Symmetry properties of spheroidal functions with respect to their parameter

Danilo Erricolo, Tadahiro Negishi

***Abstract*—Spheroidal wave functions depend on a parameter c. Their behavior with respect to changes of sign of c is investigated and explicit formulas are provided. Sample applications of the resulting symmetry rules are provided for some electromagnetic scattering problems.**

***Index Terms*—Artificial materials, Electromagnetic Radiation, Electromagnetic Scattering, Electromagnetic Analysis, Electromagnetic Fields, Electromagnetic Theory, Spheroids, Isorefractive Material**

I. Introduction

Spheroidal functions are used in problems that involve the prolate or the oblate spheroidal coordinate system. Sample application problems include, for example, the scattering of a charged particle [1] and light [2], semiconductor nanodevices [3], Schrödinger's equation [4], and various acoustics [5] and electromagnetic problems whose boundaries correspond to coordinate surfaces in the prolate or oblate spheroidal coordinate system [6], [7]. In the the solution of Helmholtz equation in electromagnetics applications, spheroidal functions depend upon a parameter $c = \beta d/2$, where β is the wavenumber and d is the focal distance. In turn, the wavenumber $\beta = \omega\sqrt{\varepsilon\mu}$ depends on the angular frequency ω, the dielectric permittivity ε, and on the magnetic permeability μ. For most materials, $\varepsilon > 0$ and $\mu > 0$ and these are called double positive or DPS. Artificial materials or metamaterials with $\varepsilon < 0$ and $\mu < 0$ are referred to as double negative or DNG and have been theoretically proposed by Veselago [8]. More recently, properties of DNG metamaterials have been investigated by many researchers, e.g. in [9], [10] because they allow applications such as perfect lenses, super-resolution and invisibility. For a DPS material, $\beta > 0$ and $c > 0$, while for a DNG material $\beta < 0$ and $c < 0$ to satisfy causality [11]. New exact solutions of electromagnetic scattering problems were obtained when the materials involved are isorefractive [12] to each other, *i.e.* in the case of two materials when

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2. \tag{1}$$

A special case occurs when the two materials are anti-isorefractive to each other; assuming medium 1 is DPS and medium 2 is DNG one obtains

$$\beta_1 = \omega\sqrt{\varepsilon_1\mu_1} > 0 \Rightarrow c = c_1 = \frac{d}{2}\beta_1 > 0 \tag{2}$$

$$\beta_2 = \omega\sqrt{\varepsilon_2\mu_2} < 0 \Rightarrow -c = c_2 = \frac{d}{2}\beta_2 < 0 \tag{3}$$

As a result, when anti-isorefractive metamaterials are involved [13]–[18] it is necessary to know the behavior of spheroidal functions for $c < 0$.

The symmetry properties of the spheroidal functions with respect to the parameter c are not provided anywhere to the best of these authors knowledge, including classical reference such as [19]–[24] or recent articles such as [25] or the NIST Digital Library [26].Spheroidal functions may be computed, for example, using Fortran [27] or Mathematica [28], which provide values that are in agreement with these new symmetry properties.

Manuscript received date1; revised date2; accepted date3. Date of publication date4; date of current version date5

D. Erricolo and T. Negishi are with the Department of Electrical and Computer Engineering, University of Illinois at Chicago, Chicago, IL, 60607, USA. (email:derric1@uic.edu; t.negishi@gmail.com). This research was partially supported by the US Air Force Research Laboratory with grant FA9550-12-1-0174.

