Technical Brief

GEOMETRICALLY ACCURATE INFINITESIMAL-ROTATION SPATIAL FINITE ELEMENTS

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ABSTRACT

In this paper, a general procedure is used to develop new geometrically accurate infinitesimal-rotation finite elements (FE). New spatial beam, plate, and solid elements are developed in terms of constant geometric coefficients obtained using the matrix of position vector gradients. The spatial beam, plate, and solid element shape functions are developed using the displacement field of the absolute nodal coordinate formulation (ANCF). These elements are suited for developing reduced-order models for structural and multibody system (MBS) analyses, particularly when the floating frame of reference (FFR) formulation is used. The initial geometry of the new elements, referred to as the ANCF/FFR elements, is related to B-splines and NURBS (Non-uniform rational B-splines) by a linear mapping, and therefore, their use does not lead to geometry distortion when geometry models are converted to analysis meshes. The main contribution of this technical note is the proposed general procedure for the development of the geometrically accurate spatial ANCF/FFR elements and demonstrating the use of this procedure by developing the ANCF/FFR displacement field of three different spatial elements.

Keywords: Structural analysis; infinitesimal-rotation finite elements; absolute nodal coordinate formulation; floating frame of reference formulation; consistent rotation-based formulation.

1. INTRODUCTION

In the classical FE literature, different displacement fields are used for straight and curved beams and for plate and shells [1 – 3]. Furthermore, the development of cubic shape functions for the representation of bending deformation of some elements such as plates can be challenging. More importantly, the initial geometry of existing infinitesimal-rotation-based elements is not related by a linear mapping to B-splines and NURBS [4]. As a result, converting solid models to FE analysis meshes leads to geometry distortion, resulting in significant economic loss [5]. Having geometrically accurate rotation-based finite elements is necessary for developing reduced-order models for structural and MBS applications [6 – 9]. In this paper, a general procedure is proposed for developing new geometrically accurate spatial infinitesimal-rotation finite elements (FE). New spatial beam, plate, and solid (brick) elements are developed in terms of constant geometric coefficients obtained using the elements of the matrix of position gradients defined in the reference configuration [10]. The new spatial beam, plate, and solid element shape functions are obtained from the ANCF kinematic description. These elements can be used for developing reduced-order models for structural and MBS/FFR analysis. This particularly important when the FFR formulation is used in MBS analysis. The new ANCF/FFR initial geometry is identical to the ANCF geometry, which is related to B-splines and NURBS by a constant transformation, and therefore, the use of the new elements does not lead to geometry distortion when computer-aided design (CAD) models are converted to analysis meshes [11]. The velocity transformation of the consistent rotation-based formulation (CRBF) is used to develop a constant velocity transformation that allows writing systematically the ANCF position gradients in terms of the infinitesimal rotations while preserving the geometry in the reference configuration [12, 13].

2. GEOMETRY AND POSITION VECTOR GRADIENTS

Three configurations are often used to describe the kinematics of continuum with complex geometries. These are the *straight configuration*, the stress-free curved *reference configuration*, and the *current deformed configuration* described, respectively, by the parameters or coordinates $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$, $\mathbf{X} = [X_1 \quad X_2 \quad X_3]^T$, and $\mathbf{r} = [r_1 \quad r_2 \quad r_3]^T$ [10]. The position vector $\mathbf{r}$ of the material points can be written in the form $\mathbf{r} = \mathbf{X} + \mathbf{u}$, where the vector $\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$ describes the displacement. The position gradient matrix $\mathbf{J}$ is defined as $\mathbf{J} = \partial\mathbf{r}/\partial\mathbf{X} = (\partial\mathbf{r}/\partial\mathbf{x})(\partial\mathbf{x}/\partial\mathbf{X}) = \mathbf{J}_e\mathbf{J}_o^{-1}$, with $\mathbf{J}_e = \partial\mathbf{r}/\partial\mathbf{x} = [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad \mathbf{r}_{x_3}]$, and $\mathbf{J}_o = \partial\mathbf{X}/\partial\mathbf{x} = [\mathbf{J}_{o1} \quad \mathbf{J}_{o2} \quad \mathbf{J}_{o3}]$.

