Mathematical Results for Nonlinear Equations of Schrödinger Type

by

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Contribution of Authors

The content of this thesis comprises joint work with my advisor Christof Sparber, and in parts together with Paolo Antonelli, Christian Klein, and Irina Nenciu:

[5] P. Antonelli, J. Arbunich and C. Sparber. *Regularizing nonlinear Schrödinger equations through partial off-axis variations.* SIAM J. Math. Anal. **51** (2019), no. 1, pp. 110–130.

[7] J. Arbunich, C. Klein and C. Sparber. *On a class of derivative nonlinear Schrödinger equations in two spatial dimensions.* To appear in ESAIM Math. Model. Numer. Anal.

[9] J. Arbunich and C. Sparber *Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures.* J. Math. Phys. **59** (2018), 011509.

[8] J. Arbunich, I. Nenciu and C. Sparber *Stability and instability properties of rotating Bose-Einstein condensates.* Lett. Math. Phys (2019) 109: 1415.

I have contributed in equal parts to all four works included in this thesis. Christof Sparber suggested the problems where amongst collaboration, converse and research, I would then produce initial iterations of our manuscripts. Here I would prove the relevant results and dialogue to develop a framework under which we could all begin to modify and finalize our results. Much of the thesis is excerpted directly from the published works above with modifications in certain instances to either physically motivate our mathematical results or to expand in directions of which the details were not necessary for the published manuscripts. In addition, certain commonalities were merged and redacted, however the overall organization of the thesis is ordered in chapters in accordance with how they appear listed above.

Summary

In this thesis, we shall consider a variety of dispersive semilinear partial differential equations of Schrödinger type that serve as models for different physical phenomena in laser optics and quantum physics. Our results highlight a number of mathematical techniques that are employed in order to understand emergent properties within these physical systems. The introduction is divided between models of nonlinear optics and quantum physics.

In the first two chapters, we study two models that have recently been derived by Dumas, Lannes, and Szeftel as an effective description for the propagation of laser beams at high intensities. This was part of an effort to remedy some shortcomings of the classical focusing *Nonlinear Schrödinger equation* (NLS) in regimes where physical observations indicate that the behavior of these beams depart from the predictions based on the NLS. Dumas et al. derive these new extended NLS type models from the underlying Maxwell equations by employing a *slowly varying envelope approximation.* In particular, when a laser pulse propagates in a nonlinear medium, the refractive index changes according to its intensity. This change in index induces a regularizing effect in the off-axis directions of the beam, which creates asymmetry whilst widening or smoothing the profile of the beam. In addition, depending on the medium, asymmetry can occur due to possible nonlinear effects of the profile associated to the off-axis directions, causing the peak and edges of the pulse envelope to travel with different velocities. It was posed as separate open problems by Dumas et al. whether these effects, separately or together, provide an adequate description of these nonlinear optical phenomenon.

To this end, we shall verify that these equations are mathematically *(globally) well-posed* and prove a result concerning the long time behavior of small solutions in a particular regime. In the two-dimensional setting we shall attempt to verify that the analysis we perform is in a sense sharp by providing several numerical simulations, even in the case when these effects above act in opposing directions and where there are no analytical results available at present. These simulations rely on numerically constructing nonlinear ground states as initial data, in order to determine whether their time evolution is stable under Gaussian perturbations for given nonlinear strengths. Moreover, we

shall compare our numerical simulations with those where these new effects are not present, i.e. the classical focusing NLS.

In the last two chapters we shall be concerned with two mathematical models of quantum mechanics. First, the focus is on a macroscopic model for weakly nonlinear wave packets in honeycomb structures. As a means of exploiting the interesting features that arise from the geometry of these structures, we shall spectrally localize around a conical singularity in the dispersion surface called a *Dirac point*. We shall be interested in high frequency wave propagation where the wavelength is comparable to the period of a honeycomb lattice, both of which are small relative to the length scale of the lattice. Thus, there are two natural scales we consider, the fast scale measuring the variations within a (microscopic) fundamental cell and the slow scale measuring variations within the macroscopic region of interest. Starting from a macroscopic description given by a semi-classical NLS, we derive an approximate macroscopic solution via a rigorous multi-scale expansion. We shall see that solutions to the semi-classical NLS can be approximated by slowly modulated plane waves whose leading order amplitudes satisfy a nonlinear Dirac system. We then prove a nonlinear stability result which verifies that our approximate solution is indeed accurate up to small errors.

The second work concerns a model for rotating bosonic particles at temperatures near absolute zero confined by a magnetic trap. In this setting these quantum gases are known to condense into a single macroscopic quantum state called a Bose-Einstein condensate. In this state the gas is superfluid and exhibits the nucleation of quantized vortices. We first show for rotational speeds that do not exceed the trapping frequencies that there exists time-periodic solutions called *ground states*. Formally, these solutions are minimizers of an associated energy functional and we prove *orbital stability* of the set of ground states. In short, this says that if we prepare initial data close to a ground state, then global solutions will remain close to some ground state solution for all time. The other aspect of this work is on the instability properties associated to rotational speeds that exceed the smallest trapping frequency. Here we show that the quantum mechanical mean position and momentum satisfy the Ehrenfest equations – a coupled system of ordinary differential equations. In the case of non-isotropic potentials, we show that solutions can develop frequencies that cause the solution to escape the confines of the trap, in the sense that asymptotically the mass gets transferred to infinity as time goes to infinity.

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CHAPTER 1

Introduction[1]

1. Models of nonlinear laser optics

1.1. Well-posedness for the NLS. Consider the initial value problem or *Cauchy problem* for a general (focusing) *nonlinear Schrödinger equation* (NLS) in $d \geqslant 1$ spatial dimensions, i.e.,

$$\begin{cases} i\partial_t u + \Delta u + |u|^{2\sigma} u = 0, \ t \in \mathbb{R}, \ \mathbf{x} \in \mathbb{R}^d, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), \end{cases} \tag{1}$$

where $\sigma > 0$ is a parameter describing nonlinear effects. The NLS is a canonical model for slowly modulated, self-focusing wave propagation in a weakly nonlinear dispersive medium. The choice of $\sigma = 1$ is well studied in the context of nonlinear laser optics, and thereby corresponds to the physically most relevant case of a *Kerr nonlinearity*, cf. [**48, 101**]. The NLS thereby describes diffractive effects which modify the propagation of slowly modulated light rays of geometrical optics over large times. In this context, the variable "t" should not be thought of as time, but rather as the main spatial direction of propagation of the ray.

It should be noted that solutions $u = u(t, \mathbf{x})$ to (1) admit several conservation laws, in particular one finds that

$$\|u(t, \cdot)\|^2_{L^2_{\mathbf{x}}} = \|u_0\|^2_{L^2_{\mathbf{x}}}, \tag{2}$$

which corresponds to the conservation of the (total) power, or intensity of the wave train. In general, these conservation laws correspond to symmetries of the equation, i.e., phase rotation (intensity), time translation (energy) and space translation (momentum). However, from a mathematical point of view, solutions u to the initial value problem (1) also satisfy a certain scaling symmetry in that

[1]The Introduction has been expanded in places but in general is excerpted from the following works:
[**5**] P. Antonelli, J. Arbunich and C. Sparber. *Regularizing nonlinear Schrödinger equations through partial off-axis variations.* SIAM J. Math. Anal. **51** (2019), no. 1, pp. 110–130.
[**7**] J. Arbunich, C. Klein and C. Sparber. *On a class of derivative nonlinear Schrödinger equations in two spatial dimensions.* To appear in ESAIM Math. Model. Numer. Anal.
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[**8**] J. Arbunich, I. Nenciu and C. Sparber *Stability and instability properties of rotating Bose-Einstein condensates.* Lett. Math. Phys (2019) 109: 1415.

for $\lambda > 0$ we have

$$u_\lambda(t, \mathbf{x}) = \lambda^{-1/\sigma} u\left(\frac{t}{\lambda^2}, \frac{\mathbf{x}}{\lambda}\right) \tag{3}$$

also solves the initial value problem (1). This particular symmetry is characteristic of a variety of physical models, in that quite frequently, these models are derived from physical considerations whereby physical quantities are added or equated with each other in terms of the same physical units. However, we shall see below that there exists models that do not exhibit this scaling symmetry.

When one encounters a time evolution equation such as (1), a natural question might be whether this is indeed a *well-posed* problem. In the context of partial differential equations of evolutionary type, this implies that once initial data is assigned (in a suitable function space) there must *exist* a *unique* solution evolving from the data. Moreover, the behavior of the solution should vary *continuously* with respect to changes in the initial data. As we will investigate a few variants of (1) and study the well-posedness of such models, we shall present here a formal definition (from [31]) of well-posedness for such initial value problems.

DEFINITION 1.1 (Well-posedness). *We say that the initial value problem (1) is locally well-posed in the Sobolev space $H^s(\mathbb{R}^d)$, if for any ball $B \subset H^s(\mathbb{R}^d)$, there exists a time $T > 0$ and a Banach space $X \subset L^\infty([-T, T]; H^s(\mathbb{R}^d))$, such that for each initial data $u_0 \in B$ there exists a unique solution $u \in X \cap C([-T, T], H^s(\mathbb{R}^d))$ to the integral formulation*

$$u(t) = S_0(t)u_0 - i\int_0^t S_0(t-s)(|u|^{2\sigma}u)(s)\, ds \equiv \Phi(u)(t),$$

where $S_0(t) = e^{it\Delta}$ is the linear propagator associated to (1). Moreover, the map $u_0 \mapsto u$ is a continuous map from H^s to $C([-T, T], H^s(\mathbb{R}^d))$. If the above holds for all $T \in \mathbb{R}$, then we say that the initial value problem is globally well-posed.

In order to show that the map $\Phi(u)(t) \equiv u$ in accordance with the definition above, the general strategy to prove well-posedness follows by showing the map $\Phi(u)$ is a contraction in a suitable ball and invoke Banach's fixed point theorem. In performing this type of analysis, the time of existence is chosen with respect to the initial data in order to close the fixed point argument, which in general is informed by the scaling symmetry (3). In particular, let us consider the homogeneous H^s-norm of the scaled initial data $u_{\lambda,0}(\mathbf{x}) = \lambda^{-1/\sigma}u_0(\lambda^{-1}\mathbf{x})$ then

$$\|u_{\lambda,0}\|_{\dot{H}^s} := \||\cdot|^s\widehat{u_{\lambda,0}}(\cdot)\|_{L^2_{\mathbf{x}}} = \lambda^{-s+(\frac{d}{2}-\frac{1}{\sigma})}\|u_0\|_{\dot{H}^s}. \tag{4}$$

The key point is that if we send $\lambda \to +\infty$, then whenever $s > \frac{d}{2} - \frac{1}{\sigma}$, called the H^s*-subcritical regime*, the H^s-norm of the initial data u_0 can be made small while the interval of existence is made longer. This is so because $u_\lambda(t, \mathbf{x})$ lives on the time interval $[-\lambda^2 T, \lambda^2 T]$. In the H^s*-critical regime*, i.e. $s = \frac{d}{2} - \frac{1}{\sigma}$, the norm is left invariant by the scaling while the interval of existence gets longer. Finally, in the H^s*-supercritical regime* when $s < \frac{d}{2} - \frac{1}{\sigma}$ the norms grow as the existence time decreases, which prevents one from closing a fixed point argument.

With the above dialogue concerning scaling, it is seen that (1) is L^2-subcritical ($s = 0$) provided $\sigma < \frac{2}{d}$. In this regime, one can employ the dispersive properties (i.e. Strichartz estimates) of the NLS and the conserved quantity (2) to obtain a global solution $u \in C(\mathbb{R}_t; L^2(\mathbb{R}^d))$, see e.g. [**25**]. In the L^2-critical case, i.e. $\sigma = \frac{2}{d}$, there is a sharp dichotomy characterizing the possibility of this blow-up and a rather complete description of this phenomenon is available in this case. In particular, it is known that global solutions exist for intensities $\|u_0\|_{L^2} < \|Q\|_{L^2}$, where Q denotes the nonzero least energy solution called the *nonlinear ground state*, which is a stationary solution associated to (1) of the form $u(t, \mathbf{x}) = e^{it} Q(\mathbf{x})$. Above this threshold, finite-time blow-up appears and solutions will in general exhibit self-similar blow-up with a profile given by Q (up to symmetries). This behavior has been analyzed in a series of works, see [**85, 86, 87**] and the references therein. In turn, this also implies that stationary states of the form $e^{it} Q(\mathbf{x})$ are *strongly unstable*.

For $\frac{2}{d} \leqslant \sigma < \frac{2}{(d-2)_+}$ one seeks solutions in $u(t, \cdot) \in H^1(\mathbb{R}^d)$ as the problem is H^1-subcritical ($s = 1$), in particular this includes the cubic case in dimensions $d = 2, 3$. However such a solution may not exist for all times $t \in \mathbb{R}$, due to the possibility of *finite-time blow-up* in the sense that

$$\lim_{t \to T_-} \|\nabla u(t, \cdot)\|_{L^2_{\mathbf{x}}} = +\infty$$

for some $T < \infty$, depending on the initial data. It should be noted that this type of blow-up is not a feature of the physical phenomena, but rather an indication that the model breaks down.

1.2. Regularizard NLS with off-axis variations. From the point of view of laser physics, blow-up is usually referred to as *optical collapse*. However, it is known from physics experiments that higher order effects, neglected in the derivation of (1), can arrest such a collapse and instead yield a process called *filamentation*. The latter corresponds to a complicated interplay between diffraction, self-focusing, and defocusing mechanisms present at high intensities which allow the beam to propagate beyond the theoretical predicted blow-up point, see [**48**].

In their recent mathematical study [39], Dumas, Lannes and Szeftel derive several new variants of the NLS from the underlying Maxwell equations of electromagnetism, in an effort to incorporate additional physical effects not present in (1). One of the new NLS type models derived in [39] allows for the possibility of an *off-axis variation of the group velocity*. It takes into account the fact that self-focusing pulses usually become asymmetric due to variations of the group velocity within off-axis rays, a phenomenon referred to as *space-time focusing* in the optics literature, cf. [94].

