Regularity and Energy Laws in Hydrodynamic Models of Newtonian Fluids and Collective Behavior

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TML

CONTRIBUTION OF AUTHORS

Most of this thesis is excerpted verbatim (or nearly so) from the following four papers (numbered as in the Cited Literature below):

[**38**] T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differential Equations*, 261(6):3719–3733, 2016.

[**39**] Trevor M. Leslie and Roman Shvydkoy. Conditions Implying Energy Equality for Weak Solutions of the Navier–Stokes Equations. *SIAM J. Math. Anal.*, 50(1):870–890, 2018.

[**37**] T. M. Leslie and R. Shvydkoy. The Energy Measure for the Euler and Navier-Stokes Equations. *Arch. Ration. Mech. Anal. (to appear)*.

[**36**] T. M. Leslie. Weak and Strong Solutions to the Forced Fractional Euler Alignment System. *ArXiv e-prints*, March 2018.

Roman Shvydkoy suggested most of the problems that are treated in these works (including those of [**36**]), and in many cases identified several possible approaches to studying these problems. Trevor Leslie carried out most of the research once these were identified, and wrote most of the manuscripts of [**38, 39, 37**] (and all of [**36**]), receiving advice from Shvydkoy in the process.

In this thesis, Chapter 1 contains material from [**38**], [**37**], and [**36**]; Chapter 2 is largely excerpted from [**37**] (though the proof of Lemma 4.3 is from [**39**]); Chapter 3 is largely excerpted from [**39**]; Chapter 4 is taken from [**38**]; and Chapter 5 is taken from [**39**].

SUMMARY

In this thesis, we consider the question of sufficient conditions for energy equality for a number of systems, and from a number of different perspectives. In the first part, our main focus is on the classical 3-dimensional Navier-Stokes equations (and, to a certain extent, the n-dimensional Euler equations). We give criteria on the integrability properties of the solution which guarantee that the natural energy law holds; these integrability criteria depend on the size and structure of the singularity set. In particular, we consider the cases when the singularity set is either restricted to a single time-slice (the first possible time of blowup) or when the singularity set has (parabolic) Hausdorff dimension strictly less than 3. One important situation where we are able to prove energy balance is the case of Type-I in time blowup for the 3-dimensional Navier-Stokes Equations.

In the time-slice singularity case, we can sometimes quantify the possible failure of the energy law in those situations where we cannot prove energy balance. This is done using the so-called energy measure $\mathcal{E}$: the weak-$*$ limit of the measures $|u(t)|^2\,\mathrm{d}x$ as t approaches the first possible time of blowup. We give bounds on the lower local dimension and the concentration dimension of the energy measure associated to a given solution, in terms of the integrability class to which the solution belongs.

The idea of relating criteria for energy equality to the size of the singularity set is inspired by the celebrated Caffarelli-Kohn-Nirenberg theorem, which states that the parabolic Hausdorff dimension of the singularity set is at most 1 for suitable weak solutions of the Navier-Stokes equations. In recent years, analogues of the Caffarelli-Kohn-Nirenberg Theorem have been discovered for the fractional Navier-Stokes equations. Accordingly, we apply our method to those equations as well.

In the second part of the thesis, we treat inhomogeneous models, where the density is not assumed to be constant. First, we consider the inhomogeneous incompressible Euler and Navier-Stokes equations. We show that the natural energy balance law holds for solutions of this system which belong to a certain class of Besov spaces of smoothness 1/3; our criteria are reminiscent of those of the famous Onsager conjecture.

Next, we turn our attention to a newer model of collective dynamics, sometimes called the Fractional Euler Alignment model. This system has previously been studied only in classical regularity spaces, where the natural energy law is obvious. Therefore before considering the question of energy equality, we first define an acceptable notion of a weak solution. We develop an existence theory for L^∞ "weak" solutions, and give Besov regularity criteria sufficient to guarantee energy balance for weak solutions. We also prove existence and uniqueness of $W^{1,\infty}$ "strong" solutions. More regular solutions of the Euler Alignment model have been shown to exhibit fast alignment of the velocity, and convergence to a flocking state, in the absence of an external force. We show that fast alignment still holds for weak solutions, and that fast alignment and flocking hold for strong solutions.

TABLE OF CONTENTS

List of Figures

CHAPTER 1

Introduction[1]

In this Introduction, we give a more detailed overview of the contents of this thesis. There are essentially two main parts: The first part treats homogeneous models such as the classical Euler and Navier-Stokes equations, as well as the fractional Navier-Stokes equations. The second part treats inhomogeneous models, namely the density-dependent Euler and Navier Stokes systems, as well as the Fractional Euler Alignment model. In the first part, we treat the question of energy balance exclusively, with an approach and set of tools that are geometric in nature. In the second part, we use Littlewood-Paley theory to establish conditional energy equality for the models we consider. In the case of the Fractional Euler Alignment model, we also treat questions of well-posedness for the model, which have previously only been considered in classical regularity spaces.

1. Homogeneous Models

1.1. Energy Equality at the First Blowup Time. We begin by considering the incompressible Euler or Navier-Stokes initial value problem on $\mathbb{R}^n$:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u(t_0) = u_0. \tag{3}$$

Here we understand that $\nu = 0$ and $n \geq 3$ if we are considering the Euler Equations, and that $n = 3$ if $\nu > 0$. In either case, we assume that $u_0 \in H^{\frac{n}{2}+1+\epsilon}(\mathbb{R}^n)$ for some $\epsilon > 0$, so that there exists a

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[37] T. M. Leslie and R. Shvydkoy. The Energy Measure for the Euler and Navier-Stokes Equations. *Arch. Ration. Mech. Anal. (to appear)*.
[36] T. M. Leslie. Weak and Strong Solutions to the Forced Fractional Euler Alignment System. *ArXiv e-prints*, March 2018.

unique local-in-time solution

$$u \in C([t_0, t_1); H^{\frac{n}{2}+1+\epsilon}(\mathbb{R}^n)), \tag{4}$$

for some $t_1 > 0$, with the associated pressure given by

$$p = R_i R_j (u_i u_j), \tag{5}$$

where R_i, R_j denote the classical Riesz transforms. We assume that (u, p) can be extended to some larger time interval $[t_0, T]$, with $T \geq t_1$, with u a weak solution on the larger interval, which is weakly continuous in L^2 at $t = t_1$. If $\nu > 0$, we assume that u is a Leray-Hopf solution on $[t_0, T]$. Let $\Omega \subset \mathbb{R}^n$ be a bounded subdomain and assume $u \in L^3(t_0, t_1; L^3(\Omega))$ (this is automatic if $\nu > 0$). We will work either on Ω or on the full space $\mathbb{R}^n$; in the latter case we will assume without further comment that $u \in L^3(t_0, t_1; L^3(\mathbb{R}^n))$. We stress that even when we work on Ω, the pair (u, p) will solve (1)–(3) on the full space.

By a classical result of Leray [35], it is known that for divergence-free initial data $u_0 \in L^2$, there exists a weak solution to the Navier-Stokes system (1)–(3) on $[t_0, T]$ such that $u \in L^2 H^1 \cap L^\infty L^2$ and

$$\int_{\mathbb{R}^3} |u(t)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^3} |u(s)|^2 \,\mathrm{d}x - 2\nu \int_s^t \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}\tau \tag{6}$$

for all $t \in (t_0, T]$ and a.e. $s \in [t_0, t]$ including $s = t_0$. Moreover, strong solutions to (1)–(3) satisfy the corresponding energy equality:

$$\int_{\mathbb{R}^3} |u(t)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} |u_0|^2 \,\mathrm{d}x = -2\nu \int_{t_0}^t \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}\tau. \tag{7}$$

Since the introduction of these Leray-Hopf solutions, it has been notoriously difficult to establish energy equality for all such solutions. It is of obvious mathematical interest to resolve this question; energy equality is clearly a prerequisite for regularity, and can be a first step in proving conditional regularity results, c.f. [46]. But beyond purely mathematical interest, the question of energy balance is motivated on physical grounds as well: Knowing (7) rather than (6) rules out the presence of anomalous energy dissipation due to the nonlinearity, a phenomenon normally associated with weak solutions of the inviscid Euler system in the framework of the so-called Onsager conjecture [43] (more on this below). This allows, as stipulated, for example, in the text of Frisch [26], to precisely equate the classical Kolmogorov residual energy anomaly $\epsilon_\nu \to \epsilon_0$ of a turbulent flow to the Onsager dissipation in the limit of vanishing viscosity.

In the first part of this thesis, we will examine the following local version of the energy equality (7), obtained by trivial manipulations of (1) and (2), and automatically valid for all nonnegative $\sigma \in C_0^\infty(\Omega \times [t_0, t_1])$ and all $t \in [t_0, t_1)$:

$$\begin{aligned}\int_\Omega |u(t)|^2 \sigma(t)\,\mathrm{d}x = \int_\Omega |u(t_0)|^2 \sigma(t_0)\,\mathrm{d}x - 2\nu \int_{t_0}^t \int_\Omega |\nabla u|^2 \sigma\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{t_0}^t \int_\Omega |u|^2 (\partial_t \sigma + \nu \Delta \sigma)\,\mathrm{d}x\ \mathrm{d}\tau + \int_{t_0}^t \int_\Omega (|u|^2 + 2p) u \cdot \nabla \sigma\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{8}$$

We concern ourselves first with the question of whether (8) continues to hold when $t = t_1$. The answer is clearly affirmative if u remains regular at $t = t_1$; therefore we assume without loss of generality that u does in fact lose regularity at time $t = t_1$. In this case we can legitimately claim only that the local energy *inequality* holds at $t = t_1$ for all non-negative test-functions σ. That this holds is a simple consequence of the weak lower semicontinuity of the L^2 norm and the regularity in time of σ:

$$\begin{aligned}\int_\Omega |u(t_1)|^2 \sigma(t_1)\,\mathrm{d}x &\le \lim_{t \to t_1^-} \int_\Omega |u(t)|^2 \sigma(t)\,\mathrm{d}x \\ &= \int_\Omega |u(t_0)|^2 \sigma(t_0)\,\mathrm{d}x - 2\nu \int_{t_0}^{t_1} \int_\Omega |\nabla u|^2 \sigma\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{t_0}^{t_1} \int_\Omega |u|^2 (\partial_t \sigma + \nu \Delta \sigma)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t_1} \int_\Omega (|u|^2 + 2p) u \cdot \nabla \sigma\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{9}$$

We ask, then: under what circumstances may we conclude that (8) survives the first blowup time, i.e. (8) rather than just (9) holds at $t = t_1$?

1.1.1. *Background on the Energy Equality.* To begin with, we give one sufficient condition for the energy equality (8) to hold at time $t = t_1$, which gives a partial answer to the question above, and which we will use extensively below. For U an open subset of $\mathbb{R}^n$ and I a relatively open interval in $[t_0, T]$, we define the "Onsager regular" function class $\mathcal{OR}(\mathbb{R}^n \times I)$ and its local-in-space version $\mathcal{OR}(U \times I)$ as follows:

$$\mathcal{OR}(\mathbb{R}^n \times I) = \{f \in L^3(\mathbb{R}^n \times I) : \lim_{y \to 0} \frac{1}{|y|} \int_I \int_{\mathbb{R}^n} |f(x+y,t) - f(x,t)|^3\,\mathrm{d}x\,\mathrm{d}t = 0\}.$$

$$\mathcal{OR}(U \times I) = \{f \in L^3(U \times I) : \sigma f \in \mathcal{OR}(\mathbb{R}^n \times I),\ \text{for all } \sigma \in C_0^\infty(U)\}.$$

We sometimes omit parts of the notation for these spaces when there is no risk of sacrificing clarity. We say that a point (x_0, t_0) is Onsager regular if there exists an open set U and a relatively open interval I such that $(x_0, t_0) \in \mathcal{OR}(U \times I)$. Further, an open set $D \subset \mathbb{R}^n \times [t_0, T]$ is Onsager regular if it consists entirely of Onsager regular points. A point (x_0, t_0) is called Onsager singular if it is not Onsager regular, and the Onsager singular set Σ_{ons} is defined as the collection of all Onsager singular points. We have the following:

LEMMA 1.1 ([52]). *Let u be any weak solution of the Euler Equations on $[t_0, T]$ (not necessarily satisfying the regularity conditions described above), and let D be a regular set for u. Then whenever $\sigma \in C_0^\infty(D)$, we have*

$$\int |u(t)|^2\sigma(t)\,dx - \int |u(s)|^2\sigma(s)\,dx - \int_s^t \int |u|^2\partial_t\sigma\,dx\,d\tau = \int_s^t \int (|u|^2 + 2p)u\cdot\nabla\sigma\,dx\,d\tau \tag{10}$$

for all $s, t \in [t_0, T]$.

The proof of this Lemma extends without difficulty to Leray-Hopf solutions of the Navier-Stokes equations. Therefore, when we make the additional regularity assumptions (4) the relevant sufficient condition for (8) to survive the first blowup time $t = t_1$ is that $u \in \mathcal{OR}(\Omega \times [t_0, t_1])$.

The quoted result of [52] is a local critical version of a long list of preceding sufficient conditions documented in the extensive body of literature on the so-called Onsager conjecture. This conjecture, formulated in 1949 by Lars Onsager [43], states that $1/3$ is a critical smoothness in the sense that solutions to the Euler equations of smoothness greater than $1/3$ must conserve energy, and that solutions with smoothness less than $1/3$ might not. The positive direction of this conjecture was resolved in [12] by Constantin, E, and Titi and has been subsequently refined in, for example, Duchon, Robert [20], and Cheskidov, et al [7]. The other direction of the conjecture is not as relevant for the present work; however, we mention that it has been recently resolved by Isett [29], following a series of breakthrough ideas originating in topology by De Lellis and Székelyhidi [16, 17]. We do not attempt to give a detailed overview of this side of the subject, instead we refer the reader to [18] for more references and an extensive survey.

The question of energy equality has of course also been extensively studied for Leray-Hopf solutions of the 3-dimensional Navier-Stokes equations; we mention only a few results. Lions [40] and Ladyzhenskaja et al. [34] proved independently that such solutions satisfy the (global) energy equality under the additional assumption $u \in L^4(t_0, T; L^4)$ (see also [48], [49] for improvements in higher spatial dimensions). Actually, the L^4L^4 criterion is recoverable from that of [52] (and earlier results), since $L^4L^4 \cap L^2H^1 \subset \mathcal{OR}$ by interpolation. Later, Kukavica [33] proved sufficiency of the weaker but dimensionally equivalent criterion $p \in L^2(t_0, T; L^2)$. In [10], energy equality was proven for $u \in L^3D(A^{5/12})$ on a bounded domain; an extension to exterior domains was proved in [24]. (Here A denotes the Stokes operator.) In [46], Seregin and Šverák have proven energy equality (regularity, in fact) for suitable weak solutions whose associated pressure is bounded from below in some sense; their paper makes use of the low-dimensionality of the singular set for suitable weak solutions that is guaranteed by the celebrated Caffarelli-Kohn-Nirenberg Theorem [2]. A cutoff procedure was used in [50] to establish energy equality; there it was assumed that the singularity was confined to a curve $s \in C^{1/2}([0,T];\mathbb{R}^3)$ and additionally that $u \in L^3L^{9/2}$, $\nabla u \in L^3L^{9/5}((0,T)\times\mathbb{R}^3\backslash\mathrm{Graph}(s))_{\mathrm{loc}}$, the assumption dimensionally equivalent to the class $\mathcal{OR}$.

In the first part of the present work, we will make various integrability assumptions on our solution u and focus mainly on the first time of blowup. Of all the hypotheses on u that we consider, however, there is one that deserves special attention, namely the case where a solution u of the Navier-Stokes equations undergoes Type-I in time blowup. By this we mean that

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{\sqrt{t_1 - t}}, \tag{11}$$

for some constant $C > 0$. The Type-I assumption is of particular significance because of its invariance under the natural rescaling for the Navier-Stokes equations. See [47] for a discussion and further references. In fact, it is proved in [47] that axially symmetric solutions of the Navier-Stokes equations which experience Type-I in time blowup (and satisfy some natural technical assumptions) are regular, and therefore satisfy the energy equality.

1.1.2. *Definition of the Energy Measure.* Let us set $t_0 = -1$, $t_1 = 0$ for convenience. We refine our initial question somewhat:

Question 1.2. *Suppose u satisfies (1)–(5). Under what additional integrability assumptions on u may we conclude that (8) holds at $t = t_1 = 0$? If we cannot prove (8) for a given integrability assumption on u, how bad is the worst failure of (8) that we cannot eliminate under that same assumption?*

Note that the second part of this question presupposes that we can meaningfully and quantifiably distinguish between different instances of failure of (8). The tool that we use to justify the tacit assumption in this question (and address the question itself) is the energy measure $\mathcal{E}$, which we define to be the weak-$*$ limit of the measures $|u(t)|^2\,\mathrm{d}x \llcorner \Omega$ as $t \to 0^-$. (The symbol $\llcorner$ denotes restriction of a measure onto a given set.) To see that $\mathcal{E}$ is well-defined, note that $|u(t)|^2\,\mathrm{d}x$ is a bounded sequence of Radon measures, so that there exists a subsequence $|u(t_k)|^2\,\mathrm{d}x \llcorner \Omega$ which converges weak-$*$ to some Radon measure. Any two such measures agree as distributions by (8). Thus $\mathcal{E}$ is uniquely determined as a linear functional on $C_0(\Omega)$, by density of $C_0^\infty(\Omega)$ in $C_0(\Omega)$.

We can reinterpret (9) as saying that $\mathrm{d}\mathcal{E} \geq |u(0)|^2 \,\mathrm{d}x \llcorner \Omega$ in the sense of measures, with equality if and only if (8) is valid at $t = 0$. This fact clarifies how properties of the energy measure may be used to examine the possible failure of energy equality. In particular, we introduce the following two quantities, the lower local dimension $d(x, \mathcal{E})$ of $\mathcal{E}$ at $x \in \Omega$, and the concentration dimension D of $\mathcal{E}$ in Ω, defined respectively by

$$d(x, \mathcal{E}) = \liminf_{r \to 0} \frac{\ln \mathcal{E}(B_r(x))}{\ln r}, \tag{12}$$

$$D = \inf\{\dim_{\mathcal{H}}(S) : S \subset \Omega \text{ compact, and } \mathcal{E}(S) > 0\}, \tag{13}$$

with the convention that $D = n$ if the collection over which the infimum is taken is empty. Roughly speaking, lower values of $d(x, \mathcal{E})$ and D correspond to more severe energy concentration and thus more singular solutions u. The local dimension is a standard geometric measure theoretic quantity, see [**41**], while the concentration dimension was first introduced in [**53**], together with the energy measure itself. Originally, the energy measure was developed in conjunction with a study of energy concentration and drain phenomena, especially for the purpose of excluding certain cases of self-similar blowup.

1.1.3. *Overview of Main Results on the First Blowup Time.* Chapter 2 of this thesis, which deals with energy equality at the first possible blowup time, breaks into several pieces. In the first part, (Section 1), we give a systematic study of the energy measure. In particular, we discuss a connection between the Onsager singular set, the energy measure, and the local energy equality (8). Furthermore, we relate the concentration dimension of $\mathcal{E}$ to the phenomenon of concentration of energy, and we use basic tools of measure theory to understand the defect measure $\theta = \mathcal{E} - |u(0)|^2 \,\mathrm{d}x \llcorner \Omega$.

In the second part (Sections 2–3), we prove local energy bounds on u. Under the assumption that $u \in L^{q,*}(-1, 0; L^p)$ (and additional assumptions if $q = \infty$), our main results are stated in terms of bounds on the quantity

$$A(r, x_0) = \frac{1}{r^\beta} \sup_{-r^\alpha < t < 0} \int_{B_r(x_0)} |u(x,t)|^2 \,\mathrm{d}x, \tag{14}$$

where

$$\alpha = \frac{q}{q-1}\left(1 + \frac{n}{p}\right); \qquad \beta = \frac{q}{q-1}\left(n - \frac{2n}{p} - \frac{2+n}{q}\right).$$

The definitions of α and β are motivated by considerations of scale-invariance; see Section 2 below. (However, we note here that p and q must be such that $p \geq 3$ and $\beta \geq 0$.) We will prove that on any compact set $K \subset\subset \Omega$, there exists $R > 0$ and a constant C such that for any $r \in (0, R)$ and any $x_0 \in K$, we have $\sup\{A(r, x_0) : x_0 \in K,\ 0 < r < R\} \leq C$. If $q = \infty$, then the extra required hypothesis is either that $u \in L^\infty L^p$ (i.e. strong in time), or that u satisfies the explicit power-law bound $\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq C|t|^{-1/q}$. In the strong-in-time case, we will have the same conclusion as before; in the power-law bound case, we will prove that $\sup\{A(r, x_0) : x_0 \in \mathbb{R}^n,\ r > 0\} \leq C$. For the detailed statement of these bounds, see Section 2. Finally, we note that we can obtain similar bounds on $A(r, x_0)$ even if $p < 3$ in some cases, if u is a solution of the Navier-Stokes equations; see Section 3.

The uniform bounds on $A(r, x_0)$ just mentioned have several important consequences. First and foremost, we consider the special case $(p, q) = (\infty, 2)$ under the power-law assumption. If $n = 3$ and u solves the Navier-Stokes equations, then the hypothesis is precisely the Type-I condition (11). In this case, the bound $A(r, x_0) \leq C$ actually implies that u satisfies a certain Type-I *in space* condition, which is enough to guarantee energy equality. For details of this argument, see Section 3. For now, we record the end result as a Theorem:

THEOREM 1.3. *Let (u, p) be a solution to the Navier-Stokes initial value problem (1)–(3) which satisfies (5) and is regular on the time interval $[t_0, t_1) = [-1, 0)$. If u experiences Type-I in time blowup (11) at $t = 0$, then u still preserves the energy law on the closed interval $[-1, 0]$ including the first blowup time.*

The second consequence of our uniform bounds on $A(r, x_0)$ is that we obtain a uniform lower bound on the local dimension $d(x_0, \mathcal{E})$ of the energy measure for points $x_0 \in \Omega$ (or $x_0 \in \mathbb{R}^n$); namely $d(x_0, \mathcal{E}) \geq \beta$. This follows straightforwardly from the definitions of $A(r, x_0)$ and $d(x_0, \mathcal{E})$. Actually, we can say slightly more. We make a conclusion about not just the local dimension, but also about uniform boundedness of the upper β-density of the energy measure: $\Theta^{*\beta}(\mathcal{E}, x) = \limsup_{r \to 0} (2r)^{-\beta} \mathcal{E}(B_r(x))$, see [41]. This quantitatively expresses the fact that $\mathcal{E}$ behaves no worse than the Hausdorff β-dimensional measure under a given $L^q L^p$ condition on u. See Section 1.4.

By a covering argument, the bounds on $A(r, x_0)$ give the same lower bound for the concentration dimension as for the lower local dimension: $D \geq \beta$. For the details of this covering argument, see Section 1.4. However, if $u \in L^q L^p$ for some p and q such that $p < \infty$ and $\beta > 0$, this bound is demonstrably not optimal; in this case we give more refined bounds for D using different methods described below.

For the Euler equation, the improvement on D mentioned above is the following:

$$u \in L^q L^p(\Omega) \Rightarrow D \geq n - \frac{\frac{2}{q}}{1 - \frac{2}{p} - \frac{1}{q}}. \tag{15}$$

The latter is strictly larger than β if $p < \infty$ and $\beta > 0$. Consequently, we find that if the set of singular points at time $t = 0$ has dimension lower than stated in (15), then the energy of the solution is conserved; see Theorem 4.1 for the full statement. For the Navier-Stokes equations, the improvement is even more dramatic in view of the Caffarelli-Kohn-Nirenberg Theorem, [2], which tells us that the Hausdorff dimension d of the singularity set is at most 1 (see Section 1.3 for more details). Consequently, we argue that a certain range of $L^q L^p$ conditions implies energy equality. Theorem 4.4 states the full range of bounds and energy law criteria in this case.

1.1.4. *Additional Remarks on the First Blowup Time.* In the power-law assumption case, our bounds on $A(r, x_0)$ constitute an infinitesimal improvement over a result of [53]. In that paper, almost the same uniform bound $A(r, x_0) \leq C$ was proved, except that α and β are replaced by $\alpha + \delta$ and $\beta - \delta$ in that setting. In particular, the lower bounds we obtain on the local dimension are already known from [53], since the local dimension is insensitive to the presence of the δ's. On the other hand, removing the δ's is crucial in order to prove Theorem 1.3, which is only available with the sharper estimate. Key in obtaining the improved bound is a modified inequality for the pressure, which depends on u in a way that is essentially *local* in nature.

For all cases other than the power-law assumption, the bounds on $A(r, x_0)$ that we establish are, to the best of our knowledge, completely new. We use an iteration procedure reminiscent of the partial regularity theory for the Navier-Stokes equations, c.f. [45], [2]. Especially in the critical case $p = \infty$, our choice of the scaling α plays an important role in preserving smallness from step to step. This scaling is different, however, from the usual Navier-Stokes scaling (where $\alpha = 2$), except on the Prodi-Serrin line $\frac{3}{p} + \frac{2}{q} = 1$. Above this line (i.e. when $\frac{3}{p} + \frac{2}{q} > 1$), the dissipation is of lower order, according to our scaling. This partially explains why (when $p \geq 3$) our method gives the same bounds for the Euler and Navier-Stokes case, rather than an improved statement for Navier-Stokes due to the dissipation.

We make one more remark in order to bring attention to two recent works. First, in the paper [6] of Chae and Wolf, the authors consider Type-I blowup for the Euler equations, and it is proved that under the assumption

$$\sup_{-1<t<0}(-t)\|\nabla u(t)\|_{L^\infty}<\infty,$$

the energy measure has no atoms. Actually, their statement is more general than this, but we cite it in simplified form because their definition of the energy measure is slightly different from ours.

Second, our Theorem 1.3 has recently been generalized by Cheskidov and Luo [8]. In their paper they use completely different tools and prove that the energy equality survives a Type-I in time blowup even without the assumption that the solution is regular prior to the blowup. However, their work makes more direct use of the enstrophy than our approach, and therefore does not yield results in the inviscid case, whereas our approach does.

1.2. Some Extensions: Space-Time Singularities and Fractional Navier-Stokes.

In Chapter 3, we examine energy equality for the classical and fractional Navier-Stokes equations for more general singularity sets. That is, we consider singularity sets which are spread out in space-time. A tool like the energy measure is not obviously available in this setting, but we can still use essentially the same cutoff procedure as we use to bound the concentration dimension of the energy measure from below. For the classical Navier-Stokes equations, our method improves the Lions/Ladyzhenskaja L^4L^4 criterion if the singularity set has parabolic Hausdorff dimension strictly less than 1.

In light of recent extensions of the Caffarelli-Kohn-Nirenberg to the fractional Navier-Stokes system, c.f. [30, 61, 60, 11], it is also natural to see what our method tells us for those equations. We give our results on the fractional system in Section 2.

2. Inhomogeneous Models

2.1. The Inhomogeneous Incompressible Navier-Stokes Equations.

In the second part of the thesis, the first model we consider is the density-dependent incompressible Navier-Stokes system:

$$\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)-\mu\Delta u=-\nabla p+\rho f, \tag{16}$$

$$\partial_t\rho+\operatorname{div}(\rho u)=0, \tag{17}$$

$$\operatorname{div} u=0. \tag{18}$$

Here $u(x,t)$ represents the d-dimensional velocity, $f(x,t)$ is an external force (with values in $\mathbb{R}^d$), $p(x,t)$ is the pressure, $\rho(x,t)$ is the density, and μ is the viscosity coefficient (which we take to be constant). We consider (16)–(18) for $x \in \mathbb{T}^d$ and $t \geq 0$. It is known, see [**40, 28, 58**], that if u_0 is divergence-free and square-integrable, $\underline{\rho} \leq \rho_0 \leq \overline{\rho}$ for some positive constants $\underline{\rho}$ and $\overline{\rho}$, and if $f \in L^2([0,T]; L^2(\mathbb{T}^d))$, then there exists a Leray-Hopf type global weak solution to the system (ρ, u) such that $\underline{\rho} \leq \rho \leq \overline{\rho}$, $u \in L^2([0,T]; H^1(\mathbb{T}^d))$, and (ρ, u) satisfies the energy inequality

$$E(t) - E(0) \leq -\mu \int_0^t \|\nabla u\|^2_{L^2(\mathbb{T}^d)} \,\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} \rho u \cdot f \,\mathrm{d}x \,\mathrm{d}s, \quad \text{where} \quad E(s) = \frac{1}{2} \int_{\mathbb{T}^d \times \{s\}} \rho |u|^2 \,\mathrm{d}x. \tag{19}$$

Fluids with variable distribution of density arise in many physical contexts. In particular, they appear prominently in Rayleigh-Taylor mixing when a heavier layer fluid on top of lighter one gets mixed under the force of gravity, creating an non-homogeneous turbulent layer. Although an analogue of the classical Kolmogorov theory of turbulence for non-homogeneous fluids has not yet been developed, it appears to be evident that under proper self-similarity assumptions on the velocity increments $\delta u = u(r + \ell) - u(r)$ and density $\delta\rho$ a limited level of regularity would be expected of u and ρ in the limit of vanishing viscosity. Such regularity should allow for a residual amount of energy to be dissipated in the limit by analogy with the Kolmogorov's 0th law of turbulence, see [**26**]. Mathematical study of the question of what this critical regularity might be has been a subject of many recent publications centered around the so-called Onsager conjecture, discussed above. In this work we address the same question in the context of the full density-dependent forced system (16)–(18) with or without viscosity.

Let us recall that a weak solution to (16)–(18) is a triple $(\rho, u, p) \in L^\infty_{t,x} \times L^2_{t,x} \times \mathcal{D}'$ ($\mathcal{D}'$ is the space of distributions) such that for any triple of smooth test functions (η, ψ, γ), one has

$$\begin{aligned} \int_{\mathbb{T}^d} \rho u \cdot \psi(s) \,\mathrm{d}x \Big|_0^t - \int_0^t \int_{\mathbb{T}^d} \big(\rho u \cdot \partial_s \psi + (\rho u \otimes u) : \nabla \psi + p \operatorname{div} \psi\big) \,\mathrm{d}x \,\mathrm{d}s \\ = \mu \int_0^t \int_{\mathbb{T}^d} u \cdot \Delta \psi \,\mathrm{d}x \,\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} \rho f \cdot \psi \,\mathrm{d}x \,\mathrm{d}s, \end{aligned} \tag{20}$$

$$\int_{\mathbb{T}^d} \rho \eta(s) \,\mathrm{d}x \Big|_0^t = \int_0^t \int_{\mathbb{T}^d} \left(\rho \partial_s \eta + (\rho u \cdot \nabla)\eta\right) \,\mathrm{d}x \,\mathrm{d}s, \tag{21}$$

$$\int_{\mathbb{T}^d} u \cdot \nabla \gamma = 0. \tag{22}$$

In (20), we write $A : B$ for $\sum_{i,j=1}^d a_{ij}b_{ij}$, where $A = (a_{ij})$, $B = (b_{ij})$. If ρ and u are smooth, then using $\psi = u$ we readily obtain the energy balance relation:

$$E(t) - E(0) = -\mu \int_0^t \|\nabla u\|^2_{L^2(\mathbb{T}^d)}\, \mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} \rho u \cdot f \,\mathrm{d}x\, \mathrm{d}s. \tag{23}$$

In the context of weak solutions even in the class $u \in L^2H^1$, such a manipulation is not feasible; as usual, the problem is a lack of sufficient regularity to integrate by parts. This leaves room for additional mechanisms of energy dissipation due to the work of the nonlinear term. In the case $\mu = 0$, due to time reversibility the energy may also increase above the legitimate change resulting from the work of force. Our main result provides a sharp sufficient regularity condition on (ρ, u, p) to guarantee energy balance (23) to hold. We use Besov spaces to state our criteria as motivated by numerous previous studies on Onsager conjecture; see for example [**12, 7, 23**]. The definitions are standard and recalled in Section 1.2 of Chapter 4.

Theorem 2.1. *Let (ρ, u, p) be a weak solution to the density-dependent incompressible Navier-Stokes equations on $\mathbb{T}^d$, $d > 1$. Assume (ρ, u, p) satisfies*

$$u \in L^2([0,T]; H^1(\mathbb{T}^d)),\ 0 < \underline{\rho} \le \rho \le \overline{\rho} < \infty,\ \text{and } f \in L^2([0,T] \times \mathbb{T}^d), \tag{24}$$

$$\rho \in L^a([0,T]; B^{\frac{1}{3}}_{a,\infty}),\ u \in L^b([0,T]; B^{\frac{1}{3}}_{b,c_0}),\ p \in L^{\frac{b}{2}}([0,T]; B^{\frac{1}{3}}_{\frac{b}{2},\infty}),\quad \frac{1}{a} + \frac{3}{b} = 1,\ b \in [3,\infty]. \tag{25}$$

Then (ρ, u, p) satisfies the energy balance relation (23) on the time interval $[0,T]$.

The assumption on the pressure in (25) is in natural correspondence to the condition on velocity. In fact, it follows from the latter in the case of constant density (see Remark 1.4 of Chapter 4). Such a conclusion, however, cannot be made in the density-dependent case when the density has limited regularity as ours. In general the pressure is only known to exist as a distribution. We will see in the proof that the first line of assumptions (24) pertains to the control of the viscous and force terms in the local energy budget relation, while (25) is used to control anomalous flux due to the transport term. So, as a byproduct of the proof we obtain an energy conservation condition for the Euler equation.

Theorem 2.2. *Suppose (ρ, u, p) is a weak solution to the density-dependent incompressible Euler equations on $\mathbb{T}^d$ with zero force, the same set of assumptions (25), and $0 < \underline{\rho} \le \rho \le \overline{\rho} < \infty$. Then the energy is conserved in time.*

In the case when $b = 3$, we recover the result of [7] in the homogeneous case. However, for an inhomogeneous fluid, when $b = 3$ one must assume rather strong regularity of the density, namely $\rho \in B^{1/3}_{\infty,\infty} = C^{1/3}$, the usual Hölder class. It is shown in [22] that the Besov space $u \in B^{1/3}_{3,\infty}$ is sharp to control the energy flux in a homogeneous fluid. It is therefore not expected to be improved in the above results.

We also derive an extension to the density-dependent case of the classical Kármán-Howarth-Monin relation for the energy flux due to nonlinearity in the statistically homogeneous turbulence. It suggests that any of the conditions in the range of (25) arise naturally.

Finally, we note that an alternative version of our condition has appeared in [25]; the latter also applies to compressible Euler system. Pertaining to the incompressible case, the result claims energy conservation under the conditions

$$u \in B^{\alpha}_{p,\infty}((0,T) \times \mathbb{T}^d),\ \rho, \rho u \in B^{\beta}_{q,\infty}((0,T) \times \mathbb{T}^d),\ p \in L^{p^*}((0,T) \times \mathbb{T}^d),$$

where $1 \le p, q \le \infty$, $0 \le \alpha, \beta \le 1$, and $\frac{2}{p} + \frac{1}{q} = 1$, $\frac{1}{p} + \frac{1}{p^*} = 1$, $2\alpha + \beta > 1$. Notice that when $p = q = 3$ this gives a better base integrability, and with $\alpha = \beta = 1/3 + \epsilon$ it gives a weaker assumption on u and ρ in space. (The result is likely to be improved to $1/3$ with a vanishing c_0 assumption on the Littlewood-Paley pieces.) It also requires no regularity on the pressure. However, all of the above is assumed in time as well, and also on the product ρu. Therefore there is no direct inclusion on either side between the results of Theorems 2.1–2.2 and those of [25].

2.2. The Forced Euler-Alignment System.

In the final chapter, we consider another model, this time from collective dynamics. For some fixed $\alpha \in (0, 2)$, we treat the system

$$u_t + uu' = -\Lambda_\alpha(\rho u) + u\Lambda_\alpha \rho + f, \tag{26}$$

$$\rho_t + (\rho u)' = 0, \tag{27}$$

for $(x, t) \in \mathbb{T} \times [0, \infty)$. Here and below, we use primes $'$ to denote spatial derivatives. The torus $\mathbb{T}$ may have arbitrarily large period, but we work on the 2π-periodic torus for the sake of definiteness. Here $u = u(x, t)$ is the macroscopic velocity, $\rho = \rho(x, t)$ is the density (assumed nonnegative), and $f = f(x, t)$ is an external forcing term, assumed given. The operator $-\Lambda_\alpha$ is (up to a constant) the classical fractional Laplacian, with kernel

$$\phi_\alpha(z) = \sum_{k \in \mathbb{Z}} \frac{1}{|z + 2\pi k|^{1+\alpha}}, \qquad z \in [-\pi, \pi] \backslash \{0\}. \tag{28}$$

The action of $-\Lambda_\alpha$ on a sufficiently regular function $g:\mathbb{T}\to\mathbb{R}$ is given explicitly by

$$-\Lambda_\alpha g(x) = \int_{\mathbb{T}} (g(x+z)-g(x))\phi_\alpha(z)\,\mathrm{d}z = \int_{\mathbb{R}} (u(x+z)-u(x))\frac{\mathrm{d}z}{|z|^{1+\alpha}},$$

with the integral taken in the principal value sense. Let us temporarily consider the situation where $f\equiv 0$. In this setting, (26)–(27) becomes a special case of the system

$$u_t + uu' = \mathcal{L}_\phi(\rho u) - u\mathcal{L}_\phi\rho, \tag{29}$$

$$\rho_t + (\rho u)' = 0, \tag{30}$$

where $\mathcal{L}_\phi$ is given by

$$\mathcal{L}_\phi g(x) = \int_{\mathbb{T}} \phi(|x-y|)(g(y)-g(x))\,\mathrm{d}y.$$

The system (29)–(30) can in turn be interpreted as a macroscopic limit of the system

$$\begin{cases} \dot{x}_i &= v_i, \\ \dot{v}_i &= \frac{1}{N}\sum_{j=1}^N \phi(|x_i-x_j|)(v_j-v_i), \end{cases} \tag{31}$$

as $N\to\infty$. The system (31) is the celebrated Cucker-Smale model [15], which describes the positions x_i and velocities v_i of N agents whose binary interaction law depends on the radial influence function $\phi\geq 0$. We do not attempt an overview of the existing literature related to this model; rather we cite only a few results which are pertinent to the present context and refer the reader to, for example, [4] and references therein for a more substantial review. See also the Introduction of [19] for a useful and concise overview of some relevant results.

The system (31) and its long-time dynamics are associated with two especially notable phenomena. First, the velocities align to a constant (given by momentum divided by mass—both of these are conserved), and second, the system exhibits the so-called flocking phenomenon, whereby the agents gather into a crowd of finite diameter. However, it seems that in order for these characteristics to emerge, the kernel ϕ must involve some (non-physical) long-range interactions (c.f. [5], [15], [55], [59]). In order to emphasize rather the local interactions, one recent strategy has been to use a kernel ϕ which is singular at the origin, for example $\phi=\phi_\alpha$. This is the case we treat in the present work, at the macroscopic level of the system (26)–(27). See also [42] for another approach on the level of the agents.

Within the last few years, the system (26)–(27) has received a fair amount of attention; the papers [54], [55], [56], and [19] all give well-posedness results in classical regularity spaces in the case $f\equiv 0$. The second and third of these also show that classical solutions of the system exhibit flocking (see below for more details). In [32], the well-posedness of (26)–(27) is studied in the case where f is replaced by an attraction-repulsion interaction that depends on ρ and (the derivative of) a given kernel K.

In dimensions higher than 1, there are very few results on the analogues of the systems (26)–(27) or (29)–(30). He and Tadmor [27] have considered the analogue of (29)–(30) in two dimensions in the case of smooth kernels ϕ. And very recently, Shvydkoy [51] gave the first results to treat the analogue of the (forceless, singular) system (26)–(27) in arbitrary dimensions $n > 1$. The latter work proves a small data result for the full range $\alpha \in (0, 2)$.

The present work differs from all those cited above in that it treats well-posedness in low-regularity spaces, for an arbitrary external force f (which is sufficiently regular). Before giving more details on the results contained here and past work on the equations, however, we pause to give some definitions that will be helpful in this discussion.

2.2.1. *Auxiliary Quantities and Notation.* An interesting feature of the system (26)–(27) is that certain combinations of u and ρ formally satisfy conservation laws or transport equations. For example, define $e := u' - \Lambda_\alpha \rho$. Then the velocity equation can be rewritten as

$$u_t + ue = -\Lambda_\alpha(\rho u) + f. \tag{32}$$

Differentiating this, applying Λ_α to the density equation, and subtracting, we obtain an evolution equation for e:

$$e_t + (ue)' = f'. \tag{33}$$

Next, we define $q := e/\rho$. Taking the time derivative of q and using the density equation, we see that q satisfies

$$q_t + uq' = \frac{f'}{\rho}. \tag{34}$$

But then q' satisfies an equation like (33):

$$q'_t + (uq')' = (q_t + uq')' = \left(\frac{f'}{\rho}\right)'. \tag{35}$$

And finally, q'/ρ satisfies an equation like (34):

$$\left(\frac{q'}{\rho}\right)_t + u\left(\frac{q'}{\rho}\right)' = \frac{1}{\rho}\left(\frac{f'}{\rho}\right)'. \tag{36}$$

Obviously this process can be continued, but q'/ρ is the highest order quantity of this type that we make use of below.

We also set notation for the (conserved) mass $\mathcal{M}$ associated to the system:

$$\mathcal{M} = \int_{\mathbb{T}} \rho \, \mathrm{d}x.$$

2.2.2. *Weak, Strong, and Regular Solutions.* We now define several notions of a solution to (26)–(27). For weaker notions of a solution, we include e as part of our definitions. To write down a weak formulation, it is helpful to use (32) instead of the original velocity equation. We also include a weak form of the definition of e.

Definition 2.3. Let $(u_0, \rho_0, e_0) \in L^\infty \times L^\infty \times L^\infty$ satisfy the compatibility condition

$$\int_{\mathbb{T}} e_0\varphi + u_0\varphi' + \rho_0\Lambda_\alpha\varphi \, \mathrm{d}x = 0, \qquad \text{for all } \varphi \in C^\infty(\mathbb{T}) \tag{37}$$

We say that (u, ρ, e) is a *weak solution* on the time interval $[0, T]$, satisfying the initial data (u_0, ρ_0, e_0), if

- u, ρ, ρ^{-1}, and e all belong to $L^\infty(0, T; L^\infty)$.
- u and ρ belong to $L^2(0, T; H^{\alpha/2})$.
- (u, ρ, e) satisfies the following weak form of (26)–(27), for all $\varphi \in C^\infty(\mathbb{T} \times [0, T])$ and a.e. $t \in [0, T]$:

$$\int_{\mathbb{T}} u(t)\varphi(t) \, \mathrm{d}x - \int_{\mathbb{T}} u_0\varphi(0) \, \mathrm{d}x - \int_0^t \int_{\mathbb{T}} u\partial_t\varphi \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_{\mathbb{T}} -ue\varphi - \rho u\Lambda_\alpha\varphi + f\varphi \, \mathrm{d}x \, \mathrm{d}s, \tag{38}$$

$$\int_{\mathbb{T}} \rho(t)\varphi(t) \, \mathrm{d}x - \int_{\mathbb{T}} \rho_0\varphi(0) \, \mathrm{d}x - \int_0^t \int_{\mathbb{T}} \rho\partial_t\varphi \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_{\mathbb{T}} \rho u\varphi' \, \mathrm{d}x \, \mathrm{d}s. \tag{39}$$

- The compatibility condition (37) propagates in time, in the sense that

$$\int_0^T \int_{\mathbb{T}} e\varphi + u\varphi' + \rho\Lambda_\alpha\varphi \, \mathrm{d}x \, \mathrm{d}s = 0, \qquad \text{for all } \varphi \in C^\infty(\mathbb{T} \times [0, T]). \tag{40}$$

We say that (u, ρ, e) is a weak solution on $[0, T)$ $(0 < T \leq \infty)$ if (u, ρ, e) is a weak solution on $[0, T']$ for all $T' \in (0, T)$.

Definition 2.4. Let $(u_0, \rho_0, e_0) \in W^{1,\infty} \times W^{1,\infty} \times W^{1,\infty}$ satisfy the compatibility condition (37). We say that (u, ρ, e) is a *strong solution* on the time interval $[0, T]$, satisfying the initial data (u_0, ρ_0, e_0), if (u,ρ,e) is a weak solution such that u, ρ, and e all belong to $L^\infty(0,T;W^{1,\infty})$. We say that (u,ρ,e) is a strong solution on $[0,T)$ $(0<T\le\infty)$ if (u,ρ,e) is a strong solution on $[0,T']$ for all $T'\in(0,T)$.

The quantity e need not play a role in the definition of higher-regularity solutions, though it does remain an important quantity for the analysis of such solutions.

Definition 2.5. We say that (u,ρ) is a *regular solution* of (26)–(27), on the time interval $[0,T]$, satisfying the initial condition $(u_0,\rho_0)\in H^4\times H^{3+\alpha}$, if

- (u,ρ) satisfies (26)–(27) in the classical sense.
- $(u,\rho)\in C([0,T];H^4\times H^{3+\alpha})$,
- $\rho(x,t)\ge c$ for some $c>0$, for all $(x,t)\in\mathbb{T}\times[0,T]$, and
- $u(0)=u_0$ and $\rho(0)=\rho_0$ a.e. in $\mathbb{T}$.

