Shape Theory in Homotopy Theory and Algebraic Geometry

BY

Joseph Berner
B.S., University of Illinois at Chicago, 2012

THESIS
Submitted as partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2018

Chicago, Illinois

Defense Committee:
- Henri Gillet, Chair and Advisor
- Brooke Shipley,
- Ben Antieau
- John Lesieutre
- David Gepner, Purdue University

DEDICATION

para Gary, Nati, Jimmy, y Josue

ACKNOWLEDGEMENTS

Many people have assisted me through my education. Principally I must thank my wife Crystal, without her this work never have been completed. Second I must thank my doctoral adviser who skillfully guided me through the difficult process. I must also thank the many professional mathematicians who have given me helpful technical feedback, motivation, and who have collaborated, both directly and indirectly, to build a beautiful body of research. I must specifically thank Vladimir Berkovich for several useful exchanges, and for the reduction step in the proof of smooth-proper base change, Marc Hoyois who gave me helpful advice and several useful conversations, Yoichi Mieda who gave some helpful comments on Huber's work. I must also give a heartfelt thanks and acknowledgement to my fellow graduate students at UIC. They made a difficult and painful process less painful and less difficult.

JB

SUMMARY

This work defines the étale homotopy type in the context of non-archimedean geometry, in both Berkovich's and Huber's formalisms. To do this we take the shape of a site's associated hypercomplete ∞-topos. This naturally leads to discussing localizations of the category of pro-spaces. For a prime number p, we introduce a new localization intermediate between profinite spaces and ℓ-profinite spaces. This new category is well suited for comparison theorems when working over a discrete valuation ring of mixed characteristic. We prove a new comparison theorem on the level of topoi for the formalisms of Berkovich and Huber, and prove an analog of smooth-proper base change for non-archimedean analytic spaces. This provides a necessary result for the non-archimedean analog of Friedlander's homotopy fiber theorem, which we prove. For a variety over a non-archimedean field, we prove a comparison theorem between the classical étale homotopy type and our étale homotopy type of the variety's analytification. Finally, we examine certain log formal schemes over the formal spectrum of a complete discrete valuation ring, and compare their Kummer étale homotopy type with the étale homotopy type of the associated non-archimedean analytic space.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

> "I have come to believe that the whole world is an enigma, a harmless enigma that is made terrible by our own mad attempt to interpret it as though it had an underlying truth." – Umberto Eco, *Foucault's Pendulum*

A starting point for this work is the following proto-question.

QUESTION 1.1. For a scheme X, is there a space Y such that both

(1) the étale cohomology of X is the singular cohomology of Y,
(2) and the étale fundamental group of X is the fundamental group of Y?

$\triangleleft$

This is an unreasonable question since in full generality neither "étale cohomology" nor "étale fundamental group" have a unique meaning. The simplest fix to this problem is to restrict to finite coefficients.

QUESTION 1.2. For a scheme X, is there a space Y such that both

(1) the étale cohomology $H^i(X_{\text{ét}}, \mathbb{Z}/\ell)$ is the singular cohomology $H^i(Y, \mathbb{Z}/\ell)$,
(2) and there is a bijection between the categories of finite étale covers of X and the finite covering spaces of Y?

$\triangleleft$

This question is more well posed, and we can start from here.

THEOREM 1.3 (ARTIN-MAZUR'S ÉTALE HOMOTOPY TYPE, SEE[**1**]). *To a scheme X, we assign an object in the pro-category of the homotopy category of topological spaces* $\operatorname{pro}\operatorname{Ho} Top$ *denoted by* $\text{ét}\, X$ *such that the cohomology of* $\text{ét}\, X$ *agrees with sheaf cohomology,*

$$H^i(\text{ét}\, X, A) \simeq H^i(X_{\text{et}}, \underline{A})$$

and the profinite completion of the fundamental group of $\text{ét}\, X$ *agrees with the profinite étale fundamental group of X.*

Here the cohomology and fundamental groups of a pro-object are the colimit and limit of the cohomologies and fundamental groups of the elements in the diagram. The essential motivation of Artin and Mazur is the Verdier hypercovering theorem, which gives an alternate class of covers for which the analogue of Čech cohomology computes derived functor cohomology. This class is *hypercovers* which we will come back to later.

THEOREM 1.4 (VERDIER, THEOREM 01H0 OF [**2**]). *Let C be a site with fiber products. Let X be an object in C, $\mathscr{F}$ a sheaf on C, and $n \geq 0$ an integer. Then the derived functor cohomology is functorially isomorphic to the colimit of the cohomologies over all hypercovers of X*

$$H^n(X, \mathscr{F}) \simeq \operatorname*{colim}_{\mathfrak{U} \in HC(X)} \check{H}^n(\mathfrak{U}, \mathscr{F})$$

Accepting Verdier's theorem, Artin and Mazur's construction becomes quite natural. Their pro-space is essentially constructed as follows.

DEFINITION 1.5. Let X be a scheme, and $\mathfrak{U}_\bullet$ a hypercover of X. Then we write $\operatorname{re}\mathfrak{U}_\bullet$ for the geometric realization of the simplicial set determined by the assignment

$$n \mapsto \pi_0 \mathfrak{U}_n$$

Let X be a scheme, and write $HC(X)$ for the category of hypercovers of X. Then the diagram $\operatorname{re} : HC(X) \to Top$ becomes a model for computing hypercover cohomology, in the sense that for any abelian group A and hypercover $\mathfrak{U}$ of X, the two cohomologies are canonically isomorphic

$$H^n(\operatorname{re}\mathfrak{U}, A) \simeq \check{H}^n(\mathfrak{U}, \underline{A}).$$

Thus taking the colimit over $HC(X)$, the diagram of spaces $\operatorname{re} : HC(X) \to Top$ necessarily recovers sheaf cohomology. ◁

This definition is simple and satisfying. Its technical problem is that the underlying diagram is not cofiltered, which is why Artin and Mazur had to pass to the pro-category of homotopy spaces. They also upgraded that a bit to the homotopy category of pro-spaces. In [**3**], Friedlander strengthens this to an *étale topological type* construction which lands directly in pro-simplicial sets. To do this he uses a rigidification idea, which originates with Lubkin, essentially taking covers which have a chosen minimal étale neighborhood of every geometric point they contain. This fixes the fact that the hypercover category is only cofiltered up to homotopy, but breaks functoriality.

THEOREM 1.6 (FRIEDLANDER, SEE **[3]**). *Let X be a scheme. Then there is a pro-simplicial set Y whose cohomology agrees with sheaf cohomology on the étale site of X, and whose profinitely completed fundamental group agrees with the profinite étale fundamental group of X.*

Investigations into the correct notion of $\mathrm{Ho}\,\mathrm{pro}\,\mathcal{S}$ go back to the 1960's, however **[4]** first enriched this from a homotopy category to a model category. The étale homotopy type was then revisited from a purely toposic perspective, by Toën-Vezzosi in **[5]**, Lurie in **[6]**, and Hoyois in **[7]**. Their construction is simple, and instead of describing the diagram of simplicial sets, uses a functor of points approach.

THEOREM 1.7. *Let X be a scheme. Then to X we can assign the hypercomplete ∞-topos of sheaves on the étale site of X. Then the étale homotopy type of X is the pro-space which is corepresented by the functor*

$$\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$$

which takes a space $U \in \mathcal{S}$ to the constant U-valued sheaf $\underline{U}$ and then performs a pushforward back to $\mathcal{S}$.

This definition lifts the étale homotopy type of Artin and Mazur while still being functorial. However we wish to localize the étale homotopy types to make them more manageable. The full étale homotopy type is difficult to work with, and it's not clear how to distinguish which morphisms of schemes induce equivalences of the étale homotopy type. For a fixed prime p, the four localizations we use here are

(1) the category of pro-truncated spaces,
(2) the category of profinite spaces,
(3) the category of $\{p\}^c$-profinite spaces,
(4) and the category of ℓ-profinite spaces.

The category of pro-truncated spaces is the least destructive localization of pro-spaces. A pro-space's image in pro-truncated spaces still 'remembers' all the information of how the original space mapped to truncated spaces. This is the minimal localization to force the Whitehead theorem to be true. In pro-truncated spaces, a morphism of pro-truncated spaces inducing an equivalence on all homotopy group(-oid)s is already an equivalence. The same statement is not true of pro-spaces, the ur-example is the morphism of a space to its Postnikov tower. In fact the localization of pro-spaces to pro-truncated spaces is essentially localizing at the class of all morphisms of pro-spaces to their Postnikov towers. This is due to Isaksen, **[8]**.

The category of profinite spaces is the next least destructive localization. The model categorical underpinnings of this are due to Gereon Quick in **[9]** and **[10]**. However one can do this directly with

∞-categories, as Lurie does in [**11**]. The image in profinite spaces of a pro-space 'remembers' all the information of how the original space mapped to π-finite spaces, that is spaces with finitely many non-zero homotopy groups, which are all finite. Equivalences between profinite spaces are detected by their (naturally profinite) fundamental group, and cohomology with finite coefficients.

We will come back to $\{p\}^c$-profinite spaces later, these are new to this work. They are an intermediate localization between profinite spaces and ℓ-profinite spaces.

Finally the most destructive is the category of ℓ-profinite spaces. The model categorical underpinnings of this are due to Morel, in [**12**]. As before, you can find an ∞-categorical account in [**11**]. The image of a pro-space in ℓ-profinite spaces 'remembers' all the information of how the original space mapped to ℓ-finite spaces, that is spaces with finitely many non-zero homotopy groups, which are all ℓ-primary groups. Since finite ℓ-groups are nilpotent, one can show that this is a homological localization.

The new content is the following.

(1) The model category and associated ∞-category of $\{p\}^c$-profinite spaces. This is a localization of the category of profinite spaces which is not as destructive as passing to ℓ-profinite spaces. As the name suggests, the image of a pro-space in $\{p\}^c$-profinite spaces 'remembers' how the pro-space maps to classifying spaces BG and Eilenberg-MacLane spaces $K(\mathbb{Z}/\ell, n)$ for all primes $\ell \neq p$, all $n \geq 0$ and all finite groups G with order coprime to p. This forces the fundamental group of such a space to be its maximal prime-to-p quotient, and it still has the correct ℓ-adic cohomology. For a fixed p, a morphism being a $\{p\}^c$-profinite equivalence is strictly stronger than being an ℓ-profinite equivalence for every $\ell \neq p$, since the fundamental groups do not need to be nilpotent.

(2) We re-introduce the Kummer étale homotopy type of a log scheme, and introduce the adically Kummer étale homotopy type of certain log formal schemes. The usual analogy of a formal scheme over the formal spectrum of a complete dvr is as a fibration over an ϵ-ball. Under this analogy the Kummer étale topology is something like the Milnor tube over the punctured ball.

(3) We apply the developed techniques to non-archimedean geometry, both in the sense of Berkovich and of Huber. This is essentially setting up a dictionary to pass properties of spaces and morphisms between the two categories. Many geometric theorems admit an easy proof in one of the formalisms, and so we are able to easily reduce smooth-proper base change for Berkovich's formalism to the analogous and proved statement in Huber's formalism. Finally, we consider the case where a non-archimedean analytic space admits a model by a suitable formal scheme,

and show that the étale homotopy type of the analytic space agrees with the adically Kummer étale homotopy type of its formal model.

The main results are as follows.

THEOREM 1.8 (THEOREM 2.81). *Let $f : D \to C$ be a functor between locally connected 1-site with finite limits inducing a geometric morphism $f_* : \mathcal{S}h(D) \to \mathcal{S}(C)$. Write $\mathrm{sh}^\star$ for the localization the shape in one of:*

(1) *pro-truncated spaces,*
(2) *profinite spaces,*
(3) *$\{p\}^c$-profinite spaces,*
(4) *or ℓ-profinite spaces.*

Then the induced map of shapes

$$\mathrm{sh}^\star(\mathcal{S}h(D)) \to \mathrm{sh}^\star(\mathcal{S}h(C)$$

is an equivalence if and only if the corresponding condition is met.

(1) *The functor f^* induces*
  (a) *equivalences on non-abelian cohomology for every group $\mathscr{G}$*
$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$
  (b) *and equivalences on cohomology for every abelian group $\mathscr{L}$*
$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathscr{L})$$
(2) *The functor f^* induces*
  (a) *equivalences on non-abelian cohomology for every locally constant sheaf of finite groups $\mathscr{G}$*
$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$
  (b) *and equivalences on cohomology for all $n \geq 0$ and every constant sheaf of finite abelian groups $\mathscr{L}$*
$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathscr{L})$$
(3) *The functor f^* induces*
  (a) *equivalences on non-abelian cohomology for every locally constant sheaf of groups of order not divisible by p $\mathscr{G}$*
$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

(b) *and equivalences on cohomology for every prime $\ell \neq p$ and every $n \geq 0$,*

$$H^n(D, \mathbb{Z}/\ell) \simeq H^n(C, \mathbb{Z}/\ell).$$

(4) *The functor f^* induces*

(a) *equivalences on non-abelian cohomology for every locally constant sheaf of ℓ-primary groups $\mathscr{G}$*

$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

(b) *and equivalences on cohomology for all $n \geq 0$, $\mathbb{Z}/\ell$*

$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathbb{Z}/\ell)$$

PROPOSITION 1.9 (PROPOSITION 4.57). *Let $\mathfrak{X}$ be an fs log special formal scheme over $\operatorname{spf} R$, with $\mathfrak{X}^\circ$ affine. Assume that the log structure is given by a global chart $\mathscr{P} \to \mathscr{O}_{\mathfrak{X}}$, then $\mathfrak{X}$ is adically log regular if and only if the scheme $X = \operatorname{spec} \mathscr{O}_{\mathfrak{X}}(\mathfrak{X})$ is log regular.*

LEMMA 1.10 (LEMMA 4.59). *Let A be a Noetherian adic ring with ideal of definition $I \triangleleft A$. Write $X := \operatorname{spec} A$ and $X_{\mathrm{red}} = \operatorname{spec} A/I$. Then the morphisms of sites*

$$\mathbf{site}\, X_{\mathrm{red,\,\acute{e}t}} \to \mathbf{site}\, \mathfrak{X}_{\mathrm{a\acute{e}t}} \to \mathbf{site}\, X_{\acute{e}t}$$

induces equivalences on the profinitely completed shapes of the associated ∞-topoi,

$$\widehat{\mathrm{\acute{e}t}}\, X_{\mathrm{red}} \to \mathrm{a\acute{e}t}\ \mathfrak{X} \to \widehat{\mathrm{\acute{e}t}}\, X_{\acute{e}t}.$$

COROLLARY 1.11 (COROLLARY 5.27). *Let K be a non-archimedean normed field. Let X be a scheme locally of finite type over K, then*

(1) *if the characteristic of K is zero or X is proper, we have an equivalence of profinitely completed étale homotopy types*

$$\widehat{\mathrm{\acute{e}t}}\, X \simeq \widehat{\mathrm{\acute{e}t}}\, X^{\mathrm{an}}$$

(2) *if instead the characteristic of K is a positive prime p, then we have an equivalence of $\{p\}^c$-profinitely completed étale homotopy types,*

$$\widehat{\mathrm{\acute{e}t}}_{\{p\}^c}\, X \simeq \widehat{\mathrm{\acute{e}t}}_{\{p\}^c}\, X^{\mathrm{an}}$$

Theorem 1.12 (Theorem 4.42). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth and proper map of compact and Hausdorff K-analytic spaces, and y a geometric point in $\mathcal{Y}$. Pick any prime ℓ coprime to the characteristic of the residue field k. Assume that*

(1) *the morphism has geometrically connected fibers,*
(2) *the étale fundamental group $\pi_1^{\text{ét}}(\mathcal{Y}, y)$ acts trivially on the ℓ-adic cohomology of the fibers,*
(3) *and that $\mathcal{Y}$ is connected.*

Then we have a homotopy fiber sequence of ℓ-profinite spaces

$$\widehat{\text{ét}}_\ell\, \mathcal{X}_y \to \widehat{\text{ét}}_\ell\, \mathcal{X} \to \widehat{\text{ét}}_\ell\, \mathcal{Y}$$

Theorem 1.13 (Theorem 5.40). *Let $\mathcal{X}$ be a Hausdorff strictly K-analytic space. Then the quasi-étale 1-topos of $\mathcal{X}$ is equivalent to the étale 1-topos of $\mathcal{X}^{\text{ad}}$*

$$\tau\, \mathcal{X}_{\text{qét}} \simeq \tau\, \mathcal{X}^{\text{ad}}_{\text{ét}}$$

The local version of the main result is the following.

Lemma 1.14 (Lemma 5.47). *Let $\mathfrak{X}$ be a fs vertical log special formal scheme, and assume*

(1) *that $\mathfrak{X}^\circ$ is affine,*
(2) *there exists a strict and log smooth* spec R*-scheme $V \to$* spec R*,*
(3) *it admits a global chart $P \to \mathscr{O}_V$ for a fs monoid P,*
(4) *and there exists a* strict *log morphism $\mathfrak{X} \to \hat{V}_{V_s}$ making $\mathfrak{X}$ isomorphic to the completion of $\hat{V}_{V_s}$ along some closed subset with the inverse image log structure.*

Then the ℓ-profinitely completed adically Kummer étale homotopy type of $\mathfrak{X}$ is equivalent with the ℓ-profinitely completed étale homotopy type of its generic fiber $\mathcal{X} = \mathfrak{X}_\eta$.

We can glue this together into two global results. The weak one applies for any algebraizably log smooth formal model, whereas the stronger one applies to topologically of finite type formal models.

Theorem 1.15 (Theorem 5.49). *Let $\mathfrak{X}$ be a locally Noetherian and separated log formal scheme, algebraizably log smooth over* spf R*. Then there is an equivalence of ℓ-profinitely completed shapes*

$$\widehat{\text{qét}}_\ell\, \mathfrak{X}_\eta \simeq \widehat{\text{akét}}_\ell\, \mathfrak{X}$$

THEOREM 1.16 (THEOREM 5.50). *Let $\mathfrak{X}$ be a locally Noetherian and separated log formal scheme, adically log smooth over* $\operatorname{spf} R$. *Then there is an equivalence of* $\{p\}^c$*-profinitely completed shapes*

$$\widehat{\text{qét}}_{\{p\}^c}\,\mathfrak{X}_\eta \simeq \widehat{\text{akét}}_{\{p\}^c}\,\mathfrak{X}$$

CHAPTER 2

HOMOTOPICAL PRELIMINARIES

In this chapter we will cover the homotopical background we need for later results.

1. Simplicial objects and Model Categories

We will do this in generality.

DEFINITION 2.1. The category of finite ordered sets with order preserving (but not strictly order preserving) functions between them will be denoted by Δ. ◁

One of the well known structure results is the following, which gives an explicit set of generating morphisms and relations between them.

THEOREM 2.2. *The category Δ is determined by the objects $[n]$ where n ranges over non-negative integers and $[n] = \{0, 1, \ldots, n\}$ is ordered in the usual way. The morphisms are generated by the degeneracy maps $\delta_i^n : [n] \to [n-1]$, for all n and $0 \leq i < n$ which are the unique maps from $[n]$ to $[n-1]$ which are surjective and the preimage of i is $\{i, i+1\}$, along with the face maps $\sigma_i^n : [n] \to [n+1]$ which are the unique injective maps whose image does not contain i. In fact Δ is equivalent to the free category generated by the families δ and σ subject to the relations that*

(1) *If $i < j$, then $\delta_i^n \circ \delta_j^{n+1} = \delta_{j-}^n \circ \delta_i^{n+1}$,*
(2) *if $i < j$, then $\sigma_i^{n+1} \circ \sigma_j^n = \sigma_j^{n+1} \circ \sigma_{i-1}^n$,*
(3) *the compositions of δ and σ satisfy,*

$$\delta_i^{n-1} \circ \sigma_j^n = \begin{cases} \sigma_{j-1}^{n+1} \circ \delta_i^n & \text{if } i < j \\ \mathrm{id} & \text{if } i = j \\ \sigma_j^{n+1} \circ \delta_{i-1}^n & \text{if } j < i \end{cases}$$

We briefly state the definition and then give some motivation.

DEFINITION 2.3. A simplicial object in a category C is a functor $X_\bullet : \Delta^{op} \to C$. ◁

An excellent introduction to model categories is [**13**]. We will review some of the definitions and major results of these foundations.

DEFINITION 2.4. Let M be a category.

(1) We call M a *model category* if it is equipped with the data of three classes of morphisms, (W, C, F) which satisfy the following axioms.

As a matter of terminology, the morphisms in W are called *weak equivalences*, those in C called *cofibrations*, those in F called *fibrations*, those in $W \cap C$ called acyclic cofibrations, and those in $W \cap F$ called acyclic fibrations.

MC1 The ambient category M admits all small limits and colimits,

MC2 the class of weak equivalences satisfies the 2-of-3 property with respect to composition,

MC3 all three classes W, C, and F are closed under retracts in M,

MC4 the class of fibrations is exactly the class of morphisms satisfying the right-lifting property with respect to morphisms in $C \cap W$, and the class of cofibrations is exactly the class of morphisms satisfying the left-lifting property with respect to morphisms in $F \cap W$. That is the context of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dashed, "e"] & D \end{tikzcd}$$

the morphism p is a fibration exactly when a dashed lift exists for any i in $C \cap W$, and the morphism i is a cofibration exactly when a dashed lift exists for any p in $F \cap W$.

MC5 Every morphism can functorially be factorized as a cofibration followed by an acyclic fibration, or as an acyclic cofibration followed by a fibration.

(2) If M is enriched in simplicial sets, we call M a *simplicial scant model category* if it also satisfies

MC6 the category M is tensored and cotensored over simplicial sets, that is for every simplicial set K, and any two objects X and Y of M there are objects $X \otimes K$ and Y^K along with functorial isomorphisms of simplicial sets

$$\mathrm{Map}_M(X \otimes K, Y) \cong \mathrm{Map}_{sSet}(K, \mathrm{Map}_M(X, Y)) \cong \mathrm{Map}_M(X, Y^K)$$

MC7 if $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration, then the map

$$\mathrm{Map}_M(B, X) \to \mathrm{Map}_M(A, X) \times_{\mathrm{Map}_M(A,Y)} \mathrm{Map}_M(B, Y)$$

is a fibration, and is an acyclic fibration as soon as either i or p is acyclic.

$\triangleleft$

As promised, the category of simplicial sets is the most basic example of a model category, and satisfies numerous properties one might ask of a model category. We first make one definition.

DEFINITION 2.5. A morphism $X_\bullet \to Y_\bullet$ is called a *Kan fibration* when it satisfies the right lifting property with respect to the inclusion of the inner horns $\Lambda^n_i \subset \Delta^n$ for all non-negative integers n and all integers $0 \leq i \leq n$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, "i"] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "e"] & Y_\bullet \end{tikzcd}$$

◁

PROPOSITION 2.6. *The category of simplicial sets $sSets$ is a simplicial model category when equipped with the classes of morphisms as follows.*

(1) *Weak equivalences are homotopy equivalences,*
(2) *cofibrations are inclusions,*
(3) *and fibrations are Kan fibrations.*

This is called the Kan-Quillen model structure.

LEMMA 2.7. *Assume that M is a simplicial model category. Then if X is cofibrant and Y is fibrant, the simplicial set*

$$\mathrm{Map}_M(X, Y)$$

is a Kan complex.

PROOF. This follows immediately due to MC7, as we may take $i : \emptyset \to X$ and $p : Y \to *$. The fiber product of mapping spaces must be trivial, since the term on the right is a point. That the map $\mathrm{Map}_M(B, X) \to *$ is fibrant is exactly that the object $\mathrm{Map}_M(B, X)$ is a Kan complex. □

As a caution to the reader, the original definition of a model category was weaker than what we call a scant model category. However our definition of model category agrees with the majority of recent literature.

DEFINITION 2.8. We pose the following definitions.

(1) A model category is *left proper* when weak equivalences are closed under pushout by cofibrations.

(2) A model category is *right proper* when weak equivalences are closed under pullback by fibrations.

(3) A model category is *proper* when it is both left proper and right proper.

(4) If $f : X \to Y$ and $g : X \to Z$ are morphisms in a category with pushouts, then the pushouts of f and g are the two morphisms $f^g : X \coprod_Y Z \to X$ and $g^f X \to \coprod_X Z$.

(5) If $f : Y \to X$ and $g : Z \to X$ are morphisms in a category with pullbacks, then the pullbacks of f and g are the two morphisms $f_g : Z \times_X Y \to Z$ and $g_f : Z \times_X Y \to Y$.

(6) Let α be a limit ordinal and C a category , a *α-composable family* of morphisms in C is a diagram in C indexed over the category of elements of α with inclusion as morphisms.

(7) Let C be a category, α a limit ordinal $\alpha \to M$ an α-composable family of morphisms. If the colimit $X_0 \to \operatorname{colim}_{\beta<\alpha} X_\beta$ exists, we call it the *transfinite composition* of the α-composable family.

(8) Let M be a model category, and $I \subset \operatorname{Mor}(M)$ a set of morphisms. We denote by $I_{\coprod}$ the closure of I under pushouts

(9) Let M be a model category, and $I \subset \operatorname{Mor}(M)$ a set of morphisms. We denote by $I_{\prod}$ the closure of I under pullbacks.

(10) Let M be a model category, and $I \subset \operatorname{Mor}(M)$ a set of morphisms. We denote by $\operatorname{trfin}(I)$ to be the closure of I under transfinite composition.

(11) Let M be a model category. We say that M is *cofibrantly generated* if there are small sets $I, I^{\text{acyc}} \subset \operatorname{Mor}(M)$ of morphisms such that
   (a) the class of fibrations of M is exactly those morphisms which have the right lifting property with respect to all morphisms in $\operatorname{trfin}(I^{\text{acyc}}_{\coprod})$,
   (b) and the class of acyclic fibrations is exactly those morphisms which have the right lifting property with respect to all morphisms in $\operatorname{trfin}(I_{\coprod})$.

(12) Let M be a model category. We say that M is *fibrantly generated* if there are small sets $J, J^{\text{acyc}} \subset \operatorname{Mor}(M)$ of morphisms such that
   (a) the class of cofibrations in M is exactly those morphisms which have the left lifting property with respect to all morphisms in $\operatorname{trfin}(J^{\text{acyc}}_{\prod}$,
   (b) and the class of acyclic cofibrations in M is exactly those morphisms which have the left lifting property with respect to all morphisms in $\operatorname{trfin}(J_{\prod})$.

$\triangleleft$

THEOREM 2.9. *The model category $sSet$ is proper and cofibrantly generated.*

PROOF. That $sSet$ is proper is Theorem 13.1.13 of [**14**], that it is cofibrantly generated is Example 11.1.6 of [**14**]. □

2. Model Structures on pro-Spaces

There are two general model structures on pro-categories, both due to Isaksen. The first comes from [**4**], and we call it the strict model structure.

DEFINITION 2.10. Let C be a category. The ***pro-category*** of C, denoted $\operatorname{pro} C$, is the category whose objects are filtered diagrams in C, and whose homomorphisms are defined via limit and colimit in sets

$$\operatorname{Hom}_{\operatorname{pro} C}(X_\bullet, Y_\bullet) := \lim_j \operatorname*{colim}_i \operatorname{Hom}_C(X_i, Y_j)$$

Composition of morphisms is defined via diagram chasing and universal properties. ◁

When a category M is given the structure of a model category, it was a long standing question of how to give $\operatorname{pro} C$ a model structure so that $\operatorname{Ho}\operatorname{pro} M$ is categorically equivalent to $\operatorname{pro}\operatorname{Ho} M$. The earliest this was considered is arguably [**1**], and the question was answered for reasonable model categories M by Isaksen in [**4**].

DEFINITION 2.11. Let M be a proper model category. Then we define the weak equivalences W of $\operatorname{pro} M$ to be morphisms $f : X_\bullet \to Y_\bullet$ which are ***essentially levelwise weak equivalences***, meaning that f factors as

$$Y_\bullet \xrightarrow{\sim} Y'_\bullet \xrightarrow{f'} X_\bullet$$

where the first morphism is an isomorphism in the pro-category and the second is a levelwise weak equivalence. ◁

This definition already becomes insufficient without the proper hypothesis, as one cannot show the 2-of-3 property, or even that compositions of these weak equivalences are still weak equivalences.

The definition for cofibrations is similar,

DEFINITION 2.12. Let M be a proper model category. Then we define the cofibrations of $\operatorname{pro} M$ to be morphisms $f : X_\bullet \to Y_\bullet$ which are essentially levelwise cofibrations, meaning that f factors as

$$Y_\bullet \xrightarrow{\sim} Y'_\bullet \xrightarrow{f'} X_\bullet$$

where the first morphism is an isomorphism in the pro-category and the second is a levelwise cofibration. ◁

The class of fibrations has a fairly explicit description in terms of matching spaces, see Definition 4.2 of [**4**], but is unnecessary for our purposes.

THEOREM 2.13 (THEOREMS 4.15 AND 4.17 OF [**4**]). *The above weak equivalences and cofibrations determine a model structure on* $\mathrm{pro}\, M$. *If* M *is simplicial, then so is* $\mathrm{pro}\, M$.

Although not formally stated, it is a simple consequence of Isaksen's definitions that $\mathrm{Ho}\,\mathrm{pro}\, M$ will be $\mathrm{pro}\,\mathrm{Ho}\, M$:

LEMMA 2.14. *The natural functor taking levelwise homotopy types* $\mathrm{Ho} : \mathrm{pro}\, M \to \mathrm{pro}\,\mathrm{Ho}\, M$ *witnesses the latter category as the homotopy category of the former.*

PROOF. The functor Ho takes a morphism to an equivalence if and only if it is essentially a levelwise weak equivalence in $\mathrm{pro}\,\mathrm{Ho}\, M$, which happens if and only if it is an essentially levelwise weak equivalence. Thus we are exactly inverting the class of weak equivalences W. □

The final theorem of this section is the existence of left Bousfield localizations at a set of objects for a pro–category.

THEOREM 2.15 (THEOREM 2.4 OF [**15**]). *Let* M *be a left proper fibrantly generated model category with all small limits. Then, if* $K \subset M$ *is a set of fibrant objects the following holds.*

(1) *The left Bousfield localization* $L_K M$ *of* M *at* K *exists,*
(2) *the localization* $L_K M$ *is left proper,*
(3) *if all objects in* M *are cofibrant then* $L_K M$ *is fibrantly generated,*
(4) *and if* M *is also a simplicial model category, then* $L_K M$ *is also simplicial.*

This will be a critical tool in the next section.

In [**8**], Isaksen defines the following model structure on pro-simplicial sets.

DEFINITION 2.16. Let $\mathrm{pro}\, sSet$ be the category of pro-simplicial sets. We give it a model structure given by the following.

(1) The weak equivalences are morphisms $f : X_\bullet \to Y_\bullet$ where $\pi_0 f$ is an isomorphism of pro-sets, and f^* induces an isomorphism of homotopy groupoids for $n \geq 1$

$$f^* : \Pi_n X_\bullet \to \Pi_n Y_\bullet$$

here the homotopy groupoid is the pro-groupoid with $(\Pi_n X_\bullet)_i = \Pi_n(X_i)$ and the same index category as $X_\bullet$.

(2) The cofibrations are morphisms that are essentially levelwise injections.

(3) The fibrations are maps with the right lifting property with respect to all acyclic cofibrations.

We call this the ***non-strict model structure*** on $\mathrm{pro}\, sSet$. ◁

In **[8]**, Isaksen shows that this gives a model category on pro-simplicial sets, and that it is the localization of his strict model structure at the functor sending a pro-simplicial set to its Postnikov tower.

3. The Model Category of profinite spaces and p-profinite spaces

We will first review the model category of profinite spaces due to **[9]**.

Definition 2.17. The category of profinite spaces $\widehat{sSet}$ is the category of simplicial objects in profinite sets. ◁

In fact the underlying category was first used by **[12]**, who equipped it with a model structure modeling p-profinite completion of spaces. We will soon see that Morel's p-profinite model structure is a left Bousfield localization of the profinite model structure. We can mimic many of the basic definitions from abstract homotopy theory in the category of simlicial profinite sets.

Definition 2.18. We define the following for a pointed simplicial profinite set (X, x).

(1) The pointed profinite set $\pi_0(X, x)$ which is the equalizer in the category of profinite sets of $d_0, d_1 : X_0 \to X_1$, with chosen element the component containing $x \in X$.

(2) The profinite fundamental group $\pi_1(X, x)$ defined in terms of the Galois category in the sense of Grothendieck, of finite covers of the pair (X, x).

(3) The continuous singular chains on $X_\bullet$, where $C_n X_\bullet = \hat{\mathbb{Z}}[\mathrm{Map}(\Delta^n, X_\bullet)] = \hat{\mathbb{Z}}[X_n]$, and its homology the continuous homology $H^\bullet(X_\bullet, \hat{\mathbb{Z}})$.

(4) The continuous singular cochains as the continuous dual of the above, and continuous cohomology as the cohomology of that cochain complex.

(5) Given a continuous representation of the profinite fundamental group $\pi_1(X, x)$, we can construct the corresponding local system on the profinite space X.

◁

We can pretty quickly deduce some results.

