# CONTINUUM-BASED GEOMETRY/ANALYSIS APPROACH FOR FLEXIBLE AND SOFT ROBOTIC SYSTEMS (FSRS)

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## ABSTRACT

Control and stability of *flexible and soft robotic systems* (FSRS), which have complex geometry and experience desirable and undesirable deformations, are of major concern, particularly when light weight soft materials are used. Nonetheless, there is no unified continuum-based geometry/analysis approach that can be used for the efficient FSRS virtual prototyping and design. The goal of this paper is to propose a new FSRS geometric modeling and analysis methodology by addressing fundamental virtual prototyping challenges that include: (1) integration of the robot geometry and analysis; (2) implementation of general and unconventional material models and actuation forces; (3) use of new concepts for modelling FSRS joints; and development of efficient and robust algorithms for the FSRS virtual prototyping. In order to address these challenges, the *finite element* (FE) *absolute nodal coordinate formulation* (ANCF) and *multibody system* (MBS) computational algorithms are used. ANCF finite elements allow for modeling arbitrarily large and coupled displacements, correctly capture complex geometries, allow for implementing general and nonconventional material models, provide accurate definitions of conventional and non-conventional actuation forces, lead to a constant inertia matrix that defines optimally sparse matrix structure of the dynamic equations, and allow for exploiting new geometry concepts to define linear and more general joint constraints instead of the less general and nonlinear joint constraints currently used for robot systems. Using the general structure of the FSRS nonlinear dynamic equations of motion, a non-modal continuum-based approach can be developed and used to correctly capture the FSRS complex geometry and large deformations.



**Keywords:** Robot geometry; soft robot analysis; soft robot actuation; large deformation; absolute nodal coordinate formulation.

## 1. INTRODUCTION

Flexible and soft robotic systems (FSRS) are becoming increasingly important in many applications including medicine, space, agricultural, automotive, manufacturing, nuclear, health care and rehabilitation, locomotion, environmental conservation, and marine exploration applications. In many of these applications, robots work cooperatively with humans to increase productivity, ensure safety, and extend capabilities beyond human limitations. Advancing robot technology, therefore, will have significant economic, environmental, and security impact. While robots are made of flexible links or soft bodies that can have complex geometric shapes, our knowledge of the FSRS mechanics remains remarkably incomplete. The mathematical foundation of these technologically important systems, particularly soft robotic systems, has not been established, making it difficult to further develop, advance, and achieve breakthroughs in such an important area of science and technology. Existing robot formulations, including conventional FE, rigid body, and modal-based small deformation multibody system (MBS) approaches, are not suited for capturing the FSRS complex geometry and dynamics [1 – 11]. This is evident by the fact that there is no in existence today an agreed upon approach for modeling soft robot systems. Research in this field has been, for the most part, experimental and based on a trial and error approach for building costly prototypes or on using simplified discrete analytical models that fail to capture the effect of the FSRS distributed inertia and elasticity. Without a sound mathematical foundation, the development of reliable, efficient, and general computational framework and virtual prototyping tools cannot be realized. This obvious limitation is despite the fact that virtual prototyping tools, necessary for easily and economically experimenting with different new design configurations, are currently an integral part of the analysis, design, and performance evaluation of many other physics and engineering systems.

For this reason, there is a need for new FSRS approaches in order to capture correctly the effect of large deformation, allow for implementing general nonlinear and nonconventional material models, and define accurately the actuation forces which can be produced using both conventional and nonconventional methods.

Unlike other engineering systems, the FSRS design requires the use of complex geometry, unconventional soft materials, and unconventional actuation forces that are not necessarily produced by conventional actuators or motors. Accurate description of the geometry and its integration with the analysis is necessary because of the significant changes in shapes during the FSRS functional operations, as in the case of soft robots squeezing through gaps and grooves. In such cases, a continuum-based analysis approach is required to capture correctly the original as well as the change in the robot geometry as well as capture accurately the large deformations necessary for the stress calculations and design. The accurate CAD geometry cannot be preserved when converting a CAD solid model to an FE analysis mesh [12] because there is no linear mapping between the CAD B-splines (**B**asis **S**pline) and NURBS (**N**on-**U**niform **R**ational **B**-**S**plines) and the kinematics of structural finite elements (beams, plates, and shells) that employ rotations as nodal coordinates. The use of these finite elements or a modal MBS approach in the FSRS analysis leads to geometry distortion that makes the design process less reliable and efficient, time consuming, and costly.