The simplest mathematical model incorporating off-axis variations is given by

$$iP_\varepsilon \partial_t u + \Delta u + |u|^2 u = 0, \tag{5}$$

where $P_\varepsilon \equiv P_\varepsilon(\nabla)$ is a linear, second order, self-adjoint operator such that

$$\langle P_\varepsilon u, u\rangle_{L^2} \gtrsim \|u\|_{L^2}^2 + \varepsilon^2 \sum_{j=1}^{k} \|\boldsymbol{\nu}_j \cdot \nabla u\|_{L^2}^2.$$

Here, $\langle \cdot, \cdot \rangle_{L^2_{\mathbf{x}}}$ denotes the usual $L^2(\mathbb{R}^d)$ inner product, $0 < \varepsilon \leqslant 1$ is a small (dimensionless) parameter, and $\{\boldsymbol{\nu}_j\}_{j=1}^k \in \mathbb{R}^d$ with $k \leqslant d$ are some given (linearly independent) vectors representing the off-axis directions. The case $k = d$ thereby corresponds to a *full off-axis dependence* of the group velocity, whereas $k < d$ is referred to as *partial off-axis dependence*. In the former case, the authors of [39] have shown that solutions $u(t, \cdot) \in H^1(\mathbb{R}^d)$ to (5) exist for all $t \in \mathbb{R}$, and hence no finite-time blow-up occurs. However, the situation involving only a partial off-axis dependence is much more involved and it is an open problem posed in [39] to prove global well-posedness in this case.

In the joint work [5] with P. Anontelli and C. Sparber, we provide an answer to this question. To this end, we consider the following initial value problem:

$$\begin{cases} iP_\varepsilon \partial_t u + \Delta u + |u|^{2\sigma} u = 0, \ t \in \mathbb{R}, \ \mathbf{x} \in \mathbb{R}^d, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}). \end{cases} \tag{6}$$

From now on, we shall split the spatial coordinates into $\mathbf{x} = (x, y) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$ for $k \leqslant d$, with the understanding that if $k = d$, we again identify $y \equiv \mathbf{x} \in \mathbb{R}^d$. In addition, we choose without loss of generality $\boldsymbol{\nu}_j$ to be the j-th standard basis vectors in $\mathbb{R}^d$. Explicitly, we then have

$$P_\varepsilon = 1 - \varepsilon^2 \Delta_y = 1 - \varepsilon^2 \sum_{j=1}^{k} \frac{\partial^2}{\partial y_j^2}, \quad 0 \leqslant k \leqslant d. \tag{7}$$

The case $k = 0$ thereby corresponds to the situation with no off-axis variation, for which we will recover (as we shall see below) the usual L^2 well-posedness theory for the NLS.

Here and in the following, P_ε^s, for any $s \in \mathbb{R}$, is the non-local operator defined through multiplication in Fourier space using the symbol

$$\widehat{P}_\varepsilon^s(\boldsymbol{\xi}) = \Big(1 + \varepsilon^2 \sum_{j=1}^{k} \xi_j^2\Big)^s, \quad 0 \leqslant k \leqslant d,$$

where $\boldsymbol{\xi} = (\xi_1, .., \xi_d) \in \mathbb{R}^d$ is the Fourier variable dual to $\mathbf{x} = (x_1, .., x_d)$. Moreover when $\varepsilon = 1$ and $k = d$, note we can define the L^2-based Sobolev spaces $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ via the norm

$$\|f\|_{H^s} = \big\|P_1^{s/2} f\big\|_{L^2_{\mathbf{x}}} := \left(\int_{\mathbb{R}^2} |\widehat{P}_1^{s/2} \widehat{f}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}\right)^{\frac{1}{2}}.$$

Mathematically, (6) is related to (1), in the same way the *Benjamin–Bona–Mahoney equation* is related to the celebrated *Korteweg–de Vries equation* for shallow, unidirectional water waves in $d = 1$, see [**15, 18**]. The difference when compared to our case is that we are not confined to work in only one spatial dimension. Thus we can allow for a partial regularization in $k < d$ directions, a possibility which seems to have not been considered for BBM-type equations in higher dimensions, see [**55**]. When comparing (6) to (1), one checks that at least formally, both equations are Hamiltonian systems which conserve the same energy functional, i.e.,

$$E(t) = \frac{1}{2}\|\nabla u(t,\cdot)\|^2_{L^2_{\mathbf{x}}} - \frac{1}{2(\sigma+1)}\|u(t,\cdot)\|^{2\sigma+2}_{L^{2\sigma+2}_{\mathbf{x}}} = E(0). \tag{8}$$

However, instead of the usual L^2 conservation law (2), one finds

$$\|P_\varepsilon^{1/2} u(t,\cdot)\|^2_{L^2_{\mathbf{x}}} = \|P_\varepsilon^{1/2} u_0\|^2_{L^2_{\mathbf{x}}} = \int_{\mathbb{R}^d} |u_0|^2 + \varepsilon^2 |\nabla_y u_0|^2 \, d\mathbf{x} \tag{9}$$

in the case of (6). The identity (9) corresponds to a conservation law for (the square of) the mixed $L^2(\mathbb{R}_x^{d-k}; H^1(\mathbb{R}_y^k))$-norm of u, whenever $\varepsilon > 0$. In order to understand the influence of partial off-axis variations, it is therefore natural to set up a well-posedness theory in this mixed Sobolev-type space. With this in mind, we can now state the main results of this work.

THEOREM 1.1 (Partial off-axis variation; subcritical case). *Let $d > k \geqslant 0$ and*

- *either $k \leqslant 2$ and $0 \leqslant \sigma < \frac{2}{d-k}$,*
- *or $k > 2$ and $0 \leqslant \sigma \leqslant \frac{2}{d-2}$.*

Then for any $u_0 \in L^2(\mathbb{R}_x^{d-k}; H^1(\mathbb{R}_y^k))$ there exists a unique global-in-time solution to (6) such that $u \in C(\mathbb{R}_t; L^2(\mathbb{R}_x^{d-k}; H^1(\mathbb{R}_y^k)))$ depends continuously on the initial data and satisfies the conservation law (9) for all $t \in \mathbb{R}$.

In the result above, we have to exclude the choice $k \leqslant 2$ and $\sigma = \frac{2}{d-k}$, which corresponds to a critical case that needs to be dealt with separately (see below). Regardless of that, we see that as soon as $k > 0$, i.e., as soon as some partial off-axis variation is present, we can allow for L^2-*supercritical* powers $\sigma > \frac{2}{d}$ and still retain global-in-time solutions u. In other words, no finite-time blow-up appears in the case with partial off-axis variations, and we can even allow for initial data u_0 in a space slightly larger than $H^1(\mathbb{R}^d)$.

We now turn to the case of partial off-axis dependence with critical nonlinearity, for which we can prove an analogue of the well-posedness results given in [27]. Note that for $k = 0$ (no off-axis variation) we recover the usual L^2-critical case $\sigma = \frac{2}{d}$.

THEOREM 1.2 (Partial off-axis variation; critical case). *Let $0 \leqslant k \leqslant 2$, and $\sigma = \frac{2}{d-k}$. Then for any $u_0 \in L^2(\mathbb{R}^{d-k}_x; H^1(\mathbb{R}^k_y))$ there exist times $0 < T_{\max}, T_{\min} \leqslant \infty$ and a unique maximal solution $u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^{d-k}_x; H^1(\mathbb{R}^k_y)))$, satisfying (9) for all $t \in (-T_{\min}, T_{\max})$. In addition, we have the following blow-up alternative: If $T_{\max} < \infty$ if and only if*

$$\|u\|_{L^{\frac{2(d-k+2)}{d-k}}\left([0,T_{\max})\times\mathbb{R}^{d-k}_x; H^{\frac{2}{d-k+2}}(\mathbb{R}^k_y)\right)} = \infty,$$

and analogously for $T_{\min}$. Finally, if the $L^2(\mathbb{R}^{d-k}_x; H^1(\mathbb{R}^k_y))$-norm of the initial datum is sufficiently small, then the solution u exists for all $t \in \mathbb{R}$.

For completeness, we shall also state a result in the case of full off-axis dependence. Note when $k = d$, the mixed Sobolev space above simply becomes $H^1(\mathbb{R}^d)$ as (9) yields an a-priori bound on the H^1-norm of u, ruling out the possibility of finite-time blow-up.

THEOREM 1.3 (Full off-axis variation). *Let $k = d$ and $0 \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$. Then for any $u_0 \in H^1(\mathbb{R}^d)$ there exists a unique global-in-time solution $u \in C(\mathbb{R}_t; H^1(\mathbb{R}^d))$ to (6), depending continuously on the initial data and satisfying the conservation laws (8) and (9) for all $t \in \mathbb{R}$.*

This is a slight generalization of the result given in [39], where only the cubic case is treated. Note that we can allow for $\sigma = \frac{2}{(d-2)_+}$, i.e., the H^1-critical power, in contrast to the usual theory of NLS without off-axis variations, cf. [68].

In order to prove all of these theorems, we shall employ the following change of unknown

$$v(t, \mathbf{x}) := P_\varepsilon^{1/2} u(t, \mathbf{x}), \tag{10}$$

and rewrite the Cauchy problem (6) in the form

$$(11) \qquad \begin{cases} i\partial_t v + P_\varepsilon^{-1}\Delta v + P_\varepsilon^{-1/2}\big(|P_\varepsilon^{-1/2}v|^{2\sigma}P_\varepsilon^{-1/2}v\big) = 0, \ t\in\mathbb{R}, \ \mathbf{x}\in\mathbb{R}^d, \\ v(0,\mathbf{x}) = P_\varepsilon^{1/2}u_0(\mathbf{x}) \equiv v_0(\mathbf{x}). \end{cases}$$

Instead of (9), this new equation conserves

$$\|v(t,\cdot)\|^2_{L^2_{\mathbf{x}}} = \|P_\varepsilon^{1/2}u(t,\cdot)\|^2_{L^2_{\mathbf{x}}} = \|P_\varepsilon^{1/2}u_0\|^2_{L^2_{\mathbf{x}}} = \|v_0\|^2_{L^2_{\mathbf{x}}},$$

i.e., the usual L^2 conservation law. We therefore aim to set-up an L^2-based well-posedness theory for (11), written from Duhamel's intrgral formulation as

$$v(t) = e^{itP_\varepsilon^{-1}\Delta}v_0 + i\int_0^t e^{i(t-s)P_\varepsilon^{-1}\Delta}P_\varepsilon^{-1/2}(|P_\varepsilon^{-1/2}v|^2P_\varepsilon^{-1/2}v)(s)\,ds.$$

The advantage of working with v instead of u lies in the fact that it allows us to exploit the regularizing properties of the operator $P_\varepsilon^{-1/2}$ acting on the nonlinearity. Roughly speaking, the action of $P_\varepsilon^{-1/2}$ allows us to gain a derivative in $y\in\mathbb{R}^k$. However, we also note that the linear semi-group

$$(12) \qquad S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$$

is no longer dispersive in the same way as the usual Schrödinger group $S_0(t) = e^{it\Delta}$. Indeed, we can only expect "nice" dispersive properties in the spatial directions $x\in\mathbb{R}^{d-k}$, where P_ε does not act, which will play an important role in the derivation of suitable Strichartz estimates. It has been proved in [19] that in the case of full off-axis dependence, $S_\varepsilon(t)$ does not admit Strichartz estimates. Note that this issue is not simply an artifact of our change of unknown $u\mapsto v$, since $S_\varepsilon(t)$ also describes the dispersive properties of (the linear part of) the original equation (6) for u.

We shall also present the following result which extends Theorem 1.1.

THEOREM 1.4 (Small data scattering). *Let $k+4\geqslant d>k$ with $k=1,2$ and $\frac{2}{d-k}\leqslant\sigma\leqslant\frac{2}{d-2}$. There exists $\lambda_\varepsilon>0$ such that for every initial data with $\|P_\varepsilon^{1/2}u_0\|_{H^1}\leqslant\lambda_\varepsilon$ there exists a unique global-in-time solution such that $P_\varepsilon^{1/2}u\in C(\mathbb{R}_t;H^1(\mathbb{R}^d))$. Moreover there exists scattering states $u_\pm$ satisfying*

$$\big\|P_\varepsilon^{1/2}\big(e^{-itP_\varepsilon^{-1}\Delta}u(t)-u_\pm\big)\big\|_{H^1}\to 0 \quad \text{as} \quad t\to\pm\infty.$$

We should also mention that the sign of the nonlinearity (which is focusing) does not play a role in the proofs given below, and hence all of our results also remain true in the defocusing case.

1.3. Two-dimensional derivative NLS with partial off-axis variations. The next work is devoted to the analysis and numerical simulations for the following class of nonlinear dispersive equations in two spatial dimensions:

$$iP_\varepsilon \partial_t u + \Delta u + (1 + i\boldsymbol{\delta} \cdot \nabla)\big(|u|^{2\sigma} u\big) = 0, \quad u_{|t=0} = u_0(\mathbf{x}), \tag{13}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\boldsymbol{\delta} = (\delta_1, \delta_2)^\top \in \mathbb{R}^2$ is a given vector with $|\boldsymbol{\delta}| \leqslant 1$, and $\sigma > 0$ is a parameter describing the strength of the nonlinearity. In addition, for $0 < \varepsilon \leqslant 1$, the linear differential operator P_ε is defined in (7) for $k = 1, 2$. Indeed, we shall mainly be concerned with (13) rewritten in its *evolutionary form*:

$$i\partial_t u + P_\varepsilon^{-1} \Delta u + P_\varepsilon^{-1}(1 + i\boldsymbol{\delta} \cdot \nabla)\big(|u|^{2\sigma} u\big) = 0, \quad u_{|t=0} = u_0(x_1, x_2). \tag{14}$$

In comparison to (6), i.e. $\delta_1 = \delta_2 = 0$, this new two-dimensional model (14) includes an additional physical effect. We see here the addition of a nonlinearity of derivative type which incorporates possible *self-steepening* of the laser pulse in the direction $\boldsymbol{\delta} \in \mathbb{R}^2$. As in the previous subsection involving (6), the operator P_ε describes *off-axis variations* of the group velocity of the beam. In dimension two, the case $k = 2$ corresponds to a *full off-axis dependence*, whereas for $k = 1$ the model incorporates only a *partial off-axis dependence*. It is important to note that (14), as well as (6), does not admit the scaling invariance analogous to (1). Hence, there is no clear indication of subcritical or supercritical regimes. At least formally though, equation (14) admits the same conservation law (9) as equation (6). However, the situation is more complicated in the case with only a partial off-axis variation.