THEOREM 2.19 (SEE [**9**] AND [**10**]). *Write $F : \widehat{sSet} \to sSet$ for the levelwise forgetful functor from simplicial profinite sets to simplicial sets, and by abuse of notation write F for the forgetful functor from profinite sets to sets and from profinite groups to groups. Then the following hold.*

(1) *The underlying set of connected components of a profinite space agrees with the connected components of the underlying simplicial set,*

$$F(\pi_0(X,x)) \cong \pi_0(F(X), F(x)).$$

(2) *The the fundamental group of a profinite space agrees with the profinite completion of the fundamental group of the underlying simplicial set*

$$\pi_1(X,x) \cong \widehat{\pi_1}(F(X), F(x)).$$

(3) *at least with finite coefficients, the continuous homology of X agrees with the homology of $F(X)$,*

$$H_*(X, \mathbb{Z}/n) \cong H_*(F(X), \mathbb{Z}/n).$$

(4) *at least with finite coefficients, the continuous cohomology of X agrees with the cohomology of $F(X)$,*

$$H^*(X, \mathbb{Z}/n) \cong H^*(F(X), \mathbb{Z}/n).$$

Given this set up, we define the model structure on $\widehat{sSet}$ as follows.

DEFINITION 2.20. A morphism $f : X \to Y$ of profinite spaces is a weak equivalence when it induces an isomorphism on π_0 and on homology coefficients in arbitrary local systems. A morphism of profinite spaces is a cofibration when it is a levelwise monomorphism. $\triangleleft$

As before, we can describe a generating class of fibrations explicitly, however it is not necessary for our use case. What we do need is the nice properties of this model structure.

THEOREM 2.21 (THEOREM 2.12 OF [**9**]). *The above gives a left proper fibrantly generated model structure on $\widehat{sSet}$.*

COROLLARY 2.22. *Let $K \subset \widehat{sSet}$ be any set of objects. Then the left Bousfield localization $L_K\widehat{sSet}$ exists and is left proper, fibrantly generated, and simplicial.*

PROOF. We just apply Theorem 2.15. $\square$

To finish this section we will first enumerate some results about this model category.

THEOREM 2.23 (THEOREM 2.28 OF [**9**]). *There is a Quillen adjunction* $\widehat{(-)} : sSet \to \widehat{sSet}$ *whose right adjoint is the levelwise forgetful functor.*

PROOF. It is a well known fact that to check that an adjoint pair is a Quillen adjunction, it is sufficient to check that the left adjoint preserves cofibrations and acyclic cofibrations. This is one of a few equivalent formulations, see Theorem 8.5.3 of [**14**]. Since cofibrations in both categories are levelwise monomorphisms, checking that $\widehat{(-)}$ preserves cofibrations simply means checking that the profinite completion functor of sets preserves injections.

Injections are monomorphisms in sets, and monomorphisms are preserved by limits. Since the profinite completion functor is determined by taking the limit over all finite sets surjected upon by the given set, it preserves injections. □

DEFINITION 2.24. Let p be a prime number.

Taking K_p to be the set of $K(\mathbb{Z}/p, n)$ for all non-negative integers n gives us Morel's p-profinite spaces of [**12**].

Taking $K_{\{p\}^c}$ to be the set of spaces $K(\mathbb{Z}/\ell, n)$ for all primes $\ell \neq p$ and non-negative n, and BG where G is a finite group whose order is coprime to p gives ***the model category of*** $\{p\}^c$***-profinite spaces*** $L_{K_{\{p\}^c}}\widehat{sSet}$. ◁

LEMMA 2.25. *Let* ℓ *and* p *be distinct prime numbers. Then the category* $L_{K_p}\widehat{sSet}$ *is a left Bousfield localization of* $L_{K_{\{p\}^c}}\widehat{sSet}$.

PROOF. Since the localization $\widehat{sSet} \to L_{K_\ell}\widehat{sSet}$ inverts $L_{K_{\{p\}^c}}\widehat{sSet}$-local morphisms, we obtain a Quillen adjunction $L_{K_{\{p\}^c}}\widehat{sSet} \to L_{K_\ell}\widehat{sSet}$. By Theorem 2.15 $L_{K_{\{p\}^c}}\widehat{sSet}$ is left proper, fibrantly generated, and simplicial. It again satisfies the assumptions for Theorem 2.15, and so the localization at the images of the objects in K_ℓ exists. By the universal properties of $L_{K_\ell}L_{K_{\{p\}^c}}\widehat{sSet}$ and $L_{K_\ell}\widehat{sSet}$ we see that the two are Quillen equivalent. □

3.1. Model structures on simplicial (pre)sheaves. In this section we will recall some of the definitions and relations of the various model structures on simplicial sheaves and simplicial presheaves over a site. We will first give an exceedingly brief review of sheaves on sites.

DEFINITION 2.26. Let C be a category with fibered products. A *Grothendieck pretopology* on C is a class of families $Cov(U)$, where each $Cov(U)$ is a family of morphisms to U

$$Cov(U) = \{\{U_i \to U\}_{i \in I}\}$$

satisfying the following properties:

(1) For every isomorphism $\alpha : U' \to U$ in C, the one element covering family $\{\alpha\}$ is in $Cov(U)$.
(2) For every covering family $\{U_i \to U\}_{i \in I}$ and every arrow $f : V \to U$ the pullback of the covering family of U gives a covering family of V,

$$\{U_i \times_U V \to V\}_{i \in I} \in Cov(V)$$

(3) For every covering family $\{U_i \to U\}_{i \in I}$, and every family of covering families $\{U_{ij} \to U_i\} \in Cov(U_i)$, the composition $\{U_{ij} \to U\}_{i \in I, j \in J}$ is a covering family.

$\triangleleft$

This lays the minimum structure with which to define sheaves. The second axiom is essentially about intersections, and the third is about refinements of open coverings. Given a presheaf on C, we can use the above to make sense of whether or not it is a sheaf.

Definition 2.27. Let $\mathscr{F}$ be a presheaf on a category C that is equipped with a Grothendieck pretopology. We declare $\mathscr{F}$ a *sheaf* when for every object $U \in C$, and every covering family $\{U_i \to U\}_{i \in I}$ the diagram

$$\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_U U_j)$$

is an equalizer diagram. $\triangleleft$

The above definition is not special to presheaves of sets, and makes sense for presheaves valued in any category with equalizers. We have one more technical definition we need, that of hypercovers.

Definition 2.28. Let C be a category with fibered products and equipped with a Grothendieck pretopology. For an object X in C, a *hypercover* $U_\bullet$ of X is a simplicial object in C with an augmentation to X such that all of the maps

$$\operatorname{cosk}_n(U_\bullet/X) \to U_\bullet$$

are levelwise coverings of each U_i.

The *category of hypercoverings* $HCov(X)$ of an object X is the category with objects hypercoverings of X, and whose morphisms are morphisms of simplicial sets. $\triangleleft$

A major technical issue with the category of hypercoverings is that it is not a cofiltered category. However the associated ∞-category is cofiltered.

Lemma 2.29. *When viewed as a simplicially enriched category, the ∞-category of hypercovers $N^s(HCov(X))$ is cofiltered.*

Proof. See the discussion preceeding Proposition 4.1 of [**7**]. □

We will use this fact after we have defined ∞-categories and reviewed some basic results. We now specialize to the case where the target category is simplicial sets. We have the following lemma, whose proof is essentially trivial.

Lemma 2.30. *Let C be a category with a Grothendieck pretopology. The category of simplicial objects in the category of presheaves (resp. sheaves) of sets is equivalent to the category of presheaves (resp. sheaves) of simplicial sets.*

The major issue is that there is an abundance of model category structures which model "sheaves of spaces on C", which have different underlying categories but simplify to a few Quillen equivalent classes.

For the underlying category, we have two options.

(1) The category of simplicial presheaves of sets on C, $sPre(C)$.
(2) The category of simplicial sheaves of sets on C, $sSh(C)$.

Historically, the category of simplicial sheaves attracted more interest than the category of simplicial presheaves. Since the 2000's the simplicial presheaf category has ascended to prominence, due to the simplicity of Dugger's projective model structure.

On both $sPre(C)$ and $sSh(C)$ we have a choice of six model structures.

(1) The *injective model structure*, also called the global injective model structure,
(2) the *Čech-local injective model structure*,
(3) the *hyper-local injective* model structure,
(4) the *projective model structure*, also called the global projective model structure,
(5) the *Čech-local projective model structure*,
(6) and the *hyper-local projective model structure.*

Since the Čech-local and hyper-local model structures are just Bousfield localizations of the base ones, we will first define the injective and projective model structures. Given the previous lemma, this is essentially just restating a case of the definition of the injective and projective model structures for functor categories.

Definition 2.31. Let C be a category with a Grothendieck pretopology.

Then the injective model structure on the category of simplicial presheaves (resp. simplicial sheaves) is the model structure whose weak equivalences are sectionwise weak equivalences, and whose cofibrations are sectionwise cofibrations of simplicial sets.

The projective model structure on the category of simplicial presheaves (resp. simplicial sheaves) is the model structure whose weak equivalences are sectionwise weak equivalences, and whose fibrations are sectionwise fibrations of simplicial sets. ◁

Lemma 2.32. *The identity induces a Quillen equivalence between the global injective model structure and global projective model struture on simplicial presheaves.*

The same is true for the restrictions of those two model structures to simplicial sheaves.

Proof. This follows from immediately from the definition of a Quillen equivalence. □

We now define the local and hyper-local variants of the above.

Definition 2.33. The local model structure is obtained by performing a left Bousfield localization at the class of "Čech descent morphisms"

$$S = \{ \ \{ \operatorname*{hocolim}_n \text{čech}(U/X)_n \ \}_{U/X \in Cov(X)} \ \}_{X \in C}$$

where $\text{čech}(U/X)_n$ is the simplicial Čech complex of the covering U/X. The homotopy colimit is taken in the ambient model structure.

The hyper-local model structure is obtained by performing a left Bousfield localization at the class of "hyperdescent morphisms"

$$S = \{ \ \{ \operatorname*{hocolim}_n (U/X)_n \ \}_{U/X \in HCov(X)} \ \}_{X \in C}$$

where $HCov(X)$ is the class of hypercovers of X in the pretopology on C. ◁

Beware that this is not entrenched terminology. The term "local model structure" is typically defined in terms of stalkwise equivalences. When a site has enough points, being local for stalkwise equivalences is equivalent to a morphism being local for hyperdescent morphisms. By using Čech-local and hyper-local we avoid any ambiguity. We now state the relationships between these twelve model categories.

Theorem 2.34. *Let S be a site. Then the following four model categories are Quillen equivalent.*

(1) *The Čech-local projective model structure on simplicial presheaves,*

(2) *the Čech-local projective model structure on simplicial sheaves,*

(3) *the Čech-local injective model structure on simplicial presheaves,*

(4) *and the Čech-local injective model structure on simplicial sheaves.*

Proof. This follows from the preceding lemma and the appendix of [**16**], which shows that after Čech-localization the model structures on simplicial presheaves and simplicial sheaves are Quillen equivalenta. □

Corollary 2.35. *Let S be a site. Then the following four model categories are Quillen equivalent.*

(1) *The hyper-local projective model structure on simplicial presheaves,*

(2) *the hyper-local projective model structure on simplicial sheaves,*

(3) *the hyper-local injective model structure on simplicial presheaves,*

(4) *and the hyper-local injective model structure on simplicial sheaves.*

In this work, we will always use the projective model structures on simplicial presheaves unless otherwise specified.

Definition 2.36. A presheaf of simplicial sets $\mathscr{F}$ on a site will be called a *sheaf of spaces* when it is fibrant for the Čech-local projective model structure on simplicial presheaves. ◁

Note the corresponding sheaf in the corresponding ∞-category will be the cofibrant replacement of the given sheaf of spaces. However the cofibrant replacement cannot be too different from a fibrant presheaf due to the following lemma.

Lemma 2.37. *If $\mathscr{F}$ is a fibrant simplicial presheaf, then it is sectionwise weakly equivalent to a fibrant and cofibrant simplicial presheaf.*

Proof. There are multiple prototypes for this result in the literature, and so we make no claim to the originality of this proof. Being fibrant for the local projective model structure implies that the simpicial presheaf $\mathscr{F}$ is fibrant for the global projective model structure. Taking a cofibrant replacement there, we find a $\mathscr{F}'$ which is sectionwise equivalent to $\mathscr{F}$ and is cofibrant. However since it is sectionwise equivalent, the descent diagram for $\mathscr{F}$ and $\mathscr{F}'$ with respect to any Čech cover are levelwise equivalent. This implies their homotopy limits are the same on an arbitrary hypercover. By the Yoneda lemma, this is equivalent to saying that $\mathscr{F}'$ is local with respect to hypercovers, and is thus fibrant for the local projective model structure. □

4. Quasicategories

The notion of a higher category has been around for quite some time. Arguably, the first concrete definition of a 2-category is due to Bourbaki. The core of the analogy is that higher categories are to 1-categories as homotopy types are to sets. Just as there are multiple notions and special classes of homotopy types, there is a wealth of definitions of higher categories.

The model of ∞-categories we will use is that of quasi-categories. Denote by Λ^n_i the "i-th horn of the n-simplex", which is the n-simplex Δ^n minus both its unique non-degenerate n-simplex and the $n-1$-simplex opposite the i-th vertex.

Definition 2.38. A simplicial set $X_\bullet$ is a ***weak Kan complex***, or synonymously a ***quasi-category***, if for every positive integer n, every integer $0 < i < n$, and every commutative diagram,

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, "i"] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "e"] & \star \end{tikzcd}$$

a filling e exists which makes the resulting diagram commute. ◁

Originally studied by Boardman and Vogt in [**17**], these simplicial sets were not widely acknowledged as a model for higher category until the late 1990s. A careful and detailed treatment is available in [**6**]. We will review some of the key results and ideas of it, along with newer results used in this work. In this work ∞-category will always be interpreted in the model of quasi-categories.

The motivating theorem for the base definition of an ∞-category is the following theorem, which is originally due to Segal.

Theorem 2.39. *Let Cat be the category of small categories, and $sSet$ the category of simplicial sets. Then the nerve construction $N : Cat \to sSet$ induces a fully faithful map whose essential image is the subcategory of weak Kan complexes whose fillings in the above diagram are unique.*

The heuristic behind this is the following correspondence.

(1) A 0-simplex is an object.
(2) A 1-simplex is a morphism from its source to its target. The degenerate 1-simplices are the identity maps of the objects.
(3) A 2-simplex has three 1-simplices corresponding to morphisms f, g, and h, and the 2-cell tells us that $h = g \circ f$. The uniqueness of fillings for the horn Λ^2_1 tells us that there are unique compositions.

(4) An 3-simplex corresponds to four objects $\{x, y, z, w\}$, six morphisms

$$\Big\{f : x \to y, g : y \to z, h : z \to w, i : x \to z, j : y \to w, k : x \to w\Big\}$$

four 2-simplicies which declare that $i = g \circ f$, $j = h \circ g$, $k = j \circ f$, and $k = h \circ i$. In other words, we see that the two compositions agree.

$$h \circ (g \circ f) = (h \circ g) \circ f$$

The 3-cell is interpreted as a composable triple of morphisms, and the unique filling condition gives the associativity of morphism composition.

(5) An n-simplex corresponds to composable n-tuples of morphisms and guarantees the existence of a unique composition and that all possible parentheses placements agree.

This motivates the following definition, of a deceptively simple nature.

Definition 2.40. A ∞-category is a weak Kan complex, and a morphism of ∞-categories is simply a map of simplicial sets. $\triangleleft$

By relaxing the uniqueness condition on the simplicial set, we allow the possibility that there may be multiple compositions of two composable functions. If there are two candidates for $g \circ f$, then we expect that there are even more candidates for a triple composition $h \circ g \circ f$, and we no longer know that $(h \circ g) \circ f = h \circ (g \circ f)$. The situation rapidly becomes impossible to track by hand. The higher inner horn filling condition in fact guarantees that while there is no uniqueness of composition, nor is there associativity of composition, these are rectifiable by some higher homotopy. This higher homotopy itself is unique up to a non-unique higher homotopy.

This definition is convenient foundationally, but it is less convenient for actually producing examples of ∞-categories. The upshot is that every 1-category C is naturally an ∞-category through the nerve construction. We will take care to write $N(C)$ for the ∞-category associated with a 1-category. There are a few more families we may construct ∞-categories from.

Definition 2.41. For each non-negative integer $n \geq 0$, define the categorical n-simplex $\mathfrak{C}(n)$ to be the simplicially enriched 1-category whose objects are the elements of $\{0, 1, \ldots, n\}$, and where the homomorphisms are defined to be the nerve of the poset of subsets of $\{i + 1, \ldots, j - 1\}$ ordered by inclusion.

$$\mathrm{Hom}_{\mathfrak{C}(n)}(i, j) := N(\{S \subset (i, j) | \text{ for all } k \in S, i < k < j\})$$

Composition of such subsets is just the union.

Now let C be an arbitrary 1-category enriched in simplicial sets. The *homotopy coherent nerve* $N^s(C)$ is the geometric realization of the bisimplicial set determined by the assignment

$$\Delta^n \mapsto \operatorname{Hom}(\mathfrak{C}(n), C)$$

$\triangleleft$

The key result is that the simplicial nerve of suitable simplicially enriched categories gives us an ∞-category. Since such categories are abundant, this is quite convenient.

THEOREM 2.42. *Let C be a simplicially enriched 1-category whose homomorphism simplicial sets are Kan complexes. Then $N^s(C)$ is an ∞-category.*

See Proposition 1.1.5.10 of **[6]** for a proof.

COROLLARY 2.43. *Let M be a simplicial model category, and write M_{fc} for the full simplicially enriched subcategory of fibrant-and-cofibrant objects. Then the simplicial nerve $N^s(M_{fc})$ is an ∞-category, and furthermore we have a natural equivalence of homotopy categories $\operatorname{Ho} N^s(M_{fc}) \simeq \operatorname{Ho} M$.*

In this generality we also recover the result that homotopy (co)limits in the simplicial scant model category exactly agree with (co)limits in the corresponding ∞-category.

THEOREM 2.44 (THEOREM 4.2.4.1 OF **[6]**). *Let I and C be simplicially enriched categories such that all mapping spaces in them are Kan complexes. Further assume there is a functor $d : I \to C$, then an object $X \in C$ is the homotopy (co)limit of d if and only if $X \in N^s(C)$ is the (co)limit of the functor $d : N^s(I) \to N^s(C)$.*

We now define some of the key ∞-categories we will use.

DEFINITION 2.45. The simplicial nerve of the fibrant-cofibrant objects in the simplicial model category of simplicial sets with the Kan-Quillen model structure is the ∞-category of ∞-groupoids. We call its objects spaces or sometimes ∞-groupoids and denote the category $\mathcal{S}$.

The simplicial nerve of the fibrant-cofibrant objects in the simplicial model category of simplicial sets with the Joyal model structure is the ∞-category of ∞-categories, called $\mathcal{C}at_\infty$. $\triangleleft$

Since we have essentially defined that all categories are small, functor categories are relatively easy to construct. We simply define the functor category to be the mapping space in the simplicially enriched category of simplicial sets.

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) := \mathrm{Map}_{sSet}(\mathcal{C}, \mathcal{D})$$

This doesn't quite avoid set-theoretic difficulties, since it just pushes them into the theory of universes. However, the difficulties may be overcome and we ignore them.

5. Accessibility and Presentability

Accessibility and presentability are technical conditions which allow us to work with large categories. Since largeness and smallness are set-theoretic notions, it is no surprise that these are tied to set theory. Let us begin recalling some basic definitions. We will assume the axiom of choice.

DEFINITION 2.46. We pose the following definitions.

(1) An *ordinal* is a set α such that if $\beta \in \alpha$ is an element, then $\beta \subset \alpha$ is also a subset, and the elements of α are well ordered by inclusion.
(2) Given the axiom of choice, all ordinals are comparable by inclusion. Explicitly, if α and β are ordinals, either $\beta \subset \alpha$, $\beta = \alpha$, or $\alpha \subset \beta$.
(3) A *cardinal* is the least ordinal number in its bijection class.
(4) The *cofinality* of an ordinal X is the minimum of the order types of subsets S of X whose supremum $\sup S$ is X. It is immediate that the cofinality of an ordinal is less than or equal to itself.
(5) A cardinal is *regular* if its cofinality is equal to itself, and it is uncountable.

◁

This is the core terminology we will need to discuss these ideas. Now recall the following definition.

DEFINITION 2.47. A category C is *filtered* if for any finite diagram $d : I \to C$, we have that d has a cocone in C. ◁

The generalization we will make here is by relaxing the finiteness condition on the diagram.

DEFINITION 2.48. Let κ be a cardinal. Then a category (resp. an ∞-category) C is κ-filtered if any diagram $d : I \to C$ has a cocone in C, whenever the cardinality of

$$\mathrm{Ob}\, I \cup \bigcup_{x,y \in \mathrm{Ob}\, I} \mathrm{Hom}_I(x, y)$$

is less than κ. ◁

The intuition is that this is a highly filtered category. If $\kappa < \kappa'$ are regular cardinals, then C being κ'-filtered implies that C is also κ-filtered. Since the usual definition of filtered is just ω-filtered, we see that this is a special case of being filtered. Just as filtered colimits commute with finite limits, we generalize this idea to uncountable regular cardinals.

THEOREM 2.49. *Let $\mathcal{S}$ be the ∞-category of ∞-groupoids. Then for any cardinal κ, κ-filtered colimits in $\mathcal{S}$ commute with κ-small limits.*

PROOF. The key observation is that a κ-filtered diagram refines to a κ-filtered poset. This is Proposition 5.3.1.16 of [**6**]. Once we are taking the limit over a poset, we compute the colimit and limit as homotopy colimits and homotopy limits in the model category of simplicial sets. Then the desired result follows from the fact that κ-filtered colimits commute with κ-small limits in $Sets$. □

With this notion, we can define ind-categories.

DEFINITION 2.50. Let $\mathcal{C}$ be an ∞-category. The *κ-ind-category* of $\mathcal{C}$, denoted $\operatorname{ind}_\kappa \mathcal{C}$ as the subcategory of the functor category $\operatorname{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by functors which are κ-filtered colimits of representables.

The *ind-category* of $\mathcal{C}$, denoted $\operatorname{ind} \mathcal{C}$ is the subcategory of functors which are filtered colimits of representables. ◁

This is an appropriate definition, albeit abstract. If $X_\bullet$ is a filtered diagram of objects in $\mathcal{C}$, we can take the colimit of the filtered diagram of representables $h_{X_\bullet}$. We can construct $X_\bullet$ from the presheaf $h_{X_\bullet}$, using Grothendieck's famous result that all presheaves are colimits of representables. At least when $\mathcal{C}$ has enough limits, we can actually characterize them directly in terms of categorical properties of the corresponding presheaves.

LEMMA 2.51 (COROLLARY 5.3.5.4 OF [**6**]). *If $\mathcal{C}$ admits all κ-small limits for a regular cardinal κ, then $\operatorname{ind}_\kappa \mathcal{C}$ can be identified as functors in $\operatorname{Fun}(\mathcal{C}^{op}, \mathcal{S})$ which preserve all κ-small limits.*

With this we can now define accessible ∞-categories, which are large ∞-categories generated by some small amount of data.

DEFINITION 2.52. An ∞-category $\mathcal{C}$ is *κ-accessible* if there is a small subcategory $\mathcal{C}_0 \subset \mathcal{C}$ such that $\mathcal{C}$ is equivalent to $\operatorname{ind}_\kappa \mathcal{C}_0$.

An ∞-category $\mathcal{C}$ is *accessible* if it is κ-accessible for some regular cardinal κ. Note that such κ will not be unique. ◁

Accessible categories are ubiquitous, and often accessibility is an important technical property in proofs. A slight strengthening of this notion is that of presentability. Together, accessibility and presentability are fundamental technical concepts in ∞-category theory.

DEFINITION 2.53. An ∞-category $\mathcal{C}$ is presentable when it is accessible and closed under small colimits.

$\triangleleft$

The most useful handle on such objects is that presentable ∞-categories always come from model categories.

THEOREM 2.54 (PROPOSITION A.3.7.6 OF [**6**]). *The following are equivalent for an ∞-category $\mathcal{C}$.*

(1) *The ∞-category $\mathcal{C}$ is presentable,*
(2) *there is a combinatorial simplicial model category M such that $\mathcal{C}$ is equivalent to the simplicial nerve of the fibrant-cofibrant objects of M*

$$\mathcal{C} \simeq N^s(M_{fc}).$$

6. Basics of ∞-topoi

Classically there are two ways to think about topoi. One is geometric, where a topos is the category of sheaves on some site and is the minimal formalism one wants to do geometry. One is more foundational, where a topos is a category which behaves sufficiently much like the category of sets. The ∞-categorical perspective is closer to the former, where ∞-topoi are now the minimal formalism one wants to do homotopical geometry.

Just as topoi were categories of sheaves on some site, ∞-topoi are in a suitable sense built out of sheaves of spaces on ∞-sites which we first define. Recall that a *sieve* on a category $\mathcal{C}$ is a subcategory $\mathcal{C}' \subset \mathcal{C}$ whose morphisms are closed under precomposition by arbitrary morphisms in $\mathcal{C}$. Given a morphism $f : X \to Y$ in $\mathcal{C}$, the *sieve generated by* f is the subcategory of $\mathcal{C}_{/Y}$ whose morphisms are morphisms factoring through f.

DEFINITION 2.55. An ∞-category $\mathcal{C}$ is called an ∞-site when it is equipped with a class of *covering sieves* $Cov(U)$ of arrows

$$Cov(U) = \{\mathcal{U}_i \subset \mathcal{C}_{/U}\}_{i \in I}$$

index over the objects U of $\mathcal{C}$ satisfying the following properties:

(1) For every object U the slice category $\mathcal{C}_{/U}$ is a covering sieve.

(2) For every object U, morphism $f : V \to U$, and covering sieve $\mathcal{U}_0$ on U, the pullback $f^*\mathcal{U}_0$ is a covering sieve of V.

(3) For every object U, covering sieve $\mathcal{U}_0$, and arbitrary sieve $\mathcal{D}$, if for every morhpism f in $\mathcal{U}_0$ the pullback $f^*\mathcal{D}$ is a covering sieve, then $\mathcal{D}$ is a covering sieve.

$\triangleleft$

Given the definition of an ∞-site, we can already give some examples of ∞-topoi in analogy to the classical case.

Definition 2.56. Let $\mathcal{C}$ be an ∞-site. The *∞-topos of sheaves of spaces* on $\mathcal{C}$ is the full subcategory of $\mathrm{Fun}_{\mathcal{C}at_\infty}(\mathcal{C}^{op}, \mathcal{S})$ spanned by functors that invert monomorphisms which generate covering sieves of their target. $\triangleleft$

Theorem 2.57. *If $\mathcal{C}$ is the nerve of a 1-category, then the topos of sheaves of spaces on $\mathcal{C}$ is equivalent to the simplicial nerve of the simplicial model category of simplicial presheaves with the Čech-local projective model structure.*

Proof. Inverting those monomorphisms is the definition being local with respect to those selfsame monomorphisms. The way sites are defined, this is exactly being local with respect to Čech covers, which is what makes a presheaf fibrant in the Čech-local projective model structure. □

This does not construct all ∞-topoi. As we already saw, the hyper-local projective model structure is typically not Quillen equivalent to the local projective model structure. The corresponding property in terms of ∞-topoi is that of *hypercompleteness*. We will first discuss the notion of cotopological localizations, it will turn out that the hypercompletion is the maximal cotopological localization.

Definition 2.58. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{Y} \subset \mathcal{X}$ a subcategory, and assume the inclusion has a left adjoint $L : \mathcal{X} \to \mathcal{Y}$ which is accessible and left exact. We call the inclusion $\subset$ a *cotopological localization* if its left adjoint L does not invert any monomorphisms in $\mathcal{X}$ that are not already equivalences. $\triangleleft$

We open with the definition, in the vein of Giraud's axioms.

Definition 2.59. An ∞-category $\mathcal{X}$ is said to be an ∞-topos when

(1) the ∞-category $\mathcal{X}$ is presentable,

(2) colimits in $\mathcal{X}$ are universal,

(3) coproducts in $\mathcal{X}$ are disjoint,

(4) and every groupoid in $\mathcal{X}$ is effective.

The correct class of morphisms of ∞-topoi are geometric morphisms, which as in the 1-categorical case are those morphisms with left exact left adjoints.

A *geometric morphism* of ∞-topoi $f : \mathcal{X} \to \mathcal{Y}$ is an adjoint pair (f^*, f_*) with the additional property that $f^* : \mathcal{Y} \to \mathcal{X}$ preserves finite limits. We call the left adjoint f^* the *inverse image functor*, and the right adjoint f_* the *direct image functor*.

Note that we have chosen the morphism to be covariant with the direct image functor, as is common. $\triangleleft$

We will not use Giraud's axioms, we will always use the following alternative characterization of ∞-topoi.

Theorem 2.60 (Remark 6.5.2.20 of **[6]**). *Let $\mathcal{X}$ be an ∞-category. Then the following are equivalent.*

(1) *$\mathcal{X}$ is an ∞-topos,*
(2) *there is an ∞-site $\mathcal{C}$, and a geometric morphism $\mathcal{X} \to \mathcal{S}h(\mathcal{C})$ which witnesses $\mathcal{X}$ as a cotopological localization of the category of sheaves on $\mathcal{C}$.*

The purpose of the cotopological localization is that it inverts "∞-connective" morphisms, e.g. morphisms whose fibers have trivial homotopy group sheaves. The issue is that satisfying Čech descent is strictly weaker than satisfying hypercover descent in general, and so there are sheaves that are stalkwise equivalent but not equivalent in the non-hypercompleted topos. This is highly analogous to Whitehead's Theorem, and in fact one may interpret Whitehead's Theorem as proving that the ∞-topos $\mathcal{S}$ is hypercomplete.

Definition 2.61. (1) Let $\mathcal{C}$ be an ∞-category, and k a non-negative integer. Then an object X in $\mathcal{C}$ is called *k-truncated* if for all objects Y of $\mathcal{C}$ the mapping space $\mathrm{Map}_{\mathcal{C}}(Y, X)$ is k-truncated as a Kan complex. It is called ∞-truncated if it is k-truncated for all non-negative integers k.
(2) Let $\mathcal{C}$ be an ∞-category with finite limits, then a morphism $f : X \to Y$ in $\mathcal{C}$ is called *∞-connective* if the fiber $0 \times_Y X$ is ∞-truncated.
(3) Let $\mathcal{X}$ be an ∞-topos, then the *hypercompletion* $\mathcal{X}^\wedge$ is the cotopological localization of $\mathcal{X}$ at the class of all ∞-connective morphisms.
(4) Let $\mathcal{C}$ be an ∞-site, then the hypercompletion of the topos of sheaves of spaces on $\mathcal{C}$ is called *the hypercomplete topos of sheaves on $\mathcal{C}$.*

$\triangleleft$

7. Pro-objects in ∞-categories

We now come to notion of pro-objects in an ∞-category. While formally dual to the notion of a ind-object, the major technical difficulty is that one does not have good control of non-accessible categories. If $\mathcal{C}$ is accessible, it is almost never the case that $\mathcal{C}^{op}$ is accessible, and so pro-categories are a bit difficult to work with. This can be overcome when we have some control over $\mathcal{C}$, and we can make an alternate definition in two special cases.

Definition 2.62. Let $\mathcal{C}$ be either a small category or an accessible category, which has all finite limits in either case. The *pro-category of* $\mathcal{C}$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$ spanned by functors which are ω-accessible and preserve finite limits. $\triangleleft$

Lemma 2.63. *The following are equivalent for an ∞-category $\mathcal{C}$ that is small (resp. accessible), and a functor F in* $\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{op}$.

(1) *The functor preserves finite limits(resp. preserves finite limits, and is also accessible).*
(2) *There is a cofiltered ∞-category $\mathcal{I}$ and a diagram $G : \mathcal{I} \to \mathcal{C}$ such that the functor G corepresents on $\mathcal{C}$ is equivalent to F.*

Proof. When $\mathcal{C}$ is small, we can directly apply Corollary 5.3.5.4 of [**6**].

The case when $\mathcal{C}$ is accessible is also due to Lurie. The essential argument goes as follows, given a diagram G it corepresents the following functor.

$$X \mapsto \operatorname*{colim}_{i \in I} \mathrm{Map}_{\mathcal{C}}(G(i), X)$$

We need to show that for some regular cardinal κ that this preserves finite limits and κ-filtered colimits in its argument X. Finite limits follows since the Yoneda functors all preserve limits, the colimit above is filtered, and filtered colimits commute with finite limits in $\mathcal{S}$. We may take κ to be any regular cardinal larger than both the size of $\mathcal{I}$ and the regular cardinal that $\mathcal{C}$ is accessible with respect to. If $\mathcal{C}$ is λ-accessible, then all objects in $\mathcal{C}$ are λ-compact, and so the above mapping spaces commute with colimits who are more than λ-filtered.

