Diophantine Properties of Groups of Toral Automorphisms

by

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Thesis submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2017

Chicago, Illinois

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ACKNOWLEDGMENTS

I want to deeply thank my advisor Alex Furman for all the time he spent with me, for countless mathematical discussions and for invaluable moral support. Additionally, I would like to thank the members of my thesis committee for carefully reading the thesis and valuable remarks.

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SUMMARY

In this dissertation we study a shrinking target problem for the action of an arbitrary subgroup of $\mathrm{SL}_d(\mathbb{Z})$ on the d-torus. This can also be viewed as a non-commutative Diophantine approximation problem. Main result establishes an analogue of Khintchine's theorem from the theory of Diophantine approximations for $d = 2$. Consider the action of $\mathrm{SL}_2(\mathbb{Z})$ on the torus $\mathbb{T}^2$. For arbitrary nonelementary subgroup of $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ we show that for $\alpha > \delta_\Gamma$(where δ_Γ is the critical exponent of Γ) for any $y \in \mathbb{T}^2$, for Lebesgue a.e. $x \in \mathbb{T}^2$ there are infinitely many $g \in \Gamma$ satisfying

$$|g.x - y| \leq \|g\|^{-\alpha} \tag{0.1}$$

This extends and improves previously known results in this setting, which were only established for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Our methods are different from previous results concerning similar problems. We use hyperbolicity of convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{Z})$ to obtain sharp spectral estimates for certain Markov operators on $\ell^2(\Gamma)$ (which are better than the ones known from the property of rapid decay, and similar to those obtained by Bader-Muchnik and Boyer). This also leads to construction of optimal random walks on convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and to an effective ergodic theorem for linear action on the 2-torus.

We develop a tool to approximate nonelementary subgroups in $\mathrm{SL}_2(\mathbb{Z})$ by convex cocompact subgroup, which allows us to transfer the results to non convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

We also address similar problem for specific points on the $d-$torus($d \geq 2$), where we prove that $M-$Diophantine points on the torus also admit infinitely many solutions to (Equation 0.1) for some explicit(not sharp) α that depends only on the group and on M. This also allows to obtain estimates of Hausdorff dimension of sets of points of the torus with certain approximation properties. We use harmonic analysis and effective classification of Γ-stationary measures on the $d-$torus.

CHAPTER 1

INTRODUCTION AND STATEMENT OF MAIN RESULTS

Our motivation is to study a shrinking target problem for a group $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ with its natural action by automorphisms on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. For $g \in \mathrm{SL}_d(\mathbb{Z})$ and $x \in \mathbb{R}^d$ the action is given by $g.(x + \mathbb{Z}^d) = gx + \mathbb{Z}^d$ (where gx is just the multiplication of a $d \times d$ matrix g by a column vector x). Specifically, given a subgroup $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ we are interested in finding infinitely many solutions to

$$\{g \in \Gamma \ : \ \|g.x - y\| < \psi(\|g\|)\}$$

for a monotonically decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, e.g. $\psi(R) = R^{-\alpha}$. Here, $\|x - y\|$ is the distance coming from the Euclidean norm on $\mathbb{R}^d$, and $\|g\|$ is the corresponding operator norm on $\mathrm{SL}_d(\mathbb{R})$.

This problem is a noncommutative version of Diophantine approximation. Our goal is to establish a number of results that resemble some of the classical theorems from Diophantine approximation theory.

1.1 Diophantine Approximation Theory

We briefly recall some of the results in Diophantine approximation theory. The goal here is not to survey the field, as it would be a long detour, but rather to focus on results analogues to which we wish to prove in the noncommutative case. Additionally, most of the results below are not stated in the most general known form.

The first result associated with the theory of Diophantine approximations is due to Dirichlet. He showed that for any $x \in \mathbb{R}$ and any $N \in \mathbb{N}$ there exist integers p and q with $1 \leq q \leq N$ satisfying

$$|qx - p| < N^{-1} \qquad \textit{(Dirichlet's inequality)} \tag{1.1}$$

which for irrational α implies the existence of infinitely many pairs p, q satisfying

$$|qx - p| < q^{-1}$$

Following Dirichlet's result, the basic question in Diophantine approximation theory asks about the existence of infinitely many integer solutions $q, p \in \mathbb{Z}$ to the Diophantine inequality

$$|qx - p| < \Psi(q) \qquad \textit{(Diophantine inequality)} \tag{1.2}$$

where $x \in \mathbb{R}$ and a nonincreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given. This was answered by Khintchine in the following dichotomy.

Theorem 1.1.1 *([35], Khintchine)*

Let $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function. Then, for Lebesgue a.e. $x \in \mathbb{R}$, there are infinitely many solutions to $|qx - p| < \Psi(q)$ if and only if the series $\sum_{q=1}^{\infty} \Psi(q)$ diverges

We must point out here the difference between Dirichlet's and Khintchine theorems. Having a solution to (Equation 1.1) with $q < N$ for each sufficiently large integer N implies that there are infinitely many solutions to (Equation 1.2), but not vice versa. For example, Davenport and

Schmidt([16]) showed that if we replace N^{-1} in (Equation 1.1) by ϵN^{-1} with any $\epsilon < 1$, then Lebesgue a.e. α does not satisfy Dirichlet's theorem. On the other hand, the study of continued fractions and their relation to Diophantine approximations ([34]) showed that (Equation 1.2) has infinitely many solutions for a.e. x for $\Psi(q) = \epsilon q^{-1}$ for any $\epsilon > 0$.

Groshev ([26]) proved similar theorem for the simultaneous approximation problem (there are also simultaneous appoximation versions of Dirichlet's theorem due to Davenport and Schmidt), establishing that for almost every $\alpha \in \mathbb{R}^d$ there are infinitely many solutions $q \in \mathbb{Z}^d, p \in \mathbb{Z}$ to $|\langle q, \alpha \rangle - p| < \Psi(|q|_\infty)$ if and only if $\sum_{n=1}^{\infty} n^{d-1}\Psi(n)$ diverges. In the case $d > 1$, the monotonicity assumption on Ψ is not necessary (due to Schmidt). The convergence case follows from an easy application of the Borel-Cantelli lemma, and the main difficulty is in proving the divergence case.

Khintchine's theorem gave rise to many questions. Firstly, one can introduce an extra variable to make the approximation problem inhomogeneous. Namely, one can consider the inequality

$$|qx - y - p| < \Psi(q) \tag{1.3}$$

where $x, y \in \mathbb{R}$ and Ψ are fixed. In this case, Cassels ([13]) showed that an analogue of Dirichlet's theorem fails, i.e. for any function Ψ such that $\Psi(n) \to_{n\to\infty} 0$ there exist x, y, such that for infinitely many integers N, for any integers p and $|q| \leq N$ we have $|qx - y - p| > \Psi(N)$. On the other hand, inhomogeneous analogues of Khintchine-Groshev theorem are true ([13]).

Secondly, in the case when the series $\sum_{q=1}^{\infty} \Psi(q)$ convergences, there is still an infinite set of points $x \in \mathbb{R}$ for which the Diophantine inequality (Equation 1.2) (or Dirichlet inequality (Equation 1.1)) has infinitely many integer solutions. Although this set has measure 0, it might still be quite large. For $M \in \mathbb{R}_+$, call $x \in \mathbb{R}$ an *M-Diophantine number* if there are infinitely many integer solutions to $|qx - p| < q^{-M}$. The set of $M-$Diophantine numbers is large in some sense. Jarnik ([30]) and Besicovitch ([5]) showed that for $M \geq 1$, the Hausdorff dimension of the set of $M-$Diophantine numbers is $2/(M+1)$. Levesley ([40]) proved an inhomogeneous version of the Jarnik-Besicovitch theorem. Call $x \in \mathbb{R}$ a *Liouville number* if it is not $M-$Diophantine for any $M > 0$. In particular, the set of Liouville numbers has zero Hausdorff dimension.

We should mention that from this point Diophantine approximation theory branches into many different directions that have interconnections with numerous areas of mathematics. We would like to explore its connection to the theory of shrinking targets, as explained below.

1.1.1 Diophantine approximation viewed as shrinking target problem

Classical Diophantine approximation problem can be viewed through a prism of the following dynamical system. Consider multiplicative semigroup $\mathbb{N}$ acting on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by $n.(x + \mathbb{Z}) = nx + \mathbb{Z}$. $\mathbb{T}$ has a natural metric, and inhomogeneous Diophantine approximation is concerned with solutions to $d_{\mathbb{T}}(q.x, y) < \Psi(q)$ for given x and y. We can now formulate Diophantine approximation for actions of other semigroups(or groups). For abelian semigroups, this is known as simultaneous approximation, and our main interest is in noncommutative case.

Consider a countable semigroup (or a group) Γ, with submultiplicative norm $\|\cdot\|$. Assume Γ acts on a metric space (X, d). Denote by $B(y, r)$ the open ball of radius r around $y \in X$. Given nonincreasing $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $x, y \in X$ we would like to study whether the Diophantine inequality $g.x \in B(y, \Psi(\|g\|))$ has infinitely many solutions $g \in \Gamma$. Alternatively, we can investigate the Dirichlet question, namely, whether for any sufficiently large integer N, there exists $g \in \Gamma$ with $\|g\| \leq N$ satisfying $g.x \in B(y, \Psi(N))$.

It should be obvious now how to restate most of the questions from classical Diophantine approximation theory in the language of shrinking target problems. In fact, we can consider a probability measure space (X, m), and replace the balls $B(y, r)$ by a family of arbitrary measurable targets B_r, with $m(B_r) = \Psi(r)$. So, loosely speaking, our targets may not only be shrinking with the time, but moving and changing their shapes as well.

We should list examples of the actions as above that were investigated in the past. Action of $\mathbb{N}$ on the $d-$torus generated by an integer valued $d \times d$ matrix was studied by Hill and Velani ([28]). There are quite a few examples of actions of noncommutative groups that were studied in this context: actions of finitely generated subgroups of a Lie group acting on the Lie group itself ($SU(2)$ in ([21]), affine group of the line ([48]), nilpotent Lie groups ([1])), actions of automorphism groups on Heisenberg nilmanifolds ([4]), actions of lattices in semisimple Lie groups on suitable homogeneous spaces G/H ([23]). It is possible to translate results about quantifying residual finiteness ([29]) as shrinking target results for the action of a group on its profinite completion with a suitable metric.

This thesis focuses on the action of subgroups of $SL_d(\mathbb{Z})$ on the $d-$dimensional torus.

1.2 Main results

1.2.1 Approximation by Lebesgue a.e. point on the two torus

We start with the discussion of the Γ-Diophantine properties of Lebesgue almost every point. Our results have a sharp form in dimension $d = 2$. Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts on the hyperbolic plane $(\mathbf{H}^2, d_{\mathbf{H}^2})$. Fix a point $x_0 \in \mathbf{H}^2$, and denote

$$B_n = \{g \in \Gamma \ : \ d_{\mathbf{H}^2}(g.x_0, x_0) \leq n\}, \qquad \delta_\Gamma = \limsup_{n\to\infty} \frac{1}{n} \cdot \log \#B_n.$$

Then δ_Γ is called the critical exponent of Γ. We say that a subgroup of $\mathrm{SL}_2(\mathbb{R})$ is elementary if it is virtually abelian, otherwise it is nonelementary. Now we are ready to state the main result.

Theorem A

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a non-elementary subgroup. For any $y \in \mathbb{T}^2$, for Lebesgue a.e. $x \in \mathbb{T}^2$, the set

$$\left\{g \in \Gamma \ : \ \|g.x - y\| < \|g\|^{-\alpha}\right\} \qquad \textit{is}$$

1. *finite for every $\alpha > \delta_\Gamma$,*

2. *infinite for every $\alpha < \delta_\Gamma$.*

The proof proceeds via a reduction to subgroups of $\mathrm{SL}_2(\mathbb{Z})$ whose action on the hyperbolic plane is convex cocompact(i.e. groups that act properly discontinously and cocompactly by isometries on some convex subset of the hyperbolic plane). For such groups we have even sharper estimates below.

Let us now replace the balls around $y \in \mathbb{T}^2$ by an arbitrary family of targets $\{\mathrm{Targ}_r\}_{r>0}$ of Lebesgue subsets of the torus with measure $m(\mathrm{Targ}_r) = \pi r^2$.

Theorem B

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a subgroup whose action on the hyperbolic plane is convex cocompact. Let $\{\mathrm{Targ}_r\}_{r>0}$ be a family of Lebesgue subsets of the torus of measure $m(\mathrm{Targ}_r) = \pi r^2$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function as $r \to \infty$. Then for Lebesgue-a.e. $x \in \mathbb{T}^2$ the set

$$\left\{ g \in \Gamma \;:\; g.x \in \mathrm{Targ}_{\psi(\|g\|)} \right\} \qquad \text{is}$$

1. *finite, if*

$$\sum_{n=1}^{\infty} n^{2\delta_\Gamma - 1} \psi(n)^2 < \infty,$$

2. *infinite if*

$$\sum_{n=1}^{\infty} (\log n)^2 n^{-2\delta_\Gamma - 1} \psi(n)^{-2} < \infty.$$

Remark 1.2.1 *The rates in Theorem B are sharper than in Theorem A. For example, (1) holds for $\psi(n) = n^{-\delta_\Gamma} \log^{-0.5-\epsilon} n$, while (2) holds for $\psi(n) = n^{-\delta_\Gamma} \log^{1.5+\epsilon} n$ for any $\epsilon > 0$.*

Remark 1.2.2 *In fact, Theorem B can be stated as Dirichlet's theorem, namely then for Ψ as in part (2) and any $y \in \mathbb{T}^2$, for Lebesgue a.e. $x \in \mathbb{T}^2$, for any sufficiently large T, there exists $g \in \Gamma$ satisfying*

$$\|g\| \leq T, \qquad \|g.x - y\| \leq \psi(T).$$

This can be seen immediately from the proof of Theorem B(see Remark 4.1.1).

