+1}$ since $-q \equiv 1 \mod (q+1)$. Notice also that $T_{\mathrm{U}_{n_2-r'}(q)}(c_2)$ is a stabilizer in a unitary group of smaller dimension than n_2 and as mentioned $\mathcal{D}'$ depends on the embedding in $\mathrm{U}_n(q)$. Later we will make use of an inductive argument.

5 Central Modules

In this section we first summarize Ku's results and then briefly examine a map which is related to the usual determinant map. At the end of this section we also examine how the useful cancellation applies to the central module case. We will be examining parabolic subgroups P_J in $\mathrm{U}_{n'}(q)$, where $n' \leq n$, as embedded in $\mathrm{U}_n(q)$. Let l be the maximal member of J. Write $Z_l = Z(U_l)$. Then the normal series for U_J terminates with $U_l \geq Z_l > 1$. Thus far the modules we have considered arise when $\chi \in \mathrm{Irr}(P_J)$ contains $Z(U_l)$ in its kernel. We will consider now the case when $\chi \in \mathrm{Irr}(P_J)$ does not contain $Z(U_l)$ in its kernel.

The parabolic group P_J has the following decomposition

$$P_J = \left(P^{+l}_{J(<l)} \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l$$

and acts on U_l by conjugation. This induces an action of the quotient group P_J/U_l on Z_l which is isomorphic as an abelian group to $M_{l,l}(q)$ which we denote by V. Indeed, since P_J is upper triangular we may write elements of V as matrices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & & & d_1 \\ a_{2,1} & & & d_2 & \\ & & & & \\ & & & & -a_{1,2}^q \\ d_l & & & -a_{2,1}^q & -a_{1,1}^q \end{pmatrix} \text{ in } M_{l,l}(q^2) \text{ where } d_i^q + d_i = 0.$$

Let V^l be the usual module for $\mathrm{GL}_l(q^2)$ then $V^l \otimes (V^l)^* \cong M_{l,l}(q^2)$ is a module for $P^{+l}_{J(<l)}$. Observe that $V \leq M_{l,l}(q^2)$. The quotient P_J/U_l acts on V as follows. For matrix $A \in P^{+l}_{J(<l)}$, $B \in \mathrm{U}_{n'-2l}(q)$, and $v \in V$, $(A,B) \cdot v = Av(\widetilde{A})^{-1}$ where $\widetilde{A}$ is as defined at the beginning of Chapter 5, i.e. if $A = (a_{i,j})$, then $\widetilde{A} = M((a^q_{j,i}))^{-1}M^{-1}$ where M is the matrix with ones on the reverse diagonal. This induces an action on the subset $\mathrm{Irr}(V) \leq \mathrm{Irr}(V^l \otimes (V^l)^*) \cong \mathrm{Hom}(V^l, V^l)$ which is invariant on the rank of $\tau \in \mathrm{Irr}(V)$ which recall is the co-dimension of the kernel of τ viewed as a homomorphism from V^l to itself. Notice that the unitary factor acts trivially.

Case 1. We begin with the special case $n' = 2m$ and $J = \{m\}$ so that $P_J = \mathrm{GL}_m(q^2) \ltimes U_m$ and $U_m = Z_m \cong M_{m,m}(q)$.

Let $X = \mathrm{Irr}(V, r)$ the subset of characters in $\mathrm{Irr}(V)$ of rank r. Fix non-zero ϵ in the algebraic closure of F_q satisfying $\epsilon^q + \epsilon = 0$ and define the $m \times m$ matrix

$$x_r = (a_{i,j}) \text{ where } a_{i,j} = \begin{cases} \epsilon, & j - i = m - r; \\ 0, & \text{otherwise.} \end{cases}$$

As a matrix x_r has rank r.

Proposition 6.5.1 *((15), Chapter 7) The group* $\mathrm{GL}_m(q^2)$ *is transitive on* X*. The set* $\{0, x_r \mid 1 \leq r \leq m\}$ *is a complete set of representatives for the* $\mathrm{GL}_m(q^2)$*-orbits on* $\mathrm{Irr}(V)$*, where* 0 *denotes the zero matrix. Moreover*

$$T_{\mathrm{GL}_m(q^2)}(\tau_r) = T_{\mathrm{GL}_m(q^2)}(x_r) = \big(\mathrm{U}_r(q) \times \mathrm{GL}_{m-r}(q^2)\big) \ltimes M_{r,m-r}(q^2)$$

as in ((15), Lemma 7.3.1).

Case 2. Now consider the case where $U_l \neq Z_l$. Let $J = \{l\}$ so that

$$P_J = P_l \cong \big(\mathrm{GL}_l(q^2) \times \mathrm{U}_{n'-2l}(q)\big) \ltimes U_l$$

where $U_l/Z_l \cong M_{l,n'-2l}(q^2)$ and $Z_l \cong M_{l,l}(q)$. Let τ_r be identified with x_r. Since $\mathrm{U}_{n'-2l}(q)$ acts trivially on Z_l, by the first special case the set $\{1, \tau_r \mid 1 \leq r \leq l\}$ is a complete set of representatives for the P_J-orbits on $\mathrm{Irr}(Z_l)$, where 1 is the trivial character. Let

$$\mathrm{Irr}(U_l, \tau_r) = \{\phi \in \mathrm{Irr}(U_l) \mid \phi \text{ lies over } \tau_r\}.$$

If $\chi \in \mathrm{Irr}(P_l)$ does not contain Z_l in its kernel then χ lies over $\phi \in \mathrm{Irr}(U_l, \tau_r)$ where ϕ restricted to Z_l is a multiple of τ_r, for some $1 \leq r \leq l$. In this case, τ_r is not extendible to its stabilizer in P_l. However, it turns out that ϕ is extendible to $T_{P_l}(\phi)$.

Given $\phi \in \mathrm{Irr}(U_l, \tau_r)$, $\ker(\tau_r) \leq \ker(\phi)$. We may consider ϕ as a character of the quotient group $U_l/\ker(\tau_r)$. We may consider τ_r as a character of the quotient group $Z_l/\ker(\tau_r)$. The center Z_l is elementary abelian so $\mathrm{Irr}(Z_l) \cong \mathrm{Hom}_{F_p}(Z_l, F_p)$. Thus $Z_l/\ker(\tau_r)$ is cyclic of order p. The group $U_l/\ker(\phi)$ has an irreducible faithful representation and hence has cyclic center. Moreover, $Z(U_l/\ker(\phi))$ is a homomorphic image of $Z(U_l/\ker(\tau_r))$ which is elementary abelian ((15), p.101). Thus $Z(U_l/\ker(\phi))$ must have order p. We have

$$Z(U_l/\ker(\phi)) \geq Z_l\ker(\phi) \Big/ \ker(\phi) \cong Z_l \Big/ (\ker(\phi) \cap Z_l) = Z_l/\ker(\tau).$$

Hence

$$Z(U_l/\ker(\phi)) \cong Z_l/\ker(\tau).$$

Moreover

$$U_l/\ker(\phi) \Big/ Z\left(U_l/\ker(\phi)\right) \text{ is an elementary abelian } p\text{-group.}$$

Hence $U_l/\ker(\phi)$ is an extraspecial p-group. The ordinary character theory of such groups is well known. In summary, we have the following:

1. The order of $U_l/\ker(\phi)$ is p^{1+2a} for some integer a.
2. There are exactly p^{2a} linear characters, each corresponding to a character of the quotient

$$U_l/\ker(\phi) \Big/ Z\left(U_l/\ker(\phi)\right).$$

3. There are exactly $p-1$ non-linear characters each of dimension p^a. There is one of these characters χ for each non-trivial irreducible character θ of $Z(U_l/\ker(\phi))$ with character values given by $\chi(x) = p^a\theta(x)$ for x in $Z(U_l/\ker(\phi))$ and $\chi(x) = 0$ for x not in $Z(U_l/\ker(\phi))$.

Considered as a character of $U_l/\ker(\tau_r)$, ϕ is uniquely determined by its kernel. We summarize the properties of ϕ.

Proposition 6.5.2 *((15), Lemma 7.2.4) Let $1 \leq r \leq l$ and $\phi \in \mathrm{Irr}(U_l, \tau_r)$.*

1. ϕ is extendible to $T_{P_l}(\phi)$

2. $\phi(1) = q^{r(n-2l)}$

3. If $r = l$*, then* $\mathrm{Irr}(U_l, \tau_l)$ *contains a unique member and*

$$T_{P_l}(\phi) = T_{P_l}(\tau_l) \cong \big(\mathrm{U}_l(q) \times \mathrm{GL}_{n'-2l}(q^2)\big) \ltimes U_l.$$

4. If $1 \leq r < l$*, then* ϕ *is uniquely determined by its kernel.*

$$T_{P_l}(\phi) = T_{L_l}(\phi) \ltimes U_l = (T_{L_l}(\tau_r) \cap T_{L_l}(\ker(\phi))) \ltimes U_l.$$

For a non-negative integer k, let V^k be the natural module for $\mathrm{GL}_k(q^2)$. Let $R \leq V^l$ be a subspace of codimension r stabilized by $T_{L_l}(\tau_r)$. Then

$$V^{l-r} \otimes (V^{n'-2l})^*$$

is a module for $\mathrm{GL}_{l-r}(q^2) \times \mathrm{U}_{n'-2l}(q)$. Moreover, there is a $1-1$ correspondence

$$\mathrm{Irr}(U_l, \tau_r) \longleftrightarrow \mathrm{Irr}(V^{l-r} \otimes (V^{n'-2l})^*).$$

Case 3. In order to count characters of P_J that do not contain Z_l in their kernel, we observe that

$$\begin{aligned} k_d^1(P_J, Z_l, \rho, \det, j) &= \sum_{\tau \in Z_l/P_J} k_d(P_J, \tau, \rho, \det, j) \\ &= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \mathrm{Irr}(U_l, \tau)} k_d(P_J, \phi, \rho, \det, j) \\ &= \sum_{\tau \in Z_l/P_J} \sum_{\phi \in \mathrm{Irr}(U_l, \tau)} k_{d-d'}(T_{P_J}(\phi)/U_l, \rho, \det, j'). \end{aligned}$$

where $d' = r(n' - 2l) - d''$ for d'' equal to the exponent of q in $|T_{P_J}(\phi)\backslash P_J|$ and j' is the least positive integer such that

$$j \text{ divides } j' \cdot \mid T_{P_J}(\phi)\ker(\det)\backslash P_J \mid .$$

We use Proposition 6.5.2 to describe P_J orbits on $X = \mathrm{Irr}(V, r)$ for the general case where J is any subset in I with maximal element l.

Definition 6.5.3 *For fixed r, let $\mathcal{K} = T_{\mathrm{GL}_l(q^2)}(\tau_r)$.*

Let w be zero if $r = l$, or if $r < l$ a complement in V^l to the r dimensional subspace stabilized by P_r^{+l}. We have the structure of $\mathcal{K}$ given above by Proposition 6.5.1.

Proposition 6.5.4 *There is a 1-1 correspondence between the P_J-orbits on X and the $\mathcal{K}$-orbits on the set of chains of type $J(< l)$ in $\mathcal{P}(V^l)$. If τ corresponds to c then up to conjugation*

$$T_{P^{+l}_{J(<l)}}(\tau) = T_{\mathcal{K}}(c).$$

Let $r = l$ so that $\mathcal{K} = T_{\mathrm{GL}_l(q^2)}(\tau_l) = \mathrm{U}_l(q)$.

Definition 6.5.5 *Let $S^z(V, J, l)$ denote the P_J-orbits in X labeled by a normal chain in $\mathcal{P}(V^l)$ of type $J(< l)$.*

Let $r < l$ so that w is a complement in V^l to the r dimensional subspace stabilized by P_r^{+l}. We can assume that $\mathcal{K} \leq T_{\mathrm{GL}_l(q^2)}(w)$. Then $\mathcal{K}$ is transitive on complements of w. Moreover if $\tau \in X$ corresponds to c of type $J(< l)$ with

$$c : 0 < V_1 < \cdots < V_i < \cdots < V_s$$

where V_i is a complement to w then we may assign to c a pair (c_1, c_2) of shorter chains much as we did in the previous section. As $(V_i)^\perp = w$ we may define

$$c_1 : 0 < V_1 < \cdots < V_{i-1}$$

$$c_2 : 0 < (V_{i+1} \cap w) < \cdots < (V_s \cap w).$$

Notice that c_1 is a chain of type $J(< r)$ in $\mathcal{P}(V^r)$ and c_2 of of type $\{j - r | j \in J(> r)\}$ in $\mathcal{P}(V^{l-r})$. We have the stabilizer of c in $\mathcal{K}$ given by

$$T_{\mathcal{K}}(c) = T_{\mathrm{U}_r(q)}(c_1) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2).$$

With this identification of c with the pair (c_1, c_2), we make the following definition.

Definition 6.5.6 *Let $S^z(V,J,r)$ denote the P_J-orbits in X labeled by a normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$.*

By construction $S^z(V,J,r)$ is in 1-1 correspondence with the $\mathrm{U}_r(q)$-orbits on the set of chains of type $J(<r)$ in $\mathcal{P}(V^r)$.

Definition 6.5.7 *We define the following subsets of $S^z(V,J,r)$.*

1. *Let $S^z_r(V,J,r)$ be the subset labeled by a singular normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$.*
2. *For $r' < r$ let $S^z_{r'}(V,J,r)$ be the subset labeled by a normal chain c_1 of type $J(<r)$ in $\mathcal{P}(V^r)$ with non-singular rank r'.*

We may now assign to c_1 a pair of even shorter chains based on the dimension of the minimal non-degenerate subspace in c_1. In the manner of the previous section c_1 corresponds to the pair (c_{11}, c_{12}) where c_{11} is a totally isotropic chain in $\mathcal{P}(V^{r'})$ and c_{12} is a normal chain in $\mathcal{P}(V^{r-r'})$.

In summary for J with maximal member l, $r \in J$ and $\tau \in X = \mathrm{Irr}(V,r)$ corresponding to (c_1, c_2) where c_1 corresponds to (c_{11}, c_{12}) we have

$$\begin{aligned}
T_{P_J}(\tau) = & \left(T_{P^{+l}_{J(<l)}}(\tau) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\
& \left(T_{\mathrm{U}_r(q)}(c_1) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\
& \left(T_{\mathrm{U}_{r'}(q)}(c_{11}) \times T_{\mathrm{U}_{r-r'}(q)}(c_{12}) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l \\
& \left(P^{r'}_{J(<r')} \times T_{\mathrm{U}_{r-r'}(q)}(c_{12}) \times T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q)\right) \ltimes U_l.
\end{aligned}$$

where if $r = l$ then we take $c_2 = 0$. Now let $\phi \in \mathrm{Irr}(U_l, \tau_r)$ correspond to $\psi \in \mathrm{Irr}(V^{l-r} \otimes (V^{n'-2l})^*)$ and let

$$D = T_{\mathrm{GL}_{l-r}(q^2)}(c_2) \times \mathrm{U}_{n'-2l}(q).$$

Then

$$T_{P_J}(\phi) = (T_{P_J}(\tau) \times T_D(\psi)) \ltimes U_l.$$

5.1 <u>The determinant map in the central module context</u>

We would like to determine the determinant map on quotients of the form $P_J/U_l \leq \mathrm{U}_{n'}(q)$, as embedded in the larger unitary group $\mathrm{U}_n(q)$. Here P_J is a parabolic subgroup of $\mathrm{U}_{n'}(q)$. We have

$$P_J/U_l \cong P^{+l}_{J(<l)} \times \mathrm{U}_{n'-2l}(q).$$

At this stage we consider the same map $\mathcal{D}$ from section 4.1 on unitary linear modules.

$$\begin{array}{rccc} \mathcal{D}: & P^{+l}_{J(<l)} \times \mathrm{U}_{n'-2l}(q) & \longrightarrow & F^*_{q^2} \\ & (x,y) & \mapsto & \det(x)^{i(1-q)}\det(y), \end{array}$$

for some positive integer i. The map $\mathcal{D}$ is constructed to be the restriction to P_J/U_l of the usual determinant map on $\mathrm{U}_n(q)$. The integer i depends on the embedding of P_J/U_l in $\mathrm{U}_n(q)$.

Observe the image of P_J/U_l under this map is all of $\mathbb{C}_{(q+1)}$ if $n' - 2l \neq 0$. On the other hand if $n' = 2l$ then the image is $\mathbb{C}_{(q+1)/\gcd(q+1,i)}$.

We briefly remark on the restriction of the map $\mathcal{D}$ to the factor $P^{r'}_{J(<r')}$ in the stabilizer $T_{P_J}(\tau)$. For $A \in P^{r'}_{J(<r')}$, $\mathcal{D}(A) = \det(A)^{2i}$ since $-q \equiv 1 \mod (q+1)$.

