A PATHOLOGY OF ASYMPTOTIC MULTIPLICITY IN THE RELATIVE SETTING

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Abstract. We point out an example of a projective family $\pi : X \to S$, a π-pseudoeffective divisor D on X, and a subvariety $V \subset X$ for which the asymptotic multiplicity $\sigma_V(D; X/S)$ is infinite. This shows that the divisorial Zariski decomposition is not always defined for pseudoeffective divisors in the relative setting.

1. Introduction

We work throughout over $\mathbb{C}$. Suppose that X is a smooth projective variety and D is a pseudoeffective $\mathbb{R}$-divisor on X. The asymptotic multiplicity of D along a subvariety $V \subset X$, studied by Nakayama [11] and Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa [5], has proved to be a fundamental tool in understanding the properties of the divisor D. For big divisors D, the definition of the asymptotic multiplicity is straightforward: roughly, one considers the linear series $|mD|$ for larger and larger values of m, and takes $\sigma_V(D) = \lim_{m\to\infty} \frac{1}{m} \operatorname{mult}_V |mD|$, where the multiplicity of a linear series along a subvariety is defined to be the multiplicity of a general member.

Complications arise, however, in carrying out this construction for divisors D which are pseudoeffective but not big, i.e. for divisors on the boundary of the pseudoeffective cone $\overline{\operatorname{Eff}}(X) \subset N^1(X)$. Nakayama realized that $\sigma_V(D)$ can be extended to a lower semicontinuous function on $\overline{\operatorname{Eff}}(X)$ by setting

$$\sigma_V(D) = \lim_{\epsilon\to 0} \sigma_V(D + \epsilon A),$$

where A is a fixed ample divisor. In some applications (e.g. in the construction of Zariski decompositions), it is important to know that the limit in question takes a finite value. While it is clear that the quantity on the right is nondecreasing as ϵ is made smaller, it might *a priori* be unbounded in the limit. That this does not happen in the non-relative setting was observed by Nakayama.

Our aim in this note is to demonstrate by example that when asymptotic multiplicity invariants are considered in the greater generality of divisors on a projective family $\pi : X \to S$, this finiteness need not hold: for a π-pseudoeffective divisor, the limit defining $\sigma_V(D; X/S)$ can indeed be infinite. This answers a question of Nakayama [11, pg. 33]. The example itself is familiar, a divisor on the versal deformation space of a fiber of Kodaira type I_2, which has been considered in related contexts by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)],[9, Example 9].

Theorem 1. *There exists a projective family $\pi : X \to S$, a π-pseudoeffective divisor D, and a subvariety $V \subset X$ for which $\sigma_V(D; X/S)$ is infinite.*

An important use of asymptotic multiplicity invariants is in the construction of the divisorial Zariski decomposition, a higher-dimensional analog of the usual Zariski decomposition on surfaces. The example here shows that trouble arises if one generalizes this construction to pseudoeffective classes in the relative setting: after passing to a blow-up on which the valuation corresponding to V is divisorial, we obtain an example in which the decomposition is not defined.

Corollary 2. *Let $\pi : X \to S$ be as in Theorem 1. If $f : W \to X$ is the blow-up along V with exceptional divisor E, then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_\sigma(\tilde{D}; W/S)$ is not defined.*

Moreover, the divisor $\tilde{D}$ does not admit any Zariski decomposition in a very strong sense:

Corollary 3. *There does not exist a birational model $g : Z \to W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$-movable divisor and N effective.*

In Section 2 we recall the basic definitions and properties of the invariants $\sigma_V(D; X/S)$ and $N_\sigma(D; X/S)$ appearing in Theorem 1 and Corollary 2, before establishing the claims in Section 3. In Section 4, we describe a more general setting for making computations in a similar spirit.

2. Preliminaries

Suppose that $\pi : X \to S$ is a projective, surjective morphism with connected fibers, with X and S smooth (hereafter, a *nice family*). We will find it convenient to allow the base S to be a surface germ, following [7]. The proofs of the results in this section hold either when S is a quasiprojective variety or a germ. In Section 3, it will be convenient to make computations with the base a germ. However, the germ we consider is algebraizable, and it follows that the same pathology occurs when the base is extended to be an affine scheme; this is discussed in Remark 1.

Two divisors D and D' on X are said to be numerically equivalent over S, or π-numerically equivalent, if $D \cdot C = D' \cdot C$ for any curve C that is contracted by π; write $D \equiv_\pi D'$ for the relation of numerical equivalence over S, and $N^1(X/S)$ for the vector space of $\mathbb{R}$-divisors on X, modulo this equivalence.

The familiar cones of positive divisors on a projective variety all have analogs in the relative setting: a divisor D on X is said to be

(1) π-ample if D_s is ample on every fiber $X_s = \pi^{-1}(s)$;
(2) π-strongly movable if the support of the cokernel of $f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ has codimension at least 2;
(3) π-big if the restriction of D to the generic fiber is big.

Corresponding to these classes of divisors are cones inside $N^1(X/S)$:

$$\mathrm{Amp}(X/S) \subseteq \mathrm{Mov}(X/S) \subseteq \mathrm{Big}(X/S).$$

The cones $\mathrm{Amp}(X/S)$ and $\mathrm{Big}(X/S)$ are both open. We will also consider the closures of these three cones inside $N^1(X/S)$, which are, respectively:

- $\mathrm{Nef}(X/S)$. A divisor is π-nef if D_s is nef on every fiber X_s (i.e. if $D \cdot C \geq 0$ for every curve C contracted by π);
- $\overline{\mathrm{Mov}}(X/S)$, the cone of π-movable divisors.
- $\overline{\mathrm{Eff}}(X/S)$, the cone of π-pseudoeffective divisors. A divisor D is π-pseudoeffective if the restriction of D to the generic fiber is pseudoeffective.