The finiteness part follows from the first Borel-Cantelli lemma, which says that if B_n is a sequence of measurable subsets in a probability space (X, m) such that $\sum_{n=1}^{\infty} m(B_n) < \infty$, then $m(\limsup B_n) = 0$. Second Borel-Cantelli lemma states that the partial converse is true, namely if the sets are independent and $\sum_{n=1}^{\infty} B_n = \infty$ then $m(\limsup B_n) = 1$. The classical independence assumption in the second Borel-Cantelli lemma may be replaced by decay of correlations conditions. In our case, this role is played by spectral estimates for the Γ-action on the torus $\mathbb{T}^2$ as discussed in § 1.2.3.

1.2.2 Approximation by Diophantine points

For a subgroup $\Gamma < \mathrm{SL}_d(\mathbb{Z})$, $\alpha > 0$, and points $x, y \in \mathbb{T}^d$, we say that y admits (Γ, α)-*fast approximation by* x if

$$\left\{g \in \Gamma \ : \ \|g.x - y\| < \|g\|^{-\alpha}\right\} \qquad \text{is infinite.}$$

Theorem A shows that every $y \in \mathbb{T}^2$ is $(\Gamma, \delta_\Gamma - \epsilon)$-fast approximable by Lebesgue a.e. $x \in \mathbb{T}^2$ for any $\epsilon > 0$. In this section we try to analyze how large is the exceptional set of $x \in \mathbb{T}^2$, which fail to provide fast approximations for all points y on the torus. We work with subgroups $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ acting on the d-torus $\mathbb{T}^d$ with $d \geq 2$.

For $q \in \mathbb{N}$, denote by $R_q \subset \mathbb{T}^d$ the set $\frac{1}{q} \cdot \mathbb{Z}^d + \mathbb{Z}^d$ of points with rational coordinates with denominators dividing q, and by $R = \bigcup R_q$ the set of all rational points. We say that a point $x \in \mathbb{T}^d$ is *M-Diophantine* if there are only finitely many $q \in \mathbb{N}$ so that x is q^{-M}-close to a point in R_q. Points that are not M-Diophantine for any M, are called *Liouville*.

Note that rational points $x \in \mathbb{T}^d$ have finite Γ-orbits on the torus (because each R_q is $\mathrm{SL}_d(\mathbb{Z})$-invariant), and so any $y \in \mathbb{T}^d \setminus R$ does not admit (Γ, ϵ)-fast approximation by $x \in R$ for any $\epsilon > 0$.

We want to establish a relation between $(\Gamma, \alpha)-$fast approximability by $x \in \mathbb{T}^d$ and Diophantine properties of x. We use the results of [8]. For $d \geq 3$ we need to impose the following conditions on $\Gamma < \mathrm{SL}_d(\mathbb{Z})$:

(SI) Γ acts strongly irreducibly on $\mathbb{R}^d$, i.e. every subgroup of finite index in Γ preserves no non-trivial vector subspaces.

(PE) Γ has a proximal element, i.e. an element with a simple dominant eigenvalue.

For example, existence of a proximal element is guaranteed if Γ is Zariski dense ([45]). Both conditions are automatically satisfied by any nonelementary subgroup of $\Gamma < \mathrm{SL}_2(\mathbb{Z})$.

Theorem C

Let $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ satisfy (SI) and (PE). Then there exists $C_\Gamma > 0$, such that for every $M > 0$, every point $y \in \mathbb{T}^d$ is $(\Gamma, \frac{C_\Gamma}{M})$-fast approximable by any M-Diophantine point $x \in \mathbb{T}^d$.

We can deduce as a corollary an analogue of Jarnik-Besicovitch Theorem.

Corollary 1.2.3

Let Γ be as in Theorem C and let $y \in \mathbb{T}^d$. The set of points $x \in \mathbb{T}^d$ that do not give (Γ, ϵ)-fast approximation of y for any $\epsilon > 0$ consists only of Liouville points and, in particular, has zero Hausdorff dimension.

1.2.3 Spectral estimates

Let us now state the main spectral estimate needed for the proof of Theorems A and B. Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a subgroup whose action on the hyperbolic plane is convex cocompact. Let $\pi : \Gamma \to U(\mathcal{H})$ be a unitary Γ-representation on Hilbert space $\mathcal{H}$, and μ a probability measure on Γ. Define the Markov operator on $\mathcal{H}$

$$\pi(\mu) \leq \sum_{g \in \Gamma} \mu(g) \cdot \pi(g).$$

Note that it always satisfies $\|\pi(\mu)\| \leq 1$ and if μ is symmetric, then $\pi(\mu)$ is self-adjoint. We shall denote by π the unitary Γ-representation on $L^2(\mathbb{T}^2)$, and π_0 the sub-representation on $L^2_0(\mathbb{T}^2)$. In the proof of Theorem B we need the estimate provided in the following result.

Theorem 1.2.4

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a convex cocompact subgroup. There exists a sequence of symmetric probability measures μ_n on Γ with $\mathrm{supp}(\mu_n) \subset B_n$ and

$$\|\pi_0(\mu_n)\| \leq e^{-\frac{1}{2}\delta_\Gamma \cdot n + \log n + O(1)}.$$

In fact, the above estimate holds for μ_n being uniform measures on the shells $S_n = B_n \setminus B_{n-k}$ for some fixed k, that depends only on Γ.

We can view the spectral estimates we obtained as a quantitative ergodic theorem for the linear action on the 2−torus.

Corollary 1.2.5

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a convex cocompact subgroup, and let $S_n = B_n \setminus B_{n-k} \subset \Gamma$ be shells as above. Then for any $f \in L^2(\mathbb{T}^2, m)$ we have

$$\left\| \frac{1}{|S_n|} \sum_{g \in S_n} f(g.x) - \int_{\mathbb{T}^2} f dm \right\|_2 \leq n e^{-\frac{1}{2}\delta_\Gamma \cdot n + O(1)} \cdot \|f\|_2.$$

The constant δ_Γ in the above rate of convergence cannot be improved by choosing a different averaging family S_n, or different averaging weights (see § 6).

1.2.4 Spectrally optimal random walks

For weakly equivalent unitary Γ-representations $\pi' \sim \pi''$, one has $\|\pi'(\mu)\| = \|\pi''(\mu)\|$ for any probability measure μ on Γ. Hence π_0 in the above theorem can be replaced by any weakly equivalent unitary representation, and it is known (Proposition 2.2.7) that the left regular representation $\lambda : \Gamma \to U(\ell^2\Gamma)$ is such. So Theorem 1.2.4 is a special case of the following more general result, in which convex cocompact subgroup of $\mathrm{Isom}(\mathbf{H}^2)$ is replaced by a group Γ acting properly and cocompactly on a proper quasiruled hyperbolic space (X, d). The notion of quasiruled hyperbolic spaces is defined in § 2. We remark that geodesic Gromov hyperbolic spaces are examples of proper quasiruled hyperbolic spaces. We have the following general form of Theorem 1.2.4:

Theorem D

Let (X, d) be a proper quasiruled hyperbolic space, Γ a finitely generated group, acting properly cocompacty by isometries on (X, d). Then for some k and all n, the uniform distributions μ_n on the shells $S_n = B_n \setminus B_{n-k}$ satisfy

$$\|\lambda(\mu_n)\| \leq e^{-\frac{1}{2}\delta_\Gamma \cdot n + \log n + O(1)}$$

where λ is the regular representation on $\ell^2(\Gamma)$.

In fact, in our proof we replace the regular representation λ by the quasi-regular representation on the boundary of Γ endowed with the Patterson-Sullivan measure, which satisfies the same estimate.

Let us put Theorem D in a broader perspective. Let Γ be a group with proper left invariant metric d, and let us denote by B_n the ball of radius n in Γ. Given a unitary Γ-representation π define the function $\rho_\pi : \mathbb{N} \to \mathbb{R}_+$ by

$$\rho_\pi(n) := \min \{\|\pi(\mu)\| \ : \ \mathrm{supp}(\mu) \subset B_n\},$$

where $\|\cdot\|$ is the operator norm. Since $B_n \cdot B_m \subset B_{n+m}$, and the operator norm is submultiplicative, one has $\rho_\pi(n+m) \le \rho_\pi(n) \cdot \rho_\pi(m)$. Therefore the limit

$$\lim_{n\to\infty} \frac{1}{n} \cdot \log \rho_\pi(n)$$

exists. It might be of interest to investigate $\rho_\pi(n)$ for a given Γ, d, π as above.

For a finitely supported probability measure μ on Γ we recall the definitions of the **drift** and the **asymptotic entropy**

$$\ell(\mu) := \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{g\in\Gamma} d(g,e) \cdot \mu^{*n}(g),$$

$$h(\mu) := \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{g\in\Gamma} -\log \mu^{*n}(g) \cdot \mu^{*n}(g).$$

Let λ be the left regular representation of Γ. The following inequalities are well known and hold for any finitely supported symmetric probability measure

$$-2\log(\|\lambda(\mu)\|) \le h(\mu) \le \delta_\Gamma \cdot \ell(\mu).$$

If $\mathrm{supp}(\mu) \subset B_n$, one has the trivial estimate $\ell(\mu) \le n$, that gives the upper bound $\le \delta_\Gamma \cdot n$ for th right hand side above.

Theorem D describes a situation that allows one to choose a symmetric μ_n supported in B_n so that the above sequence of inequalities is asymptotically tight

$$\delta_\Gamma \cdot n - 2\log n + O(1) \leq -2\log(\|\lambda(\mu_n)\|) \leq h(\mu_n) \leq \delta_\Gamma \cdot \ell(\mu_n) \leq \delta_\Gamma \cdot n.$$

1.3 Existing results

1.3.1 Diophantine properties of noncommutative groups

Similar shrinking target problems were previously studied by multiple authors.

Laurent and Nogueira in [39] considered the natural $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{R}^2$. They showed (by explicit construction) that for $x \in \mathbb{R}^2$ with irrational slope, there are infinitely many solutions $g \in \mathrm{SL}_2(\mathbb{Z})$ to $|g.x - y| \leq \Psi(\|g\|)$ in the following situations:

1. $\Psi(R) = R^{-1}, y = (0,0)$
2. $\Psi(R) = cR^{-1/2}, y = (y_1, y_2)$, where either y_1/y_2 is rational or $y_2 = 0$, and c is explicit constant depending on x, y.
3. $\Psi(R) = c'R^{-1/3}, y = (y_1, y_2)$, where either y_1/y_2 is irrational, and c' is explicit constant depending on x, y.

Recall that $\delta_{\mathrm{SL}_2(\mathbb{Z})} = 1$, so Theorem A gives rate of $\Psi(R) = R^{-1+\epsilon}$ for any $y \in \mathbb{T}^2$ and for a.e. $x \in \mathbb{T}^2$

Similar results were given for cocompact lattices of $\mathrm{SL}_2(\mathbb{R})$ (and $\mathrm{SL}_2(\mathbb{Z})$ with extra Diophantine conditions on starting points) acting on the plane by Maucourant and Weiss in [41] and for the $SL_2(\mathbb{C})$ action on the complex plane by Policott in [44] using effective equidistribution results. Both of those have explicit (but far from optimal exponents in the approximation rates).

In contrast to Theorem A, the above results fix the starting point x and consider different targets y that $SL_2(\mathbb{Z})$ can effectively approximate, whereas in our case, we fix the target y and study the set of starting points which give successful approximation.

Ghosh, Gorodnik and Nevo in [23] considered a more general setting, where a lattice $\Gamma < G$ acts on a homogenous space G/H, with a dense Γ-orbit. As a corollary, they established Theorem A for $\Gamma = SL_2(\mathbb{Z})$ for a.e. $y \in \mathbb{T}^2$.

All of the results listed above assume the acting group to be a lattice. So the main novelty in our work is in treating arbitrary, in particular, thin subgroups Γ in $SL_2(\mathbb{Z})$(i.e. discrete Zariski dense subgroups wiht infinite covolume).

1.3.2 Solutions to Diophantine inequality in proper subsets

Even though, a lot is known about solutions to Diophantine inequality in the classical setting, it seems a much harder problem if we require additional properties of the solutions.

For example, the only analogue to Theorem C in the classical Diophantine approximation theory of which the author is aware is the following. Bourgain, Lindenstrauss, Michel and Venkatesh were looking for solutions to classical Diophantine inequality that lie in multiplicative semigroup of $\mathbb{N}$ generated by two elements. In ([10], Theorem 1.8) they proved that for any multiplicatively independent integers a, b (i.e. not powers of the same integer) there exist

$k = k(a,b), N_0 = N_0(M,a,b)$ so that for any $N > N_0, y \in \mathbb{R}/\mathbb{Z}$, and any $M-$Diophantine $x \in \mathbb{R}/\mathbb{Z}$ there exist $s,t \leq N$ satisfying

$$\left|a^s b^t x - y\right| \leq (\log\log N)^{-k}.$$

In the non-commutative case, Kirsebom in [36] considered orbits in $\mathbb{T}^d$ of random walks of general subgroups of $\mathrm{SL}_d(\mathbb{Z})$, (subgroups that do not fix a proper subtorus). Using extreme value theory, with $u_n = n^{-d}e^{-r}$, he showed that for a.e. starting point $x \in \mathbb{T}^d$, the probability of random walk not returning to u_n-neighborhood of x after less than n steps converges (as $n \to \infty$) to Ce^{-dr} where the constant $C > 0$ is explicit and depends only on the dimension d. In particular, this probability is bounded away from 1 for large n, hence it is easy to deduce that in this case the Diophantine inequality has infinitely many solutions for $\Psi(R) = \log(R)^{-\lambda}$ (where $\lambda > 0$ depends on the subgroup Γ). Theorem A gives a much better rate $\Psi(R) = R^{-\alpha}$ for explicit and sharp exponent α.