To construct the diagram, we note that F can be realized by some colimit of corepresentables. The properties of limits and colimits in $\mathcal{S}$ force the underlying diagram to be cofiltered. $\square$

LEMMA 2.64. *Let $\mathcal{C}$ be an ∞-category, then if $X_\bullet : \mathcal{I} \to \mathcal{C}$ and $Y_\bullet : \mathcal{J} \to \mathcal{C}$ are cofiltered diagrams, the mapping space in the pro-category is equivalent to the expected formula.*

$$\mathrm{Map}_{\mathrm{pro}\,\mathcal{C}}(X_\bullet, Y_\bullet) \simeq \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Map}_{\mathcal{C}}(X(i), Y(j))$$

PROOF. Our definition of pro-objects is as special functors. To pass from a diagram to a functor, we take the functor it corepresents on $\mathcal{C}$. Given $X_\bullet$, we simply send it to the functor

$$X_\bullet \mapsto \left[c \mapsto \lim_{i\in\mathcal{I}} \mathrm{Map}_{\mathcal{C}}(X(i), c)\right]$$

which will satisfy the assumptions of preserving finite limits (resp. preserving finite limits and κ-filtered colimits for κ larger than the size of $\mathcal{I}$) by virtue of the fact that $\mathcal{I}$ is a cofiltered category. We can instead write $X_\bullet$ as the formal colimit in functors to $\mathcal{S}$ of $h^{X(i)}$ so that we have the equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\operatorname*{colim}_i h^{X(i)}, \operatorname*{colim}_j h^{Y(j)}) \simeq \lim_i \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(h^{X(i)}, \operatorname*{colim}_j Y(j))$$

By the Yoneda lemma, we can simplify the right hand side

$$\lim_i (\operatorname*{colim}_j Y(j)) \circ X(i) \simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_{\mathcal{C}}(Y(j), X(i))$$

Of course, the pro-category is the opposite category of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$, and so we conclude the desired lemma. $\square$

We can now state the universal property of the pro-category as we have constructed it.

DEFINITION 2.65 (CF. **[11]** AND **[6]**). Let $\mathcal{C}$ be an arbitrary ∞-category. A *pro-category construction for* $\mathcal{C}$ is a functor $c : \mathcal{C} \to \mathcal{P}$ such for any other ∞-category $\mathcal{D}$ admitting all cofiltered limits, there is an equivalence of functor categories given by precomposition

$$- \circ c : \mathrm{Fun}'(\mathcal{P}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where Fun' is the subcategory of functors that preserve all cofiltered limits. $\triangleleft$

As a trivial consequence of this universal property, pro-category constructions for a fixed ∞-category are unique up to equivalence.

THEOREM 2.66 (PROPOSITION 3.1.6 OF **[11]**). *The first definition of* $\mathrm{pro}\,\mathcal{C}$ *gives a pro-category construction for small categories and for accessible categories.*

The above gives the universal property of the pro-category, however we still need some more functoriality properties of pro-categories. Given a functor $\mathcal{C} \to \mathcal{D}$ we get a functor (up to contractible space of indeterminacy) $\operatorname{pro}\mathcal{C} \to \operatorname{pro}\mathcal{D}$, and one of the more important questions for us is when it admits a left adjoint. The following lemma suffices for what we need.

LEMMA 2.67. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with all finite limits, and $f : \mathcal{C} \to \mathcal{D}$ a functor that preserves finite limits. Assume that one of the following holds,*

(1) *the former ∞-category $\mathcal{C}$ is small,*
(2) *or both $\mathcal{C}$ and $\mathcal{D}$ are accessible, and f is accessible.*

Then the following statements hold.

(1) *The induced functor $F : \operatorname{pro} C \to \operatorname{pro} D$ admits a left adjoint L_f.*
(2) *If f is fully faithful, so is F.*
(3) *If f is fully faithful, the functor L_f is a localization.*
(4) *A morphism $X_\bullet \to Y_\bullet$ in $\operatorname{pro}\mathcal{D}$ is taken to an equivalence in $\mathcal{C}$ if and only if*

$$\operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Map}_{\mathcal{C}}(fY(j), Z) \to \operatorname*{colim}_{i \in \mathcal{I}} \operatorname{Map}_{\mathcal{C}}(fX(i), Z)$$

is an equivalence for every Z in $\mathcal{C}$.

(5) *If $\mathcal{C}$ is generated under finite limits by a subcategory $\mathcal{C}'$, then a morphism $X_\bullet \to Y_\bullet$ in $\operatorname{pro}\mathcal{D}$ is taken to an equivalence in $\mathcal{C}$ if and only if*

$$\operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Map}_{\mathcal{C}}(fY(j), Z) \to \operatorname*{colim}_{i \in \mathcal{I}} \operatorname{Map}_{\mathcal{C}}(fX(i), Z)$$

is an equivalence for every Z in $\mathcal{C}'$.

PROOF. We separate the proofs of the statements.

(1) This is a mild extension of similar results in **[11]** and **[18]**.

The functor $L_f : \operatorname{pro}\mathcal{D} \to \operatorname{pro}\mathcal{C}$ has a simple definition.

$$L_f : \Big(c \in \operatorname{Fun}(\mathcal{C}, \mathcal{S})^{op}\Big) \mapsto \Big(c \circ f \in \operatorname{Fun}(\mathcal{D}, \mathcal{S})^{op}\Big)$$

If $\mathcal{C}$ is small, then this automatically lands in the full subcategory of pro-objects. If $\mathcal{C}$ is accessible, then the assumption that f is accessible and left exact precisely implies that it maps functors in $\operatorname{pro}\mathcal{D}$ to functors in $\operatorname{pro}\mathcal{C}$. To put F and L_f on the same footing, we need to give explicit and easily comparable expressions. Recall that the easiest way to do this is with diagram notation. If $X_\bullet : \mathcal{I} \to \mathcal{C}$ is a cofiltered diagram, then we have

$$FX_\bullet \simeq F \circ X_\bullet$$

unraveling the functor this corepresents leads to

$$FX_\bullet \simeq \operatorname*{colim}_{i \in \mathcal{I}} (f(X_i), -) = \operatorname*{colim}_{i \in \mathcal{I}} h^{fX(i)} : \mathcal{D} \to \mathcal{S}$$

and if $Y_\bullet : \mathcal{J} \to \mathcal{D}$

$$L_f Y_\bullet \simeq \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{D}}(Y_j, f(-)) : \mathcal{C} \to \mathcal{S}$$

We have to check that the following are naturally equivalent simplicial sets whenever $X_\bullet$ and $Y_\bullet$ are left exact (and if $\mathcal{C}$ is accessible, also accessible) functors

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{S})}(FX_\bullet, Y_\bullet) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(X_\bullet, L_f Y_\bullet)$$

By Lemma 2.64,

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{S})}(FX_\bullet, Y_\bullet) \simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_{\mathcal{D}}(Y(j), fX(i))$$

but the mapping space there is exactly $(L_f h^{Y(j)})(X(i))$.

$$\lim_i \operatorname*{colim}_j \mathrm{Map}_{\mathcal{D}}(Y(j), fX(i)) \simeq \lim_i L_f Y_j(X(i))$$

and by the Yoneda lemma for $\mathcal{C}$,

$$\lim_i L_f Y_j(X(i)) \simeq \lim_i \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S}}(X(i), Ł_f Y_\bullet)$$

finally we bring the limit inside and deduce

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{D},\mathcal{S})}(FX_\bullet, Y_\bullet) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(X_\bullet, L_f Y_\bullet)$$

In fact from the above proof we see that the only place we needed accessibility/smallness was in understanding when the functor L_f takes pro-objects to pro-objects. In general we do not have a neat condition that guarantees L_f takes pro-objects to pro-objects.

(2) To see that F is fully faithful if f is, we use the explicit representation of the mapping spaces

$$\mathrm{Map}_{\mathrm{pro}\,\mathcal{C}}(X_\bullet, Y_\bullet) \simeq \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{C}}(X(i), Y(j)) \simeq \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{C}}(fX(i), fY(j))$$

and

$$\mathrm{Map}_{\mathrm{pro}\,\mathcal{C}}(X_\bullet, Y_\bullet) \simeq \lim_{i\in\mathcal{I}} \operatorname*{colim}_{j\in\mathcal{J}} \mathrm{Map}_{\mathcal{C}}(fX(i), fY(j)) \simeq \mathrm{Map}_{\mathrm{pro}\,\mathcal{D}}(FX_\bullet, FY_\bullet)$$

(3) A functor is defined to be a localization when it is left-exact and has a fully faithful right adjoint.

(4) This follows directly by the definition of L_f, any morphism of pro-objects satisfying this property becomes a natural equivalence of functors after composition by L_f.

(5) Since pro-objects are left-exact, checking that the two functors corepresented by two pro-objects are equivalent can be checked on any subcategory generating $\mathcal{C}$ under finite limits.

□

We close the section with the following identifications.

THEOREM 2.68. *The following pairs of ∞-categories are equivalent.*

(1) *The pro-∞-category of spaces and the simplicial nerve of Isaksen's strict model structure on pro-simplicial sets,*

$$\mathrm{pro}\ \mathcal{S} \simeq N^s(\mathrm{pro}\, sSet, \mathrm{strict}).$$

(2) *The ∞-category of pro-truncated spaces and the simplicial nerve of Isaksen's non-strict model structure on pro-simplicial sets*

$$\mathrm{pro}^{\natural}\mathcal{S} \simeq N^s(\mathrm{pro}\, sSet, \mathrm{non}-\mathrm{strict}).$$

(3) *The ∞-category of profinite spaces and the simplicial nerve of Quick's model structure on simplicial profinite sets,*

$$\mathcal{S}^\wedge \simeq N^s(\widehat{sSet}, \mathrm{Quick}).$$

(4) *The ∞-category of $\{p\}^c$-profinite spaces and the simplicial nerve of the localization of Quick's model structure at $\{p\}^c$-profinite equivalences,*

$$\mathcal{S}^\wedge_{\{p\}^c} \simeq N^s(L_{K_{\{p\}^c}}\widehat{sSet}, \mathrm{Quick}).$$

(5) *The ∞-category of ℓ-profinite spaces and the simplicial nerve of Morel's model structure on simplicial profinite sets,*

$$\mathcal{S}^\wedge_\ell \simeq N^s(L_{K_\ell}\widehat{sSet}, \mathrm{Quick}) \simeq N^s(\widehat{sSet}, \mathrm{Morel}).$$

PROOF. We split up the claims.

(1) This is Theorem 5.2.1 of [**18**].
(2) This follows since both are the localization at the Postnikov tower construction.
(3) This is Theorem 7.4.5 of [**18**].
(4) This follows from the previous since we are localizing both sides at a set of elements.
(5) This is Theorem 7.4.8 of [**18**].

□

8. The ∞-category of ∞-topoi and shapes

Modulo some set theoretic issues, the full subcategory of $\mathcal{C}at_\infty$ spanned by ∞-topoi is a good first approximation to the category of ∞-topoi, however it has too many morphisms. The correct notion of a morphism of ∞-topoi is that of a *geometric morphism*.

Now we come to a potentially tricky definition.

DEFINITION 2.69. Write $\mathcal{L}\mathcal{T}op$ for the ∞-category of ∞-topoi whose morphisms are the inverse image functors. As shorthand, we write

$$\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$$

for the mapping space between two ∞-topoi in $\mathcal{L}\mathcal{T}op$, e.g. geometric morphisms from $\mathcal{Y}$ to $\mathcal{X}$.

Write $\mathcal{R}\mathcal{T}top$ for the ∞-category of ∞-topoi whose morphisms are the direct image functors. As shorthand, we write

$$\mathrm{Fun}_*(\mathcal{X}, \mathcal{Y})$$

for the mapping space between two ∞-topoi in $\mathcal{R}\mathcal{T}op$, e.g. geometric morphisms from $\mathcal{X}$ to $\mathcal{Y}$.

Beware that we will mostly use $\mathcal{L}\mathcal{T}op$ whose hom-sets are going to be contravariant in the adjunction. ◁

The first lemma we cite is the expected result that these are opposite categories of each other.

LEMMA 2.70 (COROLLARY 6.3.1.8 OF [**6**]). *The category $\mathcal{L}\mathcal{T}op$ is canonically anti-equivalent to $\mathcal{R}\mathcal{T}op$.*

We have the following structural results on the category of ∞-topoi.

THEOREM 2.71. *The category of ∞-topoi has the following properties.*

(1) *The category $\mathcal{R}\mathcal{T}op$ admits an initial object given by $\mathcal{S}$.*

(2) *The category* $\mathcal{RT}op$ *admits small colimits, and they agree with their underlying colimits in* $\mathcal{C}at_\infty$.

(3) *The category* $\mathcal{RT}op$ *admits filtered limits, and they agree with their underlying limits in* $\mathcal{C}at_\infty$.

(4) *The category* $\mathcal{RT}op$ *admits fibered products, and they do not agree with their underlying fibered product in* $\mathcal{C}at_\infty$.

Proof. We separate the claims.

(1) See Proposition 6.3.4.1 of [**6**].
(2) See Proposition 6.3.2.3 of [**6**].
(3) See Theorem 6.3.3.1 of [**6**].
(4) See Proposition 6.3.4.6 of [**6**]

□

Now given a geometric morphism of topoi $f : \mathcal{X} \to \mathcal{Y}$, we can use Lemma 2.67.

Definition 2.72. Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of topoi. Then the left adjoint f^* satisfies the assumptions of Lemma 2.67 and thus its extension $f^* : \operatorname{pro} \mathcal{Y} \to \operatorname{pro} \mathcal{X}$ admits a left adjoint we denote by $f_! : \operatorname{pro} \mathcal{X} \to \operatorname{pro} \mathcal{Y}$ and call *the relative shape functor.*

In the case of $\pi : \mathcal{X} \to \mathcal{S}$, we instead call $\pi_! : \operatorname{pro} \mathcal{X} \to \operatorname{pro} \mathcal{S}$ as *the shape functor internal to* $\mathcal{X}$. ◁

Lemma 2.73. *Given two composable morphisms of topoi* $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$, *the two compositions* $g_! \circ f_!$ *and* $(g \circ f)_!$ *are equivalent.*

Proof. This is essentially just the uniqueness of adjoints, cf. the dual of Proposition 5.2.2.6 of [**6**]. □

This motivates the following terminology.

Definition 2.74. Let $\operatorname{sh} : \mathcal{RT}op \to \operatorname{pro} \mathcal{S}$ be the functor induces by sending a topos $\mathcal{X}$ to $\pi_! \mathbb{1}_{\mathcal{X}}$, where $\pi : \mathcal{X} \to \mathcal{S}$ is the terminal morphism in $\mathcal{RT}op$. This will be called the *shape functor.* ◁

Lemma 2.75. *Let* $\mathcal{X}$ *be an* ∞*-topos, and* $\operatorname{sh} \mathcal{X}$ *its shape. Writing* $\pi : \mathcal{X} \to \mathcal{S}$, *the functor*

$$U \mapsto \pi_* \pi^* U$$

naturally corepresents $\operatorname{sh} \mathcal{X}$ *in* $\operatorname{pro} \mathcal{S}$.

PROOF. The formal left adjoint to $\pi^* : \mathrm{pro}\,\mathcal{S} \to \mathrm{pro}\,\mathcal{X}$ is given by precomposition by π^*, that is

$$\pi_! \mathscr{F} \in \mathrm{Map}_{\mathrm{Fun}(\mathcal{S},\mathcal{S})} : U \mapsto \mathrm{Map}_{\mathcal{X}}(\mathscr{F}, \pi^* U)$$

Since we are taking the sheaf to be $\mathbb{1}_{\mathcal{X}} \simeq \pi^* \star$, we get the desired identity.

$$\mathrm{sh}\,\mathcal{X} : U \mapsto \mathrm{Map}_{\mathcal{X}}(\mathbb{1}_{\mathcal{X}}, \pi^* U) \simeq \mathrm{Map}_{\mathcal{S}}(\star, \pi_* \pi^* U)$$

□

We can also use the generalized Verdier theorem of [**16**] to deduce

LEMMA 2.76. *Let C be a 1-site with a terminal object, and $HCov$ be the hyper covers of the terminal object. Then*

$$\mathrm{sh}\,\tau_{\infty}^{\wedge} C \simeq \underset{U_{\bullet} \in HCov}{\mathrm{colim}}\ \underset{n}{\mathrm{colim}}\, \pi_!(U_{[n]})$$

PROOF. This follows since the inner colimits are all equivalent to $\pi_! \mathbb{1}_C$. □

THEOREM 2.77. *Let C be a locally connected 1-site with finite limits, so in particular there is a well defined functor $\pi_0 : C \to Set$. Then the shape of the hypercomplete topos of sheaves of spaces on C is naturally corepresented by the diagram of π_0 indexed by hypercovers*

$$\pi_0 : HCov(\mathbb{1}_C) \to \mathcal{S}$$

PROOF. This follows by the generalized Verdier theorem. □

We immediately deduce the following corollary.

COROLLARY 2.78. *Let $\mathcal{X}$ be an ∞-topos and assume that $\mathcal{X}$ is the topos of sheaves on some 1-categorical site C. Then for any abelian group A, and non-negative integer i the singular cohomology of the shape agrees with the derived functor cohomology of the corresponding constant sheaf.*

$$H^i(\mathrm{sh}\,\mathcal{X}, A) := \mathrm{Map}_{\mathrm{pro}\ \mathcal{S}}(\mathrm{sh}\,\mathcal{X}, K(A,i)) \simeq H^i(C, \underline{A})$$

For any group G, the pointed set of principal G-torsors over the shape agrees with the pointed set of non-abelian cohomology of the corresponding constant sheaf.

$$H^1(\mathrm{sh}\,\mathcal{X}, G) := \mathrm{Map}_{\mathrm{pro}\ \mathcal{S}}(\mathrm{sh}\,\mathcal{X}, BG) \simeq H^1(C, \underline{G})$$

Proof. This follows by the above and the original Verdier theorem relating hypercover cohomology with sheaf cohomology. □

In fact we could essentially take this to be a definition of the shape associated with a site. The historical issue with this is that the category $HCov$ is typically not a cofiltered category. Instead, one can show that the simplicial nerve of the simplicially enriched version of $HCov$ is a cofiltered ∞-category. See the discussion preceeding Proposition 4.1 of [**7**] for further details.

Definition 2.79. Let X be a topological space. The *shape of* X, $\operatorname{sh} X$, is the shape of the hypercompletion of its ∞-topos of sheaves of spaces.

$$\operatorname{sh} X := \operatorname{sh} \mathcal{S}h(X)^{\wedge}$$

The *Čech shape of* X is the shape of its ∞-topos.

$$\check{\operatorname{sh}} X := \operatorname{sh} \mathcal{S}h(X)$$

◁

Lemma 2.80. *Let $\mathcal{X}$ be an ∞-topos. Then the geometric morphism $h : \mathcal{X}^{\wedge} \to \mathcal{X}$ induces an equivalence of pro-truncated shapes*

$$\operatorname{sh}^{\natural} \mathcal{X}^{\wedge} \simeq \operatorname{sh}^{\natural} \mathcal{X}$$

Proof. Let $\pi : \mathcal{X} \to \mathcal{S}$ be the geometric morphism to the terminal ∞-topos. The morphism on shapes is the opposite arrow of the natural tranformation $\pi_*\pi^* \rightsquigarrow \pi_* h_* h^* \pi^*$. We have to show that this is an equivalence on the subcategory of truncated spaces. If $X \in \mathcal{S}$ is a truncated space, then $\pi^* X$ is truncated as well, in which case it automatically satisfies hyperdescent. Thus $\pi^* X \to h_* h^* \pi^* X$ is an equivalence, and the pro-truncated shapes of $\mathcal{X}$ and $\mathcal{X}^{\wedge}$ are naturally equivalent. □

We can give a streamlined recognition principle, which gives us a convenient way of recognizing which morphisms of sites lead to shape equivalences, possibly after some localization of the shapes.

Theorem 2.81. *Let $f : D \to C$ be a functor between locally connected 1-site with finite limits inducing a geometric morphism $f_* : \mathcal{S}h(D) \to \mathcal{S}(C)$. Write $\operatorname{sh}^{\star}$ for the localization the shape in one of:*

(1) *pro-truncated spaces,*
(2) *profinite spaces,*
(3) $\{p\}^c$*-profinite spaces,*

(4) *or ℓ-profinite spaces.*

Then the induced map of shapes

$$\mathrm{sh}^{\star}(\mathcal{S}h(D)) \to \mathrm{sh}^{\star}(\mathcal{S}h(C)$$

is an equivalence if and only if the corresponding condition is met.

(1) *The functor f^* induces*

  (a) *equivalences on non-abelian cohomology for every group $\mathscr{G}$*

$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

  (b) *and equivalences on cohomology for every abelian group $\mathscr{L}$*

$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathscr{L})$$

(2) *The functor f^* induces*

  (a) *equivalences on non-abelian cohomology for every locally constant sheaf of finite groups $\mathscr{G}$*

$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

  (b) *and equivalences on cohomology for all $n \geq 0$ and every constant sheaf of finite abelian groups $\mathscr{L}$*

$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathscr{L})$$

(3) *The functor f^* induces*

  (a) *equivalences on non-abelian cohomology for every locally constant sheaf of groups of order not divisible by p $\mathscr{G}$*

$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

  (b) *and equivalences on cohomology for every prime $\ell \neq p$ and every $n \geq 0$,*

$$H^n(D, \mathbb{Z}/\ell) \simeq H^n(C, \mathbb{Z}/\ell).$$

(4) *The functor f^* induces*

  (a) *equivalences on non-abelian cohomology for every locally constant sheaf of ℓ-primary groups $\mathscr{G}$*

$$\check{H}^1(D, f^*\mathscr{G}) \simeq \check{H}^1(C, \mathscr{G}),$$

(b) *and equivalences on cohomology for all* $n \geq 0$, $\mathbb{Z}/\ell$

$$H^n(D, f^*\mathscr{L}) \simeq H^n(C, \mathbb{Z}/\ell)$$

PROOF. (1) We are working with the pro-truncated shapes, so equivalence of the pro-spaces is defined in terms of homotopy groupoids, however Theorem 7.3.c of [**8**] gives a cohomological criterion for a map to be a weak equivalence. By 2.67, the pro-truncated shape is determined by its maps to truncated spaces. Since the functor it co-represents preserves finite limits and coproducts of truncated spaces, we use homological induction to reduce to the case where the space to be mapped to is an Eilenberg-MacLane space. Once in that case, this is just Theorem 2.77.

(2) Profinite spaces are the localization of pro-truncated spaces at the class of π-finite spaces. By the same Postnikov tower argument as in (1), we can reduce to the Eilenberg-MacLane case. However, the only Eilenberg-MacLane spaces satisfying the criteria to be a π-finite space are those with finite homotopy group.

(3) This follows from (2) as the ∞-category of $\{p\}^c$-profinite spaces is the left localization of profinite spaces at the corresponding collection of objects.

(4) This follows from (2) as the ∞-category of ℓ-profinite spaces is the left localization of profinite spaces at the corresponding collection of objects.

□

CHAPTER 3

ALGEBRAIC PRELIMINARIES

This chapter recalls classical notions of algebra: rings, groups, monoids, valuations. It further gives basic development of topological monoids.

1. Monoids

DEFINITION 3.1. We pose the following definitions.

(1) A *monoid* is a set M along with a binary operation $\cdot : M \times M \to M$ which is associative and admits an identity element. The operation will typically be referred to as the *multiplication of the monoid* M.
(2) A *commutative monoid* is a monoid whose binary operation is commutative.
(3) A *morphism of monoids* is any morphism which preserves the binary operation and the identity.
(4) The *category of monoids* is the category Mon whose objects are monoids, and whose morphisms are homomorphisms of monoids.
(5) The *category of commutative monoids* is the full subcategory $CMon$ of Mon spanned by commutative monoids.
(6) The *trivial monoid*, denoted by 0 or 1 depending on context, is any monoid isomorphic to one whose underlying set is a singleton, with trivial multiplication. It is both the initial and terminal monoid.
(7) A *group* is a monoid where every element has an inverse.
(8) The *product* of any collection of monoids is the product of the underlying sets along with coordinate-wise multiplication.
(9) A *congruence* $\sim$ on a monoid M is an equivalence relation on M whose graph $\sim \subset M \times M$ is a submonoid of the product of M with itself.

$\triangleleft$

Construction 3.2. *Let M be a monoid and a congruence $\sim$ on M. Then we construct the* quotient of M by $\sim$ *as follows. Denote the set of equivalence classes of elements of M under the equivalence relation $\sim$ by $M/\sim$. Then we can define multiplication as follows.*

For elements $m, m' \in M/\sim$, represented by elements $[a]$ and $[b]$ we define $m \cdot_{M/\sim} m'$ to be the class of $[a \cdot_M b]$.

We present the following lemma without proof. It lists the expected properties of the construction.

Lemma 3.3. *Let M and $\sim$ be as in the construction above. Then the following statements hold.*

(1) *The multiplication in the above construction is well defined.*
(2) *The equivalence class of the identity element of M is an identity for $M/\sim$.*
(3) *The set function $M \to M/\sim$ given by sending $m \to [m]$ is a homomorphism of monoids.*
(4) *The homomorphism $M \to M/\sim$ is initial among morphisms from M which identify all elements equivalent under $\sim$.*
(5) *The diagram below is a pushout in the category of monoids.*

$$\begin{array}{ccc} \sim & \longrightarrow & M \times M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M/\sim \end{array}$$

Definition 3.4. We pose the following definitions.

(1) The *groupification* of a monoid M is the quotient of $M \times M$ by the congruence that $(m, m) \sim (0, 0)$ for all $m \in M$. This is a group since $[(m, n)] \cdot [(n, m)]$ is equivalent to the identity, and admits an monoid homomorphism from M by the function $m \mapsto [(m, 0)]$.
(2) If M is a monoid and G is a submonoid of M which is a group, we write M/G for the quotient of M by G. This is constructed by the equivalence relation that $m \cdot g \sim m$ and $g \cdot m \sim m$ for every $m \in M$ and $g \in G$.
(3) A monoid M is called *integral* if it satisfies the cancellative property $(ab = ac) \Rightarrow (b = c)$ for any three elements in M.
(4) A commutative monoid M is called *fine* if it is finitely generated and integral.
(5) A monoid M is called *saturated* if it is integral, and for any $g \in M^{\mathrm{gp}}$, if g^n lies in the image of M in M^{gp} for any positive natural number n then g is also in the image of M in M^{gp}.
(6) A monoid is called *sharp* if it has no units besides its identity.
(7) For a monoid M the *integration* M^{int} of M is the image of M in M^{gp}.
(8) For a monoid M the *sharpening* $\overline{M}$ of M is the quotient $M/M^{\times}$ of M by its subgroup of invertible elements.

(9) For a monoid M the ***saturation*** M^{sat} of M is the smallest submonoid of M^{gp} generated by the image of M which is also saturated.

(10) The ***category of integral commutative monoids*** is the full subcategory $CMon^{\text{int}}$ of $CMon$ spanned by integral monoids.

(11) The ***category of saturated commutative monoids*** is the full subcategory $CMon^{\text{sat}}$ of $CMon$ spanned by saturated monoids.

(12) The ***category of sharp commutative monoids*** is the full subcategory $\overline{CMon}$ of $CMon$ spanned by sharp monoids.

$\triangleleft$

From here all monoids will be commutative unless stated otherwise. However we will still use $CMon$ instead of the more common Mon to denote the category of commutative monoids. We state some basic categorical results.

Lemma 3.5. *The category $CMon$ admits all small limits and colimits.*

The following adjunctions hold.

(1) *Groupification, integration, saturation, and sharpening extend to functors.*
(2) *The inclusion $CGrp \subset CMon$ is right adjoint to the groupification functor.*
(3) *The inclusion $CMon^{\text{int}} \subset CMon$ is right adjoint to the integration functor.*
(4) *The inclusion $CMon^{\text{sat}} \subset CMon$ is right adjoint to the saturation functor.*
(5) *The inclusion $\overline{CMon} \subset CMon$ is right adjoint to the sharpening functor.*
(6) *Integration, saturation, and sharpening preserve small colimits.*

The following operations commute

(1) *sharpening and saturation,*
(2) *saturation and integration,*
(3) *and finally sharpening and integration.*

Proof. We have already constructed products in $CMon$, so we must only construct equalizers to deduce that it admits all small limits. However we can just take the set-theoretic equalizer which is forced to be a submonoid by the multiplicativity of homomorphisms of monoids.

We next prove that $CMon$ is cocomplete. It is a well known result that a category is cocomplete if and only if it admits all small coproducts and coequalizers. Coproducts for commutative monoids are constructed as follows, let $\{M_i\}_{i\in I}$ be a set of monoids. Then the coproduct

$$\coprod M_i = \{(m_i) \mid m_i = 0_i \text{ for all but finitely many } i\}$$

with coordinate-wise multiplication. Given two morphisms of monoids $\phi, \psi : M \rightrightarrows M'$, we form the congruence on M' generated by $\phi(m) \sim \psi(m)$ for all $m \in M$. One may easily show this is a submonoid and an equivalence relation and thus a congruence. Taking the quotient $M'/\sim$ satisfies the categorical condition to be a coequalizer.

Functoriality follows quickly from the objectwise definitions. If $M \to M'$ is a morphism of monoids, we have an induced morphisms $M^{\text{gp}} \to M'^{\text{gp}}$, and an induced morphism $M^\times \to M'^\times$. The former follows by the universal property of quotienting by a congruence, and the fact that the congruence for groupification is preserved under homomorphisms of monoids. The second follows since a homomorphism of monoids preserves units.

Now we prove the case for integration. If we have a morphism $f : M \to M'$ we get an associated commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \downarrow{\scriptstyle \gamma_M} & & \downarrow{\scriptstyle \gamma_{M'}} \\ M^{\text{gp}} & \xrightarrow{f^{\text{gp}}} & M'^{\text{gp}} \end{array}$$

the commutativity forces that $f^{\text{gp}}(\gamma_M(M)) \subset \gamma_{M'}(f(M))$. Thus the restriction of f^{gp} to $\operatorname{im}\gamma_M$ is the morphism we want. For saturation, if $m \in M^{\text{gp}}$ is such that m^n lies in the image of γ_M, then of course $f^{\text{gp}}(m)^n$ must lie in the image of $\gamma_{M'}$ as desired. Thus $f^{\text{gp}} : M^{\text{sat}} \to M'^{\text{gp}}$ factors through the inclusion $M'^{\text{sat}} \subset M'^{\text{gp}}$ as desired. For sharpening, the universal property of $\overline{M}$ is that it is initial among monoids equipped with maps from M which identify everything in $M^\times$ with 0_M. The map $M \to \overline{M'}$ certainly satisfies this, and so we get a unique morphism $\overline{M} \to \overline{M'}$ as desired.

That groupification is a left adjoint to the inclusion is a well known result, so we omit a proof. For integration, we must show that if $M \to M'$ is a morphism from M to an integral monoid that the morphism factors through the integration M^{int}. However since M' is its own integration, the fact that we have a commutative square

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \downarrow & & \downarrow{\scriptstyle \cong} \\ M^{\text{int}} & \longrightarrow & M' = M'^{\text{int}} \end{array}$$

forces the morphism $M \to M'$ to factor through M^{int} as desired. For saturation the reasoning is essentially the same, M'^{sat} is its own saturation and a slightly modified diagram shows the same result. For sharpening we simply note that a morphism $M \to M'$ where M' is sharp must kill off $M^\times$ by necessity, and so it must factor through $\overline{M}$.

The claim that the three functors preserve colimits follows by being left adjoints. To show that the three pairs of functors commute, we simply repeat the above arguments to show that both compositions give left adjoints to the inclusions of categories of sharp and saturated monoids, saturated monoids, and sharp and integral monoids respectively. $\square$

Now we turn to monoids internal to topoi.

Definition 3.6. We pose the following definitions.

(1) Let S be a site. A *sheaf of monoids on* S is a sheaf of sets $\mathscr{M}$ plus a binary operator $\cdot : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ such that for every object $s \in S$ every $\mathscr{F}(s)$ is a monoid and for every $i : s \to s'$ the set-theoretic morphism $\mathscr{F}(i)$ is a morphism of monoids.

(2) Let T be a topos and S a site defining T. We write $CMon(T)$ for the category of *commutative monoid objects of* T.

    By the adjunction of functor categories and the product of categories, $CMon(T)$ is isomorphic to the category of sheaves of commutative monoids on S.

(3) We write $Ab(T)$ and the category of *commutative abelian group objects of* T.

(4) We write $CMon^{\text{int}}(T)$ for the category of *integral commutative monoid objects of* T, that is sheaves of commutative monoids whose sections are all integral.

(5) We write $\overline{CMon}(T)$ for the category of *sharp commutative monoid objects of* T, that is sheaves of commutative monoids whose sections are all sharp monoids.

(6) When T has enough points, we write $CMon^{\text{sat}}(T)$ for the category of *saturated commutative monoid objects of* T, that is sheaves of commutative monoids whose stalks are all saturated.

(7) We can define the groupification functor as follows:, to a sheaf of commutative monoids we first perform groupification sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\text{gp}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\text{gp}}\right]^+$$

(8) We can define the integration functor as follows: to a sheaf of commutative monoids we first perform integration sectionwise, then we integrate.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{int}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{int}}\right]^{+}$$

(9) We can define the sharpening functor as follows: to a sheaf of commutative monoids we first perform sharpening sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \overline{\mathscr{F}(U)}\right] \mapsto \left[U \mapsto \overline{\mathscr{F}(U)}\right]^{+}$$

(10) We can define the saturation functor as follows: to a sheaf of commutative monoids we first perform saturation sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{sat}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{sat}}\right]^{+}$$

◁

The following lemma follows from the analogous cases for $CMon$, except that the saturation functor must be treated slightly specially. The subtlety for saturation is that sheafification breaks the condition, since an element is in $\mathscr{F}^{\mathrm{sat}}(U)$ if and only if it locally satisfies that some power is in the groupification. However there may not be a supremum if there are infinitary covers of U that don't refine to finite ones.