II. Spheroidal functions

According to Flammer [22], the prolate spheroidal coordinates are related to the cartesian coordinates by

$$x = \frac{d}{2}\sqrt{\xi^2-1}\sqrt{1-\eta^2}\cos\varphi \tag{4}$$

$$y = \frac{d}{2}\sqrt{\xi^2-1}\sqrt{1-\eta^2}\sin\varphi \tag{5}$$

$$z = \frac{d}{2}\xi\eta \tag{6}$$

where $\xi \geq 1$, $-1 \leq \eta \leq 1$, and $0 \leq \varphi \leq 2\pi$. Surfaces with $\xi =$ constant are confocal prolate spheroids, surfaces with $\eta = \mathrm{constant}$ are confocal hyperboloids with two sheets, and surfaces with $\varphi = \mathrm{constant}$ are planes originating in the z axis. On the other hand, the oblate spheroidal coordinate system is related to the cartesian coordinate system by

$$x = \frac{d}{2}\sqrt{\xi^2+1}\sqrt{1-\eta^2}\cos\varphi \tag{7}$$

$$y = \frac{d}{2}\sqrt{\xi^2+1}\sqrt{1-\eta^2}\sin\varphi \tag{8}$$

$$z = \frac{d}{2}\xi\eta \tag{9}$$

where $\xi \geq 0$, $-1 \leq \eta \leq 1$. Surfaces with $\xi =$ constant are confocal oblate spheroids, surfaces with $\eta = \mathrm{constant}$ are confocal hyperboloids with one sheet, and surfaces with $\varphi =$ constant are planes originating in the z axis. When the scalar wave equation

$$\nabla^2\psi + \beta^2\psi = 0, \tag{10}$$

is solved in the prolate spheroidal coordinate system with the method of separation of variables, the solution is written in the form

$$\psi_{mn} = S_{mn}(c,\eta)R_{mn}(c,\xi)\begin{matrix}\cos\\ \sin\end{matrix} m\varphi \tag{11}$$

where $S_{mn}(c,\eta)$ are prolate spheroidal angular functions and $R_{mn}(c,\xi)$ are prolate spheroidal radial functions. These functions satisfy

$$\begin{aligned}&\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]\\ &+\left[\lambda_{mn} - c^2\eta^2 - \frac{m^2}{1-\eta^2}\right]S_{mn}(c,\eta) = 0\end{aligned} \tag{12}$$

$$\begin{aligned}&\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]\\ &-\left[\lambda_{mn} - c^2\xi^2 + \frac{m^2}{\xi^2-1}\right]R_{mn}(c,\xi) = 0\end{aligned} \tag{13}$$

where $m = 0, 1, 2, ...,$ $n \geq m$ and λ_{mn} are separation constants of the original scalar Helmholtz equation (10). In addition, λ_{mn} is the eigenvalue of the differential equations (12) and (13) . One should notice that both angular and radial functions satisfy the same type of differential equation.

The solution of the differential equation (10) in the oblate coordinate system may be obtained by the following transformation

$$c \to \mp jc, \qquad \xi \to \pm j\xi, \tag{14}$$

so that it is sufficient to discuss in detail the prolate case and apply the previous transformation to address the oblate case.

Most physical problems require angular spheroidal functions that are finite for all possible values $-1 \le \eta \le 1$, thus limiting solutions of the differential equation (12) to the angular functions of the first kind $S^{(1)}_{mn}(c,\eta)$ or simply $S_{mn}(c,\eta)$, which may be written as a series expansion

$$S_{mn}(c,\eta) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) P^m_{m+r}(\eta), \tag{15}$$

where $P^m_{m+r}(\eta)$ are associated Legendre functions of the first kind and $d_r^{mn}(c)$ are expansion coefficients. In this expression and later on, the prime over the summation symbol means that r must take even values when $n-m$ is even and odd values when $n-m$ is odd, respectively. The normalization coefficient of the prolate angular function is

$$N_{mn}(c) = 2\sum_{r=0,1}^{\infty}{}' \frac{(r+2m)!(d_r^{mn}(c))^2}{(2r+2m+1)r!}. \tag{16}$$

Radial spheroidal functions are defined as

$$R^{(\ell)}_{mn}(c,\xi) = \left(\sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}\right)^{-1}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2} \times \sum_{r=0,1}^{\infty}{}' j^{r+m-n} d_r^{mn}(c)\frac{(2m+r)!}{r!} b^{(\ell)}_{m+r}(c\xi), \ell = 1,...,4. \tag{17}$$

where

$$b_n^{(\ell)}(z) = \sqrt{\frac{\pi}{2z}} B^{(\ell)}_{n+1/2}(z) \tag{18}$$

and $B_n^{(\ell)}(z)$ are Bessel functions of the first and second kind, respectively, for $\ell = 1, 2$, and they are Hankel functions of the first and second kind, respectively, for $\ell = 3, 4$.