1.3.3 Spectral estimates

Spectral estimates similar to ones we obtain in Theorem 3.0.9 can be deduced from Bader-Muchnick in [3] and by Boyer in [11] for groups acting properly cocompactly by isometries on CAT(-1) space. They were interested in the irreducibility of the boundary representation.

Slightly weaker estimates on the norm of $\lambda_\Gamma(\mu_n)$ can be obtained from the property of Rapid Decay. Let Γ be a discrete group, and l a length function (i.e. $l : \Gamma \to \mathbb{R}_+$, with $l(e) = 0, l(g) = l(g^{-1})$, and $l(gh) \leq l(g) + l(h)$ for any $g, h \in \Gamma$. We say that Γ has the *property* *of Rapid Decay(RD) with respect to l* if there exists a polynomial P such that for any f in the complex group algebra $\mathbb{C}\Gamma$ supported on elements of length shorter than n, the following inequality holds:

$$\|f\|_* \leq P(n)\|f\|_2$$

where $\|f\|_*$ denotes the operator norm of f acting by left convolution on $l^2(\Gamma)$.

If d is a left invariant pseudo-metric on Γ, one can consider $l(g) = d(g, e)$ as the length function. Property RD was first established for free groups by Haagerup (with respect to word metric), and later Jollisant and de La Harpe ([31], [17]) proved it for geometrically finite Kleinian groups without parabolic elements, with the length function coming from the action on the hyperbolic space. In fact, in both of the above cases they also show that the polynomial P in the definition is of degree 2. In particular, for convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{Z})$, let $f(g) = \frac{1}{\# S_n}\chi_{S_n}(g)$, where χ_{S_n} is the characteristic function of the shell $S_n = B_n \setminus B_{n-k}$. The convolution by f is the operator $\lambda_\Gamma(\mu_n)$ where μ_n is the uniform distribution on S_n. Using Coornaert's estimate on the cardinality of S_n (Lemma 2.1.8), we have

$$\|f\|_2^2 = \sum_{g\in B_n} \frac{1}{|B_n|^2} = \frac{1}{|B_n|} \leq e^{-\delta_\Gamma n + O(1)},$$

hence,

$$\|\lambda_\Gamma(\mu_n)\| = \|f\|_* \leq Cn^2\|f\|_2 \leq e^{-\frac{1}{2}\delta n + 2\log n + O(1)}.$$

Note that Theorem D gives $\log n$ in the exponent, versus $2\log n$ obtained from property RD.

CHAPTER 2

BACKGROUND AND NOTATIONS

We will use Landau's asymptotic notation: $f(x) = O(g(x))$ means that there exists constant $K > 0$, so that $|f(x)| \leq Kg(x)$. For a function $h : X \to \mathbb{R}$(where X is a general space), we will write $h = O(1)$ meaning that h is a bounded function.

2.1 Quasi-ruled hyperbolic spaces

2.1.1 Basic definitions

Let (X, d) be a metric space. For $x, y, z \in X$ the Gromov product is defined by

$$(x|y)_z := \frac{1}{2}\left(d(x, z) + d(y, z) - d(x, y)\right)$$

The notion of hyperbolicity is usually studied in the setting of complete geodesic spaces. In this paper we are interested to exploit the hyperbolicity of non-geodesic metric spaces. For our purposes we want a notion for which the boundary theory and the theory of quasiconformal measures still exist. We recall the theory of quasiruled hyperbolic spaces (see appendix of [6] for more details)

Definition 2.1.1 *Let X be a proper metric space.*

(1) A (λ, c)-quasigeodesic curve (resp. ray, segment) is the image of $\mathbb{R}$ (resp. $\mathbb{R}_+$, a compact interval of $\mathbb{R}$) by a (λ, c)-quasi-isometric embedding.

(2) *A τ-quasiruler is a quasigeodesic $g : \mathbb{R} \to X$ (resp. quasisegment $g : I \to X$, quasiray $g : \mathbb{R}+ \to X$) such that, for any $s < t < u$, we have*

$$(g(s)|g(u))_{g(t)} \leq \tau$$

(3) *We say that X is quasi-ruled if there exist constants $\lambda \geq 1$ and $\tau, c \geq 0$ such that any two points in X can be joined by a (λ, c)-quasigeodesic, and every (λ, c)-quasigeodesic is a τ-quasiruler.*

(4) *A quasitriangle is given by three points $x, y, z \in X$ together with three quasirulers(edges) joining them.*

(5) *A quasitriangle is $\delta-$thin if any of its edges is in the δ-neighborhood of the union of two other edges.*

(6) *A quasiruled metric space X is called hyperbolic if it satsifies the Rips condition for some $\delta \geq 0$, i.e. every quasitriangle is $\delta-$thin.*

Example 2.1.2 *An important example of quasiruled hyperbolic spaces is the class of convex cocompact subgroups of* $\mathrm{Isom}(\mathbf{H}^n)$*. They act properly cocompactly by isometries on their convex core in $(\mathbf{H}^n, d_{\mathbf{H}^n})$. Fix $x_0 \in \mathbf{H}^n$ a basepoint. We can define the left invariant metric on Γ: for $g, h \in \Gamma$, $d(g, h) := d_{\mathbf{H}^n}(g.x_0, h.x_0)$. This might be a pseudo-metric, but properness of the action ensures that a stabilizer of a point if finite, which will not matter for us for the purpose of asymptotic computations. If the stabilizer is trivial, this is a honest metric, which is quasi-*

isometric to the word metric on Γ, and with respect to this metric, Γ is itself a proper quasiruled hyperbolic space.

One of the useful features of thin triangles is that they admit a centroid. More precisely, given three points x, y, z, there is a tripod T(a tree with three leaves) and an isometric embedding $f : \{x, y, z\} \to T$ such that the images are the endpoints of T. We denote by C_T the center of T. If x, y, z all lie on the same geodesic, the associated tripod T is degenerate (it has only 2 leaves). In such a situation he center of T is one of the points x, y, z that is not a leaf.

Lemma 2.1.3 *(Tripod lemma)([6]Lemma A.3)*

Let Δ be a δ-thin quasitriangle with vertices x, y, z in a quasiruled hyperbolic space X. There is a $(1, c_0)$-quasiisometry $f_\Delta : \Delta \to T$, where T is the tripod associated with x, y, z and c_0 depends only on the data $(\delta, \lambda, c, \tau)$.

We call $f_\Delta^{-1}(C_T)$ a centroid of Δ. Of course, the map f_Δ, and thus the centroid are not unique, but there exists a constant c_1 depending on the space only, such that for every quasitriangle $\Delta \subset X$, every 2 centroids of Δ are at most at distance c_1.

2.1.2 Visual boundary and Patterson-Sullivan measures

Geodesic hyperbolic spaces admit a visual boundary and conformal densities on it. In a similar fashion, proper quasiruled hyperbolic metric spaces admit a natural boundary, called the *visual boundary* associated to (X, d, x_0)

$$\partial X := \left\{ (x_i)_{i=1}^{\infty} \ : \ x_i \in X, \lim_{i,j\to\infty} (x_i|x_j)_{x_0} = \infty \right\} / \sim$$

where

$$(x_i) \sim (y_i) \Leftrightarrow (x_i|y_i)_{x_0} \underset{i\to\infty}{\longrightarrow} \infty$$

The visual boundary is the set of equivalence classes of infinite quasiruler rays, where two rays are equivalent if they are at bounded Hausdorff distance from each other. The boundary ∂X doesn't depend on the choice of basepoint x_0.

Similarly to geodesic hyperbolic spaces, in a quasiruled hyperbolic space there exists a quasiruled curve between any two points in the boundary.

The boundary ∂X may be equipped with the topology, whose basis is given by shadows. For $y \in X$ and $C \geq 0$, the *shadow* $O_C(x_0, y)$ is

$$O_C(x_0, y) := \left\{ [(z_i)] \in \partial X \ : \ \liminf_{j\to\infty} (z_j|y)_{x_0} \geq d(x_0, y) - C \right\}$$

Alternatively, a point $\xi \in \partial X$ belongs to the shadow $O_C(x_0, y)$ if some quasiruler ray from x_0 to ξ intersects the closed C-ball around y.

Sometimes we would like to think of shadows as subsets of X, in this case

$$\bar{O}_C(x_0, y) := \{z \in X | (y|z)_{x_0} \geq d(x_0, y) - C\}$$

For $z \in X$, the *Busemann function at* z, $\beta_z : X \times X \to \mathbb{R}$ is

$$\beta_z(x, y) := d(z, x) - d(z, y)$$

For $\xi \in \partial X$, we define Busemann function at ξ by

$$\beta_\xi(x,y) := \sup_{z_t \to \xi} \limsup_{t\to\infty} \{d(z(t),x) - d(z(t),y)\}$$

The above sup should be taken along all possible quasiruler rays $z(t)$ from y to ξ.

Recall that for $\Gamma < Isom(X,d)$, with a chosen basepoint $x_0 \in X$, the *critical exponent* for Γ is given by

$$\delta_\Gamma := \limsup_{R\to\infty} \frac{\log \#\{g \in \Gamma \ : \ d(g.x_0, x_0) \leq R\}}{R}$$

The Γ action on X induces natural action on ∂X and on the space of Busemann functions.

$$g.\beta_\xi(x,y) := \beta_{g.\xi}(x,y) = \beta_\xi(g^{-1}.x, g^{-1}.y)$$

The next theorem summarizes the main properties of quasiconformal measures on the boundary of X. It was proved by Coornaert in [14] for geodesic hyperbolic spaces, and by Blachere-Haissinsky-Mathieu in [6] for proper quasiruled hyperbolic spaces.

Theorem 2.1.4 *([6], Theorem 2.3)*

Let Γ be a finitely generated group acting properly cocompactly by isometries on a pointed proper quasiruled hyperbolic space (X,d,x_0). For any small enough $\epsilon > 0$

(1) There exists a visual metric d_ϵ on the boundary ∂X, its Hausdorff dimension is given by

$\dim_H(\partial X, d_\epsilon) = \delta_\Gamma/\epsilon$

(2) There exists a $\Gamma-$*equivariant family* $\{\rho_x\}_{x\in X}$ *of Radon probability measures on* ∂X*, i.e. for any* $g\in\Gamma, x\in X$ *we have* $g_*\rho_x=\rho_{g.x}$*. Moreover, the entire family* ρ_x *is in the same measure class.*

(3) The distortion of a measure by the Γ *action is measured by the Busemann functions, namely for any* $\xi\in\partial X$

$$\frac{d\rho_y}{d\rho_x}(\xi)=e^{-\delta_\Gamma\beta_\xi(y,x)+O(1)}$$

(4) ρ_x *are Ahlfors-regular of dimension* δ_Γ/ϵ*, i.e. for any* $\xi\in\partial X$*, for any* $r\in(0,diam_\epsilon(\partial X))$*, we have*

$$\rho_x(B_{d_\epsilon}(a,r))=r^{\delta_\Gamma/\epsilon+O(1)}$$

(5) Γ *action on* $(\partial X,\rho_x)$ *is ergodic for any* $x\in X$

This class of measures is called the *Patterson-Sullivan measure class.* It does not depend on the choice of ϵ. Denote $\rho:=\rho_{x_0}$.

In fact, the metric d_ϵ is given in the following way. First one extends the Gromov product to the boundary by defining

$$([x_i]|[y_i])_{x_0}:=\limsup_{i\to\infty}(x_i|y_i)_{x_0}$$

where lim sup is taken over all quasiruled rays in the equivaence classes. There exists $\epsilon_0 > 0$, such that for any $0 < \epsilon < \epsilon_0$ there exists a metric on ∂X satisfying

$$d_\epsilon(\xi, \eta) := O(1)e^{-\epsilon(\xi|\eta)_0}$$

Such metric d_ϵ induces the boundary topology described above. Moreover, the shadows are related to the balls in metric d_ϵ.

Proposition 2.1.5 *([6], Proposition 2.1)*

There exists $C_0 \geq 0$, such that for any $C \geq C_0$ and any $x \in X$

$$diam_\epsilon(O_C(x_0, x)) = e^{-\epsilon d(x,x_0)+O(1)}$$

Combining the fact that Patterson-Sullivan measures are Ahlfors regular with respect to this metric and the description of shadows we can conlude the following corollary known as the lemma of the shadow,

Corollary 2.1.6 *(Lemma of the shadow, [6], Lemma 2.4)*

There exists $C \geq 0$, such that for any $x \in X$

$$\rho(O_C(x_0, x)) = e^{-\delta_\Gamma d(x,x_0)+O(1)}$$

The Γ action on (X, d) induces the left invariant metric $d_0 := d(g.x_0, h.x_0)$. If the action is proper and cocompact, (Γ, d_0) is itself a proper quasiruled hyperbolic space. We denote by B_n the $n-$ball in Γ with respect to d_0 and define the $k-$shell:

$$S_{n,k} := B_n \setminus B_{n-k}$$

The shadows of the shells $S_{n,k}$ cover the boundary with finitely many overlaps (with the bound uniform in n). More precisely,

Lemma 2.1.7 *(*[14]*, Lemma 6.5)*

There exist $C, k \geq 0$ such that for any $n \in \mathbb{N}$

$$\bigcup_{g \in S_{n,k}} O_C(e, g) \supseteq \partial\Gamma$$

Moreover, there exists L(depending only on C and k) such that for any n and any $\xi \in \partial G$

$$\#\{g \in S_{n,k} \ : \ \xi \in O_C(e, g)\} \leq L$$

i.e. every $\xi \in \partial\Gamma$ is covered by at most L shadows of elements in the shell $S_{n,k}$

We also have precise asymptotics of the growth of balls and shells

Lemma 2.1.8 *(*[14]*, Theorem 7.2)*

There exists $k > 0$, such that

(1) $\#S_{n,k} = e^{\delta_\Gamma n + O(1)}$

(2) $\#B_n = e^{\delta_\Gamma n + O(1)}$

Two above lemmas are stated for geodesic hyperbolic spaces in [14], but the same proofs will work for quasiruled hyperbolic spaces.