We say that (u,ρ) is a regular solution on the time interval $[0,T)$ $(0<T\le\infty)$ if (u,ρ) is a regular solution on $[0,T']$ for all $T'\in(0,T)$.

2.2.3. *Alignment and Flocking.* In the discussion above, we have already mentioned the phenomena of alignment and flocking in the context of agent-based models. We now give the more precise definitions associated to the macroscopic system (26)–(27).

Definition 2.6. A solution (u,ρ,e) is said to experience *alignment* if the diameter of the velocities tends to zero as $t\to\infty$:

$$A(t):=\operatorname*{ess\,sup}_{x,y\in\mathbb{T}}|u(x,t)-u(y,t)|\to 0,\qquad \text{as } t\to\infty.$$

We say that the solution undergoes *fast alignment* if the convergence $A(t)\to 0$ is exponentially fast.

Definition 2.7. We define the set of *flocking states* $\mathcal{F}$ as in [55]:

$$\mathcal{F}:=\{(\overline{u},\overline{\rho}):\overline{u}\text{ is constant, }\ \overline{\rho}(x,t)=\rho_\infty(x-t\overline{u})\}.$$

We say that (u,ρ) converges to a flocking state $(\overline{u},\overline{\rho})\in\mathcal{F}$ in the space $X\times Y$ if

$$\|u(\cdot,t)-\overline{u}\|_X+\|\rho(\cdot,t)-\overline{\rho}(\cdot,t)\|_Y\to 0\qquad\text{as } t\to\infty. \tag{41}$$

And furthermore, we say that (u,ρ) experiences *fast flocking* in $X\times Y$ if the convergence rate of (41) is exponentially fast.

2.2.4. *Previous and Present Results.* The existing well-posedness theory for the system (26)–(27) mostly concerns the special case $f \equiv 0$. In [54], a priori estimates implying the local existence of regular solutions for the case $1 \leq \alpha < 2$ and $f \equiv 0$ were given. With local existence in hand, the authors refined these estimates and proved global in time existence of solutions. Further refinements were given in [55], which proves that regular solutions undergo fast alignment and converge exponentially quickly to a flocking state in $H^3 \times H^{3-\epsilon}$, for any $\epsilon > 0$.

The first treatment of well-posedness for the case $0 < \alpha < 1$, $f \equiv 0$ appeared in [19]. Later, the results of [54], [55] were extended to the case $0 < \alpha < 1$ in [56], which obtains local and global existence as a byproduct of the proof of flocking and fast alignment. In [32] the authors prove results analogous to those of [19], in the presence of an additional force of the form $-\partial_x K * \rho$.

The techniques used in the two groups of papers [54], [55], [56] and [19], [32] are quite different from one another. The first group uses regularity theory for fractional parabolic equations and relies extensively on the nonlinear maximum principle of [14]; the nonlinear maximum principle was originally used to prove a well-posedness result for the critical SQG equation. The papers [19], [32] use instead the modulus of continuity method, which has also been used to treat (for example) the SQG equation in [31].

In this work, we consider the case of a general external force f which is regular enough for our computations to go through. In principle, we could include the force considered in [32] in our existence results, but to do so we would need to repeat several of the arguments from [32], rendering the inclusion somewhat artificial. The problem is that the density in [32] is not obviously bounded a priori, and in fact may grow exponentially in time. Since their force in turn depends on the density, we would need to make quite a few adjustments to our arguments (and intermediate conclusions) in order to include this case. To simplify our arguments, we assume that our force f and a sufficient number of its spatial derivatives are uniformly bounded in $\mathbb{T} \times [0, \infty)$. In particular, our arguments ultimately do not apply to the force considered in [32]. Rather, we extend the result of [54], [55], and [56], concerning existence of regular solutions, to the forced case (for nice enough f). We construct both weak and strong solutions as limits of regular solutions. These solutions are slightly more regular (in the Hölder sense) than one can conclude a priori using only the definitions of weak and strong solutions. However, the strong solutions we construct are in fact unique within their class; therefore, the regularity properties obtained by the method of construction are enjoyed by all strong solutions.

The results described thus far are all basically in the spirit of [54], [55], [56] (and, to a certain extent, [32]). We also include, however, a discussion of the natural energy laws of the system (26)–(27) which has no counterpart in any of the aforementioned papers. (The energy equalities are obvious for solutions in classical regularity spaces, so there was no need for such a discussion in those contexts.) We propose Onsager-type criteria that guarantee that these energy laws hold for weak solutions. We emphasize that these criteria are valid for *any* weak solutions, not just the ones we construct as limits of regular solutions. To treat the nonlinear term, we rely on the techniques of [38] (which in turn relies on [7], [12]). However, existing commutator estimates seem to be insufficient to treat the dissipation term directly, and we therefore devote a fair amount of effort to showing that the dissipation cannot cause any problems. It turns out that our Onsager-type criteria are satisfied for all weak solutions in the case where $\alpha \in [1,2)$. For smaller α one can prove the analogous energy *in*equalities for the constructed solutions, even if our Onsager-type criteria are not satisfied.

We state our main results in the following four theorems.

THEOREM 2.8. *Let $(u_0, \rho_0) \in H^4 \times H^{3+\alpha}$, with $\rho_0^{-1} \in L^\infty$, and assume that $f \in L^\infty(0,\infty; H^4)$. Then there exists a global-in-time regular solution of (26)–(27) associated to the initial data (u_0, ρ_0).*

THEOREM 2.9. *Let $(u_0, \rho_0, e_0) \in L^\infty \times L^\infty \times L^\infty$ satisfy the compatibility condition (37). Assume additionally that $\rho_0^{-1} \in L^\infty$ and that $f \in L^\infty(0,\infty; W^{1,\infty})$. Then there exists a global-in-time weak solution (u, ρ, e) associated to the initial data (u_0, ρ_0, e_0), which satisfies the following energy inequalities:*

$$\frac{1}{2}\int_{\mathbb{T}} \rho u^2(t)\, dx + \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}\, dy\, dx\, ds \leq \frac{1}{2}\int_{\mathbb{T}} \rho_0 u_0^2\, dx + \int_0^t \int_{\mathbb{T}} \rho u f\, dx\, ds \tag{42}$$

$$\int_{\mathbb{T}} \rho(t)^2\, dx + \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} (\rho(x)+\rho(y))\frac{|\rho(x)-\rho(y)|^2}{|x-y|^{1+\alpha}}\, dy\, dx\, ds \leq \int_{\mathbb{T}} \rho_0^2\, dx - \int_0^t \int_{\mathbb{T}} e\rho^2\, dx\, ds. \tag{43}$$

For this solution, u and ρ are Hölder continuous on compact sets of $\mathbb{T}\times(0,\infty)$ (with Hölder exponent depending on the compact set). Moreover, in the case where f is compactly supported in time, the velocity field u exhibits fast alignment to a constant.

THEOREM 2.10. *Let (u, ρ, e) be any weak solution on $[0,T]$, with $f \in L^2(\mathbb{T}\times[0,T])$. If $\alpha \in (0,1)$, we assume additionally that $u \in L^3(0,T; B^{1/3}_{3,c_0})$ and $\rho \in L^3(0,T; B^{1/3}_{3,\infty})$. If $\alpha \in [1,2)$, no additional assumption is needed. Then (u,ρ,e) satisfies the following energy equalities for a.e. $t \in [0,T]$:*

$$\frac{1}{2}\int_{\mathbb{T}} \rho u^2(t)\, dx + \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}\, dy\, dx\, ds = \frac{1}{2}\int_{\mathbb{T}} \rho_0 u_0^2\, dx + \int_0^t \int_{\mathbb{T}} \rho u f\, dx\, ds \tag{44}$$

$$\int_{\mathbb{T}} \rho(t)^2\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} (\rho(x)+\rho(y))\frac{|\rho(x)-\rho(y)|^2}{|x-y|^{1+\alpha}}\,dy\,dx\,ds = \int_{\mathbb{T}} \rho_0^2\,dx - \int_0^t \int_{\mathbb{T}} e\rho^2\,dx\,ds. \tag{45}$$

THEOREM 2.11. *Let $(u_0, \rho_0, e_0) \in W^{1,\infty} \times W^{1,\infty} \times W^{1,\infty}$ satisfy the compatibility condition (37), and assume additionally that $\rho_0^{-1} \in L^\infty$, that $f \in L^\infty(0,\infty; W^{2,\infty})$, and that $\alpha \neq 1$. Then there exists a unique global-in-time strong solution (u, ρ, e) associated to the initial data (u_0, ρ_0, e_0). For this solution, u and ρ belong to $C_{\text{loc}}((0,\infty); C^1)$. Moreover, in the case where f is compactly supported in time, (u, ρ) undergoes fast flocking in $W^{1,\infty} \times L^\infty$ to some $(\overline{u}, \overline{\rho}) \in \mathcal{F}$. In fact, the convergence occurs in $C^{1,\epsilon} \times C^{1,\epsilon}$ for some $\epsilon > 0$ (though we make no statement on the rate of convergence in this space). Finally, even in the case $\alpha = 1$, any strong solution is unique if it exists.*

REMARK 2.12. It may seem somewhat strange that the case $\alpha = 1$ should be excluded from our existence result on strong solutions. The reason why our method does not yield existence of strong solutions in this case will be clear later from the estimates in Section 4.1; for now we simply note that the exclusion of the case $\alpha = 1$ has some precedent. In fact, the arguments of [54], [56] prove existence of solutions in $H^3 \times H^{2+\alpha}$ for all $\alpha \in (0,2)\backslash\{1\}$; going up one more derivative is necessary only for $\alpha = 1$. It seems likely that our method could be applied to the case $\alpha = 1$ (or other α, for that matter) to yield solutions in $W^{2,\infty}$; however, we prefer to leave this case for future research.

The outline of Chapter 5 is as follows. In Section 1, we prove a priori bounds at the L^∞ level for regular solutions. Once these are established, the rest of the proof of Theorem 2.8 follows the same steps as in [54], [56], with trivial modifications. We keep careful track of the dependencies of constants involved in these a priori bounds, in order to prove that they survive the limiting procedure we use to construct weak solutions in Section 2. Some additional bounds beyond those required for Theorem 2.8 are needed to pass to the limit; we also include these in Section 1. In Section 3, we prove Theorem 2.10. In Section 4, we continue proving bounds on regular solutions at the $W^{1,\infty}$ level, and in Section 5 we use these bounds to prove the existence of strong solutions when $\alpha \neq 1$. Section 5 also contains the proof of the rest of Theorem 2.11.

CHAPTER 2

Energy Equality at the First Blowup Time: Considerations for the Classical Euler and Navier-Stokes Equations[1]

1. The Energy Measure

1.1. Energy Measure and the Local Energy Equality. Let us look at the classical Lebesgue decomposition of the energy measure relative to $\mathrm{d}x \llcorner \Omega$:

$$\mathrm{d}\mathcal{E} = f\, \mathrm{d}x \llcorner \Omega + \mathrm{d}\mu, \quad \mathrm{d}x \perp \mathrm{d}\mu.$$

According to the discussion in the Introduction, the defect measure $\mathrm{d}\theta = \mathrm{d}\mathcal{E} - |u(0)|^2\, \mathrm{d}x \llcorner \Omega$ is nonnegative. Therefore we have $f \geq |u(0)|^2$ a.e., and $\mathrm{d}\mu \geq 0$ in general. In light of this, it is natural to attribute a possible failure of the local energy equality to two phenomena:

- Concentration: $\mathrm{d}\mu > 0$;
- Oscillation: $f > |u(0)|^2$.

It is easy to give one sufficient condition to rule out oscillation. We recall the definition of the Onsager singular set Σ_{ons} from the Introduction (and abuse notation throughout this chapter by identifying Σ_{ons} with its time-slice at $t = 0$). Let U be the largest open set in Ω for which $u \in \mathcal{OR}(U)$ (i.e., let U be the union of all such sets). Define the set of Onsager singular points by $\Sigma_{ons} = \Omega \backslash U$; this set is relatively closed in Ω. According to the previous lemma, the defect measure θ is supported on Σ_{ons}. So, if $|\Sigma_{ons}| = 0$, then the defect measure is mutually singular to $\mathrm{d}x$. (Here and below, we use $|A|$ to denote the Lebesgue measure of a set $A \subset \mathbb{R}^n$.) Thus the above Lebesgue decomposition becomes

$$\mathrm{d}\mathcal{E} = |u(0)|^2\, \mathrm{d}x \llcorner \Omega + \mathrm{d}\theta, \quad \mathrm{d}x \perp \mathrm{d}\theta,$$

i.e. $f = |u(0)|^2$ and $\mu = \theta$. The size of the set Σ_{ons} is related to the phenomenon of intermittency in fully developed turbulence and is out of scope of this present work.

[1]This chapter is largely excerpted from:
[37] T. M. Leslie and R. Shvydkoy. The Energy Measure for the Euler and Navier-Stokes Equations. *Arch. Ration. Mech. Anal. (to appear).*

1.2. Concentration Dimension. Generally, the smaller the set on which $\mathcal{E}$ is concentrated, the more severe we view the blowup. The concentration dimension assigns a numerical value to the concentration of the energy measure, namely the smallest Hausdorff dimension of a set of positive $\mathcal{E}$-measure:

$$D = \inf\{\dim_{\mathcal{H}}(S) : S \subset \Omega \text{ compact, and } \mathcal{E}(S) > 0\}.$$

We recall that if the above family of sets is empty, then we set $D = n$ by convention. This situation occurs when the energy is drained from the domain Ω, a scenario not excluded at the time of blowup. Generally, if $D = n$ one might say that the measure has no lower dimensional concentration. This, however, does not rule out the presence of a singular component $\mathrm{d}\mu$. It can still be concentrated on a set of Lebesgue measure zero, but of dimension 3. If, however, we have $D < n$, then the concentration pertains to the singular part $\mathrm{d}\mu$ only, since obviously $f\,\mathrm{d}x$ vanishes on any subset of Ω with dimension less than n. It is in the case $D < n$ only where we can properly address the concentration issue.

By analogy with the set of Onsager-singular points, which encompasses the maximal set on which the energy equality may fail, we introduce a corresponding set of singularities which encompasses any possible concentration of the energy measure. Again, we define a set Σ as complementary to

$$\mathbb{R}^n \backslash \Sigma = \{x \in \mathbb{R}^n : \exists \text{ open } U, x \in U, \exists p > 2, \exists \epsilon > 0 : u \in L^\infty(-\epsilon, 0; L^p(U))\}. \tag{46}$$

Clearly $\mathbb{R}^n \backslash \Sigma$ is open, so Σ is closed.

LEMMA 1.1. *The energy measure* $\mathrm{d}\mathcal{E}$ *is absolutely continuous with respect to Lebesgue measure* $\mathrm{d}x$ *on* $\Omega \backslash \Sigma$. *Hence,* $\operatorname{supp} \mathrm{d}\mu \subset \Sigma$.

PROOF. Let $A \subset \Omega \backslash \Sigma$ be a set of Lebesgue measure zero. We need to show $\mathcal{E}(A) = 0$. By considering the sequence $A \cap \{x \in A : \operatorname{dist}(x, \partial\Omega \backslash \Sigma) > 1/k\}$ we may assume without loss of generality that A has a positive distance to $\partial\Omega \backslash \Sigma$. Moreover, by inner regularity we may assume that A is compact. Thus, A is compactly embedded into $\Omega \backslash \Sigma$. For every point $x \in A$ we can find an open neighborhood U_x, $\epsilon_x > 0$ and $p_x > 2$ as in the definition (46). By compactness there is a finite subcover, and hence we can pick the smallest of all ϵ's and p's to find a compactly embedded open neighborhood U of A such that $u \in L^\infty(-\epsilon, 0; L^p(U))$. We further reduce U to $V \subset U$ (still containing A) with $|V| < \delta$. Find a function $\sigma \in C_0(V)$, $0 \le \sigma \le 1$, and $\sigma|_A = 1$. Then

$$\mathcal{E}(A) \le \int \sigma \,\mathrm{d}\mathcal{E} = \lim_{t \to 0} \int |u(t)|^2 \sigma \,\mathrm{d}x \le \|u\|^2_{L^\infty(-\epsilon,0;L^p(U))} |V|^{\frac{p-2}{p}} < C\delta^{\frac{p-2}{p}}.$$

This shows that $\mathcal{E}(A) = 0$, and the lemma follows. $\square$

Let us note that since in general there is no relationship between the sets Σ and Σ_{ons}, we cannot claim that the local energy equality necessarily holds on the set $\Omega\backslash\Sigma$. Instead, we only rule out the concentration phenomenon, while oscillation may still occur. To summarize, the lemma claims

$$\mathrm{d}\mathcal{E} \llcorner (\Omega\backslash\Sigma) = f\,\mathrm{d}x \llcorner (\Omega\backslash\Sigma), \quad f \geq |u(0)|^2.$$

Lemma 1.1 has two additional immediate consequences. By definition of Σ, we have $\Sigma = \emptyset$ if $u \in L^\infty(-1,0;L^p(\Omega))$ for some $p > 2$, in which case μ is trivial (since $\operatorname{supp}\mu \subset \Sigma$ by Lemma 1.1). This rules out any concentration and allows us to conclude that $D = n$ in this case. The second consequence is that we may take the infimum in the definition of D over sets that are contained in Σ, rather than over general compact subsets of Ω. We record these two corollaries for reference:

COROLLARY 1.2. *If $u \in L^\infty(-1,0;L^p(\Omega))$ for some $p > 2$, then the energy measure suffers no concentration. That is, $\Sigma = \emptyset$, and therefore $D = n$.*

COROLLARY 1.3. *The dimension of concentration is equal to*

$$D = \inf\{\dim_{\mathcal{H}}(S) : S \subset \Omega \cap \Sigma \text{ compact, and } \mathcal{E}(S) > 0\}.$$

PROOF. Let us denote the new dimension D' for reference. Clearly, $D' \geq D$, since the new infimum is taken over a smaller family. Let us address the case $D = n$ separately. In this case $D' = n$, either by convention (if no sets S are available), or because $D' \geq D$. If $D < n$, we can pick $\epsilon > 0$ and a set S with $\dim_{\mathcal{H}}(S) \leq D + \epsilon < n$ such that $\mathcal{E}(S) > 0$. However, $|S| = 0$, and hence by Lemma 1.1 we have $\mathcal{E}(S\backslash\Sigma) = 0$. We can then replace S with $S \cap \Sigma$ without changing its $\mathcal{E}$-measure. But then $\dim_{\mathcal{H}}(S \cap \Sigma) \leq \dim_{\mathcal{H}}(S)$, while $\mathcal{E}(S \cap \Sigma) > 0$; thus $D' \leq \dim_{\mathcal{H}}(S) < D + \epsilon$. This proves the statement. $\square$

1.3. Navier-Stokes and Suitable Weak Solutions. In the case of the NSE, the partial regularity theory of Caffarelli, Kohn, and Nirenberg [2] allows us to restrict attention to lower-dimensional singular sets at time $t = 0$, even though we have not assumed that our solution u is suitable. Indeed, assume (u,p) satisfies (1)–(5), and assume u be a Leray-Hopf weak solution on $[-1,0]$ (which is regular on $[-1,0)$). Let $(\widetilde{u},\widetilde{p})$ be a suitable weak solution on $[-\frac{1}{2},\infty)$, with initial data $\widetilde{u}(-\frac{1}{2}) := u(-\frac{1}{2})$ and pressure $\widetilde{p} = R_i R_j(u_i u_j)$. Assume without loss of generality that $\widetilde{u}$ is weakly continuous in time; this can be achieved by modifying $\widetilde{u}$ on a Lebesgue null set of times. Then By weak-strong uniqueness, we have $(u,p) = (\widetilde{u},\widetilde{p})$ on $[-\frac{1}{2},0)$. Then, by weak continuity in time, we have $u(0) = \widetilde{u}(0)$. Since $\widetilde{u}$ is suitable, the Caffarelli-Kohn-Nirenberg Theorem implies that the parabolic 1-dimensional Hausdorff measure of $\widetilde{S}$ is 0, where $\widetilde{S}$ is the set of singular points of $\widetilde{u}$. Note that by the Prodi-Serrin criterion, $\widetilde{u}$ is C^∞ in the spatial variables on the complement of $\widetilde{S}$. This immediately implies that $\Sigma_{ons} \cup \Sigma \subset \widetilde{S} \cap \{t = 0\}$, and hence that the dimensions of Σ_{ons} and Σ are both at most 1. Combining the fact that $\dim_{\mathcal{H}}(\Sigma) \leq 1$ with Corollary 1.3, we may also conclude that $D \geq 1$. These facts will be used in Section 4.2.

1.4. Upper Densities, Local Dimension, and Concentration Dimension. As mentioned in the Introduction, the uniform bounds $A(r, x_0) \leq C$, or

$$\sup_{-r^\alpha < t < 0} \int_{B_r(x_0)} |u(x,t)|^2 \,\mathrm{d}x \leq C r^\beta, \tag{47}$$

imply slightly more than just a lower local dimension of at least β. Once we know that β is a lower bound, we can refine our geometric measure-theoretic statement by asserting the finiteness of the upper β-density of $\mathcal{E}$. Let us recall that for $0 \leq s < \infty$ and μ a Radon measure, the upper s-density of μ at $x \in \mathbb{R}^n$ is given by

$$\Theta^{*s}(\mu, x) = \limsup_{r \to 0} (2r)^{-s} \mu(B_r(x)). \tag{48}$$

If μ has finite s-density at x, then, roughly speaking, μ behaves near x like s-dimensional Hausdorff measure on an s-dimensional set: $\mu(B_r(x)) \lesssim r^s$.

Let us also give the covering argument alluded to in the Introduction, which relates bounds of the form (47) to lower bounds on D. Suppose we have $\mu(B_r(x)) \leq C(K) r^s$, for all $x \in K$ and all sufficiently small $r > 0$. Then for any set $S \subset \Omega$ with $\dim_{\mathcal{H}}(S) < s$ and for any compact subset $K \subset S$, we have $\mu(K) \leq \sum_i \mu(B_{r_i}(x_i)) \leq C \sum r_i^s \to 0$, as the cover closes on K. So, $\mu(K) = 0$, and hence $\mu(S) = 0$ by inner regularity. This shows that $D \geq s$.

Interestingly, the above argument does not proved a sharp bound on D from below, due to the fact that the covering argument using additivity of $\mathcal{E}$ is simply not optimal. One obtains a better estimate by examining the cover in its entirety via the local energy inequality, c.f. Section 4.

2. Local Dimension of the Energy measure

Let u be a classical solution to the Euler equation on time interval $[-1, 0)$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Suppose that $u \in L^{q,*}(-1, 0; L^p(\Omega))$ for some $p \geq 3$ and $q > 1$. Out of the classical two parameter family of scaling symmetries of the Euler equation there is one that leaves the L^qL^p-condition invariant, namely

$$u(x,t) \mapsto \lambda^{\alpha-1}u(\lambda x, \lambda^\alpha t), \text{ where } \alpha = \frac{q}{q-1}(1+\frac{n}{p}). \tag{49}$$

With the scaling (49) in mind, we state our main results for this section as follows.

PROPOSITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open domain and $K \subset \Omega$ a compact subset. Suppose u is a solution to the Euler equations satisfying $u \in L^{q,*}(-1,0;L^p(\Omega))$ with $3 \le p < \infty$, or satisfying $u \in L^q(-1,0;L^\infty(\Omega))$, and in both cases*

$$\frac{2n}{p} + \frac{2+n}{q} \le n. \tag{50}$$

Then there exist positive constants $R = R(n,p,q,u,K)$ and $C_0 = C_0(n,p,q,u,K)$ such that for all $r \in (0,R)$ we have

$$\sup_{-r^\alpha<t<0,\, x_0\in K} \int_{B_r(x_0)} |u(x,t)|^2 \,\mathrm{d}x \le C_0\, r^\beta, \tag{51}$$

where $\alpha = \frac{q}{q-1}\left(1+\frac{n}{p}\right)$ and $\beta = \frac{q}{q-1}\left(n - \frac{2n}{p} - \frac{2+n}{q}\right)$.

Note that (50) is precisely equivalent to the condition $\beta \ge 0$. Our other main result of this section is the following:

PROPOSITION 2.2. *Suppose u is a solution to the Euler equation which is regular on $[-1,0)$ and satisfies the bound $\|u(t,\cdot)\|_{L^\infty} \le c_0|t|^{-1/q}$, $\frac{n+2}{n} \le q$. Then there exists a constant $C = C(u,n,q)$ such that*

$$\sup_{-1<t<0,\, x_0\in\mathbb{R}^n} \int_{|x-x_0|<r} |u(x,t)|^2 \,\mathrm{d}x \le Cr^{n-\frac{2}{q-1}}. \tag{52}$$

We define several scale-invariant quantities relating to the scaling (49), which will be used in the proof of Proposition 2.1. First, denote $Q_r := B_r \times (-r^\alpha, 0)$, and let $(p)_r = \frac{1}{|B_r|}\int_{B_r} p(x)\,\mathrm{d}x$ denote the average of p on B_r. We define

$$A(r) = \frac{1}{r^\beta} \sup_{-r^\alpha<t<0} \int_{B_r} |u(x,t)|^2 \,\mathrm{d}x, \qquad \text{(energy)}$$

$$G(r) = \frac{1}{r^{\beta+1}} \int_{Q_r} |u(x,t)|^3 \,\mathrm{d}x\,\mathrm{d}t, \qquad \text{(flux)}$$

$$P(r) = \frac{1}{r^{\beta+1}} \int_{Q_r} |p-(p)_r||u| \,\mathrm{d}x\,\mathrm{d}t, \qquad \text{(pressure).}$$

REMARK 2.3. The inequality (52) can be expressed as $u \in L^\infty \mathcal{M}^{2,n-\frac{2}{q-1}}$, where $\mathcal{M}^{p,\lambda}$ is the Morrey space with integrability p and rate index λ. This observation plays an important role in the proof of Theorem 1.3; see the end of Section 3.

We devote the next three subsections to the proof of Proposition 2.1; Proposition 2.2 is proved in Section 2.4.

2.1. Essential estimates. The proof of Proposition 2.1 is executed by induction on scales according to the sequence of bounds $A(r) \to G(r) \to P(r) \to A(r/2)$. Although the details of the iteration procedure depend on which hypothesis is used, the proofs of both cases rely on common estimates on the quantities A, G, P. We start with an elementary L^3L^3 estimate away from the boundary.

CLAIM 2.4. Suppose $u \in L^{q,*}(-1,0;L^p(\Omega))$, where (p,q) satisfies $3 \le p \le \infty$ and (50). Then $u \in L^3L^3(\Omega)$ and $p \in L^{3/2}L^{3/2}(\Omega_\epsilon)$ for any $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \epsilon\}$.

PROOF. Note that $q > p/(p-2)$, by (50). By reducing q we can assume without loss of generality that $u \in L^qL^p$, i.e. strong in q, yet $q > p/(p-2)$ still holds. Then finiteness of $\|u\|_{L^3L^3(\Omega)}$ follows easily by interpolation and Hölder's inequality:

$$\int_0^T \|u\|_{L^3(\Omega)}^3 \,\mathrm{d}t \le \int_0^T \|u\|_{L^2(\Omega)}^{\frac{2(p-3)}{p-2}} \|u\|_{L^p(\Omega)}^{\frac{p}{p-2}} \,\mathrm{d}t \le \|u\|_{L^\infty L^2(\Omega)}^{\frac{2(p-3)}{p-2}} \|u\|_{L^qL^p(\Omega)}^{\frac{p}{p-2}} T^{1-\frac{p}{q(p-2)}}.$$

Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\eta \equiv 1$ on $\Omega_{\epsilon/2}$ and $\operatorname{supp}\eta \subset \Omega$. Let R_i, R_j denote the Riesz transforms on $\mathbb{R}^n$, and let

$$K_{ij}(y) = \frac{ny_iy_j - \delta_{ij}|y|^2}{n\omega_n|y|^{n+2}}$$

denote the kernel of R_iR_j. (Here δ_{ij} is the Kronecker delta, and ω_n is the volume of the unit ball in $\mathbb{R}^n$.) Since $p = R_iR_j(u_iu_j)$, we can use the boundedness of the Riesz transforms on $L^{3/2}$ to estimate $\|p\|_{L^{3/2}(\Omega_\epsilon)}$ as follows:

$$\begin{aligned}
\|p\|_{L^{3/2}(\Omega_\epsilon)} &\le \|R_iR_j(\eta u_iu_j)\|_{L^{3/2}(\Omega_\epsilon)} + \|R_iR_j((1-\eta)u_iu_j)\|_{L^{3/2}(\Omega_\epsilon)} \\
&\le C\|\eta u_iu_j\|_{L^{3/2}(\mathbb{R}^n)} + \left\| \int_{\mathbb{R}^n} K_{ij}(\cdot - y)(1-\eta(y))u_i(y)u_j(y)\,\mathrm{d}y \right\|_{L^{3/2}(\Omega_\epsilon)} \\
&\le C\|u\|_{L^3(\Omega)}^2 + C\left\| \int_{\Omega_{\epsilon/2}^c} \frac{|u(y)|^2}{|\cdot - y|^n}\,\mathrm{d}y \right\|_{L^{3/2}(\Omega_\epsilon)} \le C\|u\|_{L^3(\Omega)}^2 + C\epsilon^{-n}\|u\|_{L^2(\Omega)}^2|\Omega_\epsilon|^{2/3}.
\end{aligned}$$

From here it is obvious that taking the $L^{3/2}$ norm in time yields a finite quantity. $\square$

In what follows we assume without loss of generality that $0 \in \Omega$ and $r < \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$. We start with local energy equality

$$\int |u(t)|^2 \sigma(t)\,\mathrm{d}x = \int |u(s)|^2\sigma(s)\,\mathrm{d}x + \int_s^t \int |u|^2 \partial_t \sigma \,\mathrm{d}x\,\mathrm{d}\tau + \int_s^t \int (|u|^2 + 2p) u\cdot\nabla\sigma\,\mathrm{d}x\,\mathrm{d}\tau, \tag{53}$$

valid for any $\sigma \in C_0^\infty([-1,0]\times\Omega)$ and $-1 \le s \le t < 0$. Let $\psi : [0,\infty) \to \mathbb{R}$ be a smooth nonincreasing function such that $\psi(z) = 1$ for $z \le 1$, and $\psi(z) = 0$ for $z \ge 2$. Define $\phi_r(x,t) = \psi(|x|/r)\psi(|t|/r^\alpha)$, so that ϕ_r is 1 on Q_r and zero outside Q_{2r}. Putting $\sigma = \phi_r$ in the local energy equality yields

$$\sup_{-r^\alpha \le t \le 0} \int_{B_r} |u(t)|^2\,\mathrm{d}x \le \int |u|^2 |\partial_t\phi_r|\,\mathrm{d}x\,\mathrm{d}\tau + \int |u|^3|\nabla\phi_r|\,\mathrm{d}x\,\mathrm{d}\tau + 2\int |p-(p)_r||u||\nabla\phi_r|\,\mathrm{d}x\,\mathrm{d}\tau, \tag{54}$$

Note that

$$|\phi_r| \le \chi_{Q_{2r}}, \quad |\nabla\phi_r| \le Cr^{-1}\chi_{Q_{2r}}, \quad |\partial_t\phi_r| \le Cr^{-\alpha}\chi_{Q_{2r}}.$$

Evaluating the above at half the radius $r \to r/2$ and dividing through by r^β yields [2]

$$\begin{aligned}
A(r/2) &\le \frac{1}{r^{\alpha+\beta}} \int_{Q_r} |u|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{r^{1+\beta}}\int_{Q_r} |u|^3\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{r^{1+\beta}}\int_{Q_r}|p-(p)_r||u|\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{r^{2(1+\beta)/3}}{r^{\alpha+\beta}} \cdot (r^{n+\alpha})^{1/3}\left(\frac{1}{r^{1+\beta}}\int_{Q_r}|u|^3\,\mathrm{d}x\,\mathrm{d}\tau\right)^{2/3} + G(r) + P(r) \\
&\le G(r)^{2/3} + G(r) + P(r).
\end{aligned}$$

We have obtained

$$A(r/2) \le C[G(r)^{2/3} + G(r) + P(r)]. \tag{55}$$

Next, we establish a bound on the flux $G(r)$ in terms of $A(r)$.

$$\begin{aligned}
G(r) &= r^{-\beta-1}\int_{-r^\alpha}^0 \int_{B_r} |u(x,t)|^3\,\mathrm{d}x\,\mathrm{d}t \\
&\le r^{-\beta-1}\int_{-r^\alpha}^0 \left(\int_{B_r}|u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p-3}{p-2}} \left(\int_{B_r}|u(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p-2}}\,\mathrm{d}t \\
&\le r^{-\frac{\beta}{p-2}-1}\int_{-r^\alpha}^0 \left(\frac{1}{r^\beta}\int_{B_r}|u(x,t)|^2\,\mathrm{d}x\right)^{\frac{p-3}{p-2}} \left(\int_{B_r}|u(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p-2}}\,\mathrm{d}t \\
&\le r^{-\frac{\beta}{p-2}-1} A(r)^{\frac{p-3}{p-2}} \int_{-r^\alpha}^0 \left(\int_{B_r}|u(x,t)|^p\,\mathrm{d}x\right)^{\frac{1}{p-2}}\,\mathrm{d}t.
\end{aligned}$$

Denote $f(t) = \|u\|_{L^p}$. Under the time integral we have a quantity bounded by $f^{\frac{p}{p-2}}$. We know, however, that $f \in L^{q,*}$, and that under our assumption (50) we have $\frac{p}{p-2} < q$. This allows us to

[2]In all intermediate estimates we omit constants C which are independent of the radius.

extract the same asymptotic behavior in r as if f were in the strong L^q-space. Indeed,

$$\int_{-r^\alpha}^{0} f(t)^{\frac{p}{p-2}}\,\mathrm{d}t \le \frac{p}{p-2}\int_0^\infty \lambda^{\frac{2}{p-2}} \min\{|\{f>\lambda\}|, r^\alpha\}\,\mathrm{d}\lambda.$$

Using that $|\{f>\lambda\}| \le C/\lambda^q$ and splitting the integral, we obtain a bound by $r^{\alpha-\frac{\alpha p}{q(p-2)}}$. Adding this power of r to the already present power $-\frac{\beta}{p-2}-1$ gives a net power 0. Thus, we obtain

$$G(r) \le CA(r)^{\frac{p-3}{p-2}}. \tag{56}$$

The case $p<\infty$ has a clear advantage of yielding a power of A smaller than 1, while the case $p=\infty$ is critical. The latter can be handled in a similar way under the strong L^q in time condition: making the obvious adjustments for $p=\infty$ in the estimates on $G(r)$ above, we obtain the alternative bound

$$G(r) \le C\epsilon(r)A(r), \tag{57}$$

where

$$\epsilon(r) = \|u\|_{L^q(-r^\alpha,0;L^\infty)}. \tag{58}$$

The small parameter $\epsilon(r)$, which vanishes as $r\to 0$, allows us to compensate for the accrued constant C and close the circle of bounds $A(r)\to G(r)\to P(r)\to A(r/2)$ by induction. (See below for more details.)

In order to handle the weak case of $L^{q,*}L^\infty$, we need an explicit power bound in time, and we use a more subtle argument, c.f. Section 2.4.

Turning now to the pressure term, we recall the following local pressure inequality.

Lemma 2.5. *There exists an absolute constant c such that whenever $p\in L^{3/2}(B_\rho)$ and $-\Delta p = \partial_i\partial_j(u_iu_j)$ a.e. on B_ρ, then for any $r\in(0,\rho/2]$ we have*

$$\begin{aligned}\|p-(p)_r\|_{L^{3/2}(B_r)} \le c\|u\|^2_{L^3(B_{2r})} + cr^{\frac{2}{3}n+1}\int_{2r<|y|<\rho}\frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y \\ + c\frac{r^{\frac{2}{3}n+1}}{\rho^{\frac{2}{3}n+1}}\left(\int_{B_\rho}|u|^3+|p|^{3/2}\,\mathrm{d}y\right)^{\frac{2}{3}}.\end{aligned} \tag{59}$$

This inequality is proven for $n=3$ in in Lemma 15.12 in [44]. (Actually, the inequality is stated there with a time integral; (59) is obtained as an intermediate step in their proof.) The n-dimensional case is adaptable by simply replacing 3 with n in the appropriate places. We provide the details in Appendix A. Since we are considering only times prior to the first possible blowup, the hypotheses are valid for the pair $(u(t), p(t))$ in either the Euler or Navier-Stokes case. Therefore we may use this estimate for either set of equations. Note that subtracting off the average on B_{r_k} is crucial in order to obtain this local estimate, since p depends nonlocally on u.

Choose $R < \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$ and write $r_j = R/2^j$ for all $j \in \mathbb{N}$. Then using the local pressure inequality with $r = r_k$ and $\rho = R/2 = r_1$, we will obtain an estimate on $P(r_k)$. First, we split the integral in the second term into dyadic shells $r_{j+1} \leq |y| < r_j$ and estimate $|y|^{-n-1}$ pointwise on each of these shells before replacing the shells with balls. The result is

$$\|p - (p)_{r_k}\|_{L^{3/2}(B_{r_k})} \leq \|u\|^2_{L^3(B_{r_{k-1}})} + r_k^{\frac{2}{3}n+1} \sum_{j=1}^{k-2} r_j^{-(n+1)} \|u\|^2_{L^2(B_{r_j})} + \frac{1}{2^{(\frac{2}{3}n+1)k}} g(t),$$

where $g(t)$ is some function belonging to $L^{3/2}_t$, by Claim 2.4. We turn this into a bound on $P(r_k)$ as follows:

$$\begin{aligned}
P(r_k) \leq{}& \frac{1}{r_k^{1+\beta}} \int_{-r_k^\alpha}^0 \|u\|^2_{L^3(B_{r_{k-1}})} \|u\|_{L^3(B_{r_k})}\,\mathrm{d}t + r_k^{\frac{2}{3}n-\beta} \sum_{j=1}^{k-2} r_j^{-n-1} \int_{-r_k^\alpha}^0 \|u\|^2_{L^2(B_{r_j})} \|u\|_{L^3(B_{r_k})}\,\mathrm{d}t \\
&+ \frac{1}{2^{(\frac{2}{3}n+1)k}\ r_k^{1+\beta}} \int_{-r_k^\alpha}^0 g(t) \|u\|_{L^3(B_{r_k})}\,\mathrm{d}t \\
\leq{}& G(r_{k-1}) + r_k^{\frac{2}{3}n-\beta} \sum_{j=1}^{k-2} r_j^{-n-1} r_j^{\frac{n}{3}} \int_{-r_k^\alpha}^0 \|u\|_{L^3(B_{r_k})} \|u\|^2_{L^3(B_{r_j})}\,\mathrm{d}t + \frac{1}{2^{(\frac{2}{3}n+1)k}\ r_k^{2(1+\beta)/3}} G(r_k)^{1/3} \\
\leq{}& G(r_{k-1}) + r_k^{\frac{2}{3}n-\beta} \sum_{j=1}^{k-2} r_j^{-n-1} r_j^{\frac{n}{3}} r_j^{\beta+1} G(r_k)^{1/3} G(r_j)^{2/3} + \frac{1}{2^{(\frac{2}{3}n+\frac{1}{3}-\frac{2}{3}\beta)k}\ R^{2(1+\beta)/3}} G(r_k)^{1/3}
\end{aligned}$$

Using the fact that the powers of r_j in the sum add up to β, and the fact that $\frac{2}{3}n + \frac{1}{3} - \frac{2}{3}\beta > 0$, we have obtained the following:

$$P(r_k) \leq C \max\{G(r_1), \dots, G(r_k)\} + CR^{-2(1+\beta)/3} G(r_k)^{1/3}, \tag{60}$$

with C independent of k and R in the range $R < \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$.

2.2. Case $u \in L^{q,*}L^p$, $3 \leq p < \infty$. Let us fix an arbitrary initial radius $R < \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$, and set a constant $A > 1$ to be determined later but so that

$$A(R) < A.$$

This sets the initial step in the induction on $k = 0, 1, \dots$. Suppose we have

$$A(r_j) < A,$$

for all $j \leq k$. By (56) we have

$$G(r_j) \leq C_1 A^{1-\delta}, \quad \delta = \frac{1}{p-2},$$

for all $j \leq k$. In view of (60),

$$P(r_k) \leq C_2 A^{1-\delta},$$

where C_2 depends on the (fixed) constant R. (Note that we used that $A > 1$ to bound of $A^{(1-\delta)/3}$ by $A^{1-\delta}$.) Returning to (55), we obtain

$$A(r_{k+1}) \leq C_3 A^{1-\delta},$$

where still C_3 depends only on R. By setting $A > \max\{1, C_3^{1/\delta}\}$ initially, we have achieved the bound

$$A(r_{k+1}) < A,$$

which finishes the induction.

2.3. Case $u \in L^q L^\infty$. Let us fix $R < \frac{1}{2}\mathrm{dist}(0, \partial\Omega)$, $R < 1$, so that $\epsilon(R) < 1$. (Recall that $\epsilon(r)$ is defined by (58).) Let E denote the total energy $\|u\|_{L^2}^2$, which is independent of time on the interval $[-1, 0)$. We aim to show that the bound

$$A(r) < R^{-\beta} E + R^{-1-\beta} := A \tag{61}$$

propagates through scales for initial R sufficiently small. Clearly it holds for $r_0 = R$. Suppose we have

$$A(r_j) < A$$

for all $j \leq k$. Denote $\epsilon = \epsilon(R)$ for convenience. Since $\epsilon(r) \leq \epsilon$ for $r \leq R$, the bound (57) gives us

$$G(r_j) < C_1 \epsilon A$$

for all $j \leq k$ as well. The pressure bound (60) yields

$$P(r_k) < C_2 \epsilon A + C_3 \epsilon^{1/3} A^{1/3} R^{-2(1+\beta)/3}.$$

However, $R^{-2(1+\beta)/3} < A^{2/3}$, by (61). So,

$$P(r_k) < C_4 \epsilon^{1/3} A.$$

Returning to (55) again, we find that

$$A(r_{k+1}) < C_5(\epsilon^{2/3}A^{2/3} + \epsilon A + \epsilon^{1/3}A) \leq C_6\epsilon^{1/3}A,$$

where C_6 is independent of R. Picking R so that $C_6\epsilon^{1/3}(R) < 1$ finishes the induction.

2.4. Case $\|u(t)\|_{L^\infty} \leq c_0|t|^{-1/q}$. Assume $f(t) := \|u(t,\cdot)\|_{L^\infty} \leq \frac{c_0}{|t|^{1/q}}$, where $\frac{n+2}{n} \leq q$. In this case we disregard the subdomain Ω and work on the full space only. Let $\psi : [0,\infty) \to \mathbb{R}$ be a standard bump function—equal to 1 on $\{|x| \leq 1/2\}$ and supported inside $\{|x| < 1\}$. Denote $\phi_r(x) = \psi(|x|/r)$ and define

$$E(t,r) = \int |u(x,t)|^2\phi_r(x)\,\mathrm{d}x, \qquad E_k(t,r) = \frac{E(t,2^k r)}{2^{kn}}.$$

Note that by definition of $E(t,r)$ we have

$$\|u(t)\|^2_{L^2(B_r)} \leq E(t,2r) \leq \|u(t)\|^2_{L^2(B_{2r})}.$$

We have the following Lemma:

Lemma 2.6. *There exists a constant $C_0 = C_0(u,n,q)$ such that for any $s < t < 0$ and $r > 0$, we have*

$$E(t,r) \leq r^n f(s)^2 + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j\in\mathbb{N}} 2^{-j}E_j(\tau,r)\,\mathrm{d}\tau. \tag{62}$$

Before giving the proof, let us make a few remarks. Fix $r > 0$ and then set

$$t_0 = -r^{q'}, \quad C_1 = C_0c_0q', \tag{63}$$

where q' is the Hölder conjugate of q. Iteration of the Lemma will eventually allow us to prove the bound

$$E(t,r) \leq r^n f(t_0)^2 e^{C_1} + \frac{C_1^M}{M!}\|u\|^2_{L^\infty L^2}, \tag{64}$$

valid for $t \in (t_0, 0)$ and all $M \in \mathbb{N}$. The constant C_1 is chosen so that

$$\frac{C_0}{r}\int_{t_0}^0 f(\tau)\,\mathrm{d}\tau \leq C_0c_0q't_0^{1/q'}r^{-1} = C_1. \tag{65}$$

Taking $M \to \infty$ in (64), we obtain

$$E(t,r) \leq Cr^n f(t_0)^2 \leq Cr^{n-\frac{2}{q-1}}, \quad t \in (-r^{q'}, 0), \tag{66}$$

where the second inequality follows from the assumed bound on f and the definition of t_0. This is almost the desired bound in Proposition 2.2, except that the time interval does not extend to -1 in the negative direction. However, the bound $E(t,r) \le Cr^{n-\frac{2}{q-1}}$ follows automatically from the assumption $\|u(t)\|_{L^\infty} \le C|t|^{-1/q}$ when $t \in [-1, r^{q'}]$. Therefore in order to prove Proposition 2.2, it suffices to prove the Lemma (and the fact that the bound (64) follows).