2.1 Position Vector Gradients and Finite Rotations (ANCF/CRBF Elements)

The recently proposed ANCF/CRBF elements can have a number of finite-rotation nodal coordinates equal to those of the conventional elements [12, 13]. For the ANCF/CRBF elements, the nodal position gradients can be defined as $\mathbf{J}_e = \partial\mathbf{r}/\partial\mathbf{x} = \mathbf{J}\mathbf{J}_o = \mathbf{A}(\theta_1, \theta_2, \theta_3)\mathbf{J}_o$. In this equation, $\mathbf{A}$ is an orthogonal matrix that depends on three finite-rotation parameters θ_1, θ_2, and θ_3. The matrix $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{J} = [\mathbf{a}_1(\theta_1, \theta_2, \theta_3) \quad \mathbf{a}_2(\theta_1, \theta_2, \theta_3) \quad \mathbf{a}_3(\theta_1, \theta_2, \theta_3)]$. The columns $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ of this matrix are orthogonal unit vectors.

2.2 Position Gradients and Infinitesimal Rotations (ANCF/FFR Elements)

If the assumption of infinitesimal nodal rotations is used, one obtains the ANCF/FFR element formulation [10]. Using this assumption, the matrix $\mathbf{A}$ is approximated as $\mathbf{A} = \mathbf{I} + \tilde{\boldsymbol{\theta}}$, and $\tilde{\boldsymbol{\theta}}$ is the skew-symmetric matrix associated with the vector $\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$ when the Euler angle sequence $X_1 - X_2 - X_3$ is used. One, therefore, has

$$\mathbf{J}_e = \mathbf{A}\mathbf{J}_o = \mathbf{J}_o + \tilde{\boldsymbol{\theta}}\mathbf{J}_o \tag{1}$$

One can write this equation in the alternate form

$$\mathbf{J}_e = \begin{bmatrix} \mathbf{r}_{x_1} & \mathbf{r}_{x_2} & \mathbf{r}_{x_3} \end{bmatrix} = \mathbf{J}_o + \begin{bmatrix} -\tilde{\mathbf{J}}_{o1} & -\tilde{\mathbf{J}}_{o2} & -\tilde{\mathbf{J}}_{o3} \end{bmatrix}\boldsymbol{\theta} \tag{2}$$

where $\tilde{\mathbf{J}}_{o1}, \tilde{\mathbf{J}}_{o2}$, and $\tilde{\mathbf{J}}_{o3}$ are the skew symmetric matrices associated with the columns $\mathbf{J}_{o1}, \mathbf{J}_{o2}$, and $\mathbf{J}_{o3}$ of $\mathbf{J}_o$. Therefore, the vector $\mathbf{r}$ and the position gradient vectors $\mathbf{r}_{x_1}, \mathbf{r}_{x_2}$, and $\mathbf{r}_{x_3}$ at an arbitrary point, including an FE nodal point, on a continuum can be written as

$$\begin{bmatrix} \mathbf{r} \\ \mathbf{r}_{x_1} \\ \mathbf{r}_{x_2} \\ \mathbf{r}_{x_3} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_o \\ \mathbf{J}_{o1} \\ \mathbf{J}_{o2} \\ \mathbf{J}_{o3} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o1} \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o2} \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o3} \end{bmatrix} \begin{bmatrix} \mathbf{r}_d \\ \boldsymbol{\theta} \end{bmatrix} \tag{3}$$

where $\mathbf{r}_o$ and $\mathbf{r}_d$ are, respectively, the vector $\mathbf{r}$ in the reference configuration and the displacement vector. The transformation given by Eq. 3 is used in this paper to develop the new spatial ANCF/FFR elements.

3. SPATIAL ANCF/FFR ELEMENTS

The ANCF kinematics and geometry is defined using the equation $\bar{\mathbf{u}}(\mathbf{x},t) = \mathbf{S}(\mathbf{x})\mathbf{e}(t)$. In this kinematic description, $\mathbf{S}$ is the element shape-function matrix, $\mathbf{e}$ is the vector of element nodal coordinates, and t is time. Figure 1 shows the ANCF nodal coordinates in the case of a fully parameterized spatial beam element. The same coordinates for each node are used for the plate and solid elements used in this paper. Because of the use of the position vector gradients, the ANCF kinematic description is related to the CAD B-splines and NURBS by a linear mapping [11]. At a given node k, the nodal coordinates are defined by the vector