Lemma 3.7. *The following properties hold for the above constructions.*

(1) *The given construction of groupification satisfies the following.*
   (a) *It gives a functor* $(-)^{\mathrm{gp}} : \mathrm{Mon}(T) \to \mathrm{Ab}(T)$,
   (b) *the functor* $(-)^{\mathrm{gp}}$ *is a left adjoint to the forgetful functor* $\mathrm{Ab}(T) \subset \mathrm{Mon}(T)$,
   (c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{Mon}(T)$ *the induced morphism*

$$(\mathscr{M}_x)^{\mathrm{gp}} \to (\mathscr{M}^{\mathrm{gp}})_x$$

   is an isomorphism of abelian groups.

(2) *The given construction of sharpening satisfies the following.*
   (a) *It gives a functor* $\overline{(-)} : \mathrm{Mon}(T) \to \mathrm{Mon}(T)$,
   (b) *the functor* $\overline{(-)}$ *is a left adjoint to the forgetful functor* $\overline{\mathrm{Mon}}(T) \subset \mathrm{Mon}(T)$,
   (c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{Mon}(T)$ *the induced morphism*

$$\overline{(\mathscr{M}_x)} \to (\overline{\mathscr{M}})_x$$

is an isomorphism of sharp commutative monoids.

(3) *The given construction of saturation satisfies the following when T has enough points.*

(a) *It gives a functor $(-)^{\mathrm{sat}} : \mathrm{Mon}(T) \to \mathrm{Mon}(T)$.*

(b) *the functor $(-)^{\mathrm{sat}}$ is a left adjoint to the forgetful functor $\mathrm{Mon}^{\mathrm{sat}}(T) \subset \mathrm{Mon}(T)$,*

(c) *and for any stalk functor $x : Sets \to T$ and any sheaf $\mathscr{M}$ in $\mathrm{Mon}(T)$ the induced morphism*

$$(\mathscr{M}_x)^{\mathrm{sat}} \to (\mathscr{M}^{\mathrm{sat}})_x$$

is an isomorphism of saturated commutative monoids.

Proof. We separate the parts first.

(1) *Groupification*

(a) To see that this gives a functor, we simply note that groupification is functorial.

(b) Sectionwise groupification is the left adjoint on the level of presheaves, since a morphism from a sheaf of commutative monoids $\mathscr{M}$ to a sheaf of abelian groups $\mathscr{A}$ induces a map $\mathscr{M}(U) \to \mathscr{A}(U)$ for every U, which by the universal property of groupification factors through $\mathscr{M}(U)^{\mathrm{gp}}$. The universal property also forces a unique system of restriction maps $\mathscr{M}(U)^{\mathrm{gp}} \to \mathscr{M}(V)^{\mathrm{gp}}$ whenever $V \to U$ is a morphism in a site of definition. The uniqueness guarantees these glue into a presheaf. The sheafification is further universal with respect to morphisms from the presheaf groupification to sheaves, and so we conclude that

$$\mathrm{Hom}_{\mathrm{Mon}(T)}(\mathscr{M}, \mathscr{A}) \simeq \mathrm{Hom}_{\mathrm{Ab}(T)}(\mathscr{M}^{\mathrm{gp}}, \mathrm{A})$$

as desired.

(c) Stalk functors are functors from $Sets$ which are preserve finite limits and all small colimits. We may even write the stalk functor as a (possibly non-filtered!) colimit over neighborhoods of the point in a suitable sense, see the discussion at the beginning of 7.31 of **[2]**. Groupification is the left adjoint to the forgetful functor, and so it commutes with colimits. Thus the stalk of sectionwise groupification is the groupification of the stalk. Since the sectionwise groupification and its sheafification have naturally isomorphic stalks, we conclude the desired result.

(2) *Sharpening*

(a) The sharpening of a monoid is functorial.

(b) This follows formally as in argument (b) for groupification, we are instead using the fact that the sharpening funtor is the left adjoint to the inclusion of sharp commutative monoids into all commutative monoids.

(c) This follows formally as in argument (c) for groupification.

(3) *Saturation*

(a) The saturation of a monoid is functorial.

(b) This follows formally as in argument (b) for groupification, we are instead using the fact that the saturation functor is the left adjoint to the inclusion of saturated commutative monoids into all commutative monoids.

(c) This follows formally as in argument (c) for groupification.

$\square$

THEOREM 3.8. *Let S and S' be sites admitting finite limits, and $F : S \to S'$ a functor inducing a geometric morphism of topoi $F_* : \tau S' \to \tau S$. Also assume both sites have enough points.*

(1) *The functors F^* and F_* commute with groupification of sheaves of commutative monoids.*
(2) *The functors F^* and F_* commute with integration of sheaves of commutative monoids.*
(3) *The functors F^* and F_* commute with sharpening of sheaves of commutative monoids.*
(4) *Assume that both sites have enough points. Then F_* commutes with saturation of sheaves of commutative monoids.*

PROOF. Since the groupification, integration, sharpening, and saturation satisfy a universal property with respect to mapping into groups, integral monoids, sharp monoids, and saturated monoids respectively, it is enough to show that both F_* and F^* preserve the properties of being a group, being integral, sharp, and saturated respectively.

(1) The claim is essentially obvious for F_*, since a sheaf is a group when all sections are groups. Since the sections of the direct-image are always sections of the original sheaf the result is immediate. For F^*, this follows via the description of F^* as a (possibly non-filtered) colimit, which groupification commutes with.
(2) The claim is again obvious for F_*, since a sheaf is integral when all sections are integral. For F^*, this follows again via the description of F^* as a colimit, which integration commutes with.
(3) The claim is again obvious for F_*, since a sheaf is sharp when all sections are sharp. For F^*, this follows again via the description of F^* as a colimit, which sharpening commutes with.

(4) We need the assumption on points to have a easy description of which sheaves are saturated. Since the stalks of the inverse image sheaf are the stalks of the original sheaf, it is automatic that F^* preserves saturation.

□

2. Commutative rings and completions

DEFINITION 3.9. We pose the following definitions.

(1) Let R be a Noetherian commutative ring, and $I \triangleleft R$ an ideal. Then $\hat{R}_I$ the *I-adic completion of R* is defined to be the limit in $CRing$

$$\hat{R}_I := \lim_n R/I^n.$$

(2) Let R be a Noetherian commutative ring, $I \triangleleft R$ an ideal, and M an R-module. Then $\hat{M}_I$ the *I-adic completion of M* is defined to be the limit in $R\mathrm{-Mod}$

$$\hat{M}_I := \lim_n M/I^n M$$

(3) Let R be a Noetherian commutative ring, and $I \triangleleft R$ an ideal. For any R-module M we can equip $\hat{M}_I$ with the topology determined by being invariant under addition and scalar multiplication, and that each $(I^n\hat{R}_I) \cdot \hat{M}_I$ is a neighborhood of zero. This is called the *I-adic topology on $\hat{M}_I$*.

◁

LEMMA 3.10. *Let R be a Noetherian commutative ring and $I \triangleleft R$ an ideal. For any R-module M*

(1) *The ring $\hat{R}_I$ is a flat R-algebra.*
(2) *For any R-module M, the completion $\hat{M}_I$ is complete with respect to the $I\hat{R}$-adic topology.*
(3) *The functor $M \mapsto \hat{M}_I$ is right exact.*
(4) *If M is finitely generated, the morphism $M \otimes_R \hat{R}_I \to \hat{M}_I$ is an isomorphism.*
(5) *Completion is exact when restricted to finitely generated R-modules.*

3. Norms and valuations

DEFINITION 3.11. (1) A *seminorm* on a ring R is a set function to the semi-ring $\|-\|_R : R \to [0,\infty)$ such that $\|0\|_R = 0$, $\|x \cdot y\|_R \leq \|x\|_R\|y\|_R$, and satisfies the triangle inequality $\|x+y\|_R \leq \|x\|_R + \|y\|_R$.

(2) A seminorm on a ring R is called *non-archimedean* if it satisfies the ultrametric inequality, $\|x+y\|_R \leq \max\{\|x\|_R, \|y\|_R\}$.

(3) A seminorm on a ring is a *norm* when the only ring element with seminorm zero is the zero element of the ring.

(4) Two seminorms on a ring are *equivalent* when they are bounded with respect to each other, that is there are positive real numbers $C_1, C_2 > 0$ such that $C_1 \cdot \| - \|_1 < \| - \|_2 < C_2 \cdot \| - \|_1$.

(5) A *Banach ring* is a ring R along with a , and R is complete with respect to the metric topology given by $d(x,y) = \|x-y\|_R$.

(6) If R is a Banach ring, a *Banach algebra* under R is a R-algebra $i : R \to S$ such that S is a Banach ring, and the algebra map is bounded, i.e. that there is a real number $C > 0$ such that $\|i(-)\|_S \leq C \cdot \| - \|_R$ as functions on R.

(7) The *category* $BRing$ *of commutative Banach rings* is the category whose objects are Banach rings, and whose morphisms are Banach algebra maps, that is bounded ring homomorphisms.

(8) A ring homomorphism $\phi : S \to T$ of Banach algebras over a base Banach ring R is *admissible* if the induced norm on $S/\ker\phi$ is equivalent to that of the restricted norm on $\operatorname{im}\phi$, explicitly

$$\|\phi(f)\|_T = \inf_{\phi(f')=\phi(f)} \|f'\|_S$$

(9) A *seminorm on a Banach ring* R is a seminorm on the underlying ring that is bounded with respect to the Banach ring's norm, that is there is a positive real number $C > 0$ such that $| - | < C \cdot \| - \|_R$.

(10) Let M be a module over a normed ring R. Then a *seminorm on the module* M is a set function $\| - \|_M : M \to [0,\infty)$ satisfying that $\|0_M\|_M = 0$, the triangle inequality with respect to the group law on M, and that multiplication by elements in the ring R is bounded with respect to the seminorm on M, i.e. there is a positive real number $C > 0$ such that $\|rm\|_M \leq C \cdot \|r\|_R\|m\|_M$. By dividing $\| - \|_M$ by such a C, we can construct an equivalent seminorm with $C = 1$.

(11) Let M be a module over a normed ring R. Then a *norm on the module* M is a seminorm with the further property that the only element of the module with seminorm zero is the additive identity of the module.

(12) Given two seminormed modules M and N over an arbitraryu Banach ring R, we can form the completed tensor product of the two by completing the algebraic tensor product with respect to a seminorm we define as follows.

$$\| - \|_{M \otimes_R N} : x \in M \otimes_R N \mapsto \inf_{x = \sum m_i \otimes n_i} \left\{ \sum \|m_i\|_M \|n_i\|_N \right\}$$

(13) Given two seminormed modules M and N over a non-archimedean Banach ring R, we can insteadform the *completed tensor product* of the two by completing the algebraic tensor product with respect to the non-archimedean seminorm we define as follows.

$$\| - \|_{M \otimes_R N} : x \in M \otimes_R N \mapsto \inf_{x = \sum m_i \otimes n_i} \left\{ \max_i \{ \|m_i\|_M \|n_i\|_N \} \right\}$$

(14) Given a Banach ring R and Banach algebras S/R and T/R, a morphism of Banach R-algebras $\phi : S \to T$ is called *inner with respect to* R when there exists an admissible epimorphism of Banach algebras

$$\pi : R\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to S$$

such that the morphism ϕ does not increase the spectral radius of the T_i, that is the spectral radius of $\phi\pi(T_i)$ is less than r_i.

$\triangleleft$

LEMMA 3.12. *The following statements hold.*

(1) *The completed tensor product $S \hat{\otimes}_R T$ is the coproduct of $S \backslash R$ and $T \backslash R$ in the category of Banach R-algebras.*

(2) *For a Banach ring R, the category of finite Banach modules with bounded homomorphisms is equivalent to the category of finite R-modules. This is Proposition 2.1.9 of* [**19**].

(3) *The category of finite Banach algebras over a fixed Banach ring R is equivalent to the category of finite R-algebras via the forgetful functor. This is Proposition 2.1.12 of* [**19**].

LEMMA 3.13 (THEOREM A.3.7.I OF [**20**]). *If S, S', T, and T' are all Banach algebras over a non-archimedean Banach field R, and further $i : S \to S'$ and $j : T \to T'$ are admissible algebra maps, then the induced map of completed tensor products*

$$i \otimes_R j : S \hat{\otimes}_R T \to S' \hat{\otimes}_R T'$$

is also an admissible algebra map.

Now we switch gears from normed rings to valued rings. The theories give the algebraic underpinnings of non-archimedean analytic spaces in the sense of Berkovich and of adic spaces of Huber.

DEFINITION 3.14. We pose the following definitions.

(1) Let Γ be a totally ordered abelian group, and for convenience assume it is also written multiplicatively. Then with Γ we associate the totally ordered, associative, and commutative magma $\Gamma \cup \{0\}$ extending the multiplication on Γ by declaring $x \cdot 0 = 0$ for any $x \in \Gamma \cup \{0\}$. For brevity we will call this the *value magma of* Γ.

(2) Let R be a topological ring, then a subset $T \subset R$ is *bounded* if for every open neighborhood U of 0_R, there is another open neighborhood V of 0_R such that $T \cdot V \subset U$.

(3) For a topological ring R, we write $R^{\circ\circ}$ for the subset of *topologically nilpotent elements in* R.

(4) For a topological ring R, we write R° for the subset of *power-bounded elements in* R, that is elements $r \in R$ where the set $\{r^n | n \geq 0\}$ is bounded.

(5) We call a topological ring R *adic* if there is some ideal $I \lhd R$ such that the powers of I form a fundamental system of neighborhoods of 0_R. Such an ideal is called an *ideal of definition for* R.

(6) We call a topological ring R f*-adic* if there is an open subring R_0 such that R_0 is an adic ring for the subspace topology and some ideal of definition is finitely generated. Any open subring R_0 which is adic will be called a *ring of definition for* R.

(7) We call a topological ring R *Tate* if it is both f-adic and has a unit that is topologically nilpotent.

(8) For a Tate ring R, the *ring of convergent power series in* n *variables* $R\langle T_1, \ldots, T_n\rangle$ is the ring whose elements are formal power series $\sum r_I T^I$ satisfying that $r_I \to 0$ as $I \to \infty$.

(9) A Tate ring R is called *strongly Noetherian* if rings of convergent power series in any number of variables are always Noetherian.

(10) Let R be an f-adic ring, then a *ring of integral elements* is an open subring $S \subset R$ consisting of power-bounded elements $S \subset R^{\circ}$, which is further integrally closed in R.

(11) An *affinoid pair* R is a pair of topological rings $(R^{\triangleright}, R^{+})$ such that $R^{\triangleright}$ is an f-adic ring and R^{+} is a ring of integral elements for it. Beware that this is weaker than Huber's definition in **[21]**.

(12) A *adic morphism* between affinoid pairs $R = (R^{\triangleright}, R^{+})$ and $S = (S^{\triangleright}, S^{+})$ is a ring homomorphism $f : R^{\triangleright} \to S^{\triangleright}$ such that $f(R^{+}) \subset S^{+}$ and for some ideal of definition $I \triangleleft R^{\triangleright}$, the $f(I)$ generates an ideal of definition for S^{+}.

(13) Let R be a ring. Then a *valuation* on R is a multiplicative map to a value magma $v : R \to \Gamma \cup \{-\infty\}$, where Γ is some totally ordered abelian group, v is subadditive

$$v(r_1 + r_2) \leq \max\{v(r_1), v(r_2)\}$$

and finally $v(0_R) = 0$ and $v(1_R) = 1_\Gamma$.

(14) Let $v : R \to \Gamma \cup \{0\}$ be a valuation on a ring R. Then the *tidying of the group* Γ *with respect to the valuation* v is the subgroup spanned by the image of v, $\Gamma^{\mathrm{tdy}} = \langle \operatorname{im} v \setminus \{0\} \rangle$. The *tidying of the valuation* is the valuation $v^{\mathrm{tdy}} : R \to \Gamma^{\mathrm{tdy}} \cup \{0\}$ obtained by coastriction. A valuation is called *tidy* if the tidying of the valuation is itself, equivalently if the image of the valuation generates the value magma under multiplication.

(15) Two valuations $v_1 : R \to \Gamma_1 \cup \{0\}$ and $v_2 : R \to \Gamma_2 \cup \{0\}$ on a ring R are said to be *equivalent* if there is an isomorphism of groups $\phi : \Gamma_1^{\mathrm{tdy}} \to \Gamma_2^{\mathrm{tdy}}$ such that $v_2^{\mathrm{tdy}} = \phi \circ v_1^{\mathrm{tdy}}$.

(16) Given a topological ring R, a tidy valuation $v : R \to \Gamma \cup \{0\}$ is *continuous*, if the preimage of any half-open intervals $[0, \gamma) \subset \Gamma \cup \{0\}$ is open in R. A valuation is called *continuous* if its tidying is.

(17) Given a ring R the *raw valuative spectrum of* R, $\operatorname{spv} R$, is the set of equivalence classes of valuations on R, given the topology generated by sets $U(r_1, r_2) = \{v \in \operatorname{spv} R \,|\, v(r_1) < v(r_2) \text{ and } v(r_2) \neq 0\}$.

(18) Given a topological ring R the *valuative spectrum of* R, $\operatorname{spv} R$, is the set of equivalence classes of continuous valuations on R, given the subspace topology from the raw valuative spectrum.

(19) Given an affinoid pair $R = (R^{\triangleright}, R^{+})$, the *valuative spectrum of* R, $\operatorname{spa} R$, is the subspace of $\operatorname{spv} R^{\triangleright}$ determined by

$$\operatorname{spa} R = \{x \in \operatorname{spv} R^{\triangleright} \,|\, |R^{+}|_x \subset [0, 1]\}$$

(20) Given a complete topological ring R, the *ring of restricted power series* $R\langle T_1, \ldots, T_n \rangle$ is the ring $R\langle T_1, \ldots, T_n \rangle$, defined as the subset

$$\{f \in R[[T_1, \ldots, T_n]] \,|\, \text{all neighborhoods } U \text{ of } 0, \text{ only finitely many coefficients aren't in } U\}$$

This is the completion of the usual polynomial ring given a topology as follows. For an open neighborhood $U \subset R^{\triangleright}$ of 0 we take the subset of all power series whose coefficients all lie in U, and as U varies we let this be a basis of neighborhoods of 0 in $R^{\triangleright}\langle T_1, \ldots, T_n\rangle$.

(21) Given a complete topological ring R and a list of finite subsets $M_1, \ldots, M_n \subset R$ such that each M_i generates an open ideal, we can define the ring $R\langle T_1, \ldots, T_n\rangle_{M_1,\ldots,M_n}$

$$\{f \in R[[T_1, \ldots, T_n]] \mid \text{for open } U \ni 0,\ \text{only finitely many coefficients } a_I \text{ are outside of } M^I \cdot U.\}$$

where for a multi-index $I = (i_1, \ldots, i_n)$, M^I is defined to be $M_1^{i_1} \cdot \ldots \cdot M_n^{i_n}$. We give this the topology where to an open subset $U \subset R^{\triangleright}$ we take the subset of all formal power series whose coefficients all lie in $M^I U$ for the corresponding multi-index. This gives a basis of neighborhoods of 0.

This satisfies a universal property with respect to algebras where $m_i X_i$ is power bounded for every $1 \leq i \leq n$ and every $m_i \in M_i$, see Lemma 3.1 (i) of **[22]**. Given that property, we will call this construction the ***ring of power-bounded power series with respect to*** $M_1, \ldots, M_n$.

(22) A homomorphism of affinoid pairs $f : R \to S$ is called ***of topologically finite type***, or ***finite type in the sense of Huber*** if there is danger of confusion, if there are finite subsets $M_1, \ldots, M_n$ of R and there is a continuous, open, and surjective homomorphism $R\langle T_1, \ldots, T_n\rangle_{M_1,\ldots,M_n} \to S$ factoring the algebra map.

$\triangleleft$

LEMMA 3.15. *The following hold.*

(1) *Let R be a ring. Then the raw valuative spectrum* spv R *is a spectral space.*
(2) *Let R be a topological ring. Then the valuative spectrum* spv R *is a spectral space.*
(3) *Let $R = (R^{\triangleright}, R^+)$ be an affinoid pair. Then the valuative spectrum of R is a spectral space.*

4. Topological monoids

Before we begin on topological monoids, recall the definition of the initial and final topologies on a set.

DEFINITION 3.16. Let X be a set and $\{Y_i\}_{i\in I}$ a collection of topological spaces. For a collection of set functions $\{f_i : X \to Y_i\}_{i\in I}$ the ***initial topology on*** X ***with respect to the family*** $\{f_i\}$ is the coarsest topology on X such that each f_i is continuous.

Let Y be a set and $\{X_i\}_{i\in I}$ a collection of topological spaces. For a collection of set-functions $\{f_i : X_i \to Y\}_{i\in I}$ the ***final topology on Y with respect to the family $\{f_i\}$*** is the finest topology on Y such that each f_i is continuous. $\triangleleft$

The final topology tends to be difficult to interpret, however in the cases we care about it will behave. A concrete example of it is the quotient topology of a space with respect to an equivalence relation on it. The final topology satisfies the following universal property.

LEMMA 3.17. *Let Y be a set and $\{f_i : X_i \to Y\}_{i\in I}$ a family of set-functions from topological spaces X_i. Then a set-function $g : Y \to Z$ is continuous if and only if every $g \circ f_i$ is continuous.*

Now we introduce the basic definitions of topological monoids.

DEFINITION 3.18.

(1) A ***topological monoid*** is a monoid given a topology such that its multiplication is continuous.
(2) If M is a topological monoid, we write $\operatorname{obliv} M$ for the ***underlying monoid of M*** obtained by forgetting the topology.
(3) A ***commutative topological monoid*** is a topological monoid whose multiplication is also commutative.
(4) A ***morphism of topological monoids*** is a morphism of the underlying monoids which is also continuous.
(5) The ***category of topological monoids*** is the category $TMon$ whose objects are topological monoids, and whose morphisms are morphisms of topological monoids.
(6) The ***category of commutative topological monoids*** is the category $CTMon$ whose objects are commutative topological monoids, and whose morphisms are morphisms of topological monoids.
(7) By abuse of notation we will denote both forgetful functors $CTMon \to CMon$ and $TMon \to Mon$ via obliv, using context to make sense of which we are using.
(8) The ***trivial topological monoid*** 0 is the trivial monoid equipped with the trivial topology. It is both the initial and terminal topological monoid.
(9) A ***topological group*** is a topological monoid where every element has a multiplicative inverse.
(10) The ***product*** of a set of topological monoids is the product of the underlying monoids given the product topology.
(11) A ***congruence*** of a topological monoid M is a congruence on the underlying monoid of M.
(12) The ***quotient of a topological monoid by a congruence*** is the quotient of the underlying monoid given the quotient topology.

(13) The *groupification* of a topological monoid M is the quotient of $M \times M$ with respect to the congruence $\sim$ generated by the image of $\Delta \times 0$ in $(M \times M) \times (M \times M)$, where $\Delta : M \to M \times M$ is the diagonal map.

(14) If M is a topological monoid, and $G \subset M$ is a submonoid which is a topological group, then the quotient of underlying monoid and group M/G is given the quotient topology to define the *quotient of the topological monoid M by the topological subgroup G.*

(15) A topological monoid will be called *sharp*, *integral*, or *saturated* if its underlying monoid is.

(16) We will call a topological monoid *fine* when it is integral, finitely generated, and the topology is discrete.

(17) We will write $CTMon^{\mathrm{int}}$, $\overline{CTMon}$, and $CTMon^{\mathrm{sat}}$ for the full subcategories of $CTMon$ spanned by topological monoids that are respectively integral, sharp, or saturated.

(18) For a commutative topological monoid M, we define the *integration* of M to be the image of M in M^{gp} given the quotient topology.

(19) For a commutative topological monoid M, we define the *sharpening* of M to be the quotient of M by the subgroup $M^{\times}$.

(20) For a commutative topological monoid M, we define the *saturation* of M to be the saturation of M given the final topology with respect to the map $M \to M^{\mathrm{sat}}$.

(21) For a monoid M, we write M^{disc} for the topological monoid resulting from equipping M with the discrete topology.

$\triangleleft$

From here we will only consider commutative topological monoids unless specifically stated otherwise. Note that the topological aspect becomes difficult to work with in some ways. For example he author does not believe that M^{int} has the subspace topology when considered as a subspace of M^{gp} for general topological monoids M. However the topologies involved still satisfy some of the basic properties monoids do.

LEMMA 3.19. *Let M be a topological monoid.*

(1) *The morphism $M \to M^{\mathrm{gp}}$ is initial among morphisms from M to a topological group.*

(2) *The morphism $M \to M^{\mathrm{int}}$ is initial among morphisms from M to an integral topological monoid.*

(3) *The morphism $M \to \overline{M}$ is initial among morphisms from M to a sharp topological monoid.*

(4) *The morphism $M \to M^{\mathrm{sat}}$ is initial among morphisms from M to a saturated topological monoid.*

Proof. That any morphism M to a integral (resp. sharp, resp. saturated) monoid factors through the underlying monoid of the topological integration (resp. sharpening, resp. saturation) is purely algebraic and was proved in the section on monoids. That the new morphism is continuous follows by the universal property of the quotient topology(resp. quotient topology, resp. final topology). □

We can immediately deduce the following corollary.

Corollary 3.20. *The constructions above are functorial.*

(1) *Integration is functorial.*
(2) *Sharpening is functorial.*
(3) *Saturation is functorial.*

Further they are left adjoints to forgetful functors.

(1) *The discrete functor* $\mathrm{disc} : Mon \to TMon$ *is right adjoint to the forgetful functor* obliv.
(2) *The discrete functor* $\mathrm{disc} : CMon \to CTMon$ *is right adjoint to the forgetful functor* obliv.
(3) *The forgetful functor* $TAb \to CTMon$ *is right adjoint to groupification*
(4) *The forgetful functor* $CTMon^{\mathrm{int}} \to CTMon$ *is right adjoint to integration.*
(5) *The forgetful functor* $\overline{CTMon} \to CTMon$ *is right adjoint to sharpening.*
(6) *The forgetful functor* $CTMon^{\mathrm{sat}} \to CTMon$ *is right adjoint to saturation.*

Proof. The only claims that do not follow from the previous lemma are those about the adjunction between the discrete and forgetful functors. To see this we must check that

$$\mathrm{Hom}(M^{\mathrm{disc}}, M') \cong \mathrm{Hom}(M, \mathrm{obliv}\, M')$$

however this follows immediately since all morphisms of the underlying monoids $M \to \mathrm{obliv}\, M'$ are continuous when M is given the discrete topology, irregardless of what topology M' has. □

Theorem 3.21. *The category of commutative topological monoids admits all small limits and colimits.*

Proof. To compute a limit of commutative topological monoids, we take the limit of the underlying diagram monoids and equip it with the initial topology with respect to the maps from the limit to the elements of the diagram.

To compute a colimit of commutative topological monoids, we take the colimit of the underlying diagram of monoids and equip it with the final topology with respect to the maps from the elements of the diagram to the colimit. □

Now the topologies we have put on groupification, integration, sharpening, and saturation are in general badly behaved. However for fine monoids things work nicely.

Lemma 3.22. *Let X be a discrete topological space, and $X \to Y$ a set-theoretic function. Then the final topology on Y with respect to the family of maps $\{X \to Y\}$ is the discrete topology.*

Proof. The universal property for the final topology with respect to $\{X \to Y\}$ is that a map $Y \to Z$ is continuous if and only if its precomposition $X \to Z$ is continuous. However, all set functions out of a discrete topological space are continuous so every set function $Y \to Z$ is also continuous. This forces Y to be discrete as well. □

Corollary 3.23. *Let M be a commutative topological monoid. If M is discrete, so are*

(1) *its groupification M^{gp},*
(2) *its integration M^{int},*
(3) *its sharpening $\overline{M}$,*
(4) *and its saturation M^{sat}.*

Proof. Since M is discrete, so is $M \times M$. All four constructions are quotients of either M or $M \times M$, so by the immediately preceding lemma they are all also discrete. □

Corollary 3.24. *Let M be a commutative topological monoid. If M is fine, then so are*

(1) *its groupification M^{gp},*
(2) *its sharpening $\overline{M}$,*
(3) *and its saturation M^{sat}.*

Definition 3.25. We pose the following definitions.

(1) Let S be a site. A *sheaf of topological monoids on S* is a sheaf of monoids $\mathscr{M}$ along with the data of a topology on every $\mathscr{M}(s)$ such that for every map $s \to s'$ in S the restriction morphism $\mathscr{M}(s') \to \mathscr{M}(s)$ is continuous.
(2) Let T be a topos and S a site defining T. We write $TCMon(T)$ for the category of *commutative topological monoid objects of T.*

    By the adjunction of functor categories and the product of categories, $TCMon(T)$ is isomorphic to the category of sheaves of commutative monoids on S.
(3) We write $TAb(T)$ and the category of *commutative topological abelian group objects of T.*

(4) We write $TCMon^{\mathrm{int}}(T)$ for the category of *integral commutative topological monoid objects of* T, that is sheaves of commutative monoids whose sections are all integral.

(5) We write $\overline{TCMon}(T)$ for the category of *sharp commutative topological monoid objects of* T, that is sheaves of commutative monoids whose sections are all sharp monoids.

(6) When T has enough points, we write $TCMon^{\mathrm{sat}}(T)$ for the category of *saturated commutative topological monoid objects of* T, that is sheaves of commutative monoids whose stalks are all saturated.

(7) We can define the groupification functor as follow:, to a sheaf of commutative topological monoids we first perform groupification sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{gp}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{gp}}\right]^{+}$$

(8) We can define the integration functor as follows: to a sheaf of commutative topological monoids we first perform integration sectionwise, then we integrate.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{int}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{int}}\right]^{+}$$

(9) We can define the sharpening functor as follows: to a sheaf of commutative topological monoids we first perform sharpening sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \overline{\mathscr{F}(U)}\right] \mapsto \left[U \mapsto \overline{\mathscr{F}(U)}\right]^{+}$$

(10) We can define the saturation functor as follows: to a sheaf of commutative topological monoids we first perform saturation sectionwise, then we sheafify.

$$\mathscr{F} \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{sat}}\right] \mapsto \left[U \mapsto \mathscr{F}(U)^{\mathrm{sat}}\right]^{+}$$

$\triangleleft$

EXAMPLE 3.26. Let X be a space, and $\mathscr{F}$ any sheaf of abelian groups. Giving all sections the discrete topology gives $\mathscr{F}^{\mathrm{disc}}$ the structure of a sheaf of commutative topological monoids.

Let $\mathfrak{X}$ be a formal scheme, then $\mathscr{O}_{\mathfrak{X}}$ with its multiplication law is a sheaf of commutative monoids. $\triangleleft$

The following lemma follows from the analogous one for sheaves of monoids plus the fact that all the topologies we are using are final topologies.

LEMMA 3.27. *The following properties hold for the above constructions.*

(1) *The given construction of groupification satisfies the following.*

(a) *It gives a functor* $(-)^{\mathrm{gp}} : \mathrm{CTMon}(T) \to \mathrm{TAb}(T)$,

(b) *the functor* $(-)^{\mathrm{gp}}$ *is a left adjoint to the forgetful functor* $\mathrm{TAb}(T) \subset \mathrm{CTMon}(T)$,

(c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{CTMon}(T)$ *the induced morphism*

$$(\mathscr{M}_x)^{\mathrm{gp}} \to (\mathscr{M}^{\mathrm{gp}})_x$$

is an isomorphism of topological abelian groups.

(2) *The given construction of integration satisfies the following.*

(a) *It gives a functor* $(-)^{\mathrm{gp}} : \mathrm{CTMon}(T) \to \mathrm{CTMon}^{\mathrm{int}}(T)$,

(b) *the functor* $(-)^{\mathrm{gp}}$ *is a left adjoint to the forgetful functor* $\mathrm{CTMon}^{\mathrm{int}}(T) \subset \mathrm{CTMon}(T)$,

(c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{CTMon}(T)$ *the induced morphism*

$$(\mathscr{M}_x)^{\mathrm{int}} \to (\mathscr{M}^{\mathrm{int}})_x$$

is an isomorphism of integral commutative topological monoids.

(3) *The given construction of sharpening satisfies the following.*

(a) *It gives a functor* $\overline{(-)} : \mathrm{TMon}(T) \to \overline{\mathrm{TMon}}(T)$,

(b) *the functor* $\overline{(-)}$ *is a left adjoint to the forgetful functor* $\overline{\mathrm{CTMon}}(T) \subset \mathrm{CTMon}(T)$,

(c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{CTMon}(T)$ *the induced morphism*

$$\overline{(\mathscr{M}_x)} \to (\overline{\mathscr{M}})_x$$

is an isomorphism of sharp commutative topological monoids.

(4) *The given construction of saturation satisfies the following when* T *has enough points.*

(a) *It gives a functor* $(-)^{\mathrm{sat}} : \mathrm{CTMon}(T) \to \mathrm{CTMon}(T)$.

(b) *the functor* $(-)^{\mathrm{sat}}$ *is a left adjoint to the forgetful functor* $\mathrm{CTMon}^{\mathrm{sat}}(T) \subset \mathrm{CTMon}(T)$,

(c) *and for any stalk functor* $x : Sets \to T$ *and any sheaf* $\mathscr{M}$ *in* $\mathrm{CTMon}(T)$ *the induced morphism*

$$(\mathscr{M}_x)^{\mathrm{sat}} \to (\mathscr{M}^{\mathrm{sat}})_x$$

is an isomorphism of saturated commutative topological monoids.

The following results follow just as in the usual sheaves of commutative monoid case.