Remark: A version of the useful cancellation discussed in section 2 applies to the central modules in the following way: Let G be the subgroup of $\mathrm{GL}_l(q^2) \times \mathrm{GL}_l(q^2)$ defined

$$G = \{(A, \widetilde{A}) | A \in \mathrm{GL}_l(q^2)\}.$$

Let V^l be the natural module for $\mathrm{GL}_l(q^2)$. Set $\mathbf{V} = V^l \otimes (V^l)^*$. Let G act on $\mathbf{V}$ via $A \cdot v = Av\widetilde{A}^{-1}$. In keeping with the notation used in section 1, $l = n_1 = n_2$ and G_2 is an isomorphic copy of G_1. As an F_{q^2}-vector space, recall a basis for $\mathbf{V}$ was given by $\{E_{i,j}\}$. View F_{q^2} as an extension of F_q. Let ϑ be a root of the irreducible polynomial

$$x^2 - (\vartheta + \vartheta^q) + \vartheta^{q+1} \text{ in } F_q[x]$$

so that $F_{q^2} = F_q(\vartheta)$. Let V be the F_q-subspace of $\mathbf{V}$ with basis given by

$$\{E_{i,j} - E_{l-j+1,l-i+1}\} \bigcup \{\vartheta E_{i,j} - \vartheta^q E_{l-j+1,l-i+1}\}.$$

The subspace V, and hence $\mathrm{Irr}(V)$, is closed under the action of G. Let $X = \mathrm{Irr}(V, r)$. Then X is a subset of $\mathrm{Irr}(\mathbf{V}, r)$. For $J \subseteq [l-1]$, define the subgroup

$$G_J = \{(A, \widetilde{A}) | A \in P_J^{+l}\} \leq G.$$

Let $H = G \ltimes V$ and $H_J = G_J \ltimes V$. In this case for $(A, \widetilde{A}) \in G_J$,

$$\mathcal{D}(A, \widetilde{A}) = \det(A)^{i(1-q)}$$

for some integer i.

Then $Y = \{y \leq V^l | \text{codim(y) = r}\}$ is a transitive H-set. If J has maximal member l then

$$P_J/U_l \cong G_{J(<l)} \times \mathrm{U}_{n'-2l}(q) \text{ and } Z_l \cong V.$$

If $r = l$ then $X(0) = X$ and $T_{G_J}(0) = G_J$. Now suppose that $r < l$. Set w to be a complement to the r-dimensional subspace R of V^l stabilized by P_r^{+l}. The vector space $V^l = R \oplus w$. Take $\tau \in X(w)$ so that viewed as a linear transformation of V^l, τ has kernel equal to w. Now even if r is not a member of $J(< l)$ we have

$$T_{G_J}(w) = M_{r,l-r}(q^2) \rtimes \left(P_{J(<r)}^{+r} \times P_{J_1}^{+(l-r)}\right)$$

where $J_1 = \{j - r | j \in J(> r)\}$. For $y \in Y$, $y = Aw$ for some $A \in \mathrm{GL}_l(q^2)$, hence $T_{G_J}(y)$ is $\mathrm{GL}_l(q^2)$-conjugate to $T_{G_J}(w)$.

The main difference in applying the useful cancellation to the central module context is in the definition of a G-stable function. We define

$$f : \Delta(P) \times Y \longrightarrow \mathbb{Z}$$

$$(c_J, y) \mapsto k_d(P_{J\cup\{l\}}, X(y), \rho, \mathcal{D}, j).$$

Notice that $P_{J\cup\{l\}}$ is not H_J. Also notice that $l - r \neq 0$ in this case so that

$$|T_{P_J}(y)\ker(\mathcal{D})\backslash P_J| = 1 \text{ holds.}$$

With these changes in the proof of Proposition 6.2.4 the useful cancellation applies to the the central module case.

Proposition 6.5.8 *Let $Z \leq Z(G)$ and $\rho \in Irr(Z)$. Fix $1 \leq r \leq l$. Let $V = Z_l$ and let $X = \mathrm{Irr}(V, r)$. Let w be a complement in V^l to the r-dimensional subspace stabilized by P_r^{+l}, in its action on the natural module for $\mathrm{GL}_l(q^2)$. Let $X(w) = \{\tau \in X \mid \ker(\tau) = w\}$. Let $\mathcal{D}$ be the map on P_J defined above. For $d \geq 0$ the following holds*

1. *If $r = l$, then $Y = 0$ and $X = X(0)$.*

2. If $r < l$, then

$$\sum_{J\subseteq[l-1]} (-1)^{|J|} k_d(P_{J\cup\{l\}}, X, \rho, \mathcal{D}, j) = \sum_{\substack{J\subseteq[l-1] \\ r\in J}} (-1)^{|J|} k_d(P_{J\cup\{l\}}, X(w), \rho, \mathcal{D}, j).$$

Representatives of $T_{G_J}(w)$-orbits in $X(w)$ are given by chains of type $J(< l)\cup\{r\}$ in $\mathcal{P}(V^l)$ where c corresponds to (c_1, c_2) as described earlier in this section.

Having examined the orbits, stabilizers, and maps $\mathcal{D}$ at the level of H_J in the linear cases (both general and unitary) and $P_J^{n'}$ in the central case, we are ready to proceed up to the level of P_J^n. We will do so in the following chapter where we begin a systematic codification of the alternating sum that occurs on the left hand side of Equation 4.6b.

CHAPTER 7

COUNTING CHARACTERS OF PARABOLIC SUBGROUPS NOT TRIVIAL ON THE UNIPOTENT RADICAL

For fixed $J \subset I$ write $J = \{j_1, j_2, \ldots, j_s\}$ in increasing order. Then U_J has the following normal series:

$$U_J = U_{J(\geq j_1)} > U_{J(\geq j_2)} > \cdots > U_{J(\geq j_s)} = U_{j_s} \geq Z(U_{j_s}) > 1 \tag{7.1}$$

The quotient groups in this series are abelian. These are the modules described as general linear, unitary linear, and central modules in the previous chapter where recall we examined their H_J orbits. If $\chi \in \mathrm{Irr}(P_J)$ does not contain U_J in its kernel then there exists a term in the above series which is contained in the kernel of χ, but the previous term is not in the kernel of χ.

Recall $V(j_i, j_{i+1})$ denotes the quotient group $U_{J(\geq j_i)}/U_{J(\geq j_{i+1})}$. We note that in the following definition, *1.* is a slight modification of definition 4.1.2 but that *2.* has already been defined and is only restated for clarity.

Definition 7.0.9 *For $\rho \in \mathrm{Irr}(Z)$:*

1. *Let $k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j)$ be the number of $\chi \in \mathrm{Irr}(P_J)$ which are trivial on $U_{J(\geq j_{i+1})}$ but not trivial on $U_{J(\geq j_i)}$, lie over ρ, and upon restriction to $\ker(\det)$ have j' irreducible constituents where $j \mid j'$.*

2. *Let* $k_d^1(P_J, U_{j_s}, \rho, \det, j)$ *be the number of* $\chi \in \mathrm{Irr}(P_J)$ *which are not trivial on* U_{j_s}*, lie over* ρ*, and upon restriction to* $\ker(\det)$ *have* j' *irreducible constituents where* $j \mid j'$*.*

Case 1. If $\chi \in \mathrm{Irr}(P_J)$ is trivial on $U_{J(\geq j_{i+1})}$ but not trivial on $U_{J(\geq j_i)}$, then we may consider χ as a character of

$$\overline{P_J} = P_J/U_{J(\geq j_{i+1})} \cong P^{+j_{i+1}}_{J(\leq j_i)} \times L_{J'}$$

where $J' = \{j - j_{i+1} | j \in J(> j_{i+1})\}$ and $L_{J'}$ is isomorphic to a Levi subgroup in $\mathrm{U}_{n-2j_{i+1}}(q)$. Then

$$P^{+j_{i+1}}_{J(\leq j_i)} \cong \left(P^{+j_i}_{J(<j_i)} \times \mathrm{GL}_{j_{i+1}-j_i}(q)\right) \ltimes V(j_i, j_{i+1}), \tag{7.2}$$

as in section 3 of chapter 6, on general linear modules, where $P^{+j_{i+1}}_{J(\leq j_i)}$ plays the role of H_J. Hence χ is of the following form:

$$\chi = \chi'(\tilde{\tau}\psi)^{P^{+j_{i+1}}_{J(\leq j_i)}}$$

where $\tau \in \mathrm{Irr}(V(j_i, j_{i+1}))$ is linear and hence extendible to $\tilde{\tau} \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}(T/V(j_i, j_{i+1}))$ where T is the stabilizer of τ in $P^{+j_{i+1}}_{J(\leq j_i)}$ and χ' is an irreducible character of the factor $L_{J'}$. It follows that χ corresponds to τ and that

$$k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \det, j)$$

where this sum is taken over representatives τ of P_J-orbits in $\mathrm{Irr}(V(j_i, j_{i+1}))$.

Case 2. Suppose that $\chi \in \mathrm{Irr}(P_J)$ is not trivial on U_{j_s}. There are two possibilities: χ is trivial on $Z(U_{j_s}) = Z_{j_s}$ or not.

Case 2a. If χ is trivial on Z_{j_s}, then we may consider χ as a character of

$$P_J/Z_{j_s} \cong \left(P_{J(<j_s)}^{+j_s} \times \mathrm{U}_{n-2j_s}(q)\right) \ltimes (U_{j_s}/Z_{j_s}), \tag{7.3}$$

as in section 4 of chapter 6, on unitary linear modules, where P_J/Z_{j_s} plays the role of H_J. Hence χ is of the following form:

$$\chi = (\tilde{\tau}\psi)^{P_J/Z_{j_s}}$$

where $\tau \in \mathrm{Irr}(U_{j_s}/Z_{j_s})$ linear and hence extendible to $\tilde{\tau} \in \mathrm{Irr}(T)$ and $\psi \in \mathrm{Irr}(T/(U_{j_s}/Z_{j_s}))$ where T is the stabilizer of τ in P_J/Z_{j_s}. It follows that χ corresponds to an irreducible character of the quotient U_{j_s}/Z_{j_s}, and that

$$k_d^1(P_J, U_{j_s}/Z_{j_s}, \rho, \det, j) = \sum_{\tau} k_d(P_J, \tau, \rho, \det, j)$$

where this sum is taken over nontrivial representatives τ of P_J-orbits in $\mathrm{Irr}(U_{j_s}/Z_{j_s})$.

Case 2b. Suppose χ is not trivial on Z_{j_s}. As discussed in section 5 of chapter 6, χ corresponds to $\phi \in \mathrm{Irr}(U_{j_s})$ where ϕ lies over a non-trivial character $\tau_r \in \mathrm{Irr}(Z_{j_s})$. Set $N = \ker(\phi)$ then $K = N \cap Z_{j_s}$ is non-trivial. Then U_{j_s}/N is an extra special p-group and hence the non-linear $\phi \in \mathrm{Irr}(U_{j_s}/N)$ is extendible to $\tilde{\phi} \in \mathrm{Irr}(T)$ where T is the stabilizer of ϕ in P_J/N. Thus χ is of the form:

$$\chi = (\tilde{\phi}\psi)^{P_J}$$

where $\phi \in \mathrm{Irr}(U_{j_s}/N)$ is the unique character whose restriction to Z_{j_s}/K is a multiple of non-trivial $\tau \in \mathrm{Irr}(Z_{j_s}/K)$. The character ψ is the lift to T of an irreducible character of $T/(U_{j_s}/N)$. The character ϕ lifts to an irreducible character in $\mathrm{Irr}(U_{j_s})$ and τ lifts to an irreducible character in $\mathrm{Irr}(Z_{j_s})$. It follows that χ corresponds to an irreducible character of Z_{j_s} and that

$$k_d^1(P_J, Z_{j_s}, \rho, \det, j) = \sum_{\tau} \sum_{\substack{\phi \\ \phi \in \mathrm{Irr}(U_{j_s}, \tau)}} k_d(P_J, \phi, \rho, \det, j)$$

where this sum is taken over nontrivial representatives τ of P_J orbits in $\mathrm{Irr}(Z_{j_s})$.

We have the following decomposition:

$$k_d^1(P_J, U_J, \rho, \det, j) = \sum_{i=1}^{s-1} k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j) + k_d^1(P_J, U_{j_s}, \rho, \det, j). \tag{7.4}$$

For fixed J containing adjacent elements l and l', let $V = V(l, l') = U_{J(\geq l)}/U_{J(\geq l')} \cong V_1 \otimes V_2$, where we recall that V_1 is the natural module for $\mathrm{GL}_l(q^2)$ and V_2 is the dual of the natural module for $\mathrm{GL}_{l'-l}(q^2)$. Then

$$\overline{P}_J = P_J/U_{J(\geq l')} \text{ contains a submodule isomorphic to } V \text{ (see Equation 7.2).}$$

The useful cancellation from chapter 6 applies in this situation. We will sum over all $J \subseteq I$ of the form $J = J' \cup J''$ where $J' \subseteq [l-1]$ varies and $J'' \subset I$ is fixed with minimal member l'. We will only be concerned with calculating $T_{H_J}(\tau)$ where τ has rank $r \in J$ and $\ker(\tau)$ is a complement w in V_1 to the r-dimensional subspace stabilized by P_r^{+l}. Then, for $r \in J$ and $\tau \in \mathrm{Irr}(V, r)$ with $\ker(\tau) = w$

$$k_d(P_J, \tau, \rho, \det, j) = k_d(\overline{P}_J/V, \tau, \rho, \det, j) = k_{d-d'}(T_{\overline{P}_J/V}(\tau), \rho, \det, j') \tag{7.5}$$

where d' is the power of q in the index of $T_{\overline{P}_J/V}(\tau)$ in $\overline{P}_J/V$and j' is the smallest positive integer such that

$$j \mid j' \cdot \left| T_{\overline{P}_J/V}(\tau) \cdot \ker(\det) \backslash (\overline{P}_J/V) \right|.$$

It turns out that $T_{\overline{P}_J/V}(\tau)$ contains a subgroup which itself contains a submodule isomorphic to a general linear module and hence we may further expand Equation 7.5.

Let J have maximal element l. Let $V = U_l/Z_l \cong V_1 \otimes V_2$, where recall V_1 is the natural module for $\mathrm{GL}_l(q^2)$ as above and V_2 is the dual of the natural module for $\mathrm{U}_{n-2l}(q)$. Then

$$\overline{P}_J = P_J/Z_l \text{ contains a submodule isomorphic to } V \text{ (see Equation 7.3).}$$

As above, the useful cancellation from chapter 6 applies to this situation. We will when sum over all $J \subseteq I$ with maximal element l. We will only be concerned with calculating $T_{H_J}(\tau)$

where τ has rank $r \in J$ and $\ker(\tau)$ is a complement w in V_1 to the r-dimensional subspace stabilized by P_r^{+l}. Then, for $r \in J$ and $\tau \in \mathrm{Irr}(V, r)$ with $\ker(\tau) = w$,

$$k_d(P_J, \tau, \rho, \det, j) = k_d(\overline{P}_J/V, \tau, \rho, \det, j) = k_{d-d'}(T_{\overline{P}_J/V}(\tau), \rho, \det, j') \tag{7.6}$$

where d' and j' are as given above in Equation 7.5. It turns out that when τ corresponds to a singular chain in the unitary vector space V_2^*, $T_{\overline{P}_J/V}(\tau)$ contains a subgroup which itself contains submodule isomorphic to a (unitary) quotient module and hence we may further expand Equation 7.6.

The main goal in this chapter is to unravel Equation 7.4 via Equation 7.5 and Equation 7.6. Ku has introduced two sets of triples E and F with related objects which codify this unraveling for the unitary case but without regard to the determinant map or any splitting. For an $e \in E$ or an $f \in F$ we will present Ku's objects including a length, a parity, a group, and a normal subgroup. For our purposes we will also define a map related to the determinant and an integer related to splitting. We remark that while the definitions of these two new objects are very natural extensions of Ku's existing objects, the computations involved in producing their definitions are far from trivial. The set E will codify parabolic characters corresponding to internal general linear modules, i.e. those counted in the alternating sum

$$\sum_{J \subseteq I} (-1)^{|J|} \sum_{i=1}^{s-1} k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j),$$

whereas the set F will codify the parabolic characters corresponding to internal unitary linear or central modules, i.e. those characters counted in the alternating sum

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_{j_s}, \rho, \det, j).$$

Before proceeding to the rather technical definitions, we present the idea. Fix nonempty $J \subseteq I$.