In the case *without* self-steepening, i.e. when $\delta_1 = \delta_2 = 0$, it is seen from Theorem 1.1 that even a partial off-axis variation (mediated by P_ε with $k = 1$) can arrest the blow-up for all $\sigma < 2$. In particular, this allows for nonlinearities larger than the L^2-critical case, cf. Chapter 2 for more details. One motivation for the present work is to give numerical evidence for the fact that the results of Chapter 2 are indeed sharp, and that in dimension two, we can expect finite-time blow-up as soon as $\sigma \geqslant 2$.

The aim of Chapter 3 is to extend the analysis of Chapter 2 in the two-dimensional situations involving self-steepening ($\boldsymbol{\delta} \neq 0$), and to provide further insight into the qualitative interplay between this effect and the one stemming from P_ε. From a mathematical point of view, the addition of a derivative nonlinearity makes the question of global well-posedness versus finite-time blow-up much more involved. Derivative NLS and their corresponding ground states are usually studied in one spatial dimension only, see [4, 30, 57, 61, 81, 82, 103, 106] and references therein. For $\sigma = 1$, the classical one-dimensional derivative NLS is known to be completely integrable. Furthermore, there has only very recently been a breakthrough in the proof of global-in-time existence for this case, see [65, 66]. In contrast to that, [82] gives strong numerical indications for a self-similar finite-time blow-up in derivative NLS with $\sigma > 1$. The blow-up thereby seems to be a result of the self-steepening effect in the density $\rho = |u|^2$, which generically undergoes a time evolution similar to a dispersive shock wave formation in Burgers' equation. To our knowledge, however, no rigorous proof of this phenomenon is currently available.

In two and higher dimensions, even the local-in-time existence of solutions to derivative NLS type equations seems to be largely unknown, let alone any further qualitative properties of their solutions. In view of this, the present work aims to shine some light on the specific variant of two-dimensional derivative NLS given by (14). Except for its physical significance, this class of models also has the advantage that the inclusion of (partial) off-axis variations via P_ε are expected to have a strong regularizing effect on the solution, and thus allow for several stable situations without blow-up. Indeed we shall prove:

THEOREM 1.5. *Let* $P_\varepsilon = 1 - \varepsilon^2 \partial_{x_1}^2$, $\sigma = 1$ *and* $\boldsymbol{\delta} = (\delta, 0)^\top$ *for some* $\delta \neq 0$. *Then for any* $u_0 \in L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1}))$ *there exists a unique global solution* $u \in C(\mathbb{R}_t; L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1})))$ *to* (13).

We also prove global well-posedness in the case of full off-axis dependence, i.e. $P_\varepsilon = 1 - \varepsilon^2 \Delta$, and general self-steepening with $\boldsymbol{\delta} \neq \mathbf{0}$.

2. Models of quantum physics

2.1. Modulated wave dynamics in honeycomb structures. Two-dimensional honeycomb lattice structures have attracted considerable interest, both in the physics and applied mathematics communities, due to the unusual transport properties associated with materials exhibiting this structural symmetry. This curiosity has been stimulated by the somewhat recent fabrication of graphene – an allotrope of carbon consisting of a single layer of atoms arranged in a hexagonal lattice. Most notedly, this material is known to exhibit many unusual thermal and electronic properties, a result of the dynamics of excited electrons which behave like two–dimensional massless *Dirac fermions*, cf. [24]. Honeycomb structures also appear in nonlinear optics, modeling laser beam propagation in certain types of photonic crystals, see for instance [11, 60].

To physically motivate the following result, we shall consider the microscopic and non-relativistic (single electron) model for wave transport in a honeycomb structure subject to self-interaction. In this way, the wave function $\Psi = \Psi(t, \mathbf{x})$ satisfies the following Schrödinger equation

$$i\partial_t \Psi = -\Delta\Psi + V_{\text{per}}(\mathbf{x})\Psi + \kappa|\Psi|^2\Psi, \quad \mathbf{x} \in \mathbb{R}^2, t \in \mathbb{R}, \tag{15}$$

subject to initial data $\Psi_0 = \Psi(0, \mathbf{x})$, where $V_{\text{per}} \in C^\infty(\mathbb{R}^2; \mathbb{R})$ denotes a smooth potential which is periodic with respect to a honeycomb lattice Λ, with a corresponding fundamental cell $Y \subset \Lambda$ (see Section 1 in Chapter 4 below for more details). Hence in order to study wave dynamics in honeycomb structures, the starting point is a two-dimensional periodic Schrödinger operator

$$H = -\Delta + V_{\text{per}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{16}$$

In [43], C. L. Fefferman and M. I. Weinstein prove that the associated quasi-particle dispersion relation generically exhibits conical singularities at the points of degeneracy, at the so-called *Dirac points*. In turn, this yields effective equations of (massless) Dirac type for wave packets spectrally localized around these singularities, see [1, 44, 45]. To be more precise, let us recall from *Bloch-Floquet theory* that the spectrum $\sigma(H) \subset \mathbb{R}$ is given by a union of spectral bands, which is obtained through the following $\mathbf{k}$–pseudo periodic boundary value problem:

$$\begin{cases} H\Phi(\mathbf{y}; \mathbf{k}) = \mu(\mathbf{k})\Phi(\mathbf{y}; \mathbf{k}), \ \mathbf{y} \in Y, \\[2ex] \Phi(\mathbf{y} + \mathbf{v}; \mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{v}}\Phi(\mathbf{y}; \mathbf{k}), \ \mathbf{v} \in \Lambda, \end{cases} \tag{17}$$

where $\mathbf{k} \in Y^*$ denotes the wave-vector (or quasi-momentum) varying within the Brioullin zone – the fundamental cell of the dual lattice Λ^*. For each $k \in Y^*$, this consequently yields a countable sequence of discrete eigenvalues

$$\mu_0(\mathbf{k}) \leqslant \mu_1(\mathbf{k}) \leqslant \mu_2(\mathbf{k}) \leqslant ...,$$

which are ordered including multiplicity and which tend to infinity. The corresponding pseudo-periodic eigenfunctions $\Phi_m(\cdot\ ; \mathbf{k})$ associated to the E_m-th band are known as *Bloch waves*, see Subsection 1.2 of Chapter 4. In the following, suppose there exists an energy band E_m such that at $\mathbf{k} = \mathbf{K}_*$, the eigenvalue $\mu_* \equiv \mu(\mathbf{K}_*)$ has a Dirac point such that $\mu(\mathbf{k}) \sim |\mathbf{k}|$ near $\mathbf{k} \sim \mathbf{K}_*$ and

$$\text{Nullspace}(H - \mu_*) = \text{span}\left\{\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x})\right\},$$

where $\Phi_{1,2}(\mathbf{x}) = \Phi_{1,2}(\mathbf{x}, \mathbf{K}_*)$, see Subsection 1.3 of Chapter 4.

Furthermore, let $0 < \varepsilon \ll 1$ be a small (dimensionless) parameter, and assume that initially $\Psi_0 = \Psi_0^\varepsilon$ is a wave packet, spectrally concentrated around the Dirac point, i.e.,

$$\Psi_0^\varepsilon(\mathbf{x}) = \varepsilon\alpha_{0,1}(\varepsilon\mathbf{x})\Phi_1(\mathbf{x}) + \varepsilon\alpha_{0,2}(\varepsilon\mathbf{x})\Phi_2(\mathbf{x})$$

where $\alpha_{0,1}, \alpha_{0,2} \in \mathcal{S}(\mathbb{R}^2;\mathbb{C})$ are some slowly varying and rapidly decaying amplitudes. The overall factor ε is thereby introduced to ensure that $\|\Psi_0^\varepsilon\|_{L^2} = 1$. It is proved in [45] that the corresponding solution $\Psi^\varepsilon(t,\cdot)$ to the linear part of the Schrödinger equation (15) satisfies,

$$\Psi^\varepsilon(t,\mathbf{x}) \underset{\varepsilon\to 0}{\sim} \varepsilon e^{-i\mu_* t}\Big(\alpha_1(\varepsilon t,\varepsilon\mathbf{x})\Phi_1(\mathbf{x}) + \alpha_2(\varepsilon t,\varepsilon\mathbf{x})\Phi_2(\mathbf{x})\Big) + \mathcal{O}(\varepsilon),$$

provided $\alpha_{1,2}$ satisfy the following massless Dirac system:

$$\begin{cases} \partial_t\alpha_1 + \overline{\lambda}_\#\big(\partial_{x_1} + i\partial_{x_2}\big)\alpha_2 = 0, \quad \alpha_{1|t=0} = \alpha_{0,1}, \\ \partial_t\alpha_2 + \lambda_\#\big(\partial_{x_1} - i\partial_{x_2}\big)\alpha_1 = 0, \quad \alpha_{2|t=0} = \alpha_{0,2}, \end{cases} \tag{18}$$

where $0 \neq \lambda_\# \in \mathbb{C}$ is some constant depending on Bloch waves $\Phi_{1,2}$. This approximation is shown to hold up to small errors in $H^s(\mathbb{R}^2)$, over time intervals of order $T \sim \mathcal{O}(\varepsilon^{-2+\delta})$, for $\delta > 0$. The Dirac system (18) consequently describes the dynamics of the slowly varying amplitudes on large time scales. Of course, since the problem is linear, solutions of any size (with respect to ε) will satisfy the same asymptotic behavior.

From the author's joint work with C. Sparber in [9], our aim is to generalize this type of result to the case of *weakly nonlinear* wave packets. Formally, this nonlinear problem was described in [44], but without providing any rigorous error estimates. Similarly, in [1, 2] the authors derive several nonlinear Dirac type models by formal multi-scale expansions in the case of *deformed* and *shallow* honeycomb lattice structures, respectively. In the latter case, the size of the lattice potential serves as the small parameter in the expansion. In contrast, we shall not assume that the periodic potential V_{per} is small, but rather keep it of a fixed size of order $\mathcal{O}(1)$ in $L^\infty(\mathbb{R}^2)$ and independent of epsilon. However, as always in nonlinear problems, the size of the solution becomes important, depending on the power of the nonlinearity. We restrict ourselves to the case of a cubic nonlinearity, since this is the most important example within the context of nonlinear wave propagation with self-interaction (a generalization to other power law nonlinearities is straightforward). We shall also remark on the case of *Hartree nonlinearities* in Section 5 below, as they are more natural in the description of the mean-field dynamics of electrons in graphene [59, 60].

To this end, we find it more convenient to put ourselves in a *macroscopic reference frame*, which means that we rescale (15) by the change of coordinates

$$t \mapsto \tilde{t} = \varepsilon t, \quad \mathbf{x} \mapsto \tilde{\mathbf{x}} = \varepsilon \mathbf{x}, \quad \Psi^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}) = \varepsilon \Psi(\varepsilon t, \varepsilon \mathbf{x}),$$

with $\|\Psi_0^\varepsilon(t, \cdot)\|_{L^2} = 1$. We consequently consider the following *semi-classical* nonlinear Schrödinger equation, after having dropped the tildes on the variables, to give

$$i\varepsilon \partial_t \Psi^\varepsilon = -\varepsilon^2 \Delta \Psi^\varepsilon + V_{\text{per}}\left(\frac{\mathbf{x}}{\varepsilon}\right)\Psi^\varepsilon + \kappa^\varepsilon |\Psi^\varepsilon|^2 \Psi^\varepsilon, \quad \Psi^\varepsilon_{t=0} = \Psi_0^\varepsilon(\mathbf{x}). \tag{19}$$

Notice that in this reference frame the honeycomb lattice potential V_{per} becomes *highly oscillatory*. Furthermore, the nonlinear coupling constant $\kappa^\varepsilon \in \mathbb{R}$, is assumed to be of the form

$$\kappa^\varepsilon = \varepsilon \kappa, \quad \text{with } \kappa = \pm 1.$$

As will become clear, this size is *critical* with respect to our asymptotic expansion, in the sense that the modulated amplitudes $\alpha_{1,2}$ will satisfy a nonlinear analog of (18). It should be noted that for values smaller than κ^ε, the nonlinear effects will not be present in leading order as $\varepsilon \to 0$. Whereas for κ^ε larger than $\mathcal{O}(\varepsilon)$, we do not expect the Dirac model to be valid any longer. Alternatively, this can be reformulated as saying that we consider asymptotically small solutions of critical size $\mathcal{O}(\sqrt{\varepsilon})$ in L^∞ to (19) but with fixed coupling strength $\kappa = \mathcal{O}(1)$. The advantage of our scaling is that it yields an asymptotic description for solutions of order $\mathcal{O}(1)$ in L^∞ on macroscopic space-time scales, in contrast to the setting of [**1**, **44**], in which the size of the solution varies as $\varepsilon \to 0$. Another advantage is that this scaling puts us firmly in the regime of weakly nonlinear, semi-classical NLS with highly oscillatory periodic potentials, which have been extensively studied in the works [**16**, **20**, **21**, **22**, **53**], albeit not for the case of honeycomb lattices. From the mathematical point of view, the present work will follow the ideas presented in [**53**] and the strategy will be adapted to the current situation.