In addition to the challenge of accurately capturing the original and deformed geometry, the continuum-based approach used must allow for using general and/or unconventional large deformation material models including composites [13]. Simple discrete spring-damper models currently used are not adequate for describing the distributed elasticity or for implementing general constitutive models. In FSRS applications, it is necessary to be able to use continuum-based, conventional and non-conventional, and compressible and incompressible material models that capture the elastic nonlinearities resulting from the large displacements. The use of such an approach will enable adopting a general continuum mechanics theory capable of describing a wide range of materials that cannot be accurately represented using the simplified discrete models. A continuum-based material approach will also allow for systematically modeling thermal effects, fluids, muscles, and pressurized air used as FSRS actuation forces.

Furthermore, conventional actuators and motors, used in rigid or nearly rigid robotic manipulators, may not be suited for the motion control of very soft robot manipulators whose control may require the use of *unconventional actuation forces* such as pressurized air, fluidic, and muscle-like actuations used in the control of soft robotic systems [2, 14 - 26]. Unlike rigid body dynamics, in flexible body dynamics, the force is not a sliding vector and the moment is not a free vector. A Cartesian force or a moment produces generalized forces associated with the deformation degrees of freedom [27]. Therefore, new definitions of non-modal actuation forces are necessary in order to be able to develop the mathematical foundation that governs the FSRS motion and to develop effective FSRS control algorithms.

Accounting for the effect of the robot deformation significantly increases the complexity and degree of nonlinearity of the *joint formulations*. Such complexity and nonlinearity adversely affect the robustness and efficiency of the solution algorithms. A new approach, that employs new geometry concepts in the formulation of the FSRS equations of motion, will allow for developing new linear FSRS joint formulations instead of using the conventional approaches which lead to highly nonlinear joint algebraic equations [28, 29]. Using the new concepts, FSRS joints, including revolute joints, can be formulated using linear algebraic equations that can be eliminated with the redundant variables at a preprocessing stage, thereby significantly reducing the problem dimension and achieving an optimum sparse matrix structure that can be effectively exploited in order to develop efficient and robust algorithms for the FSRS virtual prototyping and design.

(a) (b)

Figure 1 Robot Geometry and Analysis

It is the objective of this investigation to propose an approach for developing the FSRS mathematical foundation and new computational framework that will allow defining the structure of the FSRS nonlinear dynamic equations, successfully integrating the geometry and analysis meshes, exploiting new concepts in joint modeling, developing accurate definition of conventional and nonconventional material models and actuation forces, and establishing efficient computational procedures for the simulation and virtual prototyping of such new technologically important systems. Figure 1a shows the overall technical goals of this study, while Figure 2b shows how the approaches developed can lead to new virtual prototyping algorithms based on successful geometry/analysis integration. In order to achieve the goals of this study, the nonlinear FE absolute nodal coordinate formulation (ANCF) is used to provide a mechanics-based non-modal framework for both the geometry and analysis, allowing for dealing with all the system components and joints from the outset at the geometry creation stage. Using ANCF elements, conventional and nonconventional continuum-based material models and actuation forces can be systematically defined, large deformations of flexible links and soft bodies can be accurately represented, and geometric and material nonlinearities can be captured.

## 2. LIMITATIONS OF EXISTING METODS

Most of the robot analytical and computational studies have been concerned with rigid-link or small deformation problems [30 - 41]. The most widely used approach to study the small deformations of such systems is the *floating frame of reference* (FFR) formulation [39, 42]. In this formulation, two sets of coordinates are used; the reference and elastic coordinates. The reference coordinates are used to define the position and orientation of a robot link coordinate system, while the elastic coordinates are used to define the link deformation with respect to its coordinate system. The large displacement of the robot link is described using the reference coordinates, while its small deformation can be defined using approximation methods. In the FFR formulation, the global position of an arbitrary point on a deformable robot link can be written as $\mathbf{r} = \mathbf{R} + \mathbf{A}\left(\bar{\mathbf{u}}_o + \bar{\mathbf{u}}_f\right)$, where $\mathbf{R}$ is the global position vector of the origin of the body reference, $\mathbf{A}$ is the transformation matrix that defines the body orientation in terms of a set of orientation parameters $\boldsymbol{\theta}$ such as Euler angles or Euler parameters, $\bar{\mathbf{u}}_o$ is the local position vector of the arbitrary point with respect to the body coordinate system in the reference configuration, and $\bar{\mathbf{u}}_f$ is the deformation vector defined in the body coordinate system. The deformation vector $\bar{\mathbf{u}}_f$ can be defined using approximation techniques including the separation of variable techniques and the FE methods. The use of the local reference frame leads to a highly nonlinear inertia matrix and to complex expressions for the Coriolis and centrifugal forces (Shabana, 2013). Nonetheless, if the assumptions of linear theory of elasticity are used, the stiffness matrix becomes constant in the FFR formulation.