- For each pair of adjacent members l and l' of J, we will define an initial triple e so that its associated group

  $$P(e) \cong P_J/U_{l'} \cong \left(\left(P^{+l}_{J(<l)} \times \mathrm{GL}_{l'-l}(q^2)\right) \ltimes V(l,l')\right) \times L_{J'}$$

  where $J' = \{j - l' | j \in J(> l')\}$ and $L_{J'}$ is a Levi subgroup in $\mathrm{U}_{n-2l'}(q)$. Moreover, for $\tau \in \mathrm{Irr}(V(l,l'))$ of rank r where $r \in J$ and $\ker(\tau)$ is a complement w to the r-dimensional subspace stabilized by P^{+l}_r, we will define a subsequent triple e' related to e such that

  $$T_{P(e)}(\tau) = P(e') \ltimes V(l,l').$$

  If $P(e')$ itself contains a non trivial general linear module we will define a further triple e'' related to e' in a similar way. In this way J, l, and l' give rise to a sequence e, e', e'',... of elements in E.

- Now let l be the maximal member of J. We will define an initial triple f so that its associated group

$$P(f) \cong P_J \cong \left(P^{+l}_{J(<l)} \times \mathrm{U}_{n-2l}(q)\right) \ltimes U_l$$

Moreover, for $\tau \in \mathrm{Irr}(U_l/Z_l)$ of rank r where $r \in J$, $\ker(\tau)$ is a complement w to the r-dimensional subspace stabilized by P^{+l}_r, and τ corresponds to a singular flag in a unitary space of suitable dimention, we will define a subsequent triple f' related to f such that

$$T_{P(f)}(\tau) = P(f') \ltimes U_l.$$

If $P(f')$ itself contains a non trivial unitary linear module with an irreducible singular character (i.e. corresponds to a singular flag in a unitary space of suitable dimension) we will define a further triple f'' related to f' in a similar way. In this way J and l give rise to a sequence f, f', f'',... of elements in F.

In this fashion, we will use the elements of E and F to reformulate Equation 4.6b in a systematic way by unraveling Equation 7.5 and Equation 7.6 which leads to a second reduction of Equation 4.6b.

The notation used in the following sections is primarily due to Ku. In particular he defines the following: E, $P(e)$, $V(e)$, $l(e)$, $|e|$, and $d(e)$ used in section 1; F, $P(f)$, $U(f)$, $Z(f)$, $V(f)$, $l(f)$, $|f|$, and $d(f)$ used in section 2; $S^u(f)$, $S^{su}(f)$, $S^{nu}(f)$, $S^z(f)$ used in section 3.

1 The elements of E and their related objects

Let e be an ordered triple $(J, C, (l, l'))$, where either $e = (\emptyset, \emptyset, \emptyset)$ or J, C, and (l, l') satisfy the following conditions:

1. $J = \{j_1, j_2, \ldots, j_r\}$ is a subset of $I = [m]$. We will assume that J is enumerated in increasing order.
2. (l, l') is a pair of consecutive members of J.
3. $C = \{l_1, l_2, \ldots, l_s\}$ is a subset of $I = [m]$ also enumerated in increasing order. The sequence C must be convex. By this we mean that the related sequence $\partial C = (l_1, l_2 - l_1, \ldots, l_s - l_{s-1})$ is non-increasing. We require that the members of this related sequence appear in J. We further require that C and (l, l') are related via $l = l_1 < l_2 < \ldots < l_s \leq l'$.

Let E be the set of all such triples e. The length of e is $l(e) = |C| = s$ and the parity of e is $|e| = |J|$. Before we proceed to the rest of the objects related to $e \in E$, we make some observations.

Definition 7.1.1 *Given $C = \{l_1, l_2, \ldots, l_s\}$, we define $\partial l_1 = l_1$ and $\partial l_i = l_i - l_{i-1}$ for $2 \leq i \leq s$ so that ∂C is the partition $(\partial l_1, \partial l_2, \ldots, \partial l_s)$.*

Observe that a convex sequence C corresponds to a unique partition $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s) \vdash l_s$. Notice that the ∂l_i's, where $l = \partial l_1 = l_1$, and l' are contained in J. Moreover, l and l' are consecutive members of J. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l < l'$ divides J into a collection of $s+1$ disjoint subsets

$$\{j \in J | j < \partial l_s\}, \quad \cdots, \quad \{j \in J | \partial l_2 < j < \partial l_1\}, \quad \{j \in J | l' < j\}.$$

Definition 7.1.2 *We define*

1. $J_s = \{j \mid j \in J \text{ and } j < \partial l_s\}$,
2. $J_i = \{j - \partial l_{i+1} \mid j \in J \text{ and } \partial l_{i+1} < j < \partial l_i\}$ *for* $1 \leq i < s$, *and*
3. $J_0 = \{j - l' \mid j \in J \text{ and } l' < j\}$.

Notice that we have the containment $J_s \subset [\partial l_s - 1]$, $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$, and $J_0 \subset [m - l']$. We are now ready to define the groups related to the triple $e = (J, C, t)$.

Definition 7.1.3 *For* $e \in E$ *we define the following objects.*

1. *Let*

$$P(e) = L_{J_0(e)}^{n_0(e)} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times P_{J_{s-1}(e)}^{+n_{s-1}(e)} \times P_{J_s(e)}^{+n_s(e)}$$

where

(a) $n_0(e) = n - 2l'$, $J_0(e) = J_0$

(b) $n_i(e) = \partial l_i - \partial l_{i+1}$ *and* $J_i(e) = J_i$ *for* $1 \leq i < s$

(c)

$$n_s(e) = l' - l_{s-1} \text{ and } J_s(e) = \begin{cases} J_s, & \text{if } l_s = l'; \\ J_s \cup \{\partial l_s\}, & \text{if } l_s < l'. \end{cases}$$

Notice that the first factor is a Levi subgroup of a unitary group whereas the remaining s factors are parabolic subgroups of general linear groups.

2. *We define an abelian normal subgroup of $P(e)$ in the following way. If $l_s < l'$ then the group $P^{+n_s(e)}_{J_s(e)}$ is contained in the maximal parabolic $P^{+n_s(e)}_{\partial l_s}$ which has unipotent radical $U^{+n_s(e)}_{\partial l_s}$, a normal subgroup of $P^{+n_s(e)}_{J_s(e)}$ and hence of $P(e)$. Set*

$$V(e) = \begin{cases} U^{+n_s(e)}_{\partial l_s}, & \textit{if } l_s < l'; \\ 1, & \textit{if } l_s = l'. \end{cases}$$

Observe that

$$P^{+n_s(e)}_{J_s(e)} \cong \left(P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l'-l_s}(q^2)\right) \ltimes V(e). \tag{7.7}$$

Moreover $V(e)$ is a general linear module for $P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l'-l_s}(q^2)$ as discussed in Chapter 6 Section 3.

3. *We define*

$$d(e) = 2\sum_{i=1}^{s-1}\left(\binom{\partial l_i}{2} - \binom{\partial l_i - l_{i+1}}{2}\right).$$

Notice that $d(e)$ depends only on the sequence C.

4. *Finally we define a map* $\phi_e : P(e) \to F_{q^2}$. *For* $g \in P(e)$, *write* $g = A_0 A_1 \cdots A_{s-1} A_s$ *where* $A_0 \in L_{J_0(e)}^{n_0(e)}$ *and* $A_i \in P_{J_i(e)}^{+n_i(e)}$. *We have a decomposition of* $A_s \in P_{J_s(e)}^{+n_s(e)}$. *Write* $A_s \equiv A_{s,1} A_{s,2} \mathrm{mod}(V(e))$ *where* $A_{s,1} \in P_{J_s}^{+\partial l_s}$ *and* $A_{s,2} \in \mathrm{GL}_{l'-l_s}(q^2)$. *Let*

$$\phi_e(g) = \det A_0 \left[\left(\prod_{i=1}^{s-1} (\det A_i)^i \right) (\det A_{s,1})^s \det A_{s,2} \right]^{1-q} \tag{7.8}$$

where det *denotes the usual determinant map. Notice* $\phi_e(P(e)) \le \mathbb{C}_{q+1}$.

In the context of general linear modules, the map $\mathcal{D}$ *defined Chapter 6 Section 3 on* $P_{J_s(e)}^{+n_s(e)}$ *corresponds to* $i = s$ *and is the restriction of* ϕ_e *to* $P_{J_s(e)}^{+n_s(e)}$.

An example of E: Fix J with adjacent members l and l'. Set the initial $e = (J, \{l\}, (l, l'))$. Then $\partial C = (l)$

$$\begin{aligned} P(e) \cong P_J/U_{J(<l')} &\cong L_{J_0}^{n-2l'} \times P_{J(\le l)}^{+l'} \\ &\cong L_{J_0}^{n-2l'} \times \left(P_{J(<l)}^{+l} \times \mathrm{GL}_{l'-l}(q^2) \right) \ltimes V(e) \end{aligned}$$

and $V(e) \cong V(l, l') \cong M_{l,l'-l}(q^2)$. Under this isomorphism the map ϕ_e is the determinant map on $P_J/U_{J(<l')}$.

For $\tau \in \mathrm{Irr}(V(l,l'))$ of rank r where $r \in J$ and $\ker(\tau) = w$, where w is a complement to the r-dimensional subspace stabilized by P_r^{+l}, there is a subsequent $e' = (J, \{l, l+r\}, (l,l'))$ with $\partial C' = (l,r)$. Then by definition

$$\begin{aligned} P(e') \cong & L_{J_0}^{n-2l'} \times P_{J_1(e')}^{+(l-r)} \times P_{J(\le r)}^{+(l'-l)} \\ \cong & L_{J_0}^{n-2l'} \times P_{J_1(e')}^{+(l-r)} \times \left(P_{J(<r)}^{+r} \times \mathrm{GL}_{l'-(l+r)}(q^2) \right) \ltimes V(e') \end{aligned}$$

We have

$$T_{P(e)}(\tau) \cong P(e') \ltimes V(e).$$

As a block subgroup embedded in $P(e)/V(e)$, the $P_{J(<r)}^{+r}$ term in the last factor $P_{J(\le r)}^{+(l'-l)}$ of $P(e')$ occurs twice. Thus the map $\phi_{e'}$ is the determinant on $P(e')$ as a subgroup in P_J.

If $l+r < l'$, then $P_{J(\le r)}^{+(l'-l)}$ contains a nontrivial submodule isomorphic to $V(e') \cong M_{r,l'-(l+r)}(q^2)$. Suppose $J(\le r)$ is nonempty. Then there exists $s \in J(\le r)$ and $\tau' \in \mathrm{Irr}(V(e'))$ of rank s and $\ker(\tau')$ is a complement w' to the s-dimensional subspace stabilized by P_s^{+r}. There is a further triple $e'' = (J, \{l, l+r, l+r+s\}, (l,l'))$ with $\partial C'' = (l,r,s)$. By definition

$$\begin{aligned} P(e'') \cong & L_{J_0}^{n-2l'} \times P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times P_{J(\le s)}^{+(l'-(l+r))} \\ \cong & L_{J_0}^{n-2l'} \times P_{J_1}^{+(l-r)} \times P_{J_2}^{+(r-s)} \times \left(P_{J(<s)}^{+s} \times \mathrm{GL}_{l'-(l+r+s)}(q^2) \right) \ltimes V(e''). \end{aligned}$$

We have

$$T_{P(e')}(\tau') \cong P(e'') \ltimes V(e').$$

As a block subgroup embedded in $P(e)/V(e)$, the $P^{+s}_{J(<s)}$ term in the last factor $P^{+(l'-l)}_{J(\leq r)}$ of $P(e'')$ occurs three times. Thus the map $\phi_{e''}$ is the determinant on $P(e'')$ as a subgroup in P_J.

Remark: The collection of all $e \in E$ with fixed J and (l, l') unravels the alternating sum involving characters $\chi \in \mathrm{Irr}(P_J)$ that correspond to characters in $\mathrm{Irr}(V(l, l'))$. Informally, we describe the method as taking stabilizers in stabilizers in stabilizers, etc. At each step we mod out the involved interior (general linear) module until the only interior (general linear) module is trivial. This occurs when the last element of C is l'. Of course this description misses all the important details. The length of C keeps track of how many times we iterate this process. The partition ∂C keeps track of the rank of the characters for which we calculate stabilizers.

2 The elements of F and their related objects

Let f be an ordered triple (J, C, l) where either $f = (\emptyset, \emptyset, 0)$ or J, C, and l satisfy the following conditions:

1. The sequence $J = \{j_1, j_2, \ldots, j_r\}$ is a subset of $I = [m]$. We will assume that J is enumerated in increasing order.
2. The sequence $C = \{l_1, l_2, \ldots, l_s\}$ is a subset of $I = [m]$ also enumerated in increasing order. The sequence C must satisfy the same conditions as listed in the previous section. So C must be convex and the members of the related sequence ∂C appear in J.
3. The integer $l = j_r$ and $l = l_1$, so that l is the maximal member of J and the minimal member of C.

Notice that J and C are defined as they were for the triples $e \in E$. Let F be the set of all such f. The length of f is $l(f) = |C| = s$ and the parity of f is $|f| = |J|$ just as we did for $e \in E$.

Write $\partial C = (\partial l_1, \partial l_2, \ldots, \partial l_s)$ as in definition 7.1.1. Notice that the ∂l_i's are contained in J with $l = \partial l_1 = l_1$. Moreover l is the largest member of J. Thus the sequence $\partial l_s \leq \partial l_{s-1} \leq \cdots \leq \partial l_2 \leq \partial l_1 = l$ divides J into a collection of s disjoint subsets

$$\{j \in J | j < \partial l_s\}, \quad \cdots, \quad \{j \in J | \partial l_2 < j < \partial l_1 = l\}.$$

Definition 7.2.1 *We define*

1. $J_s = \{j \mid j \in J$ *and* $j < \partial l_s\}$, *and*
2. $J_i = \{j - \partial l_{i+1} \mid j \in J$ *and* $\partial l_{i+1} < j < \partial l_i\}$ *for* $1 \leq i < s$.

Notice that we have the containment $J_s \subset [\partial l_s - 1]$ and $J_i \subset [\partial l_i - \partial l_{i+1} - 1]$ for $1 \leq i < s$. We are now ready to define the groups related to the triple $f = (J, C, l)$.

Definition 7.2.2 *For $f \in F$ we define the following objects.*

1. *Let*

$$P(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}$$

   where

   (a) $n_i(f) = \partial l_i - \partial l_{i+1}$ *and* $J_i(f) = J_i$ *for* $1 \leq i < s$

(b) $n_s(f) = n - 2l_{s-1}$ *and* $J_s(f) = J_s \cup \{\partial l_s\} = J(\leq \partial l_s)$

Notice that the last factor is a parabolic subgroup of a unitary group, whereas the first $s-1$ *factors are parabolic subgroups of general linear groups.*

2. *We define a normal subgroup of* $P(f)$. *The group* $P^{n_s(f)}_{J_s(f)}$ *is contained in the maximal parabolic* $P^{n_s(f)}_{\partial l_s}$ *which has unipotent radical* $U^{n_s(f)}_{\partial l_s}$, *a normal subgroup of* $P^{n_s(f)}_{J_s(f)}$ *and hence of* $P(f)$. *We set* $U(f) = U^{n_s(f)}_{\partial l_s}$. *Let* $Z(f) = Z(U(f))$ *and the quotient* $V(f) = U(f)/Z(f)$. *Observe that*

$$P^{n_s(f)}_{J_s(f)} \cong \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)\right) \ltimes U(f) \tag{7.9}$$

Moreover $V(f)$ *is a unitary linear module for* $P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)$ *as discussed in Chapter 6 Section 4 and* $Z(f)$ *is a central module for* $P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l_s}(q)$ *as discussed in Chapter 6 Section 5.*

3. *We define*

$$d(f) = 2\sum_{i=1}^{s-1}\left(\binom{\partial l_i}{2} - \binom{\partial l_i - l_{i+1}}{2}\right).$$

Notice that $d(f)$ *depends only on the sequence* C.