$\mathbf{e}^k = \begin{bmatrix} \mathbf{r}^{k^T} & \mathbf{r}_{x_1}^{k^T} & \mathbf{r}_{x_2}^{k^T} & \mathbf{r}_{x_3}^{k^T} \end{bmatrix}^T$, which can be written as $\mathbf{e}^k = \mathbf{e}_o^k + \mathbf{B}^k \mathbf{e}_d^k$, where $\mathbf{e}_o^k$, $\mathbf{e}_d^k$, and $\mathbf{B}^k$ are defined, respectively, using Eq. 3 as

$$\mathbf{e}_o^k = \begin{bmatrix} \mathbf{r}_o^k \\ \mathbf{J}_{o1}^k \\ \mathbf{J}_{o2}^k \\ \mathbf{J}_{o3}^k \end{bmatrix}, \quad \mathbf{e}_d^k = \begin{bmatrix} \mathbf{r}_d^k \\ \boldsymbol{\theta}^k \end{bmatrix}, \quad \mathbf{B}^k = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o1}^k \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o2}^k \\ \mathbf{0} & -\tilde{\mathbf{J}}_{o3}^k \end{bmatrix} \tag{4}$$

Using these definitions and the ANCF kinematic description, one has [10]

$$\bar{\mathbf{u}}(\mathbf{x},t) = \mathbf{S}(\mathbf{x})\mathbf{e}(t) = \mathbf{S}(\mathbf{x})\mathbf{e}_0(t) + \mathbf{S}_d(\mathbf{x})\mathbf{e}_d(t) \tag{5}$$

In this equation, $\mathbf{S}_d = \mathbf{SB}$ is a new element *displacement shape function* that will be defined in Section 4 for different element types, $\mathbf{e}(t) = \mathbf{e}_0(t) + \mathbf{B}\mathbf{e}_d(t)$, and $\mathbf{B}$ is a block diagonal matrix whose blocks $\mathbf{B}^k$ correspond to the element nodes. The nodal position vector $\mathbf{e}_o$ in the reference configuration defines the initial stress-free geometry, while the element nodal vector $\mathbf{e}_d$ describes the displacements. The two terms on the right-hand side of Eq. 5 are functions of $\mathbf{J}_o$, which defines the initial element geometry. Therefore, the elements of $\mathbf{J}_o$ can be used to define new geometric coefficients, allowing for accurately capturing the initial element shape [10]. Equation 5 differs significantly from the displacement field used for conventional infinitesimal-rotation finite elements because this equation can be used for both straight and curved elements to capture accurately initial curvatures using the new geometric coefficients. Using Eq. 5, there is no need to distinguish between plate and shell elements, which are treated differently in the FE literature. One can also show that the matrix $\mathbf{S}_d = \mathbf{SB}$ has a proper set of rigid body modes at the velocity level [10]. This property is important since the velocity equation $\dot{\bar{\mathbf{u}}}(\mathbf{x},t) = \mathbf{S}_d(\mathbf{x})\dot{\mathbf{e}}_d(t)$ is used to define the element inertia. One can use a procedure similar to the one used in [10] to verify that all the elements presented in this paper have six rigid body modes that describe arbitrarily large translations and infinitesimal rigid body rotations.

4. CHARACTERIZATION OF SPATIAL ANCF/FFR ELEMENTS

In this section, several new spatial ANCF/FFR elements are developed, including beam, plate, and solid (brick) elements. It is assumed that each node of these elements has three position and nine gradient coordinates. For a node k of the element, the coordinate vector is defined as $\mathbf{e}^k = \begin{bmatrix} \mathbf{r}^{k^T} & \mathbf{r}_{x_1}^{k^T} & \mathbf{r}_{x_2}^{k^T} & \mathbf{r}_{x_3}^{k^T} \end{bmatrix}^T$. The procedure presented in this section can also be used for higher-order ANCF elements since curvature coordinates can be replaced by position coordinates as discussed in the literature. In order to generalize the development presented in this section, we assume that the partition of the ANCF shape function matrix $\mathbf{S}$ associated with node k can be written as

$$\mathbf{S}^k = \begin{bmatrix} s_1^k \mathbf{I} & s_2^k \mathbf{I} & s_3^k \mathbf{I} & s_4^k \mathbf{I} \end{bmatrix} \tag{6}$$

In this equation, $s_j^k,\ j = 1,2,3,4$, are the shape functions associated with the coordinate vectors of node k. The corresponding partition of the ANCF/FFR shape function matrix $\mathbf{S}_d^k = \mathbf{S}^k \mathbf{B}^k$ associated with the reduced number of coordinates is defined as

$$\mathbf{S}_d^k = \mathbf{S}^k \mathbf{B}^k = \begin{bmatrix} s_1^k \mathbf{I} & -\left(s_2^k \tilde{\mathbf{J}}_{o1} + s_3^k \tilde{\mathbf{J}}_{o2} + s_4^k \tilde{\mathbf{J}}_{o3} \right) \end{bmatrix} \tag{7}$$