The use of the FFR formulation allows creating a local linear problem that can be exploited to significantly reduce the number of the degrees of freedom. Standard component mode synthesis techniques are often used to eliminate high frequency modes which have insignificant effect on the accuracy of the solution. Because conventional structural finite elements such as beams are used with the FFR formulation, the FFR formulation is not suited for the analysis of general large deformation problems. Furthermore, the geometry of conventional finite elements such as beam and plate elements which employ infinitesimal rotation as nodal coordinates are not invariant under an orthogonal rigid body transformation [43, 44]. Therefore, there is no linear mapping between the displacement field of these elements and the more geometrically accurate CAD methods such as B-splines and NURBS. Because of the lack of such a linear mapping, the conversion of CAD model to FE analysis mesh is very costly and time consuming and often leads to geometry distortion [12]. The FFR formulation, therefore, has two serious limitations that make it unsuitable as the basis for the proposed FSRS formulation and computational framework. First, the FFR linear modes limit its use, for the most part, to the small deformation problem. Second, the modal-based FFR formulation cannot be the basis for successful geometry/analysis integration; necessary in order to be able to develop an approach that allows using efficient and general virtual prototyping techniques to explore new FSRS design configurations.

## 3. LARGE DISPLACEMENTS AND DEFORMATIONS

As reported by Hod [1], other existing FE formulations are not suited for the analysis of the FSRS large deformations. In this section, a justification for using and the basic equations used in the proposed ANCF approach for the FRSR geometry and analysis are provided.

### 3.1 Conventional FE Approaches

In general purpose FE software, an incremental rotation procedure based on the co-rotational approach is used [45]. Such an approach has proven to be inaccurate in the analysis of multibody system (MBS) applications [46]. The use of the co-rotational approach with conventional FE beams, plates, and shells leads to linearization of the kinematic equations, making such an approach unsuitable for the analysis of arbitrarily large rigid body displacements that characterize robot systems. For this reason, such an approach has not been widely used for FSRS applications. Furthermore, as in the case of the FFR formulation, structural finite elements used with the co-rotational approach have geometry that is not invariant under an arbitrary orthogonal rigid body transformation, and therefore, such elements cannot be used as the basis for the geometry/analysis integration. Other serious drawbacks of this FE approach are the difficulties of formulating the robot joint constraints and actuation forces, and difficulties of exploiting new geometry concepts to obtain accurate and new definitions of the robot joints and the generalized actuation forces.

### 3.2 Large Rotation Vector Formulation (LRVF)

Another large displacement FE formulation is the large rotation vector formulation (LRVF), in which, independent interpolations are used for the position and rotation fields [47 – 49]. As reported in the literature, such a formulation leads to fundamental problems including coordinate redundancy and violation of basic mechanics principles [46]. Finite rotations that characterize FSRS applications are not commutative, and therefore, cannot be treated as vectors, added, or interpolated [50, 51]. When Euler parameters are used as orientation coordinates, quaternion algebra, instead of conventional vector algebra, is used, that is, Euler parameters like any other orientation parameters cannot be interpolated or added. Furthermore, as in the case of the methods previously discussed, there is no linear mapping between LRVF finite elements and the more geometrically accurate B-splines and NURBS [52 – 54], and therefore, LRVF elements cannot be used to develop a unified geometry/analysis approach to eliminate the need for the costly conversion of solid models to FE analysis meshes [12]. Additionally, LRVF elements lead to nonlinear inertia matrix in the spatial analysis, making it difficult to obtain an optimum sparse matrix structure of the FSRS dynamic equations.