Definition 2.1.9 *Fix $k > 0$ for which Lemmas 2.1.7 and 2.1.8 hold. Denote the shell $S_n := S_{n,k}$.*

Definition 2.1.10 *Let Γ as above. Let $C \geq 0$ be large enough to satisfy Corollary 2.1.6 and Lemma 2.1.7. For $g \in \Gamma$, the $g-$shadow in Γ is a subset of $\partial\Gamma$ given by*

$$O(g) := O_C(e, g)$$

2.2 Some Unitary Representations

A discrete group Γ acts on itself by left multiplication which induces the *left regular representation* $\lambda_\Gamma : \Gamma \to U(l^2(\Gamma))$ given by:

$$\lambda_\Gamma(g)f(h) = f(g^{-1}h) \quad \text{for } f \in l^2(\Gamma), g \in \Gamma$$

If Γ acts by measure preserving transformations on a probability space (X, m) we can associate with the action the *Koopman representation* $\pi : \Gamma \to U(L^2(X, m))$, which is given by

$$\pi(g)f(x) = f(g^{-1}.x) \quad \text{for } f \in L^2(X), g \in \Gamma$$

The constant functions are invariant, hence we denote by π_0 the restriction of π to the orthogonal complement of the constant functions $L^2_0(X,m) = \left\{f \in L^2(X,m) \ : \ \int_X f dm = 0\right\}$.

If, however, the action only preserves the measure class, we can modify the Koopman representaion to become a unitary representation $\pi_X : G \to U(L^2(X,\nu))$:

$$\pi_X(g)f(x) = f(g^{-1}.x)\sqrt{\frac{dg_*\nu}{d\nu}(x)}$$

Such π_X is called the *quasi-regular representation.*

For example, if Γ is as in § 2.1.2, Γ acts on its visual boundary equipped with Patterson Sullivan measure. We call the associated quasi-regular representation the *boundary representation* and denote it by $\pi_{\partial\Gamma}$.

Given finitely supported probability measure μ on Γ and a unitary representation $\sigma : \Gamma \to U(H)$ we can average the representation to get a Markov operator $\sigma(\mu) : H \to H$ by

$$\sigma(\mu) = \sum_{g\in\Gamma} \mu(g)\sigma(g)$$

Example 2.2.1 *$\lambda_\Gamma(\mu)$ is the Markov operator associated with the random walk on Γ with law μ. It is known that $\|\lambda_\Gamma(\mu)\| < 1$ if and only if Γ is nonamenable.*

Example 2.2.2 *Let $H < \Gamma$ a subgroup. Γ acts on Γ/H by left multiplication, which induces the representation $\pi_{\Gamma/H} : \Gamma \to U(l^2(\Gamma/H))$.*

Theorem 2.2.3 *(Kesten,* [33]*)*

Let μ be a uniform measure on some generating set S of Γ. If H is amenable, then

$$\|\lambda_\Gamma(\mu)\| = \|\pi_{\Gamma/H}(\mu)\|$$

In fact Kesten proved that, in the case where H is a normal subgroup, the converse is also true. A generalized version of this is the following:

Theorem 2.2.4 *(Kuhn,* [38]*)*

Let Γ be a discrete group, $\mu \in Prob(\Gamma)$, and let Γ act ergodically preserving the measure class on a probability space (X, ν). Assume the action is amenable in the sense of Zimmer, and let π_X the corresponding quasi-regular representation. Then,

$$\|\lambda_\Gamma(\mu)\| \geq \|\pi_X(\mu)\|$$

This lemma by Shalom gives a useful condition for an opposite inequality

Lemma 2.2.5 *(*[46]*, Lemma 2.3)*

Let π be a unitary Γ-representation, with a positive Γ-vector, that is nonzero vector $v \in \mathcal{H}$, such that $\langle \pi(g)v, v\rangle \geq 0$ for all $g \in \Gamma$. Then for any finitely supported probability measure μ on Γ

$$\|\lambda_\Gamma(\mu)\| \leq \|\pi_X(\mu)\|$$

Example 2.2.6 *An example of an ergodic amenable action is the action of convex cocompact subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ on its Poisson boundary (which can be identified with the visual boundary*

$\partial\Gamma$) equipped with Patterson Sullivan measure([49]). Moreover, $\pi_{\partial\Gamma}$ has a positive Γ-vector(e.g. a constant function), thus we can deduce that for any probability measure μ on Γ we have

$$\|\lambda_\Gamma(\mu)\| = \|\pi_{\partial\Gamma}(\mu)\|$$

This also follows from spectral transfer principle(see [42], Theorem 1 or [43] for more general statement)

The following proposition is well known, but doesn't seem to appear in the literature. It relates the left regular representation and the Koopman representation on the two torus.

Proposition 2.2.7

Let $\Gamma < SL_2(\mathbb{Z})$ act on the torus $\mathbb{T}^2$ equipped with Lebesgue measure m, π_0 be the Koopman representation on $L^2_0(\mathbb{T}^2)$. Then, for any probability measure μ on Γ

$$\|\pi_0(\mu)\| = \|\lambda_\Gamma(\mu)\|$$

Proof:

Recall that the Fourier transform is an isometry between

$$\widehat{\ } \ : L^2_0(\mathbb{T}^2) \to \ell^2(\mathbb{Z}^2 \setminus 0)$$

defined as following: for $f \in L_0^2(\mathbb{T}^2)$

$$\widehat{f}(\vec{n}) = \int_{\mathbb{T}^2} f(x) e^{2\pi i \langle \vec{n}, x \rangle} dm(x)$$

Γ acts on $\mathbb{Z}^2 \setminus 0$ via left multiplication by transpose matrix. This induces a representation $\widehat{\pi_0}$ on $\ell^2(\mathbb{Z}^2 \setminus 0)$ given by

$$\widehat{\pi_0}(g)\widehat{f}(\vec{n}) = \widehat{f}\left(g^T \vec{n}\right)$$

The following diagram commutes

$$\begin{tikzcd} L_0^2(\mathbb{T}^2) \arrow[r, "\widehat{}"] \arrow[d, "\pi_0(g)"'] & L_0^2(\mathbb{Z}^2 \setminus 0) \arrow[d, "\widehat{\pi_0(g)}"] \\ L_0^2(\mathbb{T}^2) \arrow[r, "\widehat{}"] & L_0^2(\mathbb{Z}^2 \setminus 0) \end{tikzcd}$$

The Fourier transform intertwines the representations. Hence, $\|\pi_0(\mu)\| = \|\widehat{\pi_0(\mu)}\|$.

Pick representatives from each $\Gamma-$orbit of $\widehat{\pi_0}$: $D = \{v_1, v_2, v_3, ...\}$. Then,

$$\mathbb{Z}^2 \setminus 0 \cong \bigcup_i \Gamma / Stab(v_i)$$

and

$$\widehat{\pi_0} = \bigoplus_i \pi_{\Gamma / Stab(v_i)}$$

hence, $\|\widehat{\pi_0}(\mu)\| = \sup_i \|\pi_{\Gamma/Stab_{v_i}}(\mu)\|$. The stabilizers of vectors in $\mathbb{Z}^2 \setminus 0$ are amenable (conjugate to the group of upper triangular matrices), thus by Kesten's theorem we have $\|\pi_{\Gamma/Stab(v_i)}(\mu)\| = \|\lambda_\Gamma(\mu)\|$ for every i, and hence $\|\pi_0(\mu)\| = \|\widehat{\pi_0}(\mu)\| = \|\lambda_\Gamma(\mu)\|$ $\square$

Combining results from this section we have

Corollary 2.2.8

Let $\Gamma < SL_2(\mathbb{Z})$ be convex cocompact. Let λ be the left regular representation, $\pi_{\partial\Gamma}$ the boundary representation as described in § 2.1.2 and π_0 the Koopman representation on the torus. Let $\mu \in Prob(\Gamma)$ be a finitely supported measure, such that the support generates the entire group, then

$$\|\pi_{\partial\Gamma}(\mu)\| = \|\pi_0(\mu)\| = \|\lambda(\mu)\|$$

CHAPTER 3

THE SPECTRAL ESTIMATE FOR THE BOUNDARY REPRESENTATION

In this section we prove Theorem D. Consider a group Γ that acts by isometries properly cocompactly on a proper quasiruled hyperbolic space (X, d). Fix $x_0 \in X$ a basepoint. We will abuse the notation and use d as a metric on a group, i.e. $d(g, h) := d(g.x_0, h.x_0)$. With this metric, (Γ, d) is a proper quasiruled hyperbolic space. Let δ_Γ be the critical exponent of Γ. For every $n \in \mathbb{N}$, let μ_n be a uniform probability measure on the shell S_n (as defined in 2.1.9).

By Corollary 2.2.8, Theorem 1.2.4 and Theorem D follow immediately from the theorem below.

Theorem 3.0.9

Let (Γ, d) and μ_n as above. Let ρ be Patterson-Sullivan measure on $\partial\Gamma$ and $\pi_{\partial\Gamma} : \Gamma \to U(L^2(\partial\Gamma), \rho)$ the corresponding quasiregular representation of Γ on the boundary. Then

$$\|\pi_{\partial\Gamma}(\mu_n)\| \leq e^{-\frac{1}{2}\delta_\Gamma n + \log n + O(1)}$$

We will call $\pi_{\partial\Gamma}(\mu_n)$ the boundary operators. We fix n throughout the proof. We will bound the operator norm of the boundary operator by testing it on a dense set of simple functions. For each $r \in \mathbb{N}$ we will construct a finite dimensional operator Π_r that mimics the application of $\pi_{\partial\Gamma}(\mu_n)$ to a step function f (the complexity of f will determine how large should be r). We will then study Π_r and relate their operator norms to the operator norm of $\pi_{\partial\Gamma}(\mu_n)$.

Let $r \in \mathbb{N}$. Enumerate the elements $\{g_j\}$ in the shell $S_r \subset \Gamma$. Denote by $O_j = O(g_j)$ the shadows as defined in 2.1.10, and their characteristic functions by $\chi_j = \chi_{O_j}$. Define a $|S_r| \times |S_r|$ matrix $\Pi_r(\mu_n)$ by

$$(\Pi_r(\mu_n))_{ij} := \langle \pi_{\partial\Gamma}(\mu_n)\chi_i, \chi_j\rangle = \int_{\partial\Gamma} \left(\pi_{\partial\Gamma}(\mu_n)\chi_i\right)(\xi)\chi_j(\xi)d\rho(\xi)$$

The main step will be estimating the operator norms of finite dimensional operators $\Pi_r(\mu_n)$

Theorem 3.0.10

For $\Pi_r(\mu_n)$ *as above we have*

$$\|\Pi_r(\mu_n)\| \leq e^{-\delta_\Gamma r - \frac{1}{2}\delta_\Gamma n + \log n + O(1)}$$

In § 3.1 we will show that Theorem 3.0.10 implies Theorem 3.0.9. In § 3.3 we will prove Theorem 3.0.10

3.1 Reduction to linear algebra

Proof:

(Theorem 3.0.10 $\Longrightarrow$ Theorem 3.0.9)

Recall that

$$\|\pi_{\partial\Gamma}(\mu_n)\| = \sup_{\|f\|=1} \langle \pi_{\partial\Gamma}(\mu_n)f, f\rangle$$

Since $\pi_{\partial\Gamma}(\mu_n)$ is an operator preserving the cone of positive functions, it is sufficient to take the supremum only over non-negative functions(or a dense subset of it).

We fix a visual metric d_ϵ for some small enough $\epsilon > 0$. Recall that the balls in the visual metric generate the topology. We consider

$$H_+ := \left\{ f = \sum_{i=1}^{t} a_i \chi_{I_i} \ : \ a_i > 0, I_i \subseteq \partial\Gamma \text{ disjoint closed balls}, \|f\| = 1 \right\}$$

H_+ is clearly dense in the set of non-negative functions of norm 1.

Our strategy will be to show that for each $f \in H_+$ there exists $r > 0$ and a vector $\vec{v} \in \mathbb{R}^{|S_r|}$ such that

(M1) $\langle \pi_{\partial\Gamma}(\mu_n) f, f \rangle \leq \vec{v}^T \Pi_r(\mu_n) \vec{v}$

(M2) $\|\vec{v}\|^2 \leq e^{\delta_\Gamma r + O(1)}$

where $\|\vec{v}\|$ is the Euclidean norm on $\mathbb{R}^{|S_r|}$.

This, combining with Theorem 3.0.10 will imply that for each $f \in H_+$ we have some $\vec{v}$ and r satisfying

$$\begin{aligned} \langle \pi_{\partial\Gamma}(\mu_n) f, f \rangle \leq e^{\delta_\Gamma r + O(1)} \frac{\vec{v}^T \Pi_r(\mu_n) \vec{v}}{\|\vec{v}\|^2} & \leq e^{\delta_\Gamma r + O(1)} \|\Pi_r(\mu_n)\| \leq \\ \leq e^{-\frac{1}{2}\delta_\Gamma n + \log n + O(1)} & \end{aligned}$$

Taking the supremum over $f \in H_+$ will finish the proof of Theorem 3.0.9.