The shape function $\mathbf{S}_d$ associated with the element coordinates $\mathbf{e}_d = \begin{bmatrix} \mathbf{e}_d^{1^T} & \mathbf{e}_d^{2^T} & \dots & \mathbf{e}_d^{n\,T} \end{bmatrix}^T$, where n is the number of nodes and $\mathbf{e}_d^k = \begin{bmatrix} \mathbf{r}^{k^T} & \boldsymbol{\theta}^{k^T} \end{bmatrix}^T$, can then be written as

$$\mathbf{S}_d = \begin{bmatrix} \mathbf{S}_d^1 & \mathbf{S}_d^2 & \dots & \mathbf{S}_d^n \end{bmatrix} \tag{8}$$

This simple procedure will be used to define the shape function matrix $\mathbf{S}_d$ of different spatial ANCF/FFR beam, plate, and solid (brick) elements as shown in the remainder of this section. This shape function matrix as described in [10] is used to formulate a local linear problem that allows for systematically reducing the order of the model using component-mode synthesis methods.

4.1 Spatial ANCF/FFR Beam Element

The spatial ANCF fully parameterized beam element considered in this section has two nodes and twelve coordinates per node [11]. The shape functions of this spatial beam element are

$$\left.\begin{aligned} & s_1^1 = 1-3\xi^2+2\xi^3, \quad s_2^1 = l\left(\xi-2\xi^2+\xi^3\right), \quad s_3^1 = l\left(\eta-\xi\eta\right), \quad s_4^1 = l\left(\varsigma-\xi\varsigma\right) \\ & s_1^2 = 3\xi^2-2\xi^3, \quad s_2^2 = l\left(-\xi^2+\xi^3\right), \quad s_3^2 = l\xi\eta, \quad s_4^2 = l\xi\varsigma \end{aligned}\right\} \tag{9}$$

with $\xi = x_1/l$, $\eta = x_2/l$, and $\varsigma = x_3/l$. In this case, the displacement shape matrix $\mathbf{S}_d$ is defined as $\mathbf{S}_d = \begin{bmatrix} \mathbf{S}_d^1 & \mathbf{S}_d^2 \end{bmatrix}$. Using this shape function matrix, no distinction is made between the displacement fields of straight and curved beams. By properly selecting the geometric coefficients, which are the position gradients in the reference configuration, the same shape function matrix can be used for both straight and curved beam elements.

4.2 Spatial ANCF/FFR Plate Element

Using the conventional FE approach, it is challenging to develop a plate element using a cubic interpolation for the bending deformation. Furthermore, in the conventional FE approach, one always distinguishes between the kinematics and geometries of plates and shells. These problems can be addressed using the approach discussed in this paper. The ANCF fully parameterized plate/shell element considered in this section has four nodes, each of which has twelve nodal coordinates. The element shape functions are defined as [11]

$$
\left.\begin{array}{l}
s_1^1 = -(\xi-1)(\eta-1)\left(2\eta^2-\eta+2\xi^2-\xi-1\right), \quad s_2^1 = -a\xi(\xi-1)^2(\eta-1), \\
s_3^1 = -b\eta(\eta-1)^2(\xi-1), \quad s_4^1 = t\varsigma(\xi-1)(\eta-1), \\
s_1^2 = \xi\left(2\eta^2-\eta-3\xi+2\xi^2\right)(\eta-1), \quad s_2^2 = -a\xi^2(\xi-1)(\eta-1), \\
s_3^2 = b\xi\eta(\eta-1)^2, \quad s_4^2 = -t\xi\varsigma(\eta-1), \\
s_1^3 = -\xi\eta\left(1-3\xi-3\eta+2\eta^2+2\xi^2\right), \quad s_2^3 = a\xi^2\eta(\xi-1), \\
s_3^3 = b\xi\eta^2(\eta-1), \quad s_4^3 = t\xi\eta\varsigma, \\
s_1^4 = \eta(\xi-1)\left(2\xi^2-\xi-3\eta+2\eta^2\right), \quad s_2^4 = a\xi\eta(\xi-1)^2, \\
s_3^4 = -b\eta^2(\xi-1)(\eta-1), \quad s_4^4 = -t\eta\varsigma(\xi-1)
\end{array}\right\} \qquad (10)
$$