### 3.3 FSRS Continuum-Based ANCF Approach

Because existing FE formulations and software cannot be used in the analysis and design of FSRS applications, one resorted to using discrete spring-damper elements to capture the large FSRS displacements [1]. This simplified approach, however, does not capture correctly the distributed inertia and elasticity of flexible or soft robots, does not allow for systematically implementing general continuum-based material models, cannot predict accurately the actuation forces, and cannot be used as the basis for developing effective virtual prototyping tools for the design of such technologically and economically important systems. In this study, a continuum-based approach based on the *absolute nodal coordinate formulation* (ANCF) is proposed to address these deficiencies [55 - 65]. The displacement field of ANCF finite elements, which can describe arbitrarily large displacements including finite rigid body rotations and large deformations, is expressed in terms of absolute position and position vector gradient coordinates as $\mathbf{r}(\mathbf{x},t)=\mathbf{S}(\mathbf{x})\mathbf{e}(t)$. In this equation, $\mathbf{r}$ is the global position vector of an arbitrary point on the element, $\mathbf{S}$ is the shape function matrix, $\mathbf{e}$ is the vector of the element nodal coordinates, and $\mathbf{x}=\begin{bmatrix} x & y & z \end{bmatrix}^T$ is the vector of the element spatial coordinates. In the case of fully parameterized ANCF finite elements, the vector of nodal coordinates can be defined using the position and gradient coordinates as $\mathbf{e}=\begin{bmatrix} \mathbf{r}^T & \mathbf{r}_x^T & \mathbf{r}_y^T & \mathbf{r}_z^T \end{bmatrix}^T$, where $\mathbf{r}_x=\partial\mathbf{r}/\partial x$, $\mathbf{r}_y=\partial\mathbf{r}/\partial y$, and $\mathbf{r}_z=\partial\mathbf{r}/\partial z$ are the position vector gradients, which in general do not remain orthogonal unit vectors. The ANCF kinematic description is shown in Fig. 2 for a three-dimensional beam element. In this figure, the superscript indicates the node number. ANCF Plate, solid, and tetrahedral elements employ the similar coordinates and have more nodal points (Shabana, 2018). Using this ANCF description, there is no restriction on the amount of rotation or deformation within the element. ANCF finite elements can have higher order of interpolations, allowing capturing complex shapes and large deformations without the need for using the co-rotational approach. Use of position vector gradients as nodal coordinates ensures proper description of the finite rotation and therefore there is no need for using incremental-rotation procedures. Figure 3 shows the geometry which can be captured using a single ANCF triangular plate element [66].

Figure 2  ANCF Kinematic Description

Figure 3 Geometry Captured by Single ANCF Triangular Plate Element [66]

Using the ANCF displacement field, $\mathbf{r}(\mathbf{x},t)=\mathbf{S}(\mathbf{x})\mathbf{e}(t)$, which can describe arbitrarily large displacements, the inertia forces can be defined using the virtual work as $\delta W_i = \int_V \rho \ddot{\mathbf{r}}^T \delta \mathbf{r} dV$, where $\rho$ and $V$ are, respectively, the mass density and volume of the ANCF element. One can show that this expression of the virtual work of the inertia forces leads to a symmetric and constant mass matrix defined as $\mathbf{M} = \int_V \rho \mathbf{S}^T \mathbf{S} dV$, and consequently, Coriolis and centrifugal forces are identically equal to zero, a unique ANCF feature that leads to significant simplification of the dynamic equations of motion. Furthermore, a Cholesky coordinate transformation can be used to define an identity generalized mass matrix associated with the ANCF Cholesky Coordinates [63]. This leads to an optimum sparse matrix structure of the FSRS dynamic equations of motion.

ANCF finite elements allow for implementing general material models, including Neo-Hookean and incompressible Mooney-Rivlin material models, and for adopting a general continuum mechanics approach when structural elements such as beams, plates and shells are used. Continuum-based composite materials, fluid constitutive equations, and viscoelastic and plastic material models can also be used systematically with ANCF elements [63]. A continuum-based approach can be used to formulate the FSRS *elastic forces*. The virtual wok of the elastic forces can be written as $\delta W_s = -\int_V \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon} dV$ , where $\boldsymbol{\varepsilon}$ is the Green-Lagrange strain tensor which can be evaluated using the ANCF position gradient vectors and $\boldsymbol{\sigma}$ is the second Piola-Kirchhoff stress tensor. This general continuum mechanics approach allows for systematically using nonlinear and nonconventional material models including composite and fluid materials. The ANCF results have been validated experimentally and numerically by different researchers; an example is a recent study published in the ASME Transactions [67].