PROOF OF LEMMA 2.6. Without loss of generality we assume $x_0 = 0$. We use our time-independent test function ϕ_r (53), dropping the subscript for convenience:

$$\int |u(x,t)|^2\phi(x)\,\mathrm{d}x = \int |u(x,s)|^2\phi(x)\,\mathrm{d}x + \int_s^t \int (|u|^2 + 2p)u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau. \tag{67}$$

Applying the obvious pointwise bounds, we get

$$\begin{aligned} E(t,r) &\le E(s,r) + \frac{C}{r}\int_s^t \int_{B_r} |u|^3 + |p-(p)_r||u|\,\mathrm{d}x\,\mathrm{d}\tau \\ &\le E(s,r) + \frac{C}{r}\int_s^t \|u\|^3_{L^3(B_r)} + \|p-(p)_r\|_{L^{3/2}(B_r)}\|u\|_{L^3(B_r)}\,\mathrm{d}\tau. \end{aligned}$$

Take $\rho \to \infty$ in (59); the last term tends to zero because $u \in L^3(-1,0;L^3(\mathbb{R}^n))$.

$$\|p-(p)_r\|_{L^{3/2}(B_r)} \le c\|u\|^2_{L^3(B_{2r})} + cr^{\frac{2n}{3}+1}\int_{2r<|y|<\infty} \frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y. \tag{68}$$

As before, we split the remaining integral into dyadic shells, estimate $|y|^{-n-1}$ on each shell, and then replace the shells with balls. We obtain

$$r^{\frac{2n}{3}+1}\int_{2r<|y|<\infty} \frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y \le r^{\frac{2n}{3}+1}\sum_{j=1}^{\infty}(2^j r)^{-n-1}\int_{B_{2^{j+1}r}} |u|^2\,\mathrm{d}y \le Cr^{-\frac{n}{3}}\sum_{j=3}^{\infty} 2^{-j}E_j(t,r).$$

So

$$\|p-(p)_r\|_{L^{3/2}(B_r)}\|u\|_{L^3(B_r)} \le c\|u\|^3_{L^3(B_{2r})} + Cr^{-\frac{n}{3}}\|u\|_{L^3(B_r)}\sum_{j=3}^{\infty} 2^{-j}E_j(t,r).$$

Next,

$$\|u(\tau)\|_{L^3(B_r)} \le Cf(\tau)^{\frac{1}{3}}E(\tau,2r)^{\frac{1}{3}} \le Cf(\tau)r^{\frac{n}{3}}.$$

Therefore

$$\begin{aligned} \int_s^t \|p-(p)_r\|_{L^{3/2}(B_r)}\|u\|_{L^3(B_r)}\,\mathrm{d}\tau &\le C\int_s^t f(\tau)E(\tau,4r) + f(\tau)\sum_{j=3}^{\infty} 2^{-j}E_j(\tau,r)\,\mathrm{d}\tau \\ &\le C\int_s^t f(\tau)\sum_{j=2}^{\infty} 2^{-j}E_j(\tau,r)\,\mathrm{d}\tau, \end{aligned}$$

and consequently,

$$E(t,r) \le E(s,r) + \frac{C}{r}\int_s^t \|u\|^3_{L^3(B_r)} + \|p-(p)_r\|_{L^{3/2}(B_r)}\|u\|_{L^3(B_r)}\,\mathrm{d}\tau$$
$$\le r^n f(s)^2 + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j=1}^{\infty} 2^{-j}E_j(\tau,r)\,\mathrm{d}\tau,$$

where C_0 is defined so that the last inequality holds. □

The final step in the proof of Proposition 2.2 consists of showing that iteration of (62) gives (64).

Proof of (64). Notice that the quantities $E_k(t,r)$ possess the following scaling property:

$$\frac{E_j(t,2^k r)}{2^{kn}} = E_{j+k}(t,r), \quad j,k\in\mathbb{N}\cup\{0\}.$$

We can therefore rescale the bound (62) as follows:

$$E_k(t,r) = \frac{E(t,2^k r)}{2^{kn}} \le r^n f(s)^2 + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j=1}^{\infty}2^{-j-k}\cdot\frac{E_j(\tau,2^k r))}{2^{kn}}\,\mathrm{d}\tau$$
$$\le r^n f(s)^2 + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j=k+1}^{\infty}2^{-j}E_j(\tau,r)\,\mathrm{d}\tau.$$

We don't actually use the fact that this last sum starts from $j=k+1$; for our purposes it suffices to use a rougher bound, where we trivially replace the sum above with a sum over all of $\mathbb{N}$:

$$E_k(t,r) \le r^n f(s)^2 + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j\in\mathbb{N}}^{\infty}2^{-j}E_j(\tau,r)\,\mathrm{d}\tau. \tag{69}$$

Next, we iterate to obtain (64). By (69), we immediately have

$$E(t,r) = E_0(t,r) \le r^n f(t_0)^2 + \frac{C_0}{r}\int_{t_0}^t f(t_1)\sum_{k_1\in\mathbb{N}}2^{-k_1}E_{k_1}(t_1,r)\,\mathrm{d}t_1, \tag{70}$$

$$E_{k_1}(t_1,r) \le r^n f(t_0)^2 + \frac{C_0}{r}\int_{t_0}^{t_1} f(t_2)\sum_{k_2\in\mathbb{N}}2^{-k_2}E_{k_2}(t_2,r)\,\mathrm{d}t_2. \tag{71}$$

Substituting (71) into (70), we get

$$E(t,r) \le r^n f(t_0)^2\left[1+\frac{C_0}{r}\int_{t_0}^t f(t_1)\,\mathrm{d}t_1\right] + \frac{C_0}{r}\int_{t_0}^t f(t_1)\sum_{k_1\in\mathbb{N}}2^{-k_1}\frac{C_0}{r}\int_{t_0}^{t_1} f(t_2)\sum_{k_2\in\mathbb{N}}2^{-k_2}E_{k_2}(t_2,r)\,\mathrm{d}t_2\,\mathrm{d}t_1$$
$$\le r^n f(t_0)^2\,[1+C_1] + \frac{C_0^2}{r^2}\int_{t_0}^t f(t_1)\int_{t_0}^{t_1} f(t_2)\sum_{k_1,k_2\in\mathbb{N}}2^{-(k_1+k_2)}E_{k_2}(t_2,r)\,\mathrm{d}t_2\,\mathrm{d}t_1,$$

where we have applied (65) in order to reach the last inequality. This completes our second iteration. We claim that at the Mth step, we will have the bound

$$E(t,r) \le r^n f(t_0)^2 \sum_{j=0}^{M-1} \frac{C_1^j}{j!} + \frac{C_0^M}{r^M} \int_{t_0}^{t} f(t_1) \cdots \int_{t_0}^{t_{M-1}} f(t_M) \sum_{k_1,\dots,k_M \in \mathbb{N}} \frac{E_{k_M}(t_M,r)}{2^{k_1+\cdots+k_M}} \, \mathrm{d}t_M \cdots \mathrm{d}t_1. \tag{72}$$

We have shown this is true for $M = 1, 2$. Now we induct, using this bound to derive step $M+1$, which we will see has the same form. Indeed, Lemma 2.6 gives us

$$E_{k_M}(t_M,r) \le r^n f(t_0)^2 + \frac{C_0}{r} \int_{t_0}^{t_M} f(t_{M+1}) \sum_{k_{M+1} \in \mathbb{N}} 2^{-k_{M+1}} E_{k_{M+1}}(t_{M+1},r) \, \mathrm{d}t_{M+1}; \tag{73}$$

substituting this into our inductive hypothesis, we get

$$\begin{aligned} E(t,r) \le r^n f(t_0)^2 & \left[\sum_{j=0}^{M-1} \frac{C_1^j}{j!} + \frac{C_0^M}{r^M} \int_{t_0}^{t} f(t_1) \cdots \int_{t_0}^{t_{M-1}} f(t_M) \, \mathrm{d}t_M \cdots \mathrm{d}t_1 \right] \\ & + \frac{C_0^{M+1}}{r^{M+1}} \int_{t_0}^{t} f(t_1) \cdots \int_{t_0}^{t_M} f(t_{M+1}) \sum_{k_1,\dots,k_{M+1} \in \mathbb{N}} \frac{E_{k_{M+1}}(t_{M+1},r)}{2^{k_1+\cdots+k_{M+1}}} \, \mathrm{d}t_{M+1} \cdots \mathrm{d}t_1. \end{aligned}$$

Since

$$\frac{C_0^M}{r^M} \int_{t_0}^{t} f(t_1) \cdots \int_{t_0}^{t_{M-1}} f(t_M) \, \mathrm{d}t_M \cdots \mathrm{d}t_1 = \frac{1}{M!} \left[\frac{C_0}{r} \int_{t_0}^{t} f(\tau) \, \mathrm{d}\tau \right]^M \le \frac{C_1^M}{M!}, \tag{74}$$

we have now proved (72). Having established (72), we can prove (64) quickly. First, we clearly have

$$\sum_{j=0}^{M-1} \frac{C_1^j}{j!} < e^{C_1}.$$

To deal with the other term in (72), we estimate each $E_{k_M}(t_M,r)$ trivially by $\|u\|_{L^\infty L^2}$ (so the entire sum can be bounded by $\|u\|_{L^\infty L^2}$). Then we use (74) to take care of the nested integrals.

Altogether we have

$$E(t,r) \le r^n f(t_0)^2 e^{C_1} + \frac{C_1^M}{M!} \|u\|_{L^\infty L^2}^2,$$

which is (64). □

Corollary 2.7. *Under the assumptions of either of Propositions 2.1 or 2.2, we have the bound $d(x,\mathcal{E}) \ge \beta$ for all $x \in \Omega$. Furthermore, the β-density of $\mathcal{E}$ is uniformly bounded on Ω.*

3. Applications to the Navier-Stokes Equations

If we consider the 3-dimensional Navier-Stokes equations instead of the n-dimensional Euler equations, we can reach conclusions in the same spirit as those above. We describe the necessary modifications below. Adding

$$-2\nu\int|\nabla u|^2\sigma\,\mathrm{d}x\,\mathrm{d}\tau+\nu\int|u|^2\Delta\sigma\,\mathrm{d}x\,\mathrm{d}\tau \tag{75}$$

to the right side of (53), we obtain the energy equality for the Navier-Stokes equations. However, the first of these terms can be dropped without affecting the inequality. The way we deal with the second term depends on the method and test function used for the Euler case; these depend in turn on the assumptions made on u.

We consider together the cases $u\in L^{q,*}L^p$ or $u\in L^qL^\infty$ (subject of course to the restrictions on p and q described above). We take $\sigma=\phi_r$ as constructed above when considering these cases. The second term of (75) can clearly be bounded above by

$$\frac{C}{r^2}\int_{Q_{2r}}|u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

We claim that we can ignore this term as well. Indeed, $\beta>2$ whenever

$$\frac{3}{p}+\frac{2}{q}>1.$$

The negation of this inequality is precisely the Prodi-Serrin condition. Therefore we may assume without loss of generality that $\beta>2$, so that

$$\frac{C}{r^2}\int_{Q_{2r}}|u|^2\,\mathrm{d}x\,\mathrm{d}\tau\le\frac{C}{r^\beta}\int_{Q_{2r}}|u|^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

The right side of this inequality is the same quantity we use to bound the term $\int|u|^2|\partial_t\phi_r|\,\mathrm{d}x\,\mathrm{d}\tau$ that appears in (54); therefore it is clear that the addition of the viscous term can cause no trouble. That is, (51) holds whenever the Prodi-Serrin condition fails, while it is obsolete whenever the Prodi-Serrin condition holds.

Remark 3.1. We mention one other extension of Proposition 2.1 before moving on, which is applicable only to the Navier-Stokes system. We can obtain a condition similar to (51) under the assumption $u\in L^qL^p$ for some pairs (p,q) with $p<3$, simply by interpolation with the enstrophy space L^2H^1. In particular, this is possible when

$$\frac{9}{p} + \frac{5}{q} \leq 4, \quad 2 < p < 3.$$

If (p, q) satisfies this condition and $u \in L^qL^p$, then u also belongs to L^aL^3, where

$$a = \frac{\frac{6}{p} - 1}{\frac{3}{p} + \frac{1}{q} - 1}$$

We can apply Proposition 2.1 to $u \in L^aL^3$, yielding

$$\sup_{-r^{\widetilde{\alpha}} < t < 0,\, x_0 \in K} \int_{B_r(x_0)} |u(x,t)|^2 \, \mathrm{d}x \leq C_0 \, r^{\widetilde{\beta}},$$

where

$$\widetilde{\alpha} = \frac{2(\frac{6}{p} - 1)}{\frac{3}{p} - \frac{1}{q}}, \quad \widetilde{\beta} = \frac{4 - \frac{9}{p} - \frac{5}{q}}{\frac{3}{p} - \frac{1}{q}}.$$

When $\|u(t)\|_{L^\infty} \leq c_0|t|^{-1/q}$ (and we use the corresponding time-independent test function $\sigma = \phi_r = \phi$ from Section 2.4), we may estimate the second term of (75) as follows:

$$\begin{aligned}
\int_{t_0}^{t} \int |u|^2 \Delta\phi \, \mathrm{d}x \, \mathrm{d}\tau &\leq \left[\int_{t_0}^{t} \int_{B_r} |u|^3 \, \mathrm{d}x \, \mathrm{d}\tau \right]^{\frac{2}{3}} \cdot \frac{C}{r^2} \cdot [|t_0| r^3]^{\frac{1}{3}} \\
&\leq \left(\frac{|t_0|}{r} \right)^{\frac{1}{3}} \left[\frac{C}{r} \int_{t_0}^{t} f(\tau) E(\tau, 2r) \, \mathrm{d}\tau \right]^{\frac{2}{3}} \\
&\leq C|t_0|r^{-1} + \frac{C}{r} \int_{t_0}^{t} f(\tau) E(\tau, 2r) \, \mathrm{d}\tau.
\end{aligned}$$

The second term can be absorbed into a term already existing in our energy estimates. We claim that running the first term through the iteration scheme yields a quantity which can be bounded above by $C|t_0|r^{-1}e^{C_1}$, which is of the same order as $r^{q'-1}$. Note that this is at least the required order $r^{3-\frac{2q'}{q}}$ whenever $q \leq 2$, therefore Proposition 2.2 holds for the Navier-Stokes equations when $n = 3$ and $5/3 \leq q \leq 2$. On the other hand, the conclusion is trivial whenever $q > 2$, by the Prodi-Serrin criterion.

We now sketch the argument needed to substantiate our claim regarding the term $C|t_0|r^{-1}$. By making straightforward adjustments to the proof of Lemma 2.6, we can write

$$E(t,r) \leq r^3 f(s)^2 + \frac{C|s|}{r} + \frac{C_0}{r} \int_s^t f(\tau) \sum_{j \in \mathbb{N}} 2^{-j} E_j(\tau, r) \, \mathrm{d}\tau, \qquad -1 \leq s \leq t < 0, \tag{76}$$

together with its rescaled version

$$E_k(t,r) \le r^3 f(s)^2 + \frac{C|s|}{2^{k(n+1)}r} + \frac{C_0}{r}\int_s^t f(\tau)\sum_{j\in\mathbb{N}} 2^{-j}E_j(\tau,r)\,\mathrm{d}\tau, \qquad -1\le s\le t<0, \tag{77}$$

the analog of (69). Setting $s = t_0$, we have the analogs of (70) and (71):

$$E(t,r) = E_0(t,r) \le r^3 f(t_0)^2 + \frac{C|t_0|}{r} + \frac{C_0}{r}\int_{t_0}^t f(t_1)\sum_{k_1\in\mathbb{N}} 2^{-k_1}E_{k_1}(t_1,r)\,\mathrm{d}t_1, \tag{78}$$

$$E_{k_1}(t_1,r) \le r^3 f(t_0)^2 + \frac{C|t_0|}{r} + \frac{C_0}{r}\int_{t_0}^{t_1} f(t_2)\sum_{k_2\in\mathbb{N}} 2^{-k_2}E_{k_2}(t_2,r)\,\mathrm{d}t_2. \tag{79}$$

So our second iterative step becomes

$$E(t,r) \le \left[r^3 f(t_0)^2 + \frac{C|t_0|}{r}\right][1+C_1] + \frac{C_0^2}{r^2}\int_{t_0}^t f(t_1)\int_{t_0}^{t_1} f(t_2)\sum_{k_1,k_2\in\mathbb{N}} 2^{-(k_1+k_2)}E_{k_2}(t_2,r)\,\mathrm{d}t_2\,\mathrm{d}t_1,$$

whereas our Mth step yields

$$\begin{aligned} E(t,r) \le{}& \left[r^3 f(t_0)^2 + \frac{C|t_0|}{r}\right]\sum_{j=0}^{M-1}\frac{C_1^j}{j!} \\ &+ \frac{C_0^M}{r^M}\int_{t_0}^t f(t_1)\cdots\int_{t_0}^{t_{M-1}} f(t_M)\sum_{k_1,\dots,k_M\in\mathbb{N}}\frac{E_{k_M}(t_M,r)}{2^{k_1+\cdots+k_M}}\,\mathrm{d}t_M\cdots\mathrm{d}t_1. \end{aligned}$$

Bounding the two sums and the nested integrals as before, then taking $M\to\infty$, we obtain the bound

$$E(t,r) \le [r^3 f(t_0)^2 + C|t_0|r^{-1}]e^{C_1} \le Cr^{3-\frac{2}{q-1}},$$

justifying our claim. We pause to record this as a Proposition:

PROPOSITION 3.2. *Propositions 2.1 and 2.2 remain valid for solutions of 3D Navier-Stokes equation, where* 0 *is the first time of blowup.*

We are now in a position to prove Theorem 1.3.

PROOF. Recall (c.f. Remark 2.3) that Proposition 2.2 can be reframed as the implication

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \le \frac{c_0}{|t|^{1/q}} \implies u\in L^\infty\mathcal{M}^{2,n-\frac{2}{q-1}}.$$

In the Type-I case for the 3-dimensional Navier-Stokes equations, we have $q=2$, $n=3$, so that the above becomes

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{c_0}{\sqrt{t}} \implies u\in L^\infty\mathcal{M}^{2,1}.$$

Before proceeding with the proof, we note that this is the implication "Type-I in time implies Type-I in space" alluded to in the Introduction. By "Type-I in space," we mean we mean a blowup which occurs under control of a scaling invariant norm in space—in this case the Morrey space $\mathcal{M}^{2,1}$ with integrability 2 and rate 1. So it remains to show that the Type-I in space condition implies energy equality. We argue as follows: Since $\mathcal{M}^{1,2}$ is invariant under shifts $f \mapsto f(\cdot - x_0)$ and the rescaling $f(x) \mapsto \lambda f(\lambda x)$, we have by Cannone's Theorem [3] that $u \in L^\infty B^{-1}_{\infty,\infty}$. Consequently, interpolation with the enstrophy space $L^2H^1 = L^2B^1_{2,2}$ puts the solution into the Onsager-critical class $L^3B^{1/3}_{3,3} \subset \mathcal{OR}$, from which we conclude energy equality. $\square$

4. Concentration Dimension of the Energy Measure

As explained in Section 1, the results above directly imply a lower bound on D. For example, if u belongs to L^qL^p and (p,q) satisfies (50), $p \geq 3$, and $q < \infty$, then we have

$$D \geq \beta = \frac{q}{q-1}\left(n - \frac{2n}{p} - \frac{2+n}{q}\right). \tag{80}$$

It turns out that for pairs (p,q) such that $p < \infty$ and $\beta > 0$, one can obtain a sharper bound by exploiting the local energy inequality directly for the entire cover of a concentration set.

In what follows, we make use of the following alternate form of the local energy equality in terms of the energy measure. For any $\sigma \in C_0^\infty(\Omega \times (-1,0])$, we have

$$\int \sigma(0)\,\mathrm{d}\mathcal{E} = \int_{-1}^0 \int_\Omega |u|^2\partial_t\sigma + (|u|^2 + 2p)u \cdot \nabla\sigma - 2\nu(|\nabla u|^2\sigma - u \otimes \nabla\sigma : \nabla u)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{81}$$

where as usual we understand that $\nu = 0$ for the Euler Equations. Notice that we have killed off the initial data by requiring support in $\Omega \times (-1,0]$ rather than $\Omega \times [-1,0]$. By approximation, (81) holds for all $\sigma \in \mathrm{Lip}_0((-1,0] \times \Omega)$.

4.1. Euler Equations.

Theorem 4.1. *Define a function $f(p,q)$ by*

$$f(p,q) = n - \frac{\frac{2}{q}}{1 - \frac{2}{p} - \frac{1}{q}}$$

where we interpret $\frac{1}{\infty} = 0$*. Suppose* $u \in L^q L^p(\Omega)$ *for some* (p,q) *satisfying* (50)*, and suppose* $d \geq 0$ *satisfies*

$$\begin{aligned} d &\leq f(p,q), \quad \text{if } q \leq p \leq \infty,\ q < \infty \\ d &< f(p,q), \quad \text{if } 3 \leq p < q < \infty,\ \text{or if } 2 < p \leq \infty = q. \end{aligned} \tag{82}$$

Then $\mathcal{E}(S) = 0$ *for every* $S \subset \Omega$ *with finite d-dimensional Hausdorff measure. In particular, if* $\dim_{\mathcal{H}}(\Sigma_{ons})$ *satisfies* (82)*, then* u *satisfies the local energy equality on* $[-1,0]$*. Regardless of the size of* Σ_{ons}*, the right side of* (82) *gives a lower bound on the concentration dimension* D*. Similarly, if* $\dim_{\mathcal{H}}(\Sigma)$ *satisfies* (82)*, then* $D = n$*.*

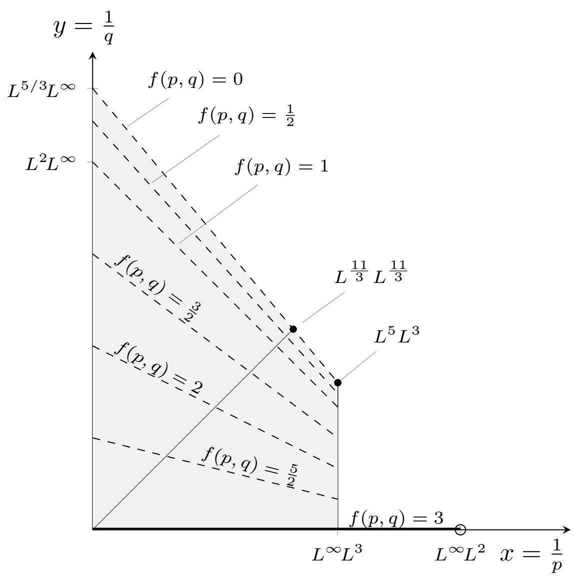

FIGURE 1. Level curves of the lower bound on the concentration dimension for the Euler equations, $n = 3$.

REMARK 4.2. Notice that

$$n - \frac{\frac{2}{q}}{1 - \frac{2}{p} - \frac{1}{q}} = \frac{n - \frac{2n}{p} - \frac{2+n}{q}}{1 - \frac{2}{p} - \frac{1}{q}} \geq \frac{n - \frac{2n}{p} - \frac{2+n}{q}}{1 - \frac{1}{q}} = \beta,$$

with equality precisely when $p = \infty$. So the lower bound on D given by the present theorem is indeed better than (80) except when $p = \infty$, when it is the same.

PROOF. The statement regarding the concentration set Σ is a direct consequence of Corollary 1.3. Next, recall from Section 1 that the two conditions $|\Sigma_{ons}| = 0$ and $\mathcal{E}(\Sigma_{ons}) = 0$ together imply energy equality. Of course, if $\dim_{\mathcal{H}}(\Sigma_{ons}) < n$ (which occurs whenever $\dim_{\mathcal{H}}(\Sigma_{ons})$ satisfies (82)), then $|\Sigma_{ons}| = 0$ trivially and we need only prove $\mathcal{E}(\Sigma_{ons}) = 0$ in order to conclude energy equality.

Therefore the conclusion holds trivially for $u \in L^\infty L^p(\Omega)$, $p > 2$, since we have already proven $D = n$ in this case (see Corollary 1.2).

It remains to show that $\mathcal{E}(S) = 0$ whenever $\dim_{\mathcal{H}}(S)$ satisfies (82). Let us first reduce to the case $p, q \in [3, \infty)$. This can fail for three reasons: $q < 3$, $p = \infty$, or $q = \infty$. We have already dealt with the last case; the other two are covered by the following interpolation argument. Suppose $q < p \le \infty$ and put $r = 2 + q - \frac{2q}{p}$. Then (r, r) satisfies (50), $r \in [3, \infty)$, and $u \in L^r L^r$, as it lies along the line segment joining $L^q L^p$ with $L^\infty L^2$. (That is, $(\frac{1}{r}, \frac{1}{r})$ lies between $(\frac{1}{p}, \frac{1}{q})$ and $(\frac{1}{2}, 0)$ on the line $2px + q(p-2)y = p$.) Furthermore, it is easy to check that

$$n - \frac{2}{r - 1 - \frac{2r}{r}} = n - \frac{2}{q - 1 - \frac{2q}{p}},$$

and therefore that $d \le n - \frac{2}{r-1-\frac{2r}{r}}$, so that $\mathcal{E}(S) = 0$, as desired. For the remainder of the proof, we assume that $p, q \in [3, \infty)$.

Choose $\delta \in (0, \epsilon/3)$, then choose $x_i \in \Omega_\epsilon$, $r_i \in (0, \delta)$ for all i, such that $S \subset \bigcup_i B_{r_i}(x_i)$ and $\sum_{i=1}^\infty r_i^d \lesssim \mathcal{H}_d(S) + 1$. Denote $I_i = (-2r_i^\alpha, 0)$ (where α is determined below). Let $\psi(s)$ be the usual (symmetric, radially decreasing) cut-off function on the line with $\psi(s) = 1$ on $|s| < 1.1$ and $\psi(s)$ vanishing on $|s| > 1.9$. Let $\phi_i(x,t) = \psi(|x - x_i|/r_i)\psi(t/r_i^\alpha)$. Define

$$\phi^N = \sup_{1 \le i \le N} \phi_i, \qquad \phi = \sup_{i \in \mathbb{N}} \phi_i.$$

Then each ϕ^N is continuous with support in $\Omega_\epsilon \times (0, T]$, $0 \le \phi^N \le 1$, and ϕ^N increases pointwise to ϕ, which is identically 1 on $S \times \{0\}$. So

$$\mathcal{E}(S) \le \int \phi(0)\, \mathrm{d}\mathcal{E} = \lim_{N \to \infty} \int \phi^N(0)\, \mathrm{d}\mathcal{E},$$

by monotone convergence. Furthermore, each ϕ^N is differentiable a.e., with

$$|\partial \phi^N(x,t)| \le \sup_{1 \le i \le N} |\partial \phi_i(x,t)|, \quad \text{a.e., see [21, Theorem 4.13].} \tag{83}$$

(In fact, we even have $|\partial \phi(x,t)| \le \sup_{i \in \mathbb{N}} |\partial \phi_i(x,t)|$, though we don't use it.) Therefore, an approximation argument shows that we can put ϕ^N in the local energy equality:

$$\lim_{t \to 0} \int_\Omega |u(t)|^2 \phi^N(t)\, \mathrm{d}x = \int_{-1}^0 \int_\Omega |u|^2 \partial_t \phi^N + (|u(t)|^2 + 2p) u \cdot \nabla \phi^N \, \mathrm{d}x\, \mathrm{d}\tau.$$

Putting all this together, we obtain

$$\mathcal{E}(S) \leq \lim_{N\to\infty} \int_{-1}^{0} \int_{\Omega} |u|^2 \partial_t \phi^N + (|u|^2 + 2p) u \cdot \nabla \phi^N \, \mathrm{d}x \, \mathrm{d}\tau. \tag{84}$$

For d small enough, we will obtain uniform bounds on

$$C_N = \int_{-1}^{0} \int_{\Omega} |u|^2 \partial_t \phi^N \, \mathrm{d}x \, \mathrm{d}\tau, \quad D_N + P_N = \int_{-1}^{0} \int_{\Omega} (|u|^2 + 2p) u \cdot \nabla \phi^N \, \mathrm{d}x \, \mathrm{d}\tau.$$

Using Hölder's inequality and (83), we have

$$C_N \leq C \|u\|^2_{L^q(I, L^p(\Omega))} \left(\int_{-1}^{0} \left(\sum_{i=1}^{N} r_i^{-\frac{\alpha p}{p-2} + n} \chi_{I_i}(t) \right)^{\frac{p-2}{p} \frac{q}{q-2}} \mathrm{d}t \right)^{\frac{q-2}{q}}$$

$$D_N + P_N \leq C \|u\|^3_{L^q(I, L^p(\Omega))} \left(\int_{-1}^{0} \left(\sum_{i=1}^{N} r_i^{-\frac{p}{p-3} + n} \chi_{I_i}(t) \right)^{\frac{p-3}{p} \frac{q}{q-3}} \mathrm{d}t \right)^{\frac{q-3}{q}}$$

Note that to bound P_N, we have also used boundedness of the Riesz transforms on $L^{p/2}$ (recall that $p \in [3, \infty)$). That is, we use the bound $\|p\|_{L^{p/2}} \leq C \|u\|^2_{L^p}$ before exhausting the remaining integrability on $\nabla \phi^N$. The following lemma allows us to bound the quantities C_N, $D_N + P_N$ for small enough d:

LEMMA 4.3 ([39]). *Let d, δ, r_i, I_i be as above, and let σ, s be positive numbers. Suppose the sum $H = \sum_i r_i^d$ is finite. Then the inequality*

$$\int \left(\sum_i r_i^{-\sigma} \chi_{I_i}(t) \right)^s \mathrm{d}t \lesssim H^s \tag{85}$$

holds whenever $s \geq 1$ and $d \leq \frac{\alpha}{s} - \sigma$, or $s < 1$ and $d < \frac{\alpha}{s} - \sigma$. When $d = 0$, the above holds (trivially) under the non-strict assumption $0 \leq \frac{\alpha}{s} - \sigma$.

PROOF. Case 1. $s \geq 1$. By Hölder's inequality, we have

$$\left(\sum_i r_i^{-\sigma} \chi_{I_i}(t) \right)^s = \left(\sum_i r_i^d r_i^{-\sigma - d} \chi_{I_i}(t) \right)^s \leq H^{s-1} \sum_i r_i^{d - (\sigma + d)s} \chi_{I_i}(t). \tag{86}$$

Integrating in time, we obtain

$$\int \left(\sum_i r_i^{-\sigma} \chi_{I_i}(t) \right)^s \mathrm{d}t \lesssim H^{s-1} \sum_i r_i^{d - (\sigma + d)s + \alpha}.$$

The sum is at most H whenever the condition stated in the lemma is satisfied.

Case 2. $s<1$. For each $j \in \mathbb{Z}$, define $R_j := \{r_i : r_i \in [2^{-j},\, 2^{-j+1})\}$, and let N_j denote the cardinality of R_j. Clearly, $N_j \lesssim 2^{jd} H$ and $N_j = 0$ for $j \leq 0$. Also denote $J_j = [-2^{(-j+1)\alpha},\, 2^{(-j+1)\alpha}]$. So if $r_i \in R_j$, then $r_i^{-\sigma} \chi_{I_i}(t) \leq 2^{j\sigma} \chi_{J_j}(t)$. Therefore,

$$\int \left(\sum_i r_i^{-\sigma} \chi_{I_i}(t) \right)^s \mathrm{d}t \leq \int \left(\sum_j N_j 2^{j\sigma} \chi_{J_j}(t) \right)^s \mathrm{d}t \lesssim H^s \int \left(\sum_j 2^{j(\sigma+d)} \chi_{J_j}(t) \right)^s \mathrm{d}t$$
$$\leq H^s \int \sum_{j=1}^{\infty} 2^{j(\sigma+d)s} \chi_{J_j}(t) \, \mathrm{d}t \lesssim H^s \sum_{j=1}^{\infty} 2^{j((\sigma+d)s-\alpha)}.$$

The final sum converges to an adimensional number by the assumption of the lemma. □

We translate the hypotheses and conclusion of the Lemma into statements involving p, q, α, and d; then we optimize in α. When dealing with C_N, we set $\sigma = \frac{\alpha p}{p-2} - n$ and $s = \frac{p-2}{p}\frac{q}{q-2}$. Denoting $H_N = \sum_{i=1}^N r_i^d$, we conclude that

$$C_N \leq \|u\|^2_{L^q(I,L^p(\Omega))} H_N^{1-\frac{2}{p}} \quad \text{whenever} \quad \begin{cases} d \leq n - \frac{\frac{2}{q}\alpha}{1-\frac{2}{p}}, & 2 \leq q \leq p \leq \infty, \\ d < n - \frac{\frac{2}{q}\alpha}{1-\frac{2}{p}}, & 2 \leq p < q \leq \infty. \end{cases} \tag{87}$$

Dealing with $D_N + P_N$ is essentially the same: we set $\sigma = \frac{p}{p-3} - n$ and $s = \frac{p-3}{p}\frac{q}{q-3}$ and apply the lemma to conclude that

$$D_N + P_N \leq \|u\|^3_{L^q(I,L^p(\Omega))} H_N^{1-\frac{3}{p}} \quad \text{whenever} \quad \begin{cases} d \leq n - \frac{1-\alpha(1-\frac{3}{q})}{1-\frac{3}{p}}, & 3 \leq q \leq p < \infty, \\ d < n - \frac{1-\alpha(1-\frac{3}{q})}{1-\frac{3}{p}}, & 3 \leq p < q \leq \infty. \end{cases} \tag{88}$$

In light of the bound $H_N \leq \mathcal{H}_d(S) + 1$, which is uniform in both N and δ, we have

$$\mathcal{E}(S) \leq \|u\|^2_{L^q(I,L^p(\Omega))} (\mathcal{H}_d(S)+1)^{1-\frac{2}{p}} + \|u\|^3_{L^q(I,L^p(\Omega))} (\mathcal{H}_d(S)+1)^{1-\frac{3}{p}},$$

by (84), whenever the conditions on d from (87), (88) are satisfied for some α. (Note that, while the estimates on C_N and $D_N + P_N$ are valid for the ranges of p and q stated above, we continue to work under the assumption that p and q both lie in the range $p, q \in [3, \infty)$.) Since $|I| \to 0$ as $\delta \to 0$, we have $\|u\|_{L^q(I,L^p(\Omega))} \to 0$ as well (as $q < \infty$), and therefore $\mathcal{E}(S) = 0$. The choice of α which maximizes

$$\min \left\{ n - \frac{1-\alpha(1-\frac{3}{q})}{1-\frac{3}{p}},\ n - \frac{\frac{2}{q}\alpha}{1-\frac{2}{p}} \right\}$$

(and therefore gives the optimal range for d) is given by

$$\alpha = \frac{1-\frac{2}{p}}{1-\frac{2}{p}-\frac{1}{q}}. \tag{89}$$

Substituting this value of α into the conditions on d derived above, we conclude that $\mathcal{E}(S) = 0$ whenever

$$\begin{aligned} d &\le n - \frac{\frac{2}{q}}{1-\frac{2}{p}-\frac{1}{q}}, \quad 3 \le q \le p < \infty \\ d &< n - \frac{\frac{2}{q}}{1-\frac{2}{p}-\frac{1}{q}}, \quad 3 \le p < q < \infty. \end{aligned} \tag{90}$$

This completes the proof. $\square$

4.2. Navier-Stokes Equations. In the case of the Navier-Stokes equations, the optimal condition on d analogous to (82) breaks into many different parts, depending on p and q. To streamline the statement of the theorem, let us introduce notation for the various regions involved. Each region defined below consists of pairs $(p,q) \subset [1,\infty]^2$.

$$\begin{aligned} &\mathrm{I} := \left\{p \ge q,\ \ \frac{1}{p}+\frac{1}{q} > \frac{1}{2},\ \ \frac{6}{p}+\frac{5}{q} \le 3\right\}, \quad \mathrm{II} := \left\{3 \le p < q,\ \ \frac{1}{p}+\frac{1}{q} \ge \frac{1}{2},\ \ \frac{6}{p}+\frac{5}{q} \le 3\right\}, \\ &\mathrm{III} := \left\{(p,q) : 2 < p < 3,\ \ \left(\frac{1}{2}-\frac{1}{p}\right)\left(2-\frac{3}{p}\right) \le \frac{1}{q} \le \left(\frac{1}{2}-\frac{1}{p}\right)\left(2-\frac{3}{p}\right)\left(\frac{7}{6}-\frac{1}{p}\right)^{-1}\right\}, \\ &\mathrm{IV} := \left\{\frac{1}{p}+\frac{1}{q} \le \frac{1}{2},\ \ \frac{3}{p}+\frac{1}{q} \le 1\right\}, \quad \mathrm{V} := \left\{\frac{1}{p}+\frac{1}{q} < \frac{1}{2},\ \ \frac{1}{q} < \left(\frac{1}{2}-\frac{1}{p}\right)\left(2-\frac{3}{p}\right)\right\} \setminus \mathrm{IV}. \end{aligned} \tag{91}$$

Let us introduce also a piecewise function defined on these regions, which will serve as a sort of threshold dimension in what follows:

$$f(p,q) = 3 - \frac{\frac{2}{q}}{1-\frac{2}{p}-\frac{1}{q}}, \text{ in } \mathrm{I}\cup\mathrm{II}, \qquad f(p,q) = 3 - \frac{\frac{2}{q}(\frac{6}{p}-1)}{(2-\frac{3}{p}-\frac{3}{q})(1-\frac{2}{p})}, \text{ in } \mathrm{III},$$

$$f(p,q) = 3, \text{ in } \mathrm{IV}\cup\mathrm{V}.$$

THEOREM 4.4. *Suppose $u \in L^qL^p(\Omega)$ for some (p,q) satisfying (50), and suppose $d \ge 0$ satisfies*

$$\begin{aligned} d &\le f(p,q), \quad (p,q) \in \mathrm{I} \\ d &< f(p,q), \quad (p,q) \in \mathrm{II}\cup\mathrm{III}\cup\mathrm{IV}\cup\mathrm{V}. \end{aligned} \tag{92}$$

Then $\mathcal{E}(S) = 0$ for every $S \subset \Omega$ with finite d-dimensional Hausdorff measure. In particular, if $\dim_{\mathcal{H}}(\Sigma_{ons})$ satisfies (92), then u satisfies the local energy equality. Regardless of the size of Σ_{ons},

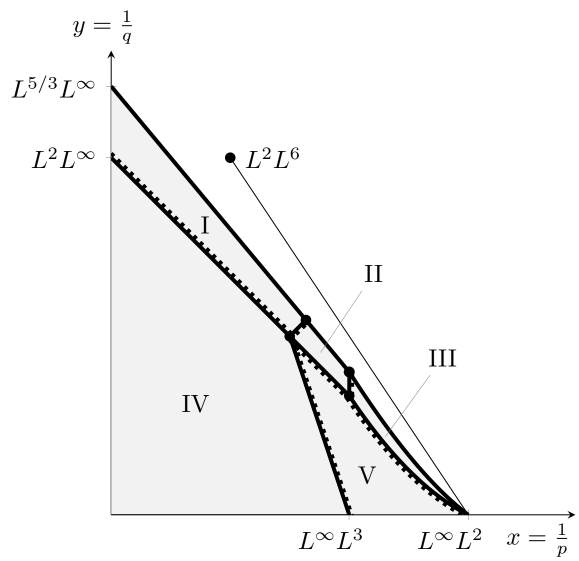

FIGURE 2. The regions of (p,q)-space involved in the statement of Theorem 4.4.

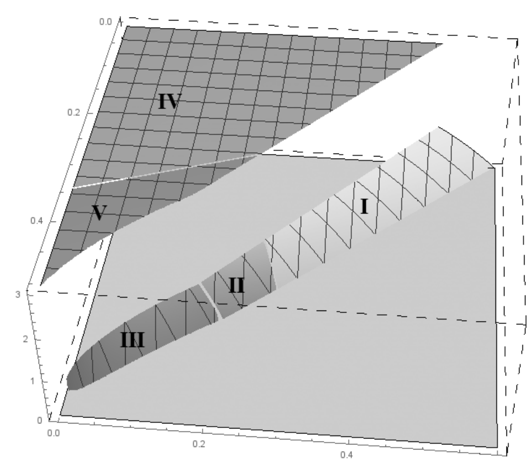

FIGURE 3. A 3D plot of the lower bound on the concentration dimension for the Navier-Stokes equations.

the right side of (92) *gives a lower bound on the concentration dimension D. Similarly, if* $\dim_{\mathcal{H}}(\Sigma)$ *satisfies* (92)*, then $D = 3$.*

REMARK 4.5. For $(p,q) \in \text{III}$, the formula defining $f(p,q)$ can be written as a deviation from the formula in the neighboring region II:

$$3 - \frac{\frac{2}{q}(\frac{6}{p} - 1)}{(2 - \frac{3}{p} - \frac{3}{q})(1 - \frac{2}{p})} = 3 - \frac{\frac{2}{q}}{1 - \frac{2}{p} - \frac{1}{q}} - \frac{(\frac{3}{p} - 1)(3 - \frac{6}{p} - \frac{4}{q})}{(1 - \frac{2}{p})(1 - \frac{2}{p} - \frac{1}{q})(2 - \frac{3}{p} - \frac{3}{q})}.$$

REMARK 4.6. In Figure 2, to highlight values of $f(p,q)$ along jump discontinuities the boundary segments are dotted according to which of two adjoining regions contains the segment in question.

For example, Region IV contains the segment joining L^2L^∞ with L^4L^4 along which $f = 3$, but Region II contains the segment joining L^4L^4 with L^6L^3.

Proof. The claims regarding Σ and Σ_{ons} follow by the same reasoning as in the proof of Theorem 4.1. Also, the local energy equality for the space L^4L^4 gives the result in region IV. To treat the remaining regions, let us use the same setup and test function as in the proof of Theorem 4.1, assuming $n = 3$. Since $\nu > 0$ now, we do have to consider the two additional terms

$$E_N = \int_{-1}^{0}\int_{\Omega} |\nabla u|^2 \phi^N \,\mathrm{d}x\,\mathrm{d}\tau, \qquad F_N = \int_{-1}^{0}\int_{\Omega} u \otimes \nabla\phi^N : \nabla u \,\mathrm{d}x\,\mathrm{d}\tau.$$

Taking $N \to \infty$ and then $\delta \to 0$, it is clear that E_N vanishes in the limit regardless of (p, q). It turns out that F_N is also never limiting with respect to the best possible value of D, but it takes a bit of work to see this.

We treat the remaining four regions in turn as follows:

- In Regions I and II, we reuse the bounds (87) and (88) (with $n = 3$), and we show that the analogous bounds for F_N are strictly better in these two regions. Strictly speaking, this argument only works for $q \geq 3$, but we can use the same logic as in the previous proof to cover the missing region $\mathrm{I} \cap \{q < 3\}$.
- In Region III, we reuse (87) once again, but (88) is no longer valid for any pair (p, q) under consideration. We give a replacement bound using the enstrophy, which is valid for $2 < p < 3$, then we optimize as in the previous theorem.
- In Region V, we use a sort of bootstrap argument. First, we construct a function $g(p, q)$ defined on Region V such that $\mathcal{E}(S) = 0$ whenever $\dim_{\mathcal{H}}(S) < g(p, q)$. Then, we show that $g(p, q) > 1$ everywhere on V. By the discussion in Section 1.3, we know that $\dim_{\mathcal{H}}(\Sigma_{ons}) \leq 1 < g(p, q)$, and therefore that $\mathcal{E}(\Sigma_{ons}) = 0$. But this implies that $\mathrm{d}\mathcal{E} = |u(0)|^2\,\mathrm{d}x$ and that the local energy equality holds for $t = 0$. This obviously implies $D = 3$, which is the desired conclusion for Region V.

Step 1: Regions I and II. In accordance with the outline above, we assume without loss of generality that $q \geq 3$. Estimating

$$F_N \leq C\|u\|^2_{L^q(I,L^p(\Omega))}\|\nabla u\|_{L^2L^2}\left(\int_{-1}^{0}\left(\sum_{i=1}^{N} r_i^{-\frac{2p}{p-2}+3}\chi_{I_i}(t)\right)^{\frac{p-2}{p}\frac{q}{q-2}}\mathrm{d}t\right)^{\frac{q-2}{2q}}$$

and applying Lemma 4.3, we conclude that

$$F_N \le \|u\|_{L^q(I,L^p(\Omega))}\|\nabla u\|_{L^2L^2} H_N^{\frac{1}{2}-\frac{1}{p}} \quad \text{whenever} \quad \begin{cases} d \le 3 - \frac{\frac{2}{q}\alpha-(\alpha-2)}{1-\frac{2}{p}},\ 2 \le q \le p \le \infty, \\ \\ d < 3 - \frac{\frac{2}{q}\alpha-(\alpha-2)}{1-\frac{2}{p}},\ 2 \le p < q \le \infty. \end{cases} \tag{93}$$

Comparison with (87) shows that our condition on F_N is superfluous if $\alpha \ge 2$, which is satisfied by (89) exactly when $\frac{1}{p}+\frac{1}{q} \ge \frac{1}{2}$. But $\frac{1}{p}+\frac{1}{q} \ge \frac{1}{2}$ holds for all $(p,q) \in \mathrm{I} \cup \mathrm{II}$. Therefore we can ignore F_N in Regions I and II and read off the relevant conclusion from the previous theorem in these regions (with $n = 3$). That is, $\mathcal{E}(S) = 0$ holds if S has finite d-dimensional Hausdorff measure, where d satisfies

$$\begin{aligned} d &\le 3 - \frac{\frac{2}{q}}{1-\frac{2}{p}-\frac{1}{q}}, \quad (p,q) \in \mathrm{I}, \\ d &< 3 - \frac{\frac{2}{q}}{1-\frac{2}{p}-\frac{1}{q}}, \quad (p,q) \in \mathrm{II}. \end{aligned} \tag{94}$$

This takes care of Regions I and II completely. However, before proceeding we note that our condition cannot be improved by considering $\alpha < 2$: in this case, the bounds on d from (88) and (93) can both be improved by increasing α, while the bound on d from (87) is superfluous.