Our main result, described in more detail in Section 2 of Chapter 4, rigorously shows that solutions to (19) with initial data Ψ_0^ε spectrally localized around a Dirac point, can be approximated via

$$\Psi^\varepsilon(t, \mathbf{x}) \underset{\varepsilon \to 0}{\sim} \Psi^\varepsilon_{\text{app}}(t, \mathbf{x}) = e^{-i\mu_* t/\varepsilon}\left(\alpha_1(t, \mathbf{x})\Phi_1\left(\frac{\mathbf{x}}{\varepsilon}\right) + \alpha_2(t, \mathbf{x})\Phi_2\left(\frac{\mathbf{x}}{\varepsilon}\right)\right) + \mathcal{O}(\varepsilon), \tag{20}$$

where the amplitudes $\alpha_{1,2}$ solve the following nonlinear Dirac system

$$\begin{cases} \partial_t \alpha_1 + \overline{\lambda}_{\#}(\partial_{x_1} + i\partial_{x_2})\alpha_2 = -i\kappa\Big(b_1|\alpha_1|^2 + 2b_2|\alpha_2|^2\Big)\alpha_1, \\ \partial_t \alpha_2 + \lambda_{\#}(\partial_{x_1} - i\partial_{x_2})\alpha_1 = -i\kappa\Big(b_1|\alpha_2|^2 + 2b_2|\alpha_1|^2\Big)\alpha_2, \end{cases} \tag{21}$$

subject to initial data $\alpha_{0,1}(\mathbf{x})$, $\alpha_{0,2}(\mathbf{x})$, respectively, and with coefficients $b_{1,2} > 0$ given by

$$b_1 = \int_Y |\Phi_1(\mathbf{y})|^4 \, d\mathbf{y} = \int_Y |\Phi_2(\mathbf{y})|^4 \, d\mathbf{y}, \quad b_2 = \int_Y |\Phi_1(\mathbf{y})|^2 |\Phi_2(\mathbf{y})|^2 \, d\mathbf{y}.$$

We now state a simplified version of our main result, i.e. Theorem 4.1, where it should be noted that this approximation result holds on the existence time of the nonlinear Dirac system (21).

THEOREM 1.6. *Let V_{per} be a smooth honeycomb-lattice potential and $\boldsymbol{\alpha} \in C([0,T), H^s(\mathbb{R}^2))$ be the solution to (21) for some $s > 4$. Assume that the initial data Ψ_0^ε of (19) satisfies*

$$\left\| \Psi_0^\varepsilon(\cdot) - \left(\alpha_{0,1}(\cdot)\Phi_1\left(\frac{\cdot}{\varepsilon}\right) + \alpha_{0,2}(\cdot)\Phi_2\left(\frac{\cdot}{\varepsilon}\right) + \varepsilon u_1\left(0, \cdot, \frac{\cdot}{\varepsilon}\right)\right)\right\|_{L^2} \lesssim \varepsilon^2,$$

where u_1 is a corrector such that $\langle \Phi_{1,2}, u_1\rangle_{L^2(Y)} = 0$. Then, for any $T_* \in [0,T)$ there exists an $\varepsilon_0 = \varepsilon_0(T_*) \in (0,1)$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the solution $\Psi^\varepsilon \in C([0,T_*); H^s_\varepsilon(\mathbb{R}^2))$ of (19) exists and moreover

$$\sup_{0 \leqslant t \leqslant T_*} \left\| \Psi^\varepsilon(t,\cdot) - \Psi^\varepsilon_{\text{app}}(t,\cdot)\right\|_{L^2} \lesssim \varepsilon^2.$$

In order to prove this result, we first derive sufficient regularity estimates for the approximate solution $\Psi^\varepsilon_{\text{app}}$ and the resulting remainder of order $\mathcal{O}(\varepsilon^2)$, see Proposition 4.10. These estimates then allow us to perform energy estimates for the difference $\Psi^\varepsilon - \Psi^\varepsilon_{\text{app}}$. Using a Moser-type lemma (Lemma 4.13) in combination with a continuity argument then allows us to control the growth of the error under the nonlinear time evolution.

In terms of the unscaled variables used in [**44**, **45**], our approximation result is seen to hold on times of order $\mathcal{O}(\varepsilon^{-1})$, which is considerably shorter than the time-scale $O(\varepsilon^{-2+\delta})$ obtained in [**44**]. However, this drawback is expected due to the influence of our nonlinear perturbation and consistent with earlier results for linear semi-classical Schrödinger equations with periodic potentials and additional, slowly varying non-periodic perturbations $U = U(t,x)$, see [**16**]. In the case of the latter, one also expects the appearance of purely geometric effects, such as the celebrated Berry phase term (cf. [**21**, **20**]). It would certainly be interesting to understand the corresponding Dirac-type dynamics already on the linear level, when such geometric effects are present (but this is beyond the scope of the current work). We finally note that other examples of coupled mode equations have been derived in [36, 37, 56, 91, 92]. However, in these models the resulting mode equations are of transport type, and any coupling between the amplitudes stems purely from the nonlinearity, in contrast to the Dirac model. We finally remark that discrete mode equations, valid in the tight binding regime, have recently been studied in [2, 42].

2.2. Orbital stability and instability of rotating Bose-Einstein Gases. In the joint work [8] with I. Nenciu and C. Sparber, we consider the dynamics of (harmonically) trapped *Bose-Einstein condensates* (BEC), subject to an external rotating force. Because of their ability to display quantum effects at the macroscopic scale, BEC have become an important subject of research, both experimentally and theoretically. In particular, the expression of quantum vortices in rapidly rotating BEC has been an ongoing topic of interest over the last few decades, see, e.g., [3, 13, 23, 33, 34, 47, 96] and the references therein. It is well-known that in the mean-field regime, BEC can be accurately described by the celebrated *Gross-Pitaevskii equation* (GP) for ψ, the macroscopic wave function of the condensate, see [80, 97, 98]. In dimensionless units, the GP equation with general nonlinearity reads

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\mathbf{x})\psi + a|\psi|^{2\sigma}\psi - (\mathbf{\Omega}\cdot\mathbf{L})\psi\,, \quad \psi_{|t=0} = \psi_0(\mathbf{x})\,. \tag{22}$$

Here, $a \in \mathbb{R}$, $\sigma > 0$ and $(t, \mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$ with $d = 2$, or 3, respectively. The former situation thereby corresponds to the case of an *effective* two-dimensional BEC, obtained via strong confining forces, see [84] for more details. The external potential $V(\mathbf{x}) \in \mathbb{R}$ is assumed to be *harmonic*, i.e.,

$$V(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{d}\omega_j^2 x_j^2, \tag{23}$$

where the parameters $\omega_j \in \mathbb{R}\setminus\{0\}$ represent the respective trapping frequencies in each spatial direction. As we shall see, the smallest trapping frequency denoted by $\omega \equiv \min_{j=1,\dots,d}\{\omega_j\}$ will play a particular role in our analysis. We shall further assume that the BEC is subject to a rotating force along a given *rotation axis* $\mathbf{\Omega} \in \mathbb{R}^3$ and denote by

$$\mathbf{L} = -i\mathbf{x}\wedge\nabla,$$

the quantum mechanical angular momentum operator. Note that in dimension $d = 2$, we always have

$$\mathbf{\Omega}\cdot\mathbf{L} = -i|\mathbf{\Omega}|\big(x_1\partial_{x_2} - x_2\partial_{x_1}\big), \tag{24}$$

corresponding to the case where $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|) \in \mathbb{R}^3$. The nonlinearity in (22) describes the mean-field self-interaction of the condensate particles. The physically most relevant case is given by a cubic nonlinearity, i.e. $\sigma = 1$, but for the sake of generality we shall in the following allow for more general $\sigma > 0$. We shall also allow for both *attractive* $a < 0$ and *repulsive* $a > 0$ interactions, satisfying Assumption 1 below. Vortices are generally believed to be unstable in the former case (see, e.g., [**23, 32, 96**]), while they are known to form stable lattice configurations in the latter [**3, 33, 47**].

In this work, we shall not be interested in the dynamical features of individual vortices, but rather study bulk properties of the condensate, as described by (22). To this end, we recall that the natural energy space associated to (22) is given by

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \, : \, |\mathbf{x}|u \in L^2(\mathbb{R}^d)\},$$

equipped with the norm

$$\|u\|_\Sigma^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \||\mathbf{x}|u\|_{L^2}^2 \,.$$

We also impose the following subcriticality condition on the nonlinearity:

ASSUMPTION 1. *One of the following holds:*

- $a > 0$ *(defocusing) and* $0 < \sigma < \frac{2}{(d-2)_+}$, *or*
- $a < 0$ *(focusing) and* $0 < \sigma < \frac{2}{d}$.

Under these hypotheses, the existence of a unique global-in-time solution $\psi \in C(\mathbb{R}_t; \Sigma)$ to (22) has been proved in [**6**]. In particular, the restriction $\sigma < \frac{2}{d}$ in the focusing case ($a < 0$) ensures that *no finite-time blow-up* can occur. In addition, the global solution $\psi(t, \cdot) \in \Sigma$ is known to conserve the *total mass*, given by

$$N(\psi(t, \cdot)) = \int_{\mathbb{R}^d} |\psi(t, \mathbf{x})|^2 \, d\mathbf{x} = N(\psi_0) \quad \forall \, t \in \mathbb{R},$$

as well as the associated *Gross-Pitaevskii energy*

$$E_\Omega(\psi(t, \cdot)) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{a}{\sigma+1}|\psi|^{2\sigma+2} - \overline{\psi}(\mathbf{\Omega} \cdot \mathbf{L})\psi \, d\mathbf{x} = E_\Omega(\psi_0), \quad \forall \, t \in \mathbb{R}$$

Note that the last term within E_Ω is sign indefinite.

In the following, we shall focus on various stability and/or instability properties of solutions ψ to (22). Our first task will be to study the orbital stability of nonlinear *ground states*. These are solutions associated to (22) given by

$$\psi(t, \mathbf{x}) = e^{-i\mu t}\varphi(\mathbf{x}), \quad \mu \in \mathbb{R},$$

where φ is obtained as a constrained minimizer of the energy functional $E_\Omega(\varphi)$. In [**64, 97, 98**], the onset of vortex nucleation is linked to a symmetry breaking phenomenon for minimizers of $E_\Omega(\varphi)$, which is proved to happen for $|\mathbf{\Omega}|$ above a certain critical speed $\mathbf{\Omega}_{\text{crit}} > 0$, even in the case of radially symmetric traps V with $\omega_1 = \omega_2 = \omega_3$ (see Section 2 of Chapter 5 for more details). In our first main result below, we shall prove that under Assumption 1 and for $|\mathbf{\Omega}| < \omega$, the set of all energy minimizers is indeed orbitally stable under the time evolution of (22).

THEOREM 1.7. *Let $|\mathbf{\Omega}| < \omega = \min\{\omega_j\}$ and suppose the nonlinearity satisfies Assumption 1. Then, for any given mass $N > 0$, there exists at least one energy minimizer $\varphi \neq 0$, such that the associated set of stationary solutions*

$$\mathcal{G}_\Omega = \{\varphi \in \Sigma \ : \ \varphi \textit{ an energy minimizer with mass } N\},$$

is orbitally stable in Σ. More precisely, for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\psi_0 \in \Sigma$ satisfies

$$\inf_{\varphi} \|\psi_0(\cdot) - \varphi(\cdot)\|_\Sigma < \delta,$$

then the solution $\psi \in C(\mathbb{R}_t, \Sigma)$ to (137) with $\psi(0, \mathbf{x}) = \psi_0 \in \Sigma$ satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\varphi} \|\psi(t, \cdot) - \varphi(\cdot)\|_\Sigma < \varepsilon.$$

In turn, this will allow us to conclude several new results of orbital stability for a class of rotating solutions to nonlinear Schrödinger equations *without* the angular momentum term, see for instance Corollary 5.6 and 5.7.

The question of whether the condition $|\mathbf{\Omega}| < \omega$ is only needed for the existence of ground states, or also has a nontrivial effect in the solution of the time-dependent equation (22), then leads us to our second line of investigation. A theorem based on the Ehrenfest equations associated to (22), i.e. the quantum mechanical mean position and momentum

$$\mathbf{X}(t) := \langle \psi(t, \cdot), \mathbf{x}\psi(t, \cdot)\rangle_{L^2} \quad \text{and} \quad \mathbf{P}(t) := -i\langle \psi(t, \cdot), \nabla\psi(t, \cdot)\rangle_{L^2},$$

solve a coupled system of ordinary differential equations, shows that in the case of *non-istotropic potentials V*, a resonance-type phenomenon can occur for $|\mathbf{\Omega}| \geqslant \omega$.

THEOREM 1.8. *Suppose the nonlinearity satisfies Assumption 1 and* $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|)^\top$*. If the following condition holds*

$$\omega_1 \neq \omega_2 \text{ and } \min\{\omega_1, \omega_2\} \leqslant |\mathbf{\Omega}| \leqslant \max\{\omega_1, \omega_2\},$$

and if $\psi_0 \in \Sigma$ *is such that the associated averages* $(\mathbf{X}_0, \mathbf{P}_0) \notin \mathcal{H} = \mathcal{H}(\omega_1, \dots, \omega_d, \mathbf{\Omega})$ *a certain linear subspace of* $\mathbb{R}^{2d}$*. Then the solution* $\psi \in C(\mathbb{R}_t; \Sigma)$ *satisfies*

$$\lim_{t \to +\infty} \|\psi(t, \cdot)\|_\Sigma = +\infty, \text{ or } \lim_{t \to -\infty} \|\psi(t, \cdot)\|_\Sigma = +\infty.$$

This leads to solutions ψ whose Σ-norm is growing (forward or backward) in time with a rate that can even be exponential, depending on the choice of $\mathbf{\Omega}$ and ω_j. Physically, this can be interpreted as a manifestation of non-trapped solutions of (22) whose mass is pushed towards spatial infinity.

CHAPTER 2

Regularizing nonlinear Schrödinger equations through partial off-axis variations [1]

The outline of Chapter 2 is organized as follows. In the first section we introduce some notations and definitions, which shall hold throughout the entire thesis when applicable. Then in Section 2, we shall study the dispersive properties of $S_\varepsilon(t)$ and derive appropriate Strichartz estimates in the case of partial off-axis dispersion. These will then be used in Section 3 to prove global well-posedness of (11) in the subcritical case. The critical case, and the case of full off-axis dispersion, will be treated in Section 4. In addition, included in Section 5 we include an extension to the well-posedness theory in [5] and prove small data scattering. Lastly, we present an approximation result in Section 6 showing that smooth global solutions to (11) converge to solutions to (1) as the parameter epsilon goes to zero.