The meaning of "movable" is unfortunately not entirely uniform in the literature, and we stress that here an $\mathbb{R}$-divisor is called π-movable if it lies in the closed cone $\overline{\mathrm{Mov}}(X/S)$; this is sometimes called π-nef in codimension 1. We note too that the cone $\overline{\mathrm{Eff}}(X/S)$ is not necessarily a strictly convex cone, in that it might contain an entire line through the origin; this contrasts with the familiar case when S is a point. For example, if D restricts to 0 on a general fiber of π, then D and $-D$ are both π-pseudoeffective.

For simplicity, we will assume that the base space S is either affine or a germ. This is not really necessary, but the invariants under consideration can be computed when the base is projective simply by restricting to the preimage of a suitable affine open set; we refer to [11, §3.2] for details. The existence of a π-ample divisor A on X is automatic in this setting. If D is a π-big divisor, then $f_*\mathcal{O}_X(mD) \neq 0$ for sufficiently large and divisible m, and so $H^0(X, \mathcal{O}_X(mD)) = f_*(\mathcal{O}_X(mD))$ is nonzero as well. Hence in these settings, any π-big class has an effective representative.

Definition 1. Suppose that X is smooth. Given an irreducible subvariety $V \subset X$ and a π-big $\mathbb{R}$-divisor D, set

$$\sigma_V(D; X/S) = \inf_{\substack{D' \equiv_\pi D \\ D' \geq 0}} \mathrm{mult}_V(D').$$

Since D is π-big, there exists an effective $\mathbb{R}$-divisor D' that is π-numerically equivalent to D, and this infimum is taken over a nonempty set.

When D is a big integral divisor, a sequence D'_m of effective $\mathbb{R}$-divisors with multiplicities converging to the infimum can be found by taking $D'_m \in \frac{1}{m}|mD|$, where we choose a general element of the linear system $|mD|$.

We next extend the definition of the asymptotic multiplicity from π-big divisors to π-pseudoeffective divisors.

Definition 2. Given a π-pseudoeffective $\mathbb{R}$-divisor D, set

$$\sigma_V(D; X/S) = \lim_{\epsilon \to 0} \sigma_V(D + \epsilon A; X/S).$$

This is evidently a nondecreasing function as ϵ approaches 0, but it might have infinite limit. To show that it has a finite limit, it suffices to bound $\sigma_V(D + \epsilon A; X/S)$ above, independent of ϵ. Nakayama gives several conditions under which this can be achieved.

Theorem 4 ([11], Lemmas 2.1.2, 3.2.6)**.** *If any of the following holds, then $\sigma_V(D; X/S)$ is finite.*

(1) $S = \operatorname{Spec}\mathbb{C}$ *is a point;*
(2) D *is numerically equivalent over* S *to an effective* $\mathbb{R}$*-divisor* Δ*;*
(3) $\operatorname{codim}\pi(V) < 2$.

We recall the proof in case (1), perhaps the most important in practice. Case (2) is immediate from the definition, and we refer to [11] for (3). Assume for a moment that $V \subset X$ is an irreducible divisor; that this implies the general statement will follow from Theorem 5(2) below.

Proof of (1). For any ϵ, $(D + \epsilon A) - \sigma_V(D + \epsilon A)V$ is pseudoeffective, and so

$$((D + \epsilon A) - \sigma_V(D + \epsilon A)V) \cdot A^{n-1} \geq 0.$$

As long as $\epsilon < 1$ it follows that

$$\sigma_V(D+\epsilon A) \leq \frac{(D+\epsilon A)\cdot A^{n-1}}{V\cdot A^{n-1}} \leq \frac{(D+A)\cdot A^{n-1}}{V\cdot A^{n-1}}$$

is bounded above as ϵ decreases to 0. □

This argument relies in a crucial way on the properness of X to carry out intersection theory, and is not applicable in the relative setting in general.

Proposition 5 ([11], Lemmas 2.1.4, 2.2.2, 2.1.7)**.** *Suppose that $\pi : X \to S$ is a nice family and $V \subset X$ is an irreducible subvariety.*

(1) *If F is any π-pseudoeffective divisor on X, then*

$$\lim_{\epsilon\to 0} \sigma_V(D+\epsilon F; X/S) = \sigma_V(D; X/S).$$

(2) *Let $f : W \to X$ be the normalized blow-up of X along V, and let E be the exceptional divisor over V. Then $\sigma_E(f^*D; W/S) = \sigma_V(D; X/S)$.*

(3) *The number of prime divisors Γ for which $\sigma_\Gamma(D; X/S) > 0$ is finite.*

The first of these shows that Definition 1 is independent of the choice of π-ample divisor A, while the second completes the proof of Theorem 4(1) in the case that V has codimension greater than 1.

Definition 3 ([11], [4])**.** Suppose that $\pi : X \to S$ is a nice family and that D is a π-pseudoeffective divisor such that $\sigma_\Gamma(D; X/S)$ is finite for every prime divisor Γ. Then set

$$N_\sigma(D; X/S) = \sum_\Gamma \sigma_\Gamma(D; X/S)\,\Gamma,$$
$$P_\sigma(D; X/S) = D - N_\sigma(D; X/S).$$

It follows from Proposition 5(3) that there are only finitely many nonzero terms in the sum defining $N_\sigma(D; X/S)$.