Step 2: Region III. Region III lies entirely in the range $\{2 < p < 3\}$, where (88) is not applicable. Therefore we estimate $D_N + P_N$ differently:

$$D_N + P_N \le \|u\|_{L^2H^1}^{3\beta}\|u\|_{L^qL^p}^{3(1-\beta)} \left(\int \sup_i r_i^{-\sigma}\chi_{I_i}(t)\,\mathrm{d}t\right)^{\frac{1}{\sigma}}, \tag{95}$$

where

$$\frac{1}{3} = \frac{\beta}{6} + \frac{1-\beta}{p} \implies \beta = \frac{\frac{6}{p}-2}{\frac{6}{p}-1}; \quad \frac{1}{\sigma} = 1 - \frac{3\beta}{2} - \frac{3(1-\beta)}{q} = \frac{2-\frac{3}{p}-\frac{3}{q}}{\frac{6}{p}-1}. \tag{96}$$

Now using the notation of Lemma 4.3, we have

$$\int \sup_i r_i^{-\sigma}\chi_{I_i}(t)\,\mathrm{d}t \le \int \sup_j 2^{j\sigma}\chi_{J_j}(t)\,\mathrm{d}t \lesssim \sum_j (2^{\alpha-\sigma})^{-j},$$

which is bounded whenever $\sigma < \alpha$. Note that this condition is *independent of* d, so we formulate it as a bound on α:

$$\alpha > \frac{\frac{6}{p} - 1}{2 - \frac{3}{p} - \frac{3}{q}}. \tag{97}$$

Reasoning as before, we have control over both C_N and F_N whenever

$$d < 3 - \frac{\frac{2}{q}\alpha}{1 - \frac{2}{p}} \tag{98}$$

and $\alpha \geq 2$. Now

$$\frac{\frac{6}{p} - 1}{2 - \frac{3}{p} - \frac{3}{q}} > 2 \iff \frac{2}{p} + \frac{1}{q} > \frac{5}{6}, \tag{99}$$

and every pair (p, q) in Region III satisfies $\frac{2}{p} + \frac{1}{q} > \frac{5}{6}$. (This is not difficult to show algebraically, but it is even easier to see geometrically by noting that the line $\frac{2}{p} + \frac{1}{q} = \frac{5}{6}$ passes through L^6L^3 and $L^\infty L^{12/5}$.)

Therefore we can substitute (97) into (98) in order to conclude that $\mathcal{E}(S) = 0$ whenever $d = \dim_{\mathcal{H}}(S)$ satisfies

$$d < 3 - \frac{\frac{2}{q}(\frac{6}{p} - 1)}{(1 - \frac{2}{p})(2 - \frac{3}{p} - \frac{3}{q})}, \quad (p, q) \in \mathrm{III}. \tag{100}$$

This is the desired conclusion for Region III.

Step 3: Region V. The general strategy for dealing with Region V is outlined at the beginning of the proof. We recall that to complete the proof, it suffices to find a function $g(p, q) > 1$ on V such that $\dim_{\mathcal{H}}(S) < g(p, q)$ implies that $\mathcal{E}(S) = 0$. In order to define this function $g(p, q)$, we first split the region V further into three pieces:

$$\mathrm{V}_a := \mathrm{V} \cap \{p \geq 3\}, \quad \mathrm{V}_b := \mathrm{V} \cap \{p < 3\} \cap \left\{\frac{2}{p} + \frac{1}{q} \leq \frac{5}{6}\right\}, \quad \mathrm{V}_c := \mathrm{V} \cap \left\{\frac{2}{p} + \frac{1}{q} > \frac{5}{6}\right\}. \tag{101}$$

In Region V_c, we can reason as in Step 2 and define $g(p, q)$ by the right side of (100). Furthermore,

$$\begin{aligned} g(p, q) = 3 - \frac{\frac{2}{q}(\frac{6}{p} - 1)}{(1 - \frac{2}{p})(2 - \frac{3}{p} - \frac{3}{q})} &= 3 - \frac{2(1 - \frac{2}{p})(2 - \frac{3}{p} - \frac{3}{q}) - 4((2 - \frac{3}{p})(\frac{1}{2} - \frac{1}{p}) - \frac{1}{q})}{(1 - \frac{2}{p})(2 - \frac{3}{p} - \frac{3}{q})} \\ &= 1 + \frac{4[(2 - \frac{3}{p})(\frac{1}{2} - \frac{1}{p}) - \frac{1}{q}]}{(1 - \frac{2}{p})(2 - \frac{3}{p} - \frac{3}{q})} > 1, \end{aligned}$$

since $\frac{1}{q} < (2 - \frac{3}{p})(\frac{1}{2} - \frac{1}{p})$ for $(p, q) \in \mathrm{V}$.

In the Regions V_a and V_b, we set $\alpha = 2$ and note that the restrictions on d due to (87) and (93) coincide in this case. On the other hand, the restriction due to $D_N + P_N$ becomes superfluous here. This is easy to see for $(p,q) \in \mathrm{V}_b$, since (97) is satisfied in this region by (99). For $(p,q) \in \mathrm{V}_a$, one can also compare (87) and (88) directly, but the following argument is perhaps more insightful: Notice that, as we increase α, the requirements on d become more stringent for (87) and less stringent for (88). The two conditions coincide when α is given by (89). As discussed in Step 1, this value of α is less than 2 if $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, which is satisfied for all pairs $(p,q) \in \mathrm{V}$. Therefore, (88) is superfluous when $(p,q) \in \mathrm{V}_a$ and $\alpha = 2$. We therefore define

$$g(p,q) = 3 - \frac{\frac{4}{q}}{1 - \frac{2}{p}}, \quad (p,q) \in \mathrm{V}_a \cup \mathrm{V}_b.$$

Since $\frac{2}{p} + \frac{2}{q} < 1$ in V, we have

$$3 - \frac{\frac{4}{q}}{1 - \frac{2}{p}} = \frac{2(1 - \frac{2}{p} - \frac{2}{q}) + (1 - \frac{2}{p})}{1 - \frac{2}{p}} > 1, \quad (p,q) \in \mathrm{V}, \tag{102}$$

and therefore $g(p,q) > 1$ in $\mathrm{V}_a \cup \mathrm{V}_b$, and therefore in all of V. This completes the proof. $\square$

CHAPTER 3

More on Energy Equality: General Singularity Sets, Fractional Navier-Stokes[1]

1. General singularities

Even if the energy equality is known on each time interval of regularity including at the critical time, it is unknown whether energy equality holds globally on the time interval of existence of the weak solution. This is due to lack of a proper gluing procedure that could restore energy equality from pieces. In this section, we therefore address the question of energy balance when the singularity set Σ_{ons} is spread in space-time. We consider only the case when $\nu > 0$.

When dealing with a singularity set that is spread in space-time, we must change our approach from that of the previous chapter. Indeed, our method in the setting of the first blowup time was built on the fact that no term of (8) 'sees' the singularity set before the end of the limiting procedure. We considered (8) as a sequence of local energy equalities, indexed by time, and we made our conclusions by taking limits. Once we fix a time interval on which singularities may be present, we immediately lose our natural sequence of equalities; it is not even clear what a replacement family of equalities would have as its index! Fortunately, Lemma 1.1 from the Introduction suggests a collection of equalities to consider, and our experience in the previous chapter with the concentration dimension suggests how we might index these equalities.[2] We can actually preserve the spirit of the argument used to deal with the concentration dimensions, with a few important modifications. However, to the author's knowledge, there is not a natural analogue of the energy measure in this situation.[3] Therefore, we can only answer in a binary way the question of whether it is possible to conclude energy balance under a given integrability condition on the solution.

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[1]This chapter is largely excerpted from:
[39] Trevor M. Leslie and Roman Shvydkoy. Conditions Implying Energy Equality for Weak Solutions of the Navier–Stokes Equations. *SIAM J. Math. Anal.*, 50(1):870–890, 2018.
In particular, the cited article is the original source of all figures contained in this chapter.

[2]This way of framing things is, in the author's opinion, logically correct; however, it is anachronistic. In fact, the technique of 'cutting out' the singularity set as described in this chapter predates the arguments in the previous chapter which give bounds on the concentration dimension.

[3]One might try, though, to see what is possible with the Duchon-Robert distribution of [20]. We leave the exploration of this topic to future research.

In light of these differences, we alter our approach somewhat. First, we content ourselves with studying the *global* energy equality

$$\frac{1}{2}\int |u(t)|^2\,\mathrm{d}x - \frac{1}{2}\int |u_0|^2\,\mathrm{d}x = -\nu\int_0^t\int |\nabla u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s. \tag{103}$$

In the classical Navier-Stokes case, our method can easily be adjusted to prove the local energy equality whenever we can establish the global one. However, this is not true of the fractional Navier-Stokes equations, which we consider below. We prefer to keep our approach to the two equations as unified as is practical, and we therefore restrict attention to (103) (and later its fractional analogue).

We now describe the modified method in full detail. The analogue of (10) in the present context is

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u|^2\phi(t)\,\mathrm{d}x - \int_{\mathbb{R}^3} |u|^2\phi(s)\,\mathrm{d}x - \int_s^t\int_{\mathbb{R}^3} |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad = \int_s^t\int_{\mathbb{R}^3} |u|^2 u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau + 2\int_s^t\int_{\mathbb{R}^3} pu\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad\quad - 2\nu\int_s^t\int_{\mathbb{R}^3} |\nabla u|^2\phi - 2\nu\int_s^t\int_{\mathbb{R}^3} u\otimes\nabla\phi : \nabla u\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned} \tag{104}$$

which is valid for all $\phi \in C_0^\infty((\mathbb{R}^3\times[0,T])\backslash\Sigma_{ons})$. An approximation argument shows that (53) remains valid for functions ϕ (supported outside Σ_{ons}, as before) which belong only to $W^{1,\infty}$ rather than C^∞. The main idea of this section is to construct a sequence of test functions which satisfy this equality and to show that passing to the limit causes it to reduce to (103).

Recall that Leray–Hopf solutions satisfy $u(t)\to u(0)$ strongly in $L^2(\mathbb{R}^3)$ as $t\to 0^+$. Therefore, in order to establish (103), it suffices to prove energy balance on the time interval $[s,T]$ for each $s\in(0,T)$; the Onsager singularity set at the initial time is irrelevant for our analysis. Therefore, we introduce the following singularity set, which we call the *postinitial singularity set* S (or simply the *singularity set* when it will cause no confusion), defined by

$$S = \Sigma_{ons}\backslash(\mathbb{R}^3\times\{0\}).$$

Working with S rather than all of Σ_{ons} allows us to obtain better conditions guaranteeing energy balance for solutions which have arbitrary divergence free initial condition $u_0\in L^2$ (but which have small postinitial singularity sets). A priori, this replacement requires us to assume $s>0$ rather than $s\geq 0$ in (53). However, as pointed out above, we may extend to $s=0$ by continuity, so that we may consider S instead of Σ_{ons} at no real cost. We will make the standing assumption that the Lebesgue measure $|S|$ of S in $\mathbb{R}^3\times[0,T]$ is equal to zero.

Let us label each of the terms in (104) and rewrite the equation as

$$A - B - C = D + 2P - 2\nu E - 2\nu F. \tag{105}$$

Having established the above considerations and notation, we can now describe the main idea more clearly and succinctly. Given a Leray–Hopf solution u and its (postinitial) singularity set S, we seek a sequence $\{\phi_\delta\}_{\delta>0}$ of test functions such that

- $\operatorname{supp}\phi_\delta \subset (\mathbb{R}^3 \times [0,T])\backslash S$ and $\phi_\delta \in W^{1,\infty}(\mathbb{R}^3 \times [0,T])$ (so (104) is valid for $0 < s < t \le T$);
- $0 \le \phi_\delta \le 1$ and $\phi_\delta \to 1$ pointwise a.e. as $\delta \to 0$ (which is possible since $|S| = 0$), guaranteeing the convergence of the terms A, B, and E to their natural limits

  $$\int_{\mathbb{R}^3} |u(t)|^2 \,\mathrm{d}x, \int_{\mathbb{R}^3} |u(s)|^2 \,\mathrm{d}x, \int_s^t \int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}\tau,$$

  respectively. These convergences follow from the fact that $u \in L^\infty L^2 \cap L^2 H^1$, together with the dominated convergence theorem.

When A, B, and E tend to their natural limits, we see that in order to establish energy balance on $[s,T]$, it suffices to prove that the other terms C, $D + 2P$, and F vanish as $\delta \to 0$. In order to ensure this, we make integrability assumptions on the solution u, i.e., $u \in L^q(0,T,L^p(\mathbb{R}^3))$ for some pair (p,q) of integrability exponents. The set of admissible values for p and q, which will make the terms C, $D + 2P$ and F vanish, depend on the integrability properties of the functions ϕ_δ, which in turn depend on the size of S. Below we will generally suppress the notation δ from the subscript of our sequence of test functions.

We make one more remark before proceeding to construct the test functions. One remaining technical difference between the setting of Chapter 2 and the present one is that we have no freedom in choosing the time scale of the covering cylinders; rather, the scale should already be built into the definition of the Hausdorff dimension. We choose to work with the classical parabolic dimension, i.e., $\alpha = 2$ in our terms. Consequently, the conclusion of Lemma 4.3 may not be valid when $s < 1$. Instead, we can only prove that the left side of (85) is bounded above by H (multiplied by some constant which is independent of δ) under the stronger assumption $\sigma s + d \le \alpha = 2$. This is achieved simply by bringing the exponent s inside the sum. However, the condition $\sigma s + d \le \alpha = 2$ is the sharpest possible under which the conclusion of the lemma holds, as one can see by considering an example of the opposite extreme, where all the intervals I_i are disjoint. However, the proof of the lemma in the case $s \geq 1$ does not depend on the intervals I_i being nested; the proof and conclusion remain valid in this case.

Assume then that S has finite d-dimensional parabolic Hausdorff measure for some $d \in (0,1]$ but no other special properties.[4] Our method does not yield anything new for $d > 1$, so we do not treat these values of d. Let $\delta > 0$; then choose finitely many $(x_i, t_i) \in \mathbb{R}^3 \times (0,T]$ and $r_i \in (0,\delta)$ such that $S \subset Q := \bigcup_i Q_i$, where $Q_i = B_{r_i}(x_i) \times (t_i - r_i^2,\ t_i + r_i^2)$. Write $I_i = (t_i - 2r_i^2,\ t_i + 2r_i^2)$. Let Q^* denote the union of the double-dilated cylinders and $I = \bigcup_i I_i$. Let ψ be the usual (symmetric, radially decreasing) cutoff function with $\psi(s) = 1$ on $|s| < 1.1$ and $\psi(s) = 0$ for $s > 1.9$; put $\phi_i = \psi(|x - x_i|/r_i)\psi(|t - t_i|/r_i^2)$ and $\phi = 1 - \sup_i \phi_i$.

We note that we have $|I| \to 0$ as $\delta \to 0$, even though the intervals I_i are no longer nested. This is because

$$|I| \leq \sum_i |I_i| \lesssim \sum_i r_i^{d+(2-d)} < \delta^{2-d} \sum_i r_i^d \tag{106}$$

and because $d < 2$ in all cases considered in this section.

Assume $p \leq q$. Using bounds analogous to (87), (93), we see that $C, F \to 0$ whenever

$$\left(\frac{2p}{p-2} - 3\right) \frac{p-2}{p} \frac{q}{q-2} + d \leq 2,$$

or, simplifying,

$$\frac{3}{p} + \frac{2-d}{q} \leq \frac{3-d}{2} \quad (p \leq q). \tag{107}$$

Similarly, if $\infty > q \geq p \geq 3$, then $D, P \to 0$ whenever

$$\frac{3}{p} + \frac{2-d}{q} \leq \frac{4-d}{3} \quad (3 \leq p \leq q < \infty). \tag{108}$$

Of course, when $d \in [0,1]$, we have $\frac{4-d}{3} \leq \frac{3-d}{2}$, so the restriction (108) is limiting in this case.

On the other hand, if $p < 3$, then we use (95) and (96). Estimating

$$\|\nabla\phi\|_{L^\sigma L^\infty}^\sigma \leq \sum_i \int r_i^{-\sigma} \chi_{I_i}(t)\,\mathrm{d}t \leq \sum_i r_i^{2-\sigma},$$

[4] Let us note that in the special case $d = 0$, S is once again a finite point set. The energy balance relation holds on each of the finitely many time-slices associated to each of the points in S under the criteria of the previous section. Therefore, it holds under these criteria for a general 0-dimensional singularity set; we assume $d > 0$ without loss of generality.

we see that $D, P \to 0$ whenever and $2 - \sigma \geq d$, i.e.,

$$\frac{4-d}{p} + \frac{2-d}{q} \leq \frac{5-2d}{3}, \quad p < 3. \tag{109}$$

Notice that we could have also reached this inequality by interpolation. This argument covers all terms under consideration in the case $p \leq q$; it remains to deal with the case when $p > q$. Most of the analysis from the single time-slice situation carries over in this case since Lemma 4.3 from Chapter 2 does not require nested I_i in the case $s \geq 1$. However, the lack of freedom to choose α restricts the applicable range of pairs (p, q). After translating the condition $s(\sigma + d) \leq 2$ into conditions on C, D, P, F, we see that D, P are most stringent when $p \geq q \geq 3$ and correspond to the condition

$$\frac{3-d}{p} + \frac{2}{q} \leq \frac{4-d}{3}, \quad 3 \leq q \leq p.$$

Using interpolation to treat the cases $p = \infty$, $q < 3$, and $q = \infty$ as well, we can state our criteria for energy balance as follows:

$$\frac{2(3-d)}{p} + \frac{5-d}{q} \leq 3 - d, \quad q \leq 3 \leq p \tag{110a}$$

$$\frac{3-d}{p} + \frac{2}{q} \leq \frac{4-d}{3}, \quad 3 \leq q \leq p \tag{110b}$$

$$\frac{3}{p} + \frac{2-d}{q} \leq \frac{4-d}{3}, \quad 3 \leq p \leq q \tag{110c}$$

$$\frac{4-d}{p} + \frac{2-d}{q} \leq \frac{5-2d}{3}, \quad p \leq 3 \leq q. \tag{110d}$$

As $d \to 1^-$, these criteria collectively collapse to the region implicated by the Lions L^4L^4 condition. However, when $d \in (0,1)$ we obtain a new region bounded by the points $L^{\frac{5-d}{3-d}}L^\infty$, $L^3L^{\frac{9-3d}{2-d}}$, $L^{\frac{15-3d}{4-d}}L^{\frac{15-3d}{4-d}}$, $L^{\frac{6-3d}{1-d}}L^3$, $L^\infty L^{\frac{5-2d}{12-3d}}$. See Figure 4.

2. Fractional Navier-Stokes Equations

In this section, we present extensions of some of our results for the classical Navier-Stokes equations to the case of fractional dissipation $\gamma < 1$:

$$\partial_t u + u \cdot \nabla u + \nu \Lambda_{2\gamma} u = -\nabla p \tag{111}$$

$$\operatorname{div} u = 0 \tag{112}$$

where $\widehat{\Lambda_s u} = |\xi|^s \widehat{u}$. We define the Onsager regular and singular sets as in the classical case. In the fractional dissipation case, weak solutions belong to $L^2H^\gamma \cap L^\infty L^2$, and the analogue of (104) can

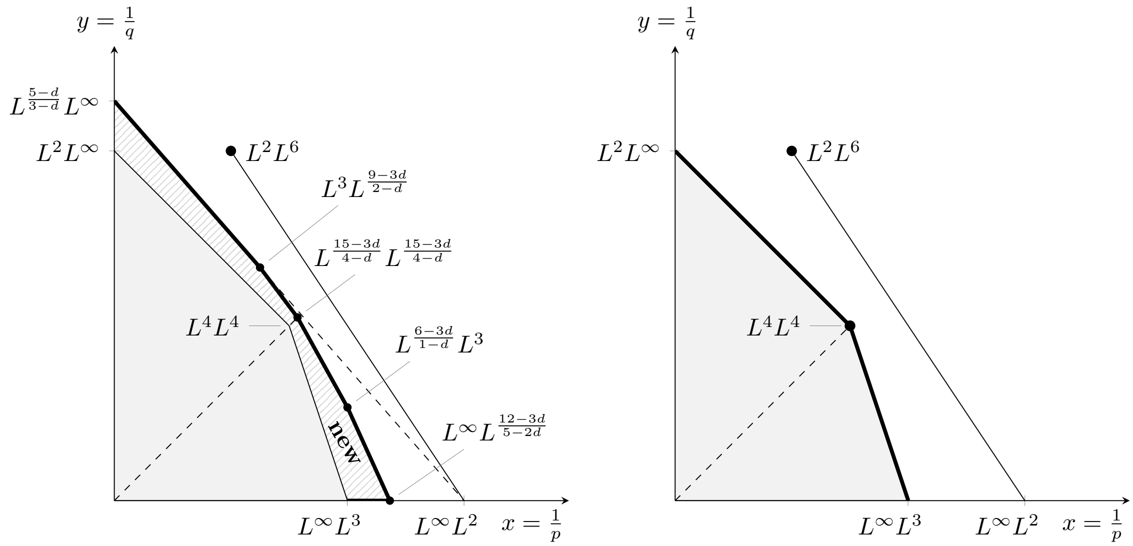

FIGURE 4. L^qL^p spaces which guarantee energy equality for the 3D Navier-Stokes equations, for a general singularity set of dimension $0 < d < 1$ (left) and $d = 1$ (right). Our method gives new results when $d < 1$.

be written

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u|^2\phi(t)\,\mathrm{d}x - \int_{\mathbb{R}^3} |u|^2\phi(s)\,\mathrm{d}x - \int_s^t\int_{\mathbb{R}^3} |u|^2\partial_t\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&= \int_s^t\int_{\mathbb{R}^3} |u|^2 u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau + 2\int_s^t\int_{\mathbb{R}^3} pu\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}\tau - 2\nu\int_s^t\int_{\mathbb{R}^3} |\Lambda_\gamma u|^2\phi\,\mathrm{d}x\,\mathrm{d}\tau \\
&- 2\nu\int_s^t\int_{\mathbb{R}^3} \Lambda_\gamma u\cdot u\Lambda_\gamma\phi - 2\nu\int_s^t\int_{\mathbb{R}^3} \Lambda_\gamma u\cdot[\Lambda_\gamma(u\phi) - (\Lambda_\gamma u)\phi - u\Lambda_\gamma\phi]\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned} \tag{113}$$

As in the classical case, this equality is valid for $\phi \in W^{1,\infty}(\mathbb{R}^3 \times [0,T])$ which are supported outside the Onsager singular set. We label our terms in the same manner as in the classical case:

$$A - B - C = D + 2P - 2\nu E - 2\nu F - 2\nu G.$$

As before, convergence of A, B, E is obvious; proving energy equality amounts to showing that the other terms vanish. We consider the global energy equality for both the time-slice and the general singularity cases. Though the calculations pertaining to the concentration dimension of the energy measure can be extended to the time-slice case in a rather straightforward manner, it is not obvious how to extend the bounds on the local dimension in a manner analogous to that of Section 3. We therefore prefer to leave the analysis of both cases for future research.

For sufficiently regular f and $\gamma \in (0,2)$, we have

$$\Lambda_\gamma f(x) = -c_\gamma \int \frac{\delta_{-z}\delta_z f(x)}{|z|^{3+\gamma}}\,\mathrm{d}z = \widetilde{c}_\gamma\ p.v.\int \frac{\delta_z f(x)}{|z|^{3+\gamma}}\,\mathrm{d}z,$$

where δ_z denotes the difference operator $\delta_z f(x) = f(x+z) - f(x)$. We also recall the bound

$$\|\Lambda_\gamma \phi\|_{L^a} \lesssim \|\phi\|_{L^a}^{1-\gamma} \|\nabla\phi\|_{L^a}^{\gamma}, \tag{114}$$

valid for $\phi \in W^{1,a}$, $a \in [1,\infty]$, $\gamma \in (0,1)$. We will use this without comment in what follows.

LEMMA 2.1. *Let $u \in H^\gamma \cap L^p$, $p > 2$, $\gamma \in (0,1)$, and $\phi \in W^{1,\frac{2p}{p-2}}$. Then*

$$\|\Lambda_\gamma(u\phi) - (\Lambda_\gamma u)\phi - u\Lambda_\gamma\phi\|_{L^2} \lesssim \|u\|_{L^p} \|\phi\|^{1-\gamma}_{L^{\frac{2p}{p-2}}} \|\nabla\phi\|^{\gamma}_{L^{\frac{2p}{p-2}}}. \tag{115}$$

The inequality continues to hold when $p = 2$ and $2p/(p-2)$ is replaced by ∞.

PROOF. We use the identity

$$[\Lambda_\gamma(u\phi) - (\Lambda_\gamma u)\phi - u\Lambda_\gamma\phi](x) = -\int \frac{\delta_z u(x)\,\delta_z\phi(x)}{|z|^{3+\gamma}}\,\mathrm{d}z$$

and estimate the right side of this equality. Let $r > 0$ be arbitrary for now. Then

$$\begin{aligned}
\left\|\int \frac{\delta_z u\,\delta_z\phi}{|z|^{3+\gamma}}\,\mathrm{d}z\right\|_{L^2} &\le \int_{|z|\le r} \frac{\|\delta_z u\,\delta_z\phi\|_{L^2}}{|z|^{3+\gamma}}\,\mathrm{d}z + \int_{|z|>r} \frac{\|\delta_z u\,\delta_z\phi\|_{L^2}}{|z|^{3+\gamma}}\,\mathrm{d}z \\
&\le 2\|u\|_{L^p}\left[\int_{|z|\le r} \frac{\|\nabla\phi\|_{L^{\frac{2p}{p-2}}}}{|z|^{2+\gamma}}\,\mathrm{d}z + \int_{|z|>r} \frac{2\|\phi\|_{L^{\frac{2p}{p-2}}}}{|z|^{3+\gamma}}\,\mathrm{d}z\right] \\
&\le 2r^{-\gamma}\|u\|_{L^p}[r\|\nabla\phi\|_{L^{\frac{2p}{p-2}}} + 2\|\phi\|_{L^{\frac{2p}{p-2}}}].
\end{aligned}$$

Put $r = \|\phi\|_{L^{\frac{2p}{p-2}}} \|\nabla\phi\|^{-1}_{L^{\frac{2p}{p-2}}}$ to complete the proof. □

We apply this Lemma to our original test function ϕ:

$$\begin{aligned}
&\int \|u\Lambda_\gamma\phi\|_{L^2}^2 + \|\Lambda_\gamma(u\phi) - (\Lambda_\gamma u)\phi - u\Lambda_\gamma\phi\|_{L^2}^2\,\mathrm{d}t \\
&\quad\lesssim \int (\|u\|_{L^p}\|\phi\|^{1-\gamma}_{L^{\frac{2p}{p-2}}}\|\nabla\phi\|^{\gamma}_{L^{\frac{2p}{p-2}}})^2\,\mathrm{d}t \\
&\quad\le \|u\|^2_{L^q L^p}\|\phi\|^{2(1-\gamma)}_{L^\infty L^{\frac{2p}{p-2}}}\|\nabla\phi\|^{2\gamma}_{L^{\frac{2q\gamma}{q-2}} L^{\frac{2p}{p-2}}}.
\end{aligned}$$

Now with the bound $|\nabla\phi(x,t)| \le \sup_i |\nabla\phi_i(x,t)|$, we obtain

$$\|\nabla\phi\|^{\frac{2q\gamma}{q-2}}_{L^{\frac{2q\gamma}{q-2}} L^{\frac{2p}{p-2}}} \le \int \left(\sum_i r_i^{-\frac{2p}{p-2}+3}\chi_{I_i}(t)\right)^{\frac{p-2}{p}\frac{q}{q-2}\gamma} \mathrm{d}t. \tag{116}$$

So we can use Lemma 4.3 from Chapter 2 to give conditions on when $|F| + |G| \to 0$, depending on whether we are dealing with the one-slice or general type singularity.

Remark 2.2. It has been shown (c.f. [30, 61, 60, 11]) that $5-4\gamma$ serves as an upper bound for the dimension of the singular set, provided that $\gamma \in (\frac{3}{4}, 1) \cup (1, \frac{5}{4})$. Interestingly, the value $\gamma = \frac{3}{4}$ plays a significant role in our analysis as well, as is perhaps evidenced by the figures below. We include some analysis relating to the empty case $\gamma \in (\frac{3}{4}, 1)$, $d > 5-4\gamma$ just to see what the method gives us as compared to when $\gamma < \frac{3}{4}$, where an analogue of the Caffarelli-Kohn-Nirenberg Theorem is not available.

2.1. One-time singularity case, $\frac{1}{2} < \gamma < 1$. We recall some of the conditions for the vanishing of C and $D+P$ and (using the lemma) add to them conditions for the vanishing of $F+G$. Note that the restriction (118) below on $D+P$ is only valid inside the square $p, q \geq 3$, just as before. We deal with this case first and investigate the case $p < 3$ separately:

$$\frac{3-d}{p} + \frac{\alpha}{q} \leq \frac{3-d}{2},\ p \geq q \geq 2; \quad \frac{3-d}{p} + \frac{\alpha}{q} < \frac{3-d}{2},\ 2 \leq p < q \tag{117}$$

$$\frac{3-d}{p} + \frac{\alpha}{q} \leq \frac{2+\alpha-d}{3},\ p \geq q \geq 3; \quad \frac{3-d}{p} + \frac{\alpha}{q} < \frac{2+\alpha-d}{3},\ 3 \leq p < q \tag{118}$$

$$\frac{(3-d)\gamma}{p} + \frac{\alpha}{q} \leq \frac{(3-d)\gamma + \alpha - 2\gamma}{2}, \quad \frac{1}{q} - \frac{\gamma}{p} \geq \frac{1-\gamma}{2},\ \ p, q \geq 2 \tag{119a}$$

$$\frac{(3-d)\gamma}{p} + \frac{\alpha}{q} < \frac{(3-d)\gamma + \alpha - 2\gamma}{2}, \quad \frac{1}{q} - \frac{\gamma}{p} < \frac{1-\gamma}{2},\ \ p, q \geq 2. \tag{119b}$$

The line $\frac{1}{q} - \frac{\gamma}{p} = \frac{1-\gamma}{2}$ joins $L^{\frac{2}{1-\gamma}} L^\infty$ with $L^2 L^2$. It plays the role for $F+G$ that the bisectrice plays for C and $D+P$. Also note that for each restriction, all inequalities are nonstrict in the special case $d = 0$, just as before.

When $d \leq 5-4\gamma$, we find using the same argument as in the classical case that $\alpha = \frac{5-d}{2}$ gives the optimal region. At this value of α, (117) and (118) coincide, and (119a) and (119b) are less restrictive than (117) and (118). Furthermore, since the line corresponding to (117) rotates about $L^\infty L^2$, we may use interpolation to remove the restriction $q \geq 3$.

In the case $p < 3$, we repeat the argument used for the classical Navier-Stokes case and make changes where necessary. Assume first that $\gamma \geq \frac{3}{4}$. Then we have

$$|D| + |P| \leq \|u\|^{3\beta}_{L^2 H^1} \|u\|^{3(1-\beta)}_{L^q L^p} \|\nabla \phi\|_{L^\sigma L^\infty}, \tag{120}$$

where

$$\frac{1}{3} = \frac{(3-2\gamma)\beta}{6} + \frac{1-\beta}{p} \implies \beta = \frac{6-2p}{6-(3-2\gamma)p};\ 1-\beta = \frac{(2\gamma-1)p}{6-(3-2\gamma)p} \tag{121}$$

$$\frac{1}{\sigma} = 1 - \frac{3\beta}{2} - \frac{3(1-\beta)}{q} = \frac{2\gamma pq - 3p(2\gamma-1) - 3q}{(6-(3-2\gamma)p)q}. \tag{122}$$

Now

$$\|\nabla\phi\|^{\sigma}_{L^\sigma L^\infty} = \int \sup_i r_i^{-\sigma}\chi_{I_i}(t)\,\mathrm{d}t \le \int \sup_j 2^{j\sigma}\chi_{J_j}(t)\,\mathrm{d}t \lesssim \sum_j (2^{\alpha-\sigma})^{-j},$$

and the sum on the right is bounded whenever $\sigma < \alpha$. Substituting in for σ and simplifying,

$$\frac{2+\alpha}{p} + \frac{(2\gamma-1)\alpha}{q} < \frac{3-2\gamma+2\alpha\gamma}{3}. \tag{123}$$

Note that as α increases, the line corresponding to equality rotates counterclockwise about $L^2L^{\frac{6}{3-2\gamma}}$. Combining (117) and (123) with inequality replaced by equality in both cases, we find the curve

$$\begin{aligned} &6(3-d)x^2 + 6(6\gamma-5-(2\gamma-1)d)xy - (4\gamma+3)(3-d)x \\ &\qquad - (22\gamma-15-(2\gamma-1)3d)y + 2\gamma(3-d) = 0, \end{aligned} \tag{124}$$

where $x = p^{-1}$ and $y = q^{-1}$. Notice that the curve contains both $L^2L^{\frac{6}{3-2\gamma}}$ and $L^\infty L^2$, as we expect. (Indeed, these points are the axes of rotation for our lines.) However, since we are restricted to the case $p < 3$, the part of the curve that we can use is limited to that connecting $L^{\frac{15-3d}{3-d}}L^3$ and $L^\infty L^2$.

If $\gamma < \frac{3}{4}$, then (120) is not valid for all values of p, q. In particular, we need $3\beta \le 2$ for the obvious application of Hölder to be valid, which translates to $p \ge \frac{3}{2\gamma}$. When $\gamma \in (\frac{1}{2}, \frac{3}{4})$, we also see that the two places where the curve (124) crosses the x-axis are at $x = \frac{1}{2}$ and $x = \frac{2\gamma}{3}$. When $\gamma \in (\frac{1}{2}, \frac{3}{4})$, we have $\frac{2\gamma}{3} \in (\frac{1}{3}, \frac{1}{2})$. So the curve still gives us a meaningful restriction up to the point where it crosses the x-axis for the first time. Once $\gamma < \frac{1}{2}$, however, we have $\frac{2\gamma}{3} < \frac{1}{3}$, so the use of enstrophy does not allow us to make any statement about the range $p < 3$.

All in all, our criteria for energy equality in the case $\frac{1}{2} < \gamma < 1$, $0 \le d \le 5 - 4\gamma$ can be stated as

$$\frac{2(3-d)}{p} + \frac{5-d}{q} \le 3-d,\ p \ge q; \qquad \frac{2(3-d)}{p} + \frac{5-d}{q} < 3-d,\ 3 \le p < q \tag{125}$$

$$\begin{aligned} &6(3-d)x^2 + 6(6\gamma-5-(2\gamma-1)d)xy - (4\gamma+3)(3-d)x \\ &\qquad - (22\gamma-15-(2\gamma-1)3d)y + 2\gamma(3-d) > 0, \quad \frac{1}{3} < x < \min\{\frac{1}{2}, \frac{2\gamma}{3}\}. \end{aligned} \tag{126}$$

Once again, strict inequalities are replaced by nonstrict ones if $d = 0$.

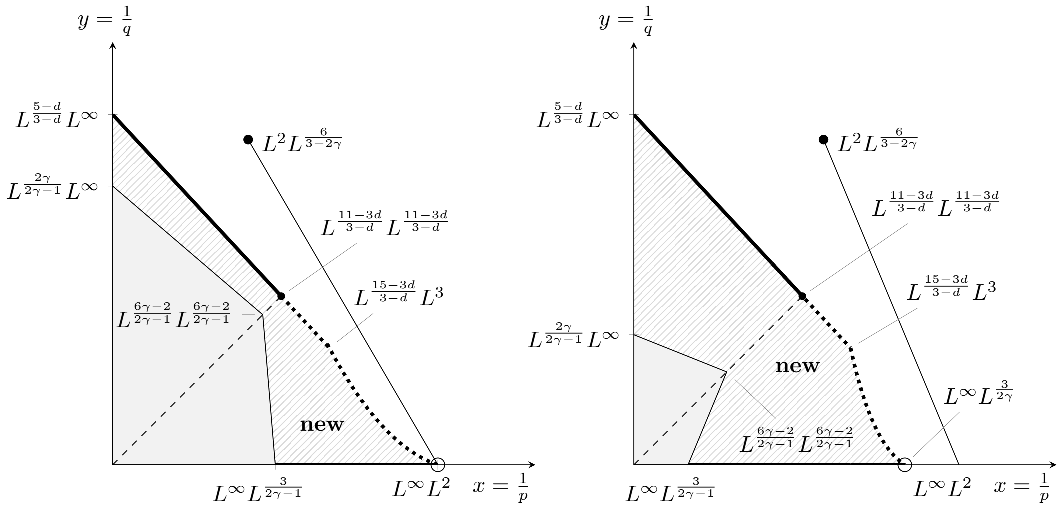

FIGURE 5. $L^q L^p$ spaces which guarantee energy equality for the 3D γ-fractional Navier-Stokes Equations, with $\frac{3}{4} \le \gamma < 1$ (left) or $\frac{1}{2} < \gamma < \frac{3}{4}$ (right) and time-slice singularity of dimension $0 < d \le 5 - 4\gamma$.

Figure 5 diagrams our results for a fixed value of $d \in (0, 5-4\gamma)$ (we use $d = \frac{2}{3}$) and varying $\gamma \in (\frac{1}{2}, 1)$. Note that $L^{\frac{6\gamma-2}{2\gamma-1}} L^{\frac{6\gamma-2}{2\gamma-1}}$ serves as the analogue of the Lions space in the present context, because interpolation between this space and $L^2 H^\gamma$ lands in the Onsager space $L^3 B^{1/3}_{3,c_0}$.

As we take $d \to 5-4\gamma$ from below, the new region above the bisectrice collapses to the segment $[L^\infty L^{\frac{2\gamma}{2\gamma-1}}, L^{\frac{6\gamma-2}{2\gamma-1}} L^{\frac{6\gamma-2}{2\gamma-1}}]$. In this respect, the value $d = 5-4\gamma$ serves a similar role to the value $d = 1$ in the classical case. Things are slightly more complicated when $5-4\gamma < d < 3$. In this case, setting α equal to its usually optimal value of $\alpha = \frac{5-d}{2}$ places too heavy a burden on $F+G$; for a fixed p, we must increase α to optimize until the restrictions on C and $F+G$ coincide. An elementary computation gives the optimal value of α to be

$$\alpha_{CF}(x) = (3-d)(1-\gamma)(1-2x) + 2\gamma \qquad (x = p^{-1}). \tag{127}$$

We see then that as x increases, the optimal value of α decreases. When $p \ge 3$, the restriction on $D+P$ is always less stringent for this value of α than the corresponding restriction for C. However, as x increases beyond $\frac{1}{3}$, $\alpha_{CF}(x)$ eventually becomes sufficiently small so that (123) becomes limiting once again. At this point, the optimal restriction is once again determined by the intersection of the C and $D+P$ lines, following the curve (124). Indeed, along the curve (124), α is given by

$$\alpha_{CDP}(x) = \frac{3[(3-d)(2\gamma-1)-2](1-2x)+4\gamma}{2(2\gamma-3x)}. \tag{128}$$

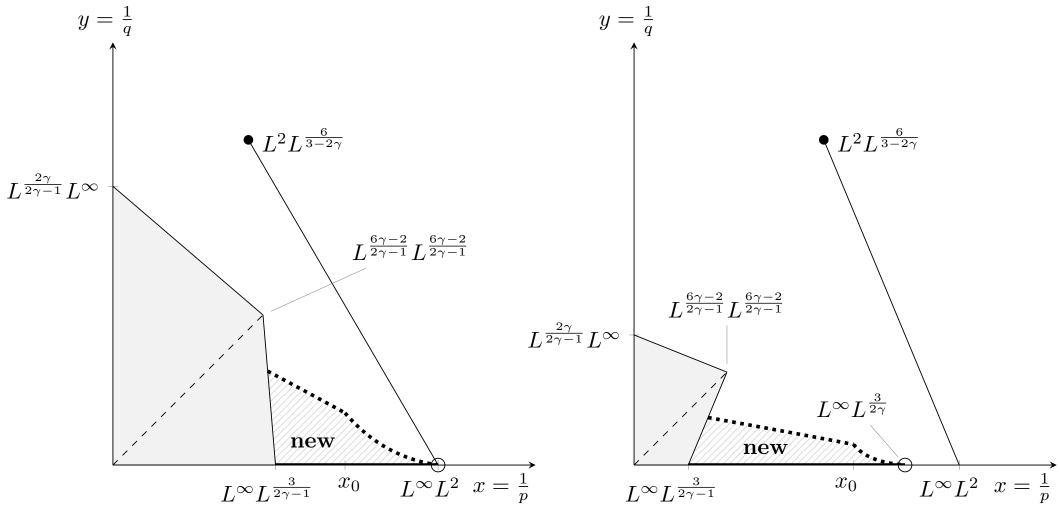

FIGURE 6. $L^q L^p$ spaces which guarantee energy equality for the 3D γ-fractional Navier-Stokes equations, with $\frac{3}{4} \le \gamma < 1$ (left) or $\frac{1}{2} < \gamma < \frac{3}{4}$ (right) and time-slice singularity of dimension $5 - 4\gamma < d < 3$.

Now

$$\alpha_{CDP}(1/3) = \frac{5-d}{2} < 2\gamma < \alpha_{CF}(1/3),$$

whereas

$$\alpha_{CDP}(1/2) = \frac{4\gamma}{4\gamma - 3} > 2\gamma = \alpha_{CF}(1/2) \quad (3/4 < \gamma < 1),$$

$$\lim_{x \to \frac{2\gamma}{3}^-} \alpha_{CDP}(x) = \infty > \alpha_{CF}(2\gamma/3) \quad (1/2 < \gamma \le 3/4).$$

So there must be some $x_0 \in (\frac{1}{3}, \min\{\frac{1}{2}, \frac{2\gamma}{3}\})$, where $\alpha_{CDP}(x_0) = \alpha_{CF}(x_0)$. The actual value of x_0 does not seem to take a particularly enlightening form in general, but it can be easily calculated given $\gamma \in (\frac{1}{2}, 1)$ and $d \in (5 - 4\gamma, 3)$. See Figure 6.

Altogether, the criteria for energy equality in the case $\gamma \in (\frac{1}{2}, 1)$ and $d \in (5 - 4\gamma, 3)$ can be stated as

$$4(1-\gamma)(3-d)xy - 2(3-d+(d-1)\gamma)y + (1-2x)(3-d) > 0, \quad x < x_0 \tag{129}$$

$$\begin{aligned} &6(3-d)x^2 + 6(6\gamma - 5 - (2\gamma-1)d)xy - (4\gamma+3)(3-d)x \\ &\qquad - (22\gamma - 15 - (2\gamma-1)3d)y + 2\gamma(3-d) > 0, \quad x_0 < x < \min\{\frac{1}{2}, \frac{2\gamma}{3}\}. \end{aligned} \tag{130}$$

2.2. One-time singularity case, $0 < \gamma \le \frac{1}{2}$**.** Much of the analysis of the previous subsection carries over to the case when $\gamma \in (0, \frac{1}{2}]$. However, there are a few important differences. For one thing, the Lions region is the single point $L^\infty L^\infty$ when $\gamma = \frac{1}{2}$ and trivial otherwise. Second, the case $d > 5 - 4\gamma$ is geometrically impossible since $5 - 4\gamma > 3$ here. Finally, we cannot say anything about the region $p < 3$. As was mentioned earlier, the enstrophy argument used to deal with this

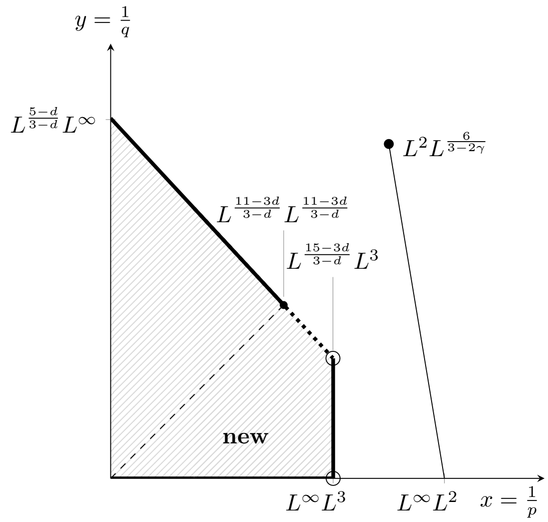

FIGURE 7. L^qL^p spaces guaranteeing energy equality for the 3D γ-fractional Navier-Stokes equations, $\gamma \in (0, \frac{1}{2})$, and time-slice singularity of dimension $d < 3$.

region for larger values of γ does not apply when $\gamma \in (0, \frac{1}{2})$. In fact, we cannot even get any new information by interpolation with the Leray–Hopf line since the point $L^2 L^{\frac{6}{3-2\gamma}}$ lies on the line $x = \frac{1}{3}$ when $\gamma = \frac{1}{2}$ and to the right of this line when $\gamma < \frac{1}{2}$. So the region for which we have proved energy equality is independent of γ for $\gamma < \frac{1}{2}$; the region depends only on d. See Figure 7.