1. Basic notations and definitions

As mentioned in the Introduction, we shall denote $\mathbf{x} = (x, y) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$ with the understanding that if either $k = 0$ (no off-axis dependence) or if $k = d$ (full off-axis dependence), the variable y does not appear. Due to the ansiotropic nature of the our problem, we will make use of the mixed Lebesgue spaces denoted by $L^p(\mathbb{R}^{d-k}_x; L^q(\mathbb{R}^k_y))$, which will be shortly denoted by $L^p_x L^q_y$. These spaces are equipped with the following norms:

$$\|f\|_{L^p_x L^q_y} := \left(\int_{\mathbb{R}^{d-k}} \left(\int_{\mathbb{R}^k} |f(x,y)|^q \, dy \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}}.$$

When $p = q$, we shall denote the usual Lebesgue space $L^p(\mathbb{R}^d)$ by $L^p_{\mathbf{x}} = L^p_x L^p_y$. Note that we shall only employ these subscript notations in Chapter 2 and 3.

We denote the usual Fourier transform of a function $f = f(x, y)$ as

$$(\mathcal{F}f)(\xi, \eta) \equiv \widehat{f}(\xi, \eta) = \frac{1}{(2\pi)^{d/2}} \iint_{\mathbb{R}^d} f(x,y) e^{-i(x\cdot\xi + y\cdot\eta)} \, dx \, dy,$$

[1]This Chapter is excerpted from the following work:
[5] P. Antonelli, J. Arbunich and C. Sparber. *Regularizing nonlinear Schrödinger equations through partial off-axis variations.* SIAM J. Math. Anal. **51** (2019), no. 1, pp. 110–130.

whereas the partial Fourier transform with respect to the y-variable only will be denoted by

$$(\mathcal{F}_{y\to\eta}f)(x,\eta) \equiv \tilde{f}(x,\eta) = \frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k} f(x,y)e^{-iy\cdot\eta}\,dy.$$

Analogously, we denote the partial Fourier transform in x by $\mathcal{F}_{x\to\xi}$.

By recalling the (family of) differential operators $P_\varepsilon = 1-\varepsilon^2\Delta_y$, defined in (7) with $0<\varepsilon\leqslant 1$, we shall introduce the class of mixed Sobolev-type spaces $L^p(\mathbb{R}^{d-k}_x; H^s(\mathbb{R}^k_y))$ of order $s\in\mathbb{R}$, via the following norm

$$\|f\|_{L^p_x H^s_y} := \big\|P_1^{s/2} f\big\|_{L^p_x L^2_y} \equiv \|(1+|\eta|^2)^{s/2}\tilde{f}\|_{L^p_x L^2_\eta}.$$

Obviously, the Fourier symbol corresponding to $P_1^{1/2}$ is nothing but the well-known Japanese bracket $\langle\eta\rangle = (1+|\eta|^2)^{1/2}$ used in the definition of H^s. Incorporating the small parameter $0<\varepsilon\leqslant 1$ comes at the expense of some (possibly) ε-dependent constants. Indeed, for $s\geqslant 0$, we have the following inequalities

$$\varepsilon^s\|f\|_{H^s} \leqslant \|P_\varepsilon^{s/2} f\|_{L^2} \leqslant \|f\|_{H^s}, \tag{25}$$

$$\|f\|_{H^{-s}} \leqslant \|P_\varepsilon^{-s/2} f\|_{L^2} \leqslant \varepsilon^{-s}\|f\|_{H^{-s}}. \tag{26}$$

From now on, we shall write $a\lesssim b$ whenever there exists a universal constant $C>0$, independent of ε, such that $a\leqslant Cb$. In general this constant C may change from inequality to inequality.

Furthermore, for any time interval $I\subset\mathbb{R}$ we will also make use of the mixed space-time spaces $L^q(I_t, L^p(\mathbb{R}^{d-k}_x; H^s(\mathbb{R}^k_y)))$ briefly denoted by $L^q_t L^p_x H^s_y(I)$, or simply $L^q_t L^p_x H^s_y$, whenever the time interval is clear. These spaces are equipped with the norm

$$\|F\|_{L^q_t L^p_x H^s_y} := \left(\int_I \|F(t)\|^q_{L^p_x H^s_y}\,dt\right)^{\frac{1}{q}}. \tag{27}$$

Associated with these spaces is the following notion of Strichartz admissibility.

DEFINITION 2.1. *Let $d>k\geqslant 0$ be given. We say that the pair (q,r) is admissible if $2\leqslant r\leqslant \infty, 2\leqslant q\leqslant\infty$, and*

$$\frac{2}{q} = (d-k)\left(\frac{1}{2}-\frac{1}{r}\right) =: \delta(r)$$

where we omit the endpoint case, i.e., $(q,r)\neq(2,\frac{2(d-k)}{(d-k-2)_+})$ for $d-k\geqslant 2$.

Clearly, if $k=0$, this is just the usual admissibility condition for nonendpoint Strichartz pairs corresponding to the Schrödinger group $S_0(t)=e^{it\Delta}$ acting on $\mathbb{R}^d$.

2. Dispersive properties with partial off-axis variation

In this section, we shall derive Strichartz estimates associated to $S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$ in the case of partial off-axis variation, i.e. $d > k$. To this end we first derive a set of basic dispersion estimates associated to this linear propagator.

2.1. Dispersion estimate for $\mathbf{S}_\varepsilon(\mathbf{t})$.

Recall the notation $\delta(r) \geqslant 0$ introduced in Definition 2.1. Then we shall prove the following set of dispersive estimates.

PROPOSITION 2.2. *Let $r \in [2, \infty]$, and $t \neq 0$. Then, for any $\varepsilon > 0$, the group of L^2-unitary operators $S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$ continuously maps*

$$L^{r'}(\mathbb{R}_x^{d-k}; H^{\delta(r)}(\mathbb{R}_y^k)) \to L^r(\mathbb{R}_x^{d-k}; H^{-\delta(r)}(\mathbb{R}_y^k)), \quad \textit{for} \ \ \frac{1}{r} + \frac{1}{r'} = 1,$$

and it holds that

$$\|S_\varepsilon(t)f\|_{L_x^r H_y^{-\delta(r)}} \leqslant |4\pi t|^{-\delta(r)} \|f\|_{L_x^{r'} H_y^{\delta(r)}}. \tag{28}$$

PROOF. The estimate (28) will in itself be a consequence of the following inequality, which is more directly linked to the explicit form of our propagator $S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$:

$$\|S_\varepsilon(t)f\|_{L_x^r L_y^2} \leqslant |4\pi t|^{-\delta(r)} \|P_\varepsilon^{\delta(r)} f\|_{L_x^{r'} L_y^2}. \tag{29}$$

Indeed, if we replace f by $P_\varepsilon^{-\frac{\delta(r)}{2}} f$ in (29) and keep in mind the basic estimates (26) and (25), we obtain (28) through the string of inequalities

$$\|S_\varepsilon(t)f\|_{L_x^r H_y^{-\delta(r)}} \leqslant \|S_\varepsilon(t)P_\varepsilon^{-\frac{\delta(r)}{2}} f\|_{L_x^r L_y^2} \leqslant |4\pi t|^{-\delta(r)} \|P_\varepsilon^{\frac{\delta(r)}{2}} f\|_{L_x^{r'} L_y^2} \leqslant |4\pi t|^{-\delta(r)} \|f\|_{L_x^{r'} H_y^{\delta(r)}},$$

which also ensures the continuity of $S_\varepsilon(t)$. We also point out that there are no ε-dependent constants involved in any of these inequalities.

In order to prove (29), we first note that by density it is enough to show this for $f \in \mathcal{S}(\mathbb{R}^d)$, the space of smooth and rapidly decaying functions. Moreover, we shall argue by duality and rather prove that for $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$|\langle S_\varepsilon(t)f, g\rangle_{L_{\mathbf{x}}^2}| \leqslant |4\pi t|^{-\delta(r)} \|P_\varepsilon^{\delta(r)} f\|_{L_x^{r'} L_y^2} \|g\|_{L_x^{r'} L_y^2}. \tag{30}$$

In the trivial case $r = 2$, $\delta(r) = 0$, this estimate directly follows by Cauchy–Schwarz and the fact that $S_\varepsilon(t)$ is unitary on L^2:

$$|\langle S_\varepsilon(t)f, g\rangle_{L^2_{\mathbf{x}}}| \leqslant \|S_\varepsilon(t)f\|_{L^2_{\mathbf{x}}}\|g\|_{L^2_{\mathbf{x}}} = \|f\|_{L^2_{\mathbf{x}}}\|g\|_{L^2_{\mathbf{x}}}. \tag{31}$$

Next, we treat the case $r = \infty$, $\delta(r) = \frac{d-k}{2}$, i.e., we want to show that for $f, g \in \mathcal{S}(\mathbb{R}^d)$ it holds that

$$|\langle S_\varepsilon(t)f, g\rangle_{L^2_{\mathbf{x}}}| \leqslant |4\pi t|^{-\frac{d-k}{2}}\|P_\varepsilon^{\frac{d-k}{2}} f\|_{L^1_x L^2_y}\|g\|_{L^1_x L^2_y}. \tag{32}$$

To this end, we use Plancherel's identity to write

$$\begin{aligned}\langle S_\varepsilon(t)f, g\rangle_{L^2_{\mathbf{x}}} = \big\langle\, \widehat{(S_\varepsilon(t)f)}, \widehat{g}\,\big\rangle_{L^2_{\mathbf{x}}} &= \iint\limits_{\mathbb{R}^{d-k}\times\mathbb{R}^k} e^{-\frac{i(|\eta|^2+|\xi|^2)t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\xi,\eta)\overline{\widehat{g}(\xi,\eta)}\, d\xi\, d\eta \\ &= \int\limits_{\mathbb{R}^k} e^{-\frac{i|\eta|^2 t}{1+\varepsilon^2|\eta|^2}} \Big(\int\limits_{\mathbb{R}^{d-k}} e^{-\frac{i|\xi|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\xi,\eta)\overline{\widehat{g}(\xi,\eta)}\, d\xi\Big) d\eta.\end{aligned}$$

Here, we first compute the inner integral by writing out the partial Fourier transform in ξ on $\widehat{g}$ to obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^{d-k}} e^{-\frac{i|\xi|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\xi,\eta)\overline{\widehat{g}(\xi,\eta)}\, d\xi &= \frac{1}{(2\pi)^{\frac{d-k}{2}}} \int\limits_{\mathbb{R}^{d-k}} e^{-\frac{i|\xi|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\xi,\eta) \int\limits_{\mathbb{R}^{d-k}} e^{ix\cdot\xi}\overline{\tilde{g}(x,\eta)}\, dx d\xi \\ &= \int\limits_{\mathbb{R}^{d-k}} \overline{\tilde{g}(x,\eta)} \Big(\frac{1}{(2\pi)^{\frac{d-k}{2}}} \int\limits_{\mathbb{R}^{d-k}} e^{ix\cdot\xi} e^{-\frac{i|\xi|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\xi,\eta)\, d\xi\Big) dx \\ &= \int\limits_{\mathbb{R}^{d-k}} \overline{\tilde{g}(x,\eta)}\mathcal{F}^{-1}_{\xi\to x}\Big(e^{-\frac{i|\cdot|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\cdot,\eta)\Big)(x) dx,\end{aligned} \tag{33}$$

where we have used Fubini's theorem to change the order of integration. We now recall that for $a \in \mathbb{R}$,

$$\mathcal{F}^{-1}_{\xi\to x}\Big(e^{-\frac{i|\cdot|^2 t}{a}}\Big)(z) = \Big(\frac{a}{2it}\Big)^{\frac{d-k}{2}} e^{\frac{ia|z|^2}{4t}}.$$

By setting $a = 1 + \varepsilon^2|\eta|^2$, we can express the integrand in the last line of (33) as

$$\begin{aligned}\mathcal{F}^{-1}_{\xi\to x}\Big(e^{-\frac{i|\cdot|^2 t}{1+\varepsilon^2|\eta|^2}}\, \widehat{f}(\cdot,\eta)\Big)(x) &= \frac{1}{(2\pi)^{\frac{d-k}{2}}} \Big(\mathcal{F}^{-1}_{\xi\to x}\Big(e^{-\frac{i|\cdot|^2 t}{1+\varepsilon^2|\eta|^2}}\Big) * \tilde{f}(\cdot,\eta)\Big)(x) \\ &= \Big(\frac{1+\varepsilon^2|\eta|^2}{4\pi i t}\Big)^{\frac{d-k}{2}} \int\limits_{\mathbb{R}^{d-k}} e^{\frac{i(1+\varepsilon^2|\eta|^2)|x-z|^2}{4t}}\, \tilde{f}(z,\eta)\, dz.\end{aligned} \tag{34}$$

Now it is clear by (33) and (34) that

$$(4\pi i t)^{\frac{d-k}{2}} \langle S_\varepsilon(t)f, g\rangle_{L^2_{\mathbf{x}}} = \iiint\limits_{\mathbb{R}^k\times(\mathbb{R}^{d-k})^2} (1+\varepsilon^2|\eta|^2)^{\frac{d-k}{2}} e^{-\frac{i|\eta|^2 t}{1+\varepsilon^2|\eta|^2}} e^{\frac{i(1+\varepsilon^2|\eta|^2)|x-z|^2}{4t}}\, \tilde{f}(z,\eta)\overline{\tilde{g}(x,\eta)}\, dz\, dx\, d\eta.$$

This implies the following estimate:

$$|\langle S_\varepsilon(t)f,g\rangle_{L^2_{\mathbf{x}}}| \leqslant |4\pi t|^{-\frac{d-k}{2}} \int\limits_{(\mathbb{R}^{d-k})^2} \Big(\int\limits_{\mathbb{R}^k} (1+\varepsilon^2|\eta|^2)^{\frac{d-k}{2}} |\tilde f(z,\eta)||\tilde g(x,\eta)|\,d\eta\Big) dx\,dz.$$

A Cauchy–Schwarz inequality in η, followed by Plancherel's identity, then gives

$$\begin{aligned}|\langle S_\varepsilon(t)f,g\rangle_{L^2_{\mathbf{x}}}| &\leqslant |4\pi t|^{-\frac{d-k}{2}} \iint\limits_{(\mathbb{R}^{d-k})^2} \Big(\|\widehat{P}_\varepsilon^{\frac{d-k}{2}}\tilde f(z,\cdot)\|_{L^2_\eta}\|\tilde g(x,\cdot)\|_{L^2_\eta}\Big) dx\,dz\\ &\leqslant |4\pi t|^{-\frac{d-k}{2}} \|P_\varepsilon^{\frac{d-k}{2}} f\|_{L^1_xL^2_y}\|g\|_{L^1_xL^2_y},\end{aligned}$$

which is the desired estimate (32).