We refer to $N_\sigma(D; X/S)$ as the negative part of the divisorial Zariski decomposition, and $P_\sigma(D; X/S)$ as the positive part. The negative part is a rigid, effective divisor. The positive part might not be nef, but it lies in the cone $\overline{\mathrm{Mov}}(X/S)$ of π-movable divisors. Corollary 2 shows that without the finiteness hypothesis on $\sigma_\Gamma(D; X/S)$, the definition is not always applicable in the relative setting.

In the non-relative setting, the divisorial Zariski decomposition is defined for any pseudo-effective class D, but it lacks certain useful properties of the classical Zariski decomposition in dimension 2: most importantly, the positive part $P_\sigma(D)$ is not nef in general. In many cases one may construct a birational model $f : W \to X$ on which $P_\sigma(f^*D)$ is actually nef, even if $P_\sigma(D)$ is not. However, a basic example of Nakayama shows that even this is not always possible [11, Theorem 5.2.6].

Another variant on Zariski decomposition in higher dimensions, the weak Zariski decomposition of Birkar, imposes fewer conditions and so exists for a larger class of divisors, including that of Nakayama's example. In this decomposition, the positive part P is allowed to be any relatively nef divisor, not necessarily the positive part $P_\sigma(f^*D)$ of the divisorial Zariski decomposition.

Definition 4 ([2])**.** Suppose that $\pi : X \to S$ is a nice family and that D is a pseudoeffective divisor on X. We say that D admits a *weak Zariski decomposition over* S if there exists a birational map $f : Y \to X$ and a decomposition $f^*D = P + N$, where P is $(f \circ \pi)$-nef and N is effective.

This condition is fairly unrestrictive, but there nevertheless exist pseudoeffective $\mathbb{R}$-divisors on smooth threefolds which do not admit a weak Zariski decomposition [10]. Corollary 3 asserts that the divisor $\tilde{D}$ provides another such example. Indeed, $\tilde{D}$ admits no Zariski decomposition in a still stronger sense: even after pulling back to a higher model, it cannot be decomposed as the sum of an effective divisor and a relatively movable divisor. The example is qualitatively rather different from that of [10]: there, a certain pseudoeffective divisor D_λ has negative intersection with infinitely many curves; here, there is a single curve on which D is negative, but the multiplicity of D along this curve is infinite.

3. Main example

The claimed pathology follows from a few calculations on an example that has been studied by Reid [12, 6.8] and Kawamata [8, Example 3.8(2)]. Let $\pi : X \to S$ be the versal deformation space of a fiber of Kodaira type I_2. The base S is smooth, 2-dimensional germ. The fiber over the central point $0 \in S$ is consists of two smooth rational curves C_1 and C_2, meeting transversely at two points p_1 and p_2. Let $C = \pi^{-1}(0)$ be the union of these two curves.

There are two divisors $\Gamma_1, \Gamma_2 \subset S$ corresponding to the smoothings of the two nodes of C. The fiber of π over a general point of Γ_i is a nodal rational curve, while the fiber over a general point of S is a smooth curve of genus 1.

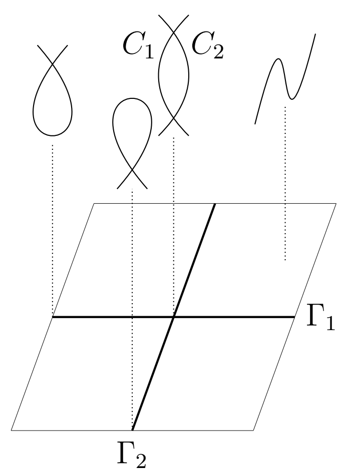

FIGURE 1. The family $\pi : X \to S$

Remark 1. The computations that follow will give an example in which some $\sigma_V(D; X/S)$ is infinite, in the case where S is a germ. The calculations rely on the fact that $\pi : X \to S$ is a versal deformation space. However, the local analytic results imply that the same pathological behavior occurs even when the base S is an affine surface. Indeed, we will see in

Lemma 7 below that there is a projective family $\bar{\pi} : \bar{X} \to \bar{S}$ where $\bar{S}$ is an affine surface, such that the restriction of $\bar{\pi}$ to the germ at a point $0 \in \bar{S}$ coincides with the map $\pi : X \to S$.

If $\bar{G}$ is a $\bar{\pi}$-big divisor, with restriction G to the germ, then $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(G; X/S)$: indeed, if $\bar{G}'$ is an effective divisor on $\bar{X}$ which is $\bar{\pi}$-numerically equivalent to $\bar{G}$, its restriction to the central germ is an effective divisor on X which is π-numerically equivalent to G. Thus the infimum defining $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S})$ in is taken over a subset of the infimum defining $\sigma_{C_1}(G; X/S)$ in Definition 1, giving the claimed inequality. It follows that in the limit at the pseudoeffective boundary, $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(D; X/S)$, and it must be that $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S})$ is infinite as well. The claims about Zariski decomposition follow in the same way.

Lemma 6. *The normal bundle $N_{C_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Suppose for instance that $i = 1$. There is an exact sequence

$$0 \longrightarrow N_{C_1/X} \longrightarrow (N_{C/X})|_{C_1} \longrightarrow T_{C_2,p_1} \oplus T_{C_2,p_2} \longrightarrow 0$$

with the property that a first-order deformation, determined by a section $s \in H^0(C, N_{C/X})$ smooths the node at p_i if and only if s has nonzero image in T_{C_2,p_i} [6, Lemma 2.6]. The sheaf in the middle is the trivial $\mathcal{O}_C \oplus \mathcal{O}_C$. In one direction p_1 is smoothed, and in another p_2 is, so the map sends $(1, 0)$ to $(1, 0)$ and $(0, 1)$ to $(0, 1)$ with respect to the direct sum decompositions. It follows that the kernel is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. □

Lemma 6 shows in particular that K_X has intersection 0 with both C_1 and C_2 and so is relatively numerically trivial.