2.3. General singularities. We fix $\alpha = 2\gamma$ in consideration of the natural scaling. The restrictions corresponding to C, $D + P$, and $F + G$ become

$$\frac{3-d}{p} + \frac{2\gamma}{q} \leq \frac{3-d}{2},\ p \geq q \geq 2; \quad \frac{3}{p} + \frac{2\gamma - d}{q} \leq \frac{3-d}{2},\ 2 \leq p < q \tag{131}$$

$$\frac{3-d}{p} + \frac{2\gamma}{q} \leq \frac{2+2\gamma-d}{3},\ p \geq q \geq 3; \quad \frac{3}{p} + \frac{2\gamma - d}{q} \leq \frac{2+2\gamma-d}{3},\ 3 \leq p < q \tag{132}$$

$$\frac{3-d}{p} + \frac{2}{q} \leq \frac{3-d}{2}, \quad \frac{1}{q} - \frac{\gamma}{p} \geq \frac{1-\gamma}{2},\ \ p, q \geq 2 \tag{133a}$$

$$\frac{3\gamma}{p} + \frac{2\gamma - d}{q} \leq \frac{3\gamma - d}{2}, \quad \frac{1}{q} - \frac{\gamma}{p} < \frac{1-\gamma}{2},\ \ p, q \geq 2. \tag{133b}$$

We will not present figures pertaining to this particular situation, as the reader can easily verify conditions above for any particular values of γ, d, p, q. However, we make several comments.

First, we note that the measure of I may not vanish for certain combinations of γ, d. Mimicking the argument of (106) only gives $|I| \to 0$ when $d \leq 2\gamma$. If $d > 2\gamma$, then we continue with the additional assumption that $\mathcal{H}_d(S)$ is actually zero (rather than merely finite, as we usually assume).

Assume first that $\gamma \in (\frac{1}{2}, 1)$. Then (132) is more stringent than (131) when $d < 5 - 4\gamma$; the two inequalities coincide when $d = 5 - 4\gamma$. At this value of d, the region satisfying (131), (132) is exactly the region already covered by the analogue of the Lions result. So only the case $d < 5 - 4\gamma$ can give new information. However, in contrast to the classical case, the restrictions (133a), (133b) are not always superfluous. If $\gamma < \frac{1}{2}$, then the Lions region is trivial, and consequently the value $d = 5 - 4\gamma$ has no special significance for our argument in the case of a general 2γ-parabolic d-dimensional singularity with $\gamma \in (0, \frac{1}{2})$.

When $d = 0$, the singularity set can be covered by finitely many time-slices, and the region covered is the same as in the one-slice case. When $d \in (0, 2\gamma - 1)$, the DP-lines are limiting, but there is still a nontrivial region covered in the range $p < 3$ by interpolation. This region disappears when $d = 2\gamma - 1$, but the DP-lines remain the limiting restriction until d surpasses the value $\frac{1}{2}(5 + \gamma - \sqrt{9\gamma^2 - 18\gamma + 25})$, at which point the lower FG-line (corresponding to (133b)) cuts into both the upper and the lower DP-lines. This situation prevails until d reaches the value $\frac{5\gamma - 4\gamma^2}{3 - 2\gamma}$, at which point the lower FG-line becomes more stringent than the lower DP-line everywhere below the bisectrice. However, at this point, the upper FG-line is still less stringent than the upper DP-line; this changes once d surpasses 1. Note that the point $L^{\frac{5-d}{3-d}} L^\infty$ is no longer included in the region covered for $d > 1$. Rather, the upper FG-line lies strictly below the interpolation line obtained in the region $q < 3$ from the uppermost point on the DP segment. When d lies in the range $d \in [1, 2\gamma + 1 - 2\sqrt{3\gamma^2 - 3\gamma + 1})$, the upper DP-line remains more stringent than the FG-lines on a small segment. However, once $d \geq 2\gamma + 1 - 2\sqrt{3\gamma^2 - 3\gamma + 1}$, the FG restrictions are limiting in all cases.

There are a few larger values of significance for d, but they involve the interaction between the FG-lines and the Lions region rather than the FG-lines and the other restrictions imposed by our method. We describe briefly the bifurcations of the diagrams. When d reaches the value $2 - \gamma$, the Lions point $L^{\frac{6\gamma-2}{2\gamma-1}} L^{\frac{6\gamma-2}{2\gamma-1}}$ lies on the lower FG segment. When $d = \gamma(5 - 4\gamma)$, the new region below the bisectrice disappears entirely (since $\frac{3\gamma - d}{6\gamma} = \frac{2\gamma - 1}{3}$ for this value of d). The new region disappears entirely into the Lions region once $d = \frac{2-\gamma}{\gamma}$. Indeed, at this value of d, we have $\frac{3-d}{4} = \frac{2\gamma-1}{2\gamma}$; furthermore, both the upper FG-line and the line containing the upper part of the boundary for the Lions region pass through $L^\infty L^2$. Therefore, the upper FG-line collapses to (a portion of) the boundary of the Lions region when $d = \frac{2-\gamma}{\gamma}$.

CHAPTER 4

Energy Equality for the Inhomogeneous Incompressible Navier-Stokes and Euler Equations[1]

In this chapter we consider the question of energy equality for the system (16)–(18), as outlined in the Introduction.

1. Preliminaries and Preparations for the Main Theorem

In [7] it was shown that if $u \in L^3(0,T;B^{1/3}_{3,c_0}(\mathbb{R}^d)) \cap C_w([0,T];L^2(\mathbb{R}^d))$ is a weak solution to the (homogeneous) incompressible Euler equations, then u conserves energy. The authors define an energy flux $\Pi_Q(t)$ describing the energy dissipated from scales associated to wave numbers $\lambda_q = 2^q$ for $-1 \le q \le Q$. To prove their result, they bound $\Pi_Q(t)$ using the convolution of a sequence involving the Littlewood-Paley projections of the solution u with a localization kernel; they conclude by noting that their bound tends to zero in the limit. We follow a similar program in this section. After motivating our use of Besov spaces by generalizing the Kármán-Howarth-Monin relation to the present context, we recall the definition of a Besov space and set some notation. Next, we derive an energy budget relation associated to the density-dependent Navier-Stokes equations. Finally, we define localization kernels and present some estimates that will streamline the proof of our theorem.

1.1. Kármán-Howarth-Monin relation. Let us motivate the use of Besov spaces and the choice of regularity classes by ideas from the turbulence theory. Our immediate goal is to extend the classical Kármán-Howarth-Monin relation to the density-dependent case, see [26]. Let us suppose that our fluid reached a state of fully developed turbulence in which statistical laws with respect to an ensemble average $\langle \cdot \rangle$ are independent of a location in space where are measured[2]. In order to measure how much regularity is needed to control the energy flux we derive a formula for the physical space energy flux due to the nonlinear transport term defined by

$$\pi(\ell) = \frac{1}{4}\partial_t \langle u(r+\ell)\cdot u(r)(\rho(r+\ell)+\rho(r))\rangle_{\mathrm{T}}.$$

[1]Most of this chapter is taken from:
[38] T. M. Leslie and R. Shvydkoy. The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations. *J. Differential Equations*, 261(6):3719–3733, 2016.

[2]The common term *homogeneous turbulence* may be misleading in our settings as our density still remains variable.

Note that it coincides with the classical flux in the case when ρ is constant, and it is symmetric with respect to $r+\ell, r$. Let us use the notation $u_i = u_i(r)$, $u_i' = u_i(r+\ell)$, $\partial_i = \frac{\partial}{\partial r_i}$, $\partial_i' = \frac{\partial}{\partial \ell_i}$. From the transport term in the momentum equation (16) we obtain

$$-4\pi(\ell) = \langle \partial_j(\rho' u_j' u_i') u_i \rangle + \langle \rho' u_i' \partial_j (u_j u_i) \rangle + \langle \partial_j(\rho u_j u_i) u_i' \rangle + \langle \rho u_i \partial_j (u_j' u_i') \rangle. \tag{134}$$

Note that $\partial_j(\rho' u_j' u_i') = \partial_j'(\rho' u_j' u_i')$, and $\langle \partial_j'(\rho' u_j' u_i') u_i \rangle = \partial_j' \langle \rho' u_j' u_i' u_i \rangle$. Similarly, $\langle \rho u_i \partial_j (u_j' u_i') \rangle = \partial_j' \langle \rho u_i u_j' u_i' \rangle$. As to the two terms in the middle we first perform integration by parts. This can be justified by first averaging over the fluid domain $\mathbb{T}^d$. Since the ensembles are independent of r, this does not change the quantities. Then switching the order of averaging, integrating by parts, switching again, and un-averaging produces the result. So, $\langle \rho' u_i' \partial_j (u_j u_i) \rangle = -\langle \partial_j (\rho' u_i') u_j u_i \rangle = -\partial_j' \langle \rho' u_i' u_j u_i \rangle$, and similarly, $\langle \partial_j (\rho u_j u_i) u_i' \rangle = -\langle \rho u_j u_i \partial_j (u_i') \rangle = -\partial_j' \langle \rho u_j u_i u_i' \rangle$. We thus obtain

$$4\pi(\ell) = -\partial_j' \langle \rho' u_j' u_i' u_i \rangle + \partial_j' \langle \rho' u_i' u_j u_i \rangle + \partial_j' \langle \rho u_j u_i u_i' \rangle - \partial_j' \langle \rho u_i u_j' u_i' \rangle. \tag{135}$$

Let us denote $\delta u(\ell) = u(r+\ell) - u(r)$, and similar for ρ. The expression on the right can be written

$$-\nabla_\ell \cdot \langle (\delta(\rho u) \cdot \delta u) \delta u \rangle. \tag{136}$$

This can be proved directly by breaking the above into individual terms and noting that $\langle \rho' u_j' u_i' u_i' \rangle = \langle \rho u_j u_i u_i \rangle$ are independent of ℓ, and $\partial_j' \langle \rho u_i u_i u_j' \rangle = 0$ by the divergence-free condition, and $\partial_j' \langle \rho' u_i' u_i' u_j \rangle = 0$ by the same reason after changing $r \to r - \ell$. Applying the algebraic identity $\delta(fg) = \frac{1}{2}[(f + f')\delta g + (g + g')\delta f]$ to (136), we obtain

$$\pi(\ell) = -\frac{1}{8} \nabla_\ell \cdot \langle \delta\rho \delta u ((u(r+\ell) + u(r)) \cdot \delta u) \rangle - \frac{1}{8} \nabla_\ell \cdot \langle (\rho(r+\ell) + \rho(r)) |\delta u|^2 \delta u \rangle. \tag{137}$$

This is a direct generalization of the classical Kármán-Howarth-Monin relation. One can see from this relation that there are a few different ways to cause the flux to vanish, in terms of distribution of smoothness and integrability between ρ and u. Given that $\rho \in L^\infty$ is a natural assumption, the last term vanishes if u is $1/3$ regular in L^3-sense. Then, for the first term to vanish one must also have that u is $1/3$ regular in L^b-sense and ρ is $1/3$ regular in L^a-sense, where $\frac{1}{a} + \frac{3}{b} = 1$. This leads to the use of Besov spaces and suggests that the set of assumptions (25) is sharp.

1.2. Besov Spaces via Littlewood-Paley Decomposition. We follow the setup of [9] and [7] in defining the Littlewood-Paley projections of the functions ρ, u, p. Fix $\chi \in C_0^\infty(B(0,1))$ such that $\chi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. Define $\phi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$. Define length scales $\lambda_q = 2^q$, and define

$\varphi_{-1}(\xi) = \chi(\xi)$, $\varphi_q(\xi) = \phi(\lambda_q^{-1}\xi)$ for $q \in \mathbb{N}\cup\{0\}$. Then $\sum_{q=-1}^{\infty} \varphi_q \equiv 1$; in particular $\sum_{q=-1}^{\infty} \varphi_q(k) = 1$ for all $k \in \mathbb{Z}^d$. We do not distinguish notationally between φ_q and its restriction to the integer lattice, but occasionally it will be necessary to interpret φ_q in the latter sense. Note that φ_q, φ_r have disjoint supports unless $r \in \{q-1, q, q+1\}$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse transform for $\mathbb{T}^d$: $\mathcal{F}(f)(k) = \int_{\mathbb{T}^d} f(x)e^{-2\pi ik\cdot x}\,\mathrm{d}x$, $\mathcal{F}^{-1}(g)(x) = \sum_{k\in\mathbb{Z}^d} g(k)e^{2\pi ik\cdot x}$.

Define the following functions:

$$
\begin{aligned}
&h_q = \mathcal{F}^{-1}(\varphi_q), \qquad \widetilde{h}_Q = \mathcal{F}^{-1}(\chi(\lambda_{Q+1}^{-1}\,\cdot\,)),\\
&u_q = \mathcal{F}^{-1}(\varphi_q \mathcal{F}u) = h_q * u, \qquad u_{\le Q} = \sum_{q=-1}^{Q} u_q = \mathcal{F}^{-1}(\chi(\lambda_{Q+1}^{-1}\,\cdot\,)\mathcal{F}u) = \widetilde{h}_Q * u,\\
&u_{\sim Q} = \sum_{q=Q-2}^{Q+2} u_q, \qquad u_{>Q} = \sum_{q=Q+1}^{\infty} u_q.
\end{aligned}
$$

Write $A := \mathbb{N} \cup \{0, -1\}$. The Besov space $B^s_{p,r}(\mathbb{T}^d)$ ($s \in \mathbb{R}$, $p, r \in [1,\infty]$) is the space of tempered distributions u whose corresponding norm, defined by

$$
\|u\|_{B^s_{p,r}(\mathbb{T}^d)} = \left\|(\lambda_q^s \|u_q\|_{L^p(\mathbb{T}^d)})_{q\in A}\right\|_{\ell^r(A)},
$$

is finite. Clearly $B^s_{p,r}(\mathbb{T}^d) \subset B^{s'}_{p',r'}(\mathbb{T}^d)$ for $s' \le s$, $p' \le p$, $r' \ge r$. Furthermore, $B^s_{a,\infty} \subset L^a$ for all $a \in [1,\infty)$, $s > 0$. We define $B^s_{p,c_0}(\mathbb{T}^d)$ as the space of tempered distributions u such that $\lambda_q^s \|u_q\|_{L^p(\mathbb{T}^d)} \xrightarrow{q\to\infty} 0$, together with the norm inherited from $B^s_{p,\infty}(\mathbb{T}^d)$. This space contains $B^s_{p,r}(\mathbb{T}^d)$ for all $r \in [1,\infty)$. We will write $B^s_{p,r}$ for $B^s_{p,r}(\mathbb{T}^d)$ unless the abbreviation could cause confusion.

1.3. Derivation of the Energy Budget Relation. Define $E_{\le Q}(s) := \frac{1}{2}\int_{\mathbb{T}^d} \frac{(\rho u)^2_{\le Q}}{\rho_{\le Q}}(s)\,\mathrm{d}x$, the energy associated to scales λ_q for $q \le Q$. Define $U = \frac{(\rho u)_{\le Q}}{\rho_{\le Q}}$ and put $\psi = U_{\le Q}$ in (20), to yield

$$
\begin{aligned}
2E_{\le Q}(s)\big|_0^t = &\int_0^t \int ((\rho u)_{\le Q}\cdot \partial_s U + (\rho u \otimes u)_{\le Q} : \nabla U + p_{\le Q}\operatorname{div} U)\,\mathrm{d}x\,\mathrm{d}s\\
&- \mu \int_0^t \int \nabla u_{\le Q} : \nabla U \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int (\rho f)_{\le Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
\tag{138}
$$

On the other hand, we can rewrite the definition of $E_{\le Q}$ using the weak form of the density equation:

$$
\begin{aligned}
E_{\le Q}(s)\big|_0^t &= \frac{1}{2}\int_{\mathbb{T}^d\times\{s\}} \rho_{\le Q} U^2\,\mathrm{d}x\Big|_0^t = \frac{1}{2}\int_{\mathbb{T}^d\times\{s\}} \rho (U^2)_{\le Q}\,\mathrm{d}x\Big|_0^t\\
&= \frac{1}{2}\int_0^t \int \left(\rho\partial_s (U^2)_{\le Q} + (\rho u\cdot\nabla)(U^2)_{\le Q}\right)\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Then, we easily see that

$$E_{\le Q}(s)\big|_0^t = \frac{1}{2}\int_0^t \int \left(\rho_{\le Q}\partial_s(U^2) + ((\rho u)_{\le Q}\cdot\nabla)(U^2)\right)\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_0^t \int \left((\rho u)_{\le Q}\cdot\partial_s U + ((\rho u)_{\le Q}\otimes U) : \nabla U\right)\,\mathrm{d}x\,\mathrm{d}s.$$

Subtracting the result from (138), we obtain the energy budget relation at scales $q \le Q$:

$$E_{\le Q}(t) - E_{\le Q}(0) = \int_0^t \Pi_Q(s)\,\mathrm{d}s - \varepsilon_Q(t) + \int_0^t \int (\rho f)_{\le Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s. \tag{139}$$

Here $\Pi_Q(s)$ is the flux through scales of order Q due to the nonlinearity and the pressure:

$$\Pi_Q = \int F_Q(\rho,u) : \nabla U\,\mathrm{d}x + \int p_{\le Q}\operatorname{div} U\,\mathrm{d}x; \tag{140}$$

$$F_Q(\rho,u) = (\rho u\otimes u)_{\le Q} - U\otimes(\rho u)_{\le Q}. \tag{141}$$

Also, ε_Q and $\int_0^t\int(\rho f)_{\le Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s$ represent the energy dissipation due to local interactions and the external force, respectively, at scales $q\le Q$. Now ε_Q is given by

$$\varepsilon_Q(t) = \mu\int_0^t\int \nabla u_{\le Q} : \nabla U\,\mathrm{d}x\,\mathrm{d}s.$$

Also denote

$$\varepsilon(t) = \mu\int_0^t \|\nabla u\|_{L^2}^2\,\mathrm{d}s.$$

We aim to show that for appropriate (ρ,u,p) and all $t\in[0,T]$, we have (as $Q\to\infty$) that $E_{\le Q}(t)\to E(t)$, $\int_0^t\Pi_Q(s)\,\mathrm{d}s\to 0$, $\varepsilon_Q(t)\to\varepsilon(t)$, and $\int_0^t\int(\rho f)_{\le Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t\int \rho u\cdot f\,\mathrm{d}x\,\mathrm{d}s$. These convergences will immediately imply that (23) holds for (ρ,u,p).

1.4. The Localization Kernel and Estimates on the Littlewood-Paley Projections.

Let $a,b\in[1,\infty]$, $s\in(0,1]$, and let f be a real-valued function. Define the following:

$$K_q^s = \begin{cases} \lambda_q^{s-1}, & q\ge 0;\\ \lambda_q^s, & q<0;\end{cases} \qquad d_{a,q}^s(f) = \lambda_q^s\|f_q\|_{L^a}; \qquad D_{a,Q}^s(f) = \sum_{q=-1}^{\infty} K_{Q-q}^s d_{a,q}^s(f).$$

We can define these expressions analogously for the vector-valued f. Note that in view of summability of the kernel we have

$$\limsup_{Q\to\infty} D_{a,Q}^s(f) \sim \limsup_{q\to\infty} d_{a,q}^s(f) \tag{142}$$

where the similarity constant depends only on s.

Proposition 1.1. *For $f \in B^s_{a,\infty}$, $g \in B^t_{b,\infty}$, $a, b \in [1,\infty]$, $s, t \in (0,1)$, we have the following estimates:*

$$\|(fg)_{\le Q} - f_{\le Q} g_{\le Q}\|_{L^c} \lesssim \lambda_Q^{-s-t} D^s_{a,Q}(f) D^t_{b,Q}(g), \quad \frac{1}{c} = \frac{1}{a} + \frac{1}{b}, \tag{143}$$

$$\|(fg)_{\le Q} - f_{\le Q} g_{\le Q}\|_{L^a} \lesssim \lambda_Q^{-s} D^s_{a,Q}(f) \|g\|_{L^\infty}, \tag{144}$$

$$\|\nabla f_{\le Q}\|_{L^a} \lesssim \lambda_Q^{1-s} D^s_{a,Q}(f), \tag{145}$$

$$\|f_{>Q}\|_{L^a} \le \lambda_Q^{-s} D^s_{a,Q}(f). \tag{146}$$

Remark 1.2. *Let us note that (144) is still meaningful when $s = 1$. However, in this case, the kernel is not localized in the region $q > 0$, which meets finitely many terms in the convolution D. Nonetheless, uniform bounds on the convolution would be applicable under stronger summability assumption on Littlewood-Paley components of f. For example, when $a = 2$ and $f \in H^1$ we clearly have*

$$D^1_{2,Q}(f) \le \|f\|_{H^1}.$$

Proof. Since

$$\widetilde{h}_Q * f = f_{\le Q}, \qquad \int \widetilde{h}_Q(y)\,\mathrm{d}y = 1,$$

we can write

$$(fg)_{\le Q} - f_{\le Q} g_{\le Q} = r_Q(f,g) - f_{>Q} g_{>Q},$$

where

$$r_Q(f,g) = \int \widetilde{h}_Q(y)(f(\cdot - y) - f(\cdot))(g(\cdot - y) - g(\cdot))\,\mathrm{d}y. \tag{147}$$

Therefore, to prove (143) it suffices to estimate $r_Q(f,g)$, $f_{>Q}$, $g_{>Q}$ appropriately.

We can write

$$\|f_{>Q}\|_{L^a} \le \lambda_Q^{-s} \sum_{q>Q} \lambda^s_{Q-q} \lambda^s_q \|f_q\|_{L^a} = \lambda_Q^{-s} \sum_{q>Q} K^s_{Q-q} d^s_{a,q}(f) \le \lambda_Q^{-s} D^s_{a,Q}(f).$$

This proves (146). The same reasoning yields $\|g_{>Q}\|_{L^b} \le \lambda_Q^{-t} D^t_{b,Q}(g)$, and by Hölder,

$$\|f_{>Q} g_{>Q}\|_{L^c} \le \lambda_Q^{-s-t} D^s_{a,Q}(f) D^t_{b,Q}(g).$$

Next, we have

$$\|f_q(\cdot - y) - f_q(\cdot)\|_{L^a} = \left\| \int_0^1 (\nabla f_q)(\cdot - \theta y) \cdot y\,\mathrm{d}\theta \right\|_{L^a} \le |y| \|\nabla f_q\|_{L^a} \lesssim |y| \lambda_q \|f_q\|_{L^a}. \tag{148}$$

We use (148) for $q \leq Q$ in the following estimate:

$$
\begin{aligned}
\|f(\cdot - y) - f(\cdot)\|_{L^a} &\lesssim \lambda_Q^{1-s} \sum_{q \leq Q} \lambda_Q^{s-1} \|f_q(\cdot - y) - f_q(\cdot)\|_{L^a} + \lambda_Q^{-s} \sum_{q>Q} \lambda_Q^s \|f_q(\cdot - y) - f_q(\cdot)\|_{L^a} \\
&\lesssim \lambda_Q^{1-s} \sum_{q \leq Q} \lambda_{Q-q}^{s-1} \lambda_q^{s-1} \cdot |y| \lambda_q \|f_q\|_{L^a} + \lambda_Q^{-s} \sum_{q>Q} \lambda_{Q-q}^s \lambda_q^s \|f_q\|_{L^a} \\
&= \lambda_Q^{1-s} |y| \sum_{q \leq Q} K_{Q-q}^s d_{a,q}^s(f) + \lambda_Q^{-s} \sum_{q>Q} K_{Q-q}^s d_{a,q}^s(f) \\
&\leq (\lambda_Q |y| + 1) \lambda_Q^{-s} D_{a,Q}^s(f).
\end{aligned}
$$

Clearly $\|g(\cdot - y) - g(\cdot)\|_{L^b} \leq (\lambda_Q |y| + 1) \lambda_Q^{-t} D_{b,Q}^t(g)$, by the same argument. Now we can easily estimate $r_Q(f,g)$:

$$
\begin{aligned}
\|r_Q(f,g)\|_{L^c} &\leq \int |\widetilde{h}_Q(y)| \|f(\cdot - y) - f(\cdot)\|_{L^a} \|g(\cdot - y) - g(\cdot)\|_{L^b} \, \mathrm{d}y \\
&\lesssim \left(\int |\widetilde{h}_Q(y)| (\lambda_Q |y| + 1)^2 \, \mathrm{d}y \right) \lambda_Q^{-s-t} D_{a,Q}^s(f) D_{b,Q}^t(g) \\
&\lesssim \lambda_Q^{-s-t} D_{a,Q}^s(f) D_{b,Q}^t(g).
\end{aligned}
$$

This proves (143). The proof of (144) follows the same lines, except we apply $\|g_{>Q}\|_{L^\infty} \leq \|g\|_{L^\infty}$, and $\|g(\cdot - y) - g(\cdot)\|_{L^\infty} \leq 2\|g\|_{L^\infty}$. The latter results in the term $(\lambda_Q |y| + 1)$ with power 1 inside the h_Q-integral, which is also bounded uniformly in Q.

Finally, we write

$$
\begin{aligned}
\|\nabla f_{\leq Q}\|_{L^a} \lesssim \lambda_Q^{1-s} \sum_{q \leq Q} \lambda_Q^{s-1} \|\nabla f_q\|_{L^a} &\lesssim \lambda_Q^{1-s} \sum_{q \leq Q} \lambda_{Q-q}^{s-1} \lambda_q^{s-1} \cdot \lambda_q \|f_q\|_{L^a} \\
= \lambda_Q^{1-s} \sum_{Q-q \geq 0} K_{Q-q}^s d_{a,q}^s(f) &\leq \lambda_Q^{1-s} D_{a,Q}^s(f).
\end{aligned}
$$

□

PROPOSITION 1.3. *Let* $f \in B_{a,\infty}^s$, $g \in B_{b,\infty}^s$, $a, b \in [1, \infty]$, $s \in (0,1)$, $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. *Then*

$$
\|\nabla (fg)_{\leq Q}\|_{L^c} \lesssim \lambda_Q^{1-s} (D_{a,Q}^s(f) \|g\|_{L^b} + D_{b,Q}^s(g) \|f\|_{L^a}). \tag{149}
$$

PROOF. First, notice that if p or r is greater than $Q + 2$ and $|p - r| > 2$, then the Fourier support of $f_p g_r$ lies outside the ball of radius λ_{Q+1} centered at 0. In particular, $(f_p g_r)_{\leq Q}$ vanishes. Therefore

$$
(fg)_{\leq Q} = (f_{\leq Q+2} g_{\leq Q+2})_{\leq Q} + \sum_{\substack{\max\{p,r\} > Q+2 \\ |p-r| \leq 2}} (f_p g_r)_{\leq Q},
$$

so we have

$$(150)\quad \|\nabla(fg)_{\le Q}\|_{L^c} \le \|\nabla(f_{\le Q}g_{\le Q})\|_{L^c} + \|\nabla(f_{\sim Q}g_{\le Q} + f_{\le Q}g_{\sim Q})\|_{L^c} + \sum_{\substack{p,r>Q\\|p-r|\le 2}} \|\nabla(f_p g_r)_{\le Q}\|_{L^c}.$$

We estimate each of the terms on the right side of this inequality. First, we have

$$(151)\quad \|\nabla(f_{\le Q}g_{\le Q})\|_{L^c} \le \|\nabla f_{\le Q}\|_{L^a}\|g\|_{L^b} + \|\nabla g_{\le Q}\|_{L^b}\|f\|_{L^a} \lesssim \lambda_Q^{1-s}\big(D^s_{a,Q}(f)\|g\|_{L^b} + D^s_{b,Q}(g)\|f\|_{L^a}\big).$$

Next,

$$\begin{aligned}\|\nabla(f_{\sim Q}g_{\le Q})\|_{L^c} &\lesssim \|\nabla f_{\sim Q}\|_{L^a}\|g\|_{L^b} + \|\nabla g_{\le Q}\|_{L^b}\|f\|_{L^a}\\ &\lesssim \lambda_Q^{1-s}\big(D^s_{a,Q}(f)\|g\|_{L^b} + D^s_{b,Q}(g)\|f\|_{L^a}\big),\end{aligned}$$

where we note that $\|\nabla f_{\sim Q}\|_{L^a} \sim \lambda_Q^{1-s} D^s_{a,Q}(f)$ and use (145) in order to obtain the second inequality. We can estimate $\|\nabla(f_{\le Q}g_{\sim Q})\|_{L^c}$ similarly, concluding that

$$(152)\qquad \|\nabla(f_{\sim Q}g_{\le Q} + f_{\le Q}g_{\sim Q})\|_{L^c} \lesssim \lambda_Q^{1-s}(D^s_{a,Q}(f)\|g\|_{L^b} + D^s_{b,Q}(g)\|f\|_{L^a}).$$

By differential Bernstein's and Hölder inequalities we have

$$\|\nabla(f_p g_r)_{\le Q}\|_{L^c} \lesssim \lambda_Q \|f_p\|_{L^a}\|g_r\|_{L^b}$$

Using this we obtain

$$\sum_{\substack{p,r>Q\\|p-r|\le 2}} \|\nabla(f_p g_r)_{\le Q}\|_{L^c} \lesssim \lambda_Q^{1-s}\sum_{p>Q} \lambda^s_{Q-q}\lambda^s_q \|f_p\|_{L^a}\|g\|_{L^b} \le \lambda_Q^{1-s} D^s_{a,Q}(f)\|g\|_{L^b}.$$

Combining this estimate with (150), (151), and (152) immediately yields the desired statement. □

REMARK 1.4. *One can also show (by a proof nearly identical to the above) that if $f, g \in B^s_{a,\infty} \cap L^b$ with $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $a, b \in [1,\infty]$, then $\|\nabla(fg)_{\le Q}\|_{L^c} \lesssim \lambda_Q^{1-s}(D^s_{a,Q}(f)\|g\|_{L^b} + D^s_{a,Q}(g)\|f\|_{L^b})$.*

REMARK 1.5. *Recall the following result for the classical Navier-Stokes equations (i.e. (16) and (18), with $\rho \equiv 1$, $f \equiv 0$): If (u,p) is a weak solution, with $u \in C^\alpha$ for some $\alpha \in (0,1)$, then $p = \Delta^{-1}(\operatorname{div}\operatorname{div}(u\otimes u)) \in C^\alpha$. We can generalize this result using Proposition 1.3: Assume $u \in B^s_{a,\infty}$, with $a \in [2,\infty]$ and $s \in (0,1)$; then $p \in B^s_{a/2,\infty}$. Indeed, we have*

$$\lambda_Q^s\|p_Q\|_{L^{a/2}} \sim \lambda_Q^{-(1-s)}\|\operatorname{div}(u\otimes u)_Q\|_{L^{a/2}} \lesssim (D^s_{a,Q}(u))^2.$$

This observation motivates our integrability assumption on p in Theorem 2.1 of the Introduction.

2. Estimates on the Flux

First, we give a decomposition of $F_Q(\rho, u)$ which is more conducive to estimates. In order to do so we define, in analogy with (147), the quantity

$$r_Q(\rho, u, u) = \int \widetilde{h}_Q(y)[\rho(x-y) - \rho(x)][u(x-y) - u(x)] \otimes [u(x-y) - u(x)]\,\mathrm{d}y.$$

Lemma 2.1. *$F_Q(\rho, u)$ can be written as*

$$\begin{aligned} F_Q(\rho, u) =& r_Q(\rho, u, u) - \frac{1}{\rho_{\leq Q}}[(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \otimes [(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] + \rho_{>Q} u_{>Q} \otimes u_{>Q} \\ & + 2Sym([(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \otimes u_{>Q}) + \rho[(u \otimes u)_{\leq Q} - u_{\leq Q} \otimes u_{\leq Q}]. \end{aligned} \tag{153}$$

Proof. We can write

$$\begin{aligned} r_Q(\rho, u, u) &= (\rho u \otimes u)_{\leq Q} - 2Sym[(\rho u)_{\leq Q} \otimes u] + \rho_{\leq Q} u \otimes u - \rho r_Q(u, u) \\ &= (\rho u \otimes u)_{\leq Q} - 2Sym[((\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}) \otimes u] \\ &\quad + \rho_{\leq Q}(u \otimes u - u_{\leq Q} \otimes u - u \otimes u_{\leq Q}) - \rho r_Q(u, u) \\ &= (\rho u \otimes u)_{\leq Q} - 2Sym[((\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}) \otimes u] \\ &\quad - \rho_{\leq Q} u_{\leq Q} \otimes u_{\leq Q} + \rho_{\leq Q} u_{>Q} \otimes u_{>Q} - \rho r_Q(u, u) \\ &= (\rho u \otimes u)_{\leq Q} - 2Sym[((\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}) \otimes u] - \rho_{\leq Q} u_{\leq Q} \otimes u_{\leq Q} \\ &\quad - \rho[(u \otimes u)_{\leq Q} - u_{\leq Q} \otimes u_{\leq Q}] - \rho_{>Q} u_{>Q} \otimes u_{>Q}, \end{aligned}$$

where Sym denotes the symmetric part. Therefore

$$\begin{aligned} (\rho u \otimes u)_{\leq Q} &= r_Q(\rho, u, u) + 2Sym([(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \otimes u) \\ &\quad + \rho[(u \otimes u)_{\leq Q} - u_{\leq Q} \otimes u_{\leq Q}] + \rho_{\leq Q} u_{\leq Q} \otimes u_{\leq Q} + \rho_{>Q} u_{>Q} \otimes u_{>Q}. \end{aligned}$$

Since we also have

$$\begin{aligned} \frac{(\rho u)_{\leq Q} \otimes (\rho u)_{\leq Q}}{\rho_{\leq Q}} &= \frac{1}{\rho_{\leq Q}}[(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \otimes [(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \\ &\quad + 2Sym([(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}] \otimes u_{\leq Q}) + \rho_{\leq Q} u_{\leq Q} \otimes u_{\leq Q}, \end{aligned}$$

subtracting the right sides of the last two equations gives the desired representation. □

Theorem 2.2. *Assume that* $0 < \underline{\rho} \le \rho \le \overline{\rho} < \infty$ *and that* (ρ, u, p) *satisfies*

$$\rho \in B^{1/3}_{a,\infty},\ u \in B^{1/3}_{b,c_0},\ p \in B^{1/3}_{b/2,\infty}, \quad \frac{1}{a} + \frac{3}{b} = 1,\ b \in [3, \infty]. \tag{154}$$

Then the flux Π_Q *defined by* (140) *tends to zero as* $Q \to \infty$.

Proof. Clearly

$$\|r_Q(\rho, u, u)\|_{L^{b/2}} \lesssim \int |\widetilde{h}_Q(y)| \|u(\cdot - y) - u(\cdot)\|^2_{L^b}\, \mathrm{d}y,$$

and we can follow the proof of Proposition 1.1 to conclude $\|r_Q(\rho, u, u)\|_{L^{b/2}} \lesssim \lambda_Q^{-2/3}(D^{1/3}_{b,Q}(u))^2$.

Using (144), we can estimate

$$\left\| \frac{1}{\rho_{\le Q}}[(\rho u)_{\le Q} - \rho_{\le Q} u_{\le Q}] \otimes [(\rho u)_{\le Q} - \rho_{\le Q} u_{\le Q}] \right\|_{L^{b/2}} \lesssim \underline{\rho}^{-1}(\lambda_Q^{-1/3} D^{1/3}_{b,Q}(u)\overline{\rho})^2 \lesssim \lambda_Q^{-2/3}(D^{1/3}_{b,Q}(u))^2$$

Using $\|\rho_{>Q}\|_{L^\infty} \le \overline{\rho}$ and (146), we get $\|\rho_{>Q} u_{>Q} \otimes u_{>Q}\|_{L^{b/2}} \lesssim \lambda_Q^{-2/3}(D^{1/3}_{b,Q}(u))^2$.

Combining (144) and (146) yields

$$\|[(\rho u)_{\le Q} - \rho_{\le Q} u_{\le Q}] \otimes u_{>Q}\|_{L^{b/2}} \le (\lambda_Q^{-1/3} D^{1/3}_{b,Q}(u)\overline{\rho})(\lambda_Q^{-1/3} D^{1/3}_{b,Q}(u)) \lesssim \lambda_Q^{-2/3}(D^{1/3}_{b,Q}(u))^2$$

Finally,

$$\|\rho[(u \otimes u)_{\le Q} - u_{\le Q} \otimes u_{\le Q}]\|_{L^{b/2}} \lesssim \overline{\rho}\lambda^{-2/3}(D^{1/3}_{b,Q}(u))^2 \lesssim \lambda^{-2/3}(D^{1/3}_{b,Q}(u))^2.$$

Therefore,

$$\|F_Q(\rho, u)\|_{L^{b/2}} \lesssim \lambda_Q^{-2/3}(D^{1/3}_{b,Q}(u))^2.$$

We also have $\nabla U = \rho^{-1}_{\le Q}\nabla(\rho u)_{\le Q} - \rho^{-2}_{\le Q}(\rho u)_{\le Q} \otimes \nabla \rho_{\le Q}$. Write $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$. Then using the two Propositions of the previous section, we estimate:

$$\|\nabla U\|_{L^c} \lesssim \|\nabla(\rho u)_{\le Q}\|_{L^c} + \|\rho u\|_{L^b}\|\nabla \rho_{\le Q}\|_{L^a} \lesssim \lambda_Q^{2/3}\big(D^{1/3}_{a,Q}(\rho)\|u\|_{L^b} + D^{1/3}_{b,Q}(u)\big).$$

Therefore

$$\int F_Q(\rho, u) : \nabla U\, \mathrm{d}x \lesssim (D^{1/3}_{b,Q}(u))^2\big(D^{1/3}_{a,Q}(\rho)\|u\|_{L^b} + D^{1/3}_{b,Q}(u)\big) \tag{155}$$

Next, we deal with the pressure term. Note that by (22), we have

$$\int p_{\le Q} \operatorname{div} U\, \mathrm{d}x = -\int \nabla p_{\le Q} \cdot (U - u_{\le Q})\, \mathrm{d}x.$$

So

$$\int_{\mathbb{T}^d} p_{\leq Q} \operatorname{div} U \,\mathrm{d}x \lesssim \|\nabla p_{\leq Q}\|_{L^{b/2}} \|(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}\|_{L^c}$$
$$\lesssim \lambda_Q^{2/3} D_{b/2,Q}^{1/3}(p) \cdot \lambda^{-2/3} D_{a,Q}^{1/3}(\rho) D_{b,Q}^{1/3}(u) = D_{a,Q}^{1/3}(\rho) D_{b,Q}^{1/3}(u) D_{b/2,Q}^{1/3}(p).$$

Thus

$$|\Pi_Q| \lesssim D_{b,Q}^{1/3}(u) \Big[D_{b,Q}^{1/3}(u) \big(D_{a,Q}^{1/3}(\rho) \|u\|_{L^b} + D_{b,Q}^{1/3}(u) \big) + D_{a,Q}^{1/3}(\rho) D_{b/2,Q}^{1/3}(p) \Big].$$

In view of (142) and our assumptions on ρ, u, p, the bracketed term in each estimate is uniformly bounded in Q, while $D_{b,Q}^{1/3}(u)$ tends to zero as $Q \to \infty$. Therefore $\lim_{Q\to\infty} \Pi_Q = 0$, as claimed. □

Note that we obtain Theorem 2.2 from the Introduction as a corollary: By Theorem 2.2 (from this chapter), as well as (142) and the dominated convergence theorem, we have

$$E_{\leq Q}(t) - E_{\leq Q}(0) = \int_0^t \Pi_Q(s) \,\mathrm{d}s \xrightarrow{Q\to\infty} 0.$$

Now we prove Theorem 2.1 from the Introduction:

Proof. It remains to show $\varepsilon_Q(t) \to \varepsilon(t)$ and $\int_0^t \int (\rho f)_{\leq Q} \cdot U \,\mathrm{d}x \,\mathrm{d}s \to \int_0^t \int \rho u \cdot f \,\mathrm{d}x \,\mathrm{d}s$. Now

$$\int \nabla u_{\leq Q} : \nabla U \,\mathrm{d}x = \int \nabla u_{\leq Q} : \nabla (U - u_{\leq Q}) \,\mathrm{d}x + \|\nabla u_{\leq Q}\|_{L^2}^2,$$

and clearly

$$\int_0^t \|\nabla u_{\leq Q}(s)\|_{L^2}^2 \,\mathrm{d}s \to \int_0^t \|\nabla u(s)\|_{L^2}^2 \,\mathrm{d}s.$$

Next,

$$\int \nabla u_{\leq Q} : \nabla (U - u_{\leq Q}) \,\mathrm{d}x = -\int \Delta u_{\leq Q} : ((\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}) \rho_{\leq Q}^{-1} \,\mathrm{d}x.$$

Using (144) and the remark following Proposition 1.1 we estimate

$$\left| \int \Delta u_{\leq Q} : ((\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}) \rho_{\leq Q}^{-1} \,\mathrm{d}x \right| \leq \|\Delta u_{\leq Q}\|_{L^2} \lambda_Q^{-1} \|u\|_{H^1} \|\rho\|_{L^\infty} \underline{\rho}^{-1}.$$

Then

$$\|\Delta u_{\leq Q}\|_{L^2} \lambda_Q^{-1} \leq \left(\sum_{q \leq Q} \lambda_{q-Q}^2 \|\nabla u_q\|_{L^2}^2 \right)^{1/2}.$$

Since the latter vanishes as $Q \to \infty$ a.e. in time and is uniformly bounded by the dominant H^1-norm of u we obtain

$$\int_0^t \|\Delta u_{\leq Q}\|_{L^2} \lambda_Q^{-1} \|u\|_{H^1} \,\mathrm{d}s \leq \|u\|_{L^2 H^1} \left(\int_0^t \sum_{q \leq Q} \lambda_{q-Q}^2 \|\nabla u_q\|_{L^2}^2 \,\mathrm{d}s \right)^{1/2} \to 0.$$

Finally, the convergence $\int_0^t \int (\rho f)_{\leq Q} \cdot U \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int \rho u \cdot f \,\mathrm{d}x\,\mathrm{d}s$ is rather straightforward. Indeed, write

$$\rho f \cdot u - (\rho f)_{\leq Q} \cdot U = (\rho f - (\rho f)_{\leq Q})u + (\rho f)_{\leq Q}(u - U).$$

Note that $u, \rho f \in L^2_{t,x}$, hence $(\rho f)_{\leq Q} \to \rho f$ strongly in $L^2_{t,x}$, and hence $\int (\rho f - (\rho f)_{\leq Q})u \to 0$. Similarly, $u - U = \frac{1}{\rho_{\leq Q}}(\rho_{\leq Q} u - (\rho u)_{\leq Q}) = \frac{1}{\rho_{\leq Q}}(\rho_{\leq Q} u_{\leq Q} - (\rho u)_{\leq Q}) + u_{>Q}$. Again, $u_{>Q} \to 0$ in $L^2_{t,x}$, while for the difference $\rho_{\leq Q} u_{\leq Q} - (\rho u)_{\leq Q}$ we can use (144) with $s = 1$, $a = 2$ to conclude that it also tends to zero in $L^2_{t,x}$. This finishes the proof. □

Remark 2.3. *Let us discuss a few extensions. First, one can see from the proof that the full strength of the integrability in time assumption on u was not used. Rather, the hypothesis $u \in L^b(0, T; B^{1/3}_{b,c_0})$ can be replaced by the weaker assumption that*

$$\lim_{q \to 0} \int_0^T \lambda_q^{b/3} \|u_q\|_{L^b}^b \,\mathrm{d}s = 0.$$

This is equivalent to a space-time averaged increment condition

$$\lim_{y \to 0} \frac{1}{|y|^{b/3}} \int_0^T \int_{\mathbb{T}^d} |u(x+y,t) - u(x,t)|^b \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Second, time integrability in (25) can be replaced with its own exponents

$$\rho \in L^{a'} B^{\frac{1}{3}}_{a,\infty},\ u \in L^{b'} B^{\frac{1}{3}}_{b,c_0},\ p \in L^{\frac{b'}{2}} B^{\frac{1}{3}}_{\frac{b}{2},\infty}, \quad \frac{1}{a} + \frac{3}{b} = 1,\ \frac{1}{a'} + \frac{3}{b'} = 1.$$

Finally, it appears possible to extend the results to the system with density-dependent kinematic viscosity $\mu = \mu(\rho)$ with sufficiently smooth μ. We leave calculations pertaining to this case to future research.