Theorem 3.28. *Let* S *and* S' *be sites with finite limits, and* $F : S \to S'$ *be a functor inducing a geometric morphism of topoi* $F_* : \tau S' \to \tau S$.

(1) *The functors* F_* *and* F^* *commute with groupification of sheaves of commutative topological monoids.*

(2) *The functors* F_* *and* F^* *commute with integration of sheaves of commutative topological monoids.*

(3) *The functors* F_* *and* F^* *commute with sharpening of sheaves of commutative topological monoids.*

(4) *When both sites have enough points, the functor* F^* *commutes with saturation of sheaves of commutative topological monoids.*

CHAPTER 4

ÉTALE HOMOTOPY THEORY OF SCHEMES, LOG SCHEMES, AND LOG FORMAL SCHEMES

In this chapter we will review étale homotopy theory for schemes. The first definition of etale homotopy type goes back to the book of Artin and Mazur [**1**], but the ideas go back further. Already in Weil's work [**23**] he refers to hypothetical Betti numbers of an algebraic variety. By the mid 1950s the Bourbaki group has defined Grothendieck topologies, sheaves, and eventually topoi. The étale homotopy type is simply another idea from topology being applied to algebraic geometry, and a direct continuation of their work.

1. Classical results in algebraic geometry

The reader is expected to be familiar with the more geometric notions of smoothness, such as those that appear in [**24**]. However we must work in extreme generality and so we review the more general definitions of the concepts we use. We assume the reader is familiar with schemes and their structure sheaves, along with notions from commutative algebra such as Krull dimension, finite presentation of algebras, flatness, and Kähler differentials. While these definitions are abstract, they are quite closely connected to the classical definitions for algebraic varieties, which provide a good base of intuition to draw on.

Definition 4.1.

(1) A morphism of schemes $f : X \to Y$ is *quasi-compact* if the preimage of a quasi-compact open in Y is quasi-compact and open in X.
(2) A morphism of schemes $f : X \to Y$ is *separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion.
(3) A morphism of schemes $f : X \to Y$ is *quasi-separated* if $\Delta_f : X \to X \times_Y X$ is a quasi-compact morphism.
(4) A morphism of schemes $f : X \to Y$ is *locally of finite presentation* if for every open affine $U \subset Y$, and for every $x \in f^{-1}(U)$, there is an open affine neighborhood V_x such that the map of affines $f : V_x \to U$ corresponds to a map of rings that is of finite presentation.

(5) A morphism of schemes is *of finite presentation* if it is quasi-compact, quasi-separated, and locally of finite presentation.

(6) A morphism of schemes $f : X \to Y$ is said to be *formally unramified* if it admits at most one lift for any diagram of the form

$$\begin{tikzcd} \operatorname{spec} T \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ \operatorname{spec} T' \arrow[r] \arrow[ur, dashed, "e"] & Y \end{tikzcd}$$

whenever $T' \to T$ is a ring map isomorphic to quotienting by a square-zero ideal.

(7) A morphism of schemes $f : X \to Y$ is said to be *formally smooth* if it admits at least one lift for any diagram of the form

$$\begin{tikzcd} \operatorname{spec} T \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ \operatorname{spec} T' \arrow[r] \arrow[ur, dashed, "e"] & Y \end{tikzcd}$$

whenever $T' \to T$ is a ring map isomorphic to quotienting by a square-zero ideal.

(8) A morphism of schemes is said to be *formally étale* if it is formally unramified and formally smooth, that is when there is a unique lift in all such diagrams.

(9) A morphism is said to be *unramified* (resp. *smooth*, resp. *étale*) if it is formally unramified (resp. formally smooth, resp. formally étale) and locally of finite presentation.

$\triangleleft$

LEMMA 4.2. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. If both f and g are any of the following*

(1) *quasi-compact,*
(2) *separated,*
(3) *quasi-separated,*
(4) *locally of finite presentation,*
(5) *of finite presentation,*
(6) *formally unramified,*
(7) *formally smooth,*
(8) *formally étale,*
(9) *unramified,*
(10) *smooth,*
(11) *or étale;*

then so is $g \circ f$.

PROOF. We separate the proofs.

(1) We immediately see that $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$, and so if f^{-1} and g^{-1} take quasi-compact opens to quasi-compact opens, then so does their composition.
(2) See Lemma 01KU of [**2**].
(3) See Lemma 01KU of [**2**].
(4) See Lemma 01TR of [**2**].
(5) See Lemma 01TR of [**2**].
(6) See Lemma 02HA of [**2**].
(7) This is stated without proof in the Stacks Project, but its proof is quite simple so we reproduce it here. We want to show there is a lifting in the diagram
$$\begin{tikzcd} \operatorname{spec} T \arrow[r] \arrow[d, "i"] & Y \arrow[d, "g"] \\ \operatorname{spec} T' \arrow[r] \arrow[ur, dashed, "e"] & Z \end{tikzcd}$$
but we simply observe that we can first lift from T' to Y, and then to X via the lifted map
$$\begin{tikzcd} \operatorname{spec} T \arrow[r] \arrow[d, "i"] & X \arrow[d, "g \circ f"] \\ \operatorname{spec} T' \arrow[r, "e"] \arrow[ur, dashed, "e'"] & Y \end{tikzcd}$$
(8) See Lemma 02HI of [**2**].

The cases of unramified, smooth, and étale follow from the cases of formally unramified, formally smooth, and formally étale plus the case of locally of finite presentation. □

THEOREM 4.3. *Let $f : X \to Y$ be a morphism of schemes, and $g : Z \to Y$ a morphism with the same target. Further, assume that f is,*

(1) *quasi-compact,*
(2) *separated,*
(3) *quasi-separated,*
(4) *locally of finite presentation,*
(5) *of finite presentation,*
(6) *formally unramified,*
(7) *formally smooth,*
(8) *formally étale,*
(9) *unramified,*
(10) *smooth,*
(11) *or étale.*

Then the base change $f' : X \times_Y Z \to Z$ also satisfies that property.

PROOF. We break up the proof again.

(1) See Lemma 01K5 of [**2**].
(2) See Lemma 01KU of [**2**].
(3) See Lemma 01KU of [**2**].
(4) See Lemma 01TS of [**2**].
(5) See Lemma 01TS of [**2**].
(6) See Lemma 02HB of [**2**].
(7) See Lemma 02H2 of [**2**].
(8) See Lemma 02HJ of [**2**].

The cases of unramified, smooth, and étale follow from the cases of formally unramified, formally smooth, and formally étale plus the case of locally of finite presentation. □

Here we define the notion of an fpqc covering.

DEFINITION 4.4. A morphism of schemes is a ***fpqc covering*** if it is flat, surjective, and quasi-compact.

The ***fpqc pretopology*** on schemes is the pretopology with covering families $\{U_i \to U\}_{i \in I}$ where the disjoint union

$$\coprod U_i \to U$$

is an fpqc covering. Beware! This is a significantly stronger condition than asking that each $U_i \to U$ is a fpqc covering. ◁

We define the little sites and then the little topoi.

DEFINITION 4.5. Let X be a scheme and P be one of Zariski, étale, smooth, fppf, or fpqc. Write $\mathbf{site}\, X_P$ for the category of morphisms of schemes with target X, which are in the class P. We put a Grothendieck pre-topology on the category as follows. For $U \to X$ an object in the category $\mathbf{site}\, X_P$, the covering families are exactly the families of morphisms $\{U_i \to U\}_{i \in I}$ such that

(1) the family is jointly surjective,
(2) and the induced morphism

$$\coprod_{i \in I} U_i \to U$$

is a fpqc covering.

The *little P site of X*, denoted $\mathbf{site}\, X_P$, is defined to be restriction of covering families from the big site to the category of morphisms to X that are in P.

The *little P topos of X*, denoted $\tau\, X_P$, is the topos of sheaves of sets on the little P site of X. ◁

We will work only with the little topoi in this document, as they are more well behaved. Two warnings are in order. The first is that not all sources define the little étale site in the same way. Some take the étale site to be jointly surjective families of étale morphisms with no condition that the disjoint union becomes fpqc over the base. By Lemma 03PH of [**2**] such covers always refine to covers in our sense. This implies the topoi are equivalent. Secondly, the fpqc topos is badly behaved due to set theoretic issues. For example there is no sheafification functor from fpqc presheaves of sets to it. There is an explicit counter-example constructed in [**25**]. We now discuss the ∞-categorical variants of the above.

Definition 4.6. Let X be a scheme and P be one of Zariski, étale, smooth, fppf, or fpqc.

The *little P ∞-topos of X*, denoted $\tau_\infty\, X_P$, is the ∞-category presented by the Čech-local projective model structure on simplicial presheaves of sets on X with respect to the P pretopology. For brevity, we will usually call this by the P ∞-topos of X.

The *hypercomplete little P ∞-topos of X*, denoted $\tau_\infty^\wedge\, X_P$, is the ∞-category presented by the hyperlocal projective model structure on simplicial presheaves of sets on X with respect to the P pretopology. For brevity, we will usually call this the hyperconmplete P ∞-topos of X. ◁

Lemma 4.7. *We have the following equivalences of étale topoi and smooth topoi,*

$$\tau\, X_{\acute{e}t} \simeq \tau\, X_{sm}$$

$$\tau_\infty\, X_{\acute{e}t} \simeq \tau_\infty\, X_{sm}$$

$$\tau_\infty^\wedge\, X_{\acute{e}t} \simeq \tau_\infty^\wedge\, X_{sm}$$

Proof. The sheaf condition for the smooth topology is actually equivalent to the sheaf condition for the étale topology. This is because every smooth cover can be refined to an étale one, see Lemma 055V of [**2**]. This implies that the collection of étale covering families is cofinal in the family of smooth covering families, so that the homotopy colimits must agree. □

2. The étale homotopy type of a scheme

We open with the stark definition.

Definition 4.8. Let X be a scheme. The ***étale homotopy type*** $\text{ét}\, X$ of X is the shape of the hypercomplete étale topos of X

$$\text{ét}\, X := \text{sh}\, \tau_{\infty}^{\wedge}\, \mathbf{site}\, X_{\text{ét}}$$

We will denote by $\text{ét}^{\natural}\, X$ (resp.$\widehat{\text{ét}}\, X$, resp. $\widehat{\text{ét}}_{\{p\}^c}\, X$, resp $\widehat{\text{ét}}_{\ell}\, X$) the pro-truncated completion (resp. the profinite completion, resp. the $\{p\}^c$-profinite completion, resp. the ℓ-profinite completion) of $\text{ét}\, X$. ◁

Theorem 4.9. *The étale homotopy types of Artin-Mazur* **[1]** *and Friedlander* **[3]***, the étale topological type of Friedlander* **[3]***, and the ∞-categorical construction above all agree; at least in the pro-category of the homotopy category of spaces.*

Proof. There are two separate claims that imply the theorem.

(1) Friedlander's étale topological type is isomorphic to the étale homotopy type in the pro-category of the homotopy category of spaces. This is essentially due to Friedlander, see Proposition 4.5 **[3]**. The key point is that Friedlander's geometrically pointed covers are cofinal in the system of all covers.
(2) Taking a fibrant-cofibrant replacement of Friedlander's étale topological type, it is equivalent to the shape of the hypercomplete ∞-topos. This follows by Thoerem 2.77.

□

Corollary 4.10. *Let A be an abelian group, and X a scheme. Then*

$$H^i(X_{\text{ét}}, \underline{A}) \simeq H^i(\text{ét}\, X, A)$$

Corollary 4.11. *Let X be a connected scheme of finite type over* $\operatorname{spec} \mathbb{C}$*. Then the profinitely completed étale homotopy type is weakly equivalent to the profinite completion of the Betti realization of X.*

$$\widehat{\text{ét}}\, X \simeq X(\mathbb{C})^{\vee}$$

Definition 4.12. The map that we claim induces an equivalence of profinite spaces is the one induced by the geometric morphism from the étale site of the scheme X to the topos of the space $X(\mathbb{C})$. To verify that it induces an equivalence of profinitely completed shapes, it is equivalent to verify that the geometric morphism preserves sheaf cohomology with locally constant coefficients and that the geometric morphism induces an equivalence of categories of finite torsors. The former is the Artin comparison theorem, see for example Theorem 3.12 of chapter III of **[26]**. The second is the Riemann existence theorem, see for example Corollaire 5.2 of Expose XII of **[27]** . ◁

THEOREM 4.13. *Let $f : (X, \overline{x}) \to (Y, \overline{y})$ be a smooth proper map of geometrically pointed, connected, Noetherian schemes such that the geometric fiber $f_{\overline{y}}$ is connected. Then for any prime $\ell > 0$ invertible on Y the ℓ-profinite completion of $\widehat{\mathrm{\acute{e}t}}_\ell\, f_{\overline{y}}$ is the homotopy fiber of the map $f : \widehat{\mathrm{\acute{e}t}}_\ell\, X \to \widehat{\mathrm{\acute{e}t}}_\ell\, Y$.*

PROOF. This is just a restatement in ∞-categorical language of Friedlander's homotopy fiber theorem. See Theorem 4.6 of [**28**] $\square$

3. Logarithmic geometry

Chapter 3 contains some results on monoids and sheaves of monoids used in this section.

DEFINITION 4.14. We pose the following definitions.

(1) A *pre-log structure* on a scheme X is a pair $(\mathscr{M}_X, \alpha)$ of a sheaf of monoids $\mathscr{M}_X$ on the étale site of X, along with a morphism of sheaves of monoids $\alpha : \mathscr{M}_X \to \mathscr{O}_X$. Here the monoid structure on the $\mathscr{O}_X$ is the multiplicative one.

(2) A *log-structure* on a scheme X is a pre-log structure $(\mathscr{M}_X, \alpha_X)$ such that the induced map $\alpha_X : \alpha_X^{-1}(\mathscr{O}_X^\times) \to \mathscr{O}_X^\times$ is an isomorphism.

(3) A *morphism of pre-log structures* from $(\mathscr{M}, \alpha)$ to $(\mathscr{M}', \alpha')$ on a scheme X is a morphism of sheaves of commutative monoids $a : \mathscr{M} \to \mathscr{M}'$ such that $\alpha' \circ a = \alpha$.

(4) A *morphism of log structures* is a morphism of the underlying pre-log structures.

(5) A *pre-log scheme* X consists of a pair $(X^\circ, \mathscr{M}_X)$ of a scheme X° with a pre-log structure $\mathscr{M}_X$ on it.

(6) A *log scheme* X consists of a pair $(X^\circ, \mathscr{M}_X)$ of a scheme X° with a log structure $\mathscr{M}_X$ on it.

(7) A *morphism of pre-log schemes* $f : (X, \mathscr{M}_X) \to (Y, \mathscr{M}_Y)$ is a pair $(f^\circ, f^\#)$ where $f^\circ : X^\circ \to Y^\circ$ is a morphism of the underlying schemes, and $f^\# : f^{\circ,-1}\mathscr{M}_Y \to \mathscr{M}_X$ is a morphism of sheaves of commutative monoids such that the following square is commutative

$$\begin{tikzcd} f^{\circ,-1}\mathscr{M}_Y \arrow[r] \arrow[d] & \mathscr{M}_X \arrow[d] \\ f^{\circ,-1}\mathscr{O}_Y \arrow[r] & \mathscr{O}_X \end{tikzcd}$$

(8) A *morphism of log schemes* is a morphism of the underlying pre-log schemes.

(9) The *trivial log structure* on a scheme X is the triple $(X, \mathscr{O}_X^\times, i : \mathscr{O}_X^\times \to \mathscr{O}_X)$. A log structure on a scheme is *trivial* if it is isomorphic to the trivial log structure.

(10) Given a log scheme $X = (X^\circ, \mathscr{M}_X, \alpha_X)$, the scheme X^{triv} is the largest open subscheme of X° such that $\mathscr{M}_X$ is trivial after restriction.

◁

Lemma 4.15. *With every pre-log structure $(\mathscr{M}', \alpha')$ on a scheme X, we may associate a log structure $(\mathscr{M}, \alpha)$ such that any morphism from the pre-log structure $(\mathscr{M}', \alpha')$ to a log structure on X uniquely factors through (M, α)*

Proof. The construction is quite straightforward, we simply take $\mathscr{M}$ to be the push-out of the following diagram.

$$\begin{array}{ccc} \alpha'^{-1}\mathscr{O}_X^\times & \longrightarrow & \mathscr{M} \\ \big\downarrow & & \\ \mathscr{O}_X^\times & & \end{array}$$

and it automatically satisfies this property. □

Corollary 4.16. *The category of log schemes admits finite limits.*

Proof. Take the limit of the underlying schemes, and give it the log structure induced by the limit of the inverse image log structures from the schemes in the diagram. □

Definition 4.17. Let X be a scheme, and $Y = (Y, \mathscr{M}_Y, \alpha_Y)$ be a log structure on Y. Given a morphism of schemes $f : X \to Y$, the *inverse-image log structure* on X is the log structure associated with the pre-log structure given by the composition $f^{-1}\mathscr{M}_Y \to f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. A morphism of log schemes is *strict* if the log structure on the domain is isomorphic to the inverse-image log structure associated with the target. ◁

Now the trouble is that while it turns out that log schemes are quite a useful concept, it is highly non-obvious how to construct them at a first glance. Luckily there is a rich set of cases where one can construct them with manageable amounts of data.

Definition 4.18. We pose the following definition.

(1) Let X be a regular Noetherian scheme and D be a reduced divisor with normal crossings on X. Then the *log structure associated with* (X, D) is the sheaf of commutative monoids given by

$$U/X \mapsto \{g \in \mathscr{O}_X \mid g \text{ is invertible outside of } D\}$$

(2) Let X be a scheme and D a Cartier divisor on X. Then we may associate with D a log structure as follows. Take any representation of D as $(U_i, f_i)_{i \in I}$ and add all powers of f_i

to $\mathscr{O}_X^\times(U_i)$. This glues to a globally defined sheaf, since on the overlaps $U_{i,j}$ there is some invertible element $u_{i,j} \in \mathscr{O}_X^\times(U_{i,j})$ with $u_{i,j} f_i = f_j$. Any two representations can be refined to a common one which differs only by an invertible element, and so the construction does not depend on the representation.

(3) Let X be a scheme, and P a monoid. Given a morphism of commutative monoids $P \to \mathscr{O}_X(X)$ we can lift this to a morphism of constant sheaves of commutative monoids $\underline{P} \to \mathscr{O}_X$ and take the associated log structure P^a. We call such a morphism $P \to \mathscr{O}_X(X)$ a ***global chart***, and the resulting P^a as the ***log structure associated with the chart.*** A log structure is said to ***admit a chart*** if there is a monoid P and a global chart for P such that the log structure is isomorphic to the log structure associated with the chart. A log structure on a scheme X is said to ***locally admit charts*** if there is a covering $\{U_i \to X^\circ\}_{i \in I}$ in the étale topology such that the inverse image log structure on each U_i admits a chart.

(4) Let P be an arbitrary commutative monoid. Then the scheme $\operatorname{spec} \mathbb{Z}[P]$ canonically admits a log structure associated with the morphism of monoids $P \to \mathbb{Z}[P]$.

◁

LEMMA 4.19. *Let X be a log scheme. The following are equivalent.*

(1) *The log scheme X admits a global chart $\underline{P} \to \mathscr{O}_X$,*

(2) *there is a strict morphism of log schemes $X \to \operatorname{spec} \mathbb{Z}[P]$.*

PROOF. This is essentially just unraveling the definitions The existence of a morphism of schemes $X^\circ \to \operatorname{spec} \mathbb{Z}[P]$ just follows from the adjunction between spec and the global sections functor. That the morphism is strict is exactly that the log structure on X is the associated log structure with $\underline{P} \to \mathscr{O}_X$. □

DEFINITION 4.20. Let X be a fine log scheme admitting a global chart $X \to \operatorname{spec} \mathbb{Z}[P]$ to some fine commutative monoid P. The ***saturation*** X^{sat} of X is given by the fiber product

$$\begin{array}{ccc} X^{\mathrm{sat}} & \longrightarrow & \operatorname{spec} \mathbb{Z}[P^{\mathrm{sat}}] \\ \downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{spec} \mathbb{Z}[P] \end{array}$$

and demanding that the map $X^{\mathrm{sat}} \to \operatorname{spec} \mathbb{Z}[P^{\mathrm{sat}}]$ is a strict morphism of log schemes.

For a general fine log scheme X, we perform the above construction locally, and then glue the result log structures to a global one resulting in X^{sat}. ◁

LEMMA 4.21 (PROPOSITION 1.8 [**29**]). *Let X be a fine log scheme. Then the above construction is well defined and independent of choices and gives X^{sat} the structure of an fs log scheme.*

In practice we will mostly use charts and log structures associated with normal crossings divisors. However the log structures used in practice are rather special, and we need a few more abstract properties for later proofs and statements.

DEFINITION 4.22. Let X be a log scheme. Then the log structure $\mathscr{M}_X \to \mathscr{O}_X$ on X is called *integral* if the geometric stalks of $\mathscr{M}_X$ are all integral monoids. It is called *fine* if it locally admits charts given by fine monoids P. It is called saturated if it locally admits charts by saturated monoids. It is called *fs* if it locally admits charts given by fine and saturated monoids P.

An *fs log scheme* is a log scheme whose log structure is fs, meaning it étale locally admits charts given by integral finitely generated monoids which are power-saturated in their groupification. $\triangleleft$

Now a word as to the motivation of this. The most powerful theorems of algebraic geometry only apply to proper morphisms. Logarithmic geometry can allow one to extend some results to non-proper morphisms by acting as an intermediate. The core geometrical idea is roughly the following. Given a non-proper variety $X/\operatorname{spec} k$, we compactify (when possible) to an $\overline{X}$ where $D = \overline{X} \setminus X$ is a normal crossing divisor. Then the logarithmic geometry of $\overline{X}$ with log structure associated with D should behave much like the usual geometry of X.

DEFINITION 4.23. We pose the following definitions.

(1) Let $f : X \to Y$ be a log morphism of log schemes, and assume that the underlying morphism of schemes f° is locally of finite presentation. Then f is said to be *formally log unramified* if for every commutative square of log schemes

$$\begin{tikzcd} \operatorname{spec} T \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ \operatorname{spec} T' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $\operatorname{spec} T \to \operatorname{spec} T'$ is a strict log morphism and a closed immersion defined by a square-zero ideal, there is at most one dashed lift.

(2) Such an f is said to be *formally log smooth* if there is at least one dashed lift in the above diagram.

(3) Such an f is said to be *formally log étale* if it is formally log unramified and formally log smooth, that is there is a unique dashed lift in the above diagram.

(4) A morphism of log schemes $f : X \to Y$ is called *Kummer* if the map $f^{-1}\mathscr{M}_Y to \mathscr{M}_X$ is injective and the cokernel of $(f^{-1}\mathscr{M}_Y)^{\mathrm{gp}} \to \mathscr{M}_X^{\mathrm{gp}}$ is torsion.

(5) A morphism of log schemes is called log unramified (resp. log smooth, resp. log étale) if the underlying morphism of schemes is locally of finite presentation and the log morphism is formally log unramified (resp. formally log smooth, resp. formally log étale).

(6) A morphism of log schemes is called Kummer étale if it is log étale and Kummer.

◁

We immediately deduce the following lemma.

LEMMA 4.24. *Let $f : X \to Y$ be a morphism of log schemes given the trivial log structure. Then f is formally log étale if and only if it is formally étale.*

PROOF. Any morphism of schemes $T'^{\circ} \to X^{\circ}$ will be a log morphism since X has the trivial log structure. Thus the condition that f is log étale is exactly the square-zero-ideal lifting condition of the formal étale property. □

The following gives some illumination as to what the local structure of log smooth and log étale morphisms looks like.

LEMMA 4.25 (THEOREM 3.5 OF [**30**]). *If $f : X \to Y$ is a morphism of fs log schemes. Then the following are equivalent.*

(1) *The morphism f is formally log smooth (resp. formally log étale),*

(2) *for any point $x \in X^{\circ}$ there are étale neighborhoods U of x and V of $f(x)$ with charts P_U and P_V such that there is a commutative diagram as follows*

$$\begin{tikzcd} U \arrow[d, "f'"] & \\ V \times_{\mathbb{Z}[P_V]} \operatorname{spec} \mathbb{Z}[P_U] \arrow[r] \arrow[d] & \operatorname{spec} \mathbb{Z}[P_U] \arrow[d] \\ V \arrow[r] & \operatorname{spec} \mathbb{Z}[P_V] \end{tikzcd}$$

such that every morphism f' is formally smooth (resp. formally étale) and the morphism of monoids $P_V \to P_U$ satisfies that the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of the morphism $P_V^{\mathrm{gp}} \to P_U^{\mathrm{gp}}$ are finitely groups of orders invertible on X.

There is a notion of logarithmic blow ups, which are special types of blow ups that have no clear analogue to the classical case.

Definition 4.26. Let X be a fs log scheme admitting a global chart by some fine and saturated monoid P. For any monoidal ideal $I \subset P$, we define the blow up of X along I as follows. The monoidal ideal I determines a ring-theoretic ideal of $\mathbb{Z}[P]$, which we can blow up to get the affine log scheme $\mathrm{bl}_I(\operatorname{spec}\mathbb{Z}[P])$. We take the logarithmic fiber product $X \times_{\mathbb{Z}[P]} \mathrm{bl}_I(\operatorname{spec}\mathbb{Z}[P])$ and then perform the saturation of this fiber product. We call the resulting fs log scheme $\mathrm{bl}_I X$.

If X is a fs log scheme and $I \triangleleft \mathscr{M}_X$ is a coherent ideal of the monoid structure, then we can perform the above construction locally and glue it together into a fs log scheme $\mathrm{bl}_I X$. ◁

Lemma 4.27. *The above construction does not depend on the choices made.*

For a proof, see the discussion in section 3.3.2 of [**31**]. There is also a notion of logarithmic regularity, which is also of critical importance.

Definition 4.28. Given a log scheme $X = (X^\circ, \mathscr{M}_X, \alpha)$ and a geometric point $x \in X^\circ$, we can form the *logarithmic ideal sheaf of* $\mathscr{M}_X$ *at* x by taking the ideal $I(\mathscr{M}_X, x)$of $\mathscr{O}_X$ generated by $\mathscr{M}_{X,x} \setminus \mathscr{O}_{X,x}^\times$. If the log structure is given by a chart in some étale neighborhood of x then by definition the ideal will be generated by the elements of the chart which are not invertible at x.

A log scheme X satisfying that X° is locally Noetherian is called *log regular* when for every geometric point $x \in X^\circ$ the ring $\mathscr{O}_{X,x}/I(\mathscr{M}_X, x)$ is regular and the following equation holds

$$\dim \mathscr{O}_{X,x} = \operatorname{rank} \overline{(\mathscr{M}_{X,x})}^{\mathrm{gp}} + \dim \mathscr{O}_{X,x}/I(\mathscr{M}_X, x)$$

◁

Lemma 4.29. *If X is an fs log scheme admitting a global chart by a fine and saturated monoid, then the above condition may be checked on the stalks for the Zariski topology instead of the étale topology.*

Proof. This result is due to Nizioł, see Lemma 2.3 of [**32**]. □

Also due to Nizioł is the following result, which is the main theorem of [**32**].

Theorem 4.30. *Log regular fs log schemes admit resolutions of singularities by logarithmic blow ups.*

Proposition 4.31. *Let Y be a log regular locally Noetherian fs log scheme, and $X \to Y$ a log smooth morphism of fs log schemes. Then X is log regular.*

Proof. This is true if the log structure is defined on the Zariski topology by Theorem 8.2 of [**33**]. Note that Kato's condition (S) is the Zariski topology analogue of our fs condition. However, being log regular étale locally implies that a scheme is log regular, since log regularity only depends on the étale local rings of the structure sheaf. Thus we conclude that such a X is log regular. □

DEFINITION 4.32. The *log étale site* of a log scheme X has the underlying category whose objects are formally log étale morphisms $U \to X$ and whose morphisms are commutative triangles. The coverings for the Grothendieck pretopology are jointly surjective families.

The *Kummer étale site* $\mathbf{site}\, X_{\text{két}}$ of a log scheme X has the underlying category whose objects are Kummer and formally log étale morphisms $U \to X$ and whose morphisms are commutative triangles. The coverings for the Grothendieck pretopology are jointly surjective families. ◁

Being Kummer and log étale has a convenient characterization due to Kisin.

THEOREM 4.33. *Let f be a morphism of fs locally Noetherian log schemes. Then the following are equivalent.*

(1) *The morphism f is Kummer étale.*
(2) *The morphism f is formally log étale, and f° is quasi-finite.*

PROOF. See Proposition 1.7 of [**34**] and the preceeding discussion. □

We can give a neat characterization of the cohomology of the Kummer étale site for log regular log schemes. This result is one of the major motivations for using log schemes.

THEOREM 4.34. *Let X be a log regular locally Noetherian fs log scheme. Then for any locally constant sheaf $\mathscr{A}$ of finite abelian groups with orders invertible on X, the inclusion $i : X^{\text{triv}} \hookleftarrow X$ induces isomorphisms*

$$H^\bullet(X_{\text{két}}, \mathscr{A}) \simeq H^\bullet(X^{\text{triv}}_{\text{ét}}, i^{-1}\mathscr{A})$$

and an isomorphism for any geometric point $x \in X^{\text{triv}}$ on pro-ℓ completions for any prime number ℓ invertible on X.

$$\pi_1^{\text{két}}(X, x)^\wedge_\ell \simeq \pi_1^{\text{ét}}\widehat{(X^{\text{triv}}, x)}_\ell.$$

More generally, the category of Kummer étale coverings of X is equivalent to the category of étale covers of X^{triv} which extend to tamely ramified covers of X.

PROOF. The if the underlying scheme is regular, then the proof is sketched following Theorem 7.4 of [**35**]. Otherwise we may resolve the singularities of X using a logarithmic blow up by the main result of [**32**]. Logarithmic blow ups do not change Kummer étale cohomology, and they do not change the open subset of triviality, so we deduce the first claim.

The second claim follows from the third one, which is proven as Proposition B.7 of [**36**]. To deduce the second claim from the third, we must simply note that finite torsors for the Kummer étale topology are representable by Kummer étale schemes. □

In fact, the entire Kummer étale topos admits a description as the étale topos of a certain stack.

THEOREM 4.35 (THEOREM 6.22 OF [**37**]). *The Kummer étale topos of a fs log scheme is equivalent to the étale topos of the associated infinite root stack* $\sqrt[\infty]{X_{\text{ét}}}$

The definition of the infinite root stack is quite elegant, but a bit too far from our applications.

In the case of a log smooth log scheme over $\operatorname{spec}\mathbb{C}$, we get the following special case.

THEOREM 4.36. *Let* $X \to \operatorname{spec}\mathbb{C}$ *be a log smooth morphism where* $\operatorname{spec}\mathbb{C}$ *is given the trivial log structure. Then the profinitely completed Kummer étale homotopy type in our sense is equivalent to the profinite completion of the Kato-Nakayama space* $X(\mathbb{C})^{\log}$ *associated with the log complex space* $X(\mathbb{C})$*,*

$$\widehat{\text{két}}\, X \simeq [X(\mathbb{C})^{\log}]^{\vee}$$

PROOF. This is the main result of [**38**], although their definition uses the étale site of the infinite root stack of Talpo-Vistoli [**37**]. By the immediately preceding theorem, these must be the same with the log smoothness assumption. □

4. Formal geometry

Open sets in schemes are too large to correctly capture all notions from geometry, as are étale neighborhoods. One concept is that of formal neighborhoods, which behave something like ϵ-neighborhoods in differential geometry.

DEFINITION 4.37. Let X be a locally Noetherian scheme and $\mathscr{I} \lhd \mathscr{O}_X$ a sheaf of ideals. The *completion of* X *along the ideal* $\mathscr{I}$, written as $\hat{X}_{\mathscr{I}}$ is the locally topologically ringed space whose underlying topological space is $V(\mathscr{I})$ with the sheaf of rings given by the limit

$$\mathscr{O}_{\hat{X}_{\mathscr{I}}} := \lim_{n\to\infty} \mathscr{O}_X/\mathscr{I}^n$$

◁

One does not need the locally Noetherian hypothesis for this locally ringed space to make sense. However completion is badly behaved for non-Noetherian rings, and consequently most of the properties we will use repeatedly will not hold without this hypothesis.

LEMMA 4.38. *Let $V \subset X$ be a closed subset of a locally Noetherian scheme X. Then for any two ideals $\mathscr{I}, \mathscr{I}' \lhd \mathscr{O}_X$ such that $V(\mathscr{I}) = V(\mathscr{I}') = V$ the completions of X along $\mathscr{I}$ and $\mathscr{I}'$ are naturally isomorphic.*

DEFINITION 4.39.