Lemma 7. *There exists a flop $\tau : X \dashrightarrow X^+/S$ with flopping curve C_1. Let $C_1^+ \subset X^+$ be the flopped curve, and $C_2' \subset X^+$ be the strict transform of C_2. There exists an isomorphism $\sigma : X^+ \to X/S$ which sends C_1^+ to C_1 and C_2' to C_2. Furthermore, there exists an involutive automorphism $\imath : X \to X/S$ which exchanges the two curves C_1 and C_2.*

Proof. The arguments here are due to Kawamata [8, Example 3.8(2)]. We make some aspects of the proof explicit by working with local defining equations given by Reid [12]. In what follows, we use the notation $\bar{\cdot}$ to denote objects on a family $\bar{\pi} : \bar{X} \to \bar{S}$ over an affine base, while objects with no bar will be the restrictions to a certain germ.

Let $\bar{S} = \mathbb{A}^2$, with coordinates t_1 and t_2. Fix two distinct complex numbers a_1 and a_2 and define $\bar{X}_0 \subset (\mathbb{A}^1 \times \mathbb{A}^1) \times \bar{S}$ by the equation

$$x_1^2 = ((x_2 - a_1)^2 - t_1)((x_2 - a_2)^2 - t_2).$$

The closure $\bar{X} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \bar{S}$ is smooth, and the second projection $\bar{\pi} : \bar{X} \to \bar{S}$ is proper. The fiber of $\bar{\pi}$ over a general point (t_1, t_2) is a smooth curve of genus 1. If exactly one of t_1 and t_2 is zero, the fiber is nodal, while if $t_1 = t_2 = 0$, the fiber is given by $x_1^2 = (x_2 - a_1)^2 (x_2 - a_2)^2$. This central fiber has two components, the rational curves C_1 defined by $x_1 = -(x_2 - a_1)(x_2 - a_2)$ and C_2 defined by $x_1 = (x_2 - a_1)(x_2 - a_2)$. The restriction of $\bar{\pi} : \bar{X} \to \bar{S}$ to the germ at $(0, 0) \in \bar{S}$ is the versal deformation space $\pi : X \to S$ considered above. The involution $\imath : \bar{X} \to \bar{X}/\bar{S}$ defined by $\imath(x_1, x_2) = (-x_1, x_2)$ exchanges the two components of the central fiber.

There is a section $\bar{\sigma} : \bar{S} \to \bar{X}$ given by

$$x_2(t_1, t_2) = \frac{a_1 + a_2}{2} - \frac{t_1 - t_2}{2(a_1 - a_2)},$$
$$x_1(t_1, t_2) = (x_2(t_1, t_2) - a_1)^2 - t_1.$$

This has $\bar{\sigma}(0,0) = \left(\frac{(a_2-a_1)^2}{4}, \frac{1}{2}(a_1+a_2)\right)$, which lies on C_1 and is disjoint from C_2.

Let $\bar{\Sigma}_1$ be the divisor $\sigma(\bar{S})$. Since $\bar{\Sigma}_1 \cdot C_1 = 1$ and $\bar{\Sigma}_1 \cdot C_2 = 0$, the curves C_1 and C_2 have distinct classes in $N_1(\bar{X}/\bar{S})$. Since all other fibers of $\bar{\pi}$ are irreducible, it must be that $N^1(\bar{X}/\bar{S})$ is has dimension 2. The divisor $2\imath_*(\bar{\Sigma}_1) - \bar{\Sigma}_1$ has positive degree on general fibers, and so is $\bar{\pi}$-big. Since $\bar{S}$ is affine, there is an effective divisor $\bar{\Delta}$ representing this class. For sufficiently small ϵ, the pair $(\bar{X}, \epsilon\bar{\Delta})$ is klt. Since $\bar{\Delta} \cdot C_1 < 0$, there exists a $(K_{\bar{X}/\bar{S}} + \epsilon\bar{\Delta})$-flip $\tau : \bar{X} \dashrightarrow \bar{X}^+$, which is a $K_{\bar{X}/\bar{S}}$-flop. The map $\bar{\pi}^+ : \bar{X}^+ \to \bar{S}$ is a minimal model of $\bar{X}^+$.

By Lemma 6, the map τ is the flop of a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It follows by looking at a resolution of τ (discussed in the proof of Lemma 8 below) that the map $\tau|_{\bar{S}}$ simply blows up the point $\bar{\Sigma}_1 \cap C_1$ and that the strict transform of $\bar{\Sigma}_1$ on $\bar{X}^+$ is smooth. This strict transform contains the curve C_1^+, and satisfies $\tau_*\bar{\Sigma}_1 \cdot C_2' = 2$.