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## 4. FSRS GEOMETRY/ANALYSIS INTEGRATION

The use of the ANCF gradient vectors as nodal coordinates allows for representing complex geometries and for accurate and unique definition of the rotation and strain fields. The position vector gradients can be used to develop an automated procedure for adjusting the degree of continuity at the nodal points [68, 69]. Another important feature is the fact that ANCF structural finite elements (beams, plates, and shells) are related to B-splines and NURBS by a linear mapping [61, 62, 68, 69]. Therefore, ANCF elements can be used as the basis for the FSRS geometry/analysis integration and for developing new mechanics-based virtual prototyping algorithms. While there is a linear mapping between ANCF displacement fields and B-splines and NURBS, B-splines and NURBS, as well as new isogeometric analysis methods [70], employ a rigid recurrence structure and were developed mainly as graphics tools without consideration of the concept of degrees of freedom which is fundamental in the analysis of robot systems [28, 29]. In B-splines and NURBS, control points which are not necessarily material points are used and continuity conditions are defined using the knot vectors and knot multiplicity [52 – 54]. The knot vector and multiplicity fail to account for the correct number of degrees of freedom when mechanical joints are present in the model as demonstrated in the literature [28, 29]. Nonetheless, the fact that there is a linear mapping between computational geometry methods such as B-splines and NURBS and ANCF elements will allow converting existing CAD models to ANCF meshes without geometry distortion, and will allow developing new ANCF mechanics-based CAD systems that can be used to develop a unified geometry/analysis mesh from the outset. Furthermore, Bezier geometry can be effectively used to develop new ANCF elements for different FSRS applications. The use of the position vector gradients as ANCF nodal coordinates allows for efficient local shape manipulations without the need for the knot vector and multiplicity rigid structure. This is particularly important in case of robot systems where the geometry of the components is selected to optimize performance and obtain the desired degree of strength or flexibility. Being able to easily experiment virtually with different geometries will facilitate exploring different design configurations.

## 5. JOINT FORMULATION AND MULTI-COMPONENT CAD SYSTEM

In robot systems, the links, or bodies in the case of soft robots, can experience large displacements including finite rotations. As a result, the formulation of the joints between different links or bodies can be highly nonlinear, and such joint formulations are often expressed in terms of orientation parameters. In this section, some concepts related to the joint formulation and the solid modeling of multi-component systems are discussed.

## 5.1 Joint Constraint Equations

In general, there are two main approaches used to study the dynamics of articulated systems subjected to joint constraints and specified motion trajectories. The first approach is the augmented Lagrangian formulation, commonly referred to as *absolute Cartesian coordinate formulation*, while the second is the embedding technique referred to as the recursive or *joint variable method*. Both formulations have their advantages and drawbacks and both can be used to systematically define the actuation forces required to produce certain motion trajectories. In the Cartesian coordinate formulation, the constraint algebraic equations are augmented to the system differential equations of motion, leading to a large system, expressed in terms of redundant coordinates, in which the actuation forces are determined using the technique of Lagrange multipliers. In the joint variable formulation, on the other hand, the nonlinear algebraic constraint equations are systematically eliminated leading to a minimum set of differential equations expressed in terms of the degrees of freedom (independent coordinates). The constraint forces can be determined using the acceleration vector and the dynamic equilibrium equations of the bodies. Both the absolute Cartesian coordinate and the joint variable formulations allow for systematically including the body deformations. While both formulations have been used in the analysis of rigid and flexible robotic manipulators, the joint variable formulation is more widely used because it allows for developing recursive approach that is more efficient for open-chain robot manipulators. The efficiency of the two approaches, however, has been debated when the system has closed kinematic chains. The recursive approach often leads to highly nonlinear inertia matrix and complex expression for the Coriolis and centrifugal forces. When the body flexibility is considered, the complexity of the inertia, centrifugal, and Coriolis forces significantly increases.

## 5.2 Joint Constraints and CAD Systems

The basic difference between the two approaches (absolute Cartesian and joint variable), as will be explained in a later section of this paper, is the method of treatment of the *joint nonlinear algebraic constraint equations*. By reducing the number of nonlinear algebraic constraint equations, the efficiencies of the two approaches become comparable. This can be achieved by exploiting new geometry concepts that can also change the solid modeling approach. Traditionally, the step of the solid modeling is viewed as separate from the step of the dynamic simulation in which the algebraic joint constraint equations are enforced. Existing CAD systems are designed to deal with one component of the robot system at a time. A new approach is needed in order to have a new CAD environment that allows for developing the solid model for a *multi-component system* from the outset. The way the joint constraint equations are formulated is crucial in the development of such a new approach. By using new geometry concepts, such an approach can be developed and used effectively for FSRS applications.