CHAPTER 5

Energy Equality and Well-Posedness for the Fractional Euler Alignment Model[1]

1. Existence of and Bounds on Regular Solutions

The proof of local-in-time existence of regular solutions of (26)–(27) follows the arguments of [54], [56] with trivial adjustments to account for the forcing term. The proof of global-in-time existence mostly carries over to the forced case. The only step that requires adjustment from the unforced argument is proving L^∞ bounds on the quantities u, ρ, ρ^{-1}, and e that do not blow up in finite time. Once such bounds are established, the proof of global-in-time existence again requires only minor adjustments to the proof in the unforced case. In what follows we assume that (u, ρ) is a regular solution and that $e = u' - \Lambda_\alpha \rho$ and $q = e/\rho$. For the purposes of proving global existence of regular solutions, we only need to show that (u, ρ) remains bounded in $H^4 \times H^{3+\alpha}$ on bounded time intervals (or, effectively, we need to show that u, ρ, and e remain bounded in L^∞ on bounded time intervals); however, we will later construct weak solutions as limits of regular solutions, and in order to ensure that our L^∞ bounds survive the limiting process, we track the dependencies of these bounds carefully and relate each to the initial data. The limiting process we use also requires some kind of compactness, which we satisfy by proving bounds in Hölder spaces. We also derive the energy laws (44) and (45) that are satisfied by regular solutions. These equalities in particular show that u and ρ are bounded in $L^2 H^{\alpha/2}$ on finite time intervals, with bounds depending only on the L^∞ norms of the initial data and some other fixed quantities.

1.1. L^∞ Bounds on First-Order Quantities.

We collect the bounds on u, ρ, and e together in the following proposition.

PROPOSITION 1.1. *Let (u, ρ) be a regular solution on the time interval $[0, \infty)$, and define e and q as above. The following bounds hold for some positive constants c_i, $i = 0, 1, \dots, 4$ and for all $x \in \mathbb{T}$,*

[1] This chapter is taken from:
[36] T. M. Leslie. Weak and Strong Solutions to the Forced Fractional Euler Alignment System. *ArXiv e-prints*, March 2018.

$t \geq 0$.

$$|u(x,t)| \leq \|u_0\|_{L^\infty} + t\|f\|_{L^\infty_{x,t}} \tag{156}$$

$$c_0 \exp(-c_1 t) \leq \rho(x,t) \leq c_3 \exp(c_4 t) \tag{157}$$

$$|q(x,t)| \leq c_2 \exp(c_1 t) \tag{158}$$

$$|e(x,t)| \leq c_2 c_3 \exp((c_1 + c_4)t) \tag{159}$$

The constants c_0 and c_2 depend only on $\|\rho_0^{-1}\|_{L^\infty}$, $\|q_0\|_{L^\infty}$, $\mathcal{M}$, and α; the constant c_1 depends only on $\|f'\|_{L^\infty_{x,t}}$, $\mathcal{M}$, and α; the constant c_3 depends only on $\|\rho_0\|_{L^\infty}$, c_2, $\mathcal{M}$, and α; and c_4 depends only on c_1 and α.

Remark 1.2. Our proof of the lower bound on the density is the same in spirit of the corresponding bound in [32] and uses a 'breakthrough scenario' type argument. We include the argument for our force f in its entirety for the sake of completeness.

Throughout the proof of Proposition 1.1 (as well as in Section 4.1 below), we will tacitly make use of the following application of the classical Rademacher Theorem:

Lemma 1.3. *Suppose $g : \mathbb{T} \times \mathbb{R}_+ \to \mathbb{R}$ is a Lipschitz function, such that for every $x \in \mathbb{T}$, the function $g(x, \cdot)$ is differentiable on all of $\mathbb{R}_+$. For each $t \in \mathbb{R}_+$, let $x_+(t)$ and $x_-(t)$ denote points in $\mathbb{T}$ where $g(\cdot, t)$ achieves its maximum and minimum, respectively, and define $g_+(t) = g(x_+(t), t)$ and $g_-(t) = g(x_-(t), t)$. Then*

- *$g_+(t)$ and $g_-(t)$ are Lipschitz, with the same Lipschitz constant as g, and*
- *$\partial_t g_+(t) = \partial_t g(x_+(t), t)$ and $\partial_t g_-(t) = \partial_t g(x_-(t), t)$ for a.e. $t \in \mathbb{R}_+$.*

For a proof of this precise statement, see for example the Appendix of [13].

Proof of Proposition 1.1. *Part 1: Bounds on u*

Let $x_+(t)$ and $x_-(t)$ denote a maximum and a minimum, respectively, for $u(\cdot, t)$.

We write the velocity equation as

$$u_t + uu' = -\Lambda_\alpha(\rho u) + u\Lambda_\alpha \rho + f = \int_{\mathbb{R}} \frac{u(\cdot + z) - u}{|z|^{1+\alpha}} \rho(\cdot + z)\, \mathrm{d}z + f. \tag{160}$$

It is clear from the form of the integral in (160) and the nonnegativity of ρ that

$$[-\Lambda_\alpha(\rho u) + u\Lambda_\alpha(\rho)](x_+(t), t) \leq 0 \leq [-\Lambda_\alpha(\rho u) + u\Lambda_\alpha(\rho)](x_-(t), t).$$

This immediately implies that

$$[\partial_t u - f](x_+(t), t) \le 0 \le [\partial_t u - f](x_-(t), t),$$

so that

$$\partial_t \|u(t)\|_{L^\infty} \le \|f\|_{L^\infty_{t,x}},$$

from which (156) follows by integration.

Part 2: Lower Bound on ρ; Bounds on q

To avoid estimating ρ in terms of derivatives of u, we rewrite the density equation as

$$\rho_t + u\rho' = -q\rho^2 - \rho\Lambda_\alpha \rho. \tag{161}$$

Evaluating at a minimum $x_-(t)$ of $\rho(\cdot, t)$, we obtain

$$\partial_t \rho(x_-(t), t) \ge -\|q(t)\|_{L^\infty} \rho_-(t)^2 + \rho_-(t)\Lambda_\alpha \rho(x_-(t), t). \tag{162}$$

Here $\rho_-(t)$ denotes the minimum of ρ at time t. (Below we will use the analogous notation $\rho_+(t)$ for the maximum at time t.) We now require a bound on q, which is feasible because of the transport equation (34) that it satisfies. We have

$$\|q(t)\|_{L^\infty} \le \|q_0\|_{L^\infty} + \|f'\|_{L^\infty_{x,t}} \int_0^t \rho_-(s)^{-1} \, \mathrm{d}s, \tag{163}$$

Now we can substitute (163) into (162) to eliminate $q(t)$. We obtain

$$\partial_t \rho(x_-(t), t) \ge -\left[\|q_0\|_{L^\infty} + \|f'\|_{L^\infty_{x,t}} \int_0^t \rho_-(s)^{-1} \, \mathrm{d}s\right] \rho_-(t)^2 + \rho_-(t)\Lambda_\alpha \rho(x_-(t), t). \tag{164}$$

In order to establish the desired lower bound on ρ, we use (164) and argue by contradiction. But let us first define the constants involved. Denote $\iota(r) := \inf_{|x|<r} \phi_\alpha(x)$, where ϕ_α is as in (28). Note that $\iota(r) < \infty$ for all $r > 0$ and that $\iota(\pi) = \inf_{x\in\mathbb{T}} \phi_\alpha(x) > 0$. Define

$$c_0 = \frac{1}{2} \min\left\{\rho_-(0),\ \frac{\iota(\pi)\mathcal{M}}{2\pi\iota(\pi) + \|q_0\|_{L^\infty}}\right\}, \quad c_1 = \frac{2\|f'\|_{L^\infty_{x,t}}}{\iota(\pi)\mathcal{M}}, \tag{165}$$

We claim that the desired lower bound on ρ holds for this choice of c_0, c_1. Indeed, suppose that the lower bound in (157) fails; then we can define $t_0 := \inf\{t \ge 0 : \rho_-(t) = c_0 \exp(-c_1 t)\}$. Clearly $t_0 > 0$, since $\rho_-(0) \ge 2c_0$ by definition of c_0. Furthermore, $\rho_-(s) \ge c_0 \exp(-c_1 s)$ for $s \in [0, t_0]$, so that

$$\begin{aligned}
\|q_0\|_{L^\infty} + \|f'\|_{L^\infty_{x,t}} \int_0^{t_0} \rho_-(s)^{-1}\,\mathrm{d}s &\le \|q_0\|_{L^\infty} + \|f'\|_{L^\infty_{x,t}} \int_0^{t} c_0^{-1}\exp(c_1 s)\,\mathrm{d}s \\
&= \|q_0\|_{L^\infty} + \frac{\iota(\pi)\mathcal{M}}{2}[\rho_-(t_0)^{-1} - c_0^{-1}] \\
&\le \|q_0\|_{L^\infty} + \frac{\iota(\pi)\mathcal{M}}{2}\left[\rho_-(t_0)^{-1} - \frac{2(2\pi\iota(\pi) + \|q_0\|_{L^\infty})}{\iota(\pi)\mathcal{M}}\right] \\
&= \frac{\iota(\pi)\mathcal{M}}{2\rho_-(t_0)} - 2\pi\iota(\pi).
\end{aligned}$$

We also have

$$\begin{aligned}
\Lambda_\alpha \rho(x_-(t_0), t_0) &= \int_{\mathbb{T}} \phi_\alpha(z)(\rho(x_-(t_0)+z, t_0) - \rho_-(t_0))\,\mathrm{d}z \\
&\ge \int_{\mathbb{T}} \iota(\pi)(\rho(x_- + z, t_0) - \rho_-(t_0))\,\mathrm{d}z \\
&= \iota(\pi)\mathcal{M} - 2\pi\iota(\pi)\rho_-(t_0).
\end{aligned}$$

Substituting the two previous estimates into (164) then gives us

$$\begin{aligned}
\partial_t \rho(x_-(t_0), t_0) &\ge -\left[\frac{\iota(\pi)\mathcal{M}}{2\rho_-(t_0)} - 2\pi\iota(\pi)\right]\rho_-(t_0)^2 + \rho_-(t_0)[\iota(\pi)\mathcal{M} - 2\pi\iota(\pi)\rho_-(t_0)] \\
&= \frac{\rho_-(t_0)\iota(\pi)\mathcal{M}}{2} > 0.
\end{aligned}$$

It follows that $\rho_-(s) < c_0\exp(-c_1 s)$ for some time $s < t_0$, contradicting our choice of t_0. This proves the lower bound on ρ.

The bound (158) on q is obtained, with $c_2 = \iota(\pi)\mathcal{M}/(2c_0)$, by substituting the lower bound on ρ into (163).

Part 3: Upper Bound on ρ; Bounds on e

For this bound, we exploit the singularity of the kernel, using the fact that $\limsup_{r\to 0} r\iota(r) = \infty$. Recall that for $z \in [-\pi,\pi]\backslash\{0\}$, $\phi_\alpha(z)$ is defined as in (28), so that for $r \in (0,\pi]$, we have

$$\iota(r) = \frac{1}{r^{1+\alpha}} + \sum_{k\in\mathbb{N}} \frac{1}{(2\pi k + r)^{1+\alpha}} + \sum_{k\in\mathbb{N}} \frac{1}{(2\pi k - r)^{1+\alpha}}.$$

Both the sums in the equation above are bounded by some constant: $\iota(r) \le r^{-1-\alpha} + C$. Let $r_0 \in (0,\pi)$ be such that $r_0^{-1-\alpha} = C$. (This is certainly possible, because taking $r = \pi$ in the second sum above shows that $C > \pi^{-1-\alpha}$.) Then

$$r^{-1-\alpha} \le \iota(r) \le 2r^{-1-\alpha}, \qquad \text{for } r \in (0, r_0). \tag{166}$$

Now we define

$$c_3 = \max\left\{2\|\rho_0\|_{L^\infty_x}, 4\mathcal{M}(2c_2)^{\frac{1}{\alpha}}, \frac{2}{c_2}\mathcal{M}r_0^{-1-\alpha}\right\}, \quad c_4 = \frac{1+\alpha}{\alpha}c_1. \tag{167}$$

We claim that the desired upper bound for ρ holds for this choice of c_3, c_4. Suppose that the upper bound of (157) fails; then we can define $t_0 := \inf\{t \geq 0 : \rho_+(t) = c_3\exp(c_4 t)\}$. Clearly $t_0 > 0$, since $\rho_+(0) \leq c_3/2$ by definition of c_3.

Let $x_+(t_0)$ denote the x-value where the maximum of $\rho(\cdot, t_0)$ is achieved and put

$$r_1 := \min\left\{(2c_2\exp(c_1 t_0))^{-\frac{1}{\alpha}}, r_0\right\}.$$

Then

$$\begin{aligned}
\partial_t\rho(x_+(t_0), t_0) &\leq \|q(t_0)\|_{L^\infty}\rho_+(t_0)^2 + \rho_+(t_0)\int_{|z|<r_1}\phi_\alpha(z)(\rho(x_+(t_0)+z, t_0) - \rho_+(t_0))\,\mathrm{d}z \\
&\leq \|q(t_0)\|_{L^\infty}\rho_+(t_0)^2 + \iota(r_1)\rho_+(t_0)(\mathcal{M} - r_1\rho_+(t_0)) \\
&= [\|q(t_0)\|_{L^\infty} - r_1\iota(r_1)]\rho_+(t_0)^2 + \iota(r_1)\mathcal{M}\rho_+(t_0).
\end{aligned}$$

By choice of r_1, we have

$$r_1\iota(r_1) \geq r_1^{-\alpha} \geq 2c_2\exp(c_1 t_0).$$

Combining this with the upper bound in (158), we continue the estimate above:

$$\begin{aligned}
\partial_t\rho(x_+(t_0), t_0) &\leq [c_2\exp(c_1 t_0) - 2c_2\exp(c_1 t_0)]\rho_+(t_0)^2 + \iota(r_1)\mathcal{M}\rho_+(t_0) \\
&= [\iota(r_1)\mathcal{M} - c_2\exp(c_1 t_0)\rho_+(t_0)]\rho_+(t_0).
\end{aligned}$$

But then by our choices of r_0, r_1, c_3, c_4, and t_0, we have

$$\begin{aligned}
\iota(r_1)\mathcal{M} &\leq 2\left[\min\left\{(2c_2\exp(c_1 t_0))^{-\frac{1}{\alpha}}, r_0\right\}\right]^{-1-\alpha}\cdot\mathcal{M} \\
&= 2\mathcal{M}\max\left\{(2c_2\cdot(2c_2)^{\frac{1}{\alpha}}\exp(c_4 t_0), r_0^{-1-\alpha}\right\} \\
&\leq c_2\max\left\{4\mathcal{M}\,(2c_2)^{\frac{1}{\alpha}}, \frac{2}{c_2}\mathcal{M}r_0^{-1-\alpha}\right\}\exp(c_4 t_0) \\
&\leq c_2 c_3\exp(c_4 t_0) = c_2\rho_+(t_0),
\end{aligned}$$

which implies that

$$\partial_t\rho(x_+(t_0), t_0) \leq c_2\rho_+(t_0)^2[1 - \exp(c_1 t_0)] < 0,$$

contradicting the definition of t_0. This proves the upper bound on ρ.

In light of the relation $e = q\rho$, the upper bound (159) on e is obtained by multiplying the upper bounds for q and ρ. $\square$

As noted above, this essentially completes the proof of the existence of a global-in-time regular solution for any initial data $(u_0, \rho_0) \in H^4 \times H^{3+\alpha}$ away from vacuum.

1.2. Energy Equality and Bounds in $L^2H^{\alpha/2}$. Next, we recall that regular solutions satisfy the certain energy equalities (44), (45). Multiplying the velocity equation by ρ and the density equation by u, then adding the results together, we obtain the momentum equation:

$$(\rho u)_t + (\rho u^2)' = -\rho\Lambda_\alpha(\rho u) + \rho u \Lambda_\alpha \rho + \rho f. \tag{168}$$

Multiply (168) by u and add ρu times the velocity equation. The result is

$$(\rho u^2)_t + (\rho u^3)' = -2\rho u[\Lambda_\alpha(\rho u) - u\Lambda_\alpha \rho] + 2\rho u f, \tag{169}$$

or, after integration,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho u^2 \,\mathrm{d}x &= -\int_{\mathbb{T}} 2\rho u \Lambda_\alpha(\rho u) - 2\rho u^2 \Lambda_\alpha \rho \,\mathrm{d}x + 2\int_{\mathbb{T}} \rho u f \,\mathrm{d}x \\
&= -\int_{\mathbb{T}} 2\rho u \Lambda_\alpha(\rho u) - \rho u^2 \Lambda_\alpha \rho - \rho \Lambda_\alpha(\rho u^2) \,\mathrm{d}x + 2\int_{\mathbb{T}} \rho u f \,\mathrm{d}x \\
&= -\int_{\mathbb{T}}\int_{\mathbb{R}} \rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}} \,\mathrm{d}y\,\mathrm{d}x + 2\int_{\mathbb{T}} \rho u f \,\mathrm{d}x.
\end{aligned}$$

We used the self-adjointness of Λ_α to pass from the first to the second line. Integrating in time, we obtain the following energy equality:

$$\frac{1}{2}\int_{\mathbb{T}} \rho u^2(t)\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} \rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_{\mathbb{T}} \rho_0 u_0^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} \rho u f \,\mathrm{d}x\,\mathrm{d}s,$$

which is (44). This equality also proves that u is bounded in $L^2(0,T;H^{\alpha/2})$ by a constant depending on $\|u_0\|_{L^\infty}$, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|e_0\|_{L^\infty}$, $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, T, and α.

The derivation of (45) is similar. We start with the density equation, multiply by ρ, and integrate, obtaining

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \rho(t)^2 \,\mathrm{d}x = -\int_{\mathbb{T}} (\rho u)'\rho \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}} (\rho^2)' u \,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{T}} \rho^2(e + \Lambda_\alpha \rho)\,\mathrm{d}x \\
= -\frac{1}{2}\int_{\mathbb{T}} e\rho^2 + \rho^2 \Lambda_\alpha \rho \,\mathrm{d}x.
\end{aligned}$$

Symmetrizing, we get

$$\int_{\mathbb{T}} \rho^2 \Lambda_\alpha \rho \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{R}} (\rho^2(x) - \rho^2(y)) \frac{\rho(x)-\rho(y)}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{R}} (\rho(x)+\rho(y)) \frac{|\rho(x)-\rho(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x.$$

Therefore

$$\int_{\mathbb{T}} \rho(t)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} (\rho(x)+\rho(y)) \frac{|\rho(x)-\rho(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{T}} \rho_0^2\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}} e\rho^2\,\mathrm{d}x\,\mathrm{d}s,$$

which is (45). Thus ρ is also bounded in $L^2(0,T;H^{\alpha/2})$ by a constant depending only on $\|\rho_0\|_{L^\infty_x}$, $\|\rho_0^{-1}\|_{L^\infty_x}$, $\|e_0\|_{L^\infty_x}$, $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, and T. Finally, since $L^\infty \cap H^{\alpha/2}$ is an algebra, we have that ρu is bounded in $L^2(0,T;H^{\alpha/2})$, with a bound that depends only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0.

1.3. Bounds in Hölder Spaces. Let $[\cdot]_{C^\gamma(\mathbb{T})}$ denote the Hölder seminorm

$$[g]_{C^\gamma(\mathbb{T})} = \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|^\gamma},$$

for any $g \in C^\gamma(\mathbb{T})$. Below we will write C^γ for $C^\gamma(\mathbb{T})$, with the understanding that this will always denote the seminorm with respect to spatial variables only (not including time). As mentioned above, our construction of weak solutions will require bounds on u and ρ in some Hölder space. The precise statement that we use is recorded below:

PROPOSITION 1.4. *Let (u,ρ) be a regular solution on the time interval $[0,\infty)$. There exists $\gamma > 0$ such that ρ, $m = \rho u$, and u satisfy bounds of the form*

$$[\rho(t)]_{C^\gamma} \le t^{-\gamma/\alpha} C_T, \quad t \in (0,T] \tag{170}$$

$$[m(t)]_{C^\gamma} \le t^{-\gamma/\alpha} C_T, \quad t \in (0,T] \tag{171}$$

$$[u(t)]_{C^\gamma} \le t^{-\gamma/\alpha} C_T, \quad t \in (0,T]. \tag{172}$$

The constants C_T may depend on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0. The number γ ultimately depends only on these same quantities.

REMARK 1.5. For the purposes of constructing a weak solution, the bounds

$$[u(t)]_{C^\gamma} \le C_{\delta,T}, \quad [\rho(t)]_{C^\gamma} \le C_{\delta,T}, \qquad t \in [\delta, T].$$

would suffice. (Here $C_{\delta,T}$ is a constant that may depend on the same quantities as C_T above, and also may depend on δ.) However, the explicit bound (170) will be used later to control ρ'.

Remark 1.6. The main Theorem of [57] plays a key role here. In the case when $\alpha \in (0,1)$, the hypothesis of [57] asks for the drift u to be $C^{1-\alpha}_{x,t}$ (in both time and space). However, all that is really used in the proof there is the Hölder regularity in space, uniformly in time, i.e. $L^\infty(0,T;C^{1-\alpha})$. This is fortunate for us, since the latter is exactly the norm we can control for u, by the equality $e = u' - \Lambda_\alpha \rho$. Therefore, when we refer to the result of [57] here and below, it should be understood that (when $\alpha \in (0,1)$) we consider the version of the statement with the relaxed hypothesis $u \in L^\infty(0,T;C^{1-\alpha})$.

Proof. The bound (172) follows from (170), (171), and the bounds from the previous subsection, since

$$u(y) - u(x) = \rho(y)^{-1}[m(y) - m(x)] + u(x)\rho(y)^{-1}[\rho(x) - \rho(y)].$$

As for (170) and (171), we begin by writing the density and momentum equations in parabolic form:

$$\rho_t + u\rho' + \rho\Lambda_\alpha \rho = -e\rho \tag{173}$$

$$m_t + um' + \rho\Lambda_\alpha m = -em + \rho f. \tag{174}$$

The diffusion operator $\rho\Lambda_\alpha$ for these equations has kernel $K(x,h,t) = \rho(x,t)|h|^{-1-\alpha}$, so they are of the type considered in [57]. The quantities $-e\rho$ and $-em + \rho f$ play the role of (bounded) forcing terms. In order to apply the main result of [57], we split into two cases. In the first case, we assume $\alpha \in (0,1)$. In this case, we apply ∂_x^{-1} to the relation $u' = e + \Lambda_\alpha \rho$, then take $C^{1-\alpha}$ norms, to conclude that $u(t)$ is bounded in $C^{1-\alpha}$, with bounds that depend only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0. (Notice that (172) is actually trivially satisfied in this case.) Therefore the hypothesis of the main theorem of [57] is satisfied, and we may conclude that that there exists $\gamma > 0$, depending only on $\|u\|_{L^\infty(0,T;C^{1-\alpha})}$, such that

$$[\rho(t)]_{C^\gamma} \le \frac{C_T}{t^{\gamma/\alpha}}(\|\rho\|_{L^\infty(\mathbb{T}\times(0,T))} + \|e\rho\|_{L^\infty(\mathbb{T}\times(0,T))}) \tag{175}$$

$$[m(t)]_{C^\gamma} \le \frac{C_T}{t^{\gamma/\alpha}}(\|m\|_{L^\infty(\mathbb{T}\times(0,T))} + \|em - \rho f\|_{L^\infty(\mathbb{T}\times(0,T))}), \tag{176}$$

where the constant C_T here depends only on T and $\|u\|_{L^\infty(0,T;C^{1-\alpha})}$. Absorbing the L^∞ norms on the right hand sides of (175) and (176), and recalling that $C^{1-\alpha}$ depends only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0, we obtain (170) and (171), with the claimed dependencies.

The situation is similar if $\alpha \in [1,2)$. In this [57] gives us $\gamma > 0$, depending only on $\|u\|_{L^\infty(\mathbb{T}\times(0,T))}$, such that (175) and (176) continue to hold, with C_T depending only on T and $\|u\|_{L^\infty(\mathbb{T}\times(0,T))}$. Since

$\|u\|_{L^\infty(\mathbb{T}\times(0,T))}$ itself depends only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0, this completes the proof of the second case. ☐

Remark 1.7. In the case $\alpha \in (1,2)$, we can apply $\partial_x^{-\alpha}$ to the relation $\Lambda_\alpha \rho = u' - e$, then take $C^{\alpha-1}$ norms, to conclude that ρ is bounded in $C^{\alpha-1}$, with bounds that depend only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0. In particular, the bound (170) is trivially satisfied in this case.

1.4. Bounds on Time Derivatives.

Proposition 1.8. *Let (u,ρ) be a regular solution on the time interval $[0,\infty)$ and let $e = u' - \Lambda_\alpha \rho$. Then for any $T > 0$, $\partial_t u$ is bounded in $L^2(0,T;H^{-\alpha/2})$. Furthermore, $\partial_t \rho$ and $\partial_t e$ are bounded in $L^\infty(0,T;H^{-1})$. In each case, the bounds depend only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, and the L^∞ norms of u_0, ρ_0, ρ_0^{-1}, and e_0.*

Proof. For $\varphi \in H^{\alpha/2}(\mathbb{T})$, we have

$$\langle u(t),\varphi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}} - \langle u(s),\varphi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}} = \int_s^t \langle g(s),\varphi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s,$$

where

$$\langle g(s),\varphi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}} = \int_{\mathbb{T}} -ue(s)\varphi - \Lambda_{\alpha/2}(\rho u)(s)\Lambda_{\alpha/2}\varphi + f(s)\varphi\,\mathrm{d}x.$$

Since

$$\|g(s)\|_{H^{-\alpha/2}} \le C[\|u(s)\|_{L^\infty}\|e(s)\|_{L^\infty} + \|\rho u(s)\|_{H^{\alpha/2}} + \|f(s)\|_{L^\infty}],$$

the desired bound on $\partial_t u$ thus follows from the results of Sections 1.1 and 1.2.

For $\varphi \in H^1(\mathbb{T})$, we have

$$\langle \rho(t),\varphi\rangle_{H^{-1}\times H^1} - \langle \rho(s),\varphi\rangle_{H^{-1}\times H^1} = \int_s^t \int_{\mathbb{T}} \rho u(s)\varphi'\,\mathrm{d}x\,\mathrm{d}s.$$

Therefore

$$\|\partial_t \rho(s)\|_{H^{-1}} \le C\|\rho u(s)\|_{L^\infty},$$

so that the desired bound on $\partial_t \rho$ follows from the results of Section 1.1. The bound for $\partial_t e$ can be proved in the same way. ☐

2. Weak Solutions

2.1. Properties of General Weak Solutions. Let (u, ρ, e) be a weak solution on the time interval $[0, T]$ associated to the initial data $(u_0, \rho_0, e_0) \in L^\infty \times L^\infty \times L^\infty$. The purpose of this section is to record three simple facts about such a general weak solution, namely

- The quantity e satisfies a weak form of (33). That is, for all $\varphi \in C^\infty(\mathbb{T} \times [0, T])$ and a.e. $t \in [0, T]$, we have

$$\int_{\mathbb{T}} e(t)\varphi(t)\,\mathrm{d}x - \int_{\mathbb{T}} e_0\varphi(0)\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}} e\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\mathbb{T}} ue\varphi' + f'\varphi\,\mathrm{d}x\,\mathrm{d}s. \tag{177}$$

- The solution (u, ρ, e) converges weak-$*$ in L^∞ to the initial data.
- The weak time derivative of u is a well-defined element of $L^2(0, T; H^{-\alpha/2})$; the weak time derivatives of ρ and e are well-defined elements of $L^\infty(0, T; H^{-1})$.

To see that the first of these is true, note first that (40) implies that for all for all $\varphi \in C^\infty(\mathbb{T} \times [0, T])$ and a.e. $t \in [0, T]$, we have

$$\int_{\mathbb{T}} e\varphi(t) + u\varphi'(t) + \rho\Lambda_\alpha\varphi(t)\,\mathrm{d}x = 0.$$

For any $t \in [0, T]$ for which the above holds and any $\varphi \in C^\infty(\mathbb{T} \times [0, T])$, we have then that

$$\begin{aligned}
&\int_{\mathbb{T}} e(t)\varphi(t)\,\mathrm{d}x - \int_{\mathbb{T}} e_0\varphi(0)\,\mathrm{d}x \\
&= -\left[\int_{\mathbb{T}} u\varphi'(t)\,\mathrm{d}x - \int_{\mathbb{T}} u_0\varphi'(0)\,\mathrm{d}x\right] - \left[\int_{\mathbb{T}} \rho\Lambda_\alpha\varphi(t)\,\mathrm{d}x - \int_{\mathbb{T}} \rho_0\Lambda_\alpha\varphi(0)\,\mathrm{d}x\right] \\
&= -\left[\int_0^t \int_{\mathbb{T}} u\partial_t\varphi'\,\mathrm{d}x\,\mathrm{d}s - ue\varphi' - \rho u\Lambda_\alpha\varphi' + f\varphi'\,\mathrm{d}x\,\mathrm{d}s\right] - \left[\int_0^t \int_{\mathbb{T}} \rho\partial_t\Lambda_\alpha\varphi\,\mathrm{d}x\,\mathrm{d}s + \rho u\Lambda_\alpha\varphi'\,\mathrm{d}x\,\mathrm{d}s\right] \\
&= -\int_0^t \int_{\mathbb{T}} u(\partial_t\varphi)' + \rho\Lambda_\alpha(\partial_t\varphi)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}} ue\varphi' + f'\varphi\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{T}} e\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}} ue\varphi' + f'\varphi\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

This proves (177), for a.e. $t \in [0, T]$ and all $\varphi \in C^\infty(\mathbb{T} \times [0, T])$.

To observe the weak-$*$ convergence to the initial data, substitute any (time-independent) $\varphi \in C^\infty(\mathbb{T})$ into the weak formulation (38), (39) or into (177). Clearly $\int_{\mathbb{T}}(u(t) - u_0)\varphi\,\mathrm{d}x \to 0$ as $t \to 0^+$, since the right side of (38) is an integral from 0 to t of an integrable quantity. Since $C^\infty(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, we conclude that $\int_{\mathbb{T}}(u(t) - u_0)\varphi\,\mathrm{d}x \to 0$ as $t \to 0^+$, for any $\varphi \in L^1(\mathbb{T})$, i.e. $u(t) \overset{*}{\rightharpoonup} u_0$ weak-$*$ in L^∞, as $t \to 0^+$. The situation is similar for ρ and e.

Finally, the statement regarding the time derivatives is proved in a manner similar to that of Section 1.4.

2.2. Construction of a Weak Solution. In this section we construct a weak solution as a subsequential limit of regular solutions with mollified initial data, as the mollification parameter tends to zero. The following version of the Aubin-Lions-Simon compactness Lemma ([58], c.f. Theorem II.5.16 in [1]) will allow us to use the bounds from Section 1 to choose an appropriate subsequence.

LEMMA 2.1. *Let $X \subset Y \subset Z$ be Banach spaces, where the embedding $X \subset Y$ is compact and the embedding $Y \subset Z$ is continuous. Assume $p, r \in [1, \infty]$, and define for $T > 0$ the following space:*

$$E = \{v \in L^p(0, T; X) : \frac{\mathrm{d}v}{\mathrm{d}t} \in L^r(0, T; Z)\}.$$

(1) *If $p < \infty$, the embedding $E \subset L^p(0, T; Y)$ is compact.*
(2) *If $p = \infty$ and $r > 1$, then the embedding $E \subset C([0, T], Y)$ is compact.*

Let γ be as in Section 1.3. In the notation of the Aubin-Lions-Simon Lemma, we set

$$X_T = C^\gamma, \quad Y = C^0, \quad Z = H^{-1}, \quad E_{\delta,T} = \{v \in L^\infty(\delta, T; C^\gamma) : \partial_t v \in L^2(\delta, T; H^{-1})\}.$$

The conclusion of the Lemma is then that the embedding $E_{\delta,T} \subset C([\delta, T]; C^0)$ is compact for any $T > \delta > 0$.

Choose $(u_0, \rho_0, e_0) \in L^\infty \times L^\infty \times L^\infty$, satisfying $\rho_0^{-1} \in L^\infty$ and the compatibility condition (37). Let $\eta \in C_c^\infty(\mathbb{R})$ be a standard mollifier ($\int \eta = 1$, $\operatorname{supp} \eta \subset \{|x| \leq 1\}$), and let f_ϵ denote the convolution of f by $\epsilon^{-1}\eta(\epsilon^{-1}\cdot)$: $f_\epsilon(x) = \epsilon^{-1} \int_{\mathbb{R}} \eta(\epsilon^{-1} y) f(x - y) \, \mathrm{d}y$. Let $(u^\epsilon, \rho^\epsilon)$ denote the global strong solution associated to the initial data $((u_0)_\epsilon, (\rho_0)_\epsilon)$ and let $e^\epsilon = (u^\epsilon)' - \Lambda_\alpha \rho^\epsilon$. Note that $(e_0)_\epsilon = (u_0)'_\epsilon - \Lambda_\alpha (\rho_0)_\epsilon = e^\epsilon(0)$ automatically.

CLAIM 2.2. The sequences u^ϵ and ρ^ϵ are bounded in $E_{\delta,T}$ for any $T > \delta > 0$.

PROOF. Fix $T > \delta > 0$. In order to prove the claim, one needs to prove the following two statements:

(1) u^ϵ and ρ^ϵ are bounded sequences of $L^\infty(\delta, T; C^\gamma)$.
(2) $\partial_t u^\epsilon$ and $\partial_t \rho^\epsilon$ are bounded sequences of $L^2(\delta, T; H^{-1})$.

We have essentially proved these statements already. We provide the remaining details for half of the first statement only; the rest follows the same reasoning.

Section 1.3 establishes that the norm of u^ϵ in $L^\infty(\delta, T; C^\gamma)$ can be bounded above by a quantity that depends only on $\|f\|_{L^\infty_t W^{1,\infty}_x}$, $\mathcal{M}$, α, T, δ, and the L^∞ norms of $(u_0)_\epsilon$, $(\rho_0)_\epsilon$, $(\rho_0)_\epsilon^{-1}$, and $(e_0)_\epsilon$. But these L^∞ norms are bounded by those of u_0, ρ_0, ρ_0^{-1}, and e_0, respectively, and the remaining quantities are fixed. Therefore u^ϵ is a bounded sequence in $L^\infty(\delta, T; C^\gamma)$. □

Applying the Aubin-Lions-Simon Lemma, we can now choose a subsequence $\{\epsilon_k\}$, tending to zero as $k \to \infty$, such that u^{ϵ_k} and ρ^{ϵ_k} converge (strongly) in $C([2^{-N}, 2^N]; C^0)$, with N any natural number. Using a standard diagonal argument, we obtain a further subsequence, which we continue to denote by ϵ_k, such that u^{ϵ_k} and ρ^{ϵ_k} converge to functions u and ρ, respectively, in $C_{\text{loc}}((0,\infty); C^0)$.

Using the same logic as in the Claim above, we also have that e^ϵ is bounded in $L^\infty(\mathbb{T} \times [0,T])$ and both u^ϵ and ρ^ϵ are bounded in $L^2(0,T; H^{\alpha/2})$. Therefore we may choose a further subsequence (still denoted ϵ_k) such that e^{ϵ_k} converges weak-$*$ in $L^\infty(\mathbb{T}\times[0,T])$ to some $e \in L^\infty(\mathbb{T}\times[0,T])$, and so that u^{ϵ_k} and ρ^{ϵ_k} converge weakly in $L^2(0,T; H^{\alpha/2})$ to u and ρ. Then we can use a diagonal argument as above to send $T \to \infty$. To summarize, there exists a subsequence $\{\epsilon_k\}$ and a triple (u, ρ, e), such that as $k \to \infty$, we have

$$
\begin{aligned}
u^{\epsilon_k} \to u \text{ and } \rho^{\epsilon_k} \to \rho \text{ strongly in } C_{\text{loc}}((0,\infty); C^0);\\
u^{\epsilon_k} \rightharpoonup u \text{ and } \rho^{\epsilon_k} \rightharpoonup \rho \text{ weakly in } L^2_{\text{loc}}(0,\infty; H^{\alpha/2});\\
e^{\epsilon_k} \overset{*}{\rightharpoonup} e \text{ weak-}* \text{ in } L^\infty_{\text{loc}}(\mathbb{T}\times[0,\infty)).
\end{aligned}
$$

Now $(u^{\epsilon_k}, \rho^{\epsilon_k})$ is a regular solution (therefore $(u^{\epsilon_k}, \rho^{\epsilon_k}, e^{\epsilon_k})$ is a weak solution) for each k. We can therefore consider each term in each equation of the weak formulation and easily see that the above convergences guarantee that (u, ρ, e) satisfies the weak formulation. This completes the existence part of Theorem 2.9. The construction gives Hölder continuity on compact sets of $\mathbb{T}\times(0,\infty)$. Indeed, if γ is the Hölder exponent associated to the interval $[0,T]$ as in Section 1.3, then for any $\widetilde{\gamma} \in (0,\gamma)$, the convergences $u^{\epsilon_k} \to u$ and $\rho^{\epsilon_k} \to \rho$ can be taken in $L^\infty(\delta, T; C^{\widetilde{\gamma}})$ for any fixed $\delta > 0$.

Remark 2.3. If $\alpha \neq 1$, slightly more information is available. If $0 < \alpha < 1$, then the above construction can be modified slightly to give $u^{\epsilon_k} \to u$ in $C([0,\infty); C^{1-\alpha-\kappa})$, for any $\kappa \in (0, 1-\alpha)$; if $1 < \alpha < 2$, then we can obtain $\rho^{\epsilon_k} \to \rho$ in $C([0,\infty); C^{\alpha-1-\kappa})$ for any $\kappa \in (0, \alpha - 1)$.

2.3. Energy Inequality for Constructed Solutions. We now prove that the solutions constructed above satisfy (42) and (43). To prove these inequalities, we essentially use the fact that they are true (with equality) for regular enough solutions, then pass to the limit $k \to \infty$ in the sequence $(u^{\epsilon_k}, \rho^{\epsilon_k}, e^{\epsilon_k})$ from the proof of existence above. However, since the solution behaves a little better away from time zero, we initially work on $[\delta, t]$ for some $\delta > 0$. We prove (42) first. We start with the equality

$$\int_{\mathbb{T}} \rho^{\epsilon_k} (u^{\epsilon_k})^2(s)\,\mathrm{d}x\Big|_\delta^t + \int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} \rho^{\epsilon_k}(x)\rho^{\epsilon_k}(y) \frac{|u^{\epsilon_k}(x) - u^{\epsilon_k}(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s = 2\int_\delta^t \int_{\mathbb{T}} \rho^{\epsilon_k} u^{\epsilon_k} f\,\mathrm{d}x\,\mathrm{d}s. \tag{178}$$

The first term and the forcing term are easily seen to converge to their natural limits, by uniform convergence of ρ^{ϵ_k}, u^{ϵ_k} on any time interval $[\delta, T]$. To deal with the second term on the left, we write

$$\int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} [\rho^{\epsilon_k}(x)\rho^{\epsilon_k}(y) - \rho(x)\rho(y)] \frac{|u^{\epsilon_k}(x) - u^{\epsilon_k}(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \to 0, \qquad \text{as } k \to \infty, \tag{179}$$

which is valid by uniform convergence of ρ^{ϵ_k} away from time zero, as well as the $L^2 H^{\alpha/2}$ bound on u^{ϵ_k}, which is uniform in k. We also have

$$\int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(x)\rho(y) \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \le \liminf_{k\to\infty} \int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(x)\rho(y) \frac{|u^{\epsilon_k}(x) - u^{\epsilon_k}(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s, \tag{180}$$

by weak lower semicontinuity. Taking limits in (178) thus yields

$$\int_{\mathbb{T}} \rho u^2(t)\,\mathrm{d}x + \int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(x)\rho(y) \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \le \int_{\mathbb{T}} \rho u^2(\delta)\,\mathrm{d}x + 2\int_\delta^t \int_{\mathbb{T}} \rho u f\,\mathrm{d}x\,\mathrm{d}s \tag{181}$$

Next, we note that

$$\int_{\mathbb{T}} \rho^{\epsilon_k} (u^{\epsilon_k})^2(\delta)\,\mathrm{d}x \le \int_{\mathbb{T}} (\rho_0)_{\epsilon_k} (u_0)^2_{\epsilon_k}\,\mathrm{d}x + 2\int_0^\delta \int_{\mathbb{T}} \rho^{\epsilon_k} u^{\epsilon_k} f\,\mathrm{d}x\,\mathrm{d}s. \tag{182}$$

This is obtained from the energy equality for $(u^{\epsilon_k}, \rho^{\epsilon_k})$ on $[0, \delta]$, by dropping the enstrophy term. We can estimate the force term on the right by $C\delta$, where C is independent of k and δ (but may depend on t), and then take $k \to \infty$. The term on the left converges to its natural limit for the same reason as above; the initial data term converges to its natural limit by standard properties of mollifiers. We are left with

$$\int_{\mathbb{T}} \rho u^2(\delta)\,\mathrm{d}x \le \int_{\mathbb{T}} \rho_0 u_0^2\,\mathrm{d}x + C\delta.$$

Combining this with (181), we obtain

$$\int_{\mathbb{T}} \rho u^2(t)\,\mathrm{d}x + \int_\delta^t \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(x)\rho(y) \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \le \int_{\mathbb{T}} \rho_0 u_0^2\,\mathrm{d}x + 2\int_\delta^t \int_{\mathbb{T}} \rho u f\,\mathrm{d}x\,\mathrm{d}s + C\delta. \tag{183}$$

And now, taking $\delta \to 0$ yields (42).

The inequality (43) is proved in a very similar way. The only difference in approach is for the last term, on $[\delta, t]$. We write

$$\int_\delta^t \int_{\mathbb{T}} e^{\epsilon_k} (\rho^{\epsilon_k})^2 \,\mathrm{d}x\,\mathrm{d}s - \int_\delta^t \int_{\mathbb{T}} e\rho^2 \,\mathrm{d}x\,\mathrm{d}s = \int_\delta^t \int_{\mathbb{T}} e^{\epsilon_k}[(\rho^{\epsilon_k})^2 - \rho^2] \,\mathrm{d}x\,\mathrm{d}s + \int_\delta^t \int_{\mathbb{T}} (e^{\epsilon_k} - e)\rho^2 \,\mathrm{d}x\,\mathrm{d}s.$$

We use uniform convergence of the ρ^{ϵ_k} on $[\delta, t]$ to treat the first term and weak-$*$ convergence of e^{ϵ_k} to treat the second. This finishes the proof of the inequalities (42), (43).

2.4. The Case of a Compactly Supported Force. If the force f is identically zero, or, more generally, if it is compactly supported in time, then there are several implications for the solutions we have constructed. We take a moment to collect a few of these.

(1) If $f \equiv 0$, then the constants c_1 and c_4 from (157)–(159) are both zero, so that u, ρ, ρ^{-1}, and e can all be bounded above for all time by constants. If f is compactly supported in time, then all these quantities are still uniformly bounded, but the constants we can use to bound them will be larger, due to the potential growth during the time interval where f is supported. These uniform bounds will survive the limiting process used to construct weak solutions.

(2) As a consequence of the uniform boundedness of u, ρ, ρ^{-1}, and e, the quantity γ from Section 1.3 can be taken to be independent of T. Thus, the Hölder regularization will survive the limiting process (with Hölder exponent $\gamma - \kappa$ for any $\kappa \in (0, \gamma)$).

(3) As soon as the force is turned off, we have a fast alignment of the velocity field; that is, the velocity amplitude $A(t) = \max_{x,y} |u(x,t) - u(y,t)|$ decays exponentially fast for regular solutions. In particular, the case of zero force gives

$$A(t) \le A(0) e^{-\mathcal{M}\iota(\pi)t},$$

where $\iota(r) = \inf_{|x|<r} \phi_\alpha(x)$ and ϕ_α is the kernel of Λ_α, as above. See Lemma 1.1 of [55] for the short proof of this statement. Therefore the alignment survives the limiting process used to construct weak solutions, so that (the constructed) weak solutions also enjoy the alignment property if the force is compactly supported.

The constructed weak solutions do not possess quite enough regularity for us to prove that they experience flocking (which also requires convergence of the density profile) in the case of a compactly supported force; however, we will see that flocking occurs for strong solutions under the assumption of compactly supported force.

3. Energy Balance for Weak Solutions

In this section, we provide conditions which guarantee that the natural energy laws hold for weak solutions. We emphasize that the criteria we consider apply to any weak solutions, not just those weak solutions which can be constructed as in the previous section.

To begin with, we note that it turns out to be easier to work with a momentum-based equation when proving (44). However, due to the limited regularity of our weak solutions, we must prove that such a formulation is valid for our solutions. In this proof and those below, we will make use of Littlewood-Paley theory, for which we some basic notions have already been introduced in Chapter 4. Actually, we will make extensive use of the commutator estimates proved there.