Now, the varieties $\bar{X}^+$ and $\bar{S}$ are smooth and $\bar{\pi}^+ : \bar{X}^+ \to \bar{S}$ has all fibers 1-dimensional, so $\bar{\pi}^+$ is flat. Since $\pi : X \to S$ is a versal deformation space, there exists an isomorphism $\beta : X^+ \to X$ over some automorphism of S. However, β might not be defined over the identity map on S. The divisor $\Sigma_2 = \beta_*(\tau_*(\Sigma_1))$ is a smooth divisor on X, containing C_1, and meeting C_2 at two points. There is a translation on the smooth fibers of π sending Σ_1 to Σ_2, which defines a birational self-map $\gamma : X \dashrightarrow X$ over the identity on S. The map $\pi \circ \gamma : X \to S$ must be isomorphic to some minimal model of X over S, and indeed must be to isomorphic to $\pi^+ : X^+ \to S$ since the strict transforms of Σ_1 under γ and τ have the same numerical classes. It follows that there exists an isomorphism $\sigma : X^+ \to X$ over the identity of S. Replacing σ with $\sigma \circ \imath$ if necessary, we may assume that $\sigma(C_1^+) = C_1$ and $\sigma(C_2') = C_2$, as required. $\square$

Each of the maps $\sigma \circ \tau$ and $\imath$ is a birational involution of X over S, but we will soon see that the composition $\phi = (\sigma \circ \tau) \circ \imath$ is of infinite order. Since $\imath(C_2) = C_1$, the effect of repeatedly applying ϕ is to flop C_1, then C_2, then C_1 again, and so on. We will denote by $\phi_* D$ the strict transform of a divisor D under a birational map ϕ, and use the same notation for the induced map on numerical groups when confusion seems unlikely.

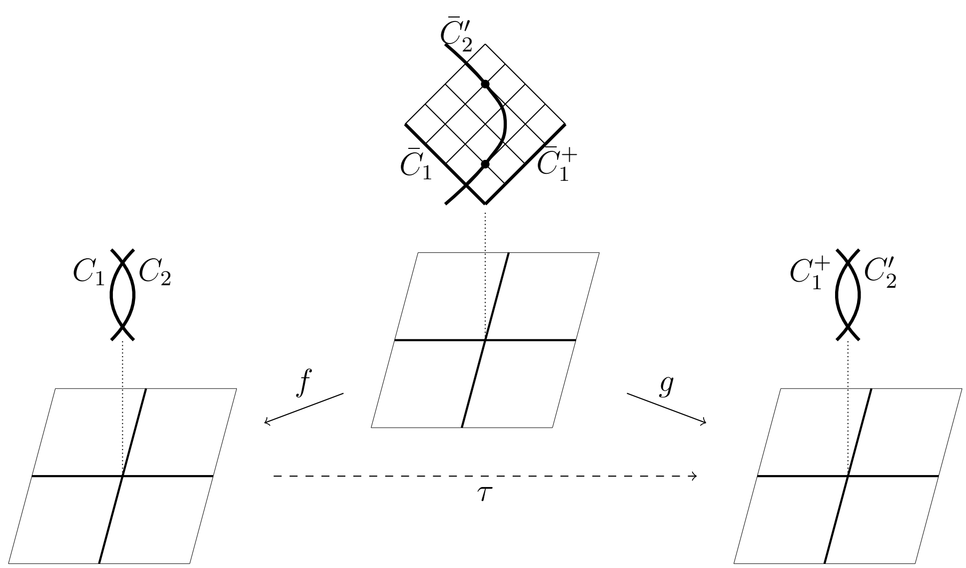

Figure 2. Resolution of the flop τ

To an effective divisor D on X, associate the 4-tuples

$$v_D = (D \cdot C_1, D \cdot C_2, \mathrm{mult}_{C_1}(D), \mathrm{mult}_{C_2}(D)),$$
$$\sigma_D = (D \cdot C_1, D \cdot C_2, \sigma_{C_1}(D; X/S), \sigma_{C_2}(D; X/S)).$$

Lemma 8. *Suppose that D is a divisor on X, and let $\tilde{D}$ denote the strict transform of D under the flop $\tau : X \dashrightarrow X^+$. Then*

(1) $\tilde{D} \cdot C_1^+ = -D \cdot C_1$,
(2) $\tilde{D} \cdot C_2' = D \cdot C_2 + 2(D \cdot C_1)$,
(3) $\mathrm{mult}_{C_1^+}(\tilde{D}) = \mathrm{mult}_{C_1}(D) + D \cdot C_1$,
(4) $\mathrm{mult}_{C_2'}(\tilde{D}) = \mathrm{mult}_{C_2}(D)$.

In matrix form, we have $v_{\phi_* D} = M v_D$ and $\sigma_{\phi_* D} = M \sigma_D$ where

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Let W be the graph of the flop τ:

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\tau"] & & X^+ \end{tikzcd}$$

Since τ is the flop of a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by Lemma 6, the map f is simply the blow-up of X along C_1, while g is the blow-up of X^+ along C_1^+. There is a single f-exceptional divisor E on W, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has normal bundle of bidegree $(-1, -1)$. Let $\bar{C}_1$ be a ruling of E contracted by g, so that f sends $\bar{C}_1$ isomorphically to C_1. Similarly, let $\bar{C}_1^+$ be a a ruling of E contracted by f, so that g maps $\bar{C}_1^+$ isomorphically onto C_1^+. Lastly, let $\bar{C}_2'$ be the strict transform of C_2 on W, a curve which meets E transversely at 2 points. This resolution is illustrated in Figure 2.