Using ANCF elements, new and more general linear formulations of joints that have been in the past formulated using nonlinear algebraic equations can be developed to allow for the first time for building multi-component solid models for the FSRS application. The geometry of the FSRS components can be simultaneously manipulated, distances between components can be adjusted while shapes are being altered, and component dimensions and properties can be changed simultaneously.

## 5.3 Linear Joint Formulations

As previously mentioned, in existing approaches, robot joints are formulated using highly nonlinear algebraic equations which are functions of the robot link orientation coordinates. In most existing formulations, the degree of nonlinearity and complexity of the joint formulations significantly increase when the deformations of the bodies or links are considered. This in turn leads to significant increase in the complexity of the FSRS nonlinear dynamic equations and solution algorithm. In this study, new geometry concepts applicable only to ANCF finite elements are used in order to allow developing new FSRS mechanics approach. Using these geometry concepts, linear and more general joints can be systematically developed and used to replace the nonlinear and less general joints currently being used. The linearity of the FSRS joints will allow eliminating the algebraic equations and the associated dependent variables at a pre-processing stage, therefore, significantly reducing the problem dimensionality.

As an example, the revolute (pin) joint, which in the classical formulations has only one degree of freedom, is based on the assumption that the joint is between two rigid coordinate systems. A more general linear revolute joint that relaxes the rigidity assumption and allows for more deformation modes can be developed. This fact can be demonstrated using the three-dimensional two-node fully parameterized ANCF beam element which has 12 nodal coordinates per node (Shabana, 2018). Using this ANCF beam element, a more general formulation for the revolute joint between two elements $i$ and $j$ can be defined using the six linear algebraic equations

$$\mathbf{r}^i = \mathbf{r}^j, \quad \left(\partial \mathbf{r}^i / \partial \beta\right) = \left(\partial \mathbf{r}^j / \partial \beta\right) \tag{1}$$

where $\beta$ is a coordinate line or parameter that defines the axis of the revolute joint. These linear revolute joint constraint equations reduce the number of degrees of freedom by six and define $C^1$ continuity along the $\beta$ coordinate line and $C^0$ continuity along the other coordinate lines.

Because of the linearity, this revolute joint can be defined at a preprocessing stage to eliminate the dependent variables, thereby defining a lower dimension geometry/analysis mesh [57]. Having linear joint constraint equations will allow developing the geometry of the FSRS system in a new multi-component CAD environment instead of the existing CAD systems which were envisioned for creating the geometry of a single component at a time.

Furthermore, newly introduced concepts such the *ANCF reference node* can be used for the robot assembly in a new mechanics-based multi-component CAD environment [71, 72]. The reference node, which is not a finite element node, and can be used to define general connectivity conditions at a preprocessing stage leading to an FSRS mesh with minimum number of nonlinear constraint equations. This method is particularly important if the FSRS application includes components that need to be modeled as rigid bodies. Kinematic constraints between a reference node $r$ and an element node $k$ of a robot link or body $i$ can be systematically developed. For example, the position constraints between the two nodes can be defined using the linear equation

$$\mathbf{r}^{ik} - \mathbf{r}^{r} = x^{ik}\mathbf{r}_x^r + y^{ik}\mathbf{r}_y^r + z^{ik}\mathbf{r}_z^r \tag{2}$$

In this equation, without any loss of generality, the reference node position gradient vectors $\mathbf{r}_x^r, \mathbf{r}_y^r$, and $\mathbf{r}_z^r$ are selected to be initially orthogonal unit vectors. The nonlinear conditions of rigidity and orthogonality of the reference node gradients can be introduced during the simulation using the technique of Lagrange multipliers. Similarly, linear constraints on the gradient coordinates can be defined at the preprocessing stage using the equation

$$\begin{bmatrix} \mathbf{r}_x^{ik} & \mathbf{r}_y^{ik} & \mathbf{r}_z^{ik} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_x^{r} & \mathbf{r}_y^{r} & \mathbf{r}_z^{r} \end{bmatrix} \mathbf{J}^{ikr} \tag{3}$$

In this equation, $\mathbf{J}^{ikr}$ is the Jacobian matrix or the matrix of position vector gradients of node or interface point $k$ in the initial reference configuration with respect to the reference node $r$. In the FSRS assembly, the reference node can represent a rigid component connected to flexible link or soft body [71, 72].