(1) A locally topologically ringed space is a *formal scheme* if it is locally isomorphic to the completion of a locally Noetherian affine scheme along a closed subset.
(2) A *morphism of formal schemes* is a map of locally ringed spaces, the maps of topological rings are automatically continuous.
(3) An *ideal of definition* for a formal scheme is any open ideal whose powers form a basis of open neighborhoods of 0 in every affine neighborhood.
(4) A formal scheme X is *regular at a point* $x \in X$ if its local ring $\mathscr{O}_{X,x}$ is a regular local ring.
(5) A formal scheme X is *regular* if it is regular at every point.
(6) Let $R = \lim_n R/I^n$ be a complete Noetherian topological ring with ideal of definition I. An R-algebra A is *topologically of finite type* if there is a finite type R-algebra B such that the completion of B with respect to IB is isomorphic to A.
(7) A morphism of formal schemes is *adic* when any ideal of definition for the target gives an ideal of definition for the source.
(8) A morphism of formal schemes $f : X \to Y$ is *locally topologically of finite type* any $x \in X$ there are affine open neighborhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and the induced map $f : \operatorname{spf} B = V \to U = \operatorname{spf} A$ comes from a algebra map $A \to B$ which is topologically of finite type.
(9) A morphism of formal schemes is *topologically of finite type* if it is locally topologically of finite type and quasi-compact.
(10) A morphism of formal schemes is *locally formally of finite type* if its reduction is locally of finite type for any compatible pairs of ideals of definition.
(11) A morphism of formal schemes is *locally adically of finite type* if it is locally formally of finite type, and the morphism is adic.
(12) A morphism of formal schemes is *formally of finite type* if its reduction is of finite type for any compatible pairs of ideals of definition.
(13) A morphism of formal schemes is *adically of finite type* if it is formally of finite type, and the morphism is adic.

(14) A morphism of formal schemes is *formally flat* if its reduction is classically flat for any compatible pair of ideals of definition.

(15) A morphism of formal schemes is *adically flat* if it is formally flat, and the morphism is adic.

(16) A morphism of formal schemes is *formally étale* if its reduction is classically étale for any ideal of definition.

(17) A morphism of formal schemes is *adically étale* if it is formally étale, and the morphism is adic.

(18) A morphism of formal schemes is *formally smooth* if its reduction is classically smooth for any ideal of definition.

(19) A morphism of formal schemes is *adically smooth* if it is formally smooth, and the morphism is adic.

$\triangleleft$

We have the following basic results.

THEOREM 4.40. *The following relations hold between the above defined classes.*

(1) *Locally adically of finite type adic morphisms are exactly the locally topologically of finite type morphisms.*

(2) *Adically of finite type adic morphisms are exactly the topologically of finite type morphisms.*

(3) *Adically étale morphisms that are adic morphisms are locally topologically of finite type.*

(4) *Adically étale morphisms are adically smooth.*

(5) *Adically smooth morphisms are adically flat.*

(6) *Adically flat morphisms $f : X \to Y$ satisfy that the induced map on all local rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ are flat ring homomorphisms.*

(7) *Let X be a locally Noetherian scheme and V a closed subset. Then $\hat{X}_V$ is regular if and only if X is regular at every point in V.*

(8) *Let $f : X \to Y$ be a morphism of finite type between integral and locally Noetherian schemes, and let $V \subset Y$ be a closed subset. Then the induced morphism $\hat{f} : \hat{X}_{f^{-1}(V)} \to \hat{Y}_V$ is adically P if and only if f is P at every point in V, when P is any of the following properties.*
   (a) *flat,*
   (b) *étale,*
   (c) *smooth*

Proof. (1) This is true since our definition of a topologically of finite type algebra requires that the ideal of definition for the base ring gives an ideal of definition for the algebra. We can apply Theorem 8.4.2 of chapter 0 of [**39**].

(2) This follows by the above plus the claim that a morphism of formal schemes is quasi-compact if and only if its reduction is. The claim is essentially trivial, since quasi-compactness of a morphism is a topological condition and the formal scheme and its reduction are homeomorphic via the reduction map.

(3) Adically étale morphisms have locally of finite type reductions, and by the first claim are necessarily locally topologically of finite type.

(4) Since the reductions of an adically étale morphism are all étale, they are also smooth.

(5) The reductions of an adically smooth morphism are of course flat, and so the morphism must be adically flat.

(6) This follows by Lemma 0912 of [**2**].

(7) It is a well known result that a local ring is regular if and only if its completion is. For any $x \in V$, we have an isomorphism on completions

$$\hat{\mathscr{O}}_{X,x} \simeq \hat{\mathscr{O}}_{\hat{X}_V,x}$$

and so one local ring is regular if and only if the other is.

(8) Since all three properties are stable under base change, the claim that f being P implies that $\hat{f}$ is adically P is immediate. We give proofs for the reverse implication.

(a) Adically flat morphisms give flat algebra maps on local rings. A local homomorphism of local rings is flat if and only if its completion is flat by Lemma 0C4G of [**2**]. But for any $x \in V$, the maps $f : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ and $\hat{f} : \mathscr{O}_{\hat{Y},f(x)} \to \mathscr{O}_{\hat{X},x}$ have isomorphic completions. Thus one is flat if and only if the other is.

(b) Since $\hat{f}$ is adically étale we deduce that for any ideal of definition I, the quotients $\mathscr{O}_{\hat{X},x}$ are all finite projective $\mathscr{O}_{\hat{Y},f(x)}/I^n$-algebras. As n-increases we can choose a compatible system of module theoretic generators which give generators for the I-adic limit, witnessing that $\mathscr{O}_{\hat{Y},f(x)}$ is a finite algebra. It is also flat and thus projective.

(c) The morphism f is flat along $f^{-1}(V)$ by the first claim, and as f is locally of finite type by assumption, Lemma 01V8 [**2**] guarantees that it is enough for the fibers of f to be smooth. However the fibers of f are just the reduced fibers of $\hat{f}$ which are smooth by assumption.

□

5. Formal geometry over a discrete valuation ring

For this subsection, let R be a complete discretely valued ring, with maximal ideal $\mathfrak{m}$, residue field k, and fraction field K. Then we can form the formal spectrum of R, $\operatorname{spf} R$, in the usual manner. This scenario is well-behaved enough that we may actually ease some of the assumptions of being topologically of finite type from our formal schemes. The benefit here is that some compact K-analytic spaces will have non-quasicompact but locally topologically of finite type formal models, and by dropping the locally topologically of finite type assumption we can get quasi-compact formal models of these non-archimedean spaces.

Definition 4.41. Let A be an R-algebra which is an adic ring with ideal of definition I. We say that A is a *special* R-algebra if A/I^n is a finite type algebra over $R/\mathfrak{m}^n$ for all positive integers $n \geq 0$. ◁

This does not depend on the ideal I, see **[40]**. This is exactly demanding that A is an algebra that is formally of finite type, without specifying that it is an adic algebra. The basic structural result is the following.

Lemma 4.42. *An R-algebra A is special if and only if it is isomorphic as a topological ring to a quotient of* $R\{T_1, \ldots, T_n\}[[S_1, \ldots, S_m]]$

Proof. See Lemma 1.2 of **[40]**. □

Definition 4.43. A *special formal scheme over* $\operatorname{spf} R$ is a locally Noetherian formal scheme and morphism $\mathfrak{X} \to \operatorname{spf} R$ that is locally formally of finite type.

The *special fiber* of a special formal scheme $\mathfrak{X}$ is the formal scheme $\tilde{\mathfrak{X}}$ over k given by the pullback $\mathfrak{X} \hat{\times}_{\operatorname{spf} R} \operatorname{spf} k$.

The *closed fiber* of a special formal scheme X is the scheme $\mathfrak{X}_s$ over k given by taking the quotient of $\mathscr{O}_{\mathfrak{X}}$ by the largest ideal of definition. ◁

Example 4.44. Let $\mathfrak{X} = \operatorname{spf} R[[x]]$, then special fiber is the formal scheme $\operatorname{spf} k[[x]]$, and the closed fiber is $\operatorname{spec} k$. ◁

Lemma 4.45. *If the structure morphism* $\mathfrak{X} \to \operatorname{spf} R$ *is adic then the closed fiber and the special fiber agree, at least after reduction.*

PROOF. Being adic over $\operatorname{spf} R$ means that the maximal ideal of R is an ideal of definition. Thus the special fiber admits a closed immersion into the closed fiber. $\square$

THEOREM 4.46 (THEOREM 2.1 (I) OF [**40**]). *Let X be a special formal scheme over $\operatorname{spf} R$, then the adically étale site of X is equivalent to the étale site of its closed fiber.*

We can set up the machinery for log geometry in this category as well.

DEFINITION 4.47. We pose the following definitions.

(1) A *pre-log structure* on a formal scheme $\mathfrak{X}$ is a sheaf of commutative topological monoids $\mathscr{M}_{\mathfrak{X}}$ for the adically étale site of $\mathfrak{X}$, and a morphism of sheaves of commutative topological monoids $\alpha_{\mathfrak{X}} : \mathscr{M}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}$.
(2) A *log structure* on a formal scheme $\mathfrak{X}$ is a pre-log structure such that the induced map $\alpha^{-1}(\mathscr{O}_{\mathfrak{X}}^{\times}) \to \mathscr{O}_{\mathfrak{X}}^{\times}$ is an isomorphism. Recall that the sheaf $\alpha^{-1}\mathscr{O}_{\mathfrak{X}}^{\times}$ carries the subspace topology.
(3) A *morphism of pre-log structures* $(\mathscr{M}, \alpha)$ and $(\mathscr{M}', \alpha')$ on a formal scheme $\mathfrak{X}$ is a morphism of sheaves of commutative topological monoids $f : \mathscr{M} \to \mathscr{M}'$ such that $\alpha' \circ f = \alpha$.
(4) A *morphism of log structures* is a morphism of pre-log structures.
(5) Given a pre-log structure $(\mathscr{M}, \alpha)$ on a formal scheme $\mathfrak{X}$, we may construct the *associated log structure* by forming the push-out as in the case for log schemes.
(6) A *log formal scheme* $\mathfrak{X} = (\mathfrak{X}^{\circ}, \mathscr{M}_{\mathfrak{X}}, \alpha_{\mathfrak{X}})$ is the data of a formal scheme $\mathscr{X}$ with a log structure $(\mathscr{M}_{\mathfrak{X}}, \alpha_{\mathfrak{X}})$ on it.
(7) A *morphism of log formal schemes* is a morphism of schemes $f : \mathfrak{X} \to \mathfrak{Y}$ and a morphism of sheaves of commutative topological monoids$f^{\#} : f^{-1}\mathscr{M}_{\mathfrak{Y}} \to \mathscr{M}_{\mathfrak{X}}$ such that the square
$$\begin{tikzcd} f^{\circ,-1}\mathscr{M}_{\mathfrak{Y}} \arrow[r] \arrow[d] & \mathscr{M}_{\mathfrak{X}} \arrow[d] \\ f^{\circ,-1}\mathscr{O}_{\mathfrak{Y}} \arrow[r] & \mathscr{O}_{\mathfrak{X}} \end{tikzcd}$$
commutes.
(8) Let $\mathscr{Y}$ be a log formal scheme and $f : \mathfrak{X} \to \mathfrak{Y}^{\circ}$ be a morphism of formal schemes. The *inverse image log structure* on $\mathfrak{X}^{\circ}$ is the log structure associated with the composition $f^{-1}\mathscr{M}_{\mathfrak{Y}} \to f^{-1}\mathscr{O}_{\mathfrak{Y}} \to \mathscr{O}_{\mathfrak{X}}$.
(9) A morphism of log formal schemes is called *strict* if the log structure on the source is isomorphic to the inverse image log structure of the underlying morphism of schemes.
(10) A log formal scheme $\mathfrak{X}$ is *quasi-coherent* if adically étale locally on $\mathfrak{X}^{\circ}$, the log structure is isomorphic to one associated with a constant sheaf of commutative monoids. It is *coherent* if

further the constant sheaves are of finitely generated and discrete commutative topological monoids.

(11) A log formal scheme is *fine* if it is coherent and the log structure is integral.

(12) A log formal scheme is *saturated* if it is coherent and the log structure is saturated.

(13) A log formal scheme is *fs* if it is fine and saturated.

◁

LEMMA 4.48. *Given a pre-log structure $\alpha : \mathscr{M} \to \mathscr{O}_{\mathfrak{X}}$ on a formal scheme $\mathfrak{X}$, the push-out of $\mathscr{M} \leftarrow \alpha^{-1}\mathscr{O}_{\mathfrak{X}}^{\times} \to \mathscr{O}_{\mathfrak{X}}^{\times}$ gives a log structure.*

PROOF. Write $\mathscr{M}^{\mathrm{a}}$ for the pushout of the diagram, and write $\alpha^{\mathrm{a}} : \mathscr{M}^{\mathrm{a}} \to \mathscr{O}_{\mathfrak{X}}$ for the associated morphism. What needs to be proven here is that the topology on the pushout agrees with the subspace topology on $\mathscr{O}_{\mathfrak{X}}^{\times}$. However a simple diagram chase shows that the pullback of $\mathscr{M}^{\mathrm{a}} \to \mathscr{O}_{\mathfrak{X}} \leftarrow \mathscr{O}_{\mathfrak{X}}^{\times}$ is a retract of $\mathscr{O}_{\mathfrak{X}}^{\times}$. Since the underlying sets are isomorphic via the retraction, it must be isomorphic as a space. □

Recall that also have a notion of saturation, sharpening, and groupification of sheaves of commutative monoids in any topos.

DEFINITION 4.49. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of coherent log formal schemes is *adically log étale* if it is locally adically of finite type, and for each $x \in \mathfrak{X}$ there are étale neighborhoods $\mathfrak{U}$ of x and $\mathfrak{V}$ of $f(x)$ with charts $P_{\mathfrak{U}}$ and $P_{\mathfrak{V}}$ such that we can factor the morphism

$$\begin{tikzcd} \mathfrak{U} \arrow[d, "f'"] & \\ \mathfrak{V} \times_{\mathbb{Z}[P_{\mathfrak{V}}]} \operatorname{spec} \mathbb{Z}[P_{\mathfrak{U}}] \arrow[r] \arrow[d] & \operatorname{spec} \mathbb{Z}[P_{\mathfrak{U}}] \arrow[d] \\ \mathfrak{V} \arrow[r] & \operatorname{spec} \mathbb{Z}[P_{\mathfrak{V}}] \end{tikzcd}$$

such that f' is adically étale, and $P_{\mathfrak{V}} \to P_{\mathfrak{U}}$ has finite kernel and cokernel with orders coprime to the characteristics of $\mathfrak{Y}$. ◁

LEMMA 4.50. *A morphism of special formal* spf R*-schemes is log étale if and only if we can adically étale locally factor the morphism as*

$$\begin{tikzcd}
 & \mathfrak{U} \arrow[d, "f'"] & \\
 & \mathfrak{V}\hat{\times}_{R[P_{\mathfrak{V}}]} \operatorname{spf} R[P_{\mathfrak{U}}] \arrow[r] \arrow[d] & \operatorname{spf} R[P_{\mathfrak{U}}] \arrow[d] \\
 & \mathfrak{V} \arrow[r] & \operatorname{spf} R[P_{\mathfrak{V}}]
\end{tikzcd}$$

where the kernel and cokernel of $P_{\mathfrak{V}} \to P_{\mathfrak{U}}$ has finite kernel and cokernel with orders coprime to the characteristics of $\mathfrak{Y}$.

Proof. The square

$$\begin{tikzcd}
\operatorname{spf} R[P_{\mathfrak{U}}] \arrow[r] \arrow[d] & \operatorname{spec} \mathbb{Z}[P_{\mathfrak{U}}] \arrow[d] \\
\operatorname{spf} R[P_{\mathfrak{V}}] \arrow[r] & \operatorname{spec} \mathbb{Z}[P_{\mathfrak{V}}]
\end{tikzcd}$$

is a pull-back square in formal schemes. Chasing the universal properties gives an bijection between commutative diagrams in the definition and in the statement of the lemma. □

Definition 4.51. We pose the following definitions.

(1) Let $\mathfrak{X}$ be a fs log formal scheme admitting a global chart $P \to \mathscr{O}_{\mathfrak{X}}$, and $I \triangleleft P$ an ideal of the monoid. Then the *log blow up of $\mathfrak{X}$ along I* is the log scheme obtained by first performing the blowup of $\operatorname{bl}_I \operatorname{spec} \mathbb{Z}[P]$, saturating the resulting log formal scheme, and then taking the fs pullback

$$\operatorname{bl}_I \mathfrak{X} := \mathfrak{X}\hat{\times}_{\mathbb{Z}[P]}(\operatorname{bl}_I \mathbb{Z}[P])^{\mathrm{sat}}$$

(2) A fs log formal scheme is *log regular* if for every geometric point $x \in \mathfrak{X}^{\circ}$ the ring $\mathscr{O}_{\mathfrak{X},x}/I(\mathscr{M}_{\mathfrak{X}}, x)$ is regular and the following equation holds

$$\dim \mathscr{O}_{\mathfrak{X},x} = \operatorname{rank} \overline{(\mathscr{M}_{\mathfrak{X},x})}^{\mathrm{gp}} + \dim \mathscr{O}_{\mathfrak{X},x}/I(\mathscr{M}_{\mathfrak{X}}, x)$$

(3) A morphism of log formal schemes is *adically Kummer étale* if it is adically log étale, and the morphisms of monoids in the diagram can be taken to be injective.

◁

Lemma 4.52. *Let $\mathfrak{X}$ be a fs log formal scheme. If the log structure on $\mathfrak{X}$ admits a chart by a fine saturated monoid P in some neighborhood of a point $x \in \mathfrak{X}$ then there are canonical isomorphisms*

$$P^{\mathrm{gp}} \xrightarrow{\sim} \overline{(\mathscr{M}_{\mathfrak{X},x})}^{\mathrm{gp}} \xleftarrow{\sim} (\mathscr{M}_{\mathfrak{X},x}/\mathscr{O}_{\mathfrak{X},x})^{\mathrm{gp}}$$

We also introduce the following notion.

Definition 4.53. A fs log special formal scheme $\mathfrak{X}$ over $\mathrm{spf}\, R$ is ***algebraizably log smooth*** if each point admits an adically étale neighborhood $\mathfrak{U} \to \mathfrak{X}$ such that

(1) there is an fs log scheme U equipped with a log smooth log morphism to $\mathrm{spec}\, R$,
(2) there is a morphism $\phi_{\mathfrak{U}} : \mathfrak{U} \to \hat{U}_{U_s}$,
(3) the morphism $\phi_{\mathfrak{U}}$ induces an isomorphism of $\mathfrak{U}$ with the completion of $\hat{U}_{U_s}$ along some closed subset,
(4) and the restricted log structure $\mathscr{M}_{\mathfrak{X}}|_{\mathfrak{U}}$ is isomorphic via the canonical homomorphism to the pull-back of the canonical log structure on $\hat{U}_{U_s}$.

◁

Lemma 4.54. *The following hold,*

(1) *The categories of log formal schemes and fs log formal schemes admit finite limits.*
(2) *Adically Kummer étale morphisms are stable under fs pullbacks.*
(3) *The distinguished formal schemes of* **[41]** *are algebraizably log smooth.*

Proof. We break up the proof.

(1) The case for log formal schemes follows as for log schemes. The case of fs log formal schemes follows from the case for fs log schemes, since the log formal scheme is determined by its diagram of reductions. We can choose $\mathbb{N}$-indexed diagrams for each of the log formal schemes, and then for each $i \in \mathbb{N}$ take the fs limit. Taking the limit of the resulting $\mathbb{N}$-indexed diagram of schemes gives a formal scheme.
(2) The reduction of an fs pullback is an fs pullback of log schemes, so this follows from the case for log schemes.
(3) The only thing that does not follow immediately is the compatibility of the pull-back of the log structure and the canonical log structure. This is a lemma in **[41]**.

□

Definition 4.55. The *adically Kummer étale site* of a fs log formal scheme $\mathfrak{X}$ is the category whose objects are adically Kummer étale morphisms $\mathfrak{U} \to \mathfrak{X}$, morphisms are commutative triangles, and the coverings for the pretopology are jointly surjective families. ◁

LEMMA 4.56. *Let $\mathfrak{X}$ be a coherent log special formal scheme over* $\operatorname{spf} R$*. Then the adically Kummer étale site of* $\mathfrak{X}$ *is equivalent to the Kummer étale site of the special fiber with induced log structure.*

PROOF. This follows from the result of Berkovich that the adically étale sites are equal, since we can reduce to the case where $\mathfrak{X}$ has a global chart by some finitely generated monoid. □

PROPOSITION 4.57. *Let $\mathfrak{X}$ be an fs log special formal scheme over* $\operatorname{spf} R$*, with $\mathfrak{X}^\circ$ affine. Assume that the log structure is given by a global chart $\mathscr{P} \to \mathscr{O}_{\mathfrak{X}}$, then $\mathfrak{X}$ is adically log regular if and only if the scheme $X = \operatorname{spec} \mathscr{O}_{\mathfrak{X}}(\mathfrak{X})$ is log regular.*

PROOF. The log structure on X is the one associated with the chart $\mathscr{P} \to \mathscr{O}_{\mathfrak{X}}(\mathfrak{X}) = \mathscr{O}_X(X)$. First, by Proposition 7.1 of [**33**], being log regular generalizes at the level of points. To prove necessity, we must then only show that X is log regular at every closed point. The closed points are naturally in bijection the closed points of $\mathfrak{X}$ via the formal immersion $\mathfrak{X} \hookrightarrow X$. Choose a closed point $x \in X$, so that we get a homomorphism of commutative monoids $P \to \mathscr{O}_{X,x}$. Log regularity for the scheme X is the condition that for the ideal $I_x = \left(P \setminus (P \cap \mathscr{O}_{X,x}^\times)\right)\mathscr{O}_{X,x}$ the following two conditions hold.

(1) The ring $\mathscr{O}_{X,x}/I_x$ is regular,
(2) and the following equation holds.

$$\dim \mathscr{O}_{X,x} = \operatorname{rank}\left(P_x^{\mathrm{gp}} \setminus (P \cap \mathscr{O}_{X,x}^\times)\right) + \dim \mathscr{O}_{X,x}/I_x$$

The proposition then reduces to show the following statements.

(1) The scheme's local ring $\mathscr{O}_{X,x}/I_x$ and the formal scheme's local ring $\mathscr{O}_{\mathfrak{X},x}/I_x$ have isomorphic completions.
(2) The groups $P_x^{\mathrm{gp}} \setminus (P \cap \mathscr{O}_{X,x}^\times)$ and $P_x^{\mathrm{gp}} \setminus (P \cap \mathscr{O}_{\mathfrak{X},x}^\times)$ are isomorphic.

Given these two statements, it becomes clear that one local ring is regular if and only if the other is, and since dimension is preserved by completion we show that the dimension terms in both equations are equal.

(1) There is a natural inclusion $\mathscr{O}_{X,x} \to \mathscr{O}_{\mathfrak{X},x}$, and since the map $P \to \mathscr{O}_{\mathfrak{X},x}$ factors through this inclusion, we determine that I_x has the same set of generators regardless of which local ring is used. We then have two short exact sequences

$$0 \to I_x\mathscr{O}_{\mathfrak{X},x} \to \mathscr{O}_{\mathfrak{X},x} \to \mathscr{O}_{\mathfrak{X},x}/I_x \to 0$$

and

$$0 \to I_x \to \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/I_x \to 0$$

where the first can be seen as an exact sequence of finitely presented $\mathscr{O}_{\mathfrak{X},x}$-modules, and the second as an exact sequence of finitely presented $\mathscr{O}_{X,x}$-modules. To conclude that the two rings have isomorphic completion, we note that the completion of $\mathscr{O}_{\mathfrak{X},x}/I_x$ as an $\mathscr{O}_{\mathfrak{X},x}$-module agrees with its completion as a local ring, and similarly the completion of $\mathscr{O}_{X,x}/I_x$ as an $\mathscr{O}_{X,x}$-module agrees with its completion as a local ring. Combining these statements and the fact that there is a natural isomorphism $\hat{\mathscr{O}}\mathfrak{X}, x \simeq \hat{\mathscr{O}}_{X,x}$ gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_x\hat{\mathscr{O}}_{\mathfrak{X},x} \arrow[r] & \hat{\mathscr{O}}_{\mathfrak{X},x} \arrow[r] & \widehat{\mathscr{O}_{\mathfrak{X},x}/I_x} \arrow[r] & 0 \\ 0 \arrow[r] & I_x \arrow[r] \arrow[u] & \hat{\mathscr{O}}_{X,x} \arrow[r] \arrow[u] & \widehat{\mathscr{O}_{X,x}/I_x} \arrow[r] & 0 \end{tikzcd}$$

where the vertical morphisms are isomorphisms. Thus by the universal property of the ring quotient, the two rings are isomorphic.

(2) The group $P \cap \mathscr{O}_{X,x}^{\times}$ is isomorphic to $P \cap \mathscr{O}_{\mathfrak{X},x}^{\times}$, since the morphism $\mathscr{O}_{X,x} \to \mathscr{O}_{\mathfrak{X},x}$ does not invert any elements that are not already invertible.

□

Corollary 4.58. *Let $f : X \to Y$ be a morphism of fs log schemes flat and finite type over* $\operatorname{spec} R$*, and $Z \subset Y_s$ a closed subset. Assume further that the log structures on both schemes are vertical. Then the induced morphism $\hat{f} : \hat{X}_{Z'} \to \hat{Y}_Z$ is adically Kummer étale if and only if f is Kummer étale.*

Lemma 4.59. *Let A be a Noetherian adic ring with ideal of definition $I \lhd A$. Write $X := \operatorname{spec} A$ and $X_{\mathrm{red}} = \operatorname{spec} A/I$. Then the morphisms of sites*

$$\mathbf{site}\, X_{\mathrm{red,\, \acute{e}t}} \to \mathbf{site}\, \mathfrak{X}_{\mathrm{a\acute{e}t}} \to \mathbf{site}\, X_{\acute{e}t}$$

induces equivalences on the profinitely completed shapes of the associated ∞-topoi,

$$\widehat{\mathrm{\acute{e}t}}\, X_{\mathrm{red}} \to \mathrm{a\acute{e}t}\ \mathfrak{X} \to \widehat{\mathrm{\acute{e}t}}\, X_{\acute{e}t}.$$

Proof. The first two topoi are actually equivalent, so the first morphism of sites is *a fortiori* a shape equivalence. We are thus reduced to showing that the usual profinitely completed étale homotopy types of X_{red} and X agree. We can assume X_{red} is connected, which forces X to be connected as well. We must first check that their étale fundamental groups agree, for which we may use any geometric point in X_{red}, since the categories of finite étale coverings are equivalent by Lemma 09ZL of [**2**].

For the cohomology, we actually have a stronger statement that the cohomology of any abelian torsion sheaf on X agrees with the cohomology of its preimage on X_{red} by Gabber's "Affine Analogue of Proper Base Change", Theorem 09ZI of [**2**]. $\square$

THEOREM 4.60. *Let A be a Noetherian adic ring with ideal of definition $I \lhd A$. Assume that there is an fs monoid $P \to A \setminus A^\times$ so that the scheme (resp. formal scheme, resp. scheme) $X = \operatorname{spec} A$ (resp. $\mathfrak{X} = \operatorname{spf} A$, resp. $X_{\mathrm{red}} = \operatorname{spec} A/I$) is an fs Noetherian log scheme (resp. fs Noetherian log formal scheme, resp. fs Noetherian log scheme). Then the morphisms of sites*

$$\mathbf{site}\, X_{\mathrm{red,\,k\acute{e}t}} \to \mathbf{site}\, \mathfrak{X}_{\mathrm{ak\acute{e}t}} \to \mathbf{site}\, X_{k\acute{e}t}$$

induce equivalences on the $\{p\}^c$-profinitely completed shapes of the associated ∞-topoi,

$$\widehat{\mathrm{k\acute{e}t}}_{\{p\}^c}\, X_{\mathrm{red}} \to \widehat{\mathrm{ak\acute{e}t}}_{\{p\}^c}\, \mathfrak{X} \to \widehat{\mathrm{k\acute{e}t}}_{\{p\}^c}\, X_{\acute{e}t}.$$

PROOF. We have a commutative diagram of morphisms of sites

$$\begin{array}{ccccc} siteX_{\mathrm{red,\,k\acute{e}t}} & \longrightarrow & \mathbf{site}\, \mathfrak{X}_{\mathrm{ak\acute{e}t}} & \longrightarrow & \mathbf{site}\, X_{k\acute{e}t} \\ \downarrow{\scriptstyle j_{\mathrm{red}}} & & \downarrow{\scriptstyle j_{\mathfrak{X}}} & & \downarrow{\scriptstyle j_X} \\ \mathbf{site}\, X_{\mathrm{red,\,\acute{e}t}} & \longrightarrow & \mathbf{site}\, \mathfrak{X}_{\mathrm{a\acute{e}t}} & \longrightarrow & \mathbf{site}\, X_{\acute{e}t} \end{array}$$

Since the morphisms of sites coming from the inclusion $X_{\mathrm{red}} \to \mathfrak{X}$ give equivalences of topoi in both the étale Kummer étale topologies, it is enough to show that the inclusion $X_{\mathrm{red}} \to X$ induces a profinite shape equivalence. We still have an equivalence of the categories of finite torsors in the adically Kummer étale topology on $\mathfrak{X}$ and the Kummer étale topology on X, and so we just need to check the cohomological statement. But we can actually use Gabber's result again.

By Theorem 2.4 of [**42**], we have functorial isomorphisms of the higher direct image functors of both the j.

$$R^q j_{\mathrm{red},*}\mathscr{L} \cong \bigwedge^q P^{\mathrm{gp}} \otimes_{\underline{\mathbb{Z}}} \mathscr{L}(-q) \qquad R^q j_{X,*}\mathscr{L} \cong \bigwedge^q P^{\mathrm{gp}} \otimes_{\underline{\mathbb{Z}}} \mathscr{L}(-q).$$

Specifically for any sheaf $\mathscr{L}$ of torsion abelian groups whose orders are invertible on X (given our assumptions this is only the prime p) we have the claimed isomorphisms, which are further functorial in the sheaf $\mathscr{L}$. Since P was a fine monoid, P^{gp} is a finitely generated group, and so its exterior powers are also finitely generated. The tensor product of a finitely generated abelian group and a torsion group whose order is coprime to p is again a torsion group of order coprime to p. Now by Gabber's result, Theorem 09XI of [**2**], the cohomology of $R^q j_{X,*}\mathscr{L}$ agrees with that of its pullback to X_{red}. But pullbacks commute with tensor products and alternating powers of sheaves, and the pullback of P^{gp} on X is exactly the groupification of the log structure on X_{red}. This implies that the higher direct images all functorially agree. Thus the $E_2^{s,t}$ terms of the Grothendieck spectral sequence for the composition of derived functors agree, and the desired cohomologies agree. $\square$

In fact the above proof gives an equivalence for any abelian torsion sheaf on the Kummer étale site with orders coprime to p.

CHAPTER 5

ÉTALE HOMOTOPY THEORY IN NON-ARCHIMEDEAN GEOMETRY

"I've found the clay!" – Andrei Tarkovsky, *Andrei Rublev*

There are several formalisms of non-archimedean geometry. The earliest formalism is that of Tate's rigid analytic spaces, first discovered via Tate's uniformization theorem which gave a p-adic uniformization of some strictly semi-stable elliptic curves. The main issue with Tate's formalism is that the underlying topological spaces are still quite pathological, and one instead has to work with a suitable Grothendieck topology. Berkovich's formalism of non-archimedean analytic spaces fixes this deficiency, as does Huber's slightly more recent notion of adic spaces. We will first review the theory of Berkovich spaces before that of adic spaces. Since adic spaces effectively subsume the theory of rigid spaces, we will not delve into the formalism of rigid spaces.

1. Non-archimedean analytic spaces

We begin with the base definitions we need, one can reference [**19**].

DEFINITION 5.1. Let R be a Banach ring. The *normative spectrum of* R is the set of all bounded multiplicative seminorms on the Banach ring R up to equivalence of seminorms,

$$\operatorname{sp} R = \{|-| : R \to [0,\infty) \mid |-| \text{ is a bounded multiplicative seminorm }\}/\sim$$

we further topologize this set by giving it the weakest topology so that the family of functions given by evaluation of the seminorms $\mathrm{eval}_r : \operatorname{sp} R \to [0,\infty)$ are all continuous.

Given a point $x \in \operatorname{sp} R$, any choice of representing seminorm $|-|_x$ determines a prime ideal $\ker|-|_x = \mathfrak{p} \triangleleft R$. This is easily checked to only depend on the equivalence class of the seminorm. Further the seminorm induces a norm on the field of fractions $FF(R/\mathfrak{p})$. Completing with respect to the norm induced by $|-|_x$ gives the *completed residue field* $\mathscr{H}(x)$. $\triangleleft$

As justification for why this is a desirable space to take as the analogue of an affine scheme, we have the following two results.

Theorem 5.2. *The normative spectrum of a Banach ring R is non-empty, Hausdorff, and compact.*

Proof. See Theorem 1.2.1 of **[19]** for details.

The general idea of the proof is that $\operatorname{sp} R$ embeds as a closed subspace of $\prod \operatorname{sp} K(x)$ where the $K(x)$ are a family of Banach fields. This reduces the claim to showing that a Banach field has a compact and Hausdorff normative spectrum. □

Lemma 5.3. *For R a Banach ring, an element r is invertible if and only if eval_r never takes the value zero on* $\operatorname{sp} A$.

Proof. A ring element r is invertible only if it lies in no proper ideal, and so it cannot be in the kernel of any valuation. Conversely, maximal ideals in Banach rings are necessarily closed. This implies that for any maximal $\mathfrak{m} \triangleleft R$, the ring $R/\mathfrak{m}$ is a non-Archimedean field and a Banach algebra over R. In particular, if r is not invertible it is contained in a maximal ideal $\mathfrak{m}$ with corresponding valuation $v_{\mathfrak{m}}$. Then $\operatorname{eval}_r(v_{\mathfrak{m}}) = 0$ as desired. □

Now we will restrict our scope quite a bit. Our base rings will be non-archimedean fields, and the basic algebras over them are the family of convergent power series rings

$$K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} = \{\sum a_I T^I \in K[[T_1, \dots, T_N]] \mid \|r^I a_I\|_K \to 0 \text{ as } I \to \infty\}$$

Note that this already adds many new rings compared to rigid analytic geometry. If any of the r_i above are not in the rational vector space spanned by the image of K^* under the norm map, then this ring is not directly accessible to rigid geometry.