Then write

$$f^* D + aE = g^* \tilde{D},$$

for some constant a. Taking the intersection of both sides with $\bar{C}_1$ yields $D \cdot C_1 + a(E \cdot \bar{C}_1) = 0$. Since $E \cdot \bar{C}_1 = -1$, we obtain $a = D \cdot C_1$. Intersecting with $\bar{C}_1^+$, we have $-a = \tilde{D} \cdot C_1^+$. Similarly, intersecting with $\bar{C}_2'$, we have $D \cdot C_2 + a(E \cdot \bar{C}_2') = \tilde{D} \cdot C_2'$, and since $E \cdot \bar{C}_2' = 2$, we have (2). It is clear that $\mathrm{mult}_{C_2'}(\tilde{D}) = \mathrm{mult}_{C_2}(D)$, since τ is an isomorphism at the generic point of C_2. Finally,

$$\mathrm{mult}_{C_1^+}(\tilde{D}) = \mathrm{mult}_E(g^* \tilde{D}) = \mathrm{mult}_E(f^* D) + a = \mathrm{mult}_{C_1}(D) + a.$$

These calculations immediately yield $v_{\phi_* D} = M v_D$, since the second map $\imath$ exchanges the two curves C_1 and C_2. Write D_m for a general divisor linearly equivalent to mD, and then

$$\begin{aligned} \sigma_{C_1}(\phi_* D) &= \lim_{m \to \infty} \frac{1}{m} \mathrm{mult}_{C_1}(\phi_* D_m) = \lim_{m \to \infty} \frac{1}{m} \left(\mathrm{mult}_{C_1} D_m + D_m \cdot C_1 \right) \\ &= \left(\lim_{m \to \infty} \frac{1}{m} \mathrm{mult}_{C_1} D_m \right) + D \cdot C_1 = \sigma_{C_1}(D) + a. \end{aligned}$$

□

We are now in position to make the main computation.

Theorem 9. *Let $\pi : X \to S$ be the versal deformation space of a singular fiber of Kodaira type I_2, and let C_1 be a component of the central fiber. There exists a divisor D on the boundary of the cone $\overline{\mathrm{Eff}}(X/S)$ with $\sigma_{C_1}(D; X/S) = \infty$.*

Proof. Fix a π-ample effective $\mathbb{Q}$-divisor $H = H_0$ on X with $H \cdot C_1 = H \cdot C_2 = 1$. Then $\sigma_{C_i}(H) = 0$ for each i. For example, we might take $H = \bar{\Sigma}_1 + i_*(\bar{\Sigma}_1)$ in the notation of Lemma 7.

Let $H_n = \phi_*^n(H)$ be the strict transform of H on X under n applications of ϕ. Using the last part of Lemma 8, and the Jordan decomposition of M, which has a 3×3 block associated to the eigenvalue 1, we compute $\sigma_{H_n} = (2n+1, -2n+1, n(n-1)/2, n(n+1)/2)$:

n	$H_n \cdot C_1$	$H_n \cdot C_2$	$\sigma_{C_1}(H_n)$	$\sigma_{C_2}(H_n)$
0	1	1	0	0
1	3	-1	0	1
2	5	-3	1	3
3	7	-5	3	6
		$\cdots$		
n	$2n+1$	$-2n+1$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$

The key feature of the example is that while $H_n \cdot C_1$ grows linearly in n, the multiplicity $\sigma_{C_1}(H_n)$ grows quadratically. Let D be the divisor class on the boundary of $\overline{\mathrm{Eff}}(X/S)$ with $D \cdot C_1 = 1$ and $D \cdot C_2 = -1$. Since C_1 and C_2 span $N_1(X/S)$, we see that

$$H_n \equiv_\pi (2n)D + H_0.$$

It follows that $\frac{1}{2n}H_n \equiv_\pi D + \frac{1}{2n}H_0$ is a sequence of divisors converging to D, whose multiplicities along the curves is known. By Definition 1, we compute

$$\sigma_{C_1}(D; X/S) = \lim_{n\to\infty} \sigma_{C_1}(D + \frac{1}{2n}H_0) = \lim_{n\to\infty} \frac{1}{2n}\sigma_{C_1}H_n = \lim_{n\to\infty} \frac{n-1}{4} = \infty. \qquad \square$$

Note that $\operatorname{codim} \pi(C_1) = 2$, so there is no contradiction with Theorem 4(3).

Corollary 10. *If $f : W \to X$ is the blow-up along C_1 with exceptional divisor E, then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_\sigma(\tilde{D}; W/S)$ contains the divisor E with infinite coefficient. In particular, there does not exist a birational model $g : Z \to W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$-movable divisor and N effective.*

Proof. By Theorem 5(2), if $f : W \to X$ is the blow-up along C_1, with exceptional divisor E, we have $\sigma_E(f^*D; W/S) = \infty$. Now, suppose that $g : Y \to W$ is any birational map, and that $g^*f^*D = P + N$, where P is a $(g \circ f \circ \pi)$-movable divisor and N is effective. Let $\tilde{E}$ denote the strict transform of E on Y. Then

$$\sigma_{\tilde{E}}(g^*f^*D; Y/S) \le \sigma_{\tilde{E}}(P; Y/S) + \sigma_{\tilde{E}}(N; Y/S) = \sigma_{\tilde{E}}(N; Y/S).$$

The last of these is finite since N is effective, while the first is infinite, a contradiction. This completes the proof. $\square$

4. A general set-up

The key feature that made possible the computation of the preceding example is that if the four numbers $D \cdot C_i$ and $\sigma_{C_i}(D)$ are all known, then the same four invariants can be computed for the strict transform of D under ϕ using Lemma 8. In this section, we give an

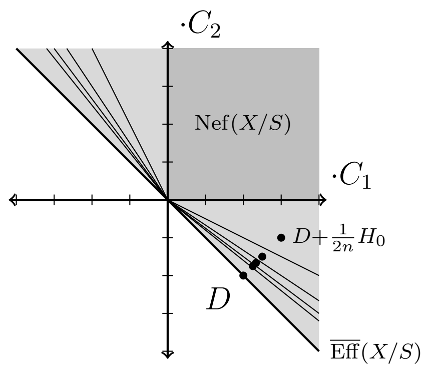

FIGURE 3. Chambers in $N^1(X/S)$

explanation for this, and describe how to make analogous computations in a more general setting.