## 6. FSRS ACTUATION FORCES

The control and stability of robots are major safety and productivity issues. To successfully control a robot and ensure its safe operation, precise definitions of the actuation forces are necessary. As previously mentioned, in flexible body dynamics, the force is not a sliding vector and the moment is not a free vector since forces and moments depend on the deformation of material points. A systematic procedure can be used to define the nonlinear expressions of the *generalized actuation forces*. In FSRS applications, both conventional and non-conventional forces are used. The conventional forces include actuator forces and motor torques, while the nonconventional actuation forces include pressurized air, fluid, and muscle-like forces.

The desired motion trajectories can be specified, and the constraint forces that produce the desired motion can be determined. Unlike rigid body dynamics in which the *inverse dynamics* leads to algebraic equations, in the dynamics of flexible bodies which have infinite number of degrees of freedom, the inverse dynamics approach does not lead to algebraic equations only, but also leads to differential equations whose solution requires the use of time integration. The desired trajectory constraints produce forces associated with the elastic degrees of freedom, a fundamental issue that must be addressed in the FSRS control [27]. For a given Cartesian force, $\mathbf{F}$, the virtual work used to define the generalized forces associated with the ANCF coordinates can be written as $\delta W = \mathbf{F}^T \delta \mathbf{r} = \mathbf{F}^T \mathbf{S} \delta \mathbf{e}$. This virtual work expression defines the generalized forces associated with the ANCF nodal coordinates as $\mathbf{S}^T \mathbf{F}$. For a given Cartesian moment, $\mathbf{M}_c$, one can also develop the generalized forces associated with the ANCF nodal coordinates using the relationship between the skew symmetric spin tensor $\mathbf{W}$ and the time derivatives of the nodal coordinates. If the skew symmetric spin tensor is formed from the elements of the vector $\mathbf{w}$, one can develop the relationship $\mathbf{w} = \mathbf{G}\dot{\mathbf{e}}$, where $\mathbf{G}$ is a velocity transformation matrix that defines the elements of the spin tensor in terms of the time derivatives of the ANCF coordinates. In this case, the generalized forces associated with the ANCF nodal coordinates as the result of the application of the moment $\mathbf{M}_c$ are defined as $\mathbf{G}^T\mathbf{M}_c$ [73]. In the case of specified motion trajectories defined by the nonlinear algebraic constraint equations $\mathbf{C} = \mathbf{0}$, one can systematically define the specified motion constraint forces in the absolute Cartesian coordinates as $-\mathbf{C}_{\mathbf{e}}^T\boldsymbol{\lambda}$, where $\mathbf{C}_{\mathbf{e}}$ is the constraint Jacobian matrix, and $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. Knowing the generalized constraint forces, the Cartesian forces and moments can be defined using the basic relationships presented in this section. In the case of the joint variable approach in which the constraint equations are systematically eliminated, one can still use the acceleration solution and the body or link equations of motion to obtain the generalized constraint forces. The same approach can also be applied in the case of nonconventional actuation forces as pressurized air. The Cartesian forces and moments obtained from the constraint forces can be used to determine the air pressure forces required to produce the desired motion trajectories.

## 7. FSRS EQUATIONS OF MOTION

Robotic manipulators are designed to perform certain tasks. One widely used approach for the design of such systems is to specify the motion trajectories and use the technique of *inverse dynamics* to determine the forces required to produce the desired motion. That is, in addition to algebraic joint constraint equations, algebraic equations that describe the desired motion trajectories need to be introduced to the dynamic formulation. The system nonlinear algebraic constraint equations, including the specified motion trajectories, can be written in a vector form at the position, velocity, and acceleration levels, respectively, as

$$\mathbf{C}(\mathbf{q},t)=\mathbf{0},\quad \mathbf{C}_{\mathbf{q}}\dot{\mathbf{q}}+\frac{\partial \mathbf{C}}{\partial t}=\mathbf{0},\quad \mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}-\mathbf{Q}_c=\mathbf{0} \tag{4}$$

where $\mathbf{q}$ is the vector of coordinates, $\mathbf{Q}_c$ is a quadratic velocity vector, and $t$ is time. In the presence of the constraint equations, the FSRS equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{q}}=\mathbf{F}+\mathbf{F}_c \tag{5}$$

where $\mathbf{M}$ is the system symmetric mass matrix, $\ddot{\mathbf{q}}$ is the acceleration vector, $\mathbf{F}$ is the vector of applied forces that include external forces and moments as well as gyroscopic moments, and $\mathbf{F}_c$ is the vector of mechanical joint and specified motion trajectory constraint forces. Depending on how the algebraic constraint equations of Eq. 4 are treated, two different general formulations can be used to formulate the FSRS equations of motion. The first approach is the *absolute Cartesian coordinate* (augmented Lagrangian) approach, while the second is the *joint variable* (recursive) approach. If most of the algebraic equations are eliminated, as previously discussed, the degrees of efficiency of the two formulations will be comparable as it is clear from their basic equations presented in the following two subsections.