Definition 5.4. We pose the following definitions.

(1) A *K-affinoid algebra* is a Banach algebra A/K such that there is an admissible epimorphism

$$K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$$

(2) In the above, if we can find an admissible epimorphism $K\{\} \to A$ every $r_i = 1$, we call A a *strictly K-affinoid algebra.*

As a warning to the reader, epimorphisms in the category of rings are not necessarily surjections.

(3) A *morphism of K-affinoid algebras* is a bounded homomorphism of Banach algebras.

(4) A morphism of K-affinoid algebras is said to be *inner* when it is an inner morphism of K-algebras.

(5) Let $V \subset \operatorname{sp} A$ be a closed subset. Then V is said to be an *affinoid domain* if there is a map of K-affinoid algebras $A \to B$ such that $\operatorname{sp} B \subset V$ and B is initial in the category of A-algebras with to this property. If B is a strictly K-affinoid algebra, we call V a *strictly affinoid domain.*

$\triangleleft$

In fact the class of affinoid domains give a Grothendieck topology, since they are closed under finite intersections. We will work up to that statement with some more basic results.

LEMMA 5.5. *Let A be a K-affinoid algebra. Then,*

(1) *the underlying ring of A is Noetherian,*

(2) *all ideals of A are topologically closed,*

(3) *the algebra A is strictly K-affinoid if and only if the spectral radius $\rho(a) = \inf_{n\geq 0} \|a^n\|_A^{1/n}$ of every element is a rational power of the norm of some element in K, i.e. $\rho(a) \in \sqrt{\|K^*\|_K}$ for every $a \in A$,*

(4) *the completed tensor product of affinoid algebras is again affinoid,*

(5) *if $V \subset \operatorname{sp} A$ is an affinoid domain with corresponding affinoid algebra A_V, then $\operatorname{sp} A_V = V$,*

(6) *if $V \subset \operatorname{sp} A$ is an affinoid domain, then A_V is a flat A-algebra,*

(7) *and if V and W are affinoid domains in $\operatorname{sp} A$, then $V \cap W$ is again an affinoid domain.*

PROOF. We separate the proofs.

(1) See Proposition 2.1.3 of [**19**].

(2) See Proposition 2.1.3 of [**19**].

(3) See Proposition 2.1.6 of [**19**].

(4) Let the two algebras be called A and B, with given admissible epimorphisms $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$ and $K\{s_1^{-1}T_1', \dots, s_m^{-1}T_m'\} \to B$. The tensor product of an epimorphism is an epimorphism, and the tensor product of admissible algebra maps is again admissible. Since

$$K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}\hat{\otimes}_K K\{s_1^{-1}T_1', \dots, s_m^{-1}T_m'\} \simeq K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, s_1^{-1}T_1', \dots, s_m^{-1}T_m'\}$$

we deduce that the map

$$K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n, s_1^{-1}T_1', \dots, s_m^{-1}T_m'\} \to A\hat{\otimes}_K B$$

(5) See Proposition 2.2.4 of [**19**].

(6) See Proposition 2.2.4 of [**19**].

(7) If $V = \operatorname{sp} A_V$ and $W = \operatorname{sp} A_W$, then we claim that $V \cap W = \operatorname{sp} A_V \hat{\otimes}_A A_W$. This follows by the universal property of affinoid domains. Assume that B is an affinoid algebra under A whose normative spectrum lies within $V \cap W$. Since A_V is initial among A-algebras whose normative spectrum lies within V, and A_W among those whose normative spectrum lies within W, we get unique algebra maps from both A_V and A_W into B, which gives a unique map from $A_V \hat{\otimes}_A A_W \to B$, thus satisfying the universal property.

□

Corollary 5.6. *The category whose objects are affinoid domains in an affinoid space, and whose morphisms are inclusions satisfies the axioms of a Grothendieck pretopology.*

Proof. The coverings are covering families of affinoid domains. The previous lemma shows that they are closed under finite fiber products, which in this category are just intersections of the affinoid domains. □

We have the germs of abstract algebraic geometry already with just this Grothendieck pretopology, as was first discovered in rigid analytic geometry.

Theorem 5.7. *Let* $\operatorname{sp} A$ *be an affinoid space.*

(1) Tate's theorem*: Let M an A-module (resp. a finite Banach A-module). Then the Čech cohomology of M with respect to the Grothendieck pretopology of affinoid domains is trivial (resp. the Čech complex is exact and has admissible boundary homomorphisms).*

(2) Kiehl's theorem*: Every descent datum for finite Banach A-modules is effective.*

Proof. (1) See Proposition 2.2.5 of [**19**].

(2) See Theorem 3 of 9.4.3 from [**43**] for the case when A is strictly analytic. If A is not, then one uses the usual trick of base change and descent.

□

We can almost define general K-analytic spaces, we need a few preparatory definitions.

Definition 5.8. We pose the following definitions,

(1) A *locally Banach ringed space* is a locally ringed space $(X, \mathscr{O}_X)$ such that $\mathscr{O}_X$ takes values in the category of Banach rings with bounded morphisms.

(2) A *morphism of locally Banach ringed spaces* is a morphism of locally ringed spaces such that the component morphisms are all bounded morphisms of Banach rings.

(3) A *quasi-net* $\{U_i\}_{i \in I}$ on a topological space X is a family of subsets of X with the property that for any $x \in X$, there is a finite subset $\{U_{i_1}, \ldots, U_{i_n}\}$ such that x is an element of the intersection $\cap_{j=1}^n U_{i_j}$, and the union $\cup_{j=1}^n U_{i_j}$ contains an open neighborhood of x.

(4) An *affinoid K-analytic space* is the locally Banach ringed space given by $(\operatorname{sp} A, \mathscr{O}_{\operatorname{sp} A})$ where the structure sheaf is defined as follows. For an open set $U \subset \operatorname{sp} A$ we set the sections over U to be the colimit over

$$\lim_{U \subset V_1 \cup \ldots \cup V_n} \ker \left(\prod A_{V_i} \to \prod A_{V_i \cap V_j} \right)$$

where every V_i is an affinoid domain.

(5) A *K-analytic space* is a locally Banach ringed space which is has a quasi-net of closed subsets isomorphic to an affinoid K-analytic space.

(6) A *morphism of K-analytic spaces* $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of locally ringed spaces f which further satisfies that for all $x \in X$, there are affinoid subspaces $V_1, \ldots, V_n$ of X and $W_1, \ldots, W_n$ of Y such that
   (a) the given point x lies in every V_i,
   (b) the point $f(x)$ lies in every W_i,
   (c) the union $\cup V_i$ contains an open neighborhood of x,
   (d) the union $\cup W_i$ contains an open neighborhood of $f(x)$,
   (e) the image $f(V_i) \subset W_i$,
   (f) the restrictions $\{f|_{V_i} \mid V_i \to W_i\}$ form a compatible family of morphisms of affinoid K-analytic spaces.

(7) The *category of K-analytic spaces*, denoted An_K, is the category whose objects are K-analytic spaces and whose morphisms are morphisms of K-analytic spaces.

(8) For convenience, we introduce this new piece of terminology. Given a morphism of K-analytic spaces $f : X \to Y$ and a point $x \in X$, an *affinoid framing of f at x* is the data of affinoid subspaces $V_1, \ldots, V_n$ of X and affinoid subspaces $W_1, \ldots, W_n$ of Y satisfying
   (a) the given point x lies in every V_i,
   (b) the point $f(x)$ lies in every W_i,
   (c) the union $\cup V_i$ contains an open neighborhood of x,

(d) the union $\cup W_i$ contains an open neighborhood of $f(x)$,

(e) the image $f(V_i) \subset W_i$,

(f) the restrictions $\{f|_{V_i} \,|\, V_i \to W_i\}$ form a compatible family of morphisms of affinoid K-analytic spaces.

(9) Given a morphism of affinoid K-analytic spaces $f : \operatorname{sp} B \to \operatorname{sp} A$, the ***relative interior of the morphism of affinoid K-analytic spaces f*** $\operatorname{Int}(\operatorname{sp} B/\operatorname{sp} A)$ is the open subset of $\operatorname{sp} B$ consisting of points $x \in \operatorname{sp} B$ such that the induced morphism to the completed local ring $B \to \mathscr{H}(x)$ is inner with respect to A.

(10) Given a morphism of K-analytic spaces $f : X \to Y$, the ***relative interior of f*** $\operatorname{Int}(X/Y)$ is the open subset of points $x \in X$ such that there is an affinoid framing of f at x with affinoid domains $V_1, \ldots, V_n \subset X$ and $W_1, \ldots, W_n \subset Y$ with the property that x is in the relative interior of each morphism of affinoid K-analytic spaces $f : V_i \to W_i$.

(11) Given a morphism of K-analytic space $f : X \to Y$, the ***relative boundary of f*** $\partial(X/Y)$ is the set-theoretic complement of the relative interior of f.

$\triangleleft$

We will return to the more subtle notions of the relative interior of a morphism after a dicussion on the less subtle notions of the Grothendieck topology generated by affinoid domains. The most fundamental result is that K-analytic spaces form a sheaf for the Grothendieck topology of affinoid domains.

Lemma 5.9. *Let X be a K-analytic space. For any quasi-net $\{Y_i\}$ of affinoid domains in X and any K-analytic space Z, the sequence*

$$\operatorname{Hom}_{\mathrm{An}_K}(X, Z) \to \prod \operatorname{Hom}_{\mathrm{An}_K}(Y_i, Z) \rightrightarrows \prod \operatorname{Hom}_{\mathrm{An}_K}(Y_i \times_X Y_j, Z)$$

is an equalizer diagram.

Proof. See Proposition 1.3.2 of [**44**]. $\square$

As noted above there is a Grothendieck topology on a given K-analytic space, and the locally compact topology. To build a morphism of their corresponding topoi, we will define an intermediate site which can be compared to both.

Definition 5.10. Let X be a K-analytic space. A subset $Y \subset X$ is an *analytic domain* in X, if every point $y \in Y$ has affinoid domains $V_1, \ldots, V_n$ of X, with each V_i contained in Y such that $y \in \cap V_i$ and $\cup V_i$ contains an open neighborhood of Y. $\triangleleft$

LEMMA 5.11. *The following statements hold.*

(1) *Arbitrary unions of analytic domains are analytic domains. Finite intersections of analytic domains are analytic domains. The family of analytic domains gives an alternate topology on X.*
(2) *The preimage of an analytic domain under a morphism of K-analytic spaces is an analytic domain.*
(3) *An analytic domain itself is a K-analytic space.*
(4) *All open subsets of a K-analytic space are analytic domains.*
(5) *The topos of sheaves τX_G on the analytic domain topology is equivalent to the topos of sheaves on the Grothendieck topology of affinoid domains.*
(6) *The obvious inclusion of the usual topology on a K-analytic space X into the analytic domain topology induces a morphism of sites $\pi_X : \mathbf{site}\, X_G \to \mathbf{site}\, X$.*

PROOF. (1) The claim for unions is clear. For intersections, we will prove the claim. Let Y and Y' be analytic domains in X, and let $y \in Y \cap Y'$. Then we have $V_1, \ldots, V_n$ in Y and $V_1', \ldots, V_m'$ in Y' satisfying the above definition. We may take all pairwise intersections between the two sets, $V_1 \cap V_1', \ldots, V_1 \cap V_m', V_2 \cap V_1' \ldots, V_n \cap V_m'$ which satisfies the above definition.
(2) We have to lift the morphism to a strict morphism of K-analytic spaces with a given atlas. Since we are not discussing atlases, we will omit the remainder of the proof.
(3) The assumptions of analytic domain are exactly that it has a quasi-net of closed affinoid analytic subspaces.
(4) The system of all affinoids is *dense*, in the sense that every point in an analytic space has a system of affinoids containing open neighborhoods which generate the topology. See the end of Remark 1.2.2 of [**44**].
(5) The only difficult observation that needs to be made is that the class of special subsets is cofinal in the class of analytic domains.
(6) Morphisms of topological spaces always induce morphisms of the underlying sites.

□

The first major result comparing the two topoi is for coherent modules in both. We first define the relevant categories.

DEFINITION 5.12. Let X be a K-analytic space. We write $Mod(X)$ for the category of $\mathscr{O}_X$-modules in τX, and $Mod(X_G)$ for the category of $\mathscr{O}_{X_G} := \pi_X^{-1}\mathscr{O}_X$-modules. We write $Coh(X)$ for the category of *coherent* $\mathscr{O}_X$ *modules* in τX and $Coh(X_G)$ for the category of coherent $\mathscr{O}_{X_G}$ modules. ◁

THEOREM 5.13. *Let X be a K-analytic space. Then,*

(1) *the functor $\pi_X^{-1} : Mod(X) \to Mod(X_G)$ is fully faithful,*
(2) *the functor $\pi_X^{-1} : Coh(X) \to Coh(X_G)$ is an equivalence,*
(3) *a coherent module over $\mathscr{O}_X$ is locally free if and only if its inverse image is a locally free $\mathscr{O}_{X_G}$-module.*
(4) *for any sheaf of abelian groups $\mathscr{F}$ in τX, we have isomorphisms on sheaf cohomology*

$$H^q(\tau X, \mathscr{F}) \simeq H^q(\tau X_G, \pi_X^{-1}\mathscr{F})$$

(5) *if X is paracompact then for any sheaf of groups $\mathscr{G}$ we have isomorphisms on non-abelian cohomology*

$$\check{H}^1(\tau X, \mathscr{G}) \simeq \check{H}^1(\tau X_G, \pi_X^{-1}\mathscr{G})$$

PROOF. (1) See Proposition 1.3.4 of [**44**].
(2) See Proposition 1.3.4 of [**44**].
(3) See Proposition 1.3.4 of [**44**].
(4) See Proposition 1.3.6 of [**44**].
(5) See Proposition 1.3.6 of [**44**].

□

We take a slight detour back to studying the category of K-analytic spaces. We first give a rigorous definition of gluing data, which is what the reader expects.

DEFINITION 5.14. ***Affinoid gluing data*** for a K-analytic space is a triple $(\{X_i\}_{i\in I}, \{X_{i,j}\}_{i,j\in I}, \{\iota_1^{i,j}, \iota_2^{i,j}\}_{i,j\in I})$ whose elements are

(1) an indexed family of affinoid K-analytic spaces, $\{X_i\}_{i\in I}$,
(2) a second family of affinoid K-analytic spaces corresponding to the pairwise intersections $\{X_{i,j}\}_{i,j\in I}$,
(3) and a family of inclusions $\iota_1^{i,j} : X_{i,j} \to X_i$ and $\iota_2^{i,j} : X_{i,j} \to X_j$ witnessing the $X_{i,j}$ as affinoid domains in the X_i and X_j.

A *gluing* of affinoid gluing data is a K-analytic space X along with a pair $(\{\gamma_i\}_{i\in I}, \{\gamma_{i,j}\}_{i,j\in I})$ consisting of

(1) morphisms of K-analytic spaces $\gamma_i : X_i \to X$ witnessing X_i as an affinoid domain in X,
(2) morphisms of K-analytic spaces $\gamma_{i,j} : X_{i,j} \to X$ witnessing $X_{i,j}$ as an affinoid domain in X,

which further satisfy that

(1) the three maps $\gamma_{i,j}$, $\gamma_i \circ \iota_1^{i,j}$ and $\gamma_j \circ \iota_2^{i,j}$ agree as maps from $X_{i,j}$ to X for every pair,
(2) the X_i cover X,
(3) and $X_i \cap X_j = X_{i,j}$ as subsets of X.

◁

There are two major cases when a gluing exists for gluing data.

LEMMA 5.15. *Let* $(\{X_i\}_{i\in I}, \{X_{i,j}\}_{i,j\in I}, \{\iota_1^{i,j}, \iota_2^{i,j}\}_{i,j\in I})$ *be affinoid gluing data. Then a gluing exists for it in either of the following cases,*

(1) *every* $X_{i,j}$ *is open in* X_i *and* X_j*,*
(2) *every* $X_{i,j}$ *is an analytic domain in* X_i *and* X_j*, and furthermore for every fixed* i *there are only finitely many* j *such that* $X_{i,j}$ *is non-empty.*

PROOF. See Proposition 1.3.3 of [**44**]. □

Using this we deduce the following proposition.

PROPOSITION 5.16. *The category* An_K *of* K*-analytic spaces admits fiber products.*

PROOF. The proof is essentially in the same spirit of the analogous result for schemes: one breaks the schemes into gluing data, performs the fiber product as a tensor product, and then reassembles the tensor products into a compatible family gluing data. The major difficulty is this only directly works for paracompact K-analytic spaces, however one can reduce the general case to this one. See the proof of Proposition 1.4.1 of [**44**] for details. □

2. Properties of morphisms of K-analytic spaces

We now come to the section where we discuss properties of morphisms of K-analytic spaces.

DEFINITION 5.17. We pose the following definitions.

(1) A morphism $f : X \to Y$ is *compact* if the underlying map of topological spaces is.
(2) A morphism $f : X \to Y$ is *separated* if the diagonal $\Delta_f : X \to X \times_Y X$ is a closed immersion.
(3) A morphism $f : X \to Y$ is *quasi-separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is compact.

(4) A morphism $f : X \to Y$ is *quasi-finite* if for any point $x \in X$, there is an open neighborhood U of x and V of $f(x)$ with the restriction $f : U \to V$ a finite morphism of K-analytic spaces. Beware! The analytic topology on normative spectra is much finer than the Zariski topology, meaning that being source-locally a finite morphism is a much weaker condition than in algebraic geometry.

(5) A morphism is *closed* or *boundaryless* if the relative interior is the entire domain, or equivalently if its relative boundary is empty.

(6) A morphism is called *naively flat* (following [**45**]) if the induced algebra maps on all local Banach rings is flat.

(7) A morphism is called *flat* (following [**45**]) if it is naively flat, and all base changes of the morphism remain naively flat.

(8) For a separated morphism $f : X \to Y$, the ideal determining the closed immersion Δ_f gives a coherent $\mathscr{O}_X$-module $\Omega_{Y/X}$, called the *sheaf of relative differentials.*

(9) A separated morphism is called *unramified* if the sheaf of relative differentials is the zero module.

(10) A separated morphism is called *étale* if it is unramified, naively flat, and quasi-finite.

(11) A separated morphism $f : X \to Y$ is called *smooth* if for every point $x \in X$, there is an open neighborhood U of x such that the induced morphism $U \to Y$ factors as $U \to Y \times \mathbb{A}^n \times Y \to Y$ where the second map is the natural projection and the first map is étale.

(12) A separated morphism $f : X \to Y$ is called *quasi-étale* if for every $x \in X$ there is a framing of f at x consisting of $V_1, \ldots, V_n \subset X$ and $W_1, \ldots, W_n \subset Y$ so that every $f : V_i \to W_i$ is étale.

(13) A separated morphism $f : X \to Y$ is called *quasi-smooth* if for every $x \in X$ there is a framing of f at x consisting of $V_1, \ldots, V_n \subset X$ and $W_1, \ldots, W_n \subset Y$ so that every $f : V_i \to W_i$ is smooth.

$\triangleleft$

We are mostly interested in the smooth, étale, quasi-smooth, and quasi-étale morphisms.

THEOREM 5.18. *The class of étale (resp. smooth, resp. quasi-étale, resp. quasi-smooth) morphisms is closed under composition, base change, and extensions of the ground field.*

Let $X \to Z$ and $Y \to Z$ be any étale (resp. quasi-étale) morphisms. Then any Z-morphism $X \to Y$ is also étale (resp. is also quasi-étale).

PROOF. For the first claims, we cite the following.

For étale morphisms, see Corollary 3.3.8 of **[44]**. The case for smooth morphisms follows essentially by the case for étale morphisms and the definition of smoothness. This is Proposition 3.5.2 of **[44]**. The case for quasi-étale morphisms is Lemma 3.1 (i) of **[46]**.

For the second claims, the case for étale morphisms follows as a corollary to Corollary 3.3.9 of **[44]**. The case of quasi-étale morphisms is Lemma 3.1 (ii) of **[46]**. □

DEFINITION 5.19. Let X be a K-analytic space. Then we can define the following sites.

(1) The *étale site* of X denoted $\mathbf{site}\, X_{\text{ét}}$, whose objects are étale morphisms to X and whose morphisms are commutative triangles between morphisms.
(2) The *quasi-étale site* of X denoted $\mathbf{site}\, X_{\text{qét}}$, whose objects are quasi-étale morphisms to X and whose morphisms are commutative triangles between morphisms.
(3) The *smooth site* of X denoted $\mathbf{site}\, X_{\text{sm}}$, whose objects are smooth morphisms to X and whose morphisms are commutative triangles of smooth morphisms.
(4) The *quasi-smooth site* of X denoted $\mathbf{site}\, X_{\text{qsm}}$, whose objects are quasi-smooth morphisms to X and whose morphisms are commutative triangles of smooth morphisms.

Since an étale morphism is automatically quasi-étale, we obtain a geometric morphism of topoi $\mu_X : \tau X_{\text{qét}} \to \tau X_{\text{ét}}$. Similarly we obtain a geometric morphism of topoi $\tau X_{\text{qsm}} \to X_{\text{sm}}$.

Lastly, we define the *étale homotopy type* (resp. *quasi-étale homotopy type*) of a K-analytic space to be the shape assigned to its étale site (resp. the quasi-étale site). ◁

The major results are the following.

THEOREM 5.20. *The étale topos (resp. quasi-étale topos) of a K-analytic space is equivalent to its smooth topos (resp. quasi-smooth topos).*

PROOF. Smooth morphisms admit étale local sections, and so étale coverings are cofinal in the class of smooth coverings. Quasi-smooth morphisms similarly admit quasi-étale local sections. □

THEOREM 5.21 (THEOREM 3.1 OF **[47]**). *Let $f : X \to Y$ be a morphism of schemes locally of finite type over K, and assume that f is of finite type. Then for any sheaf of torsion groups $\mathscr{A}$ whose orders are coprime to the characteristic of K, and any non-negative integer i.*

$$\left(R^i f_* \mathscr{A}\right)^{\text{an}} \simeq R^i f_*^{\text{an}} \mathscr{A}^{\text{an}}$$

Corollary 5.22. *Let X be a variety over K, then for any sheaf of torsion groups $\mathscr{A}$ whose orders are coprime to the characteristic of K,*

$$H^i(X_{\text{ét}}, \mathscr{A}) \simeq H^i(X^{\text{an}}_{\text{ét}}, \mathscr{A}^{\text{an}})$$

Theorem 5.23 (Theorem 3.3 (ii) of **[46]**). *Let X be a K-analytic space, and write $\mu_X^* : \tau\, X_{\text{ét}} \to \tau\, X_{\text{qét}}$ for the inverse image functor. Then for any abelian sheaf $\mathscr{A}$ on the étale topos of X and any non-negative integer i,*

$$H^i(X_{\text{ét}}, \mathscr{A}) \simeq H^i(X_{\text{qét}}, \mu_X^* \mathscr{A})$$

Theorem 5.24 (Theorems 3.1 and 4.1 of **[48]**). *Let X be a scheme locally of finite type over* $\operatorname{spec} K$. *Then the category of finite étale covers of X whose degree is coprime to the characteristic of K is equivalent to the category of finite étale covers of X^{an} whose degree is coprime to the characteristic of K.*

Theorem 5.25 (Corollary 4.1.9 of **[44]**). *For any group G, sheaf theoretic G-torsors for the étale topology are representable by K-analytic spaces.*

Corollary 5.26. *Let X be a scheme locally of finite type over* $\operatorname{spec} K$. *Then if the characteristic of K is zero, the profinite étale fundamental groups of X and X^{an} are isomorphic. If the characteristic of K is a positive prime number p, then the prime-to-p completion of the étale fundamental groups of X and X^{an} are isomorphic.*

Combining Corollary 5.22 and Theorem 5.24 gives us the following corollary.

Corollary 5.27. *Let K be a non-archimedean normed field. Let X be a scheme locally of finite type over K, then*

(1) *if the characteristic of K is zero, we have an equivalence of profinitely completed étale homotopy types*

$$\widehat{\text{ét}}\, X \simeq \widehat{\text{ét}}\, X^{\text{an}}$$

(2) *if instead the characteristic of K is a positive prime p, then we have an equivalence of $\{p\}^c$-profinitely completed étale homotopy types,*

$$\widehat{\text{ét}}_{\{p\}^c}\, X \simeq \widehat{\text{ét}}_{\{p\}^c}\, X^{\text{an}}$$

There is also a natural notion of cohomology with compact support.

Definition 5.28. Let $\mathcal{X}$ be a K-analytic space. With any abelian sheaf $\mathscr{F}$ for the étale topology on $\mathcal{X}$, we may define the *global sections with compact support*

$$\Gamma_c(\mathcal{X}, \mathscr{F}) := \{ f \in \mathscr{F}(\mathcal{X}) \mid \operatorname{supp} f \text{ is a compact subset of } \mathcal{X} \}$$

where the support of a section is the set of points where the stalk of f is zero. This is automatically left exact since morphisms of sheaves preserve sections with compact support.

The right derived functors of Γ_c are called the *cohomology with compact support* functors.

There is also a relative version of this, for a morphism $f : \mathcal{X} \to \mathcal{Y}$ and an abelian sheaf for the étale topology $\mathscr{F}$, we define the *direct image with compact support* $f_!(\mathscr{F})$ by the following

$$f_!(\mathscr{F})(\mathcal{U}/\mathcal{Y}) := \{f \in \mathscr{F}(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}) \mid \operatorname{supp} f \text{ is a compact subset of } \mathcal{X}\}$$

this extends to a functor $f_!$ and is left exact. Its derived functor is called the *higher direct image with compact support* functor and is simply written $R^i f_!$. ◁

We only use this to deduce proper-smooth base change for K-analytic spaces.

LEMMA 5.29. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a compact morphism between compact Hausdorff K-analytic spaces. Then the natural tranformation $f_! \rightsquigarrow f_*$ is an equivalence, inducing an isomorphism of derived functors.*

PROOF. The support of a section of an arbitrary sheaf is closed. Since the spaces are both compact and Hausdorff, the support will also be compact, and so all sections of all sheaves have compact support. □

THEOREM 5.30 (THEOREM 7.1.1 OF [**44**]). *Let X be a compactifiable scheme over $\operatorname{spec} K$. Then the algebraic and analytic compactly supported cohomology agree, at least for abelian torsion sheaves $\mathscr{F}$.*

$$H^i(X_{\text{ét}}, \mathscr{F}) \simeq H^i(X^{\text{an}}_{\text{ét}}, \mathscr{F}^{\text{an}})$$

COROLLARY 5.31. *Let X be a proper scheme over K. Then the profinitely completed étale homotopy type of X and X^{an} agree.*

$$\widehat{\text{ét}}\, X \simeq \widehat{\text{ét}}\, X^{\text{an}}$$

3. Adic spaces

We now come to the next formalism of non-archimedean algebraic geometry, that of adic spaces. The base theory is due to Huber. While Berkovich's non-archimedean analytic spaces give compact Hausdorff affinoids, Huber's theory gives spectral spaces. Recall that a topological space is spectral if it is homeomorphic to $\operatorname{spec} R$ for some (non-unique) ring R.

DEFINITION 5.32. We pose the following definitions.

(1) Let $R = (R^{\triangleright}, R^{+})$ be an affinoid pair and $\operatorname{spa} R$ its valuative spectrum. Then to any finite list of elements $f_1, \ldots, f_n, g \in R^{\triangleright}$ such that the ideal generated by the f_i is open in $R^{\triangleright}$ we construct the open subset

$$Rat_R(\frac{f_1, \ldots, f_n}{g}) := \{x \in \operatorname{spa} R \,|\, |f_i|_x \leq |g|_x \neq 0 \text{ for all } i\}$$

(2) A subset is called *rational* if it is of the above form for some sequence of ring elements.

(3) We define a presheaf $\mathscr{O}_{\operatorname{spa} R}$ as follows. For a rational subset $U \subset \operatorname{spa} R$, we define

$$\mathscr{O}_{\operatorname{spa} R}(U) := \widehat{R^{\triangleright}[g^{-1}]}$$

where the completion is respect to a topology which forces each f_i/g to satisfy that $\lim_{n\to\infty}(f_i/g)^n = 0_{R^{\triangleright}}$. Since the details of the topology are not important for our purposes, we will simply cite **[21]**. For a general open subset $U \subset \operatorname{spa} R$, we define is as the colimit over rational subsets of U,

$$\mathscr{O}_{\operatorname{spa} R}(U) = \operatorname*{colim}_{V \subset U} \mathscr{O}_{\operatorname{spa} R}(V)$$

To any point $x \in \operatorname{spa} R$, we get a valuation $|-|_x : \mathscr{O}_{\operatorname{spa} R, x} \to \Gamma_x$ for which the local rings are complete.

(4) We define the subpresheaf

$$\mathscr{O}^{+}_{\operatorname{spa} R} := \Big[U \mapsto \{f \in \mathscr{O}_{\operatorname{spa} R}(U) \mid |f|_x \leq 1 \text{ for all } x \in U\}\Big]$$

as the presheaf of *bounded elements*.

(5) A *valuatively ringed space* is a triple $(X, \mathscr{O}_X, Val_X)$ where X is a topological space, $\mathscr{O}_X$ is a sheaf of topological rings, and $Val_X = \{v_x : \mathscr{O}_{X,x} \to \Gamma_x\}$ is a set of valuations indexed by the points of X satisfying that
   (a) the topology on the stalk $\mathscr{O}_{X,x}$ is the same as the topology from the valuation v_x,
   (b) the ring $\mathscr{O}_{X,x}$ is a local valuation ring,
   (c) and every valuation v_x is tidy.

   Instead of the tidy condition, one may instead only consider v_x as being an equivalence class of valuations.

(6) A *morphism of valuatively ringed spaces* is a morphism of locally ringed spaces which is strictly compatible with the valuations, i.e. if $f(x) = y$ then $v_x \circ f = v_y$ as functions on $\mathscr{O}_{Y,y}$.

◁

Theorem 5.33. *Let $R = (R^{\triangleright}, R^{+})$ be an affinoid pair, satisfying one of the following assumptions.*

- **[21]** *The ring $R^{\triangleright}$ is f-adic and has some Noetherian ring of definition over which $R^{\triangleleft}$ is finitely generated,*
- **[21]** *the ring $R^{\triangleright}$ is universally Noetherian,*
- **[49]** *the ring $R^{\triangleright}$ is perfectoid,*
- **[50]** *or the ring $R^{\triangleright}$ is stably uniform.*

the pair $(\operatorname{spa} R, \mathscr{O}_{\operatorname{spa} R})$ *is a valuatively ringed space.*

Since there are two rings in play, the notion of being "of finite type" is more complicated.

Definition 5.34. We pose the following definitions.

(1) An *adic space* is a valuatively ringed space which is locally isomorphic to the valuative spectrum of some affinoid pair, and satisfying that the valuation on local rings is independent of which affine open it is computed in.

(2) A *morphism of adic spaces* is a morphism of valuatively ringed spaces.

(3) A morphism of adic spaces is *quasi-compact* when the map on topological spaces is, that is when the preimage of a quasi-compact open subset is quasi-compact.

(4) An adic space is *quasi-separated* when the intersection of any two quasi-compact open subsets is again quasi-compact.

(5) A morphism of adic spaces is *quasi-separated* when the preimage of a quasi-separated open subset is quasi-separated.

(6) A morphism $f : X \to Y$ of adic spaces is *adic* if for any affinoid opens $U \subset X$ and $V \subset Y$ with $f(U) \subset V$, the induces ring morphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is an adic morphism of pairs.

(7) A morphism $f : X \to Y$ of adic spaces is *locally weakly finite type* if for each $x \in X$ there is an affinoid open neighborhood U of x and V of $f(x)$ such that $f(U) \subset V$ and the induced morphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is topologically of finite type as a morphism of topological rings.

(8) A morphism of adic spaces is *locally $^{+}$weakly of finite type* when it is locally weakly finite type, and for any $x \in X$ and U and V as above, there exists some finite sequence of ring elements $f_1, \ldots, f_n \in \mathscr{O}_X(U)$ such that the ring $\mathscr{O}_X^{+}(U)$ is the smallest ring of integral elements for $\mathscr{O}_X(U)$ which contains both $\mathscr{O}_Y^{+}(V)$ and every f_i.

(9) A morphism $f : X \to Y$ of adic spaces is *locally of finite type* if for every $x \in X$ there are open affinoid neighborhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and the induced map on affinoid pairs

$$\left(\mathscr{O}_Y(V), \mathscr{O}_Y^{+}(V)\right) \to \left(\mathscr{O}_X(U), \mathscr{O}_X^{+}(U)\right)$$

is finite type in the sense of Huber.

(10) A morphism of adic spaces is *of weakly finite type* (resp. of ${}^+$weakly finite type, resp. of finite type) when it is quasi-compact and locally weakly finite type(resp. locally ${}^+$weakly finite type, resp. locally of finite type).

(11) A morphism of adic spaces is *of finite presentation* if it is of finite type, and if the topologies on the rings are discrete the ring morphisms should also be of finite presentation.

$\triangleleft$

The basic structural results for this category are the following.