3.1. The Weak Momentum Equation. For smooth functions f and g, we define

$$\mathcal{T}(f,g) = -\Lambda_\alpha(fg) - g\Lambda_\alpha f.$$

When (ρ, u, e) is a weak solution, we can make sense of the expression $\rho\mathcal{T}(\rho, u)$ in a weak sense. Define $X = H^{\alpha/2} \cap L^\infty$, and for each $s > 0$, let $\rho\mathcal{T}(\rho,u)(s)$ denote the element of X^* given by

$$\langle \rho\mathcal{T}(\rho,u), \varphi\rangle_{X^*,X} = \int -\Lambda_{\alpha/2}(\rho u)\Lambda_{\alpha/2}(\rho\varphi) + \Lambda_{\alpha/2}(\rho)\Lambda_{\alpha/2}(\rho u\varphi)\,\mathrm{d}x.$$

Proposition 3.1. *Let (u,ρ,e) be a weak solution on the time interval $[0,T]$. Then for each $\varphi \in C^\infty(\mathbb{T}\times[0,T])$ and a.e. $t \in [0,T]$, we have that*

$$\begin{aligned}&\int \rho u\varphi(t)\,dx - \int \rho_0 u_0\varphi(0)\,dx - \int_0^t\int \rho u\partial_t\varphi(s)\,dx\,ds \\ &\qquad = \int_0^t\int \rho u^2\varphi'\,dx\,ds + \int_0^t \langle \rho\mathcal{T}(\rho,u),\varphi\rangle_{X^*,X}\,ds + \int_0^t\int \rho f\varphi\,dx\,ds.\end{aligned} \tag{184}$$

Proof. Substitute the test function $(\rho_{\le Q}\varphi)_{\le Q}$ into the weak velocity equation. We obtain

$$\begin{aligned}&\int_{\mathbb{T}} \rho_{\le Q}u_{\le Q}(t)\varphi(t)\,\mathrm{d}x - \int_{\mathbb{T}}(\rho_0)_{\le Q}(u_0)_{\le Q}\varphi(0)\,\mathrm{d}x - \int_0^t\int_{\mathbb{T}} u_{\le Q}(\partial_t\rho_{\le Q}\varphi + \rho_{\le Q}\partial_t\varphi)\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad = \int_0^t\int_{\mathbb{T}} -(ue)_{\le Q}\rho_{\le Q}\varphi - (\rho u)_{\le Q}\Lambda_\alpha(\rho_{\le Q}\varphi) + \rho_{\le Q}f_{\le Q}\varphi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{185}$$

Then substitute $(u_{\le Q}\varphi)_{\le Q}$ into the weak density equation:

$$\begin{aligned}\int_{\mathbb{T}}\rho_{\le Q}u_{\le Q}(t)\varphi(t)\,\mathrm{d}x-\int_{\mathbb{T}}(\rho_0)_{\le Q}(u_0)_{\le Q}\varphi(0)\,\mathrm{d}x-\int_0^t\int_{\mathbb{T}}\rho_{\le Q}(\partial_t u_{\le Q}\varphi+u_{\le Q}\partial_t\varphi)\,\mathrm{d}x\,\mathrm{d}s\\ =\int_0^t\int_{\mathbb{T}}(\rho u)_{\le Q}(u'_{\le Q}\varphi+u_{\le Q}\varphi')\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{186}$$

Finally, project the compatibility condition onto the first Q modes:

$$e_{\le Q}=u'_{\le Q}-\Lambda_\alpha\rho_{\le Q}\tag{187}$$

We use (187) to eliminate $u'_{\le Q}$ from (186), then we add the result to (185). We obtain

$$\begin{aligned}\int_{\mathbb{T}}\rho_{\le Q}u_{\le Q}(t)\varphi(t)\,\mathrm{d}x-\int_{\mathbb{T}}(\rho_0)_{\le Q}(u_0)_{\le Q}\varphi(0)\,\mathrm{d}x-\int_0^t\int_{\mathbb{T}}\rho_{\le Q}u_{\le Q}\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}s\\ =\int_0^t\int_{\mathbb{T}}(\rho u)_{\le Q}u_{\le Q}\varphi'\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int[(\rho u)_{\le Q}e_{\le Q}-\rho_{\le Q}(ue)_{\le Q}]\varphi\,\mathrm{d}x\,\mathrm{d}s\\ +\int_0^t\int(\rho u)_{\le Q}[\varphi\Lambda_\alpha\rho_{\le Q}-\Lambda_\alpha(\rho_{\le Q}\varphi)]\,\mathrm{d}x\,\mathrm{d}s+\rho_{\le Q}f_{\le Q}\varphi\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Note that we have used the product rule and the fundamental theorem of calculus to simplify the left side of this equation. It should now be clear that each integral converges to its natural limit, so that the equation (184) holds. ☐

Remark 3.2. It seems likely that the converse direction is also true, i.e., that replacing (38) with (184) should give an equivalent weak formulation. To try to prove this, one might try the following strategy: Denote $U:=\rho^{-1}_{\le Q}(\rho u)_{\le Q}$ and substitute $(\rho^{-1}_{\le Q}\varphi)_{\le Q}$ into (184). Subtract from this equation the result of substituting $\left(\rho^{-1}_{\le Q}U\varphi\right)_{\le Q}$ into (39). After performing some manipulations, one obtains

$$\begin{aligned}&\int U\varphi(t)\,\mathrm{d}x-\int U\varphi(0)\,\mathrm{d}x-\int_0^t\int U\partial_t\varphi(t)\,\mathrm{d}x\\ &=\int_0^t\int[(\rho u^2)_{\le Q}-(\rho u)_{\le Q}U]\left(\frac{\varphi}{\rho_{\le Q}}\right)'\mathrm{d}x\,\mathrm{d}s+\int_0^t\int(\rho\mathcal{T}(\rho,u))_{\le Q}\frac{\varphi}{\rho_{\le Q}}-\varphi\mathcal{T}(\rho_{\le Q},u_{\le Q})\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int-\rho_{\le Q}u_{\le Q}\Lambda_\alpha\varphi-u_{\le Q}e_{\le Q}\varphi+\frac{(\rho f)_{\le Q}}{\rho_{\le Q}}\varphi\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{2}\int_0^t(U^2-u^2_{\le Q})\varphi'\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

All integrals on the left side and the last two integrals on the right side obviously converge to the natural limits. The second term on the right side also converges to zero, though this requires some work (involving computations similar to those of Section 3.4). However, it appears that the first term on the right side requires some additional smoothness in order to pass to the limit; the Onsager-type assumption (191) below is sufficient. Therefore, we can currently claim only that the two weak formulations are equivalent under this additional assumption. As noted below, (191) is automatically satisfied when $\alpha\ge 1$.

3.2. The Energy Budget. Let $E_{\leq Q}(t)$ denote the energy associated to scales λ_q for $q \leq Q$, and let $E(t)$ denote the total energy:

$$E_{\leq Q}(t) = \frac{1}{2}\int \frac{(\rho u)^2_{\leq Q}}{\rho_{\leq Q}}(t)\,\mathrm{d}x; \qquad E(t) = \frac{1}{2}\int \rho u^2(t)\,\mathrm{d}x.$$

The energy budget relation at scales $q \leq Q$ has the same structure as the one for the inhomogeneous Navier-Stokes system, which was derived in Chapter 4, Section 1.3. We do not re-derive it here; rather we simply recall the form of the equation:

$$E_{\leq Q}(t) - E_{\leq Q}(0) = \int_0^t \Pi_Q(s)\,\mathrm{d}s - \varepsilon_Q(t) + \int_0^t \int (\rho f)_{\leq Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s. \tag{188}$$

As before, $\Pi_Q(s)$ is the flux through scales of order Q due to the nonlinearity, defined by

$$\Pi_Q = \int F_Q(\rho, u) U'\,\mathrm{d}x; \tag{189}$$

$$F_Q(\rho, u) = (\rho u^2)_{\leq Q} - U(\rho u)_{\leq Q}. \tag{190}$$

The quantities ε_Q and $\int_0^t \int (\rho f)_{\leq Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s$ represent the change in energy due to the local interactions and the external force, respectively, at scales $q \leq Q$. Now ε_Q is given by a different expression than in the inhomogeneous Navier-Stokes case:

$$\varepsilon_Q(t) = -\int_0^t \int_{\mathbb{T}} (\rho \mathcal{T}(\rho, u))_{\leq Q} U\,\mathrm{d}x\,\mathrm{d}s.$$

We also denote

$$\varepsilon(t) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}} \rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$

It was already shown in Chapter 4 that $E_{\leq Q}(t) \to E(t)$ and $\int_0^t \int (\rho f)_{\leq Q}\cdot U\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int \rho u \cdot f\,\mathrm{d}x\,\mathrm{d}s$ in all cases we are interested in. We aim to show that $\int_0^t \Pi_Q(s)\,\mathrm{d}s \to 0$ and $\varepsilon_Q(t) \to \varepsilon(t)$ as well; this will immediately imply that the energy balance relation holds.

Actually, it was already shown above that $\int_0^t \Pi_Q(s)\,\mathrm{d}t \to 0$ as $Q \to \infty$ whenever

$$\rho \in L^4(0,T; B^{\frac{1}{3}}_{4,\infty}), \quad u \in L^4(0,T; B^{\frac{1}{3}}_{4,c_0}).$$

We do not expect to improve on the smoothness parameter here, but we have a bit of additional information now, namely the fact that $u \in L^\infty L^\infty$. We can consequently weaken the integrability assumptions; see below. We also claim that $\varepsilon_Q \to \varepsilon$ holds in fact for all weak solutions, since such solutions satisfy $\rho, u \in L^\infty L^\infty \cap L^2 H^{\alpha/2}$, which is really all that is needed in order to pass to the limit for this term. In the following two subsections, we will prove that the natural energy law (44) holds under the assumption that

$$\rho \in L^3(0,T; B^{\frac{1}{3}}_{3,\infty}), \quad u \in L^3(0,T; B^{\frac{1}{3}}_{3,c_0}). \tag{191}$$

Now (191) is automatically satisfied if $\alpha \in [1,2)$, since $L^\infty L^\infty \cap L^2 H^{1/2} \subset L^3 B^{1/3}_{3,3}$ by interpolation. We will prove (the more difficult half of) Theorem 2.10 over the course of the next two subsections. The proof of the other half (actually, a more general statement) is contained in Section 3.5.

3.3. Conditional Convergence of the Nonlinear Term. We recall the following from Chapter 4, Section 2.

LEMMA 3.3. *$F_Q(\rho,u)$ can be written as*

$$\begin{aligned} F_Q(\rho,u) &= r_Q(\rho,u,u) - \frac{1}{\rho_{\le Q}}[(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}]^2 + \rho_{>Q}u_{>Q} \otimes u_{>Q} \\ &\quad + 2[(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}]u_{>Q} + \rho[(u^2)_{\le Q} - u^2_{\le Q}], \end{aligned} \tag{192}$$

where

$$r_Q(\rho,u,u) = \int \widetilde{h}_Q(y)[\rho(x-y)-\rho(x)][u(x-y)-u(x)]^2\, dy,$$

and $\widetilde{h}_Q$ is a Schwartz function.

With these facts in hand, we are now in a position to prove that the nonlinear term vanishes under our hypotheses.

PROPOSITION 3.4. *The quantity $F_Q(\rho,u)$ satisfies the bound*

$$\|F_Q(\rho,u)\|_{L^{3/2}} \lesssim \lambda_Q^{-2/3}(D^{1/3}_{3,Q}(u))^2 \tag{193}$$

whenever $u \in B^{1/3}_{3,\infty}$ and $\rho \in L^\infty$.

This bound follows from the decomposition (192) and Proposition 1.1 from Chapter 4.

THEOREM 3.5. *Suppose $u \in L^3 B^{1/3}_{3,c_0}$ and $\rho \in L^3 B^{1/3}_{3,\infty}$. Then $\int_0^t \Pi_Q(s)\, ds \to 0$ as $Q \to \infty$.*

PROOF. First we write

$$U' = \frac{1}{\rho_{\le Q}}[(\rho u)'_{\le Q} - U\rho'_{\le Q}].$$

Since $L^\infty \cap B^{1/3}_{3,\infty}$ is an algebra, we have $\rho u \in L^3 B^{1/3}_{3,\infty}$. Therefore

$$\|U'\|_{L^3} \lesssim \lambda_Q^{2/3}[D^{1/3}_{3,Q}(\rho u) + D^{1/3}_{3,Q}(\rho)],$$

by (145). So

$$\int_0^t F_Q(\rho,u)U'\,\mathrm{d}s \lesssim \int_0^t (D_{3,Q}^{1/3}(u))^2[D_{3,Q}^{1/3}(\rho u)+D_{3,Q}^{1/3}(\rho)]\,\mathrm{d}s.$$

By (142), the definition of $B^{1/3}_{3,c_0}$, and the dominated convergence theorem, we conclude that the integral tends to zero, as needed. $\square$

3.4. Unconditional Convergence of the Dissipation Term.

In this subsection, we prove:

THEOREM 3.6. *Any weak solution (ρ,u) satisfies $\varepsilon_Q\to\varepsilon$, as $Q\to\infty$.*

Since the dissipation term involves fractional derivatives, we introduce a modified version of the localization kernel of Chapter 4. Define

$$\widetilde{K}_q=\begin{cases}\lambda_q^{-\alpha/2}, & q\geq 0;\\ \lambda_q^{\alpha/2}, & q<0;\end{cases}\qquad \widetilde{d}_q(f)=\lambda_q^{\alpha/2}\|f_q\|_{L^2};\qquad \widetilde{D}_Q(f)=\sum_{q=-1}^{\infty}\widetilde{K}_{Q-q}\widetilde{d}_q(f).$$

Note that

$$\limsup_{Q\to\infty}\widetilde{D}_Q(f)\sim\limsup_{q\to\infty}\widetilde{d}_q(f), \tag{194}$$

where the similarity constant depends only on α.

PROPOSITION 3.7. *For $f\in B^{\alpha/2}_{2,\infty}$, $0<\alpha<2$, we have the following estimates:*

$$\|\Lambda_\alpha f_{\leq Q}\|_2\lesssim\lambda_Q^{\alpha/2}\widetilde{D}_Q(f), \tag{195}$$

$$\|f_{>Q}\|_2\leq\lambda_Q^{-\alpha/2}\widetilde{D}_Q(f). \tag{196}$$

The proofs of (195) and (196) are similar to those of (145) and (146), respectively, and are omitted.

REMARK 3.8. We will also repeatedly use the following basic facts without comment below:

(1) If $\operatorname{supp}\widehat{f}\subset B_{\lambda_Q}(0)$, then $\|\Lambda_\alpha f\|_{L^2}\lesssim\lambda_Q^\alpha\|f\|_{L^2}$.

(2) For $f\in B^{\alpha/2}_{2,\infty}$, the inequalities in (195) and (196) continue to hold when $f_{\leq Q}$ and $f_{>Q}$ are replaced with f_Q. That is, for such f, we have

$$\|\Lambda_\alpha f_Q\|_2\lesssim\lambda_Q^{\alpha/2}\widetilde{D}_Q(f),\qquad \|f_Q\|_2\leq\lambda_Q^{-\alpha/2}\widetilde{D}_Q(f).$$

(3) For $k\in\mathbb{Z}$, we have $\widetilde{D}_Q(f)\sim\widetilde{D}_{Q+k}(f)$, with the similarity constant depending only on k. (To see this, simply note that $\widetilde{K}_{q+k}\sim\widetilde{K}_q$ for each $q\in\mathbb{Z}$, with a similarity constant that depends on k but not on q.)

Of course, analogous properties hold when we consider first derivatives instead of fractional derivatives, but the fractional case is the one which is relevant below.

To prove Theorem 3.6, we write

$$|\varepsilon_Q(t) - \varepsilon(t)| \leq \left|\varepsilon_Q(t) + \int_0^t \int \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q}) \,\mathrm{d}x \,\mathrm{d}s\right| + \left|\int_0^t \int \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q}) \,\mathrm{d}x \,\mathrm{d}s + \varepsilon(t)\right|,$$

and we show that both terms tend to zero as $Q \to \infty$. Let us take care of the (much easier) second term presently. We write

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}} \rho_{\leq Q}(x)\rho_{\leq Q}(y) \frac{|u_{\leq Q}(x) - u_{\leq Q}(y)|^2}{|x-y|^{1+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s - \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}} \rho(x)\rho(y) \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}} [\rho_{\leq Q}(x)\rho_{\leq Q}(y) - \rho(x)\rho(y)] \frac{|u(x) - u(y)|^2}{|x-y|^{1+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{T}} \rho_{\leq Q}(x)\rho_{\leq Q}(y) \frac{[(u_{\leq Q} - u)(x) - (u_{\leq Q} - u)(y)][(u_{\leq Q} + u)(x) - (u_{\leq Q} + u)(y)]}{|x-y|^{1+\alpha}} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}s.
\end{aligned}$$

The first term here tends to zero by the dominated convergence theorem (the dominating function being $C\|\rho\|_{L^\infty}^2 \frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}$), while the second term is bounded above by

$$\int_0^t \|\rho\|_{L^\infty}^2 \|u_{\leq Q} - u\|_{H^{\alpha/2}} \|u_{\leq Q} + u\|_{H^{\alpha/2}} \,\mathrm{d}s \to 0,$$

which tends to zero as $Q \to \infty$. It thus remains to show that

$$\left|\varepsilon_Q(t) + \int_0^t \int \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q}) \,\mathrm{d}x \,\mathrm{d}s\right| \to 0, \qquad \text{as } Q \to \infty.$$

We write the relevant difference as

$$\begin{aligned}
&\int \rho \mathcal{T}(\rho, u) U_{\leq Q} - \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q}) \,\mathrm{d}x \\
&= \int \rho_{\leq Q} u_{\leq Q} \Lambda_\alpha(\rho_{\leq Q} u_{\leq Q}) - (\rho \Lambda_\alpha(\rho u))_{\leq Q} U \,\mathrm{d}x + \int (\rho u \Lambda_\alpha \rho)_{\leq Q} U - \rho_{\leq Q} u_{\leq Q}^2 \Lambda_\alpha \rho_{\leq Q} \,\mathrm{d}x =: A + B.
\end{aligned}$$

Expanding further gives

$$\begin{aligned}
A &= \int [\rho_{\leq Q} u_{\leq Q} - (\rho u)_{\leq Q}] \Lambda_\alpha(\rho_{\leq Q} u_{\leq Q} + (\rho u)_{\leq Q}) \,\mathrm{d}x + \int [\rho_{\leq Q} \Lambda_\alpha (\rho u)_{\leq Q} - (\rho \Lambda_\alpha(\rho u))_{\leq Q}] U \,\mathrm{d}x \\
&=: A_1 + A_2; \\
B &= \int [(\rho u \Lambda_\alpha \rho)_{\leq Q} - (\rho u)_{\leq Q} \Lambda_\alpha \rho_{\leq Q}] U \,\mathrm{d}x + \int \rho_{\leq Q}^{-1} [(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}][(\rho u)_{\leq Q} + \rho_{\leq Q} u_{\leq Q}] \Lambda_\alpha \rho_{\leq Q} \,\mathrm{d}x \\
&=: B_1 + B_2.
\end{aligned}$$

The terms A_1 and B_2 are easy to treat:

$$\begin{aligned}
A_1 &\lesssim \|\rho_{\leq Q} u_{\leq Q} - (\rho u)_{\leq Q}\|_{L^2} \cdot \|\Lambda_\alpha(\rho_{\leq Q} u_{\leq Q} + (\rho u)_{\leq Q})\|_{L^2} \\
&\leq \lambda_Q^{-\alpha} D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u) \cdot \lambda_Q^{\alpha} \|\rho_{\leq Q} u_{\leq Q} + (\rho u)_{\leq Q}\|_{L^2} \lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u); \\
B_2 &\leq \|\rho^{-1}_{\leq Q}\|_{L^\infty} \|(\rho u)_{\leq Q} - \rho_{\leq Q} u_{\leq Q}\|_{L^2} \|(\rho u)_{\leq Q} + \rho_{\leq Q} u_{\leq Q}\|_{L^\infty} \|\Lambda_\alpha \rho_{\leq Q}\|_{L^2} \\
&\leq C \cdot \lambda_Q^{-\alpha} D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u) \cdot C \cdot \lambda_Q^{\alpha} \|\rho_{\leq Q}\|_{L^2} \lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u).
\end{aligned}$$

Thus

$$A_1 + B_2 \lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u),$$

where the implied constant is independent of Q (but may depend on the L^∞ norms of ρ and u).

To deal with A_2 and B_1, we need to work more: Since $\rho\Lambda_\alpha(\rho u)$ and $\rho u \Lambda_\alpha \rho$ are in general only $L^2 H^{-\alpha/2}$, the commutator estimate (143) does not directly apply. To overcome this difficulty, we decompose the commutator $(f\Lambda_\alpha g)_{\leq Q} - f_{\leq Q}\Lambda_\alpha g_{\leq Q}$ in such a way that repeated use of (195) and (196) (and related inequalities) becomes an adequate substitute for (143) in the treatment of A_2 and B_1. Actually, we state our decomposition for the more general commutator $(fg)_{\leq Q} - f_{\leq Q} g_{\leq Q}$, with the idea that g will be replaced by $\Lambda_\alpha g$ below.

We set the notation

$$f_{q_+} = f_{q+1} + f_{q+2}, \qquad f_{r_-} = f_{r-2} + f_{r-1} \qquad (q \geq -1,\ r \geq 1).$$

LEMMA 3.9. *The following decomposition holds:*

$$\begin{aligned}
(fg)_{\leq Q} - f_{\leq Q} g_{\leq Q} = \sum_{q > Q+2} &[f_q g_{q_-} + f_{q_-} g_q + f_q g_q]_{\leq Q} + [f_{Q_+} g_{\leq Q} + f_{\leq Q+2} g_{Q_+}]_{\leq Q} \\
&- [f_{(Q-2)_+} g_{\leq Q-2} + f_{\leq Q} g_{(Q-2)_+}]_{Q+1} - [f_Q g_{\leq Q-1} + f_{\leq Q} g_Q]_{Q+2}.
\end{aligned}$$

PROOF. Notice that if p or r is greater than $Q+2$ and $|p-r| > 2$, then the Fourier support of $f_p g_r$ lies outside the ball of radius λ_{Q+1} centered at 0. In particular, $(f_p g_r)_{\leq Q}$ vanishes. Therefore

$$(fg)_{\leq Q} = (f_{\leq Q+2} g_{\leq Q+2})_{\leq Q} + \sum_{\substack{\max\{p,r\} > Q+2 \\ |p-r| \leq 2}} (f_p g_r)_{\leq Q}.$$

So

$$\begin{aligned}
(fg)_{\leq Q} - f_{\leq Q} g_{\leq Q} &= [(fg)_{\leq Q} - (f_{\leq Q+2} g_{\leq Q+2})_{\leq Q}] + [(f_{\leq Q+2} g_{\leq Q+2})_{\leq Q} - f_{\leq Q} g_{\leq Q}] \\
&= \sum_{\substack{\max\{p,r\} > Q+2 \\ |p-r| \leq 2}} (f_p g_r)_{\leq Q} + [f_{\leq Q+2} g_{\leq Q+2} - f_{\leq Q} g_{\leq Q}]_{\leq Q} - (f_{\leq Q} g_{\leq Q})_{> Q}.
\end{aligned}$$

We have the somewhat more explicit representation for the sum:

$$\sum_{\substack{\max\{p,r\}>Q+2\\|p-r|\le 2}} (f_p g_r)_{\le Q} = \sum_{q>Q+2} [f_q g_{q_-} + f_{q_-} g_q + f_q g_q]_{\le Q}. \tag{197}$$

Writing $f_{\le Q+2} = f_{\le Q} + f_{Q_+}$ (and similarly for $g_{\le Q+2}$), then expanding $f_{\le Q+2} g_{\le Q+2}$, we obtain

$$(f_{\le Q+2} g_{\le Q+2})_{\le Q} - (f_{\le Q} g_{\le Q})_{\le Q} = [f_{Q_+} g_{\le Q} + f_{\le Q+2} g_{Q_+}]_{\le Q}. \tag{198}$$

Finally, we note that $(f_p g_r)_q = 0$ whenever $\max\{p+2, r+2\} < q$. Therefore

$$(f_{\le Q} g_{\le Q})_{>Q} = [f_{(Q-2)_+} g_{\le Q-2} + f_{\le Q} g_{(Q-2)_+}]_{Q+1} + [f_Q g_{\le Q-1} + f_{\le Q} g_Q]_{Q+2}, \tag{199}$$

Summing up the right hand sides of (197) and (198), then subtracting the right hand side of (199), we thus obtain the desired decomposition. □

Proposition 3.10. *Let (f,g) be either $(\rho, \rho u)$ or $(\rho u, \rho)$. Then*

$$\left| \int [(f \Lambda_\alpha g)_{\le Q} - f_{\le Q} \Lambda_\alpha g_{\le Q}] U \, dx \right| \lesssim \left[\sum_{q>Q} \lambda_q^\alpha \|f_q\|_{L^2}^2 \right]^{\frac{1}{2}} \left[\sum_{q>Q} \lambda_q^\alpha \|g_q\|_{L^2}^2 \right]^{\frac{1}{2}} + D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u)$$
$$+ [\widetilde{D}_Q(fu) + \widetilde{D}_Q(f) + \widetilde{D}_Q(u)] \widetilde{D}_Q(g).$$

Proof. We replace and g with $\Lambda_\alpha g$ in the decomposition of the Lemma, then multiply by U and integrate.

$$\begin{aligned}
\int [(f \Lambda_\alpha g)_{\le Q} - f_{\le Q} \Lambda_\alpha g_{\le Q}] U \, \mathrm{d}x &= \int U_{\le Q} \sum_{q>Q+2} [f_q \Lambda_\alpha g_{q_-} + f_{q_-} \Lambda_\alpha g_q + f_q \Lambda_\alpha g_q] \, \mathrm{d}x \\
&\quad + \int U_{\le Q} f_{\le Q+2} \Lambda_\alpha g_{Q_+} \, \mathrm{d}x \\
&\quad + \int [U_{\le Q} f_{Q_+} \Lambda_\alpha g_{\le Q} - U_{Q+1} f_{(Q-2)_+} \Lambda_\alpha g_{\le Q-2} - U_{Q+2} f_Q \Lambda_\alpha g_{\le Q-1}] \, \mathrm{d}x \\
&\quad - \int [U_{Q+1} f_{\le Q} \Lambda_\alpha g_{(Q-2)_+} + U_{Q+2} f_{\le Q} \Lambda_\alpha g_Q] \, \mathrm{d}x \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} - \mathrm{IV}.
\end{aligned}$$

Note that we have moved the outermost Littlewood-Paley projections onto the U's and regrouped some terms.

We estimate I and II, as well as the first term in each of III and IV. The remaining terms in III and IV can be estimated similarly.

$$
\begin{aligned}
|\mathrm{I}| &\le \|U_{\le Q}\|_{L^\infty} \sum_{q>Q+2} [\|f_q\|_{L^2}\|\Lambda_\alpha g_{q_-}\|_{L^2} + \|f_{q_-}\|_{L^2}\|\Lambda_\alpha g_q\|_{L^2} + \|f_q\|_{L^2}\|\Lambda_\alpha g_q\|_{L^2}] \\
&\lesssim \sum_{q>Q+2} [\|f_{q-2}\|_{L^2} + \|f_{q-1}\|_{L^2} + \|f_q\|_{L^2}] \cdot \lambda_q^\alpha [\|g_{q-2}\|_{L^2} + \|g_{q-1}\|_{L^2} + \|g_q\|_{L^2}].
\end{aligned}
$$

Then by Cauchy-Schwarz, we conclude that

$$
|\mathrm{I}| \lesssim \left[\sum_{q>Q} \lambda_q^\alpha \|f_q\|_{L^2}^2\right]^{\frac{1}{2}} \left[\sum_{q>Q} \lambda_q^\alpha \|g_q\|_{L^2}^2\right]^{\frac{1}{2}}. \tag{200}
$$

The next term is the most troublesome. We begin by rewriting U as $\rho_{\le Q}^{-1}[(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}] + u_{\le Q}$ and splitting the integral.

$$
\mathrm{II} = \int \rho_{\le Q}^{-1}[(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}]_{\le Q} f_{\le Q+2}\Lambda_\alpha g_{Q_+}\,\mathrm{d}x + \int (u_{\le Q})_{\le Q} f_{\le Q+2}\Lambda_\alpha g_{Q_+}\,\mathrm{d}x.
$$

We bound the first term of II as follows:

$$
\begin{aligned}
\left|\int \rho_{\le Q}^{-1}[(\rho u)_{\le Q} - u_{\le Q}]_{\le Q} f_{\le Q+2}\Lambda_\alpha g_{Q_+}\,\mathrm{d}x\right| &\le \|\rho_{\le Q}\|_{L^\infty}\|(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}\|_{L^2}\|f_{\le Q+2}\|_{L^\infty}\|\Lambda_\alpha g_{Q_+}\|_{L^2} \\
&\le C \cdot C\lambda_Q^{-\alpha} D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u) \cdot C \cdot C\lambda_Q^\alpha \lesssim D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u).
\end{aligned}
$$

To estimate the second term, we recall that $g_{Q_+} = (g_{Q_+})_{>Q-1}$; we can then move the projection $> Q-1$ onto the other term $(u_{\le Q})_{\le Q} f_{\le Q+2}$ in the integrand:

$$
\int (u_{\le Q})_{\le Q} f_{\le Q+2}\Lambda_\alpha g_{Q_+}\,\mathrm{d}x = \int [(u_{\le Q})_{\le Q} f_{\le Q+2}]_{>Q-1}\Lambda_\alpha g_{Q_+}\,\mathrm{d}x
$$

To see why this is useful, we need to massage the resulting expression a bit:

$$
\begin{aligned}
[(u_{\le Q})_{\le Q} f_{\le Q+2}]_{>Q-1} &= [(u_{\le Q} - (u_{\le Q})_{>Q})(f - f_{>Q+2})]_{>Q-1} \\
&= [(u - u_{>Q} - (u_{>Q})_{\le Q})(f - f_{>Q+2})]_{>Q-1} \\
&= (fu)_{>Q-1} - (f_{>Q+2}u)_{>Q-1} - [(u_{>Q} + (u_{>Q})_{\le Q}) f_{\le Q+2}]_{>Q-1}.
\end{aligned}
$$

The point is that, taking L^2 norms, we can now apply (196) to every term in this last expression:

$$
\begin{aligned}
\|[(u_{\le Q})_{\le Q} f_{\le Q+2}]_{>Q-1}\|_{L^2} &\le \|(fu)_{>Q-1}\|_{L^2} + \|f_{>Q+2}\|_{L^2}\|u\|_{L^\infty} + 2\|u_{>Q}\|_{L^2}\|f\|_{L^\infty} \\
&\lesssim \lambda_Q^{-\alpha/2}[\widetilde{D}_Q(fu) + \widetilde{D}_Q(f) + \widetilde{D}_Q(u)]
\end{aligned}
$$

Thus

$$\left|\int [(u_{\le Q})_{\le Q} f_{\le Q+2}]\Lambda_\alpha g_{Q_+}\,\mathrm{d}x\right| \le \|[(u_{\le Q})_{\le Q} f_{\le Q+2}]_{>Q-1}\|_{L^2}\,\|\Lambda_\alpha g_{Q_+}\|_{L^2}$$
$$\lesssim [\widetilde{D}_Q(fu)+\widetilde{D}_Q(f)+\widetilde{D}_Q(u)]\widetilde{D}_Q(g).$$

Overall, II is bounded by

$$|\mathrm{II}| \lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u) + [\widetilde{D}_Q(fu)+\widetilde{D}_Q(f)+\widetilde{D}_Q(u)]\widetilde{D}_Q(g). \tag{201}$$

We estimate first term in III as follows:

$$\left|\int U_{\le Q} f_{Q_+}\Lambda_\alpha g_{\le Q}\,\mathrm{d}x\right| \le \|U\|_{L^\infty}\|f_{Q_+}\|_{L^2}\|\Lambda_\alpha g_{\le Q}\|_{L^2}$$
$$\le C\cdot C\lambda_Q^{-\alpha/2}\widetilde{D}_Q(f)\cdot \lambda_Q^{\alpha/2}\widetilde{D}_Q(g) \lesssim \widetilde{D}_Q(f)\widetilde{D}_Q(g).$$

The second and third terms in III enjoy the same bound, which is proved the same way. Thus

$$|\mathrm{III}| \lesssim \widetilde{D}_Q(f)\widetilde{D}_Q(g). \tag{202}$$

Finally, the first term in IV is bounded by

$$\left|\int U_{Q+1} f_{\le Q}\Lambda_\alpha g_{(Q-2)_+}\,\mathrm{d}x\right| = \left|\int [U-u_{\le Q}]_{Q+1} f_{\le Q}\Lambda_\alpha g_{(Q-2)_+}\,\mathrm{d}x + \int (u_{Q+1})_{\le Q} f_{\le Q}\Lambda_\alpha g_{(Q-2)_+}\,\mathrm{d}x\right|$$
$$\lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u) + \widetilde{D}_Q(u)\widetilde{D}_Q(g).$$

(The intermediate steps are all similar to those used for previous terms.) And the other term in IV enjoys the same bound, so that

$$|\mathrm{IV}| \lesssim D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u) + \widetilde{D}_Q(u)\widetilde{D}_Q(g). \tag{203}$$

Combining (200), (201), (202), and (203), we obtain the desired statement. □

Corollary 3.11. *We have*

$$\left|\int \rho\mathcal{T}(\rho,u)U_{\le Q} - \rho_{\le Q}u_{\le Q}\mathcal{T}(\rho_{\le Q},u_{\le Q})\,dx\right|$$
$$\lesssim \left[\sum_{q>Q}\lambda_q^\alpha\|\rho_q\|_{L^2}^2\right]^{\frac12}\left[\sum_{q>Q}\lambda_q^\alpha\|(\rho u)_q\|_{L^2}^2\right]^{\frac12} + D^{\alpha/2}_{2,Q}(\rho) D^{\alpha/2}_{2,Q}(u)$$
$$+ [\widetilde{D}_Q(\rho u)+\widetilde{D}_Q(\rho)+\widetilde{D}_Q(u)]\widetilde{D}_Q(\rho u) + [\widetilde{D}_Q(\rho u^2)+\widetilde{D}_Q(u)]\widetilde{D}_Q(\rho).$$

Consequently, we have

$$\int_0^t \left| \int \rho \mathcal{T}(\rho, u) U_{\leq Q} - \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q})\, dx \right| dt \to 0, \qquad \text{as } Q \to \infty. \tag{204}$$

Proof. The displayed bound follows easily from the previous Proposition and the discussion at the beginning of this subsection. Indeed, recall that, in the notation from earlier,

$$\int \rho \mathcal{T}(\rho, u) U_{\leq Q} - \rho_{\leq Q} u_{\leq Q} \mathcal{T}(\rho_{\leq Q}, u_{\leq Q})\, \mathrm{d}x = A_1 + A_2 + B_1 + B_2.$$

We have already shown above that $A_1 + B_2 \lesssim D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u)$. The Proposition gives us bounds for A_2 (with $(f, g) = (\rho, \rho u)$) and B_1 (with $(f, g) = (\rho u, \rho)$):

$$A_2 \lesssim \left[\sum_{q>Q} \lambda_q^\alpha \|\rho_q\|_{L^2}^2\right]^{\frac{1}{2}} \left[\sum_{q>Q} \lambda_q^\alpha \|(\rho u)_q\|_{L^2}^2\right]^{\frac{1}{2}} + D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u) + [\widetilde{D}_Q(\rho u) + \widetilde{D}_Q(\rho) + \widetilde{D}_Q(u)] \widetilde{D}_Q(\rho u).$$

$$B_1 \lesssim \left[\sum_{q>Q} \lambda_q^\alpha \|\rho_q\|_{L^2}^2\right]^{\frac{1}{2}} \left[\sum_{q>Q} \lambda_q^\alpha \|(\rho u)_q\|_{L^2}^2\right]^{\frac{1}{2}} + D_{2,Q}^{\alpha/2}(\rho) D_{2,Q}^{\alpha/2}(u) + [\widetilde{D}_Q(\rho u^2) + \widetilde{D}_Q(\rho u) + \widetilde{D}_Q(u)] \widetilde{D}_Q(\rho).$$

Adding up the bounds on A_1, A_2, B_1, and B_2, we obtain the displayed estimate claimed in the Corollary.

The claimed limit then follows by the dominated convergence theorem, with dominating function $C[\|\rho\|_{H^{\alpha/2}}^2 + \|u\|_{H^{\alpha/2}}^2]$. □

This completes the proof of Theorem 3.6.

3.5. Energy Balance for the ρ Equation. It is not difficult to show that (45) holds under the same assumptions as we proved for (44). Actually, something slightly more general is true:

Proposition 3.12. *Let (u, ρ, e) be a weak solution on $[0, T]$ and assume that u and ρ satisfy*

$$\rho \in L^a(0, T; B_{a,\infty}^\sigma), \quad u \in L^b(0, T; B_{b,c_0}^\tau), \qquad \frac{2}{a} + \frac{1}{b} = 2\sigma + \tau = 1. \tag{205}$$

Then (45) holds for a.e. $t \in [0, T]$.

Remark 3.13. As suggested above, (191) is a special case of (205). The reason the latter hypothesis is more flexible is that our proof of Theorem 3.5 strongly depends on the fact that $\rho u \in B_{3,\infty}^{1/3} \cap L^\infty$ (by the algebra property of this space), whereas the argument of the present Proposition above requires information only about ρ and u.

PROOF. Substitute $(\rho_{\le Q})_{\le Q}$ into the weak density equation. This gives

$$\frac{1}{2}\int_{\mathbb{T}} \rho(s)^2_{\le Q}\,\mathrm{d}x\Big|_0^t = \int_0^t \int_{\mathbb{T}} (\rho u)_{\le Q}\rho'_{\le Q}\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{T}} [(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}]\rho'_{\le Q}\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2}\int_0^t \int_{\mathbb{T}} \rho^2_{\le Q}(e_{\le Q} + \Lambda_a\rho_{\le Q})\,\mathrm{d}x\,\mathrm{d}s.$$

The first integral vanished as $Q \to \infty$, since

$$\left|\int_0^t \int_{\mathbb{T}} [(\rho u)_{\le Q} - \rho_{\le Q}u_{\le Q}]\rho'_{\le Q}\,\mathrm{d}x\,\mathrm{d}s\right| \le \int_0^t D^s_{a,Q}(\rho)^2 D^t_{b,Q}(u)\,\mathrm{d}s.$$

The other terms tend to their natural limits. The only convergence requiring justification is

$$\int_0^t \int_{\mathbb{T}} \rho^2_{\le Q}\Lambda_\alpha\rho_{\le Q}\,\mathrm{d}x\,\mathrm{d}s \to \frac{1}{2}\int_0^t \int_{\mathbb{T}}\int_{\mathbb{R}} (\rho(x)+\rho(y))\frac{|\rho(x)-\rho(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s, \text{ as } Q \to \infty.$$

But this follows from an argument entirely similar to that of the convergence (204) which is proved above. We therefore omit the proof. □

4. More Bounds on Regular Solutions: Toward a Theory of Strong Solutions

In order to construct strong solutions, we use essentially the same limiting process that we did for weak solutions. In order to carry out this procedure up one level in regularity, we also give L^∞ (and Hölder) bounds on u', ρ', and e' below, and we track dependence of these bounds on the initial data as before.

4.1. L^∞ Bounds on Derivatives. In this subsection, we prove L^∞ bounds on ρ', u', and e'. The density once again requires the most work. We begin by eliminating all derivatives of u from the ρ' equation. Recall that $u' = e + \Lambda_\alpha\rho$. Replacing u'' by $e' + \Lambda_\alpha\rho'$ does not completely eliminate the need to estimate derivatives of u, as the e' equation involves u'. Therefore we replace e' with q':

$$e' = \rho^2\left(\frac{q'}{\rho}\right) + q\rho', \tag{206}$$

and therefore

$$\rho u'' = \rho^3\left(\frac{q'}{\rho}\right) + e\rho' + \rho\Lambda_\alpha\rho'. \tag{207}$$

This is a more satisfactory replacement in light of the transport equation (36) satisfied by q'/ρ.

In light of the above considerations, we write the ρ' equation as

$$\rho'_t + u\rho'' + \rho\Lambda_\alpha\rho' = -\rho^3\left(\frac{q'}{\rho}\right) - 3e\rho' - 2\rho'\Lambda_\alpha\rho. \tag{208}$$

We multiply by ρ' and evaluate at a maximum $x_+(t)$ of $|\rho'(\cdot,t)|$, yielding

$$\begin{aligned}\frac{1}{2}\partial_t[(\rho')^2](x_+(t),t) &= -\rho^3\rho'\left(\frac{q'}{\rho}\right)(x_+(t),t) - 3e(\rho')^2(x_+(t),t)\\ &\quad - 2(\rho')^2\Lambda_\alpha\rho(x_+(t),t) - \rho\rho'\Lambda_\alpha\rho'(x_+(t),t)\\ &\le \|\rho(t)\|_{L^\infty}^3\|\rho'(t)\|_{L^\infty}\left\|\frac{q'}{\rho}(t)\right\|_{L^\infty} + 3\|e(t)\|_{L^\infty}\|\rho'(t)\|_{L^\infty}^2\\ &\quad + 2|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| - \rho\rho'\Lambda_\alpha\rho'(x_+(t),t).\end{aligned}$$

In order to bound $\|(q'/\rho)(t)\|_{L^\infty}$, we evaluate (36) at a maximum of $(q'/\rho)(\cdot,t)$, integrate in time, and substitute the previously obtained lower bound for ρ. The result is

$$\begin{aligned}\left\|\frac{q'}{\rho}(t)\right\|_{L^\infty} &\le \left\|\frac{q_0'}{\rho_0}\right\|_{L^\infty} + \int_0^t\left\|\frac{f''}{\rho^2} - \frac{f'\rho'}{\rho^3}\right\|_{L^\infty}\mathrm{d}s\\ &\le \|q_0'\|_{L^\infty}\|\rho_0^{-1}\|_{L^\infty} + \frac{\|f''\|_{L^\infty_{x,t}}}{2c_0^2c_1}[\exp(2c_1t)-1] + \frac{\|f'\|_{L^\infty_{x,t}}}{c_0^3}\int_0^t\exp(3c_1s)\|\rho'(s)\|_{L^\infty}\,\mathrm{d}s.\end{aligned}$$

For the present purposes, the following rougher bound will suffice:

$$\left\|\frac{q'}{\rho}(t)\right\|_{L^\infty} \le C_T(\sup_{s\in[0,t]}\|\rho'(s)\|_{L^\infty}+1), \quad t\in[0,T]. \tag{209}$$

Here C_T is a constant that depends on $\|q_0\|_{W^{1,\infty}}$, T, $\|f\|_{L_t^\infty W_x^{2,\infty}}$, α, $\mathcal{M}$, and the L^∞ norms of ρ_0, and ρ_0^{-1}. However, for the remainder of this subsection, we will use C_T to denote a constant that can depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0. Recalling that $|\rho(x,t)|\le C_T$ and $|e(x,t)|\le C_T$, we have proved the following bound, which we pause to record as a Lemma.

Lemma 4.1. *Let (u,ρ) be a regular solution. If $x_+(t)$ is a maximum of $|\rho'(\cdot,t)|$, then*

$$\frac{1}{2}\partial_t[(\rho')^2](x_+(t),t) \le C_T(\sup_{s\in[0,t]}\|\rho'(s)\|_{L^\infty}^2+1) + 2|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| - \rho\rho'\Lambda_\alpha\rho'(x_+(t),t), \tag{210}$$

where C_T is a constant that may depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0.

We now provide some bounds on the final two terms of the above inequality. We will use the notation

$$D_\alpha g(y) := \int_{\mathbb{R}}\frac{|g(y)-g(y+z)|^2}{|z|^{1+\alpha}}\,\mathrm{d}z \tag{211}$$

for functions g such that the integral makes sense.

Before proceeding, we make note of the following facts, which will be useful later: The following bounds hold for a maximum x of g' and some absolute constant c_5:

$$g'\Lambda_\alpha g'(x) \geq D_\alpha g'(x) \tag{212}$$

$$D_\alpha g'(x) \geq c_5 \frac{|g'(x)|^{2+\alpha}}{\|g\|_{L^\infty}^\alpha}. \tag{213}$$

Both of these follow from the nonlinear maximum principle of [14].