We are left to construct v from f satisfying the properties (M1) and (M2). Fix an element in H_+ of the form $f = \sum_{i=1}^{t} a_i \chi_{I_i}$ with $\|f\| = 1$. Denote by $I_{i+\eta}$ the closed balls having the same centers as I_i, but with radius larger by η. Fix $\eta > 0$ such that for every $1 \leq i \leq t$ we have $\rho(I_{i+\eta}) \leq 2\rho(I_i)$ and so that $I_{i+\eta}$ are pairwise disjoint for all i. Such η exists, since I_i is a finite family. By Proposition 2.1.5 bounding the diameter of the shadows we can find r large enough, so that two following conditions are satisfied:

(S1) for every $g_j \in S_r$ we have $diam(O_j) \leq \frac{1}{3} \min_{i,i'} d_\epsilon(I_i, I_{i'})$

(S2) for every $g_j \in S_r$ we have $diam(O_j) \leq \eta$

For each $1 \leq j \leq |S_r|$ define

$$v_j = \begin{cases} a_i & \text{if } \exists i \text{ s.t. } I_i \cap O_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$\vec{v} = (v_j)$ is well defined since by condition (S1) each O_j intersects at most one of the sets from the family $\{I_i\}$.

Let $f^v = \sum_{j=1}^{|S_r|} v_j \chi_j$

By Theorem 2.1.7 there exists $L \in \mathbb{N}$ so that each point in the boundary is covered by at most L different shadows of elements in S_r. Combining it with (S2) we have for each $1 \leq i \leq t$

$$\chi_{I_i} \leq \sum_{j : O_j \cap I_i \neq \emptyset} \chi_j \leq L \chi_{I+\eta} \tag{3.1}$$

In particular from the left inequality in (Equation 3.1)

$$f \le f^v$$

It follows now that $\vec{v}$ satisfies (M1), i.e.

$$\langle \pi_{\partial\Gamma}(\mu_n) f, f\rangle \quad \le \langle \pi_{\partial\Gamma}(\mu_n) f^v, f^v\rangle \quad = \vec{v}^T \Pi_r(\mu_n)\vec{v}$$

To show (M2) we are left to estimate the of $\vec{v}$

$$\begin{aligned}
\|\vec{v}\|^2 = \sum_i \sum_{j: O_j \cap I_i \neq \emptyset} a_i^2 &\overset{(1)}{=} e^{\delta_\Gamma r + O(1)} \sum_i a_i^2 \sum_{j: O_j \cap I_i \neq \emptyset} \rho(O_j) \overset{(2)}{\le} \\
&\overset{(2)}{\le} e^{\delta_\Gamma r + O(1)} \sum_i a_i^2 L \rho(I_{i+\eta}) \overset{(3)}{\le} \\
&\overset{(3)}{\le} e^{\delta_\Gamma r + O(1)} 2L \sum_i a_i^2 \rho(I_i) \le \\
&\le e^{\delta_\Gamma r} \|f\|^2 = e^{\delta_\Gamma r + O(1)}
\end{aligned}$$

The first equality follows from Corollary 2.1.6, which, if applied here, states $\rho(O_j) = e^{-\delta_\Gamma r + O(1)}$, the second follows from integrating the right inequality in Equation 3.1, the third is obtained from our choice of η(since $\rho(I_{i+\eta}) \le 2\rho(I_i)$). This finishes the proof. □

3.2 Hyperbolic geometry

The proof of Theorem 3.0.10 relies on the hyperbolicity of the metric d on Γ. We prove a sequence of technical lemmas which will be necessary in § 3.3.

Lemma 3.2.1

There exist $R, \Delta \geq 0$ depending only on (Γ, d), such that for any $r > n + \Delta$, for any $g \in S_r$, $\xi \in O(g) = O_C(g, e)$ and $h \in S_n$ we have

$$|\beta_\xi(h, e) - \beta_g(h, e)| \leq R \tag{3.2}$$

Proof:

Let Δ be the maximal thickness of quasi-triangles in Γ. We will show that $R = 4(\tau + C + \Delta) + 1$ suffices. Let $r > n + \Delta$ and choose $g \in S_r$ and $\xi \in O(g)$.

Let $z(t)$ be a quasiruler from e to ξ s.t. for some large t_0 we have $\beta_\xi(h, 0) - d(z(t_0), h) - d(z(t_0), e) \leq 1$. Note that by definition of $O(g)$ there is some quasiruler from e to ξ, that passes in a $C-$neighborhood of g. Using Δ-thinness of triangles, we can conclude that any quasiruler from e to ξ must pass in a $(C + \Delta)-$neighborhood of g. Let $s \in \mathbb{R}$, so that $z(s)$ is at distance at most $C + \Delta$ from g. We can assume $t_0 > s$. Then,

$$0 \leq d(e, z(s)) + d(z(s), z(t_0)) - d(e, z(t_0)) \leq 2\tau$$

The right hand side of the above inequality holds since $z(t)$ is a τ-quasiruler, and the left hand side is the triangle inequality.

Let $z'(t)$ be a quasiruler between $h, z(t_0)$. Similarly, $z'(t)$ has to pass through the $C+\Delta$-neighborhood of g. Let s' such that $z'(s')$ is in the $C+\Delta$ neighborhood of g. Similarly, by the property of quasiruler for $z'(t)$

$$0 \leq d(h, z'(s')) + d(z'(s'), z(t_0)) - d(h, z(t_0)) \leq 2\tau$$

Noting that $z'(s')$ and $z(s)$ are $(C+\Delta)-$close to g and $z(t_0) = z'(t'_0)$, we can substract two of the above inequalities to get

$$|\beta_\xi(h,e) - \beta_g(h,e)| \leq 4\tau + 1 + 4(C+\Delta) = R$$

$\square$

Corollary 3.2.2

With Δ as in Lemma 3.2.1, for any $r > n+\Delta$ and for each $g \in S_r, h \in S_n, \xi \in O(g)$ we have

$$\frac{dh_*\rho}{d\rho}(\xi) = e^{-\beta_g(h,e)\delta_\Gamma + O(1)}$$

Define

$$X_a(g,n) = \{h \in S_n \;:\; n - 2a - R < -\beta_g(h,e) \leq n - 2a\} \tag{3.3}$$

where R is the constant from the Lemma 3.2.1. Without loss of generality we can take R enough large, so that our estimate for the size of the shells $S_{n,R}$ from Lemma 2.1.8 holds.

Lemma 3.2.3

With n, r as above, for any $0 \leq a \leq n$ and $g \in S_r$ we have

$$\#X_a(g,n) \leq e^{\delta_\Gamma a + O(1)}$$

Proof:

Let $g \in S_r$. Fix a quasiruler between e, g. Given $h \in X_a(g,n)$ complete it to the quasitriangle e, g, h. By Lemma 2.1.3 it is $(1, c_0)$-quasiisometric to a tripod (with c_0 depending only on the global quasiruled hyperbolic structure). Hence, the following equations carry on to the tripod via the quaiisometry

$$d(e,g) = r + O(1)$$

$$d(e,h) = n + O(1)$$

$$d(g,e) - d(g,h) = n - 2a + O(1)$$

Let $y(h) \in \Gamma$ be some preimage of the closest point to the centroid of the tripod(if it is not unique, we can choose one). Solving in the tripod it is easy to see that $d(y(h), e) = n - a + O(1)$ for every $h \in X_a(g,n)$. This will ensure that the location of the centroid $y = y(h)$ doesn't depend on which quasitriangle we chose (up to bounded distance), i.e. it doesn't depend on $h \in X_a(g,n)$. Also, $d(y,h) = a + O(1)$ for every $h \in X_a(g,n)$, hence implying that $X_a(g,n) \subseteq B(y, a + O(1))$. By Lemma 2.1.8 we can estimate

$$\#X_a(g,n) \leq \#B(y, a + O(1)) \leq e^{\delta_\Gamma a + O(1)}$$

□

Lemma 3.2.4

With n, r as above, enumerate the elements of $S_r = \{g_1, g_2, ..., g_{|S_r|}\}$. Then for any $i \in \{1, ..., \#S_r\}$ and for any $h \in S_n$ we have

$$\sum_{j}^{\#S_r} \rho(O_i \cap hO_j) \le e^{-\delta_\Gamma r + O(1)} \tag{3.4}$$

Proof:

Fix $h \in S_n, g_i \in S_r$. We first characterize $g_j \in S_r$ for which $O_i \cap hO_j \neq \emptyset$ and then estimate the measures of the intersections.

Let $1 \le j \le |S_r|$ such that $O_i \cap hO_j \neq \emptyset$. We claim that there exist a constant D, that depends only on the quasiruled hyperbolic structure, such that one of the following holds:

Case 1: hg_j lies within distance D from a quasiruler between $[e, g_i]$

Case 2: g_i lies within distance D from a quasiruler between $[e, hg_j]$

Indeed, consider $z(t) \in O_i \cap hO_j$ a quasiruler ray from e. Since $z(t) \in O_i$, g_i must lie within bounded distance from $z(t)$. Since $(h^{-1}y_k) \in O_j$, g_j must lie within bounded distance from $h^{-1}z(t)$, or equivalently hg_j must lie within bounded distance from $z(t)$, hence all e, hg_j, g_i lie within bounded distance from $z(t)$.

By similar argument as above, $e, h^{-1}g_i, g_j$ all lie on the same quasiruled geodesic(consider quasiruler $z'(t) \in h^{-1}O_i \cap O_j$). Therefore,

$$d(hg_j, g_i) = d(g_j, h^{-1}g_i) = |r - d(h^{-1}g_i, e)| + O(1) = |\beta_{g_i}(h, e)| + O(1)$$

In particular, if an element g_j produces nontrivial intersection $O_i \cap hO_j$, it has to be at distance $|\beta_{g_i}(h, e)| + O(1)$ from g_i. Hence, we can count such elements g_j.

In *case 1*, the number of elements in Γ lying in a bounded distance from a quasiruler $[e, g_i]$ and being distance $|\beta_{g_i}(h, e)| + O(1)$ from g_i is $O(1)$. Thus there are at most $O(1)$ possible g_j satisfying $O_i \cap hO_j \neq \emptyset$. In this case $hO_j \cap O_i \subseteq O_i$. The contribution of $\rho(hO_j \cap O_i) = \rho(O_i)$ for each such j to the sum in the equation (Equation 3.4) is $e^{-\delta_\Gamma r + O(1)}$ by Lemma 2.1.6, and since the number of j contributing to the sum is $O(1)$, we get the desired estimate.

In *case 2*, the number of g_j producing nontrivial intersection is at most $e^{d(hg_j, g_i)\delta_\Gamma + O(1)}$, however for each such j, we have $O_i \cap hO_j \subset hO_j$, and in particular $\rho(O_i \cap hO_j) \leq \rho(hO_j)e^{-\delta_\Gamma r - d(hg_j, g_i) + O(1)}$. Therefore, the summation over j gives us the desired estimate. $\square$

3.3 Estimating the operator norms of the finite dimensional operators

In this section we estimate the norms of $\|\Pi_r(\mu_n)\|$ to prove Theorem 3.0.10. We will use well known fact, known as the Gershgorin circle theorem. It states that the spectral radius of a matrix is bounded by the maximum of the ℓ_1-norms of the columns ([22]).

Proof:

(of Theorem 3.0.10) By Gershgorin circle theorem it is sufficient to show that sum of every column in $\Pi_r(\mu_n)$ is bounded by $e^{-\delta_\Gamma r-\frac{1}{2}\delta_\Gamma n+\log n+O(1)}$.

Recall, in (Equation 3.3) we defined

$$X_a(g_i,n)=\{h\in S_n \ : \ n-2a-R\leq -\beta_{g_i}(h,e)\leq n-2a\}$$

Note that for any fixed i we have

$$S_n=\bigcup_{a=0}^{n}X_a(g_i,n)$$

We now evaluate the sum of $i-$th column

$$\sum_{j=1}^{\#S_r}\langle\pi_{\partial\Gamma}(\mu_n)\chi_j,\chi_i\rangle=\sum_{h\in S_n}\mu_n(h)\sum_{j=1}^{\#S_r}\langle\pi_{\partial\Gamma}(h)\chi_j,\chi_i\rangle\leq$$

$$\leq\sum_{a=0}^{n}\sum_{h\in X_a(g_i,n)}\mu_n(h)\sum_{j=1}^{\#S_r}\int_{\partial\Gamma}\sqrt{\frac{dh_*\rho}{d\rho}(\xi)}\chi_j(h^{-1}\xi)\chi_i(\xi)d\rho(\xi)\tag{3.5}$$

where μ_n is uniformly distributed on S_n, hence by Lemma 2.1.8 $\mu_n(h)=e^{-\delta_\Gamma n+O(1)}$. Using Corollary 3.2.2 for the Radon Nykodim derivative we continue Equation 3.5

$$\leq O(1)\sum_{a=0}^{n}e^{-\delta_\Gamma n}e^{\frac{1}{2}\delta_\Gamma(n-2a)}\sum_{h\in X_a(g_i,n)}\sum_{j=1}^{\#S_r}\rho(O_i\cap hO_j)\leq\tag{3.6}$$

We use the upper bound for the innermost sum from Lemma 3.2.4, and the size of $X_a(g_i, n)$ from Lemma 3.2.3. Hence , continuing (Equation 3.6)

$$\leq O(1)\sum_{a=0}^{n} e^{-\delta_\Gamma n} e^{\frac{1}{2}\delta_\Gamma(n-2a)} e^{\delta_\Gamma a} e^{-\delta_\Gamma r}$$

gathering terms and summing over a we get

$$\leq e^{-\frac{1}{2}\delta_\Gamma n+\log n-\delta_\Gamma r+O(1)}$$

$\square$

CHAPTER 4

DIOPHANTINE APPROXIMATION ON THE 2-TORUS

We first prove Theorem B. Then we show how to deduce Theorem A from B.