4. Finally we define a map $\phi_f : P(f) \to F_{q^2}$*. For* $g \in P(f)$*, write* $g = A_1 \cdots A_{s-1} A_s$ *where* $A_i \in P^{+n_i(f)}_{J_i(f)}$ *for* $1 \leq i < s$ *and* $A_s \in P^{n_s(f)}_{J_s(f)}$*. We have a decomposition of* A_s*. Write* $A_s \equiv A_{s,1} A_{s,2} \text{mod}(U(f))$ *where* $A_{s,1} \in P^{+\partial l_s}_{J_s}$ *and* $A_{s,2} \in \mathrm{U}_{n-2l}(q)$*. Let*

$$\phi_f(g) = \left[\left(\prod_{i=1}^{s-1} (\det A_i)^i \right) (\det A_{s,1})^s \right]^{1-q} \det A_{s,2} \tag{7.10}$$

where det *denotes the usual determinant map. Notice* $\phi_f(P(f)) \leq \mathbb{C}_{q+1}$*.*

In the context of unitary modules (both linear and central) the map $\mathcal{D}$ *defined in Chapter 6 Sections 4 and 5 on* $P^{n_s(f)}_{J_s(f)}$ *corresponds to* $i = s$ *and is the restriction of* ϕ_f *to* $P^{n_s(f)}_{J_s(f)}$*.*

An example of F: Fix J with maximal member l. Set the initial $f = (J, \{l\}, l)$. Then $\partial C = (l)$ and

$$\begin{aligned} P(f) &\cong P_J \cong (P_J/U_l) \ltimes U_l \\ &\cong \left(P^{+l}_{J(<l)} \times \mathrm{U}_{n-2l}(q) \right) \ltimes U(f) \end{aligned}$$

where $U(f) \cong U_l$, $V(f) \cong U_l/Z_l$, $Z(f) \cong Z_l$, and under this isomorphism ϕ_f is the determinant map on P_J.

If $n \neq 2l$ then U_l/Z_l is not trivial. For $\tau \in \mathrm{Irr}(U_l/Z_l) \cong \mathrm{Irr}(V(f))$ of rank r and $\ker(\tau) = w$, where w is a complement to the r-dimensional space stabilized by P^{+l}_r, corresponding to a singular chain of type $J(< r)$ where $r \in J$, there is a subsequent $f' = (J, \{l, l+r\}, l)$ with $\partial C' = (l, r)$. Then by definition

$$\begin{aligned} P(f') \cong & P^{+(l-r)}_{J_1(f')} \times P^{n-2l}_{J(\leq r)} \\ \cong & P^{+(l-r)}_{J_1(f')} \times \left(P^{+r}_{J(<r)} \times \mathrm{U}_{n-2(l+r)}(q)\right) \ltimes U(f'). \end{aligned}$$

We have

$$T_{P(f)}(\tau) \cong P(f') \ltimes U(f).$$

As a block subgroup embedded in $P(f)/U(f)$, the $P^{+r}_{J(<r)}$ term in the last factor $P^{n-2l}_{J(\leq r)}$ of $P(f')$ occurs twice. Thus the map $\phi_{f'}$ is the determinant on $P(f')$ as a subgroup in P_J.

If $2(l+r) < n$, then $P^{n-2l}_{J(\leq r)}$ contains a submodule isomorphic to $U(f')$. If $V(f')$ is not trivial and $\mathrm{Irr}(V(f'))$ contains a character τ' with $\ker(\tau') = w'$, where w' is a complement to the s-dimensional space stabilized by P^{+r}_s, and τ' corresponds to a singular chain of rank s and type $J(< s)$ then if $s \in J$ we have $f'' = (J, \{l, l+r, l+r+s\}, l)$ with $\partial C'' = (l, r, s)$. Checking with the definition for f'' we have

$$\begin{aligned} P(f'') \cong & P^{+(l-r)}_{J_1} \times P^{+(r-s)}_{J_2} \times P^{n-2(l+r)}_{J(\leq s)} \\ \cong & P^{+(l-r)}_{J_1} \times P^{+(r-s)}_{J_2} \times \left(P^{+s}_{J(<s)} \times \mathrm{U}_{n-2(l+r+s)}(q)\right) \ltimes U(f'') \end{aligned}$$

We have

$$T_{P(f')}(\tau') \cong P(f'') \ltimes U(f').$$

As a block subgroup embedded in $P(f)/V(f)$, the $P_{J(<s)}^{+s}$ term in the last factor $P_{J(\leq s)}^{n-2(l+r)}$ of $P(f'')$ occurs three times. Thus the map $\phi_{f''}$ is the determinant on $P(f'')$ as a subgroup in P_J.

Remark: The collection of all $f \in F$ with fixed J with maximal element l unravels the alternating sum involving characters $\chi \in \mathrm{Irr}(P_J)$ that correspond to characters in $\mathrm{Irr}(U_l)$. Informally, we describe the method as taking stabilizers of singular flags in stabilizers of singular flags in stabilizers of singular flags, etc. At each step we mod out the involved interior (unitary linear) module until the remaining interior (unitary linear) module is trivial. This occurs when the last element of C is n. Of course this description misses all the important details. The length of C keeps track of how many times we iterate this process. The partition ∂C keeps track of the rank of the characters corresponding to singular flags for which we calculate stabilizers.

3 Results concerning members of E and F

There are a number of results that we need regarding members of E and F. These results lead to the unraveling of our alternating sum which in turn leads to some very nice cancellation. These are primarily modifications of Ku's results ((15), Chapter 8); however we have the added parameters det and j. To that end in this section we will define integers j_e and j_f which codify the splitting of characters upon restriction to the kernel of the determinant map. Regarding convex chains C and C', if $C = C' \backslash \{\max C'\}$ we say that C' covers C and write $C \prec C'$.

We begin with members of the set E. Let C be a sequence with $|C| = s$ and $l_s < l'$. Fix $e = (J, C, (l, l'))$ and $e' = (J, C \cup \{l_s + r\}, (l, l'))$ in E, where $1 \leq r \leq \min\{\partial l_s, l' - l_s\}$, $r \in J$, and $l_s + r \leq l'$. Then $C \prec C \cup \{l_s + r\}$. Notice $l(e) = s$ and $l(e') = s + 1$. Take $\tau \in \mathrm{Irr}(V(e), r)$ where $\ker(\tau)$ is a complement w to the r-dimensional space stabilized by $P_r^{+\partial l_s}$. Then

$$T_{P(e)}(\tau) = L_{J_0(e)}^{n_0(e)} \times P_{J_1(e)}^{+n_1(e)} \times \cdots \times P_{J_{s-1}(e)}^{+n_{s-1}(e)} \times T_{P_{J_s(e)}^{+n_s(e)}}(\tau)$$

and $T_{P_{J_s(e)}^{+n_s(e)}}(\tau) \cong \left(P_{J_s(e')}^{+n_s(e')} \times P_{J_{s+1}(e')}^{+n_{s+1}(e')}\right) \ltimes V(e)$. Thus

$$T_{P(e)}(\tau) \cong P(e') \ltimes V(e)$$

and ϕ_e restricted to $T_{P(e)}(\tau)$ is $\phi_{e'}$. Hence

$$k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e')}(P(e'), \rho, \phi_{e'}, j') \tag{7.11}$$

where j' the least positive integer such that j divides $j' \cdot |T_{P(e)}(\tau) \ker(\phi_e) \backslash P(e)|$. Write $T = T_{P(e)}(\tau)$, $K = \ker(\phi_e)$. Then $K \trianglelefteq P(e)$ so

$$TK \,/K \cong T \,/T \cap K \cong T/V(e) \Big/ (T \cap K)/V(e) \cong P(e') \Big/ \ker(\phi_{e'})$$

Moreover $P(e)/K \cong \phi_e(P(e))$ and $P(e')/\ker(\phi_{e'}) \cong \phi_{e'}(P(e'))$ and hence

$$|TK \backslash P(e)| = \left| P(e) \,/TK \right| = \left| P(e)/K \Big/ TK/K \right| = \left| \phi_e(P(e)) \Big/ \phi_{e'}(P(e')) \right|$$

Since $l_s < l'$, i.e. $l' - l_s \neq 0$ the image of $P(e)$ under ϕ_e is $\mathcal{C}_{q+1}$. Thus we have

$$|TK\backslash P(e)| = \frac{|\phi_e(P(e))|}{|\phi_{e'}(P(e'))|} = \frac{q+1}{|\phi_{e'}(P(e'))|}.$$

Definition 7.3.1 *Let j_e be the smallest positive integer such that j divides $j_e \cdot \dfrac{q+1}{|\phi_e(P(e))|}$.*

Remark: Observe that for e of length 1, $j_e = j$ certainly holds. In general, for fixed $e \in E$, $|\phi_e(P(e))|$ is almost always equal to $q+1$ and hence $j_e = j$. The image

$$\phi_e(P(e)) = \prod_{i=0}^{s} (\mathbb{C}_{h_i})$$

where $\mathbb{C}_{h_i}$ is the image of the i-the factor and so depends on the $n_i(e)$. If any of $n_0(e)$, $n_1(e)$, or $l' - l_s$ are nonzero, then $|\phi_e(P(e))| = q+1$. If all three are zero then C cannot be covered and

1. $n = 2m$, $l' = m$, and $l_s = m$
2. $C = \{l, 2l, \ldots, m\}$ so that ∂C begins $(l, l, \ldots)$.

On the other hand if C can be covered, and $e = (J, C, (l, l'))$ and $e' = (J, C \cup \{l_s + r\}, (l, l'))$ are members of E as in Equation 7.11, then

$$k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j) = k_{d-d(e)}(P(e), \tau, \rho, \phi_e, j_e) = k_{d-d(e')}(P(e'), \rho, \phi_{e'}, j_{e'}).$$

Finally, for fixed e with $l(e) \geq 2$

$$k_{d-d(e)}(P(e), \rho, \phi_e, j_e) = k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e)$$

where $k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) = 0$ if $V(e) = 1$ i.e. $l_s = l'$.

By the first reduction in Proposition 6.2.4, when summing over all e of length 1 and $J \subseteq I$ of the form $J = J' \cup J''$ where $J' \subseteq [l-1]$ varies and $J'' \subset I$ is fixed with minimal member l', we need only sum over $\tau \in \mathrm{Irr}(V(e), r)$ with $r \in J'$ and $\ker(\tau) = w$, where w is a complement to the space stabilized by P_r^{+l}. By repeated application of this reasoning it follows that

Proposition 7.3.2

$$\sum_{\substack{e \in E \\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) = \sum_{\substack{e \in E \\ l(e) \geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e).$$

Now we turn to results concerning members of F. Let C be a sequence with $|C| = s$ and $l_s < m$. Fix $f = (J, C, l)$ and $f' = (J, C \cup \{l_s + r\}, l)$ in F, where $1 \leq r \leq \min\{\partial l_s, n - 2l_s\}$, $r \in J$, and $l_s + r \leq m$. Then $C \prec C \cup \{l_s + r\}$. Notice $l(f) = s$ and $l(f') = s + 1$. Take $\tau \in \mathrm{Irr}(V(f))$ corresponding to a singular chain of rank r. Then

$$T_{P(f)}(\tau) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times T_{P^{n_s(f)}_{J_s(f)}}(\tau)$$

and $T_{P^{n_s(f)}_{J_s(f)}}(\tau) \cong \left(P^{+n_s(f')}_{J_s(f')} \times P^{n_{s+1}(f')}_{J_{s+1}(f')}\right) \ltimes U(f)$. Thus

$$T_{P(f)}(\tau) \cong P(f') \ltimes U(f)$$

and ϕ_f restricted to $T_{P(f)}(\tau)$ is $\phi_{f'}$. Hence

$$k_{d-d(f)}(P(f), \tau, \rho, \phi_f, j) = k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j') \tag{7.12}$$

where j' the least positive integer such that j divides $j' \cdot |T_{P(f)}(\tau)\ker(\phi_f)\backslash P(f)|$. Write $T = T_{P(f)}(\tau)$, $K = \ker(\phi_f)$. Then $K \trianglelefteq P(f)$ so

$$TK \big/ K \cong T \big/ T \cap K \cong T/U(f) \Big/ (T \cap K)/U(f) \cong P(f') \Big/ \ker(\phi_{f'})$$

Moreover $P(f)/K \cong \phi_f(P(f))$ and $P(f')/\ker(\phi_{f'}) \cong \phi_{f'}(P(f'))$ and hence

$$|TK\backslash P(f)| = \left|P(f) \big/ TK\right| = \left|P(f)/K \Big/ TK/K\right| = \left|\phi_f(P(f)) \Big/ \phi_{f'}(P(f'))\right|$$

Since $l_s < m$ we have nonzero $n - 2l_s$, thus the image of $P(f)$ under ϕ_f is $\mathcal{C}_{q+1}$ so we have

$$|T_{P(f)}(\tau)\ker(\phi_f)\backslash P(f)| = \frac{|\phi_f(P(f))|}{|\phi_{f'}(P(f'))|} = \frac{q+1}{|\phi_{f'}(P(f'))|}.$$

Definition 7.3.3 *Let j_f be the smallest positive integer such that j divides $j_f \cdot \frac{q+1}{|\phi_f(P(f))|}$.*

Remark: Observe that for f of length 1, $j_e = j$ certainly holds. In general, for fixed $f \in F$, $|\phi_f(P(f))|$ is almost always equal to $q+1$ and hence $j_f = j$. The image

$$\phi_f(P(f)) = \prod_{i=1}^{s} (\mathbb{C}_{h_i})$$

where $\mathbb{C}_{h_i}$ is the image of the i-the factor and so depends on $n_i(f)$. For $1 \leq i < s$, if $n_i(f) = 0$ then $h_i = 1$ whereas if $n_i(f) \neq 0$, $h_i = q + 1/\gcd(q+1, i)$. In the s-th factor, if $n - 2l_s = 0$ then $h_s = q + 1/\gcd(q+1, s)$ whereas if $n - 2l_s \neq 0$ then $h_s = q + 1$.

$$|\phi_f(P(f))| = \mathbb{C}_L \text{ where } L = \operatorname{lcm}(h_i)$$

If either of $n_1(f)$ or $n - 2l_s$ are nonzero, then $|\phi_f(P(f))| = q + 1$. If both are zero then C cannot be covered and

1. $n = 2m$ and $l_s = m$
2. $C = \{l, 2l, \ldots, m\}$ so that ∂C begins $(l, l, \ldots)$.

On the other hand, if C can be covered and $f = (J, C, l)$ and $f' = (J, C \cup \{l_s + r\}, l)$ are in F as in Equation 7.12 then

$$k_{d-d(f)}(P(f), \tau, \rho, \phi_f, j) = k_{d-d(f)}(P(f), \tau, \rho, \phi_f, j_f) = k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j_{f'}).$$

We are interested in counting irreducible characters of $P(f)$ that correspond to irreducible characters of $U(f)$. The following is analogous to definitions 6.4.2, 6.4.4, and 6.5.5 in the discussion of unitary linear modules and central modules in the previous chapter.

Definition 7.3.4 *1. Let $S^u(f)$ denote the subset of nonidentity characters in* $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$.*

2. Let $S^{su}(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$ and correspond to singular flags in the unitary space V_2.*

3. Let $S^{nu}(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are trivial on $Z(f)$ and correspond to nonsingular flags in the unitary space V_2.*

4. Let $S^z(f)$ denote the subset of characters in $\mathrm{Irr}(U(f))$ *which are not trivial on $Z(f)$.*

Clearly $\mathrm{Irr}(U(f)) = 1 \cup S^u(f) \cup S^z(f) = 1 \cup S^{su}(f) \cup S^{nu}(f) \cup S^z(f)$ holds.

For fixed f with $l(f) \geq 2$

$$k_{d-d(f)}(P(f), \rho, \phi_f, j_f) = k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f)$$

and

$$\begin{aligned} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = & k_{d-d(f)}(P(f), S^{su}(f), \rho, \phi_f, j_f) \\ & + k_{d-d(f)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) \\ & + k_{d-d(f)}(P(f), S^z(f), \rho, \phi_f, j_f). \end{aligned}$$

Observe that if X is the subset of $S^{su}(f)$ containing only characters of rank r, where $r \in J$, and kernels equal to complements of spaces stabilized by $P_r^{+\partial l_s}$, then

$$k_{d-d(f)}(P(f), X, \rho, \phi_f, j_f) = \sum_{\substack{f'=(J,C',l) \\ C \prec C'}} k_{d-d(f')}(P(f'), \rho, \phi_{f'}, j_{f'}).$$

By the first reduction in Proposition 6.2.4, when summing over all f of length 1 and $J \subseteq I$ has maximal element l, we need only sum over $\tau \in S^{su}(f)$ with $r \in J$ and $\ker(\tau) = w$, where w is a complement to the space stabilized by P_r^{+l}. By repeated application of this reasoning it follows that

Proposition 7.3.5

$$\begin{aligned}
\sum_{\substack{f \in F \\ l(f)=1}} (-1)^{|e|} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j) = \\
\sum_{\substack{f \in F \\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + \\
\sum_{\substack{f \in F \\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + \\
\sum_{\substack{f \in F \\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{z}(f), \rho, \phi_f, j_f). \quad (7.13)
\end{aligned}$$

Our next result involves the members of E and F. This is a modification of Ku's result. We have the additional parameters ϕ_e, ϕ_f and j_e, j_f. This remarkable cancellation leads to the second reduction in Equation 4.6b .