Suppose that X is normal and $\mathbb{Q}$-factorial and that $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S (i.e. a birational map for which neither ϕ nor ϕ^{-1} contracts any divisors). We will say that a birational morphism $f : Y \to X$ from a normal $\mathbb{Q}$-factorial variety Y is a *small lift* of ϕ if the induced map $\psi : Y \dashrightarrow Y$ is also a pseudoautomorphism.

$$\begin{tikzcd} Y \arrow[r, dashed, "\psi"'] \arrow[d, "f"'] & Y \arrow[d, "f"'] \\ X \arrow[r, dashed, "\phi"'] & X \end{tikzcd}$$

Observe that if $f : Y \to X$ is a small lift, then the map ψ must permute the exceptional divisors of f.

Example 11. Suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism and x is a point not contained in $\operatorname{indet}\phi$. The blow-up $f : \mathrm{Bl}_x X \to X$ is a small lift of ϕ if and only if x is a fixed point of ϕ. If x is not a fixed point, then the induced map $\psi : Y \dashrightarrow Y$ contracts the exceptional divisor E, while if x is fixed, then $\psi|_E : E \to E$ is an automorphism.

The more interesting examples are those in which f contracts a divisor lying over $\operatorname{indet}\phi$.

Example 12. Next we construct a small lift of the map $\phi : X \dashrightarrow X/S$ from Section 3. Let $f : W \to X$ be the blow-up along C_1 as before, with exceptional divisor E_1, and let $h : Y \to W$ be the blow-up along $\bar{C}_2'$, with exceptional divisor E_2. The two exceptional divisors E_1 and E_2 are swapped by the induced map $\psi : Y \dashrightarrow Y$, and $h \circ f$ is a small lift.

$$\begin{tikzcd} Y \arrow[rr, dashed, "\psi"] \arrow[dr, "h"] \arrow[dd, "h\circ f"'] & & Y \arrow[dd, "h\circ f"] \\ & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi"] & & X \end{tikzcd}$$

The curves C_1 and C_2 could have been blown up in the opposite order, yielding a different small lift $f' : Y' \to X$. This makes no real difference: the threefolds Y and Y' differ only by flops, and strict transform induces an identification $N^1(Y) \xrightarrow{\sim} N^1(Y')$ with respect to which the maps ψ_* and ψ'_* coincide.

If $f : Y \to X$ is a small lift, it follows from the negativity lemma [1, Lemma 3.6.2] and the $\mathbb{Q}$-factoriality of X that there is a decomposition $N^1(Y) = f^*N^1(X) \oplus V_E$, where $V_E = \bigoplus_i \mathbb{R} \cdot [E_i]$. If D is a divisor class on X, it is not necessarily true that $f^*\phi_*D = \psi_*f^*D$. However, the difference $f^*\phi_*D - \psi_*f^*D$ is an f-exceptional divisor, since

$$f_*(f^*\phi_*D - \psi_*f^*D) = \phi_*D - f_*\psi_*f^*D = \phi_*D - \phi_*f_*f^*D = \phi_*D - \phi_*D = 0.$$

Define $K : N^1(X) \to V_E$ by $K = f^*\phi_* - \psi_*f^*$. The next lemma characterizes the action of the strict transform $\psi_* : N^1(Y) \to N^1(Y)$ with respect to this decomposition.

Lemma 13. *Suppose that X and Y are $\mathbb{Q}$-factorial and $f : Y \to X$ is a small lift of a pseudoautomorphism $\phi : X \dashrightarrow X$. With respect to the decomposition $N^1(Y) \cong f^*N^1(X) \oplus V_E$, ψ_* is given in block form as*

$$\psi_* = \left(\begin{array}{c|c} \phi_* & 0 \\ \hline -K & P \end{array} \right),$$

where P is the permutation matrix for the action of ψ_* on the E_i. The eigenvalues of ψ_* are the union of those of ϕ_* and those of P, which are roots of unity. Its eigenvectors are

1. *$f^*v_i - (\lambda_i I - P)^{-1}Kv_i$, where v_i are the eigenvectors of ϕ_*, with eigenvalues λ_i;*
2. *E_i, the exceptional divisors of f, with eigenvalues that are roots of unity.*

Proof. For a divisor D on X, $\psi_*f^*D = f^*\phi_*D - KD$, while the exceptional divisors E_i are simply permuted by ψ; this gives the block form of the map. The eigenvectors follow from elementary linear algebra. □

Note that if ψ_* fixes the exceptional divisors, which can always be arranged by replacing ϕ and ψ by suitable iterates, the permutation matrix P is the identity.

A rational map $\phi : X \dashrightarrow Y$ between $\mathbb{Q}$-factorial varieties is said to be D-non-negative for an $\mathbb{R}$-divisor D if on some common resolution $p : W \to X$, $q : W \to Y$, we have $p^*D + E = q^*(\phi_*D)$, where E is an effective q-exceptional divisor. If $\phi : X \dashrightarrow X$ is a pseudoautomorphism with a small lift $f : Y \to X$, then we may consider a resolution of the form

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y \arrow[rr, dashed, "\psi"'] \arrow[d, "f"'] & & Y \arrow[d, "f"] \\
X \arrow[rr, dashed, "\phi"'] & & X
\end{tikzcd}$$

If D is a divisor on X for which ϕ is D-non-negative, then we have $p^*f^*D + E = q^*f^*\phi_*D$ with $E \geq 0$. Pushing forward both sides by q, this gives

$$\begin{aligned} q_*p^*f^*D + q_*E &= f^*\phi_*D \\ \psi_*f^*D + E' &= f^*\phi_*D, \end{aligned}$$

where E' is an effective f-exceptional divisor. In particular, $KD = f^*\phi_*D - \psi_*f^*D = E'$ is effective.