The normalization coefficient for the radial functions is

$$\rho_{mn}(c) = \frac{j^{m-n}c^m}{\sum{}'^{\infty}_{r=0,1} d_r^{mn}(c)\frac{(2m+r)!}{r!}}. \tag{19}$$

When c is real, the substitutions $c \to -jc$ and $\xi \to j\xi$ cause changes of sign so that the differential equations for the prolate functions are transformed into the differential equations for the oblate functions. Hence, these substitutions do not cause the spheroidal functions to produce any complex number and the arguments $-jc$ and $j\xi$ should be understood as labels to identify the oblate functions in terms of the prolate functions. In the oblate case, the eigenvalues $\lambda_{mn}(-jc)$ are all real because they are the solution of an eigenvalue problem with all real values. Consequently, the expansion coefficients $d^r_{mn}(-jc)$ are all real, and the angular functions $S_{mn}(-jc,\eta)$ and the normalization coefficient $N_{mn}(-jc)$ are all real valued.

In the prolate case, the only complex quantity in addition to the radial functions $R^{(3)}_{mn}(c,\xi)$ and $R^{(4)}_{mn}(c,\xi)$ is the normalization coefficient for the radial functions, due to the term j^{m-n} appearing in (19); in particular,

$$\rho^*_{mn}(c) = (-1)^{m-n}\rho_{mn}(c) \tag{20}$$

In the oblate case, the radial functions $R^{(3)}_{mn}(-jc,j\xi)$ and $R^{(4)}_{mn}(-jc,j\xi)$ are complex valued, as well as the radial normalization coefficient $\rho_{mn}(-jc)$, for which

$$\rho^*_{mn}(-jc) = (-1)^n\rho_{mn}(-jc). \tag{21}$$

The symmetry properties of spheroidal functions with respect to the parameter c are given in Table I and derived next.

III. Derivations

1) Prolate functions: The symmetry properties of the prolate spheroidal functions require a careful review of the process that leads to their computation, since closed form expressions are not available. The starting point of this review are the differential equations (12), (13) that are of the same type and are satisfied by the angular and radial functions, respectively. The computation of the solution of the prolate spheroidal equation starts from the determination of its eigenvalue λ_{mn}. In particular, it must be noticed that the differential equation depends on the parameter c only through c^2 so that the eigenvalue λ_{mn} must satisfy

$$\lambda_{mn} = \lambda_{mn}(c^2) \Longleftrightarrow \lambda_{mn}(-c) = \lambda_{mn}(c). \tag{22}$$

Next the expansion coefficients $d_r^{mn}(c)$ of the series that define the prolate functions are evaluated. The angular functions are defined by the series expansion given in eq. (15) and when this series expansion is substituted into the differential equation (12), one obtains the recursion formula

$$\begin{aligned}&\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)}d^{mn}_{r+2}(c)+\\&\bigg[(m+r)(m+r+1)-\lambda_{mn}(c)+\\&\frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}c^2\bigg]d_r^{mn}(c)+\\&\frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}d^{mn}_{r-2}(c)=0\end{aligned} \tag{23}$$

that allows only to compute the ratios $d_r^{mn}(c)/d^{mn}_{r-2}(c)$. Unique values for the expansion coefficients are obtained by imposing a normalization condition. Using Flammer's convention, when $n-m$ is even,

$$S_{mn}(c,0) = P_n^m(0) = \frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!} \tag{24}$$

and

$$S'_{mn}(c,0) = P_n^{m\prime}(0) = \frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}, \tag{25}$$