Proposition 7.3.6

$$\sum_{\substack{e\in E\\ l(e)\geq 2}}(-1)^{|e|}k^0_{d-d(e)}(P(e),V(e),\rho,\phi_e,j_e)+\sum_{\substack{f\in F\\ l(f)\geq 2}}(-1)^{|f|}k^0_{d-d(f)}(P(f),U(f),\rho,\phi_f,j_f)=0$$

Proof: We proceed by explicitly matching pairs in the sum with opposite parity. There are four cases.

1. Match $f=(J,C,l)$ to $e=(J\cup\{l_s\},C,(l,l_s))$. This is possible since $s\geq 2$.

$$P(f)\cong P^{+n_1(f)}_{J_1(f)}\times\cdots\times P^{+n_{s-1}(f)}_{J_{s-1}(f)}\times\left(P^{+\partial l_s}_{J_s}\times \mathrm{U}_{n-2l_s}(q)\right)\ltimes U(f)$$

Then $n_0(e)=n-2l_s$ and $J_0(e)=\emptyset$, $n_s(e)=l_s-l_{s-1}=\partial l_s$ so $V(e)=1$. Moreover $n_i(f)=n_i(e)$ for $1\leq i<s$ and we have

$$P(e)\cong \mathrm{U}_{n-2l_s}(q)\times P^{+n_1(f)}_{J_1(f)}\times\cdots\times P^{+n_{s-1}(f)}_{J_{s-1}(f)}\times\left(P^{+\partial l_s}_{J_s}\times 1\right)\ltimes 1.$$

Thus

$$P(f)/U(f)\cong P(e)/V(e)\quad\text{and}\quad |\phi_f(P(f))|=|\phi_e(P(e))|$$

since ϕ_f and ϕ_e agree on isomorphic factors of the quotient groups.

2. Match $e=(J,C,(l,l'))$ to $e'=(J\cup\{l_s\},C,(l,l_s))$, if $l_s<l'$.

$$P(e)\cong \mathrm{L}^{n-2l'}_{J_0}\times P^{+n_1(e)}_{J_1(e)}\times\cdots\times\left(P^{+\partial l_s}_{J_s}\times \mathrm{GL}_{l'-l_s}(q^2)\right)\ltimes V(e).$$

$$P(e') \cong \text{Ł}^{n-2l_s}_{J_0} \times P^{+n_1(e')}_{J_1(e')} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1.$$

Moreover since there are no elements in $J \cup \{l_s\}$ between l_s and l'

$$L^{n-2l_s}_{J_0(e')} \cong L^{n-2l'}_{J_0(e)} \times \mathrm{GL}_{l'-l_s}(q^2).$$

Thus

$$P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q+1.$$

3. Match $e = (J, C, (l, l'))$ to $e' = (J\backslash\{l'\}, C, (l, l''))$, if $l_s = l' < \max J$, where l, l', l'' are consecutive elements in J.

$$P(e) \cong \text{Ł}^{n-2l'}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1.$$

$$P(e') \cong \text{Ł}^{n-2l''}_{J_0(e')} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{GL}_{l''-l_s}(q^2)\right) \ltimes V(e').$$

Moreover since there are no elements in $J\backslash\{l'\}$ between $l_s = l'$ and l''

$$L^{n-2l'}_{J_0(e)} \cong L^{n-2l''}_{J_0(e')} \times \mathrm{GL}_{l''-l_s}(q^2).$$

and $n-2l' \neq 0$. Thus

$$P(e)/V(e) \cong P(e')/V(e') \quad \text{and} \quad |\phi_e(P(e))| = |\phi_{e'}(P(e'))| = q+1.$$

4. Match $e = (J, C, (l, l'))$ to $f = (J\backslash\{l'\}, C, l)$, if $l_s = l' = \max J$.

$$P(e) \cong \mathrm{L}^{n-2l'}_{J_0(e)} \times P^{+n_1(e)}_{J_1(e)} \times \cdots \times \left(P^{+\partial l_s}_{J_s} \times 1\right) \ltimes 1$$

where $J_0(e) = \emptyset$. Then $n_i(e) = n_i(f)$ for $1 \leq i < s$

$$P(f) \cong P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times \left(P^{+\partial l_s}_{J_s} \times \mathrm{U}_{n-2l'}(q)\right) \ltimes U(f)$$

Thus

$$P(e)/V(e) \cong P(f)/U(f) \quad \text{and} \quad |\phi_e(P(e))| = |\phi_f(P(f))|$$

since ϕ_e and ϕ_f agree on isomorphic factors of the quotient groups.

And we are done.

CHAPTER 8

COMPLETION OF THE VERIFICATION

Let ρ be an irreducible character of the center of $\mathrm{U}_n(q)$. Recall $k_d^1(P_J, U_J, \rho, \det, j)$ is the number of irreducible characters $\chi \in \mathrm{Irr}(P_J)$ such that the unipotent radical U_J is not contained in $\ker \chi$, χ lies over ρ, and χ restricted to $\ker \det$ is a sum of j' irreducible characters where j divides j'.

In this chapter we prove Equation 4.6b which we restate here

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) = - \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{8.1}$$

This completes the proof of DOC for the finite special unitary groups.

We begin by reformulating the left hand side of Equation 8.1 via our parametrization using E and F from the previous chapter. After this reformulation we will refine the statement. Lastly we will need to make use of several layers of inductive arguments. To that end we introduce several propositions which are modifications of Ku's results. The extra parameters ϕ_f and j_f require even more involved combinatorial details.

1 The Reformulation

Recall, from the example on page 111 we have

$$k_d^1(P_J, V(j_i, j_{i+1}), \rho, \det, j) = k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j_e)$$

where $e = (J, \{j_i\}, (j_i, j_{i+1}))$ and from the example on page 116 we have

$$k_d^1(P_J, U_{j_s}, \rho, \det, j) = k_{d-d(f)}^1(P(f), U(f), \rho, \phi_f, j_f)$$

where $f = (J, \{j_s\}, j_s)$. Hence for fixed J we have the decomposition

$$k_d^1(P_J, U_J, \rho, \det, j) = \sum_{\substack{e=(J,C,t)\\ l(e)=1}} k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j) + k_{d-d(f)}^1(P(f), U(f), \rho, \phi_f, j). \quad (8.2)$$

Recall our definition of parity $|e|$ and $|f|$. Also note that for e and f of length 1, $d(e) = 0$, $d(f) = 0$, $j_e = j$, and $j_f = j$. Thus the left hand side of Equation 8.1 can be written

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) =$$

$$\sum_{\substack{e \in E\\ l(e)=1}} (-1)^{|e|} k_{d-d(e)}^1(P(e), V(e), \rho, \phi_e, j) + \sum_{\substack{f \in F\\ l(f)=1}} (-1)^{|f|} k_{d-d(f)}^1(P(f), U(f), \rho, \phi_f, j) \quad (8.3)$$

We have the further reduction by Proposition 7.3.2

$$\sum_{\substack{e\in E\\ l(e)=1}} (-1)^{|e|} k^1_{d-d(e)}(P(e), V(e), \rho, \phi_e, j) = \sum_{\substack{e\in E\\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e).$$

Moreover by Proposition 7.3.5,

$$\begin{aligned}\sum_{\substack{f\in F\\ l(f)=1}} (-1)^{|e|} k^1_{d-d(f)}(P(f), U(f), \rho, \phi_f, j) = \\ \sum_{\substack{f\in F\\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{nu}(f), \rho, \phi_f, j_f) + \\ \sum_{\substack{f\in F\\ l(f)\geq 1}} (-1)^{|e|} k_{d-d(e)}(P(f), S^{z}(f), \rho, \phi_f, j_f).\end{aligned} \tag{8.4}$$

Recall the cancellation of Proposition 7.3.6, i.e.

$$\sum_{\substack{e\in E\\ l(e)\geq 2}} (-1)^{|e|} k^0_{d-d(e)}(P(e), V(e), \rho, \phi_e, j_e) + \sum_{\substack{f\in F\\ l(f)\geq 2}} (-1)^{|f|} k^0_{d-d(f)}(P(f), U(f), \rho, \phi_f, j_f) = 0$$

Thus, we can now omit all the terms in our sum where the characters correspond either to characters of so called general linear modules, or to the characters of so called unitary linear modules which themselves correspond to singular flags, i.e. all the characters of the $P(e)$'s together with the characters of the $P(f)$'s which correspond to characters of $V(f)$ corresponding to singular flags in a unitary vector space. We are left with characters of the $P(f)$'s that correspond to flags of nonsingular type for $V(f)$ and characters of $Z(f)$. The left hand side of Equation 8.1 may thus be written

$$\begin{aligned}\sum_{J\subseteq I}(-1)^{|J|}k_d^1(P_J,U_J,\rho,\det,j) = \\ \sum_{\substack{f\in F\\ l(f)\geq 1}}(-1)^{|f|}k_{d-d(f)}(P(f),S^{nu}(f),\rho,\phi_f,j_f)+ \\ \sum_{\substack{f\in F\\ l(f)\geq 1}}(-1)^{|f|}k_{d-d(f)}(P(f),S^{z}(f),\rho,\phi_f,j_f).\end{aligned} \tag{8.5}$$

2 The refinement

At this stage in order to proceed in our calculations we must separate the set of $P(f)$'s according to the length of f. Recall that $l(f) = |C|$ which keeps track of the number of times we have iterated the process of taking stabilizers. In this sense C also keeps track of the structure of $P(f)$ as a product of block subgroups embedded in $\mathrm{U}_n(q)$. Recalling definition 7.2.1 of $P(f)$, C keeps track of the number of times each component of $P(f)$ appears in $\mathrm{U}_n(q)$. By collecting all the f of the same length s, we are gathering all the $P(f)$ which have the same structure as products of block subgroups embedded in $\mathrm{U}_n(q)$.

Proposition 8.2.1 *For fixed n and s, $1 \leq s \leq m$ the following hold:*

$$\sum_{\substack{f\in F\\ l(f)=s}}(-1)^{|f|}k_{d-d(f)}(P(f),S^{nu}(f),\rho,\phi_f,j_f) = -\sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s+1\\ j|\gcd(\lambda(\mu),q+1)}}\beta(\mu,a_\rho) \tag{8.6}$$

$$\sum_{\substack{f\in F\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f), S^z(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s\\ j|\gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{8.7}$$

Summing over all s, Proposition 8.2.1 certainly implies Equation 8.1. However, this is not enough. In order to prove this proposition we must further decompose the sets $S^{nu}(f)$ and $S^z(f)$. As has been discussed, originally in chapter 6 and then in chapter 7, characters in $S^{nu}(f)$ have non-singular rank by definition. Characters in $S^z(f)$ have non-singular rank if they do not correspond to singular flags. For clarity observe that in chapter 6 the roles of r and r' were reversed.

Recall ∂C kept track of the rank of singular unitary characters for which we took stabilizers. At each iteration of the process the rank of the next singular character could not exceed the rank of the previous character. In fact ∂C is precisely the list of these ranks in order reading from left to right. We are done calculating stabilizers of singular unitary characters. However, we are not done calculating stabilizers for $\tau \in S^{nu}(f)$ or $S^z(f)$. For fixed r, if the minimal element of ∂C is greater than or equal to r, then the last factor of $P(f)$ is

$$P_{J_s(f)}^{n_s(f)} \cong \left(P_{J_s}^{+\partial l_s} \times \mathrm{U}_{n-2l_s}(q) \right) \ltimes U(f)$$

where $\partial l_s \geq r$. Hence $P(f)$ is big enough to contain $\tau \in S^{nu}(f)$ or $S^z(f)$ of non-singular rank r.

Now this is the key fact: Given τ of non-singular rank $r \in J$, the stabilizer of τ in the last factor of $P(f)$ contains a subgroup isomorphic to $P^r_{J(<r)}$. By gathering together the f of the same length with $\partial C \geq r$ we are grouping all the $P(f)$ with the same block structure as embedded in $\mathrm{U}_n(q)$ which have stabilizers containing copies of $P^r_{J(<r)}$. Hence we will be able to peel off this factor and make use of a layered inductive argument.

Recall from definition 7.3.4, $S^{nu}(f)$ is the set of irreducible characters of the unitary linear module $V(f)$ that correspond to nonsingular flags.

Definition 8.2.2 *We define the following subsets.*

1. *Let $S^{nu}_r(f)$ denote the elements of $S^{nu}(f)$ with non-singular rank r.*
2. *Let $S^{nu}_r(f)(r')$ denote the elements of $S^{nu}_r(f)$ with rank r'.*

In keeping with the original notation of definition 6.4.4 from chapter 6, we have $S^{nu}_r(f)(r') = S^{nu}_r(V(f), J(< \partial l_s), r')$. Notice that

$$S^{nu}_r(f) = \bigcup_{r'=r}^{\min(\partial l_s, n-2l_s)} S^{nu}_r(f)(r').$$

Recall from definition 7.3.4, $S^z(f)$ is the set of nontrivial irreducible characters of the central module $Z(f)$.

Definition 8.2.3 *We define the following subsets.*

1. *let $S^z_r(f)$ denote the elements of $S^z(f)$ with non-singular rank r.*

2. Let $S^z_r(f)(r')$ denote the elements of $S^z_r(f)$ with rank r'.

In keeping with the original notation of definition 6.5.7 from chapter 6, we have $S^z_r(f)(r') = S^z_r(Z(f), J(< \partial l_s), r')$. Also recall that from this definition $S^z_r(f)(r)$ is the set of elements of $S^z(f)$ labeled by singular chains. Notice that

$$S^z_r(f) = \bigcup_{r'=r}^{\partial l_s} S^z_r(f)(r').$$

Summing over all r, the following proposition certainly implies Proposition 8.2.1.

Proposition 8.2.4 *For fixed n, s, and r with $1 \leq s, r \leq m$ the following hold:*

$$\sum_{\substack{f \in F \\ l(f)=s \\ \min \partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S^{nu}_r(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2s+1 \\ \min(\mu)=r \\ j | \gcd(\lambda(\mu), q+1)}} \beta(\mu, a_\rho) \tag{8.8}$$

$$\sum_{\substack{f \in F \\ l(f)=s \\ \min \partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S^z_r(f), \rho, \phi_f, j_f) = - \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2s \\ \min(\mu)=r \\ j | \gcd(\lambda(\mu), q+1)}} \beta(\mu, a_\rho) \tag{8.9}$$

However, at this point the above proposition seems somewhat convoluted. In general inductive arguments can be less than transparent. Thus, before continuing we briefly remark on why odd partitions are involved in Equation 8.8 and even partitions are involved in Equation 8.9 in the above proposition.

Remark: Take $\tau \in S_r^{nu}(f)$. Then $T_{P(f)}(\tau)$ contains an isomorphic copy of $P^r_{J(<r)}$ and the multiplicity of this factor as a block subgroup is $2s+1$. On the other hand, for $\tau \in S_r^z(f)$ $T_{P(f)}(\tau)$ contains an isomorphic copy of $P^r_{J(<r)}$ and the multiplicity of this factor as a block subgroup is $2s$. At bottom, when for example the set $J(<r)$ is empty so that $P^r_{J(<r)} = \mathrm{U}_r(q)$, we know from the remark following definition 3.2.4 in chapter 3 that irreducible characters of $\mathrm{U}_r(q)$ fall into classes of certain type given by partitions of r. These partitions encode the degree of the characters and also the splitting upon restriction to $\mathrm{SU}_r(q)$. Since $2s+1$ or $2s$ copies of $\mathrm{U}_r(q)$ (according to whether $\tau \in S_r^{nu}(f)$ or $\tau \in S_r^z(f)$) appear in $\mathrm{U}_n(q)$ it is not unreasonable to suppose that partitions of r will lead to partitions of n. For fixed C and l the left hand sums in Equation 8.8 and Equation 8.9 involve all f with $J = J' \cup J''$ where $J' \subseteq [r-1]$ varies and J'' with minimal member r is fixed. We will eventually be able to apply induction to the following sub-sums involved in Equation 8.8 and Equation 8.9

$$\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'}(P_J^r, \rho, \det{}^{2s+1}, j') \text{ in Equation 8.8}$$

$$\sum_{J \subseteq [r-1]} (-1)^{|J|} k_{d-d(f)-d'}(P_J^r, \rho, \det{}^{2s}, j'') \text{ in Equation 8.9.}$$

where $j' = \dfrac{j}{\gcd(j, q+1, 2s+1)}$ and $j'' = \dfrac{j}{\gcd(j, q+1, 2s)}$.

3 <u>The rest</u>

Before proving Proposition 8.2.4, we state and then prove an important corollary necessary for the inductive step. Moreover we present two intermediate propositions which allow us to rewrite the left hand sides of Equation 8.8 and Equation 8.9. We introduce two maps in order to keep track of the members of the involved alternating sums. These are extensions of maps defined by Ku ((15), Chapter 9). We use his notations: h, g.