Next we observe that if the divisorial Zariski decomposition of f^*D is known for some divisor D, the decomposition of $f^*\phi_*D$ can often be computed. Although in earlier sections we assumed that X was smooth, in what follows X need only be normal and $\mathbb{Q}$-factorial, as we will only consider asymptotic multiplicities along divisors, rather than higher-dimensional subvarieties (which might lie entirely in the singular locus of X). We assume for simplicity that ψ fixes each of the f-exceptional divisors E_i; this can always be arranged by replacing ϕ by a suitable iterate. This assumption implies that the permutation matrix P is the identity, and that $\psi_*(KD) = KD$ since KD is f-exceptional.

Lemma 14. *Suppose $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S, and that D is a class in $N^1(X/S)$ with $\sigma_\Gamma(D; X/S)$ finite for all divisors Γ. Then $N_\sigma(\phi_*D; X/S) = \phi_*N_\sigma(D; X/S)$ and $P_\sigma(\phi_*D; X/S) = \phi_*P_\sigma(D; X/S)$. If ϕ is D-non-negative and $N_\sigma(f^*D; Y/S)$, then $P_\sigma(f^*\phi_*D; Y/S) = \psi_*P_\sigma(f^*D; Y/S)$.*

Proof. Since ϕ neither contracts nor extracts any divisors, for any prime divisor E we have $\sigma_E(D; X/S) = \sigma_{\phi_*E}(\phi_*D; X/S)$. The claim for $N_\sigma(\phi_*D; X/S)$ follows, and that for $P_\sigma(\phi_*D; X/S)$ is immediate.

Now, by the D-non-negativity hypothesis on ϕ, KD is an effective f-exceptional divisor. By [11, Lemma 3.5.1], if E is an effective f-exceptional divisor, we have $N_\sigma(f^*D + E) = N_\sigma(f^*D) + E$. This means that

$$\begin{aligned} N_\sigma(f^*\phi_*D) &= N_\sigma(\psi_*f^*D + KD) = N_\sigma(\psi_*(f^*D + KD)) = \psi_*N_\sigma(f^*D + KD) \\ &= \psi_*N_\sigma(f^*D) + \psi_*KD = \psi_*N_\sigma(f^*D) + KD. \end{aligned}$$

We have made use of the fact that E is effective by the non-negativity hypothesis on D. It is now simple to compute the positive part of the decomposition:

$$\begin{aligned} P_\sigma(f^*\phi_*D) &= f^*\phi_*D - N_\sigma(f^*\phi_*D) = f^*\phi_*D - \psi_*N_\sigma(f^*D) - KD \\ &= \psi_*f^*D - N_\sigma(\psi_*f^*D) = P_\sigma(\psi_*f^*D) = \psi_*P_\sigma(f^*D). \end{aligned}$$ □

Remark 2. The example of Section 3 can be interpreted as an instance of the calculations in this section. A small lift of the map ϕ is constructed in Example 12. Let F_1, F_2 be a basis for $N^1(X/S)$ dual to C_1 and C_2. A basis for $N^1(Y/S)$ is given by the four classes $(h \circ f)^*F_1$, $(h \circ f)^*F_2$, E_1, and E_2. The vector v_D gives the coefficients for the class of the strict transform of D on Y with respect to this above basis. Lemma 8 is nothing more than the calculation of the induced map ψ_* of Lemma 13. The final calculation in Theorem 9 can then be carried out as a repeated application of Lemma 14.

Suppose now that $S = \operatorname{Spec}\mathbb{C}$ and $\phi : X \dashrightarrow X$ is a pseudoautomorphism whose action $\phi_* : N^1(X) \to N^1(X)$ has a unique largest eigenvalue $\lambda > 1$, and that $f : Y \to X$ is a small lift of ϕ. Since $S = \operatorname{Spec}\mathbb{C}$, the pseudoeffective cone is strictly convex, and by a version of the Perron-Frobenius theorem [3] there exists a λ-eigenvector D_ϕ which is contained in the pseudoeffective cone. Lemma 13 then provides the existence of a λ-eigenvector D_ψ for ψ_*.

We are then able to compute the Zariski decomposition of the divisor f^*D_ϕ using Lemma 14.

Corollary 15. *Let D_ϕ be the dominant eigenvector of $\phi_* : N^1(X) \to N^1(X)$, and D_ψ be the dominant eigenvector of $\psi_* : N^1(Y) \to N^1(Y)$. If D is ϕ-non-negative, then $P_\sigma(f^*D_\phi) = D_\psi$.*

Proof. If D is any pseudoeffective divisor on X, then for every n we have

$$P_\sigma(f^*(\lambda^{-n}\phi_*^n D)) = \lambda^{-n}\psi_*^n P_\sigma(f^*D).$$

Take $D = D_\phi + D_{\phi^{-1}}$, so that the above reduces to

$$P_\sigma(f^*(D_\phi + \lambda^{-2n} D_{\phi^{-1}})) = \lambda^{-n} \psi_*^n P_\sigma(f^* D).$$

The left hand side converges to $P_\sigma(f^* D_\phi)$ by Proposition 5(1). With a suitable choice of scaling, the right hand side converges to D_ψ. ☐

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