where $\xi = x_1/a, \eta = x_2/b,$ and $\varsigma = x_3/t$, and a,b, and t are, respectively, the element length, width, and thickness. In this case, the displacement shape function matrix $\mathbf{S}_d$ is defined as $\mathbf{S}_d = \begin{bmatrix} \mathbf{S}_d^1 & \mathbf{S}_d^2 & \mathbf{S}_d^3 & \mathbf{S}_d^4 \end{bmatrix}$. This element, which is based on a cubic interpolation for the mid-surface bending, includes the drilling degree of freedom. Furthermore, using this element, one does not need to distinguish between the displacement fields of plates and shells since the geometric coefficients obtained using the position gradient matrix $\mathbf{J}_o$ can be properly selected to define arbitrary geometry.

4.3 Solid (Brick) Element

The ANCF solid element considered in this paper has eight nodes. Each of the nodes has twelve nodal coordinates. The ANCF solid element shape functions are defined as [14, 11]

$$
\left.\begin{array}{l}
S_1^k = (-1)^{1+\xi_k+\eta_k+\zeta_k}(\xi+\xi_k-1)(\eta+\eta_k-1)(\zeta+\zeta_k-1)\cdot \\
\left(1+(\xi-\xi_k)(1-2\xi)+(\eta-\eta_k)(1-2\eta)+(\zeta-\zeta_k)(1-2\zeta)\right) \\
S_2^k = (-1)^{\eta_k+\zeta_k} a\xi^{\xi_k+1}(\xi-1)^{2-\xi_k}\eta^{\eta_k}(\eta-1)^{1-\eta_k}\zeta^{\zeta_k}(\zeta-1)^{1-\zeta_k} \\
S_3^k = (-1)^{\xi_k+\zeta_k} b\xi^{\xi_k}(\xi-1)^{1-\xi_k}\eta^{\eta_k+1}(\eta-1)^{2-\eta_k}\zeta^{\zeta_k}(\zeta-1)^{1-\zeta_k} \\
S_4^k = (-1)^{\xi_k+\eta_k} c\xi^{\xi_k}(\xi-1)^{1-\xi_k}\eta^{\eta_k}(\eta-1)^{1-\eta_k}\zeta^{\zeta_k+1}(\zeta-1)^{2-\zeta_k}
\end{array}\right\}, \quad k=1,\ldots,8 \qquad (11)
$$

where a, b, and c are, respectively, the element dimensions along the x_1, x_2 and x_3 directions, $\xi = x_1 / a,\ \eta = x_2 / b,\ \zeta = x_3 / c$, $\xi,\ \eta,\ \zeta \in [0,1]$, and ξ_k, η_k, ζ_k are the dimensionless nodal locations for node k. In this case, the displacement shape function matrix $\mathbf{S}_d$ is defined as $\mathbf{S}_d = \begin{bmatrix} \mathbf{S}_d^1 & \mathbf{S}_d^2 & \ldots & \mathbf{S}_d^8 \end{bmatrix}$.

4.4 Implementation Issues

Preliminary results have shown that the mass matrices obtained for the ANCF/FFR plate and solid elements can have very small determinants, making these mass matrices close to singular. This is despite the fact that the ANCF element mass matrix, associated with the ANCF position vector gradients, does not suffer from this problem for the ANCF plate and solid elements. Therefore, more research is needed in order to better understand the problems associated with transforming the gradients to infinitesimal rotations. Because this problem is not encountered with the beam elements, it is important to understand the issues involved when infinitesimal rotations are used at four nodal points to define the deformation of a surface, particularly when a drilling degree of freedom is used.