4.1 Toral Diophantine approximation for convex cocompact subgroups of $\mathrm{SL}_2(\mathbb{Z})$

Consider the natural $SL_2(\mathbb{Z})$ action on the torus $\mathbb{T}^2$, with the Lebesgue measure m. Fix a family $\{\mathrm{Targ}_r\}_{r>0}$ of Lebesgue subsets of measure $m(\mathrm{Targ}_r) = \pi r^2$. After choosing a basepoint $x_0 \in \mathbf{H}^2$, we get a metric on Γ defined by $d(g,h) := d_{\mathbf{H}^2}(g.x_0, h.x_0)$. In this section we prove Theorem B

Proof:

(of Theorem B) The first statement follows from the first Borel Cantelli lemma. Indeed,

$$\begin{aligned}
\sum_{g\in\Gamma} m(g^{-1}\,\mathrm{Targ}_{\psi(\|g\|)}) &\leq \sum_{n=1}^{\infty} \sum_{\{g\in\Gamma: e^{n-1}<\|g\|\leq e^n\}} \pi\psi(\|g\|)^2 \\
&\leq O(1)\sum_{n=1}^{\infty} e^{2\delta n}\cdot\psi(e^{n-1})^2 \\
&= O(1)\sum_{n=1}^{\infty} e^{2\delta n}\cdot\psi(e^{n})^2 < +\infty
\end{aligned}$$

The last inequality follows from Lemma 2.1.8 giving the upper bound of the cardinality of balls in convex cocompact groups, and the fact that $d(g,e) = 2\log\|g\|$. The series $\sum_{n=1}^{\infty} e^{2\delta n}\cdot\psi(e^n)^2$

converges if and only if ψ is as in (1)(by Cauchy condensation test). Therefore, m-a.e. $x \in \mathbb{T}^2$ belongs to at most finitely many of the sets $g^{-1}\operatorname{Targ}_{\psi(\|g\|)}$, as claimed.

The main point is the second statement. Let π be the Koopman Γ-representation on $L^2(\mathbb{T}^2, m)$, and π_0 the restriction to $L^2_0(\mathbb{T}^2, m)$. Let μ_n be a sequence of probability measures on Γ, as given in Theorem 3.0.9. Observe that

$$\max\{\|g\| \ : \ g \in supp(\mu_{2n})\} \le e^n$$

We denote

$$C_n := \operatorname{Targ}_{\psi(e^n)}, \qquad E_n = X \setminus \bigcup_{g\in\Gamma, \|g\|\le e^n} g^{-1}\operatorname{Targ}_{\psi(e^n)}.$$

C_n represents the targets that we are supposed to hit by applying matrices g with $\|g\| \le e^n$(or equivalently $d(g,e) \le 2n$). A point belongs to E_n if and only if none of its translates by g with $\|g\| \le e^n$ hits the target C_n. Hence, we want to show that

$$m(E) = 0 \qquad \text{where} \qquad E = \limsup_{n\to\infty} E_n.$$

The projections of characteristic functions of C_n and E_n to $L^2_0(X, m)$ are

$$h_n = 1_{C_n} - m(C_n), \qquad f_n = 1_{E_n} - m(E_n)$$

Note that

$$\|h_n\|_2^2 \le (1 - m(C_n))m(C_n) \le m(C_n)$$

Thus,

$$\|h_n\|_2 \leq m(C_n)^{\frac{1}{2}} = O(1)\psi(e^n).$$

Similarly,

$$\|f_n\|_2 \leq m(E_n)^{\frac{1}{2}}$$

For any $g \in \Gamma$

$$\langle \pi_0(g)h_n, f_n \rangle = m(C_n) \cdot m(E_n) - m(g^{-1}C_n \cap E_n).$$

Since any $g \in supp(\mu_{2n})$ satisfies $\|g\| \leq e^n$, one has $g^{-1}C_n \cap E_n = \emptyset$ and

$$\langle \pi_0(g)h_n, f_n \rangle = m(C_n) \cdot m(E_n)$$

and consequently

$$\begin{aligned} m(C_n) \cdot m(E_n) &= \langle \pi_0(\mu_{2n})h_n, f_n \rangle \leq \|\pi_0(\mu_{2n})\| \cdot \|h_n\|_2 \cdot \|f_n\|_2 \\ &\leq \|\pi_0(\mu_{2n})\| \cdot m(C_n)^{\frac{1}{2}} \cdot m(E_n)^{\frac{1}{2}}. \end{aligned}$$

By Corollary 2.2.8 and Theorem 3.0.9 we have

$$\|\pi_0(\mu_{2n})\| \leq e^{-\delta_\Gamma n + \log n + O(1)}$$

Therefore

$$m(E_n)^{\frac{1}{2}} \leq \|\pi_0(\mu_{2n})\| \cdot m(C_n)^{-\frac{1}{2}} \leq e^{-\delta_\Gamma n + \log n + O(1)} \cdot \psi(e^n)^{-1} \tag{4.1}$$

Hence,

$$\sum_{n=1}^{\infty} m(E_n) \leq O(1) \sum_{n=1}^{\infty} n^2 e^{-2\delta_\Gamma n} \cdot \psi(e^n)^{-2} < +\infty.$$

where the convergence of the above series is equivalent to convergence of $\sum_{n=1}^{\infty} (\log n)^2 n^{-2\delta_\Gamma - 1} \psi(n)^{-2}$(by Cauchy condensation test). Consequently, $m(\limsup E_n) = 0$. $\square$

Remark 4.1.1 *In fact, the statement we proved here is a bit stronger than the one that appears in the theorem. We showed that for Lebesgue a.e. point in the torus $x \notin \limsup E_n$, which means that for some large N, $x \in E_n^c$ for every $n > N$. In other words, not only we have infinitely many solutions for the problem $g.x \in \mathrm{Targ}_{\psi(\|g\|)}$, but for any $n > N$, we have such a solution $g \in \Gamma$ with $e^{n-k} \leq \|g\| \leq e^n$, for some fixed k. This justifies Remark 1.2.2.*

Remark 4.1.2 *One might formulate a simultaneous approximation problem. Given a $d-$tuple of monotonic target families $\{\mathrm{Targ}_r^1, ..., \mathrm{Targ}_r^d\}$ as before and $x_1, ..., x_d \in \mathbb{T}^2$, can one find infinitely many $g \in \Gamma$ with $g.x_i \in \mathrm{Targ}^i_{\psi(\|g\|)}$ for each $1 \leq i \leq d$? We remark that if one had sharp spectral estimates for $\|\pi_0^{\otimes d}(\mu_n)\|$, a proof similar to Theorem B would provide the rates for which the approximation is possible for a.e. $d-$tuple $(x_1, ..., x_d)$.*

4.2 Reduction to the convex cocompact case

In this section we prove Theorem A. The proof of the first statement is analogous to the proof of Theorem B. Note, that since the group is not convex cocompact, we cannot use Lemma 2.1.8 to obtain the precise asymptotics of the growth of balls. However, it is sufficient for the proof to bound the cardinality of the balls of radius n in the group by $e^{(\delta_\Gamma+\epsilon)n+O(1)}$ for arbitrarily small ϵ, and this is possible from the definition of the critical exponent.

We now show that the second part follows from Theorem B. Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a non-elementary subgroup. Fix $y \in \mathbb{T}^2$ and let $\epsilon > 0$. We want to show that there are infinitely many solutions $g \in \Gamma$ to $g.x \in \mathrm{Targ}_{\psi(\|g\|)}$ with $\psi(n) = n^{-\delta_\Gamma + \epsilon}$, where Targ_r are the Euclidean balls of radius r around y.

The goal is to construct a convex cocompact subgroup $\Gamma_\epsilon < \Gamma$, so that the $\delta_{\Gamma_\epsilon} > \delta_\Gamma - \epsilon$. Since for sufficiently large n we have $\psi(n) = n^{-\delta_\Gamma+\epsilon} > n^{-\delta_{\Gamma_\epsilon}} \log^{1.5+\epsilon} n$, we can apply Theorem B to find infinitely many solutions $g \in \Gamma_\epsilon < \Gamma$ to $g.x \in \mathrm{Targ}_{\psi(\|g\|)}$. This proves Theorem A.

We are left to describe the construction of Γ_ϵ. We are inspired by the example provided by Bourgain and Kontorovich in [9](which they attribute to Sarnak). The following trick gives us a way to remove parabolic elements from the group without losing the critical exponent.

Lemma 4.2.1 *(*[9] *Remark 1.7, also follows from* [15] *Property 3.14)*

Let $G = SL_2(\mathbb{Z})$. Let $G(2) = Ker\{G \to SL_2(\mathbb{Z}/2\mathbb{Z})\}$ be the congruence subgroup of G. Then the commutator subgroup $G(2)' = [G(2), G(2)]$ does not have parabolic elements.

Proposition 4.2.2

Given $\Gamma < SL_2(\mathbb{Z})$ and $\epsilon > 0$ there exists a convex cocompact subgroup $\Gamma_\epsilon < \Gamma$, with $\delta_{\Gamma_\epsilon} > \delta_\Gamma - \epsilon$

Proof:

By Sullivan([47], Corollary 6) we know that

$$\delta_\Gamma = \sup\{\delta_H \ : \ H < \Gamma \text{ finitely generated}\}$$

Hence, we can find $\Gamma_0 < \Gamma$ finitely generated with $\delta_{\Gamma_0} > \delta_\Gamma - \frac{1}{2}\epsilon$. For Fuchsian groups being finitely generated is equivalent to being geometrically finite (a group is *geometrically finite* if it admits a finitely sided polygon as a fundamental domain in $\mathbf{H}^2$). Both $G(2)$ and Γ_0 are such. Greenberg showed in [25] that the intersection of two finitely generated subgroups of a discrete group in $\mathrm{Isom}(\mathbf{H}^n)$ is finitely generated itself. Hence, $\Gamma_1 = G(2) \cap \Gamma_0$ is geometrically finite.

Brooks([12],Theorem 1) proved that if Γ is geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$, $N \trianglelefteq \Gamma$ is a normal subgroup and Γ/N is amenable then the bottoms of the spectra of Laplacians on $\mathbb{H}^n + 1/\Gamma$ and $\mathbb{H}^{n+1}/N$ are equal. In particular, this implies $\delta_\Gamma = \delta_N$, since $\lambda_0(\mathbb{H}^{n+1}/\Gamma) = \delta_\Gamma(n - \delta_\Gamma)$ for Γ with $\delta_\Gamma > n/2$([47]). In case of Fuchsian groups, every non-elementary Γ with parabolic elements has $\delta_\Gamma \geq 1/2$.

We can apply this observation to $\Gamma_1 = \mathrm{Ker}\,\{\Gamma_0 \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})\}$, and then to the commutator subgroup $\Gamma_1' < \Gamma_1$. Hence, $\delta_{\Gamma_1'} = \delta_{\Gamma_1} = \delta_{\Gamma_0} > \delta_\Gamma - \frac{1}{2}\epsilon$.

Now we apply Sullivan again, to extract a finitely generated subgroup $\Gamma_\epsilon < \Gamma_1'$ with $\delta_{\Gamma_\epsilon} > \delta_{\Gamma_1'} - \frac{1}{2}\epsilon > \delta_\Gamma - \epsilon$. Since $\Gamma_\epsilon < G(2)'$, by Lemma 4.2.1 it has no parabolic elements. This group is convex cocompact, since in dimension 2 a subgroup is convex cocompact if and only if it is finitely generated and contains no parabolic elements. $\square$

CHAPTER 5

APPROXIMATION OF SPECIFIC POINTS IN THE TORUS

Theorems A and B only provide us information on approximation properties of Lebesgue almost every point. In this section we wish to characterize Diophantine properties of specific points. We consider $\Gamma < SL_d(\mathbb{Z})$ (with $d \geq 2$) acting on a d-torus $\mathbb{T}^d$. For technical reasons we rather use sup-norm on $\mathbb{T}^d$ than the Euclidean one. Clearly, this does not affect the approximation properties. For $y \in \mathbb{T}^d, r > 0$ we denote by $\mathrm{Box}(y, r)$ the ball of radius $\frac{1}{2}r$ in the $d-$torus in the $\sup-$norm. Note that $m(\mathrm{Box}(y, r)) = r^d$

Naturally, we can not expect a uniform rate of approximation for all target points and all origin points in the torus. Theorem C states that under mild assumptions on the acting group, for given M, we can produce a uniform bound for the approximation rate for all targets and all $M-$Diophantine origins. The proof of Theorem C relies on two results. First result controls the Fourier coefficients of the measures obtained from a random walk μ on the torus. If the initial distribution δ_x is concentrated on a Diophantine point $x \in \mathbb{T}^d$, then the Fourier coefficients of the distribution after k steps have exponential decay in k. More precisely,

Theorem 5.0.3 [8]

Let $\Gamma < SL_d(\mathbb{Z})$ be a finitely generated group. satisfying (SI) and (PE). Let $\mu \in Prob(\Gamma)$ be a finitely supported measure, s.t. the support generates Γ. Let $x \in \mathbb{T}^d$ be $M-$Diophantine.