The 'bad' term in the inequality from the Proposition above is $|(\rho')^2\Lambda_\alpha\rho|$. In order to estimate this term, we will use the following decomposition of the fractional Laplacian Λ_α:

LEMMA 4.2. *Let $\varphi \in C_c^\infty(\mathbb{R})$ be even, identically 1 on $[-1,1]$, and supported in $(-2,2)$. The following decomposition holds for sufficiently smooth g and any $r > 0$:*

$$\Lambda_\alpha g(x) = \int_{\mathbb{R}} \frac{z}{\alpha}\varphi\left(\frac{z}{r}\right)\frac{g'(x)-g'(x+z)}{|z|^{1+\alpha}}\,dz + \int_{\mathbb{R}}\left[1-\varphi\left(\frac{z}{r}\right)+\frac{z}{\alpha r}\varphi'\left(\frac{z}{r}\right)\right]\frac{g(x)-g(x+z)}{|z|^{1+\alpha}}\,dz. \tag{214}$$

Consequently, we have the following bounds:

$$|\Lambda_\alpha g(x)| \leq Cr^{1-\frac{\alpha}{2}} D_\alpha g'(x)^{\frac{1}{2}} + Cr^{-\alpha}\|g\|_{L^\infty}. \tag{215}$$

$$|\Lambda_\alpha g(x)| \leq Cr^{1-\frac{\alpha}{2}} D_\alpha g'(x)^{\frac{1}{2}} + Cr^{\gamma-\alpha}[g]_{C^\gamma}. \tag{216}$$

PROOF. Rewrite the right side of (214) as follows:

$$\begin{aligned}\text{RHS} = &\int_{\mathbb{R}} \frac{z\varphi\left(\frac{z}{r}\right)g'(x)}{\alpha|z|^{1+\alpha}}\,\mathrm{d}z + \int_{\mathbb{R}}\frac{z}{\alpha|z|^{1+\alpha}}\left[\frac{1}{r}\varphi'\left(\frac{z}{r}\right)(g(x)-g(x+z)) - \varphi\left(\frac{z}{r}\right)g'(x+z)\right]\mathrm{d}z \\ &+ \int_{\mathbb{R}}\left[1-\varphi\left(\frac{z}{r}\right)\right]\frac{g(x)-g(x+z)}{|z|^{1+\alpha}}\,\mathrm{d}z.\end{aligned} \tag{217}$$

The first integral vanishes, while the second can be rewritten as

$$\int_{\mathbb{R}}\frac{z}{\alpha|z|^\alpha}\frac{\mathrm{d}}{\mathrm{d}z}\left[\varphi\left(\frac{z}{r}\right)(g(x)-g(x+z))\right]\mathrm{d}z = \int_{\mathbb{R}}\varphi\left(\frac{z}{r}\right)\frac{g(x)-g(x+z)}{|z|^{1+\alpha}}\,\mathrm{d}z,$$

after integrating by parts. Combining with the third term in (217), we obtain the usual integral formula for $\Lambda_\alpha g$. This completes the proof of (214). To obtain the inequality under consideration, use Cauchy-Schwarz on the first integral in (214) and pull out the L^∞ norm or C^γ seminorm in the second. ∎

LEMMA 4.3. *Let (u,ρ) be a regular solution on the time interval $[0,T]$. The following bounds holds for a maximum $x_+(t)$ of $\rho'(\cdot,t)$ and some constant C_T which may depend only on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$,*

α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0.

$$|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| \le C_T\|\rho'(t)\|_{L^\infty}^{2+\alpha} + \frac{1}{4}\rho_-(t)D_\alpha\rho'(x_+(t),t), \qquad t\in[0,T]; \tag{218}$$

$$|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| \le C_T\frac{\|\rho'(t)\|_{L^\infty}^{2+\alpha-\gamma}}{t^{\gamma/\alpha}} + \frac{1}{4}\rho_-(t)D_\alpha\rho'(x_+(t),t), \qquad t\in(0,T]. \tag{219}$$

Proof. We begin by putting $g=\rho$ in (216). We use (170) for the first term; on the second we use Young's inequality, followed by (213).

$$\begin{aligned}
|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| &\le Cr^{\gamma-\alpha}\|\rho'(t)\|_{L^\infty}^2[\rho(t)]_{C^\gamma} + D_\alpha\rho'(x_+(t),t)^{1/2}\cdot C\|\rho'(t)\|_{L^\infty}^2 r^{1-\alpha/2}\\
&\le C_T r^{\gamma-\alpha}\frac{\|\rho'(t)\|_{L^\infty}^2}{t^{\gamma/\alpha}} + \frac{1}{8}\rho_-(t)D_\alpha\rho'(x_+(t),t) + c_6\rho_-(t)^{-1}\|\rho'(t)\|_{L^\infty}^4 r^{2-\alpha}\\
&\le C_T\frac{\|\rho'\|_{L^\infty}^{2+\alpha-\gamma}}{t^{\gamma/\alpha}} + \frac{1}{8}\rho_-(t)D_\alpha\rho'(x_+(t),t) + \frac{c_5}{8}\frac{\rho_-(t)}{\rho_+(t)^\alpha}\|\rho'(t)\|_{L^\infty}^{2+\alpha}\\
&\le C_T\frac{\|\rho'\|_{L^\infty}^{2+\alpha-\gamma}}{t^{\gamma/\alpha}} + \frac{1}{4}\rho_-(t)D_\alpha\rho'(x_+(t),t),
\end{aligned}$$

provided that we choose

$$r = \left[\frac{c_5\rho_-(t)^2}{8c_6\rho_+(t)^\alpha}\right]^{\frac{1}{2-\alpha}}\|\rho'(t)\|_{L^\infty}^{-1},$$

where c_5 is the constant from (213).

The inequality (218) is established similarly, starting with (215) instead of (216). □

Theorem 4.4. *Let (u,ρ) be a regular solution on the time interval $[0,\infty)$. For each $T>0$, there exists a constant $C_T^{\rho'}$, which may depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0, such that $\|\rho'(t)\|_{L^\infty}\le C_T^{\rho'}$, for $t\in[0,T]$.*

Proof. Step 1: We find a time t_* such that $|\rho'(t)|$ does not grow too much on the interval $[0,t_*]$; more specifically,

$$\|\rho'(t)\|_{L^\infty} \le 2\|\rho_0'\|_{L^\infty}, \qquad \text{for } t\in[0,t_*]. \tag{220}$$

To this end, we apply (210), (212), (218), and (213) to conclude that we have

$$\partial_t\|\rho'(t)\|_{L^\infty}^2 = \partial_t[(\rho')^2](x_+(t),t) \le C_1[\|\rho'\|_{L^\infty(\mathbb{T}\times[0,t])}^2+1] + C_1\|\rho'(t)\|_{L^\infty}^{2+\alpha} \le C_1\|\rho'\|_{L^\infty(\mathbb{T}\times[0,t])}^{2+\alpha}$$

on the interval $t\in[0,1]$. This implies that

$$\|\rho'(t)\|_{L^\infty}^2 \le \frac{\|\rho_0'\|_{L^\infty}^2}{\left[1-\frac{C_1\alpha\|\rho_0'\|_{L^\infty}^\alpha}{2}t\right]^{2/\alpha}} \qquad \text{for } 0\le t<\frac{2}{C_1\alpha|\rho_0'|_{L^\infty}^\alpha}.$$

In particular, putting

$$t_* = \min\left\{\frac{2(1-2^{-\alpha})}{C_1\alpha\|\rho_0'\|_{L^\infty}^\alpha}, 1\right\},$$

we obtain (220), as needed. Note that C_1 depends only on $\|u_0\|_{L^\infty}$, $\|\rho_0\|_{L^\infty}$, $\|\rho_0^{-1}\|_{L^\infty}$, $\|q_0\|_{W_x^{1,\infty}}$, $\|f\|_{L_t^\infty W_x^{2,\infty}}$, α, and $\mathcal{M}$, so that t_* depends only on these quantities and $\|\rho_0'\|_{L^\infty}$.

Step 2: We obtain bounds on ρ' for $t \geq t^*$, by using (219). The point is that in light of the reduction of the power $2+\alpha$ to $2+\alpha-\gamma$, we can now absorb the bad term into the dissipation. At a maximum $x_+(t)$ of ρ', we have

$$\begin{aligned}
\frac{1}{2}\partial_t[(\rho')^2](x_+(t),t) &\leq C_T[\|\rho'\|_{L^\infty(\mathbb{T}\times[0,t])}^2 + 1] + 2|(\rho')^2\Lambda_\alpha\rho(x_+(t),t)| - \rho\rho'\Lambda_\alpha\rho'(x_+(t),t) \\
&\leq C_T[\|\rho'\|_{L^\infty(\mathbb{T}\times[0,t])}^2 + 1] + 2\left[C_T\|\rho'(t)\|_{L^\infty}^{2+\alpha-\gamma}t^{-\gamma/\alpha} + \frac{1}{4}\rho_-(t)D_\alpha\rho'(x_+(t),t)\right] \\
&\quad - \rho_-(t)D_\alpha\rho'(x_+(t),t) \\
&\leq C_T[\|\rho'\|_{L^\infty(\mathbb{T}\times[0,t])}^2 + t^{-\gamma/\alpha}\|\rho'(t)\|_{L^\infty}^{2+\alpha-\gamma} + 1] - c_T\|\rho'(t)\|_{L^\infty}^{2+\alpha}.
\end{aligned}$$

We claim that this implies

$$\|\rho'(t)\|_{L^\infty} \leq \max\left\{\left(\frac{5C_T}{c_T}\right)^{\frac{1}{\alpha}}, \left(\frac{2C_T}{t_*^{\gamma/\alpha}c_T}\right)^{\frac{1}{\gamma}}, 3\|\rho_0'\|_{L^\infty}, 1\right\} =: C_T^{\rho'}, \qquad \text{for } t \in [0,T].$$

Indeed, let t_0 be the largest possible time in the interval $[0,T]$ such that $\|\rho'(t)\|_{L^\infty} \leq C_T^{\rho'}$ for all $t \in [0,t_0]$. Then $t_0 > t_*$ by Step 1 and the definition of $C_T^{\rho'}$. Suppose that $t_* < t_0 < T$. Then $\|\rho'(t_0)\|_{L^\infty} = \sup_{t\in[0,t_0]}\|\rho'(t)\|_{L^\infty} = C_T^{\rho'}$, so that

$$\begin{aligned}
\frac{1}{2}\partial_t[(\rho')^2](x_+(t),t) &\leq C_T\|\rho(t_0)\|_{L^\infty}^2\left(2 - \frac{c_T}{2C_T}(C_T^{\rho'})^\alpha\right) + C_T\|\rho(t_0)\|_{L^\infty}^{2+\alpha-\gamma}\left(\frac{1}{t_0^{\gamma/\alpha}} - \frac{c_T}{2C_T}(C_T^{\rho'})^\gamma\right) \\
&\leq C_T\|\rho(t_0)\|_{L^\infty}^2\left(2 - \frac{5}{2}\right) + C_T\|\rho(t_0)\|_{L^\infty}^{2+\alpha-\gamma}\left(\frac{1}{t_0^{\gamma/\alpha}} - \frac{1}{t_*^{\gamma/\alpha}}\right) < 0,
\end{aligned}$$

contradicting the definition of t_0. We conclude therefore that $t_0 = T$, finishing the proof. $\square$

Now that we have this bound on ρ', it is easy to establish bounds on e'. Recall the relation (206) and the bound (209); combining these with our bounds on ρ', we conclude that e' is uniformly bounded on $\mathbb{T}\times[0,T]$ as well, by a constant C_T which is allowed to depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0.

To bound u', we need to consider two cases, depending on the values of α. When $\alpha \in (0,1)$, we simply recall that $u' = e + \Lambda_\alpha \rho$, which we now know to be bounded on $\mathbb{T} \times [0,T]$ by a constant C_T which is allowed to depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0. For $\alpha \in [1,2)$, however, we do not yet have a bound on $\|\Lambda_\alpha \rho\|_\infty$. We argue as follows in the case $\alpha \in (1,2)$. (Note that we do not include $\alpha = 1$.) We differentiate (174), then replace u' by $\frac{m'}{\rho} - \frac{u\rho'}{\rho}$ and evaluate at a maximum $x_+(t)$ of $|m'(\cdot,t)|$, to obtain

$$\partial_t m' + \frac{|m'|^2}{\rho} - \frac{u\rho' m'}{\rho} + e'm + em' = -\rho'\Lambda_\alpha m - \rho\Lambda_\alpha m' + \rho' f + \rho f' \tag{221}$$

(where we understand that all terms are evaluated at $(x_+(t),t)$. Multiplying by $m'(x_+(t),t)$, we obtain (bracketing the lower-order terms)

$$\begin{aligned}\frac{1}{2}\partial_t[(m')^2](x_+(t),t) = &-\frac{(m')^3}{\rho}(x_+(t),t) - \rho' m'\Lambda_\alpha m(x_+(t),t) - \rho m'\Lambda_\alpha m'(x_+(t),t)\\ &+ \left[\frac{u\rho'|m'|^2}{\rho} - e'mm' - e|m'|^2 + m'\rho' f + \rho m' f'\right](x_+(t),t),\end{aligned}$$

so that

$$\frac{1}{2}\partial_t[(m')^2](x_+(t),t) \le C_T[\|m'(t)\|_{L^\infty}^3 + 1] + |\rho' m'\Lambda_\alpha m(x_+(t),t)| - \rho m'\Lambda_\alpha m'(x_+(t),t),$$

with C_T depending only on the usual quantities. To estimate $|\rho' m'\Lambda_\alpha m(x_+(t),t)|$, we take $r=1$ in (215) to obtain

$$\begin{aligned}|\rho' m'\Lambda_\alpha m(x_+(t),t)| &\le C\|\rho'(t)\|_{L^\infty}\|m'(t)\|_{L^\infty}[D_\alpha m'(x_+(t),t)^{\frac{1}{2}} + \|m\|_{L^\infty}]\\ &\le \frac{1}{4}\rho_-(t) D_\alpha m'(x_+(t),t) + C_T[\|m'\|_{L^\infty}^2 + 1].\end{aligned}$$

Applying (212) and (213) once more, we obtain the following bound:

$$\begin{aligned}\frac{1}{2}\partial_t[(m')^2](x_+(t),t) &\le C_T[\|m'(t)\|_{L^\infty}^3 + 1] + |\rho' m'\Lambda_\alpha m(x_+(t),t)| - \rho m'(x_+(t),t)\Lambda_\alpha m'(x_+(t),t)\\ &\le C_T[\|m'(t)\|_{L^\infty}^3 + 1] - c_T\|m'(t)\|_{L^\infty}^{2+\alpha}.\end{aligned}$$

Since $\alpha > 1$, we can conclude using similar reasoning as in the proof of the bound on $\|\rho'(t)\|_{L^\infty}$ (though the present situation is slightly simpler, since we do not have to reason differently for small and large times). We now pause to record the obtained bounds as a Proposition:

PROPOSITION 4.5. *Let (u,ρ) be a regular solution on the time interval $[0,\infty)$. For each $T>0$, there exists a constant $C_T^{e'}$, which may depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L_x^\infty}$, and the $W^{1,\infty}$ norms* *of u_0, ρ_0, and e_0, such that $\|e'(t)\|_{L^\infty} \le C_T^{e'}$, for $t \in [0,T]$. If $\alpha \neq 1$, there exists a constant $C_T^{u'}$ depending on the same quantities, such that $\|u'(t)\|_{L^\infty} \le C_T^{u'}$, for $t \in [0,T]$.*

4.2. Bounds in Hölder Spaces. For $\alpha \neq 1$, we can now establish Hölder bounds on u' and ρ' in the same way that we treated u and ρ above. We give the details only for ρ'. The right side of (208) is bounded in $L^\infty(0,T;L^\infty)$ for any $T > 0$. Therefore we can apply [57] to (208) now, to conclude that for some $\gamma_1 > 0$, we have

(222)
$$[\rho'(t)]_{C^{\gamma_1}} \le \frac{C_T}{t^{\gamma_1/\alpha}}\left(\|\rho'\|_{L^\infty(\mathbb{T}\times(0,T))} + \left\|\rho^3\left(\frac{q'}{\rho}\right) + 3e\rho' + 2\rho'\Lambda_\alpha\rho\right\|_{L^\infty(\mathbb{T}\times(0,T))}\right) \le \frac{C_T}{t^{\gamma_1/\alpha}}, \quad t \in (0,T].$$

Here C_T is allowed to depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of the initial data. We record this, as well as the analogous bounds for m' and u', in the following Proposition:

Proposition 4.6. *Let (u,ρ) be a regular solution on the time interval $[0,\infty)$ and assume $\alpha \neq 1$. Then for any $T > 0$, there exists $\gamma_1 > 0$ such that ρ, $m = \rho u$, and u satisfy bounds of the form*

$$[\rho'(t)]_{C^{\gamma_1}} \le t^{-\gamma_1/\alpha} C_T, \quad t \in (0,T] \tag{223}$$

$$[m'(t)]_{C^{\gamma_1}} \le t^{-\gamma_1/\alpha} C_T, \quad t \in (0,T] \tag{224}$$

$$[u'(t)]_{C^{\gamma_1}} \le t^{-\gamma_1/\alpha} C_T, \quad t \in (0,T] \tag{225}$$

The constants C_T may depend on $\|f\|_{L_t^\infty W_x^{2,\infty}}$, $\mathcal{M}$, α, T, $\|\rho_0^{-1}\|_{L^\infty}$, and the $W^{1,\infty}$ norms of u_0, ρ_0, and e_0. The number γ_1 ultimately depends only on these same quantities.

5. Strong Solutions

Our next goal is to prove the existence and uniqueness of global strong solutions. We accomplish this in all cases under consideration except the case $\alpha = 1$, where we prove only uniqueness. We will proceed as follows. First, we give the proof of existence for $\alpha \neq 1$; this follows essentially the same outline as the proof of the existence part of Theorem 2.9. Next, we take a brief detour to clarify the regularity of the time derivative $\partial_t u$ in the case when $\alpha \in [1,2)$. This discussion does not contain any deep facts, but it is strictly speaking necessary in order to carry out the integration-by-parts argument in our uniqueness argument. As a byproduct of this discussion, though, we obtain a self-contained proof of the energy equality (44) for strong solutions. Finally, our proof of uniqueness follows a standard Grönwall-type argument. Note, however, that a bit of care is required in handling the dissipation term.

5.1. Existence ($\alpha \neq 1$). We have shown above that u^ϵ and ρ^ϵ are bounded sequences of $L^\infty(\delta, T; C^{1,\gamma_1})$ for each $T > \delta > 0$. Applying the Aubin-Lions-Simon Theorem as before, we can find a subsequence of the sequence ϵ_k constructed in the proof of Theorem 2.9, which we will continue to label ϵ_k, such that $u^{\epsilon_k} \to u$ and $\rho^{\epsilon_k} \to \rho$ strongly in $C_{\text{loc}}((0,\infty); C^1)$. Now $(e^{\epsilon_k})'$ is bounded in $L^\infty(\mathbb{T} \times [0,T])$; therefore it converges weak-$*$ (up to a subsequence) as $k \to \infty$ to some g in the same class. We claim that $g = e'$. Indeed, we have as $k \to \infty$ that

$$\int_0^T \int_{\mathbb{T}} (e^{\epsilon_k})'\varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{T}} u^{\epsilon_k}\varphi'' + \rho^{\epsilon_k}\Lambda_\alpha \varphi' \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{\mathbb{T}} u\varphi'' + \rho\Lambda_\alpha\varphi' \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_{\mathbb{T}} e\varphi' \, \mathrm{d}x \, \mathrm{d}t.$$

But this limit is also equal to $\int_0^T \int_{\mathbb{T}} g\varphi$ by assumption. Therefore e is weakly differentiable in space, with weak derivative $e' = g$. It follows that $(e^{\epsilon_k})'$ converges weak-$*$ to e'. Since (u, ρ, e) is already known to be a weak solution, by Theorem 2.9, this proves that (u, ρ, e) is in fact a strong solution.

5.2. Time Derivatives of Strong Solutions. We begin by noting a few properties of strong solutions. Most of these are basically obvious from the definition, but we believe it is useful to have them recorded explicitly. First, we note that the evolution equations for ρ and e are true pointwise a.e., instead of merely in the weak sense; furthermore, all terms that appear in the equation belong to $L^\infty(0, T; L^\infty)$ for any $T > 0$:

$$\rho_t + (\rho u)' = 0, \quad \text{a.e., and in } L^\infty(0,T;L^\infty); \tag{226}$$

$$e_t + (ue)' = f', \quad \text{a.e., and in } L^\infty(0,T;L^\infty). \tag{227}$$

The same is true for the u-equation if $\alpha < 1$. If $\alpha = 1$, it may not be the case that $\Lambda_\alpha(\rho u) \in L^\infty$, but it will belong to (for example) L^2. (This is a rather academic point at the moment, though, since we have not proven the existence of strong solutions for $\alpha = 1$.) If $\alpha > 1$, though, a pointwise a.e. interpretation is not available. However, we can still view the equation in $L^2 H^{-\alpha/2}$, as we did for weak solutions.

$$u_t + ue = -\Lambda_\alpha(\rho u) + f, \quad \text{a.e., and in } L^\infty(0,T;L^\infty), \text{ if } \alpha \in (0,1); \tag{228}$$

$$u_t + ue = -\Lambda_\alpha(\rho u) + f, \quad \text{a.e., and in } L^2(0,T;L^2), \text{ if } \alpha \in (0,1]; \tag{229}$$

$$u_t + ue = -\Lambda_\alpha(\rho u) + f, \quad \text{in } L^2(0,T;H^{-\alpha/2}); \quad \alpha \in (0,2). \tag{230}$$

Thus a bit of care is warranted in treating time derivatives of u. We next show that the following formula is valid even when $\alpha \in (1,2)$:

$$\frac{1}{2}\partial_t u^2 = u\partial_t u = -u(ue) - u\Lambda_\alpha(\rho u) + uf \quad \text{in } L^2 H^{-\alpha/2}. \tag{231}$$

Of course, when we write (for example) $u\partial_t u$, we mean the element of $H^{-\alpha/2}$ defined by

$$\langle u\partial_t u, \phi\rangle_{H^{-\alpha/2},H^{\alpha/2}} = \langle \partial_t u, u\phi\rangle_{H^{-\alpha/2},H^{\alpha/2}};$$

the latter is perfectly well defined for any $\alpha \in (0,2)$ (but for different reasons, depending on whether $\alpha \in (0,1]$ or $\alpha \in (1,2)$). We will use this interpretation of $u\partial_t u$ and similar elements of $H^{-\alpha/2}$ below without further comment.

Note that (231) is obvious if $\alpha \in (0,1]$; therefore we assume below that $\alpha > 1$. But then $H^{\alpha/2}(\mathbb{T})$ is an algebra, and $\|u\phi\|_{H^{\alpha/2}} \le C\|u\|_{H^{\alpha/2}}\|\phi\|_{H^{\alpha/2}}$. Thus

$$|\langle u\Lambda_\alpha(\rho u), \phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}| = \int \Lambda_{\alpha/2}(\rho u)\Lambda_{\alpha/2}(u\phi)\,\mathrm{d}x \le C\|\rho u\|_{H^{\alpha/2}}\|u\|_{H^{\alpha/2}}\|\phi\|_{H^{\alpha/2}}.$$

It follows that the right side of (231) belongs to $L^2H^{-\alpha/2}$ and is equal to $u\partial_t u$ in this sense. It remains to show that this quantity is in fact equal to $\partial_t u^2$. To do this, we write

$$\langle u(s)^2, \phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\Big|_0^t = \int_{\mathbb{T}} u(s)\cdot u(s)\phi\,\mathrm{d}x\Big|_0^t$$

for some time-independent function $\phi \in C^\infty(\mathbb{T})$ and use the weak formulation of the u-equation, with $u\phi$ serving as the test function. Note that since this weak formulation requires a very slight modification in this case to allow for the rough test function $u\phi$; to deal with this, we simply write a duality pairing in $H^{-\alpha/2}\times H^{\alpha/2}$ when necessary. We have

$$\begin{aligned}
\langle u(s)^2, \phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\Big|_0^t &= \int_0^t \left(\langle \partial_t u - \Lambda_\alpha(\rho u), u\phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}} + \int [fu\phi - (ue)(u\phi)]\,\mathrm{d}x\right)\mathrm{d}s \\
&= \int_0^t \langle u\partial_t u - u(ue) - u\Lambda_\alpha(\rho u) + fu, \phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s \\
&= \int_0^t \langle 2u\partial_t u, \phi\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s.
\end{aligned}$$

This proves the claim. Now we have $\rho, u^2 \in L^2H^1$, $\partial_t\rho, \partial_t(u^2) \in L^2H^{-1}$, so we can apply the usual integration-by-parts formula to ρu^2 as follows:

$$\begin{aligned}
\frac{1}{2}\int \rho u^2(s)\,\mathrm{d}x\Big|_0^t &= \frac{1}{2}\int_0^t \langle \partial_t\rho, u^2\rangle_{H^{-1}\times H^1} + \langle u\partial_t u, \rho\rangle_{H^{-1}\times H^1}\,\mathrm{d}s \\
&= \int_0^t \int (\rho u)(uu')\,\mathrm{d}x + \langle -ue - \Lambda_\alpha(\rho u) + f, \rho u\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s \\
&= \int_0^t \rho u^2(u' - e) + \rho uf - \langle \Lambda_\alpha(\rho u), \rho u\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}.
\end{aligned}$$

Then recalling the definitions of e and $\mathcal{T}$, this yields

$$\frac{1}{2}\int \rho u^2(s)\,\mathrm{d}x\bigg|_0^t = \int_0^t\int \rho u f\,\mathrm{d}x + \langle \mathcal{T}(\rho,u),\rho u\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s$$
$$= -\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}}\rho(x)\rho(y)\frac{|u(x)-u(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}}\rho u f\,\mathrm{d}x\,\mathrm{d}s.$$

Thus we have a self-contained proof of the validity of the energy equality (44) for strong solutions. One can prove (45) even more easily. The main point here, though, is the validity of the integration by parts. We will use this below in our proof of the uniqueness.

5.3. Uniqueness. Next, we prove uniqueness. Let (u_1,ρ_1,e_1) and (u_2,ρ_2,e_2) be two solutions to (26)–(27) with the same initial data. We assume $u_i,\rho_i,e_i\in W^{1,\infty}$ for $i=1,2$. Define

$$u_\delta = u_1-u_2,\qquad \rho_\delta=\rho_1-\rho_2,\qquad e_\delta=e_1-e_2,\qquad q_\delta=q_1-q_2,$$
$$u_\sigma = u_1+u_2,\qquad \rho_\sigma=\rho_1+\rho_2,\qquad e_\sigma=e_1+e_2,\qquad q_\sigma=q_1+q_2.$$

Then by the integration-by-parts formula (which is valid for all $\alpha\in(0,2)$ by the discussion in the previous subsection), we have

$$\int \rho_\sigma u_\delta^2(s)\,\mathrm{d}x\bigg|_0^t = \int_0^t \langle \partial_t\rho_\sigma, u_\delta^2\rangle_{H^{-1}\times H^1} + 2\langle u_\delta\partial_t u_\delta,\rho_\sigma\rangle_{H^{-1}\times H^1}\,\mathrm{d}s$$
$$= 2\int_0^t\int (\rho u)_\sigma(u_\delta u_\delta')\,\mathrm{d}x + \langle -(ue)_\delta - \Lambda_\alpha(\rho u)_\delta, \rho_\sigma u_\delta\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s.$$

Expanding $(\rho u)_\sigma$, $(ue)_\delta$ and $(\rho u)_\delta$ yields

$$\int \rho_\sigma u_\delta^2(s)\,\mathrm{d}x\bigg|_0^t = \int_0^t\int \rho_\sigma u_\sigma u_\delta(u_\delta'-e_\delta) + \rho_\delta u_\delta^2 u_\delta' - \rho_\sigma u_\delta^2 u_\sigma' + \rho_\sigma u_\delta^2\Lambda_\alpha\rho_\sigma\,\mathrm{d}x\,\mathrm{d}s$$
$$-\int_0^t \langle \Lambda_\alpha(\rho_\sigma u_\delta)+\Lambda_\alpha(\rho_\delta u_\sigma),\rho_\sigma u_\delta\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s.$$

Using the definitions of e_δ and $\mathcal{T}$, then symmetrizing, we obtain

$$\int \rho_\sigma u_\delta^2(s)\,\mathrm{d}x\bigg|_0^t = \int_0^t\int [\rho_\delta u_\delta' - \rho_\sigma u_\sigma']u_\delta^2\,\mathrm{d}x + \langle \mathcal{T}(\rho_\sigma,u_\delta)+\mathcal{T}(\rho_\delta,u_\sigma),\rho_\sigma u_\delta\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s$$
$$= \int_0^t\int [\rho_\delta u_\delta' - \rho_\sigma u_\sigma']u_\delta^2\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}}\rho_\sigma(x)\rho_\sigma(y)\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s$$
$$+\int_0^t \langle \mathcal{T}(\rho_\delta,u_\sigma),\rho_\sigma u_\delta\rangle_{H^{-\alpha/2}\times H^{\alpha/2}}\,\mathrm{d}s.$$

For convenience, let us denote

$$\varepsilon_\delta(t) := \frac{1}{2}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}}\rho_\sigma(x)\rho_\sigma(y)\frac{|u_\delta(x)-u_\delta(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.$$

We can estimate the term $\int |\langle \mathcal{T}(\rho_\delta, u_\sigma), \rho_\sigma u_\delta \rangle_{H^{-\alpha/2}\times H^{\alpha/2}}|\,\mathrm{d}x$ as follows.

$$\begin{aligned}
&\int |\Lambda_{\alpha/2}(\rho_\sigma u_\delta u_\sigma)||\Lambda_{\alpha/2}(\rho_\delta)| + |\Lambda_{\alpha/2}(\rho_\sigma u_\delta)||\Lambda_{\alpha/2}(\rho_\delta u_\sigma)|\,\mathrm{d}x \\
&\quad\le C\|\rho_\sigma u_\sigma u_\delta\|_{H^{\alpha/2}}\|\rho_\delta\|_{H^{\alpha/2}} + C\|\rho_\sigma u_\delta\|_{H^{\alpha/2}}\|\rho_\delta u_\sigma\|_{H^{\alpha/2}} \\
&\quad\le C\|\rho_\sigma u_\sigma\|_{W^{1,\infty}}\|u_\delta\|_{H^{\alpha/2}}\|\rho_\delta\|_{H^{\alpha/2}} + C\|\rho_\sigma\|_{W^{1,\infty}}\|u_\delta\|_{H^{\alpha/2}}\|\rho_\delta\|_{H^{\alpha/2}}\|u_\sigma\|_{W^{1,\infty}} \\
&\quad\le C^*\|\rho_\delta\|^2_{H^{\alpha/2}} + \varepsilon_\delta(t).
\end{aligned}$$

Above, we have repeatedly used the fact that $\|fg\|_{H^{a/2}} \le C\|f\|_{W^{1,\infty}}\|g\|_{H^{\alpha/2}}$ for $f \in W^{1,\infty}$, $g \in H^{\alpha/2}$. To see why this inequality holds, simply note that

$$\begin{aligned}
\|fg\|^2_{H^{\alpha/2}} &= \int_{\mathbb{R}}\int_{\mathbb{T}} \frac{|fg(x) - fg(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le \int_{\mathbb{R}}\int_{\mathbb{T}} \frac{2|f(x)|^2|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}}\int_{\mathbb{T}} \frac{2|f(x)-f(y)|^2|g(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le 2\|f\|^2_{L^\infty}\|g\|^2_{H^{\alpha/2}} + C\|f\|^2_{W^{1,\infty}}\|g\|^2_{L^2} \le C\|f\|^2_{W^{1,\infty}}\|g\|^2_{H^{\alpha/2}}.
\end{aligned}$$

Thus, for any $\alpha \in (0,2)$, we have

$$\|\sqrt{\rho_\sigma}u_\delta(t)\|^2_{L^2} \le C\int_0^t \|u_\delta(s)\|^2_{L^2}\,\mathrm{d}s + C^*\int_0^t \|\rho_\delta(s)\|^2_{H^{\alpha/2}}\,\mathrm{d}s. \tag{232}$$

At this stage in the proof, the (possibly quite large) constant C^* may appear worrisome. But as we will see below, the ρ_δ equation carries a term of the form $-\int_0^t \|\rho_\delta(s)\|^2_{H^{\alpha/2}}\,\mathrm{d}s$. Therefore, by multiplying the entire ρ_δ equation by C^* and adding the result to (232), we may absorb this bad term.

We now treat the ρ_δ equation. We obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\rho_\delta^2}{\rho_\sigma}\,\mathrm{d}x &= \int \frac{2\rho_\delta\partial_t\rho_\delta}{\rho_\sigma} + \rho_\delta^2\partial_t\left(\frac{1}{\rho_\sigma}\right)\mathrm{d}x \\
&\le -\int \frac{\rho_\delta(\rho'_\sigma u_\delta + \rho_\sigma u'_\delta + \rho'_\delta u_\sigma + \rho_\delta u'_\sigma)}{\rho_\sigma}\,\mathrm{d}x + C\|\rho_\delta\|^2_{L^2}.
\end{aligned}$$

Using the fact that

$$u'_\delta = \frac{1}{2}(q_\delta\rho_\sigma + q_\sigma\rho_\delta) + \Lambda_\alpha\rho_\delta,$$

we write

$$\int \rho_\delta u'_\delta\,\mathrm{d}x = \frac{1}{2}\int (q_\delta\rho_\sigma + q_\sigma\rho_\delta)\rho_\delta\,\mathrm{d}x + \int |\Lambda_{\alpha/2}\rho_\delta|^2\,\mathrm{d}x.$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\rho_\delta}{\sqrt{\rho_\sigma}}\right\|_{L^2}^2 &\le -\int \frac{\rho_\delta(\rho_\sigma' u_\delta + \rho_\sigma u_\delta' + \rho_\delta' u_\sigma + \rho_\delta u_\sigma')}{\rho_\sigma}\,\mathrm{d}x + C\|\rho_\delta\|_{L^2}^2 \\ &= -\int \rho_\delta u_\delta \frac{\rho_\sigma'}{\rho_\sigma}\,\mathrm{d}x - \frac{1}{2}\int (q_\delta\rho_\sigma + q_\sigma\rho_\delta)\rho_\delta\,\mathrm{d}x - \|\Lambda_{\alpha/2}\rho_\delta\|_{L^2}^2 \\ &\quad + \frac{1}{2}\int \rho_\delta^2\frac{(\rho_\sigma u_\sigma)'}{\rho_\sigma^2}\,\mathrm{d}x + C\|\rho_\delta\|_{L^2}^2 \\ &\le C[\|u_\delta\|_{L^2}^2 + \|\rho_\delta\|_{L^2}^2 + \|q_\delta\|_{L^2}^2] - \|\Lambda_{\alpha/2}\rho_\delta\|_{L^2}^2.\end{aligned}$$

So adding up, integrating in time, and multiplying by the constant C^* from (232), we obtain

$$C^*\left\|\frac{\rho_\delta}{\sqrt{\rho_\sigma}}(t)\right\|_{L^2}^2 \le C\int_0^t \|u_\delta(s)\|_{L^2}^2 + \|\rho_\delta(s)\|_{L^2}^2 + \|q_\delta(s)\|_{L^2}^2\,\mathrm{d}s - C^*\int_0^t \|\rho_\delta(s)\|_{H^{\alpha/2}}^2\,\mathrm{d}s. \tag{233}$$

Adding this to (232), we obtain

$$\|\sqrt{\rho_\sigma}u_\delta(t)\|_{L^2}^2 + C^*\left\|\frac{\rho_\delta}{\sqrt{\rho_\sigma}}(t)\right\|_{L^2}^2 \le C\int_0^t \|u_\delta(s)\|_{L^2}^2 + \|\rho_\delta(s)\|_{L^2}^2 + \|q_\delta(s)\|_{L^2}^2\,\mathrm{d}s. \tag{234}$$

Finally, we deal with the q_δ equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int q_\delta^2\,\mathrm{d}x = -\int u_\delta q_\sigma' q_\delta + u_\delta q_\delta' q_\delta + \frac{2f'}{\rho_1\rho_2}\rho_\delta q_\delta\,\mathrm{d}x \le C\|u_\delta\|_{L^2}^2 + C\|\rho_\delta\|_{L^2}^2 + C\|q_\delta\|_{L^2}^2. \tag{235}$$

Integrating and adding to (234), we obtain

$$\|\sqrt{\rho_\sigma}u_\delta(t)\|_{L^2}^2 + C^*\left\|\frac{\rho_\delta}{\sqrt{\rho_\sigma}}(t)\right\|_{L^2}^2 + \|q_\delta(t)\|_{L^2}^2 \le C\int_0^t \|u_\delta(s)\|_{L^2}^2 + \|\rho_\delta(s)\|_{L^2}^2 + \|q_\delta(s)\|_{L^2}^2\,\mathrm{d}s. \tag{236}$$

This proves that u_δ, ρ_δ, and q_δ are identically zero, thus establishing uniqueness.

5.4. The Case of a Compactly Supported Force ($\alpha \neq 1$)**.** We finally note that, for $\alpha \neq 1$, the construction above gives our solution sufficient regularity so that we can prove that flocking occurs in the special case where $f \equiv 0$ (or when f is compactly supported in time). The key observation is that the velocity field u is C^1 for all positive time; therefore we can apply the results of [55] (when $\alpha > 1$) and [56] (when $\alpha < 1$) to show the existence of a flocking pair. (Actually, the results of [55] are stated and proved only for the case $\alpha = 1$, but trivial adjustments give the analogous statements and proofs for $\alpha > 1$.) We quote the results of intermediate steps without proof, giving details only for the existence of a flocking state.

First, membership of $u(t)$ in $C^1(\mathbb{T})$, $t > 0$ allows us to prove the estimate

$$D_\alpha u'(x) \ge \frac{c|u'(x)|^{2+\alpha}}{A(t)^\alpha}, \tag{237}$$

where $A(t)$ denotes the diameter of the velocities, as before, and c is some positive absolute constant. Recall that in Section 2.4 we showed that $A(t)$ decays exponentially quickly in time if f is compactly supported. Thus (237) allows us to absorb powers of u' into $D_\alpha u'$ after a finite time. For a proof of (237), see Lemma 3.3 in [55] and Lemma 2.2 in [56]. The estimate (237) is used to prove that

$$\|u'(\cdot,t)\|_{L^\infty} \leq Ce^{-\delta t}, \tag{238}$$

for some $\delta > 0$; see Lemma 3.4 in [55] and Lemma 2.3 in [56]. Thus the convergence of $u(t)$ toward the constant $\overline{u}$ (c.f. Section 1.3) occurs exponentially quickly in $W^{1,\infty}$ rather than just in L^∞.

Now, in the case where f is compactly supported in time, all of the bounds of Section 4 can be taken to be constant bounds. Therefore, for large t (say $t > 1$), $u(t)$ and $\rho(t)$ are uniformly bounded in C^{1,γ_1} for some $\gamma_1 > 0$, and $\rho(t)$ is bounded in $W^{1,\infty}$ for *all* t. Defining $\widetilde{\rho}(x,t) = \rho(x - t\overline{u}, t)$ and writing the density equation in the moving reference frame, we conclude that $\|\widetilde{\rho}_t\|_{L^\infty} \leq Ce^{-\delta t}$ on account of the uniform boundedness of $\rho(t)$ in $W^{1,\infty}$ and the estimate (238). It follows that $\widetilde{\rho}$ converges in L^∞ to a limiting state ρ_∞. Defining $\overline{\rho}(x,t) = \rho_\infty(x - t\overline{u})$, this completes the proof of fast flocking in $W^{1,\infty} \times L^\infty$ toward $(\overline{u}, \overline{\rho})$.

We can conclude flocking in stronger spaces if we are willing to sacrifice the exponential rate of convergence. Since $u(t)$ and $\rho(t)$ are uniformly bounded in C^{1,γ_1}, we must in fact have convergence in $C^{1,\epsilon} \times C^{1,\epsilon}$ for every $\epsilon \in (0, \gamma_1)$ by compactness and by uniqueness of the limit in $W^{1,\infty} \times L^\infty$.

APPENDIX A

The Local Pressure Inequality

In this appendix, we prove Lemma 2.5 of Chapter 2, which we recall for the convenience of the reader.

LEMMA. *There exists an absolute constant c such that whenever $p \in L^{3/2}(B_\rho)$ and $-\Delta p = \partial_i\partial_j(u_iu_j)$ a.e. on B_ρ, then for any $r \in (0, \rho/2]$ we have*

$$\begin{aligned} \|p-(p)_r\|_{L^{3/2}(B_r)} \le c\|u\|^2_{L^3(B_{2r})} + cr^{\frac{2}{3}n+1}\int_{2r<|y|<\rho}\frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y \\ + c\frac{r^{\frac{2}{3}n+1}}{\rho^{\frac{2}{3}n+1}}\left(\int_{B_\rho}|u|^3+|p|^{3/2}\,\mathrm{d}y\right)^{\frac{2}{3}}. \end{aligned} \tag{239}$$

PROOF. We follow the proof in [44] quite closely. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \phi(x) \le 1$ for all x, $\phi(x) \equiv 1$ on $\{|x| \le 3\rho/4\}$, $\operatorname{supp}\phi \subset \{|x| < \rho\}$, and the derivatives satisfy $|\nabla\phi| \le c\rho^{-1}$ and $|\partial_i\partial_j\phi| \le c\rho^{-2}$. Then by the solution formula for the Poisson equation and an integration by parts, ϕp can be written

$$\begin{aligned} \phi p(x) &= -c_n\int_{\mathbb{R}^n}\frac{\Delta(\phi p)(y)\,\mathrm{d}y}{|x-y|^{n-2}} \\ &= c_n\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-2}}[\phi\partial_i\partial_j(u_iu_j) - 2\nabla\phi\cdot\nabla p - p\Delta\phi](y)\,\mathrm{d}y \\ &= c_n\int_{|y|<2r}\left[\partial_i\partial_j\frac{1}{|x-y|^{n-2}}\right]\phi u_iu_j(y)\,\mathrm{d}y + c_n\int_{|y|\ge 2r}\left[\partial_i\partial_j\frac{1}{|x-y|^{n-2}}\right]\phi u_iu_j(y)\,\mathrm{d}y \\ &\quad - \frac{c_n}{2}\int\frac{x_j-y_j}{|x-y|^n}(\partial_i\phi)u_iu_j(y)\,\mathrm{d}y + c_n\int\frac{(\partial_i\partial_j\phi)u_iu_j(y)\,\mathrm{d}y}{|x-y|^{n-2}} \\ &\quad - \frac{c_n}{2}\int\frac{x_i-y_i}{|x-y|^3}p\partial_i\phi(y)\,\mathrm{d}y + c_n\int\frac{p\Delta\phi(y)}{|x-y|^{n-2}}\,\mathrm{d}y. \end{aligned}$$

We write the end result as $\phi p = p_{1,1}+p_{1,2}+p_{2,1}+p_{2,2}+p_{3,1}+p_{3,2}$ (with terms appearing in the same order as before). Clearly it suffices to estimate $\|p_{i,j}-(p_{i,j})_r\|_{L^{3/2}(B_r)}$ for each i, j. Henceforth we drop the notation for the dependence of our constant on n.

By the Calderón-Zygmund Theorem,

$$\|p_{1,1}\|_{L^3(B_r)} = \left\|\left[\partial_i\partial_j\frac{c}{|\cdot|^{n-2}}\right]*\phi u_iu_j\chi_{B_{2r}}\right\|_{L^{3/2}(\mathbb{R}^n)} \le C\|\phi u_iu_j\chi_{B_{2r}}\|_{L^{3/2}(\mathbb{R}^n)} \le C\|u\|^2_{L^3(B_{2r})},$$

and therefore

$$\|p_{1,1} - (p_{1,1})\|_{L^{3/2}(B_r)} \leq 2C\|u\|^2_{L^3(B_{2r})}. \tag{240}$$

The other terms are bounded by estimating gradients and using the Mean Value Theorem. We will use the estimate

$$\|g - (g)_r\|_{L^{3/2}(B_r)} \leq cr^{\frac{2}{3}n}\|g - (g)_r\|_{L^\infty} \leq cr^{\frac{2}{3}+1}\|\nabla g\|_{L^\infty(B_r)}$$

repeatedly, without further comment. To bound $p_{1,2}$, we write

$$|\nabla p_{1,2}(x)| \leq c\int_{|y|>2r} \frac{\phi|u|^2\,\mathrm{d}y}{|x-y|^{n+1}} \leq 2^{n+1}c\int_{2r<|y|<\rho} \frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y.$$

Thus

$$\|p_{1,2} - (p_{1,2})\|_{L^{3/2}(B_r)} \leq cr^{\frac{2}{3}n+1}\int_{2r<|y|<\rho} \frac{|u|^2}{|y|^{n+1}}\,\mathrm{d}y. \tag{241}$$

Next, noting that $\partial_i\phi \equiv 0$ in $B_{3\rho/4}$, we have for $|x| < r \leq \rho/2$ that

$$\begin{aligned}|\nabla p_{2,1}(x)| &\leq c\sum_{j=1}^{3}\int_{\frac{3\rho}{4}\leq|y|\leq\rho} \frac{|(\partial_i\phi)u_iu_j|}{|x-y|^n}\,\mathrm{d}y \leq c\int_{\frac{3\rho}{4}\leq|y|\leq\rho} \frac{c\rho^{-1}|u|^2}{(\rho/4)^n}\,\mathrm{d}y \\ &\leq c\rho^{-n+1}\int_{B_\rho}|u|^2\,\mathrm{d}y \leq \rho^{-\frac{2}{3}n-1}\left(\int_{B_\rho}|u|^3\,\mathrm{d}y\right)^{2/3}.\end{aligned}$$

Thus

$$\|p_{2,1} - (p_{2,1})\|_{L^{3/2}(B_r)} \leq c\frac{r^{\frac{2}{3}n+1}}{\rho^{\frac{2}{3}n+1}}\left(\int_{B_\rho}|u|^3\,\mathrm{d}y\right)^{2/3}. \tag{242}$$

Essentially the same argument gives a bound of the same order for $p_{2,2}$. Similarly, $p_{3,1}$ and $p_{3,2}$ can be bounded by

$$\|p_{3,i} - (p_{3,i})\|_{L^{3/2}(B_r)} \leq c\frac{r^{\frac{2}{3}n+1}}{\rho^{\frac{2}{3}n+1}}\left(\int_{B_\rho}|p|^{3/2}\,\mathrm{d}y\right)^{2/3}. \tag{243}$$

Combining all these bounds yields the desired statement.

$\square$

APPENDIX B

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