Notice that by replacing $f\mapsto |4\pi t|^{\frac{(d-k)}{2}}P_\varepsilon^{-\frac{d-k}{2}}f$ in (32), this yields that the operator

$$|4\pi t|^{\frac{d-k}{2}}S_\varepsilon(t)P_\varepsilon^{-\frac{d-k}{2}}\ :\ L^1_xL^2_y\to L^\infty_xL^2_y \quad \text{is bounded,} \tag{35}$$

with norm

$$\||4\pi t|^{\frac{d-k}{2}}S_\varepsilon(t)P_\varepsilon^{-\frac{d-k}{2}}\|\leqslant 1.$$

We have thus proved (29) in the two endpoint cases $r=2$ and $r=\infty$. The intermediate cases of (29) then follow by Stein's interpolation theorem [**99**, **100**]. To this end, we consider, for any $z\in\Omega:=\{0\leqslant \mathrm{Re}z\leqslant 1\}\subset\mathbb{C}$, the family of interpolating operators T_z given by

$$\mathcal{F}(T_zf)(\xi,\eta)=|4\pi t|^{\frac{(d-k)z}{2}}(1+\varepsilon^2|\eta|^2)^{-\frac{d-k}{2}z}e^{-it(1+\varepsilon^2|\eta|^2)^{-1}(|\xi|^2+|\eta|^2)}\widehat{f}(\xi,\eta).$$

Clearly, for $z=0$, this is nothing but the Fourier transform of $S_\varepsilon(t)$, which we know to be bounded $L^2\to L^2$ in view of (31). For $z=1$, we obtain the second endpoint case given by (35). In addition, it is straightforward to check that $\{T_z\}_{z\in\Omega}$ is an admissible family of linear operators satisfying the hypotheses of Theorem V.4.1 in [**100**]. The theorem then requires us to bound T_z at the edges of the strip Ω: For $\mu\in\mathbb{R}$, the following estimate for $z=0+i\mu$ uses (31) and Plancherel in y, to give

$$\begin{aligned}|\langle T_{0+i\mu}f,g\rangle_{L^2_{\mathbf{x}}}| &= |\langle \tilde S_\varepsilon(t)\big((|4\pi t|^{-1}\widehat{P}_\varepsilon)^{\frac{-i(d-k)\mu}{2}}\tilde f\big),\tilde g\rangle_{L^2}|\\ &= \|e^{\frac{-i(d-k)\mu}{2}\ln(|4\pi t|^{-1}\widehat{P}_\varepsilon)}\tilde f\|_{L^2_xL^2_\eta}\|g\|_{L^2}=\|f\|_{L^2_xL^2_y}\|g\|_{L^2_xL^2_y}.\end{aligned}$$

The estimate for $z=1+i\mu$ follows similarly, but now using (32), so that

$$\begin{aligned}|\langle T_{1+i\mu}f,g\rangle_{L^2_{\mathbf{x}}}| &= |4\pi t|^{\frac{(d-k)}{2}}|\langle \tilde S_\varepsilon(t)\big((|4\pi t|^{-1}\widehat{P}_\varepsilon)^{-i\frac{(d-k)\mu}{2}}\widehat{P}_\varepsilon^{-\frac{(d-k)}{2}}\tilde f\big),\tilde g\rangle_{L^2}|\\ &\leqslant \|\widehat{P}_\varepsilon^{\frac{(d-k)}{2}}e^{\frac{-i(d-k)\mu}{2}\ln(|4\pi t|^{-1}\widehat{P}_\varepsilon)}\widehat{P}_\varepsilon^{-\frac{(d-k)}{2}}\tilde f\|_{L^1_xL^2_\eta}\|\tilde g\|_{L^1_xL^2_\eta}\leqslant \|f\|_{L^1_xL^2_y}\|g\|_{L^1_xL^2_y}.\end{aligned}$$

Noting that the constants produce no growth in $z \in \mathbb{C}$, then the quoted version of Stein interpolation in [100] implies for $0 \leqslant \theta = 1 - \frac{2}{r} \leqslant 1$ and $r \in [2, \infty]$ the following estimate

$$|4\pi t|^{\delta(r)} \|P_\varepsilon^{-\delta(r)} f\|_{L^r_x L^2_y} = \|T_\theta f\|_{L^r_x L^2_y} \leqslant \|f\|_{L^{r'}_x L^2_y},$$

which by replacing f by $P_\varepsilon^{\delta(r)} f$ and dividing the above inequality by $|4\pi t|^{\delta(r)}$ gives (30). Again, we note that there are no ε-dependent constants arising from this interpolation step. Moreover, since the proof of this interpolation theorem exploits a density argument using simple functions, the result directly applies also to the mixed spaces $L^r_x L^2_y$ under consideration. □

REMARK 2.3. *Note that, as $\varepsilon \to 0$, the estimate (29) converges to*

$$\big\|S_0(t) f\big\|_{L^r_x L^2_y} \leqslant |4\pi t|^{-(d-k)\left(\frac{1}{2} - \frac{1}{r}\right)} \big\|f\big\|_{L^{r'}_x L^2_y},$$

which is similar to the usual dispersion estimate for the Schrödinger group in dimension $d - k \in \mathbb{N}$ and again reflects the fact that we don't obtain dispersion in the y-coordinates when $\varepsilon > 0$. Deriving estimate (28) from (29) has the advantage that we can use standard Sobolev spaces H^s, independent of ε, to measure the regularity in y (instead of employing the operator P_ε). The price to pay is that (28) no longer converges to the classical dispersion estimate in the limit $\varepsilon \to 0$ (except in the case $r = 2$ for which $\delta(r) = 0$). But since in this thesis we are not concerned with the limit $\varepsilon \to 0$ involving directly these estimates, then we shall ignore this issue in the following and base our Strichartz estimates on (28).

2.2. Strichartz estimates. Exploiting the dispersion estimate (28), we shall now prove space-time Strichartz estimates associated to $S_\varepsilon(t)$. These estimates also follow from abstract arguments as in [**10, 54, 67**]. For the sake of concreteness and due to our somewhat unusual function spaces, we shall give their proof in the nonendpoint case.

PROPOSITION 2.4 (Strichartz estimates). *Let $S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$ and $(q, r), (\gamma, \rho)$ be two arbitrary admissible Strichartz pairs with $0 < \delta(r), \delta(\rho) < 1$. Then for any time interval I, there exist constants $C_1, C_2 > 0$, independent of ε and I, such that*

$$\|S_\varepsilon(\cdot) f\|_{L^q_t L^r_x H^{-\delta(r)}_y} \leqslant C_1 \|f\|_{L^2_{\mathbf{x}}}, \tag{36}$$

as well as

$$\left\|\int_0^t S_\varepsilon(\cdot - s) F(s)\, ds\right\|_{L^q_t L^r_x H^{-\delta(r)}_y} \leqslant C_2 \|F\|_{L^{\gamma'}_t L^{\rho'}_x H^{\delta(\rho)}_y}. \tag{37}$$

REMARK 2.5. *The case of endpoint Strichartz estimates, i.e.,* $(q,r) = \big(2, \frac{2(d-k)}{(d-k-2)_+}\big)$ *for* $d-k \geqslant 2$, *in principle could also be dealt with as in* [67], *but since we never make use of it in our analysis we do not pursue this issue any further.*

PROOF. We start by first noticing that (36) is equivalent to saying that the map $f \mapsto S_\varepsilon(t)f$ is bounded as an operator $L^2_{\mathbf{x}} \to L^q_t L^r_x H^{-\delta(r)}_y$. Let us define the operator $T_\varepsilon : L^{q'}_t L^{r'}_x H^{\delta(r)}_y \to L^2_{\mathbf{x}}$ by

$$T_\varepsilon F = \int_{\mathbb{R}} S_\varepsilon(-s)F(s)\,ds$$

and note that its formal adjoint T^*_ε is the map $f \mapsto S_\varepsilon(t)f$. Next, we shall show that

$$T^*_\varepsilon T_\varepsilon F(t) = \int_{\mathbb{R}} S_\varepsilon(t-s)F(s)\,ds$$

is bounded as an operator $L^{q'}_t L^{r'}_x H^{\delta(r)}_y \to L^q_t L^r_x H^{-\delta(r)}_y$.

By the generalized Minkowski's inequality we have

$$\left\| \int_{\mathbb{R}} S_\varepsilon(\cdot - s)F(s)\,ds \right\|_{L^q_t L^r_x H^{-\delta(r)}_y} \leqslant \left\| \int_{\mathbb{R}} \|S_\varepsilon(\cdot - s)F(s)\|_{L^r_x H^{-\delta(r)}_y}\,ds \right\|_{L^q_t},$$

and applying the dispersion estimate (28), it follows that

$$\|S_\varepsilon(t-s)F(s)\|_{L^r_x H^{-\delta(r)}_y} \leqslant |4\pi(t-s)|^{-\delta(r)} \|F(s)\|_{L^{r'}_x H^{\delta(r)}_y}.$$

Hence recalling that $\delta(r) = \frac{2}{q} < 1$, we see it is then possible to apply the Hardy–Littlewood–Sobolev inequality in order to obtain

$$\left\| \int_{\mathbb{R}} S_\varepsilon(\cdot - s)F(s)\,ds \right\|_{L^q_t L^r_x H^{-\delta(r)}_y} \leqslant \left\| \int_{\mathbb{R}} |4\pi(\cdot - s)|^{-\delta(r)} \|F(s)\|_{L^{r'}_x H^{\delta(r)}_y}\,ds \right\|_{L^q_t} \leqslant C\|F\|_{L^{q'}_t L^{r'}_x H^{\delta(r)}_y}.$$

We thus have proven that the operator $T^*_\varepsilon T_\varepsilon : L^{q'}_t L^{r'}_x H^{\delta(r)}_y \to L^q_t L^r_x H^{-\delta(r)}_y$ is bounded. A standard functional analysis result for operators on Banach spaces (see [10]) states that

$$\|T_\varepsilon\|^2_{\mathcal{L}(L^{q'}_t L^{r'}_x H^{\delta(r)}_y; L^2_{\mathbf{x}})} = \|T^*_\varepsilon\|^2_{\mathcal{L}(L^2_{\mathbf{x}}: L^q_t L^r_x H^{-\delta(r)}_y)} = \|T^*_\varepsilon T_\varepsilon\|_{\mathcal{L}(L^{q'}_t L^{r'}_x H^{\delta(r)}_y; L^q_t L^r_x H^{-\delta(r)}_y)}.$$

This consequently implies that both

$$T_\varepsilon : L^{q'}_t L^{r'}_x H^{\delta(r)}_y \to L^2_{\mathbf{x}} \quad \text{and} \quad T^*_\varepsilon : L^2_{\mathbf{x}} \to L^q_t L^r_x H^{-\delta(r)}_y$$

are bounded with norms independent of epsilon. In particular, the estimate (36) is proved. Furthermore, we note that this holds for any nonendpoint admissible pair (q,r).

Now, choose any arbitrary (nonendpoint) admissible pairs (γ,ρ) and (q,r) such that

$$T_\varepsilon : L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)} \to L^2_{\mathbf{x}} \quad \text{and} \quad T_\varepsilon^* : L^2_{\mathbf{x}} \to L_t^q L_x^r H_y^{-\delta(r)}.$$

By combining the estimates for the operators T_ε, T_ε^*, we then infer that

$$T_\varepsilon^* T_\varepsilon : L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)} \to L_t^q L_x^r H_y^{-\delta(r)}$$

is bounded, i.e.,

$$\left\| \int_{\mathbb{R}} S_\varepsilon(\cdot - s) F(s)\, ds \right\|_{L_t^q L_x^r H_y^{-\delta(r)}} \leqslant C \|F\|_{L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)}},$$

for any arbitrary $(q,r),(\gamma,\rho)$. We can then invoke Theorem 1.2 from the paper [29] by Christ and Kiselev to conclude the retarded estimate

$$\left\| \int_{s<t} S_\varepsilon(\cdot - s) F(s)\, ds \right\|_{L_t^q L_x^r H_y^{-\delta(r)}} \leqslant C \|F\|_{L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)}}.$$

In summary, this proves the desired result. ☐

3. The Cauchy problem for partial off-axis variation in the subcritical case

In this section we shall give the proof of Theorem 1.1 by proving a global L^2-based well-posedness result for (11) with subcritical nonlinearities. In a second step we shall establish the additional H^1-regularity of the solution.

3.1. Well-posedness in terms of v.

We rewrite (11) using Duhamel's formulation, i.e.,

$$v(t) = S_\varepsilon(t) v_0 + i \int_0^t S_\varepsilon(t-s) P_\varepsilon^{-1/2} (|P_\varepsilon^{-1/2} v|^{2\sigma} P_\varepsilon^{-1/2} v)(s)\, ds =: \Phi(v)(t). \tag{38}$$

For the sake of brevity, we shall also write

$$\Phi(v)(t) = S_\varepsilon(t) v_0 + \mathcal{N}(v)(t)$$

and denote

$$\mathcal{N}(v)(t) := i \int_0^t S_\varepsilon(t-s) P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v(s))\, ds, \tag{39}$$

where $g(z) = |z|^{2\sigma} z$ with $\sigma > 0$. Of course, the basic idea is to prove that $v \mapsto \Phi(v)$ is a contraction mapping in a suitable Banach space. To this end, the following lemma is key.