Let $\nu_k = \mu^{*k} * \delta_x$. Then, there exist $c_2 > 0$, depending only on Γ and μ, and $K_0 \in \mathbb{N}$ s.t. for $k > K_0$ we have for any $B \in \mathbb{N}$

$$\max_{b \in \mathbb{Z}^d \setminus 0, 0 < \|b\|_\infty < B} |\hat{\nu_k}(b)| \le B e^{-c_2 k / M}$$

Definition 5.0.4 *Let ν be a probability measure on $\mathbb{T}^d$ and m be the Lebesgue measure. The discrepancy of ν is*

$$D(\nu) := \sup_{P \in J} |\nu(P) - m(P)|$$

where J is the set of half-open boxes in $\mathbb{T}^d$

$$J := \left\{ \prod_{i=1}^{d} [x_i, y_i) : 0 \le x_i < y_i \le 1 \right\}$$

The second ingredient of the proof is the Erdos-Turan-Koksma inequality. It relates the discrepancy between the distribution ν and the Lebesgue measure on the torus to the Fourier coefficients of ν .

Theorem 5.0.5 (Erdos-Turan-Koksma inequality) [37]

Let ν be an atomic probability measure on $\mathbb{T}^d$ with rational values. Let B be an arbitrary positive integer. Then

$$D(\nu) \le C_d \left(\frac{1}{B} + \sum_{0 < \|b\|_\infty \le B} \frac{|\hat{\nu}(b)|}{r(b)} \right)$$

where C_d is some explicit constant depending on the dimension d.

$$r(b) = \prod_{i=1}^{d} \max\{1, |b_i|\} \quad \textit{for} \quad b = (b_1, \ldots, b_d) \in \mathbb{Z}^d.$$

Now we are ready to prove Theorem C

Proof:

(of Theorem C) Let μ be the uniform measure on the finite set of generators of Γ. Let $\nu_k = \mu^{*k} * \delta_x$. Let $\lambda = \max\{\log \|g\| : g \in supp(\mu)\}$. We will show that $C_\Gamma < \frac{c_2}{d(d+2)\lambda}$ satisfies the theorem, where c_2 is the constant from Theorem 5.0.3.

By submultiplicativity of matrix norm, for every $k > 0$

$$\max\left\{\|g\| \ : \ g \in supp(\mu^{*k})\right\} \leq e^{\lambda k}$$

Assume by contradiction that there exists a point $y \in \mathbb{T}^d$ which is not $(\Gamma, \frac{C_\Gamma}{M})$-fast approximable. Then, there exists $K > 0$, such that for all $k > K$ we have

$$\nu_k(\mathrm{Box}(y, e^{-\frac{\lambda k C_\Gamma}{M}})) = 0$$

This gives us a lower bound for the discrepancy of ν_k.

$$D(\nu_k) \geq m(\mathrm{Box}(y, e^{-\frac{\lambda k C_\Gamma}{M}})) = e^{-\frac{\lambda k C_\Gamma d}{M}} \tag{5.1}$$

We will now estimate the upper bound for the discrepancy. By Theorem 5.0.5, for every $B, k \in \mathbb{N}$ we have

$$D(\nu_k) \leq C_d \left(\frac{2}{B+1} + \sum_{0<\|b\|_\infty \leq B} \frac{|\hat{\nu_k}(b)|}{r(b)} \right)$$

Using $r(b) \geq 1$ and the bound of the Fourier coefficients from Theorem 5.0.3 for k large enough we have

$$D(\nu_k) \leq \frac{2C_d}{B} + C_d(2B+1)^d \cdot Be^{-\frac{c_2 k}{M}}$$

Thus, combining with the lower bound from (Equation 5.1), we have

$$e^{-\frac{\lambda k C_\Gamma d}{M}} \leq \frac{2C_d}{B} + 2^{2d} C_d B^{d+1} \cdot e^{-\frac{c_2 k}{M}} \tag{5.2}$$

The inequality (Equation 5.2) must hold for all $B \in \mathbb{N}$ and all $k > \max(K_0, K)$, in particular for $B = B(k) = 4C_d'(k)e^{\frac{\lambda k C_\Gamma d}{M}}$ (where we choose the smallest $C_d'(k) \geq C_d$, such that $B(k)$ is an integer. Note that $C_d'(k) \leq 2C_d$ for large k. Then, inequality (Equation 5.2) becomes

$$e^{-\frac{\lambda k C_\Gamma d}{M}} \leq \frac{1}{2} e^{-\frac{\lambda k C_\Gamma d}{M}} + 2^{2d} C_d (4C_d'(k))^{d+1} e^{\frac{\lambda k C_\Gamma d(d+1) - c_2 k}{M}}$$

Multiplying both sides by $e^{\frac{\lambda k C_\Gamma d}{M}}$ and using $C_d'(k) \leq 2C_d$ we get

$$1 \leq \frac{1}{2} + 2^{5d+3} (C_d)^{d+2} e^{\frac{(\lambda C_\Gamma d(d+2) - c_2)k}{M}} \tag{5.3}$$

The assumption $C_\Gamma < \frac{c_2}{d(d+2)\lambda}$ implies that the exponent in the right hand side of inequality (Equation 5.3) is negative, so the above inequality does not hold for arbitrarily large k, which gives us the contradiction. $\square$

CHAPTER 6

SPECTRAL OPTIMALITY

6.1 Random walks

Let Γ be a countable group with a subadditive norm $|gh| \leq |g| + |h|$. (e.g. $|g| = \log \|g\|$ for $\Gamma < \mathrm{GL}_d(\mathbb{R})$). One can also use a word metric on a finitely generated group, or a *Green metric* on a non-amenable group equipped with a symmetric generating random walk (see below).

Definition 6.1.1 *Let μ be a probability measure on a countable Γ. We say that μ*

1. *μ is* symmetric *if $\mu(g) = \mu(g^{-1})$ for every $g \in \Gamma$*
2. *μ has* finite entropy *if*

$$H(\mu) \stackrel{\text{def}}{=} \sum_{g \in \Gamma} -\mu(g) \cdot \log \mu(g) < +\infty.$$

3. *μ has* finite first moment *if*

$$\mathbb{E}_{\mu}(|g|) = \sum_{g \in \Gamma} |g| \cdot \mu(g) < +\infty.$$

4. μ has finite exponential moment*, if for some $a > 0$ one has*

$$\mathbb{E}_\mu(e^{a|g|}) = \sum_{g\in\Gamma} e^{a\cdot|g|}\cdot\mu(g) < +\infty.$$

5. μ is finitely supported if $\mathrm{supp}(\mu)$ *is finite.*

6. μ is generating *if is not supported on a proper subgroup of* Γ

The integrability conditions are not sensitive to bi-Lipschitz change of a norm. Clearly (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3), and for groups with finite exponential) growth (3) $\Longrightarrow$ (2).

Definition 6.1.2 *Let μ be a probability measure on a countable group Γ.*

1. If μ is symmetric, define the spectral radius

$$\rho(\mu) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty}\left(\mu^{*2n}(e)\right)^{\frac{1}{2n}}.$$

2. Assuming μ has finite entropy, define

$$h(\mu) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty}\frac{1}{n}H(\mu^{*n}).$$

3. Assuming μ has finite first moment, define

$$\lambda(\mu) \stackrel{\mathrm{def}}{=} \lim_{n\to\infty}\frac{1}{n}\sum_{g\in\Gamma}|g|\cdot\mu^{*n}(g).$$

The above limits exist and equal to the inf of the corresponding sequences, by the classical lemma about subadditve sequences

$$-\log \mu^{*2n}(e), \qquad H(\mu^{*n}), \qquad \mathbb{E}_{\mu^{*n}}(|g|).$$

In fact, the last two sequences represent average values of certain subadditive cocycles, namely functions $f_n : \Omega \to \mathbb{R}$ satisfying

$$f_{n+m}(\omega) \leq f_n(\omega) + f_m(\theta^n \omega), \qquad f_1 \in L^1$$

over the Bernoulli system (Ω, P, θ) where

$$\Omega = \Gamma^{\mathbb{N}}, \qquad P = \mu^{\mathbb{N}}, \qquad \theta : (\omega_1, \omega_2, \dots) \mapsto (\omega_2, \omega_3, \dots).$$

Here one takes

$$f_n(\omega) = |\omega_n \cdots \omega_1|, \qquad \text{or} \qquad f_n(\omega) = -\log \mu^{*n}(\omega_n \cdots \omega_1)$$

Applying Kingman's subadditive ergodic theorem to these subadditive cocycles one obtains a more refined information

Theorem 6.1.3 *(Kaimanovich, Vershik* [32]*, Derrienic* [18]*)*

Let Γ be a countable group, μ a probability measure with finite entropy on Γ, and (Ω, P, θ) as above. Then for P-a.e. $\omega \in \Omega$, and in $L^1(\Omega, P)$

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu^{*n}(\omega_n \cdots \omega_1) \;=\; h(\mu).$$

Consequently, convergence in measure gives

$$\lim_{n\to\infty} \mu^{*n} \left\{ g \in \Gamma \;:\; e^{-(h(\mu)+\epsilon)\cdot n} < \mu^{*n}(g) < e^{-(h(\mu)-\epsilon)\cdot n} \right\} \;=\; 1.$$

Theorem 6.1.4 *(Furstenberg, Kesten* [20]*)*

Let Γ be a countable group with some norm $|g|$, μ a probability measure with finite first moment, and (Ω, P, θ) as above. Then for P-a.e. $\omega \in \Omega$, and in $L^1(\Omega, P)$

$$\lim_{n\to\infty} -\frac{1}{n} |\omega_n \cdots \omega_1| \;=\; \lambda(\mu)$$

Consequently, convergence in measure gives

$$\lim_{n\to\infty} \mu^{*n} \{ g \in \Gamma \;:\; (\lambda(\mu)-\epsilon)\cdot n < |g| < (\lambda(\mu)+\epsilon)\cdot n \} \;=\; 1.$$

Theorem 6.1.5 *(Kesten* [33]*)*

Let λ_Γ denote the regular Γ-representation on $\ell^2(\Gamma)$. Then for a symmetric μ the spectral radius of the Markov operator $\lambda_\Gamma(\mu)$ is

$$\|\lambda_\Gamma(\mu)\| = \rho(\mu)$$

and $\rho(\mu) < 1$ iff $\mathrm{supp}(\mu)$ generates a non-amenable subgroup of Γ.

Proposition 6.1.6

Let μ be a probability measure on a countable group Γ.

1. *(Avez, [2]) If μ is symmetric and has finite entropy then*

$$\rho(\mu) \leq e^{-\frac{1}{2}h(\mu)}.$$

2. *(due to Guivarch, known as "the fundamental inequality of random walks" in [32]) If μ has finite first moment and Γ has finite growth δ, then*

$$h(\mu) \leq \delta \cdot \lambda(\mu).$$

Proof:

(1) Since μ is symmetric one has

$$\mu^{*2n}(e) = \sum_{g\in\Gamma} \mu^{*n}(g)\mu^{*n}(g^{-1}) = \sum_{g\in\Gamma} \mu^{*n}(g)^2.$$

Convexity of $-\log(t)$ therefore implies

$$\begin{aligned}-\log \mu^{*2n}(e) &= -\log(\sum_{g\in\Gamma} \mu^{*n}(g)\cdot\mu^{*n}(g)) \\ &\leq \sum_{g\in\Gamma} \mu^{*n}(g)\cdot(-\log\mu^{*n}(g)) = H(\mu^{*n})\end{aligned}$$

Therefore

$$\frac{1}{2n}\log\mu^{*2n}(e) \geq -\frac{1}{2n}H(\mu^{*n})$$

and taking $n\to\infty$

$$\log\rho(\mu) \geq -\frac{1}{2}h(\mu).$$

(2) Given $\epsilon > 0$ for large n the set

$$A_n = \{g\in\Gamma \,:\, |g| < (\lambda(\mu)+\epsilon)\cdot n\}$$

has μ^{*n}-mass $> 3/4$, and has size

$$|A_n| \leq (e^{(\lambda(\mu)+\epsilon)\cdot n})^{\delta+\epsilon} = e^{(\lambda(\mu)+\epsilon)(\delta+\epsilon)\cdot n}.$$

Also, the set

$$B_n = \{g\in\Gamma \,:\, \mu^{*n}(g) < e^{(-h(\mu)-\epsilon)\cdot n}\}$$

has μ^{*n}-mass $> 3/4$, hence $B_n \bigcap A_n$ has μ^{*n}-mass $> 1/2$ and has size

$$|B_n \cap A_n| \geq \frac{1}{2}e^{(h(\mu)+\epsilon)\cdot n}$$

Since $|B_n \cap A_n| \leq |A_n|$, after taking logarithms and dividing by n we have

$$h(\mu) + \epsilon - \frac{\log 2}{n} \leq (\lambda(\mu) + \epsilon)(\delta + \epsilon)$$

Since we can take ϵ arbitrarily small and n arbitrarily large, we have $h(\mu) \leq \lambda(\mu)\delta$

$\square$

A unitary representation without invariant vectors π has spectral gap if there exist a constant $c < 1$, such that for any finitely supported μ whose support generates the group we have $\|\pi(\mu)\| \leq c$. For groups with Kazhdan's property (T) (e.g. $\mathrm{SL}_d(\mathbb{Z})$ with $d \geq 3$) every unitary representation without invariant vectors admits spectral gap(with uniform constant $c < 1$). In particular, $\|\pi(\mu^{*n})\| \leq c^n$, which gives us a spectral estimate similar to one in Theorem 1.2.4. One might try to prove analogue of Theorem A for $d \geq 3$ using the spectral gap from property (T). Unfortunately, most subgroups of $\mathrm{SL}_d(\mathbb{Z})$ do not inherit property (T). Additionally, there is no hope that those estimates will lead to sharp results in the shrinking target problems. The constant c is hard to compute, but even if we had a good estimate on it, the uniform measures on the shells μ_n(as used in Theorem 1.2.4) are very different from μ^{*n}. Most of the support of μ^{*n} is indeed located around the shells, however it is usually very far from uniformly distributed and might actually miss a big piece of the shell. Therefore, if elements, which are missed by μ^{*n}, solved the Diophantine inequality, $\pi(\mu^{*n})$ would not be able to see that using this argument.