when $n-m$ is odd. Since the previous normalization condition does not depend on c and the recurrence relation (23) depends on c directly through c^2 and indirectly through λ_{mn}, which has even symmetry, one concludes that

$$d_r^{mn}(-c) = d_r^{mn}(c). \tag{26}$$

As a result, the angular functions (15) have even symmetry

$$S_{mn}(-c,\eta) = S_{mn}(c,\eta) \tag{27}$$

and so do the corresponding normalization coefficients

$$N_{mn}(-c) = N_{mn}(c). \tag{28}$$

The radial prolate spheroidal functions are defined using the same expansion coefficients as the angular prolate spheroidal functions. The radial functions of the first kind are defined according to (17), with $\ell = 1$, and in order to investigate their behavior with respect

TABLE I: Symmetry properties for prolate spheroidal functions

	Prolate	Oblate
Eigenvalues	$\lambda_{mn}(-c) = \lambda_{mn}(c)$	$\lambda_{mn}(jc) = \lambda_{mn}(-jc)$
Expansion coefficients	$d_r^{mn}(-c) = d_r^{mn}(c)$	$d_r^{mn}(jc) = d_r^{mn}(-jc)$
Angular functions	$S_{mn}(-c,\eta) = S_{mn}(c,\eta)$	$S_{mn}(jc,\eta) = S_{mn}(-jc,\eta)$
normalization coefficient	$N_{mn}(-c) = N_{mn}(c)$	$N_{mn}(jc) = N_{mn}(-jc)$
Radial functions	$R_{mn}^{(1)}(-c,\xi) = (-1)^n R_{mn}^{(1)}(c,\xi)$	$R_{mn}^{(1)}(jc,j\xi) = (-1)^n R_{mn}^{(1)}(-jc,j\xi)$
	$R_{mn}^{(2)}(-c,\xi) = -(-1)^n R_{mn}^{(2)}(c,\xi)$	$R_{mn}^{(2)}(jc,j\xi) = -(-1)^n R_{mn}^{(2)}(-jc,j\xi)$
	$R_{mn}^{(3)}(-c,\xi) = (-1)^n R_{mn}^{(4)}(c,\xi)$	$R_{mn}^{(3)}(jc,j\xi) = (-1)^n R_{mn}^{(4)}(-jc,j\xi)$
	$R_{mn}^{(4)}(-c,\xi) = (-1)^n R_{mn}^{(3)}(c,\xi)$	$R_{mn}^{(4)}(jc,j\xi) = (-1)^n R_{mn}^{(3)}(-jc,j\xi)$
normalization coefficient	$\rho_{mn}(-c) = (-1)^m \rho_{mn}(c)$	$\rho_{mn}(jc) = (-1)^m \rho_{mn}(-jc)$

to changes of sign in c, we observe that the answer depends on the behavior of the spherical Bessel function $j_{m+r}(c\xi)$. Using the property

$$j_n(-z) = (-1)^n j_n(z) \tag{29}$$

we obtain

$$R_{mn}^{(1)}(-c,\xi) = \left(\sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!}\right)^{-1}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\times \sideset{}{'}\sum_{r=0,1}^{\infty} j^{r+m-n} d_r^{mn}(c)\frac{(2m+r)!}{r!} j_{m+r}(-c\xi) \tag{30}$$

resulting in the following cases. When $n-m$ is even,

$$\begin{aligned} R_{mn}^{(1)}(-c,\xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell}^{mn}(c)\frac{(2m+2\ell)!}{(2\ell)!}}\times \sum_{\ell=0}^{\infty} j^{2\ell+m-n} d_{2\ell}^{mn}(c)\frac{(2m+2\ell)!}{(2\ell)!}(-1)^{m+2\ell} j_{m+2\ell}(c\xi) \\ &= \begin{cases} R_{mn}^{(1)}(c,\xi) & m \text{ even} \Rightarrow n \text{ even} \\ -R_{mn}^{(1)}(c,\xi) & m \text{ odd} \Rightarrow n \text{ odd} \end{cases} \end{aligned} \tag{31}$$