### 7.1 Absolute Coordinate Formulation

In the absolute Cartesian coordinate formulations, the constraint forces are expressed in terms of the constraint Jacobian matrix $\mathbf{C}_{\mathbf{q}}$ and the vector of Lagrange multipliers $\boldsymbol{\lambda}$ as $\mathbf{F}_c=-\mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda}$. The constraint equations at the acceleration level, $\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}}=\mathbf{Q}_c$, is combined with the equations of motion $\mathbf{M}\ddot{\mathbf{q}}=\mathbf{F}-\mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda}$ to form one matrix equation which can be solved for the unknown accelerations and Lagrange multipliers. While this approach requires the solution of a system of differential and algebraic equations (DAE's), it leads to a sparse matrix structure. The constraint equations at the position level must be solved iteratively for the dependent variables using a Newton-Raphson algorithm.

### 7.2 Joint Variable Formulation

In the joint variable (recursive) approach, on the other hand, the constraint equations are used to write the system accelerations in terms of the independent accelerations as $\ddot{\mathbf{q}} = \mathbf{B}_{di}\ddot{\mathbf{q}}_i + \boldsymbol{\gamma}$, where $\ddot{\mathbf{q}}_i$ is the vector of independent accelerations, $\mathbf{B}_{di}$ is a velocity transformation matrix, and $\boldsymbol{\gamma}$ is a vector that is quadratic in the velocities. Substituting this equation into the equation of motion, $\mathbf{M}\ddot{\mathbf{q}} = \mathbf{F} - \mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda}$, pre-multiplying by the transpose of the matrix $\mathbf{B}_{di}$, and using the fact that $\mathbf{B}_{di}^T\mathbf{C}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{0}$; one obtains a minimum set of ordinary differential equations from which the constraint forces are automatically eliminated. Since $\mathbf{B}_{di}^T\mathbf{C}_{\mathbf{q}}^T = \mathbf{0}$, pre-multiplying $\ddot{\mathbf{q}} = \mathbf{B}_{di}\ddot{\mathbf{q}}_i + \boldsymbol{\gamma}$ by $\mathbf{C}_{\mathbf{q}}$ and using the equation $\mathbf{C}_{\mathbf{q}}\ddot{\mathbf{q}} = \mathbf{Q}_c$ demonstrates that $\mathbf{C}_{\mathbf{q}}\boldsymbol{\gamma} = \mathbf{Q}_c$.

It is clear that, the geometry concepts discussed in this paper can be used to significantly reduce the number of FSRS algebraic constraint equations. In this case, the two approaches (absolute coordinates and joint variables) will have comparable degrees of efficiency in many FSRS applications as has been demonstrated in preliminary studies using other mechanical systems [57].

## 8. SUMMARY AND CONCLUSIONS

Robots are becoming increasingly important in many areas including manufacturing, medicine, safety, environmental protection, space, agricultural, rehabilitation and health care, nuclear, locomotion, among many others. Advancing the robot technology will have a significant economic and environmental impact. The FSRS control and stability are of major concerns because of safety considerations, particularly when such robots are used for handling hazardous materials (HAZMAT). Without developing the mathematical foundation of such technologically important systems, significant advances in the FSRS field will be very limited, slow, costly, and time consuming. The FSRS research, in particular, is still in its infancy and our knowledge of the FSRS mechanics is remarkably incomplete mainly because of the lack of a sound mathematical foundation required for developing efficient computational framework.

It is the objective of this study to address the FSRS fundamental scientific challenges by proposing an FSRS mathematical formulation and unified geometry/analysis computational framework. The new methodology proposed in this study, which is based on successful integration of CAD, FE, and robotics algorithms, employs ANCF finite elements that can be used for both the solid modeling and analysis. By using the position vector gradients as nodal coordinates, complex geometries and large deformations can be captured. The approach proposed in this investigation alleviates the problems associated with the use of the B-splines and NURBs as analysis tools.

**Conflict of Interest**
No competing financial interests exist

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