The proof of the inequality (2) in Proposition 6.1.6($h(\mu) \leq \delta\lambda(\mu)$) suggests that the difference between the "supports" of μ^{*n} and μ_n is reflected in the inequality being strict. If we were to use spectral gap, in order to get the sharpest estimates we would like to choose a measure μ, so that its $n-$convolutions closely resemble uniform measures, i.e. this inequality is as close to equality as possible.

6.2 Optimality of random walks

Assume μ has its support in the ball of radius n with respect to corresponding metric. Using Kesten's Theorem(Theorem 6.1.5) the inequalities from Proposition 6.1.6 can be put together in the following form

$$-2\log\|\lambda_\Gamma(\mu)\| \overset{(1)}{\leq} h(\mu) \overset{(2)}{\leq} \delta_\Gamma l(\mu) \overset{(3)}{\leq} \delta_\Gamma n \tag{6.1}$$

where (3) is immediate since the drift is not greater than the maximal length of the elements in the support of μ.

It was shown by Blachere, Haissinsky and Mathieu([7]) that the fundamental inequality of random walks becomes an equality for any measure with finite entropy(assuming the resulting random walk is transient) for a special choice of metric, namely the Green metric. It is defined using Martin kernel, but essentially means the following

$$d_G(x,y) := -\ln\mathbb{P}(\text{ random walk starting from } x \text{ will eventually pass through } y)$$

For this metric, $\delta = 1$. The problem is that this metric depends on μ and it is very hard to describe even for simplest measures μ.

Much work has been done to find μ that achieves equality in (2) of the inequality (Equation 6.1) for more natural metrics. It was shown in [24] that in hyperbolic case one cannot achieve the equality with a finitely supported measure, unless the group is virtually free, and the metric is the word metric.

The proof of Theorem A confirms the inequality $-2\log\|\pi_0(\mu)\| \leq \delta_\Gamma n$ for $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ and the Koopman representation on the torus for more natural metric on Γ coming from the operator norm. If it were to fail, we would be able to achieve faster approximation rates in Theorem A(2), contradicting part (1) of the theorem.

Since we fall short of achieving the equality in (2), one can ask if approaching the equality asymptotically is possible. By this we mean finding a sequence of finitely supported measures μ_n supported on ball of radius n, so that $\frac{h(\mu_n)}{l(\mu_n)} \to \delta_\Gamma$. When this happens, the random walks μ_n are thought of as well spread in the group. Theorem D shows that one can approach the equality asymptotically in a more general inequality $-2\log\|\lambda_\Gamma(\mu)\| \geq \delta_\Gamma n$, namely one can find a sequence of measures μ_n supported on B_n, so that $\frac{-2\log(\|\lambda_\Gamma(\mu_n)\|)}{n} \to \delta_\Gamma$.

We remark that the latter approximation is indeed stronger.

Remark 6.2.1 *There exists sequence of measures μ_n on a free group on two generators, such that $\frac{h(\mu_n)}{l(\mu_n)} = \delta_\Gamma$, but $\frac{-2\log(\|\lambda_\Gamma(\mu_n)\|)}{l(\mu_n)} \to 0$*

To see this, consider the simple random walk on the free group on two generators $\Gamma = \langle a^{\pm 1}, b^{\pm 1}\rangle$ with the corresponding word metric. It is an easy exercise that the equality is achieved in both inequalities simultaneously. We will perturb the law μ preserving one of the equalities but not the other.

We use the Markov stopping time (see [19]). For each $n \in \mathbb{N}$, we define the following cut set:

$$C(n) = C_n^a \cup \{a\}$$

where C_n^a is the set of all reduced words of length n that don't start with a. Markov stopping time creates a new law of random walk μ_n. Intuitively, one can think of sample paths in the new random walk being the same paths as in the old one with the same distribution, but with rescaled time. Each unit of time in the new path corresponds to starting the walk from identity and hitting the cutting set. Forghani proved in [19] that both the entropy and the drift of the new random walk are obtained by multiplication of the initial entropy and drift by the expected value of the stopping time. Therefore for each μ_n the equality in the fundamental inequality still holds.

It is easy to see that the spectral radius is bounded from below by $\frac{1}{4}$, regardless of n(test $\pi_\Gamma(\mu_n)$ against the characteristic function supported on powers of a), and since $l(\mu_n) \to \infty$ the claim follows.

6.3 Optimal ergodic theorem for linear actions on the 2-torus

Let $\Gamma < SL_2(\mathbb{Z})$, and π_0 be the Koopman representation on the 2-torus. By Theorem 2.2.7 we have $\rho_{\pi_0}(n) = \rho_\lambda(n)$. Since the measures μ_n in Theorem D are uniform measures on the shells S_n in Γ, the operators $\pi_0(\mu_n)$ can be viewed as averaging operators, and we can reformualte Theorem D as a quantitative ergodic theorem.

Corollary 6.3.1

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$*. There exists* $k > 0$*, so that if we denote the shells* $S_n = B_n \setminus B_{n-k} \subset \Gamma$*. Then for any* $f \in L^2(\mathbb{T}^2, m)$ *we have*

$$\left\| \frac{1}{|S_n|} \sum_{g \in S_n} f(g.x) - \int_{\mathbb{T}^2} f dm \right\|_2 \leq n e^{-\frac{1}{2}\delta_\Gamma n + O(1)} \|f\|_2.$$

From the inequalities in (Equation 6.1) the convergence rate can't be faster than $e^{-\frac{1}{2}\delta_\Gamma n}$. This suggests that averaging over shells in Γ produces the most optimal ergodic theorem for this action.

CITED LITERATURE

[1] M. Aka, E. Breuillard, L. Rosenzweig, and N. de Saxcé, *Diophantine properties of nilpotent Lie groups*, Compositio Mathematica **151** (June 2015), no. 6, 1157–1188.

[2] A. Avez, *Croissance des groupes de type fini et fonctions harmoniques*, Théorie Ergodique : Actes des Journées Ergodiques, Rennes 1973/1974 (1976), 35–49.

[3] U. Bader and R. Muchnik, *Boundary unitary representations - irreducibility and rigidity*, Journal of Modern Dynamics **5** (2011), no. 1, 49–69.

[4] B. Bekka and J.-R. Heu, *Random products of automorphisms of Heisenberg nilmanifolds and Weils representation*, Ergodic Theory and Dynamical Systems **31** (2011), no. 5, 12771286.

[5] A. S. Besicovitch, *Sets of fractional dimensions (IV): On rational approximation to real numbers*, J. London Math. Soc. **9** (1934), 126–131.

[6] S. Blachère, P. Haïssinsky, and P. Mathieu, *Harmonic measures versus quasiconformal measures for hyperbolic groups*, Annales scientifiques de l'Ecole Normale Superieure **44** (2011), no. 4, 683–721 (eng).

[7] ______, *Asymptotic entropy and green speed for random walks on countable groups*, Ann. Probab. **36** (2008), no. 3, 1134–1152.

[8] J. Bourgain, A. Furman, E. Lindenstrauss, and S. Mozes, *Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus*, J. Amer. Math. Soc. **24** (2011), 231–280.

[9] J. Bourgain and A. Kontorovich, *On Representations of Integers in Thin Subgroups of $SL_2(\mathbb{Z})$*, Geometric and Functional Analysis **20** (2010), no. 5, 1144–1174.

[10] J. Bourgain, E. Lindenstrauss, P. Michel, and A. Venkatesh, *Some effective results for $\times a \times b$*, Ergodic Theory and Dynamical Systems **29** (2009), no. 6, 1705–1722.

[11] A. Boyer, *Equidistribution, ergodicity and irreducibility in CAT(-1) spaces*, Geometry, Groups and Dynamics (2016). to appear.

[12] R. Brooks, *The bottom of the spectrum of a Riemannian covering*, Journal für die reine und angewandte Mathematik **357** (1985), 101–114.

[13] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Addison-Wesley, Reading, Massachusetts, 1984. Reprinted as Vol. A of *Computers & Typesetting*, 1986.

[14] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993), no. 2, 241–270.

[15] F. Dal'Bo, *Geodesic and Horocyclic Trajectories*, Universitext, Springer-Verlag London, 2011.

[16] H. Davenport and W. M. Schmidt, *Dirichlets theorem on diophantine approximation. ii*, Acta Arith. **16** (1969), 413–424.

[17] P. de la Harpe, *Groupes Hyperboliques, algebres d'operateurs et un theoreme de Jolissaint*, C. R. Acad. Sci. Paris Ser. I **307** (1988), 771–774.

[18] Y. Derriennic, *Quelques applications du théorème ergodique sous-additif*, Astérisque **74** (1980), 183–210.

[19] B. Forghani, *Asymptotic entropy of transformed random walks*, Ergodic Theory and Dynamical Systems **FirstView** (2016), 1–12.

[20] H. Furstenberg and H. Kesten, *Products of Random Matrices*, Ann. Math. Statist. **31** (196006), no. 2, 457–469.

[21] A. Gamburd, D. Jakobson, and P. Sarnak, *Spectra of elements in the group ring of SU(2)*, Journal of the European Mathematical Society **1** (1999), no. 1, 51–85.

[22] S. Gerschgorin, *Über die Abgrenzung der Eigenwerte einer Matrix.*, Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk **7** (1931), 749–754.

[23] A. Ghosh, A. Gorodnik, and A. Nevo, *Best possible rates of distribution of dense lattice orbits in homogeneous spaces*, J. Reine Angew. Math. (2016). To appear.

[24] S. Gouëzel, F. Mathéus, and F. Maucourant, *Entropy and drift in word hyperbolic groups* (2015). Preprint.

[25] L. Greenberg, *Discrete groups of motions*, Can. J. Math. **12** (1960), 415–426.

[26] A. Groshev, *A theorem on a system of linear forms*, Dokl. Akad. Nauk SSSR **19** (1938), 151–152.

[27] U. Haagerup, *An Example of a Non Nuclear C*-Algebra, which has the Metric Approximation Property*, Inventiones mathematicae **50** (1978/79), 279–293.

[28] R. Hill and S. L. Velani, *The Shrinking Target Problem for Matrix Transformations of Tori*, Journal of the London Mathematical Society **60** (1999), no. 2, 381.

[29] M. K. I. Biringer K. Bou-Rabee and F. Matucci, *Intersection growth in groups*, Trans. Am. Math. Soc.

[30] V. Jarnik, *Über die simultanen diophantischen Approximationen*, Math. Zeit. **33** (1931), 503–543.

[31] P. Jolissaint, *Rapidly decreasing functions in reduced C^*-algebras of groups*, Transactions of the American Mathematical Society **317** (1990), no. 1, 167–196.

[32] V. A. Kaimanovich and A. M. Vershik, *Random Walks on Discrete Groups: Boundary and Entropy*, The Annals of Probability **11** (1983), no. 3, 457–490.

[33] H. Kesten, *Symmetric Random Walks on Groups*, Transactions of the American Mathematical Society **92** (1959), no. 2, 336–354.

[34] A. Ya. Khinchin, *Continued fractions*, University of Chicago Press, 1935.

[35] A. J. Khintchine, *Einige Sätze über Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*, Math. Ann. **92** (1924), 115–125.

[36] M. S. Kirsebom, *Extreme value theory for random walks on homogeneous spaces*, Discrete and Continuous Dynamical Systems **34** (2014), no. 11, 4689–4717.

[37] J. F. Koksma, *Some theorems on diophantine inequalities.*, Math. Centrum Amsterdam Scriptum, 1950.

[38] G. Kuhn, *Amenable Actions and Weak Containment of Certain Representations of Discrete Groups*, Proceedings of the American Mathematical Society **122** (1994), no. 3, 751–757.

[39] M. Laurent and A. Nogueira, *Approximation to points in the plane by* $\mathrm{SL}(2,\mathbb{Z})$*-orbits*, J. Lond. Math. Soc. **85** (2012), no. 2, 409–429.

[40] J. Levesley, *A General Inhomogeneous Jarnik-Besicovitch Theorem*, Journal of Number Theory **71** (1998), no. 1, 65 –80.

[41] F. Maucourant and B. Weiss, *Lattice actions on the plane revisited*, Geometriae Dedicata **157** (2012), no. 1, 1–21.

[42] A. Nevo, *Spectral transfer and pointwise ergodic theorems for semi-simple Kazhdan groups*, Mathematical Research Letters **5** (1998), 305–325.

[43] ______, *The spectral theory of amenable actions and invariants of discrete groups*, Geometriae Dedicata **100** (2003), no. 1, 187–218.

[44] M. Pollicott, *Rates of convergence for linear actions of cocompact lattices on the complex plane*, Integers : electronic journal of combinatorial number theory **11B** (2010), Article no. A12.

[45] G. Prasad and A. Rapinchuk, *Existence of irreducible* $\mathbb{R}$*-regular elements in Zariski-dense subgroups*, Mathematical Research Letters **10** (2003), 21–32.

[46] Y. Shalom, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group.*, Annals of Mathematics. Second Series **152** (2000), no. 1, 113–182.

[47] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **50** (1979), no. 1, 171–202.

[48] P. P. Varjú, *Diophantine property in the group of affine transformations of the line*, Acta Sci. Math. (Szeged) **80** (2014), 447–458.

[49] R. Zimmer, *Amenable ergodic group actions and an application to Poisson boundaries of random walks*, Journal of Functional Analysis **27** (1978), no. 3, 350–372.

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