and when $n-m$ is odd,

$$\begin{aligned} R_{mn}^{(1)}(-c,\xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell+1}^{mn}(c)\frac{(2m+2\ell+1)!}{(2\ell+1)!}} \sum_{\ell=0}^{\infty} j^{2\ell+m-n+1}\times d_{2\ell+1}^{mn}(c)\frac{(2m+2\ell+1)!}{(2\ell+1)!}(-1)^{m+2\ell+1} j_{m+2\ell+1}(c\xi) \\ &= \begin{cases} -R_{mn}^{(1)}(c,\xi) & m \text{ even} \Rightarrow n \text{ odd} \\ R_{mn}^{(1)}(c,\xi) & m \text{ odd} \Rightarrow n \text{ even} \end{cases} \end{aligned} \tag{32}$$

Hence, by combining the results of (31) and (32) we obtain

$$R_{mn}^{(1)}(-c,\xi) = (-1)^n R_m^{(1)}(c,\xi). \tag{33}$$

For the radial functions of the second kind, we replace the spherical Bessel functions of the first kind $j_{m+r}(c\xi)$ with the spherical Bessel functions of the second kind $y_{m+r}(c\xi)$ in (17). Then using the property

$$y_n(-z) = (-1)^{n+1} y_n(z) \tag{34}$$

we have

$$R_{mn}^{(2)}(-c,\xi) = \left(\sideset{}{'}\sum_{r=0,1}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!}\right)^{-1}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\times \sideset{}{'}\sum_{r=0,1}^{\infty} j^{r+m-n} d_r^{mn}(c)\frac{(2m+r)!}{r!} y_{m+r}(-c\xi) \tag{35}$$

resulting in the following cases. When $n-m$ is even,

$$\begin{aligned} R_{mn}^{(2)}(-c,\xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell}^{mn}(c)\frac{(2m+2\ell)!}{(2\ell)!}}\times \sum_{\ell=0}^{\infty} j^{2\ell+m-n} d_{2\ell}^{mn}(c)\frac{(2m+2\ell)!}{(2\ell)!}(-1)^{m+2\ell+1} y_{m+2\ell}(c\xi) \\ &= \begin{cases} -R_{mn}^{(2)}(c,\xi) & m \text{ even} \Rightarrow n \text{ even} \\ R_{mn}^{(2)}(c,\xi) & m \text{ odd} \Rightarrow n \text{ odd} \end{cases} \end{aligned} \tag{36}$$

and when $n-m$ is odd,

$$\begin{aligned} R_{mn}^{(2)}(-c,\xi) &= \frac{\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}}{\sum_{\ell=0}^{\infty} d_{2\ell+1}^{mn}(c)\frac{(2m+2\ell+1)!}{(2\ell+1)!}} \sum_{\ell=0}^{\infty} j^{2\ell+m-n+1}\times d_{2\ell+1}^{mn}(c)\frac{(2m+2\ell+1)!}{(2\ell+1)!}(-1)^{m+2\ell+2} y_{m+2\ell+1}(c\xi) \\ &= \begin{cases} R_{mn}^{(1)}(c,\xi) & m \text{ even} \Rightarrow n \text{ odd} \\ -R_{mn}^{(1)}(c,\xi) & m \text{ odd} \Rightarrow n \text{ even} \end{cases} \end{aligned} \tag{37}$$