In order to streamline the very involved notation we define certain subsets of F^n and then introduce two $\mathbb{Z}$-valued maps h, g on those subsets.

Definition 8.3.1 *For fixed n and s,*

1. *Let $F^n(s, r)$ denote the f in F^n with $l(f) = |C| = s$ and $\min \partial C \geq r$.*
2. *Let $F^n(\leq s)$ denote the $f \in F^n$ of length $l(f) \leq s$*

We now define our first map h which is involved in expressing the left hand side of Equation 8.8

Definition 8.3.2 *Fix n and $1 \leq r \leq m$. Define $h_{n,d,\rho,s,r,j} : F^{n-(2s+1)r}(\leq s) \to \mathbb{Z}$ by*

$$h_{n,d,\rho,s,r,j}(f) = \begin{cases} k^0_{d-d(f)}(P^{n-(2s+1)r}(f), U^{n-(2s+1)r}(f), \rho, \phi_f, j_f), & \text{if } 0 \leq l(f) < s; \\ k^0_{d-d(f)}(P^{n-(2s+1)r}(f), Z^{n-(2s+1)r}(f), \rho, \phi_f, j_f), & \text{if } l(f) = s. \end{cases}$$

The following is a first crucial step which is involved in peeling off the factor $P^r_{J(<r)}$ in the left hand side of Equation 8.8.

Proposition 8.3.3 *Fix n and $1 \leq r \leq m$. Then the alternating sum*

$$\sum_{\substack{f \in F \\ l(f)=s \\ \min \partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) =$$

$$-\sum_{\rho_1,\rho_2} \sum_{d_1,d_2} \left(\sum_{J \subseteq [r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s+1}, j_1) \sum_{f \in F^{n-(2s+1)r}(\leq s)} (-1)^{|f|} h_{n,d_2,\rho_2,s,r,j}(f) \right)$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1 \rho_2 = \rho$.

Remark: If $n-(2s+1)r = 0$, then there is a unique $f = (\emptyset, \emptyset, 0) \in F^{n-(2s+1)r}(\leq s)$. Moreover, we must set $j_f = 1$ regardless of the value of j in order that our sum may accommodate this degenerate case. Thus

$$\begin{aligned} h_{n,d,\rho,s,r,j}(f) &= k^0_{d-d(f)}(P^{n-(2s+1)r}(f), U^{n-(2s+1)r}(f), \rho, \phi_f, j_f) \\ &= k^0_d(1, 1, \rho, \det, 1) \\ &= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0. \end{aligned}$$

Proof: Take $f \in F^n(s,r)$. We consider three cases.

1. Let $\partial C = \{r\}$. Then $C = (r, 2r, \ldots, sr)$ and

$$P(f) = \left(P^{+r}_{J(<r)} \times \mathrm{U}_{n-2sr}(g) \right) \ltimes U(f).$$

$S_r^{nu}(f)$ is non-empty if and only if $n-2sr \neq 0$ and $J(<r) \subset \{1, 2, \ldots, [\frac{n-2sr}{2}]\}$. If $S_r^{nu}(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S_r^{nu}(f)$, then

$$T = T_{P(f)}(\tau) = \Big(P^r_{J(<r)} \times \mathrm{U}_{n-2sr-r}(q)\Big) \ltimes U(f).$$

Thus $k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) = k_{d-d(f)-d'}(T/U(f), \rho, \phi_f, j')$ where $j = j_f$, j' is the least positive integer such that j divides $j' \cdot \dfrac{q+1}{|\phi_f(T)|}$ and d' is q-height in the index of T in $P(f)$. We have $\mathrm{U}_{n-2sr-r}(q) = P^{n-(2s+1)r}(f')$ for $f' = (\emptyset, \emptyset, 0) \in F^{n-(2s+1)r}(\leq s)$ and

$$\begin{aligned}
&k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) \\
&\qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) k_{d_2}(P^{n-(2s+1)r}(f'), \rho_2, \phi_{f'}, j_{f'}) \\
&\qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P^r_{J(<r)}, \rho_1, (\det)^{2s+1}, j_1) h_{n,d_2,\rho_2,s,r,j}(f')
\end{aligned}$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

2. Let $\partial C \supsetneq \{r\}$. Then $\partial C = (l_1, \ldots, r)$ where $l_1 > r$.

$$P(f) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times P^{n_s(f)}_{J_s(f)}$$

where $n_i(f)$ is nonzero for some $1 \leq i < s$ and the last factor decomposes

$$P^{n_s(f)}_{J_s(f)} = \Big(P^{+r}_{J(<r)} \times \mathrm{U}_{n-2l_s}(g)\Big) \ltimes U(f).$$

$S_r^{nu}(f)$ is non-empty if and only if $n-2l_s \neq 0$ and $J(<r) \subset \{1,2,\ldots,[\frac{n-2l_s}{2}]\}$. If $S_r^{nu}(f) \neq \emptyset$ then it has one orbit. Take $\tau \in S_r^{nu}(f)$, then

$$T = T_{P(f)}(\tau) = P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times \left(P_{J(<r)}^r \times \mathrm{U}_{n-2l_s-r}(q)\right) \ltimes U(f).$$

Thus $k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) = k_{d-d(f)-d'}(T/U(f), \rho, \phi_f, j')$ where $j = j_f$, j' is the least positive integer such that j divides $j' \cdot \dfrac{q+1}{|\phi_f(T)|}$ and d' is q-height in the index of T in $P(f)$. Let $J' = \{j-r | j \in J(>r)\}$ and $C' = \{l_i - ir | l_i \in C\}\ \{0\}$. For $f' = (J', C', l_1 - r) \in F^{n-(2s+1)r}(\leq s)$ we have

$$P^{n-(2s+1)r}(f') = P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-2}(f)}^{+n_{s-2}(f)} \times \left(P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times \mathrm{U}_{n-2l_s-r}(q)\right) \ltimes U^{n-(2s+1)r}(f').$$

Thus

$$\begin{aligned}
&k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) \\
&\qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P_{J(<r)}^r, \rho_1, (\det)^{2s+1}, j_1) k_{d_2}^0(P^{n-(2s+1)r}(f'), U^{n-(2s+1)r}(f'), \rho_2, \phi_{f'}, j_{f'}) \\
&\qquad = \sum_{\rho_1,\rho_2} \sum_{d_1,d_2} k_{d_1}(P_{J(<r)}^r, \rho_1, (\det)^{2s+1}, j_1) h_{n,d_2,\rho_2,s,r,j}(f')
\end{aligned}$$

where $j_1 = j/\gcd(j, q+1, 2s+1)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

3. Let $\min(\partial C) > r$. Then $\partial l_s > r$. We have

$$P(f) = P_{J_1(f)}^{+n_1(f)} \times \cdots \times P_{J_{s-1}(f)}^{+n_{s-1}(f)} \times P_{J_s(f)}^{n_s(f)}.$$

The last factor decomposes

$$P^{n_s(f)}_{J_s(f)} = \left(P^{+\partial l_s}_{J(<r)} \times \mathrm{U}_{n-2l_s}(g)\right) \ltimes U(f).$$

Assume $S^{nu}_r(f)$ is non-empty, so $n - 2l_s \neq 0$ and $J(< r) \subset \{1, 2, \ldots, [\frac{n-2l_s}{2}]\}$. Take $\tau \in S^{nu}_r(f)$ of rank r'. Then τ corresponds to a pair of chains (c_1, c_2) where c_1 is a singular chain in a unitary space of dimension r and c_2 is a flag in a unitary space of dimension $n - 2l_s - r$. Let τ' correspond to c_2, then

$$T = T_{P(f)}(\tau) = P^{+n_1(f)}_{J_1(f)} \times \cdots \times P^{+n_{s-1}(f)}_{J_{s-1}(f)} \times T_{P^{n_s(f)}_{J_s(f)}}(\tau)$$

where

$$T_{P^{n_s(f)}_{J_s(f)}}(\tau) = \left(P^{+(\partial l_s - r')}_{J_1} \times P^{r}_{J(<r)} \times T_{\mathrm{U}_{n-2l_s-r}(q)}(\tau')\right) \ltimes U(f).$$

Here $J_1 = \{j - r' | j \in J_1(f)(> r')\}$.

And we are done.

We now define our second map g which is involved in expressing the left hand side of Equation 8.9.

Definition 8.3.4 *Fix n and $1 \leq r \leq m$. Define $g_{n,d,\rho,s,r,j} : F^{n-2sr}(\leq s) \to \mathbb{Z}$ by*

$$g_{n,d,\rho,s,r,j}(f) = \begin{cases} k^0_{d-d(f)}(P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } 0 \leq l(f) < s; \\ k_{d-d(f)}(P^{n-2sr}(f), \rho, \phi_f, j_f), & \text{if } l(f) = s. \end{cases}$$

We have a corresponding first crucial step which is involved in peeling off the factor $P^r_{J(<r)}$ in the left hand side of Equation 8.9.

Proposition 8.3.5 *For fixed n and $1 \leq r \leq m$, the alternating sum*

$$\sum_{\substack{f\in F \\ l(f)=s \\ \min\partial C \geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S^z_r(f), \rho, \phi_f, j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J, \rho_1, \det^{2s}, j_1)\sum_{f\in F^{n-2sr}(\leq s)}(-1)^{|f|}g_{n,d_2,\rho_2,s,r,j}(f)\right)$$

where $j_1 = j/\gcd(j, q+1, 2s)$, $d_1 + d_2 = d - d(f)$, and $\rho_1\rho_2 = \rho$.

Remark: If $n - 2sr = 0$, then there is a unique $f = (\emptyset, \emptyset, 0) \in F^{n-2sr}(\leq s)$. Moreover, we must set $j_f = 1$ regardless of the value of j in order that our sum may accommodate this degenerate case. Thus

$$\begin{aligned} g_{n,d,\rho,s,r,j}(f) &= k^0_{d-d(f)}(P^{n-2sr}(f), U^{n-2sr}(f), \rho, \phi_f, j_f) \\ &= k^0_d(1, 1, \rho, \det, 1) \\ &= 1 \text{ if and only if } \rho = 1 \text{ and } d = 0. \end{aligned}$$

Proof: The proof is entirely analogous the proof of Proposition 8.3.3 and consists of considering the same three cases for $f \in F^n(s,r)$:

1. $\partial C = \{r\}$.
2. $\partial C \supsetneq \{r\}$, and
3. $\min(\partial C) > r$.

We omit the proof, but remark on the following difference. For $\tau \in \mathrm{Irr}(V(f))$, τ is linear and hence extendible to $T_{P(f)}(\tau)$. However this does not hold for $\tau \in \mathrm{Irr}(Z(f))$. Rather non-linear $\phi \in \mathrm{Irr}(U(f))$ is extendible to $T_{P(f)}(\phi)$ as $U(f)/(\ker(\phi))$ is an extra special p-group.

We now state the important corollary to Proposition 8.2.4 which as mentioned will be needed for the inductive case in the proof of Proposition 8.2.4.

Corollary 8.3.6 *Assume that Proposition 8.2.4 holds. Let* $r = 0$ *in the definition of* $h_{n,d,\rho,s,0,j}$ *and* $g_{n,d,\rho,s,0,j}$. *For each* $1 \leq s \leq m$ *we have the following:*

$$\sum_{f \in F^n(\leq s)} (-1)^{|f|} h_{n,d,\rho,s,0,j}(f) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu) \leq 2s \\ j \mid \gcd(\lambda(\mu), q+1)}} \beta(\mu, a_\rho) \tag{8.10}$$

$$\sum_{f \in F^n(\leq s)} (-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)\leq 2s-1 \\ j \mid \gcd(\lambda(\mu),q+1)}} \beta(\mu, a_\rho) \tag{8.11}$$

Proof: The assumption that Proposition 8.2.4 holds implies a number of results. In order of implication these are Proposition 8.2.1, Equation 8.1 (via Equation 8.5), and thus Theorem 4.1.1, the main theorem of this paper. We proceed by induction on s.

The small case: Let $s = 1$ so that $f \in F^n(\leq 1)$ and f has length 0 or 1.

If $l(f) = 0$ then $f = (\emptyset, \emptyset, 0)$ so $P(f) = \mathrm{U}_n(q)$, $U(f) = 1$, and $d(f) = 0$. The contribution to the left hand side of Equation 8.11 is $g_{n,d,\rho,s,0,j}(f) = k_d(\mathrm{U}_n(q), \rho, \det, j)$. If $l(f) = 1$ then $f = (J, \{l\}, l)$ so $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. The contribution is $(-1)^{|f|} g_{n,d,\rho,s,0,j}(f) = (-1)^{|J|} k_d(P_J, \rho, \det, j)$.

Hence the left hand side of Equation 8.11 is

$$k_d(\mathrm{U}_n(q), \rho, \det, j) + \sum_{\emptyset \neq J \subset I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j = 1; \\ 0, & \text{otherwise} \end{cases} \tag{8.12}$$

by Equation 8.1.

The only partition μ of n with $l(\mu) \leq 1$ is $\mu = (n)$. Moreover $n'((n)) = \binom{n}{2}$ and the only j dividing $\lambda((n))$ is $j = 1$. Hence the right hand side of Equation 8.11 is equal to the left. Thus Equation 8.11 holds for $s = 1$.

As for Equation 8.10 if $l(f) = 0$ then $f = (\emptyset, \emptyset, 0)$ so $P(f) = \mathrm{U}_n(q)$, $U(f) = 1$, and $d(f) = 0$. The contribution to the left hand side of Equation 8.10 is $h_{n,d,\rho,s,0,j}(f) = k_d(\mathrm{U}_n(q), \rho, \det, j)$. If $l(f) = 1$ then $f = (J, \{l\}, l)$ so $P(f) = P_J$, $U(f) = U_l$, and $d(f) = 0$. The contribution is

$$(-1)^{|J|} k_d^0(P_J, Z_l, \rho, \det, j) = (-1)^{|J|} \big(k_d(P_J, \rho, \det, j) - k_d^1(P_J, Z_l, \rho, \det, j) \big).$$

Hence the left hand side of Equation 8.10 is given by

$$\begin{aligned} k_d(\mathrm{U}_n(q), \rho, \det, j) &+ \sum_{\emptyset \neq J \subset I} (-1)^{|J|} k_d(P_J, \rho, \det, j) - \sum_{l(f)=1} (-1)^{|f|} k_d^1(P(f), Z(f), \rho, \phi_f, j_f) \\ &= \sum_{f \in F^n(\leq 1)} (-1)^{|f|} g_{n,d,\rho,s,0,j}(f) + \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)=2 \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \\ &= \sum_{\substack{\mu \vdash n \\ n'(\mu)=d \\ l(\mu)\leq 2 \\ j \mid \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \end{aligned} \tag{8.13}$$

by Equation 8.11 and Proposition 8.2.4. Hence Equation 8.11 holds for $s = 1$ and the small case is proved.

The inductive case: Let us assume that Corollary 8.3.6 holds for all $s' < s$. Our first observation is that

$$g_{n,d,\rho,s,r,j}(f) = g_{n-2sr,d,\rho,s,0,j}(f) \text{ and } h_{n,d,\rho,s,r,j}(f) = h_{n-(2s+1)r,d,\rho,s,0,j}(f).$$

Now fixing n and letting $r = 0$ we further observe that for $f \in F^n$ with $l(f) \leq s - 2$

$$g_{n,d,\rho,s,0,j}(f) = h_{n,d,\rho,s-1,0,j}(f) = k_d^0(P^n(f), U^n(f), \rho, \phi_f, j_f).$$

For the rest of the proof we will drop the superscript n and let $P(f)$ denote $P^n(f)$, $U(f)$ denote $U^n(f)$, $Z(f)$ denote $Z^n(f)$, and $V(f)$ denote $V^n(f)$.