LEMMA 2.6. *Let $d-k>0$. Fix $T>0$ and choose the admissible pair*

$$(\gamma,\rho)=\left(\frac{4(\sigma+1)}{(d-k)\sigma},2(\sigma+1)\right).$$

Then, in the space-time slab $\mathbb{R}^d\times[0,T]$ the inequality

$$\begin{aligned}&\|\mathcal{N}(v)-\mathcal{N}(v')\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}}\\ &\qquad\lesssim \varepsilon^{-2(\sigma+1)}T^{1-\frac{(d-k)\sigma}{2}}\left(\|v\|^{2\sigma}_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}}+\|v'\|^{2\sigma}_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}}\right)\|v-v'\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}},\end{aligned}$$

holds, provided $0<\sigma\leqslant\frac{2}{(d-2)_+}$.

The case $k=0$ is classical and thus we will only give the proof for $d>k>0$.

PROOF. We first note that for our pair (γ,ρ) to be non-endpoint admissible for $d-k\geqslant 2$, we require that $\gamma>2$, which in turn is equivalent to $\sigma<\frac{2}{(d-k-2)_+}$. However, this condition will always be fulfilled since

$$\sigma\leqslant\frac{2}{(d-2)_+}<\frac{2}{(d-k-2)_+}.$$

As a consequence, we also have that $\delta(\rho)=\frac{(d-k)\sigma}{2(\sigma+1)}<1$.

Now, as a first step we apply the Strichartz estimate (37) and note that

$$\begin{aligned}\|\mathcal{N}(v)-\mathcal{N}(v')\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}}&\leqslant C_2\|P_\varepsilon^{-1/2}(g(P_\varepsilon^{-1/2}v)-g(P_\varepsilon^{-1/2}v'))\|_{L_t^{\gamma'}L_x^{\rho'}H_y^{\delta(\rho)}}\\ &\leqslant \varepsilon^{-1}C_2\|g(P_\varepsilon^{-1/2}v)-g(P_\varepsilon^{-1/2}v')\|_{L_t^{\gamma'}L_x^{\rho'}H_y^{-(1-\delta(\rho))}},\end{aligned}$$

where we have also used the scaling (26) to obtain the factor ε^{-1}. Next, by a Sobolev embedding we have that $H^s(\mathbb{R}^k)\hookrightarrow L^\rho(\mathbb{R}^k)$, where

$$s=k\left(\frac{1}{2}-\frac{1}{2(\sigma+1)}\right)=\frac{k\sigma}{2(\sigma+1)}\in\left(0,\frac{k}{2}\right).$$

In turn, this also implies the dual embedding $L^{\rho'}(\mathbb{R}^k)\hookrightarrow H^{-s}(\mathbb{R}^k)$. Now, if we impose that

$$1\geqslant s+\delta(\rho)=\frac{d\sigma}{2(\sigma+1)},$$

which is so whenever $\sigma\leqslant\frac{2}{(d-2)_+}$, then $H^{-s}(\mathbb{R}^k)\hookrightarrow H^{-(1-\delta(\rho))}(\mathbb{R}^k)$. Together these allow us to estimate

$$\begin{aligned}\|g(P_\varepsilon^{-1/2}v)-g(P_\varepsilon^{-1/2}v')\|_{H_y^{-(1-\delta(\rho))}}&\leqslant\|g(P_\varepsilon^{-1/2}v)-g(P_\varepsilon^{-1/2}v')\|_{H_y^{-s}}\\ &\leqslant C_\sigma\|(|P_\varepsilon^{-1/2}v|^{2\sigma}+|P_\varepsilon^{-1/2}v'|^{2\sigma})P_\varepsilon^{-1/2}(v-v')\|_{L_y^{\rho'}}=(*),\end{aligned}$$

where we have also used that for all $z, w \in \mathbb{C}$,

$$|g(z) - g(w)| \leqslant C_\sigma(|z|^{2\sigma} + |w|^{2\sigma})|z-w|.$$

Now, recall that $\rho = 2(\sigma+1)$ and hence $\frac{1}{\rho'} = \frac{2\sigma}{\rho} + \frac{1}{\rho}$. Thus, by first applying Hölder's inequality and using (26), we obtain

$$\begin{aligned}(*) &\lesssim (\|P_\varepsilon^{-1/2}v\|_{L_y^\rho}^{2\sigma} + \|P_\varepsilon^{-1/2}v'\|_{L_y^\rho}^{2\sigma})\|P_\varepsilon^{-1/2}(v-v')\|_{L_y^\rho} \\ &\lesssim \varepsilon^{-(2\sigma+1)}(\|v\|_{H_y^{-(1-s)}}^{2\sigma} + \|v'\|_{H_y^{-(1-s)}}^{2\sigma})\|v-v'\|_{H_y^{-(1-s)}} \\ &\lesssim \varepsilon^{-(2\sigma+1)}(\|v\|_{H_y^{-\delta(\rho)}}^{2\sigma} + \|v'\|_{H_y^{-\delta(\rho)}}^{2\sigma})\|v-v'\|_{H_y^{-\delta(\rho)}},\end{aligned}$$

where the last inequality follows from $H^{-\delta(\rho)}(\mathbb{R}^k) \hookrightarrow H^{-(1-s)}(\mathbb{R}^k)$, by the same arguments as before. Employing Hölder's inequality once more in x, we consequently infer

$$\|g(P_\varepsilon^{-1/2}v) - g(P_\varepsilon^{-1/2}v')\|_{L_x^{\rho'}H_y^{-(1-\delta(\rho))}} \lesssim \varepsilon^{-(2\sigma+1)}(\|v\|_{L_x^\rho H_y^{-\delta(\rho)}}^{2\sigma} + \|v'\|_{L_x^\rho H_y^{-\delta(\rho)}}^{2\sigma})\|v-v'\|_{L_x^\rho H_y^{-\delta(\rho)}}.$$

From here we compute that

$$\frac{1}{\gamma'} = 1 - \frac{(d-k)\sigma}{2} + \frac{2\sigma}{\gamma} + \frac{1}{\gamma}.$$

Thus, taking the $L^{\gamma'}$ norm in t and applying Hölder's inequality yields the result of the lemma. $\square$

Using Lemma 2.6, we are now able to prove global well-posedness for (11) in the subcritical case. In doing so, we will require a positive exponent

$$\alpha \equiv 1 - \frac{(d-k)\sigma}{2}$$

of T^α in the estimate obtained in Lemma 2.6, i.e., we require $\sigma < \frac{2}{d-k}$. Since Lemma 2.6 holds for $\sigma \leqslant \frac{2}{(d-2)_+}$, we need to distinguish the cases $k \leqslant 2$ and $k > 2$ in the following.

One notices immediately that for $k \leqslant 2$, we have that $\frac{2}{d-k} \leqslant \frac{2}{(d-2)_+}$, which in turn implies in this case, that we require the stronger assumption $\sigma < \frac{2}{d-k}$ to ensure $\alpha > 0$. However, for $k > 2$ (and thus $d > 3$), it holds that

$$\frac{2}{d-2} < \frac{2}{d-k} < \frac{2}{(d-k-2)_+},$$

and hence no new restriction arises. We also note that for $k > 2$, the exponent of T^α is positive and is L^2-subcritical in the sense that when $\sigma = \frac{2}{d-2}$ then

$$\alpha = 1 - \frac{(d-k)\sigma}{2} = \frac{k-2}{d-2} > 0.$$

With this in mind, we can now prove the following result.

PROPOSITION 2.7. *Let $d > k \geqslant 0$ and*

- *either $k \leqslant 2$ and $0 \leqslant \sigma < \frac{2}{d-k}$*
- *or $k > 2$ and $0 \leqslant \sigma \leqslant \frac{2}{d-2}$.*

Then for any $v_0 \in L^2(\mathbb{R}^d)$, there exists a unique global solution to (11)

$$v \in C(\mathbb{R}_t, L^2(\mathbb{R}^d)) \cap L^q_{\mathrm{loc}}(\mathbb{R}_t; L^r(\mathbb{R}^{d-k}_x; H^{-\delta(r)}(\mathbb{R}^k_y)))$$

for any (nonendpoint) admissible pair (q, r). Moreover, v depends continuously on the initial data and satisfies

$$\|v(t,\cdot)\|_{L^2_{\mathbf{x}}} = \|v_0\|_{L^2_{\mathbf{x}}} \quad \forall\, t \in \mathbb{R}.$$

By identifying $v = P_\varepsilon^{1/2} u$, this directly yields a global-in-time solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^{d-k}_x; H^1(\mathbb{R}^k_y)))$ to (6) and thus proves Theorem 1.1. Note that here continuous dependence on the initial data precisely means that for $T > 0$ the map $v_0 \mapsto v|_{[-T,T]}$ is continuous as a map

$$L^2(\mathbb{R}^d) \to C([-T,T], L^2(\mathbb{R}^d)) \cap L^q([-T,T]; L^r(\mathbb{R}^{d-k}_x; H^{-\delta(r)}(\mathbb{R}^k_y))).$$

PROOF. We shall prove Proposition 2.7 in several steps.

Step 1 (Existence): Fix the admissible pair $(\gamma, \rho) = \Big(\frac{4(\sigma+1)}{(d-k)\sigma}, 2(\sigma+1)\Big)$. Let $M, T > 0$ to be determined later and denote $I = [0, T]$, and set

$$X_{T,M} = \{v \in L^\infty_t L^2_{\mathbf{x}}(I) \cap L^q_t L^r_x H^{-\delta(r)}_y(I) : \|v\|_{L^\infty_t L^2_{\mathbf{x}}} + \|v\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y} \leqslant M\}.$$

We note that $X_{T,M}$ is a complete metric space equipped with the distance

$$d(v,w) = \|v - w\|_{L^\infty_t L^2_{\mathbf{x}}} + \|v - w\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y}.$$

Let $v \in X_{T,M}$. Then the Strichartz estimates obtained in Proposition 2.4 together with Lemma 2.6 imply that

$$\begin{aligned}\|\Phi(v)\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y} &\leqslant \|S_\varepsilon(\cdot) v_0\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y} + \|\mathcal{N}(v)\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y} \\ &\leqslant C_{\sigma,\varepsilon}\Big(\|v_0\|_{L^2_{\mathbf{x}}} + T^{1-\frac{(d-k)\sigma}{2}} M^{2\sigma+1}\Big),\end{aligned}$$

as well as

$$\|\Phi(v)\|_{L_t^\infty L_{\mathbf{x}}^2} \leqslant \|v_0\|_{L_{\mathbf{x}}^2} + C_2\|P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v)\|_{L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)}}$$
$$\leqslant C_{\sigma,\varepsilon}\Big(\|v_0\|_{L_{\mathbf{x}}^2} + T^{1-\frac{(d-k)\sigma}{2}} M^{2\sigma+1}\Big).$$

Together, these yield

$$\|\Phi(v)\|_{L_t^\infty L_{\mathbf{x}}^2} + \|\Phi(v)\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}} \leqslant 2C_{\sigma,\varepsilon}\Big(\|v_0\|_{L_{\mathbf{x}}^2} + T^{1-\frac{(d-k)\sigma}{2}} M^{2\sigma+1}\Big).$$

We now choose M such that

$$3M = 8C_{\sigma,\varepsilon}\|v_0\|_{L_{\mathbf{x}}^2}$$

and choose $T > 0$ such that

$$2C_{\sigma,\varepsilon} T^{1-\frac{(d-k)\sigma}{2}} M^{2\sigma+1} \leqslant \frac{M}{4}. \tag{40}$$

Then it follows that $\Phi(v) \in X_{T,M}$ for all $v \in X_{T,M}$ so that $\Phi(X_{T,M}) \subset X_{T,M}$. Now, let $v, w \in X_{T,M}$. Then by Lemma (2.6) and using (40) we have

$$\|\mathcal{N}(v) - \mathcal{N}(w)\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}} \leqslant 2C_{\sigma,\varepsilon} M^{2\sigma} T^{1-\frac{(d-k)\sigma}{2}} \|v-w\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}} \leqslant \frac{1}{4}\|v-w\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}}, \tag{41}$$

which together with the same estimate for the $L_t^\infty H^1$-norm gives

$$d(\Phi(v), \Phi(w)) \leqslant \frac{1}{2} d(v, w), \quad \forall v, w \in X_{T,M}.$$

Thus Φ is a contraction map on $X_{T,M}$ and Banach's fixed point theorem yields the existence of a unique fixed point $v \in X_{T,M}$. Furthermore, since the solution v satisfies the integral equation (38), we infer continuity in time, i.e., $v \in C(I; L^2(\mathbb{R}^d))$.

Moreover, if $v \in X_{T,M}$, then $v \in L_t^q L_x^r H_y^{-\delta(r)}(I)$ for any admissible pair (q, r), since by our Strichartz estimates

$$\|v\|_{L_t^q L_x^r H_y^{-\delta(r)}} \equiv \|\Phi(v)\|_{L_t^q L_x^r H_y^{-\delta(r)}} \leqslant C_1\|v_0\|_{L_{\mathbf{x}}^2} + C_2\|P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v)\|_{L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)}},$$

which is estimated as in the proof of Lemma 2.6.

Step 2 (Uniqueness): Let $I = [0, T]$ and $v, w \in C(I; L_{\mathbf{x}}^2) \cap L_t^q L_x^r H_y^{-\delta(r)}(I)$ be two solutions to (38) with $v_0 = w_0 = \varphi$. Then as in Step 1, we have $v, w \in X_{T,M}$ with $3M = 8C_{\sigma,\varepsilon}\|\varphi\|_{L_{\mathbf{x}}^2}$ and T given by

(40). Since the difference of v and w is given by

$$(v-w)(t) = \mathcal{N}(v)(t) - \mathcal{N}(w)(t),$$

then we can apply (41) from Step 1 on the interval I to obtain

$$\|v-w\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I)} \leq \frac{1}{4}\|v-w\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I)}.$$

From this we conclude (local) uniqueness

$$\|v-w\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I)} = 0,$$

i.e., $v = w$ on $I = [0,T]$.