Hence, by combining (36) and (37) we obtain

$$R_{mn}^{(2)}(-c,\xi) = -(-1)^n R_m^{(2)}(c,\xi) \tag{38}$$

For the radial functions of the third and fourth kind, we replace the spherical Bessel functions of the first kind $j_{m+r}(c\xi)$ with the spherical Hankel functions of the first and second kind $h_{m+r}^{(1)}(c\xi)$ and $h_{m+r}^{(2)}(c\xi)$, respectively, in (17). Then recalling that

$$R_{mn}^{(3),(4)}(c,\xi) = R_{mn}^{(1)}(c,\xi) \pm j R_{mn}^{(2)}(c,\xi) \tag{39}$$

and using the previous results (33), (38), we obtain

$$\begin{aligned} R_{mn}^{(3)}(-c,\xi) &= (-1)^n R_{mn}^{(1)}(c,\xi) - j(-1)^n R_{mn}^{(2)}(c,\xi) \\ &= (-1)^n R_{mn}^{(4)}(c,\xi) \end{aligned} \tag{40}$$

$$\begin{aligned} R_{mn}^{(4)}(-c,\xi) &= (-1)^n R_{mn}^{(1)}(c,\xi) + j(-1)^n R_{mn}^{(2)}(c,\xi) \\ &= (-1)^n R_{mn}^{(3)}(c,\xi) \end{aligned} \tag{41}$$

where it should be noticed that $R_{mn}^{(3)}$ is expressed in terms of $R_{mn}^{(4)}(c,\xi)$ in (40) and, similarly, $R_{mn}^{(4)}$ is expressed in terms of $R_{mn}^{(3)}$ in (41). Finally, the normalization coefficient of the radial functions (19) behaves according to

$$\rho_{mn}(-c) = (-1)^m \rho_{mn}(c) \tag{42}$$

2) Oblate functions : The derivations carried out for the prolate functions are valid for the oblate functions provided that the substitutions given in equation (14) are made.

IV. Sample application

The novelty of this article is to provide rules for the evaluation of spheroidal functions when $c < 0$, which occurs when metamaterials are involved and articles describing some applications have already been published, *e.g.* [13]–[18]. However, these articles do not explain how some of the analytical expressions are obtained when spheroidal functions are involved and $c < 0$. Hence, we provide as a sample application of the symmetry properties a justification for the analytical formulas for the field due to a dipole source used at the beginning of [13]–[18]. These articles provide exact analytical solutions, which we define as series expansions requiring the explicit closed form analytical (not numerical) determination of the modal expansion coefficients. In the case of spheroidal functions, exact solutions require axial symmetry of the geometry, materials, and primary sources involved in the boundary value problem. Hence, exact solutions exist only when the source is a dipole, provided that the dipole is located along the axis of symmetry and axially oriented, as shown in Fig. 1 [6].

In the following examples, the dipole is always located at $(\xi_0, \eta = 1)$ and the material is either DPS or DNG depending upon the case considered. We consider a time-harmonic electromagnetic analysis

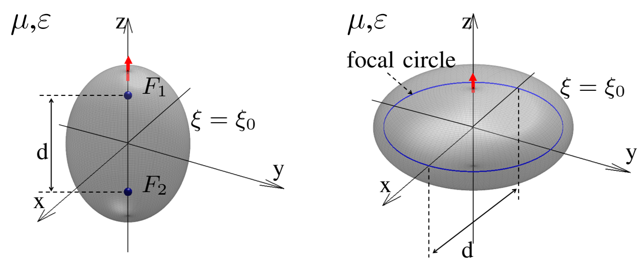

Fig. 1: A dipole source located in a prolate (left) and an oblate (right) coordinate system.

where all quantities vary in sinusoidal fashion and are expressed in terms of their phasors, using the IEEE time convention $\exp(j\omega t)$. Expressions written according to the time convention $\exp(-j\omega t)$ used in physics are obtained by taking the complex conjugate of the expressions provided in this article, which require the use of eqs. (20)-(21)

1) Prolate spheroidal coordinates:

a) DPS material: The material is characterized by the parameter c, the wavenumber $\beta = 2c/d$ and the electric dipole source has a Hertz vector potential $\mathbf{\Pi} = \hat{\mathbf{z}} \exp(-j\beta R)/(\beta R)$ resulting in the magnetic field

$$H_\varphi(\xi, \eta) = \frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{\rho_{1,n}(c) N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_<) R_{1,n}^{(4)}(c, \xi_>) S_{1,n}(c, \eta), \tag{43}$$

where $\xi_<$ and $\xi_>$ represent the smaller and the greater, respectively, between ξ_0 and ξ.