Writing $P(f) = Q(f) \times H(f)$ where $Q(f) = \prod_{i=1}^{s-1} P_{J_i(f)}^{+n_i(f)}$ and $H(f) = P_{J_s(f)}^{n_s(f)}$. Let $L(f)$ be a complement to $U(f)$ in $H(f)$. The quotient $P(f)/Z(f) \cong Q(f) \times (L(f) \ltimes V(f))$ and hence

$$k_{d-d(f)}^0(P(f), Z(f), \rho, \phi_f, j_f) = k_{d-d(f)}^0(P(f), U(f), \rho, \phi_f, j_f) + k_{d-d(f)}^1(P(f), V(f), \rho, \phi_f, j_f).$$

For brevity write $g_{d,\rho,s} = g_{n,d,\rho,s,0,j}$ and $h_{d,\rho,s-1} = h_{n,d,\rho,s-1,0,j}$. Then

$$
\begin{aligned}
&\sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) \\
&\quad = \sum_{\substack{f\in F^n \\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) + \sum_{\substack{f\in F^n \\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f),U(f),\rho,\phi_f,j_f) \\
&\qquad\quad + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f) - \sum_{\substack{f\in F^n \\ l(f)=s-1}} (-1)^{|f|} k^0_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f) \\
&\quad = \sum_{\substack{f\in F^n \\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) - \sum_{\substack{f\in F^n \\ l(f)=s-1}} (-1)^{|f|} k^1_{d-d(f)}(P(f),V(f),\rho,\phi_f,j_f) \\
&\qquad\quad + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f) \\
&\quad = - \sum_{\substack{f\in F^n \\ l(f)=s-1}} (-1)^{|f|} k_{d-d(f)}(P(f),S^{nu}(f),\rho,\phi_f,j_f) + \sum_{f\in F^n(\leq s-1)} (-1)^{|f|} h_{d,\rho,s-1}(f) \\
&\quad = + \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ l(\mu)=2s-1 \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) + \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ l(\mu)\leq 2(s-1)=2s-2 \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) \\
&\quad = \sum_{\substack{\mu\vdash n \\ n'(\mu)=d \\ l(\mu)\leq 2s-1 \\ j|\gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho)
\end{aligned}
$$

Hence Equation 8.11 holds at s.

As to Equation 8.10 notice for $f \in F^n$ $h_{d,\rho,s}(f) = g_{d,\rho,s}(f)$ if $l(f) < s$. Thus

$$
\begin{aligned}
\sum_{f\in F^n(\leq s)} (-1)^{|f|} h_{d,\rho,s}(f) &= \sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k_{d-d(f)}(P(f),\rho,\phi_f,j_f) \\
&\quad + \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k^0_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f) \\
&= \sum_{f\in F^n(\leq s)} (-1)^{|f|} g_{d,\rho,s}(f) - \sum_{\substack{f\in F^n\\ l(f)=s}} (-1)^{|f|} k^1_{d-d(f)}(P(f),Z(f),\rho,\phi_f,j_f) \\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2s-1\\ j\mid \gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) + \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)=2s\\ j\mid \gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho) \\
&= \sum_{\substack{\mu\vdash n\\ n'(\mu)=d\\ l(\mu)\leq 2s\\ j\mid \gcd(q+1,\lambda(\mu))}} \beta(\mu,a_\rho).
\end{aligned}
$$

Hence Equation 8.10 holds at s and Corollary 8.3.6 is proved.

Proof of Proposition 8.2.4: We proceed by induction on m where $n = 2m$ or $n = 2m+1$.

The small cases: Let $m = 0$ so that $n = 1$. Then both sums are empty and hence Proposition 8.2.4 is vacuously true.

Now let $m = 1$ so that $n = 2$ or $n = 3$. In either case we have $s = r = 1$. For f with $l(f) = 1$ we have $f = (J, \{l\}, l)$ where $\max J = l$. Thus $f = (\{1\}, \{1\}, 1)$ is the only $f \in F^n$ of nonzero length and $P(f) = P_{\{1\}}$ the Borel subgroup of $\mathrm{U}_n(q)$ with $U(f)$ the unipotent radical of $P_{\{1\}}$. Let $L(f)$ be a complement in $P(f)$ to $U(f)$. Notice that $d(f) = 0$ by definition.

Assume $n = 2$ so that we are working in $\mathrm{U}_2(q)$. Then $P(f) \cong \mathrm{GL}_1(q^2) \ltimes F_q$. Under this isomorphism the map ϕ_f is defined $\phi_f(A) = (\det A)^{1-q}$ where $A \in \mathrm{GL}_1(q^2)$. The group $L(f) \cong \mathrm{GL}_1(q^2)$ and $U(f) = Z(f)$ is elementary abelian of order q. We have $V(f) = U(f)/Z(f) = 1$ so that $S^{nu}(f)$ is empty. Thus the left hand side of Equation 8.8 is

$$-k_{d-d(f)}(P(f), S_1^{nu}(f), \rho, \phi_f, j_f) = 0.$$

This is equal to the right hand side of Equation 8.8 since there are no partitions $\mu \vdash 2$ of length 3.

As for Equation 8.9, note the set $S_1^z(f)$ is the collection of nontrivial irreducible characters of F_q on which $\mathrm{GL}_1(q^2) \ltimes F_q$ acts transitively. Indeed let us recall $g \in L(f)$ acts on $\tau \in S_1^z(f)$ via

$$\begin{pmatrix} a & 0 \\ 0 & a^{-q} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^q \end{pmatrix} = \begin{pmatrix} 1 & aca^q \\ 0 & 1 \end{pmatrix}.$$

Take any nontrivial $\tau \in \mathrm{Irr}(F_q)$ then $T_{GL_1(q^2)}(\tau) = \mathbb{C}_{q+1}$. Moreover the map ϕ_f restricted to $T_{\mathrm{GL}_1(q^2)}(\tau)$ is defined $\phi_f(A) = (\det A)^2$ for $A \in \mathbb{C}_{q+1}$. We have

$$k_d(\mathrm{GL}_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j) = k_d(T_{\mathrm{GL}_1(q^2)}(\tau), \rho, \det{}^2, j')$$

where j' is the least positive integer such that j divides $j' \cdot \gcd(2, q+1)$. Hence assuming that j is a divisor of 2

$$j' = j/\gcd(j, q+1, 2) = \begin{cases} 1, & \text{if } j = 2; \\ 1, & \text{if } j = 1. \end{cases}$$

Then for $j = 1$ or $j = 2$ the left hand side of Equation 8.9 is given by

$$\begin{aligned} -k_d(P(f), S_1^z(f), \rho, \phi_f, j) &= -k_d(\mathrm{GL}_1(q^2) \ltimes F_q, \tau, \rho, \phi_f, j) \\ &= -k_d(\mathbb{C}_{q+1}, \rho, \det{}^2, 1) \\ &= \begin{cases} -1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This is certainly equal to the right hand side of Equation 8.9 since the only partition $\mu \vdash 2$ with $l(\mu) = 2$ is $\mu = (1^2)$ with minimal element 1, $n'((1^2)) = 0$, and $\lambda((1^2)) = 2$ which is divisible by $j = 1$ or $j = 2$.

Assume now $n = 3$ so that we are working in $U_3(q)$. The group

$$P(f) = P_{\{1\}} \cong \big(\mathrm{GL}_1(q^2) \times \mathrm{U}_1(q)\big) \ltimes U_{\{1\}}.$$

Under this isomorphism the map ϕ_f is defined $\phi_f(A, B) = (\det A)^{1-q} \det B$ where $A \in \mathrm{GL}_1(q^2)$ and $B \in \mathrm{U}_1(q)$. In this case $U(f) = U_{\{1\}}$ is no longer equal to its center. Let $L(f)$ be a complement to $U(f)$ in $P(f)$.

The quotient $V(f) \cong M_{1,1}(F_{q^2})$ and $S_1^{nu}(f)$ has one orbit under $P_{\{1\}}$. Take $\tau \in S_1^{nu}(f)$ then $T_{L(f)}(\tau) = \mathrm{U}_1(q) \cong \mathbb{C}_{q+1}$. Moreover the map ϕ_f restricted to $T_{L(f)}(\tau)$ is defined $\phi_f(B) = (\det B)^3$ for $B \in \mathrm{U}_1(q)$

Then for $j = 1$ or $j = 3$ the left hand side of Equation 8.8 is given by

$$\begin{aligned}
-k_d(P(f), S_1^{nu}(f), \rho, \phi_f, j) &= -k_d(P(f), \tau, \rho, \phi_f, j) \\
&= -k_d(T_{L(f)}(\tau), \rho, \phi_f|, j/\gcd(j, 3, q+1)) \\
&= -k_d(\mathbb{C}_{q+1}, \rho, \det^3, 1) \\
&= \begin{cases} -1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

This is certainly equal to the right hand side of Equation 8.8 since the only partition $\mu \vdash 3$ with $l(\mu) = 2(1) + 1 = 3$ is $\mu = (1^3)$ with minimal element 1, $n'((1^3)) = 0$, and $\lambda((1^3)) = 3$ which is divisible by $j = 1$ or $j = 3$.

As for Equation 8.9, $S_1^z(f)$ is the set of non-trivial characters of $Z(f)$ on which $P(f)$ acts transitively. Take $\tau \in S_1^z(f)$. Then there exists a unique $\psi \in \mathrm{Irr}(U(f))$ lying over τ with $\psi(1) = q$. We have

$$T_P(\tau) = T_P(\psi) = \big(\mathrm{U}_1(q) \times \mathrm{U}_1(q)\big) \ltimes U_{\{1\}}$$

and q does not divide $|P|/|T_P(\psi)|$. The map ϕ_f restricted to $T_P(\psi)$ is defined $\phi_f(A,B) = (\det A)^2(\det B)$ for A in the first factor $\mathrm{U}_1(q)$ and B in the second factor $\mathrm{U}_1(q)$. Also notice that

$$\left|P(f)/T_P(\psi)\ker(\phi_f)\right| = 1.$$

Hence the left hand side of Equation 8.8 is given by

$$\begin{aligned}
-k_d(P(f), S_1^z(f), \rho, \phi_f, j) &= -k_d(P(f), \tau, \rho, \phi_f, j) \\
&= -k_d(P(f), \psi, \rho, \phi_f, j) \\
&= -k_{d-1}(T_{P(f)}(\psi)/U_{\{1\}}, \rho, \phi_f|, j) \\
&= -k_{d-1}(\mathbb{C}_{q+1} \times \mathbb{C}_{q+1}, \rho, \det{}^2 \cdot \det, j) \\
&= -\sum_{\rho_1\rho_2=\rho}\sum_{d_1+d_2=d-1} k_{d_1}(\mathbb{C}_{q+1}, \rho_1, \det{}^2, j/\gcd(j,2,q+1))k_{d_2}(\mathbb{C}_{q+1}, \rho_2, \det, j) \\
&= \begin{cases} -(q+1), & \text{if } d = 1 \text{ and } j = 1; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

This is certainly equal to the right hand side of Equation 8.9 since the only partition $\mu \vdash 3$ of length 2 is $\mu = (2,1)$ with $n'((2,1)) = 1$, $\min((2,1)) = 1$, and $\lambda((2,1)) = 1$. Indeed $-\beta((2,1), a_\rho) = -(q+1)$.

The inductive case: We assume Proposition 8.2.4 holds for all $m' < m$ where $m \geq 2$ and $n = 2m$ or $n = 2m+1$. Recall that this assumption implies that for all such m', not only does Corollary 8.3.6 hold at m', but also Proposition 8.2.1, Equation 8.1, and hence Theorem 4.1.1, the main theorem of this paper, hold at m'.

Recall our observation in the proof of 8.3.6 that

$$h_{n,d,\rho,s,r,j}(f) = h_{n-(2s+1)r,d,\rho,s,0,j}(f).$$

Moreover since $1 \leq r < n$ by induction assumption we have

$$\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det{}^{2s+1}, j_1) = \begin{cases} \beta((r), a_\rho), & \text{if } d_1 = \binom{r}{2} \text{ and } j_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that if $j_1 = 1$ we have $j = \gcd(j, 2s+1, q+1)$ in the statement of Proposition 8.3.3. Then

$$\sum_{\substack{f\in F\\ l(f)=s\\ \min\partial C\geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S_r^{nu}(f), \rho, \phi_f, j_f) =$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s+1},j_1)\sum_{f\in F^{n-(2s+1)r}(\leq s)}(-1)^{|f|}h_{n,d_2,\rho_2,s,r,j}(f)\right)$$

$$-\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]}(-1)^{|J|}k_{d_1}(P_J,\rho_1,\det^{2s+1},j_1)\sum_{f\in F^{n-(2s+1)r}(\leq s)}(-1)^{|f|}h_{n-(2s+1)r,d_2,\rho_2,s,0,j}(f)\right)$$

$$= -\sum_{\rho_1\rho_2=\rho}\left(\beta((r),a_{\rho_1})\sum_{\substack{\mu\vdash(n-(2s+1)r)\\ n'(\mu)=d_2\\ l(\mu)\leq 2s\\ j|\gcd(2s+1,q+1,\lambda(\mu))}}\beta(\mu,a_{\rho_2})\right)$$

$$= -\sum_{\substack{\tilde\mu\vdash(n)\\ n'(\tilde\mu)=d_2\\ \min(\tilde\mu)=r\\ l(\tilde\mu)\leq 2s+1\\ j|\gcd(q+1,\lambda(\tilde\mu))}}\beta(\tilde\mu,a_{\rho_2})$$

where for fixed $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_{\delta(\mu)}^{l_{\delta(\mu)}}) \vdash (n-(2s+1)r)$ the corresponding partition of n is given by

$$\tilde\mu = (r^{2s+1}) + \mu = ((a_1+r)^{l_1}, (a_2+r)^{l_2}, \ldots, (a_{\delta(\mu)}+r)^{l_{\delta(\mu)}}, r^e) \vdash n.$$

and $\sum_{i=1}^{\delta(\mu)} l_i + e = 2s+1$ so j divides e i.e. j divides $\lambda(\tilde\mu)$.

Recall our observation in the proof of 8.3.6 that

$$g_{n,d,\rho,s,r,j}(f) = g_{n-2sr,d,\rho,s,0,j}(f).$$

Moreover since $1 \leq r < n$ by induction assumption we have

$$\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s}, j_1) = \begin{cases} \beta((r), a_\rho), & \text{if } d_1 = \binom{r}{2} \text{ and } j_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that if $j_1 = 1$ we have $j = \gcd(j, 2s, q+1)$ in the statement of Proposition 8.3.5. Then

$$\begin{aligned}
&\sum_{\substack{f\in F \\ l(f)=s \\ \min\partial C\geq r}} (-1)^{|f|} k_{d-d(f)}(P(f), S_r^z(f), \rho, \phi_f, j_f) = \\
&\quad -\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s}, j_1) \sum_{f\in F^{n-2sr}(\leq s)} (-1)^{|f|} g_{n,d_2,\rho_2,s,r,j}(f)\right) \\
&\quad -\sum_{\rho_1,\rho_2}\sum_{d_1,d_2}\left(\sum_{J\subseteq[r/2]} (-1)^{|J|} k_{d_1}(P_J, \rho_1, \det^{2s}, j_1) \sum_{f\in F^{n-2sr}(\leq s)} (-1)^{|f|} g_{n-2sr,d_2,\rho_2,s,0,j}(f)\right) \\
&\qquad = -\sum_{\rho_1\rho_2=\rho}\left(\beta((r), a_{\rho_1}) \sum_{\substack{\mu\vdash(n-2sr) \\ n'(\mu)=d_2 \\ l(\mu)\leq 2s-1 \\ j|\gcd(2s,q+1,\lambda(\mu))}} \beta(\mu, a_{\rho_2})\right) \\
&\qquad = -\sum_{\substack{\tilde\mu\vdash(n) \\ n'(\tilde\mu)=d_2 \\ \min(\tilde\mu)=r \\ l(\tilde\mu)\leq 2s \\ j|\gcd(q+1,\lambda(\tilde\mu))}} \beta(\tilde\mu, a_{\rho_2})
\end{aligned}$$

where for fixed $\mu = (a_1^{l_1}, a_2^{l_2}, \ldots, a_{\delta(\mu)}^{l_{\delta(\mu)}}) \vdash (n - 2sr)$ the corresponding partition of n is given by

$$\tilde{\mu} = (r^{2s}) + \mu = ((a_1 + r)^{l_1}, (a_2 + r)^{l_2}, \ldots, (a_{\delta(\mu)} + r)^{l_{\delta(\mu)}}, r^e) \vdash n.$$

and $\sum_{i=1}^{\delta(\mu)} l_i + e = 2s$ so j divides e i.e. j divides $\lambda(\tilde{\mu})$.

And we are done, Proposition 8.2.4 is proved. Thus Proposition 8.2.1 also holds, so too does Equation 8.1 (via Equation 8.5), and thus the main result of this thesis Theorem 4.1.1 is proved.