5. SUMMARY AND CONCLUSIONS

A general procedure is used in this investigation to develop new geometrically accurate spatial infinitesimal-rotation finite elements. Using this procedure, new spatial beam, plate, and solid elements are developed in terms of constant geometric coefficients obtained using the matrix of position gradients in the reference configuration. The geometrically accurate ANCF/FFR elements allow for developing reduced order models in structural and MBS applications. Because of their compatibility with B-splines and NURBS, their use does not lead to geometry distortion when CAD models are converted to analysis meshes. The feasibility of using the approach proposed in this paper has been examined using a planar ANCF/FFR beam element [15]. The approach presented in this paper will allow for preserving the geometry in the reference configuration, and therefore, the CAD geometry is not distorted when converting the solid models to an analysis meshes. This conversion, using existing approaches, is very costly and time consuming and costs the U.S. automotive industry alone more than $600/year as reported in the literature [5]. Therefore, the development of the new ANCF/FFR elements has important practical implications for the industry. As mentioned in this paper, more research is required in order to better understand the transformation between the gradients and the infinitesimal rotations. Preliminary results showed that such a transformation can lead to very small determinant for the ANCF/FFR plate and solid elements, making the ANCF/FFR inertia matrix near singular. This is despite the fact that the ANCF element mass matrix associated with the position vector gradients does not suffer from this problem which is not encountered with beam elements that employ two nodal points. The use of infinitesimal rotations at four nodal points to describe the deformation of a surface including the effect of the drilling degree of freedom needs to be further investigated in the case of plate and solid elements.

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Notation

$\mathbf{A}$	Transformation matrix
$\mathbf{B}$	Velocity transformation matrix for the element
$\mathbf{B}^k$	Velocity transformation matrix at node k
$\mathbf{e}$	Vector of element nodal coordinates
$\mathbf{e}_o$	Vector of element nodal coordinates in the reference configuration
$\mathbf{e}_d$	Vector of element nodal displacements
$\mathbf{J}$	Matrix of position vector gradients
$\mathbf{J}_e$	Matrix of position vector gradients with respect to the straight configuration coordinates
$\mathbf{J}_o$	Matrix of position vector gradients in the reference configuration
$\mathbf{J}_{o1}, \mathbf{J}_{o2}, \mathbf{J}_{o3}$	Columns of the matrix of position vector gradients $\mathbf{J}_o$
$\mathbf{r} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^T$	The global position vector of an arbitrary point on the element
$\mathbf{r}_d$	Displacement vector
$\mathbf{r}_o$	The nodal position vector in the reference configuration
$\mathbf{S}$	Shape function matrix
$\mathbf{S}_d$	Displacement shape function matrix

$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$	Displacement vector
$\bar{\mathbf{u}}(\mathbf{x}, t)$	Assumed local position field
$\mathbf{X} = [X_1 \quad X_2 \quad X_3]^T$	Element parameters in the reference configuration
$\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$	Element coordinates in the straight configuration
$\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$	Vector of orientation parameters
ξ, η, ζ	Dimensionless parameters

List of Abbreviations

ANCF	Absolute nodal coordinate formulation
CAD	Computer-aided design
CAE	Computer-aided engineering
CRBF	Consistent rotation-based formulation
FE	Finite element
FFR	Floating frame of reference
I-CAD-A	Integration of CAD and analysis
MBS	Multibody system
NURBS	Non-uniform rational B-splines

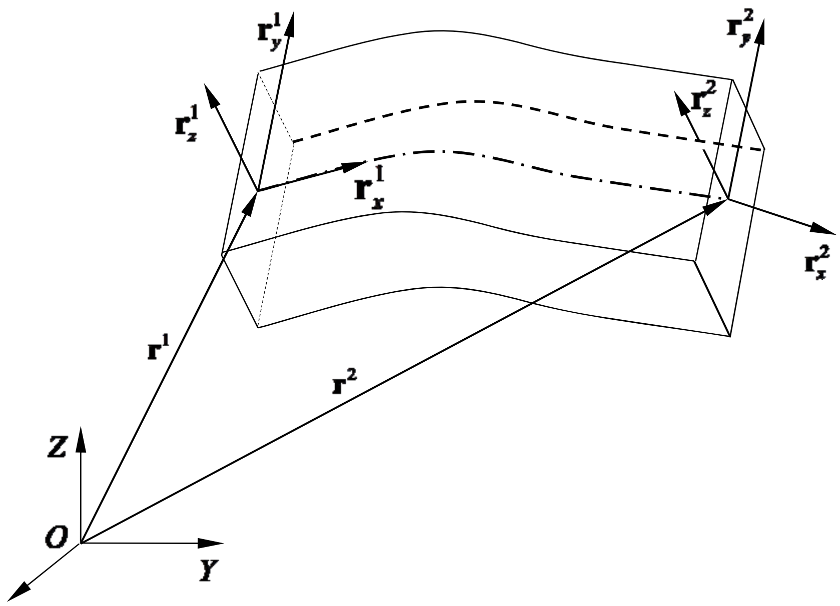

Figure 1. ANCF element nodal coordinates