b) DNG material: The material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source has a Hertz vector potential $\mathbf{\Pi} = -\hat{\mathbf{z}} \exp(j\beta R)/(\beta R)$ and applying these substitutions in (43) yields

$$H_\varphi(\xi, \eta) = -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 - 1}} \times \sum_{n=1}^{\infty} \frac{(-j)^{n-1}}{\rho_{1,n}(c) N_{1,n}(c)} R_{1,n}^{(1)}(c, \xi_<) R_{1,n}^{(3)}(c, \xi_>) S_{1,n}(c, \eta) \tag{44}$$

2) Oblate spheroidal coordinates:

a) DPS material: The material is characterized by the parameter c, the wavenumber $\beta = 2c/d$ and the electric dipole source has a Hertz vector potential $\mathbf{\Pi} = \hat{\mathbf{z}} \exp(-j\beta R)/(\beta R)$ resulting in the magnetic field

$$H_\varphi(\xi, \eta) = \frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-j)^{n}}{\rho_{1,n}(-jc) N_{1,n}(-jc)} \times R_{1,n}^{(1)}(-jc, j\xi_<) R_{1,n}^{(4)}(-jc, j\xi_>) S_{1,n}(-jc, \eta). \tag{45}$$

b) DNG material: The material is characterized by the parameter $-c$, the wavenumber $\beta = -2c/d$ and the electric dipole source has a Hertz vector potential $\mathbf{\Pi} = -\hat{\mathbf{z}} \exp(j\beta R)/(\beta R)$ and applying these substitutions in (45) yields

$$H_\varphi(\xi, \eta) = -\frac{2\beta^2 Y}{\sqrt{\xi_0^2 + 1}} \sum_{n=1}^{\infty} \frac{(-j)^{n}}{\rho_{1,n}(-jc) N_{1,n}(-jc)} \times R_{1,n}^{(1)}(-jc, j\xi_<) R_{1,n}^{(3)}(-jc, j\xi_>) S_{1,n}(-jc, \eta). \tag{46}$$

V. Conclusions

New symmetry properties of the spheroidal functions that are important for applications involving metamaterials were derived and sample applications were provided.

Acknowledgments

The authors are thankful to the reviewers for their comments that helped improve the quality of the manuscript.

References

[1] J. W. Liu, "Analytical Solutions to the Generalized Spheroidal Wave-Equation and the Green Function of One-Electron Diatomic-Molecules," *J. Math. Phys.*, vol. 33, no. 12, pp. 4026–4036, Dec. 1992.

[2] V. G. Farafonov and N. V. Voshchinnikov, "Light scattering by a multilayered spheroidal particle," *Appl. Opt.*, vol. 51, no. 10, pp. 1586–1597, Apr. 2012.

[3] W. Lin, N. Kovvali, and L. Carin, "Pseudospectral method based on prolate spheroidal wave functions for semiconductor nanodevice simulation," *Comput. Phys. Commun.*, vol. 175, no. 2, pp. 78–85, Jul 15 2006.

[4] R. Boyack and J. Lekner, "Confluent Heun functions and separation of variables in spheroidal coordinates," *J. Math. Phys.*, vol. 52, no. 7, 2011.

[5] J. Lekner and R. Boyack, "Axisymmetric scattering of scalar waves by spheroids," *J. Acoust. Soc. Am.*, vol. 129, no. 6, pp. 3465–3469, 2011.

[6] J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*. New York: Hemisphere Publishing Corporation, 1987.

[7] L.-W. Li, X.-K. Kang, M.-S. Leong, P.-S. Kooi, and T.-S. Yeo, "Electromagnetic Dyadic Greens Functions for Multilayered Spheroidal Structures I:Formulation," *IEEE Trans. Microw. Theory Tech.*, 2001.

[8] V. G. Veselago, "The electrodynamics of substances with simultaneous negative values of ε and μ," *Sov. Phys.-Usp*, vol. 10, pp. 5090–514, 1968.