CHAPTER 9

AN EXAMPLE: DIMENSION 4

Let $K = \overline{F}_q$. Let $\widetilde{G} = \mathrm{GL}_4(K)$. Under the Frobenius $F(a_{i,j}) = M(a_{j,i}^q)^{-1}M^{-1}$ we have $\widetilde{G}^F = U_4(q)$ which we denote by G. The Weyl group, $\widetilde{W}$, of $\widetilde{G}$ is of type A_3, i.e. $\widetilde{W} = S_3$, the symmetric group on three elements. Let $\{1, 2, 3\}$ be an index set for the distinguished generators of $\widetilde{W}$. The Weyl group, W, of G is of type B_2. The F-orbits on the reflections is given by $\{\{1, 3\}, \{2\}\}$. Let $I = \{1, 2\}$ index this set. Let $\widetilde{B}$ be the Borel subgroup of upper triangular matrices in $\widetilde{G}$. Observe that $\widetilde{B}$ is F-stable. Let $B = \widetilde{B}^F$. Then B is upper triangular. In keeping with the notation of this thesis we have the following parabolic subgroups,

$$P_\emptyset = G, \qquad P_{\{1\}}, \qquad P_{\{2\}}, \qquad P_{\{1,2\}} = B$$

Where each P_J has Levi decomposition $L_J U_J$. Writing the Levi subgroups as block matrices, we have the following L_J:

$$L_\emptyset = G = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\},$$

$$L_{\{1\}} = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong \mathrm{GL}_1(q^2) \times \mathrm{U}_2(q),$$

$$L_{\{2\}} = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \cong \mathrm{GL}_2(q^2),$$

$$L_{\{1,2\}} = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong \mathrm{GL}_1(q^2) \times \mathrm{GL}_1(q^2).$$

Writing the unipotent radicals of the P_J as block matrices, we have the following U_J together with their respective normal series:

$$U_\emptyset = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = 1,$$

$$U_{\{1\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > 1,$$

$$U_{\{2\}} = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1,$$

$$U_{\{1,2\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1.$$

For nonempty J, we enumerate the quotient modules for the P_J and orbit representatives.

$J = \{1\}$:

$$U_{\{1\}}/Z_{\{1\}} \cong \left\{ \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & 0 & -a^q \\ 0 & 0 & 1 & -b^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b \in F_{q^2} \right\} \cong M_{1,2}(q^2) \text{is a unitary module for } L_{\{1\}}.$$

Let τ_s correspond to a singular chain of rank 1 in unitary space of dimension 2.

Let τ_n correspond to a non-singular chain of rank 1 in unitary space of dimension 2.

$$Z_{\{1\}} \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid c + c^q = 0 \right\} \cong M_{1,1}(q) \text{is a central module for } L_{\{1\}}.$$

Let $x_1 = (\epsilon)$ be the unique non-trivial orbit representative.

$J = \{2\}$:

$$Z_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & 0 & a & d_1 \\ 0 & 1 & d_2 & -a^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, d_i \in F_{q^2} \text{ and } d_i + d_i^q = 0 \right\} \cong M_{2,2}(q)$$

is a central module for $L_{\{2\}}$. Let $x_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ be an orbit representative for elements of rank 2.

$J = \{1, 2\}$**:**

$$U_{\{1,2\}}/U_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in F_{q^2} \right\} \cong M_{1,1}(q^2) \text{ is a general linear module for } L_{\{1,2\}}.$$

$$Z_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & 0 & a & d_1 \\ 0 & 1 & d_2 & -a^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, d_i \in F_{q^2} \text{ and } d_i + d_i^q = 0 \right\} \cong M_{2,2}(q)$$

is a central module for

$$B/Z_{\{2\}} \cong \left\{ \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong P^{+2}_{\{1\}}.$$

We enumerate the members of E and F:

$$
\begin{aligned}
E = \big\{&(\emptyset, \emptyset, 0),\\
&(\{1,2\}, \{1\}, (12))\\
&(\{1,2\}, \{1,2\}, (12))\big\}, \text{ and}\\
F = \big\{&(\emptyset, \emptyset, 0),\\
&(\{1\}, \{1\}, 1), (\{2\}, \{2\}, 2), (\{1,2\}, \{2\}, 2),\\
&(\{1\}, \{1,2\}, 1)\big\}.
\end{aligned}
$$

First observation: Take $e_1 = (\{1,2\}, \{1\}, (12))$ so that $P(e_1) = P_{\{1,2\}}/U_{\{2\}} \cong P^{+2}_{\{1\}}$ with $V(e_1) = V(1,2) \cong M_{1,1}(q^2)$. Take nontrivial $\tau_g \in \mathrm{Irr}(V(1,2))$ Then

$$T_{P(e_1)}(\tau) = \mathrm{GL}_1(q^2) \ltimes V(e_1) \cong P(e_2) \ltimes V(e_1).$$

where $e_2 = (\{1,2\}, \{1,2\}, (12))$ since $P(e_2) \cong \mathrm{GL}_1(q^2)$ by definition. Also note $V(e_2) = 1$ by definition. Let $g \in T_B(\tau_g)/U_{\{1,2\}}$. As a block matrix g can be written:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix}$$

Second observation: Take $f_1 = (\{1,\}, \{1\}, 1)$ so that $P(f_1) \cong P_{\{1\}}$ with $U(f_1) = U_{\{1\}}$ Then

$$T_{P_{\{1\}}}(\tau_s)/U_{\{1\}} \cong T_{L_{\{1\}}}(\tau_s) = P^2_{\{1\}} \cong \mathrm{GL}_1(q^2) \ltimes M_{1,1}(q) \cong P(f_2) \ltimes U(f_2)$$

where $f_2 = (\{1\}, \{1,2\}, 1)$. Let $g \in T_{L_{\{1\}}}(\tau_s)$. As a block matrix g can be written:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix}, \text{ where } b + b^q = 0.$$

The elements e_2 and f_2 have opposite parity and hence lead to cancellation in the alternating sum in the statement of DOC, i.e. for $j' = j/\gcd(j, q+1, 2)$

$$k_d(B, \tau_g, \rho, \det, j) = k_{d-d'}(T_B(\tau_g)/V(1,2), \rho, \det, j') = k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2})$$

$$\begin{aligned} k_d(P_{\{1\}}, \tau_s, \rho, \det, j) &= k_{d-d''}(T_{P_{\{1\}}}(\tau_s)/U_{\{1\}}, \rho, \det, j') \\ &= k^0_{d-d(f_2)}(P(f_2), V(f_2), \rho, \phi_{f_2}, j_{f_2}) + k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) \end{aligned}$$

where d' is the power of q in $|T_B(\tau_g)\backslash B|$ and d'' is the power of q in $|T_{P_{\{1\}}}(\tau_s)\backslash P_{\{1\}}|$, i.e. $d' = d'' = 0$ Then

$$k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2}) + k^0_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) = 0.$$

With regard to the splitting of characters upon restriction to the kernel of the determinant map in general, observe that 4 is divisible by 1, 2, and 4. Moreover, assuming that $q+1$ is divisible by 4, we may consider $j = 1$, 2, or 4.

Let $j = 4$. Then $k_d(L_J, \rho, \det, 4) = 0$ for nonempty subsets J in $\{1, 2\}$ so

$$\begin{aligned}\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, 4) &= k_d(\mathrm{U}_4(q), \rho, \det, 4) \\ &= \begin{cases} \beta((1^4), a_\rho) = 1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

We have $k_d^1(P_J, U_J, \rho, \det, 4) = 0$ except for $J = \{1\}$. Take τ_s, the orbit representative discussed above, and consider

$$T_{P_{\{1\}}}(\tau_s)/U_{\{1\}} \cong P_{\{1\}}^2 \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix} \mid a, b \in F_{q^2}, b + b^q = 0 \right\}.$$

The determinant map restricted to $P(f_2) = P_{\{1\}}^2$ is $\phi_{f_2} = \det^2$. Moreover, 4_{f_2} is the least positive integer such that 4 divides

$$4 \text{ divides } 4_{f_2} \cdot \frac{q+1}{(q+1)/\gcd(q+1, 2)}.$$

so $4_{f_2} = 2$. By definition $d(f_2) = 0$ and indeed the exponent of q in $|T_{P_{\{1\}}}(\tau_s)\backslash P_{\{1\}}|$ is zero as already mentioned.

Take non trivial $x \in \mathrm{Irr}(Z^2_{\{1\}})$. Then

$$T_{P^2_{\{1\}}}(x) = U_1(q) \ltimes Z^2_{\{1\}} \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b \in F_{q^2}, a^{1+q} = 1, \text{ and } b + b^q = 0 \right\}.$$

The determinant map restricted to $T_{P^2_{\{1\}}}(x) = \det^4$. Let j' be the least positive integer such that

$$2 \text{ divides } j' \cdot \frac{\left|P^2_{\{1\}}\right|}{\left|T_{P^2_{\{1\}}}(x) \cdot \ker(\det^4)\right|},$$

then $j' = 1$. Hence

$$\begin{aligned} \sum_{J \subseteq I} (-1)^{|J|} k^1_d(P_J, U_J, \rho, \det, 4) &= -k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) \\ &= -k^1_d(P^2_{\{1\}}, Z^2_{\{1\}}, \rho, \det^2, 2) \\ &= -k_d(\mathrm{U}_1(q), \rho, \det^4, 1) \\ &= -\begin{cases} \beta((1^4), a_\rho) = 1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $j = 2$, then $k_d(L_J, \rho, \det, 2) = 0$ for subsets $J = \{1\}$ and $\{1, 2\}$. The calculations are repetitive so we will not present them for all for $d = 0$, 1, 2, and 3, but rather present some of the more interesting calculations. Consider $d = 0$. Our first observation is that this case includes the above, since 2 divides 4.

$$\begin{aligned}
\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, 4) &= k_0(\mathrm{U}_4(q), \rho, \det, 2) - k_0(\mathrm{GL}_2(q^2), \rho, \det^{1-q}, 2) \\
&= q\beta((1^4), a_\rho) - \overline{\beta}((1^2), a_\rho) \\
&= q - (q-1) \\
&= (q-1) + 1 - (q-1) \\
&= 1.
\end{aligned}$$

Notice that the remaining character splits into 4 irreducibles upon restriction to the kernel of the determinant map. We have $k_0^1(P_J, U_J, \rho, \det, 2) = 0$ except for $J = \{1\}$, which is worked out above.

Continue to assume that $j = 2$ and consider $d = 1$. Then $k_1(\mathrm{GL}_2(q^2), \rho, \det^{1-q}, 2) = 0$ since 0 and 2 are the only possible q-heights for $\chi \in \mathrm{Irr}(\mathrm{GL}_2(q^2))$. Moreover $k_1(\mathrm{U}_4(q), \rho, \det, 2) = 0$ since 2 does not divide $\lambda(\mu)$ for $\mu = (2, 1^2)$. Hence,

$$\sum_{J \subseteq I} (-1)^{|J|} k_1^0(P_J, U_J, \rho, \det, 4) = 0$$

which doesn't seem like an interesting calculation. However, examining the other side of the alternating sum is somewhat more interesting since we see cancellation. Take $x_2 \in S^z(f_2)$ and $S^z(f_3)$ as above, then

$$T_{P_{\{2\}}}(x_2)/U_{\{2\}} \cong \mathrm{U}_2(q) \text{ and } T_{P_{\{1,2\}}}(x_2)/U_{\{2\}} \cong P^2_{\{1\}}.$$

The exponent of q in

$$\left|\mathrm{U}_2(q)\backslash \mathrm{GL}_2(q^2)\right| = \left|P^2_{\{1\}}\backslash P^{+2}_{\{1\}}\right| \text{ is } 1.$$

If K is the kernel of the determinant map restricted to the stabilizers of x_2, then

$$2 \text{ divides } \left|T_{P_{\{2\}}}(x_2)K\backslash P_{\{2\}}\right| = \left|T_{P_{\{1,2\}}}(x_2)K\backslash P_{\{1,2\}}\right|.$$

Hence,

$$k_d(P_{\{2\}}, x_2, \rho, \det{}^{1-q}, 2) = k_{d-1}(\mathrm{U}_2(q), \rho, \det{}^2, 1) \text{ and}$$

$$k_d(P_{\{1,2\}}, x_2, \rho, \det{}^{1-q}, 2) = k_{d-1}(P^2_{\{1\}}, \rho, \det{}^2, 1).$$

Thus

$$\begin{aligned}\sum_{J\subseteq I}(-1)^{|J|}k^1_1(P_J, U_J, \rho, \det, 2) &= -k_0(\mathrm{U}_2(q), \rho, \det{}^2, 1) + k_0(P^2_{\{1\}}, \rho, \det{}^2, 1)\\ &= -0\end{aligned}$$

since $\binom{2}{2} = 1 > 0$.

Let $j = 1$. Since 1 divides every integer, this case is trivial. Indeed,

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho, \det, 1) = \sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho)$$

which has already been shown by Ku ((15)).

CITED LITERATURE

1. J. L. Alperin, *Local Representation Theory*, Cambridge University Press, 1986.

2. D. J. Benson, *Representations and Cohomology: I*, Cambridge University Press, 1995.

3. C. Bonnafé, *Sur Les Caractères des Groupes Réductifs Finis à Centre non Connexe : Applications aux Groupes Spéciaux Linéaires et Unitaires*, *Astérisque*, 306, 2006.

4. N. Burgoyne and C. Williamson, *On a theorem of Borel and Tits for finite Chevalley groups*, *Arch. Math.*, 27 (1976), 489–491.

5. M. Cabanes and M. Enguehard, *Representation Theory of Finite Reductive Groups*, Cambridge University Press, 2004.

6. R. W. Carter, *Finite Groups of Lie Type*, John Wiley and Sons, 1985.

7. C. W. Curtis and I. Reiner, *Methods of Representation Theory*, Vol. II, John Wiley and Sons, New York, 1981.

8. E. C. Dade, *Counting Characters in Blocks, I*, *Invent. Math.*, 109 (1992).

9. E. C. Dade, *Characters of Groups with Normal Extra Special Subgroups*, *Math. Z.*, 152 (1976), 1–31.

10. W. Feit, *The Representation Theory of Finite Groups*, North-Holland Publishing Company, 1982.

11. P. Fong and B. Srinivasan, *The Blocks of Finite General Linear and Unitary Groups*, *Invent. Math.*, 69 (1982), 109–153.

12. S. C. Huang *New Results on Dade's Conjecture*, *Sūrikaisekikenkyūsho Kōkyūroku*, 1656 (2009), 118–126.

13. J. E. Humphreys, *Defect Groups for Finite Groups of Lie Type*, *Math. Z.*, 119 (1971), 149–152.

14. R. Knörr and G. Robinson, *Some Remarks on a Conjecture of Alperin*, *J. London Math. Soc*, (2) 39 (1989), 48–60.

15. C. Ku, *Dade's Ordinary Conjecture for the Finite Unitary Groups in the Defining Characteristic*, PhD thesis, California Institute of Technology, 1999.

16. J. B. Olsson and K. Uno, *Dade's Conjecture for General Linear Groups in the Defining Characteristic*, *Proc. London Math. Soc.*, (3) 72 (1996), 359–384.

17. J. B. Olsson and K. Uno, *Results on Dade's Conjecture*, http://www.math.ku.dk/ olsson/links/dade/dade.html

18. D. Quillen, *Homotopy Properties of the Poset of Nontrival p-Subgroups of a Group*, *Advances in Math.*, 28 (1978), 101–128.

19. J. Rickard, *The Abelian Defect Group Conjecture*, *Doc. Math. J. DMV Extra Volume ICM II*, (1998), 121–128.

20. W. A. Simpson and J. S. Frame, *The Character Tables for* $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, *Can. J. Math*, vol. XXV, n. 3 (1973), 486–494.

21. S. D. Smith, *Subgroup Complexes*, *Surveys of the AMS*, 179 (2011).

22. H. Sukizaki, *Dade's Conjecture for Special Linear Groups in the Defining Characteristic*, *J. Algebra*, 220 (1999), 261–283.

23. H. Sukizaki, *The McKay Numbers of a Subgroup of* $\mathrm{GL}_n(q)$ *Containing* $\mathrm{SL}_n(q)$, *Osaka J. Math.*, 36 (1999), 177–193.

24. J. Thévenaz and P. J. Webb, *Homotopy Equivalence of Posets with a Group Action*, *J. Combinat. Theory Ser. A.*, 56 (1991), 173-181.

25. P. J. Webb, *Subgroup Complexes*, The Arcata Conference on Representations of Finite Groups, AMS Proceedings of Symposia in Pure Mathematics: ed. P. Fong, 47 (1987), 349–365.

VITA

NAME:	Katherine Anne Bird
EDUCATION:	B.S., Mathematics, University of Illinois at Chicago, 2000 M.S., Mathematics, University of Illinois at Chicago, 2004 Ph.D., Mathematics, University of Illinois at Chicago, 2012
TEACHING:	University of Illinois at Chicago, Visiting Lecturer, 2009-2010 Loyola University Chicago, Visiting Lecturer, 2010-2011 University of Illinois at Chicago, Visiting Lecturer, 2012
HONORS:	Vigre Fellowship Louis Hayes Award Herb Alexander Award
MEMBERSHIPS:	American Women in Mathematics American Mathematical Society