LEMMA 5.35. (1) *A morphism between affinoid adic spaces is locally of finite type if and only if the ring homomorphism if finite type in the sense of Huber.*

(2) *If $f : X \to Y$ and $g : Z \to Y$ are two morphisms of adic spaces, then the categorical fiber product of them exists when either of the following conditions are satisfied.*
   (a) *One of the morphisms is locally of finite type,*
   (b) *or one of the morphisms is locally weakly finite type and the other is adic.*

(3) *Locally of finite type and locally of finite presentation morphisms are stable under pullback.*

(4) *Locally of ${}^+$weakly finite type morphisms are stable under pullback by adic morphisms.*

(5) *If $f : X \to Y$ and $g : Z \to Y$ are two morphisms with f locally weakly of finite type and g adic, then*
   (a) *the pullback f' is locally weakly of finite type,*
   (b) *the pullback g' is adic,*
   (c) *if f is quasi-compact (resp. quasi-separated), then so is f',*
   (d) *if g is quasi-compact (resp. quasi-separated), then so is g',*

(6) *For any adic space Y and point $y \in Y$, the morphism $\operatorname{spa} k(y) \to Y$ is adic.*

(7) *For any $f : X \to Y$ and point $y \in Y$ the fiber X_y exists.*

4. Interplay between adic and K-analytic spaces

The theory of adic spaces is more technically convenient in some respects than the formalism of K-analytic spaces. The opposite is also true, and so it is beneficial to take results from one theory and port them to the other.

[NOTE: Have to discuss what kinds of fields this works for]

THEOREM 5.36 (THEOREM 8.3.1 OF [**21**]). *There is an equivalence of categories* $s : \mathrm{An}' \to \mathrm{Rig}$ *between the category of Hausdorff strictly* K*-analytic spaces and taut rigid analytic varieties.*

THEOREM 5.37 (DISCUSSION IN 1.1.11 OF [**21**]). *There is an equivalence of categories* $r : \mathrm{Rig}' \to \mathrm{Ad}'$ *between the category of quasi-separated rigid analytic varieties and quasi-separated and locally of finite type adic spaces over* $\mathrm{spa}\, K$.

We can combine these into the following, keeping in mind that taut implies quasi-separated.

THEOREM 5.38. *There is an equivalence of categories* $(-)^{\mathrm{ad}} : \mathrm{An} \to \mathrm{Ad}$ *between*

(1) *the category of Hausdorff strictly* K*-analytic spaces,*
(2) *and the category of taut adic spaces locally of finite type over* $\mathrm{spa}\, K$.

PROOF. Since r is fully faithful, we only need to determine the image under r of the image of s. The image of s is taut rigid analytic varieties, which by definition are quasi-separated. A morphism of rigid analytic varieties is taut if and only if the associated morphism of adic spaces is taut by Lemma 5.6.8 (i) of [**21**]. □

COROLLARY 5.39. *The functor* $(-)^{\mathrm{ad}}$ *preserves the following properties,*

(1) *it takes strictly affinoid domains to open affinoid immersions,*
(2) *it takes compact morphisms of* K*-analytic spaces to quasi-compact morphisms of the associated adic spaces,*
(3) *it takes proper morphisms of* K*-analytic spaces to proper morphisms of the associated adic spaces,*
(4) *a collection of morphisms* $\{\mathcal{U}_i \to \mathcal{X}\}_{i \in I}$ *is jointly surjective if and only if the associated collection of morphisms of adic spaces is surjective,*
(5) *it takes étale morphisms of* K*-analytic spaces to étale and partially proper morphisms of the associated adic spaces,*
(6) *it takes quasi-étale morphisms of* K*-analytic spaces to étale morphisms of the associated adic spaces,*
(7) *it takes smooth morphisms of* K*-analytic spaces to smooth morphisms of the associated adic spaces,*
(8) *and it takes quasi-smooth morphisms of* K*-analytic spaces to smooth morphisms of the associated adic spaces,*

PROOF. We split up the claims. Unless explicitly stated otherwise, all K-analytic spaces should be assumed to be Hausdorff and strictly K-analytic.

(1) It is sufficient to check this for affinoid $\mathcal{X} = \operatorname{sp} A$. In this case, a strictly analytic subdomain $\operatorname{sp} B \subset \operatorname{sp} A$ will admit a finite covering by Weierstrass domains in A. Since Weierstrass domains give open subspaces on the level of rigid varieties, such a $\operatorname{sp} B$ will give an open subspace of the adic space associated to A.

(2) See Proposition 3.3.2 of [**19**] for the associated morphism of rigid spaces, to check that the associated morphism of adic spaces remains quasi-compact, we simply note that it suffices to check that the preimage of an affinoid open is a finite union of affinoid opens, and the lattice of such affinoid opens is the same in the rigid space and the adic space.

(3) See Proposition 3.3.2 of [**19**] for the associated morphism of rigid spaces, and Remark 1.3.19 (iv) of [**21**] for the associated morphism of adic spaces.

(4) This is proved in the course of Proposition 8.3.4 of [**21**], see "Proof of (b)" on page 427.

(5) By Proposition 8.3.4 of[**21**] for a fixed base the étale site of a K-analytic space is equivalent to the étale and partially proper site of the associated rigid analytic variety. By Proposition 1.7.11 of [**21**] a morphism of rigid analytic varieties is étale if and only if the associated morphism of adic spaces is. By Proposition 1.5.9 of [**21**], a locally quasi-finite morphism of rigid analytic varieties is partially proper if and only if the associated morphism of adic spaces is. Étale morphisms of rigid analytic varieties are of course locally quasi-finite, and so we deduce that étale morphisms of K-analytic spaces are taken to étale and partially proper morphisms of adic spaces.

(6) Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-étale morphism. Then by definition at each point $x \in \mathcal{X}$, there is some affinoid framing $U_1, \ldots, U_n$ of x and $V_1, \ldots, V_n$ of $f(x)$ such that $U_i \to V_i$ is étale. The inclusions $U_i \to \mathcal{X}$ are affinoid domains, and under the construction above these map to open affinoid neighborhoods of adic spaces. Thus we can locally factor f^{ad} as an open immersion followed by an étale and partially proper map. Such a composition is still étale, and in general not partially proper.

(7) Smooth morphisms f étale locally admit factorizations as $U \to \mathbb{A}^n_K \times \mathcal{X} \to \mathcal{X}$ where the first map is étale and the second is the projection. By the previous argument, f^{ad} admits étale-and-partially-proper local factorizations into a smooth map. But smoothness is local for the étale topology on a rigid analytic variety by Theorem 4.2.7 of [**51**], and a morphism of rigid analytic varieties is smooth if and only if the associated morphism of adic spaces is smooth by Proposition 1.7.11 of [**21**].

(8) We can use the above argument but replacing étale with quasi-étale. Since the morphism of rigid spaces associated to a quasi-étale morphism is still étale, the rest of the argument goes through.

□

THEOREM 5.40 (HOMOTOPY FIBER THEOREM). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth and proper map of compact and Hausdorff K-analytic spaces, and y a geometric point in $\mathcal{Y}$. Pick any prime ℓ coprime to the characteristic of the residue field k. Assume that*

(1) *the morphism has geometrically connected fibers,*
(2) *the étale fundamental group $\pi_1^{\text{ét}}(\mathcal{Y}, y)$ acts trivially on the ℓ-adic cohomology of the fibers,*
(3) *and that $\mathcal{Y}$ is connected.*

Then we have a homotopy fiber sequence of ℓ-profinite spaces

$$\widehat{\text{ét}}_\ell\, \mathcal{X}_y \to \widehat{\text{ét}}_\ell\, \mathcal{X} \to \widehat{\text{ét}}_\ell\, \mathcal{Y}$$

We will split up the proof into several parts.

LEMMA 5.41. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth and proper map of Hausdorff and paracompact strictly K-analytic spaces, and ℓ a prime distinct from the characteristic of the residue field k. Then for each $i \geq 0$, the higher direct image functor $R^i f_*$ takes locally constant ℓ-primary torsion sheaves to locally constant ℓ-primary torsion sheaves.*

PROOF. We may take the associated adic spaces of $\mathcal{X}$ and $\mathcal{Y}$, which we write as $\mathcal{X}^{\text{ad}}$ and $\mathcal{X}^{\text{ad}}$. The map f also becomes an adic morphism $f^{\text{ad}} : \mathcal{X}^{\text{ad}} \to \mathcal{Y}^{\text{ad}}$. Smooth morphisms of K-analytic spaces become smooth morphisms of adic spaces, and proper morphisms of K-analytic spaces become proper morphisms of adic spaces. Then we have this result for f^{ad} by Corollary 6.2.3 of [**21**]. But the functors Rf_* and Rf_*^{ad} agree for ℓ-torsion sheaves. □

THEOREM 5.42. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a smooth and proper morphism of Hausdorff and paracompact K-analytic spaces, and ℓ a prime distinct from the characteristic of the residue field k. Then for each $i \geq 0$, the higher direct image functor $R^i f_*$ takes locally constant ℓ-primary torsion sheaves to locally constant ℓ-primary torsion sheaves.*

PROOF. There is a non-archimedean field K_r such that the base change of both $\mathcal{X}$ and $\mathcal{Y}$ to K_r are both strictly K_r-analytic. Write $f' : \mathcal{X}' \to \mathcal{Y}'$ We will show that the base change of $R^i f_* \mathscr{A}$ is exactly $R^i f'_*(\mathscr{A}')$. Since the morphism f is proper, the direct image functor is the same as the direct image with compact support. Thus by the Weak Base Change theorem Theorem 5.3.1 of [**44**], the stalks of $R^i f_* \mathscr{A}$

are the étale cohomology of the geometric fibers of f. However the étale cohomology of an analytic space over an algebraically closed non-archimedean field is invariant under ground field extension by Theorem 7.6.1 of [**44**]. Thus the morphism furnished by the universal δ-functor $R^i f'_* \mathscr{A} \rightsquigarrow (R^i f_* \mathscr{A})'$ is an isomorphism, and we conclude that the base change of the higher direct images are all locally constant. $\square$

Proof of Homotopy Fiber Theorem. We have three converging spectral sequences and morphisms between them

$$\begin{array}{ccccc}
E_2^{p,q}(S) & = & H^p_{\text{sing}}(\widehat{\text{ét}}_\ell\, \mathcal{Y}, H^q(F, \mathbb{Z}/\ell)) & \Rightarrow & H^{p+q}_{\text{sing}}(\widehat{\text{ét}}_\ell\, \mathcal{X}, \mathbb{Z}/\ell) \\
\uparrow & & \uparrow & & \uparrow \\
E_2^{p,q}(S') & = & H^p_{\text{sing}}(\widehat{\text{ét}}_\ell\, \mathcal{Y}, H^q(\mathcal{X}_{\overline{y}}, \mathbb{Z}/\ell)) & \Rightarrow & H^{p+q}_{\text{sing}}(\widehat{\text{ét}}_\ell\, \mathcal{X}, \mathbb{Z}/\ell) \\
\simeq\uparrow & & \simeq\uparrow & & \simeq\uparrow \\
E_2^{p,q}(V) & = & H^p_{\text{sheaf}}(\mathcal{Y}_{\text{ét}}, \underline{H^q(X_{\overline{y}}, \mathbb{Z}/\ell)}) & \Rightarrow & H^{p+q}_{\text{sheaf}}(\mathcal{X}_{\text{ét}}, \mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow \\
E_2^{p,q}(L) & = & H^p_{\text{sheaf}}(\mathcal{Y}_{\text{ét}}, R^q f_* \mathbb{Z}/\ell) & \Rightarrow & H^{p+q}_{\text{sheaf}}(\mathcal{X}_{\text{ét}}, \mathbb{Z}/\ell)
\end{array}$$

The marked isomorphisms come from Verdier's hypercover theorem. If each $R^q f_* \mathbb{Z}/\ell$ is constant, then $E_2^{p,q}(V) \to E_2^{p,q}(L)$ is also an isomorphism. The map from $E_2^{p,q}(S')$ to $E_2^{p,q}(S)$ is the one induced by the universal property of the homotopy fiber.

From here, we simply proceed as in [**3**]. We create a tower of étale coverings of $\mathcal{Y}$, and in the colimit the spectral sequences all collapse. The isomorphism on abutments forces the $E_2^{p,q}$ terms to become isomorphic, so that in particular

$$H^q(\mathcal{X}_{\overline{y}}, \mathbb{Z}/\ell) \to H^q(F, \mathbb{Z}/\ell)$$

is an isomorphism. In ℓ-profinite spaces, this says exactly that they are equivalent. $\square$

Theorem 5.43. *Let $\mathcal{X}$ be a Hausdorff strictly K-analytic space. Then the quasi-étale 1-topos of $\mathcal{X}$ is equivalent to the étale 1-topos of $\mathcal{X}^{\text{ad}}$*

$$\tau\, \mathcal{X}_{\text{qét}} \simeq \tau\, \mathcal{X}^{\text{ad}}_{\text{ét}}$$

Proof. The quasi-étale topos is equivalent to the one generated by quasi-étale morphisms whose source is affinoid. We use that $\mathcal{X}$ is Hausdorff to have fiber products in the subcategory. We may refine such a quasi-étale map $q : U \to \mathcal{X}$ so that it will factor into $q = e \circ j$ where e is étale and j is an affinoid embedding. Both e^{ad} and j^{ad} will be étale as morphisms of adic spaces, and e^{ad} is also partially proper. The composition in general is merely étale and not partially proper and étale.

We will show that such morphisms are cofinal in the category of étale maps to $\mathcal{X}^{\mathrm{ad}}$. Let $f : Y \to \mathcal{X}^{\mathrm{ad}}$ be an étale map, and we may assume without loss of generality that $Y = \operatorname{spa} A$ is affinoid and its image is contained in a partially proper affinoid of $\mathcal{X}^{\mathrm{ad}}$ as such morphisms generate a cofinal system of coverings. Theorem 2.2.8 of [**21**] gives a local compactification $f = \overline{f} \circ j$ where $\overline{f}$ is a finite étale map and j is an open embedding. Replace our affinoid cover by the local and compactifiable one. The finite étale map of course comes from a finite étale extension of the corresponding analytic domain in $\mathcal{X}$, and the open embedding may not directly come from a affinoid in $\mathcal{X}^{\mathrm{rig}}$, however it does refine to a collection of affinoids in $\mathcal{X}^{\mathrm{rig}}$ whose union is $\operatorname{spa} A$. To relate this back to $\mathcal{X}$ itself, we apply Theorem 1.6.2 of [**44**]. □

Corollary 5.44. *Let $\mathcal{X}$ be a Hausdorff strictly K-analytic space. Then the geometric morphism from the quasi-étale topos of $\mathcal{X}$ to the étale topos of $\mathcal{X}$ is a pro-truncated shape equivalence.*

Proof. This follows since the étale topos of $\mathcal{X}$ is the partially-proper étale topos of $\mathcal{X}^{\mathrm{ad}}$, and Proposition 8.12.2.i of [**21**] guarantees that on locally constant sheaves that the counit natural transformation $\theta_*\theta^* \rightsquigarrow \mathrm{id}$ is an isomorphism. This implies that it is a pro-truncated shape equivalence. □

5. Formal models and non-archimedean geometry

In this section we will explore the relationships between non-archimedean analytic spaces and their formal models. Let R be a complete discretely valued ring, with fraction field K and residue field k.

Definition 5.45. Let $\mathfrak{X}$ be a special formal scheme. Then we can construct a strictly K-analytic space $\mathcal{X}$ called the generic fiber of $\mathfrak{X}$. We follow the construction in [**40**].

(1) If $\mathfrak{X} = \operatorname{spf} A$ is affine, then we find a surjection $R[[T_1, \ldots, T_n]]\{S_1, \ldots S_m\} \to A$ with kernel I. Write $\mathcal{P} = E^m(0,1) \times D^n(0,1)$ where $E^n(c,r)$ is the closed polydisc of dimension n centered at c with radius r and $D^m(c,r)$ is the open polydisc of dimension m with center c and radius r. We take $\mathcal{X}$ to be the source of the closed immersion determined by the ideal $I\mathscr{O}_{\mathcal{P}}$.

(2) If $\mathfrak{X}$ is not affine, we take an open covering $U_0 := \{\mathfrak{U}_i\}_{i\in I}$ by affines and another affine open covering U_1 of the pairwise intersections $\{\mathfrak{U}_{i,j}\}_{i,j\in I}$. Since $\mathfrak{X}$ is locally Noetherian it is quasi-separated and U_1 is locally finite in each pairwise intersection $\mathfrak{U}_{i,j\in I}$. We can use (1) to define a diagram of strictly K-analytic spaces, where all the affinoids coming from U_1 are

analytic domains in those of U_0. This directly gives gluing data, and the gluing exists by the second criterion of Lemma 5.15.

(3) A refinement of an affinoid covering as in (2) induces a unique isomorphism of strictly K-analytic spaces. This is checked by reducing to the case where $\mathfrak{X}$ is a single affinoid. That case follows immediately from Proposition 1.3.2 of [**44**].

(4) Any two affine coverings as in (2) admit a common refinement, and so by (3) are canonically isomorphic.

(5) For any morphism of special formal schemes, we can find a compatible affine covering of both. Then the universal property of the gluing gives a unique morphism between their generic fibers.

$\triangleleft$

Lemma 5.46. (1) *The generic fiber of an adically étale morphism is quasi-étale.*

(2) *The generic fiber of an adically log étale morphism between vertical log special formal schemes is quasi-étale.*

(3) *The generic fiber of an admissible log blow up of a vertical log special formal scheme is the identity morphism.*

(4) *The generic fiber functor* $\mathbf{site}\,\mathfrak{X}_{\text{akét}} \to \mathbf{site}\,\mathfrak{X}_{\eta,\text{qét}}$ *induces a morphism of sites in the opposite direction.*

Proof. We break up the proof.

(1) This is Proposition 2.1.iii of [**40**].

(2) This appears in [**41**]. Being adically log étale means that adically locally the morphism factors as an adically étale map followed by a simple projection whose generic fiber is étale. Being quasi-étale is local for the quasi-étale topology, and so we conclude the desired statement.

(3) This reduces to the claim that admissible log blow up does not change the generic fiber, which can be checked on an affine chart.

(4) This follows since the generic fiber functor preserves fiber products, making it a continuous morphism of sites.

$\square$

Choosing a homotopy inverse of the equivalence from $\text{qét}\,\mathfrak{X}_\eta \to \text{ét}\,\mathfrak{X}_\eta$ gives the following corollary.

Lemma 5.47. *Let $\mathfrak{X}$ be a fs vertical log special formal scheme, and assume*

(1) *that* $\mathfrak{X}^\circ$ *is affine,*

(2) *there exists a strict and log smooth* $\operatorname{spec} R$*-scheme* $V \to \operatorname{spec} R$*,*

(3) *it admits a global chart* $P \to \mathscr{O}_V$ *for a fs monoid* P*,*

(4) *and there exists a* strict *log morphism* $\mathfrak{X} \to \hat{V}_{V_s}$ *making* $\mathfrak{X}$ *isomorphic to the completion of* $\hat{V}_{V_s}$ *along some closed subset with the inverse image log structure.*

Then the ℓ*-profinitely completed adically Kummer étale homotopy type of* $\mathfrak{X}$ *is equivalent with the* ℓ*-profinitely completed étale homotopy type of its generic fiber* $\mathcal{X} = \mathfrak{X}_\eta$*.*

Proof. We actually have a chain of equivalences involving a few more spaces. Since $\mathfrak{X}$ is affine, write A for its global sections, $X = \operatorname{spec} A$ and X^{ad} for the associated adic space. It is not necessarily true that X^{ad} and $\mathcal{X}^{\mathrm{ad}}$ agree, however there is a map $\mathcal{X}^{\mathrm{ad}} \to X^{\mathrm{ad}}$. We have a collection of morphisms

$$\widehat{\mathrm{\acute{e}t}}_\ell\, \mathcal{X} \xleftarrow{1} \widehat{\mathrm{q\acute{e}t}}_\ell\, \mathcal{X} \xrightarrow{2} \widehat{\mathrm{\acute{e}t}}_\ell\, \mathcal{X}^{\mathrm{ad}} \xrightarrow{3} \widehat{\mathrm{\acute{e}t}}_\ell\, X_K \xrightarrow{4} \widehat{\mathrm{k\acute{e}t}}_\ell\, X \xrightarrow{5} \widehat{\mathrm{ak\acute{e}t}}_\ell\, \mathfrak{X}.$$

We claim that each arrow is an equivalence.

(1) This is 5.44.

(2) This is 5.43.

(3) This is the most difficult argument, and relies on the other identifications. For convenience put $\mathfrak{V} = \hat{V}_{V_s}$. First we note that for $\mathfrak{V}$, the ring $\mathscr{O}_{\mathfrak{V}} \otimes_R K$ is actually isomorphic to the underlying ring of $\mathcal{X}^{\mathrm{ad}}$, since $\mathfrak{V}$ is topologically of finite type. In particular, Theorem 3.2.2 of **[21]** guarantees that the profinitely completed étale homotopy types of $\operatorname{spa} \mathscr{O}_{\mathfrak{V}} \otimes_R K$ and $\operatorname{spec} \mathscr{O}_{\mathfrak{V}} \otimes_R K$ agree. Furthermore, the morphism $i : \mathfrak{X} \to \mathfrak{V}$ satisfies the criteria of Proposition 3.15 of **[52]**. Writing $\theta_{\mathfrak{X}}$ and $\theta_{\mathfrak{V}}$ for the morphisms from the étale topoi of the corresponding adic spaces to the corresponding adically étale topoi, the referenced proposition states that $i^* R\theta_{\mathfrak{V},*}\mathscr{F} \simeq R\theta_{\mathfrak{X},*} i^{\mathrm{ad},*}\mathscr{F}$ for constructible $\mathscr{F}$. Using the isomorphism above and Theorem 4.60, we can functorially identify

$$R^i\theta_{\mathfrak{V},*}\mathbb{Z}/\ell \simeq \mathbb{Z}/\ell \otimes_{\mathbb{Z}} \wedge^i P^{\mathrm{gp}}$$

for P a chart for $\mathfrak{V}$ and now applying Proposition 3.15 of **[52]** we deduce that

$$R^i\theta_{\mathfrak{X},*}\mathbb{Z}/\ell \simeq i^*\left(\mathbb{Z}/\ell \otimes_{\mathbb{Z}} \wedge^i P^{\mathrm{gp}}\right)$$

but of course we can pull the inverse image functor inside

$$i^*\left(\mathbb{Z}/\ell \otimes_{\mathbb{Z}} \wedge^i P^{\mathrm{gp}}\right) \simeq \mathbb{Z}/\ell \otimes_{\mathbb{Z}} \wedge^i i^* P^{\mathrm{gp}}.$$

Finally, we note that the log scheme X is fs and admits a chart given by i^*P, meaning the above is exactly the description of the Kummer étale cohomology of X. However away from p, this is just the cohomology of X_K as desired.

(4) If $\mathfrak{X} \to \hat{V}_{V_s}$ is an isomorphism, we are done as X^{triv} is simply X_K. If it is not an isomorphism then *a priori* X^{triv} could be larger. However, the support of $\mathscr{M}_X/\mathscr{O}_X^\times$ must have codimension 1 if it is non-trivial and be supported on the special fiber. Thus any point on which $\mathscr{M}_X/\mathscr{O}_X^\times$ is non-trivial forces X^{triv} to be all of X_K. However any closed point of X will suffice: the chart P for $\mathfrak{X}$ automatically gives a chart for X at x.

(5) This is an equivalence by the logarithmic corollary of Gabber's Affine Analogue of Proper Base Change, Theorem 4.60.

□

Corollary 5.48. *Let $\mathfrak{X}$ be a fs log scheme over* spf R*, and assume that the structure morphism is adically log smooth and that $\mathfrak{X}$ is affine with a global chart for its log structure. Then the $\{p\}^c$-profinitely completed adically Kummer étale homotopy type of $\mathfrak{X}$ agrees with the $\{p\}^c$-profinitely completed étale homotopy type of its generic fiber.*

Proof. By Théorèm 7 of [**53**], $\mathfrak{X}$ is automatically algebraizable. Examining the proof above, the only argument that does not also apply for torsors is part (3). However in the case of a topologically of finite type affine, the generic fiber is obtained by simply applying the $-\otimes_R K$ functor to the ring of global sections. Thus the categories of torsors agree by Theorem 3.2.2 of [**21**]. □

Theorem 5.49. *Let $\mathfrak{X}$ be a locally Noetherian and separated log formal scheme, algebraizably log smooth over* spf R*. Then there is an equivalence of ℓ-profinitely completed shapes*

$$\widehat{\text{qét}}_\ell\, \mathfrak{X}_\eta \simeq \widehat{\text{akét}}_\ell\, \mathfrak{X}$$

Proof. The formal scheme admits a hypercovering by a simplicial formal scheme where each connected component in each degree is an affine étale neighborhood in $\mathfrak{X}$ and satisfies the assumptions of the previous lemma. We can now apply hypercover descent to deduce the theorem. □

THEOREM 5.50. *Let $\mathfrak{X}$ be a locally Noetherian and separated log formal scheme, adically log smooth over* $\operatorname{spf} R$. *Then there is an equivalence of* $\{p\}^c$*-profinitely completed shapes*

$$\widehat{\text{qét}}_{\{p\}^c}\,\mathfrak{X}_\eta \simeq \widehat{\text{akét}}_{\{p\}^c}\,\mathfrak{X}$$

PROOF. The formal scheme admits a hypercovering by a simplicial formal scheme where each connected component in each degree is is an affine étale neighborhood in $\mathfrak{X}$ satisfying the assumptions of the above corollary. We can now apply hypercover descent to deduce the theorem. $\square$

Bibliography

[1] Artin, M. and Mazur, B.: Etale homotopy. pages iii+169, 1969.

[2] Stacks Project Authors, T.: *stacks project.* `http://stacks.math.columbia.edu`*, 2016.*

[3] Friedlander, E. M.: Étale homotopy of simplicial schemes, volume 104 of Annals of Mathematics Studies. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[4] Isaksen, D. C.: Strict model structures for pro-categories. In Categorical decomposition techniques in algebraic topology, pages 179–198. Springer, 2003.

[5] Toën, B. and Vezzosi, G.: Segal topoi and stacks over Segal categories. arXiv preprint math/0212330, 2002.

[6] Lurie, J.: Higher topos theory. Number 170. Princeton University Press, 2009.

[7] Hoyois, M.: Higher galois theory. arXiv preprint arXiv:1506.07155, 2015.

[8] Isaksen, D.: A model structure on the category of pro-simplicial sets. Transactions of the American Mathematical Society, 353(7):2805–2841, 2001.

[9] Quick, G.: Profinite homotopy theory. Doc. Math, 13(585-612):2, 2008.

[10] Quick, G.: Continuous group actions on profinite spaces. Journal of Pure and Applied Algebra, 215(5):1024–1039, 2011.

[11] Lurie, J.: Derived Algebraic Geometry XIII: Rational and p-adic Homotopy Theory. http://math.harvard.edu/l̃urie/papers/DAG-XIII.pdf, 2012.

[12] Morel, F.: Ensembles profinis simpliciaux et interprétation géométrique du foncteur t. Bulletin de la Société Mathématique de France, 124(2):347–373, 1996.

[13] Dwyer, W. G. and Spalinski, J.: Homotopy theories and model categories. Handbook of algebraic topology, 73:126, 1995.

[14] Hirschhorn, P. S.: Model categories and their localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

[15] Quick, G.: Stable étale realization and étale cobordism. Adv. Math., 214(2):730–760, 2007.

[16] Dugger, D., Hollander, S., and Isaksen, D. C.: Hypercovers and simplicial presheaves. Math. Proc. Cambridge Philos. Soc., 136(1):9–51, 2004.

[17] Boardman, J. M. and Vogt, R. M.: Homotopy invariant algebraic structures on topological spaces, volume 347. Springer-Verlag, 1973.

[18] Barnea, I., Harpaz, Y., and Horel, G.: Pro-categories in homotopy theory. arXiv preprint arXiv:1507.01564, 2015.

[19] Berkovich, V. G.: Spectral theory and analytic geometry over non-Archimedean fields, volume 33 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990.

[20] Poineau, J. and Pulita, A.: The convergence Newton polygon of a p-adic differential equation II: Continuity and finiteness on Berkovich curves. Acta Math., 214(2):357–393, 2015.

[21] Huber, R.: Étale cohomology of rigid analytic varieties and adic spaces. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[22] Huber, R.: A generalization of formal schemes and rigid analytic varieties. Mathematische Zeitschrift, 217(1):513–551, 1994.

[23] Weil, A. et al.: Numbers of solutions of equations in finite fields. Bull. Amer. Math. Soc, 55(5):497–508, 1949.

[24] *Hartshorne, R.: Algebraic geometry, volume 52. Springer Science & Business Media, 2013.*

[25] *Waterhouse, W. C.: Basically bounded functors and flat sheaves. Pacific J. Math., 57(2):597–610, 1975.*

[26] *Milne, J. S.: Étale cohomology, volume 33 of Princeton Mathematical Series. Princeton University Press, Princeton, N.J., 1980.*

[27] *Revêtements étales et groupe fondamental (SGA 1), volume 3 of Documents Mathématiques (Paris). Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61., Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original.*

[28] *Friedlander, E. M.: The etale homotopy theorie of a geometric fibration. Manuscripta mathematica, 10:209–244, 1973.*

[29] *Nakayama, C.: Logarithmic étale cohomology. Mathematische Annalen, 308(3):365–404, 1997.*

[30] *Kato, K.: Logarithmic structures of Fontaine-Illusie. Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 191–224, 1989.*

[31] *Stix, J.: Projective anabelian curves in positive characteristic and descent theory for log-étale covers, volume 354. Mathematisches Institut der Universität, 2002.*

[32] *Niziol, W.: Toric singularities: log-blow-ups and global resolutions. Journal of algebraic geometry, 15(1):1–29, 2006.*

[33] *Kato, K.: Toric singularities. American Journal of Mathematics, 116(5):1073–1099, 1994.*

[34] *Kisin, M.: Prime to p fundamental groups and tame galois actions. In Annales de l'institut Fourier, volume 50, pages 1099–1126. Chartres: L'Institut, 1950-, 2000.*

[35] *Illusie, L.: An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. Astérisque, pages 271–322, 2002.*

[36] *Hoshi, Y. et al.: The exactness of the log homotopy sequence. Hiroshima Mathematical Journal, 39(1):61–122, 2009.*

[37] *Talpo, M. and Vistoli, A.: Infinite root stacks and quasi-coherent sheaves on logarithmic schemes. arXiv preprint arXiv:1410.1164, 2014.*

[38] *Carchedi, D., Scherotzke, S., Sibilla, N., and Talpo, M.: Kato-Nakayama spaces, infinite root stacks, and the profinite homotopy type of log schemes. arXiv preprint arXiv:1511.00037, 2015.*

[39] *Fujiwara, K. and Kato, F.: Foundations of rigid geometry i. arXiv preprint arXiv:1308.4734, 2013.*

[40] *Berkovich, V. G.: Vanishing cycles for formal schemes. II. Invent. Math., 125(2):367–390, 1996.*

[41] *Berkovich, V. G.: Complex analytic vanishing cycles for formal schemes. 2015.*

[42] *Kato, K. and Nakayama, C.: Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* $\mathbf{C}$*. Kodai Math. J., 22(2):161–186, 1999.*

[43] *Bosch, S., Güntzer, U., and Remmert, R.: Non-archimedean analysis, volume 261 of grundlehren der mathematischen wissenschaften [fundamental principles of mathematical sciences], 1984.*

[44] *Berkovich, V. G.: Étale cohomology for non-archimedean analytic spaces. Publications Mathématiques de l'IHÉS, 78(1):5–161, 1993.*

[45] *Ducros, A.: Families of berkovich spaces, 2011.*

[46] *Berkovich, V. G.: Vanishing cycles for formal schemes. Inventiones mathematicae, 115(1):539–571, 1994.*

[47] *Berkovich, V. G.: On the comparison theorem for étale cohomology of non-Archimedean analytic spaces. Israel Journal of Mathematics, 92(1-3):45–59, 1995.*

[48] *Lütkebohmert, W.: Riemann's existence problem for a p-adic field. Inventiones mathematicae, 111(1):309–330, 1993.*

[49] *Scholze, P.: Perfectoid spaces. Publications mathématiques de l'IHÉS, 116(1):245–313, 2012.*

[50] *Buzzard, K. and Verberkmoes, A.: Stably uniform affinoids are sheafy. Journal für die reine und angewandte Mathematik (Crelles Journal), 2014.*

[51] Conrad, B.: Relative ampleness in rigid geometry (amplitude relative en géométrie rigide). In Annales de l'institut Fourier, volume 56, pages 1049–126, 2006.

[52] Huber, R.: A finiteness result for direct image sheaves on the étale site of rigid analytic varieties. J. Algebraic Geom., 7(2):359–403, 1998.

[53] Elkik, R.: Solutions d'équations à coefficients dans un anneau hensélien. In Annales Scientifiques de l'Ecole Normale Supérieure, volume 6, pages 553–603, 1973.

VITA

NAME	Joseph Berner.
EDUCATION	B.S., Mathematics, University of Illinois at Chicago, Chicago, Illinois, 2012.
	Ph.D., Pure Mathematics, University of Illinois at Chicago, Chicago, Illinois, 2018.
TEACHING EXPERIENCE	Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago.
HONORS	
PROFESSIONAL MEMBERSHIP	American Mathematical Society
ABSTRACTS	
PUBLICATIONS	