[9] J. B. Pendry, "Negative refraction makes a perfect lens," *Phys. Rev. Lett.*, vol. 85, no. 18, pp. 3966–3969, Oct. 30 2000.

[10] A. Alu and N. Engheta, "Achieving transparency with plasmonic and metamaterial coatings," *Phys. Rev. E*, vol. 72, no. 1, 2, Jul. 2005.

[11] R. W. Ziolkowski and E. Heyman, "Wave propagation in media having negative permittivity and permeability," *Phys. Rev. E*, vol. 64, no. 5, p. 056625, Oct 2001.

[12] P. Uslenghi, "Exact scattering by isorefractive bodies," *IEEE Trans. Antennas Propag.*, vol. 45, no. 9, pp. 1382–1385, Sept. 1997.

[13] A. N. Askarpour and P. L. E. Uslenghi, "Exact radiation from dipole antennas on oblate spheroids coated with isorefractive and anti-isorefractive layers," *IEEE Trans. Antennas Propag.*, vol. 60, no. 11, pp. 5476–5479, Nov. 2012.

[14] ——, "Exact radiation from dipole antennas on prolate spheroids coated with isorefractive and anti-isorefractive layers," *IEEE Trans. Antennas Propag.*, vol. 60, no. 4, pp. 2129–2133, Apr. 2012.

[15] P.L.E.Uslenghi, D. Erricolo, and T. Negishi, "Radiation by a dipole antenna on the axis of a Semi-Spheroidal cavity partially filled with DNG metamaterial," in *2014 International Workshop on Antenna Technology (iWAT) (iWAT 2014)*, Sydney, Australia, Mar 2014.

[16] T. Negishi, D. Erricolo, and P. L. E. Uslenghi, "Metamaterial spheroidal cavity to enhance dipole radiation," *IEEE Trans. Antennas Propag.*, vol. 63, no. 6, pp. 2802–2807, June 2015.

[17] ——, "Radiation from an Axial Electric Dipole with Prolate Spheroidal Metamaterial Cloak Cover," in *International Conference on Electromagnetics in Advanced Applications (ICEAA)*, Turin, Italy, Sept. 7-11 2015.

[18] P. U. D. Erricolo, T. Negishi, "Radiation from an Axial Electric Dipole with Oblate Spheroidal Metamaterial Cloak Cover," in *International Symposium on Antennas and Propagation (ISAP2015)*, Hobart, Tasmania, Australia, Nov. 9-12 2015.

[19] P. Morse and H. Feshbach, *Methods of Theoretical Physics*. McGraw-Hill Book Company, Inc., 1953.

[20] J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen*. Berlin: Springer, 1954.

[21] J. A. Stratton, P. M. Morse, L. J. C. nd J. D. C. Little, and F. J. Corbató, *Spheroidal Wave Functions including Tables of Separation Constants and Coefficients*. Cambridge, Massachusetts: The M. I. T. Press, 1956.

[22] C. Flammer, *Spheroidal wave functions*. Stanford University Press, 1957.

[23] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions*. New York: Dover Publications, Inc, 1970.

[24] L.-W. Li, X.-K. Kang, and M.-S. Leong, *Spheroidal Wave Functions in Electromagnetic Theory*. John Wiley & Sons, 2001.

[25] P. E. Falloon, P. C. Abbott, and J. B. Wang, "Theory and computation of spheroidal wavefunctions," *Journal of Physics A: Mathematical and General*, vol. 36, no. 20, p. 5477, 2003. [Online]. Available: http://stacks.iop.org/0305-4470/36/i=20/a=309

[26] "NIST Digital Library of Mathematical Functions," http://dlmf.nist.gov/.

[27] S. Zhang and J.-M. Jin, *Computation of Special Functions*. New York: John Wiley & Sons, 1996.

[28] Wolfram Research Inc., *Mathematica*. Champaign, IL USA: Wolfram Research, Inc., 2017.