In addition, the solution depends continuously on the initial data, as can be seen by taking two solutions $v, \tilde{v}$ on a common time interval $I_c = \min\{I, \tilde{I}\}$. Then by what was done above, we have that $v, \tilde{v} \in X_{T,M}$ with $3M = 8\max\{\|v_0\|_{L^2_{\mathbf{x}}}, \|\tilde{v}_0\|_{L^2_{\mathbf{x}}}\}$ and $T = |I_c|$ satisfying (40) so that

$$d(v,\tilde{v}) \leqslant \|v_0 - \tilde{v}_0\|_{L^2_{\mathbf{x}}} + \frac{1}{2}d(v,\tilde{v}),$$

which proves the continuous dependence on the initial data, after extending the argument to the interval I_c.

Step 3 (Global existence): In order to show that the solution obtained in Step 1 indeed exists for all times $t \in \mathbb{R}$, let

$$T_{\max} = \sup\{T > 0 : \text{there exists a solution } v(t,\cdot) \text{ on } [0,T)\}.$$

We claim that

$$\text{if} \quad T_{\max} < +\infty, \quad \text{then} \quad \lim_{t\to T_{\max}} \|v(t)\|_{L^2_{\mathbf{x}}} = +\infty.$$

Suppose, by contradiction, that $T_{\max} < \infty$ and that there exists a sequence $t_j \to T_{\max}$ such that $\|v(t_j)\|_{L^2_{\mathbf{x}}} \leqslant M$. Now choose some integer J such that t_J is close to $T_{\max}$ where by assumption $\|v(t_J)\|_{L^2_{\mathbf{x}}} \leqslant M$. But by Step 1, using the initial data $v(t_J)$ we can extend our solution to the interval $[t_J, t_J + T]$ where we now choose t_J such that $t_J + T > T_{\max}$. This gives a contradiction to the definition of $T_{\max}$.

Next, we shall prove that the L^2-norm of v is conserved along the time-evolution. To this end, we adapt an elegant argument given in [90], which has the advantage that it does not require an approximation procedure using a sequence of sufficiently smooth solutions (as is classically done, see

[25]). First note that by Step 1 we have $v \in C([0,T]; L^2(\mathbb{R}^d))$ for any $T < T_{\max}$. We then rewrite Duhamel's formula (38), using the continuity of the semigroup S_ε to propagate backward in time

$$S_\varepsilon(-t)v(t) = v_0 + S_\varepsilon(-t)\mathcal{N}(v)(t). \tag{42}$$

The fact that $S_\varepsilon(\cdot)$ is unitary in L^2 implies $\|v(t)\|_{L^2_{\mathbf{x}}} = \|S_\varepsilon(-t)v(t)\|_{L^2_{\mathbf{x}}}$. The latter of which can be expressed using the above identity to obtain

$$\begin{aligned}\|v(t)\|_{L^2_{\mathbf{x}}}^2 &= \|v_0\|_{L^2_{\mathbf{x}}}^2 + 2\mathrm{Re}\left\langle S_\varepsilon(-t)\mathcal{N}(v)(t), v_0\right\rangle_{L^2_{\mathbf{x}}} + \|S_\varepsilon(-t)\mathcal{N}(v)(t)\|_{L^2_{\mathbf{x}}}^2 \\ &=: \|v_0\|_{L^2_{\mathbf{x}}}^2 + \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

We want to show that $\mathcal{I}_1 + \mathcal{I}_2 = 0$. In view of (39) we can rewrite

$$\begin{aligned}\mathcal{I}_1 &= -2\mathrm{Im}\left\langle \int_0^t S_\varepsilon(-s)P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2}v)(s)\, ds, v_0\right\rangle_{L^2} \\ &= -2\mathrm{Im}\int_0^t \left\langle P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2}v)(s), S_\varepsilon(s)v_0\right\rangle_{L^2} ds.\end{aligned}$$

By duality in y and Hölder's inequality in both x and t we find that this quantity is indeed finite

$$|\mathcal{I}_1| \leqslant 2\|P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2}v)\|_{L_t^{\gamma'}L_x^{\rho'}H_y^{\delta(\rho)}}\|S_\varepsilon(s)v_0\|_{L_t^{\gamma}L_x^{\rho}H_y^{-\delta(\rho)}} < \infty.$$

Denoting for simplicity $G_\varepsilon(\cdot) = P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2}v)(\cdot)$, we perform the following computation:

$$\begin{aligned}\mathcal{I}_2 &\equiv \left\langle \int_0^t S_\varepsilon(-s)G_\varepsilon(s)\, ds, \int_0^t S_\varepsilon(-s')G_\varepsilon(s')\, ds'\right\rangle_{L^2_{\mathbf{x}}} \\ &= \int_0^t \left\langle S_\varepsilon(-s)G_\varepsilon(s), \Big(\int_0^s + \int_s^t\Big) S_\varepsilon(-s')G_\varepsilon(s')\, ds'\right\rangle_{L^2_{\mathbf{x}}} ds \\ &= \int_0^t \left\langle G_\varepsilon(s), \int_0^s S_\varepsilon(s-s')G_\varepsilon(s')\, ds'\right\rangle_{L^2_{\mathbf{x}}} ds + \int_0^t\int_s^t \left\langle S_\varepsilon(s'-s)G_\varepsilon(s), G_\varepsilon(s')\right\rangle_{L^2_{\mathbf{x}}} ds'\, ds \\ &= \int_0^t \left\langle G_\varepsilon(s), -i\mathcal{N}(v)(s)\right\rangle_{L^2_{\mathbf{x}}} ds + \int_0^t \left\langle \int_0^{s'} S_\varepsilon(s'-s)G_\varepsilon(s)\, ds, G_\varepsilon(s')\right\rangle_{L^2_{\mathbf{x}}} ds' \\ &= 2\mathrm{Re}\int_0^t \left\langle G_\varepsilon(s), -i\mathcal{N}(v)(s)\right\rangle_{L^2_{\mathbf{x}}} ds.\end{aligned}$$

Using the integral formulation (38), we can express $-i\mathcal{N}(v)(s)$ and write

$$\mathcal{I}_2 = 2\mathrm{Re}\Big(\int_0^t \left\langle G_\varepsilon(s), iS_\varepsilon(s)v_0\right\rangle_{L^2_{\mathbf{x}}} ds + \int_0^t \left\langle G_\varepsilon(s), -iv(s)\right\rangle_{L^2_{\mathbf{x}}} ds\Big). \tag{43}$$

Here we note that the particular form of our nonlinearity implies

$$\mathrm{Re}\left\langle G_\varepsilon(\cdot), -iv(\cdot)\right\rangle_{L^2_{\mathbf{x}}} = \mathrm{Im}\left\langle g(P_\varepsilon^{-1/2}v)(\cdot), P_\varepsilon^{-1/2}v(\cdot)\right\rangle_{L^2_{\mathbf{x}}} = \mathrm{Im}\,\|P_\varepsilon^{-1/2}v(\cdot)\|_{L_{\mathbf{x}}^{2\sigma+2}}^{2\sigma+2} = 0,$$

and thus the second term on the right-hand side of (43) simply vanishes. In summary, we find

$$\mathcal{I}_2 = 2\mathrm{Re}\int_0^t \left\langle G_\varepsilon(s), iS_\varepsilon(s)v_0\right\rangle_{L^2_{\mathbf{x}}} ds = 2\mathrm{Im}\int_0^t \left\langle S_\varepsilon(-s)G_\varepsilon(s)\, ds, v_0\right\rangle_{L^2_{\mathbf{x}}} \equiv -\mathcal{I}_1,$$

which proves that

$$\|v(t)\|_{L^2_{\mathbf{x}}} = \|v_0\|_{L^2_{\mathbf{x}}} \quad \forall t \in [0,T].$$

This conservation law allows us to reapply Step 1 as many times as we wish, thereby preserving the length of the maximal interval in each iteration, and yielding $T_{\max} = +\infty$. Since the equation is time-reversible modulo complex conjugation, this yields a global solution for all $t \in \mathbb{R}$. □

3.2. Higher order regularity. In this subsection, we are going to prove that the global-in-time L^2-solution obtained in Proposition 2.7 enjoys persistence of regularity. Namely, if the initial datum $v_0 \in H^1$, then the corresponding solution $v(t,\cdot)$ remains in H^1 for all times $t \in \mathbb{R}$. We will prove this property by exploiting the Strichartz estimates stated in Proposition 2.4 and the global well-posedness result in L^2. Similar arguments can be used to obtain a solution $v(t,\cdot) \in H^s$, $s \geqslant 1$, provided the nonlinearity is sufficiently smooth, i.e., $\sigma \in \mathbb{N}$.

PROPOSITION 2.8. *Let $v \in C(\mathbb{R}_t, L^2(\mathbb{R}^d)) \cap L^q_{\mathrm{loc}}(\mathbb{R}_t; L^r(\mathbb{R}^{d-k}_x; H^{-\delta(r)}(\mathbb{R}^k_y)))$ be the solution obtained in Proposition 2.7 with initial data $v_0 \in L^2(\mathbb{R}^d)$. If, in addition, $v_0 \in H^1(\mathbb{R}^d)$, then $v \in C(\mathbb{R}_t; H^1(\mathbb{R}^d))$.*

PROOF. Let us fix a $0 < T < \infty$. We are going to show that

$$\|\nabla v\|_{L^\infty_t L^2_{\mathbf{x}}([0,T])} \leqslant K(T, \|\nabla v_0\|_{L^2_{\mathbf{x}}}). \tag{44}$$

Having in mind the conservation property of the L^2-norm of v, this estimate is sufficient to conclude the desired result. To obtain (44), we first recall from Proposition 2.7 that

$$\|v\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y([0,T])} \leqslant C(T, \|v_0\|_{L^2_{\mathbf{x}}}) =: C_T,$$

where $(\gamma,\rho) = \left(\frac{4(\sigma+1)}{(d-k)\sigma}, 2(\sigma+1)\right)$ is the admissible pair used in Lemma 2.6. Let $\lambda > 0$ be a small parameter to be chosen later on. We then divide $[0,T]$ into $N = N(\lambda, C_T)$ subintervals, i.e., $[0,T] = \cup_{j=1}^N I_j$, where $I_j = [t_{j-1}, t_j]$ and $0 = t_0 < t_1 < \ldots < t_N = T$, such that

$$\|v\|_{L^\gamma_t L^\rho_x H^{-\delta(\rho)}_y(I_j)} \leqslant \lambda, \quad j = 1, \ldots, N. \tag{45}$$

First we estimate the gradient of (39) by a similar strategy as in Lemma 2.6 with $v' = 0$. By applying the Strichartz estimate (37) and the appropriate embeddings in y gives

$$\|\nabla \mathcal{N}(v)\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)} \leqslant \varepsilon^{-1} C_2 \|\nabla g(P_\varepsilon^{1/2} v)\|_{L_t^{\gamma'} L_x^{\rho'} L_y^{\rho'}}.$$

Since the nonlinearity is smooth, this allows us to estimate in y as follows:

$$\begin{aligned} \|\nabla g(P_\varepsilon^{1/2} v)\|_{L_y^{\rho'}} &\leqslant (2\sigma+1) \|P_\varepsilon^{-1/2} v\|_{L_y^\rho}^{2\sigma} \|P_\varepsilon^{-1/2} \nabla v\|_{L_y^\rho} \\ &\lesssim \varepsilon^{-(2\sigma+1)} \|v\|_{H_y^{-\delta(\rho)}}^{2\sigma} \|\nabla v\|_{H_y^{-\delta(\rho)}}. \end{aligned}$$

Combining this with a Hölder estimate in x and t, similarly as in Lemma 2.6 above, we obtain

$$\|\nabla \mathcal{N}(v)\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)} \lesssim \varepsilon^{-2(\sigma+1)} |I_j|^{1-\frac{(d-k)\sigma}{2}} \lambda^{2\sigma} \|\nabla v\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)}.$$

Hence on each subinterval I_j we have that

$$\|\nabla v\|_{L_t^\infty L_{\mathbf{x}}^2(I_j)} + \|\nabla v\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)} \leqslant C_\varepsilon \Big(\|\nabla v_{j-1}\|_{L_{\mathbf{x}}^2} + |I_j|^{1-\frac{(d-k)\sigma}{2}} \lambda^{2\sigma} \|\nabla v\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)} \Big)$$

for $j = 1, \ldots, N$ where we write ∇v_{j-1} to denote $\nabla v(t_{j-1})$. Now choose $\lambda = \lambda(C_\varepsilon, T)$ such that

$$C_\varepsilon T^{1-\frac{(d-k)\sigma}{2}} \lambda^{2\sigma} < 1.$$

Since $|I_j| \leqslant T$ we infer the estimate

$$\|\nabla v\|_{L_t^\infty L_{\mathbf{x}}^2(I_j)} + \|\nabla v\|_{L_t^\gamma L_x^\rho H_y^{-\delta(\rho)}(I_j)} \leqslant K_j^\varepsilon \|\nabla v_{j-1}\|_{L_{\mathbf{x}}^2},$$

for some constant K_j^ε which depends on ε. In particular, for $j = 1, \ldots, N$ we have

$$\|\nabla v_j\|_{L_{\mathbf{x}}^2} \leqslant K_j^\varepsilon \|\nabla v_{j-1}\|_{L_{\mathbf{x}}^2}.$$

Using this, we iterate the argument on each subinterval I_j, $j = 1, \ldots, N$, to obtain the desired estimate (44). □

REMARK 2.9. *Notice that we cannot obtain uniform-in-time bounds on the H^1-norm of v by invoking the energy (8). Indeed the energy functional, written in terms of v, reads*

$$E(t) = \frac{1}{2} \|P_\varepsilon^{-1/2} \nabla v\|_{L_{\mathbf{x}}^2}^2 - \frac{1}{2(\sigma+1)} \|P_\varepsilon^{-1/2} v\|_{L_{\mathbf{x}}^{2\sigma+2}}^{2\sigma+2},$$

which cannot provide a uniform bound on the full gradient of v.

The proposition above yields a solution u to (6) such that $v(t,\cdot) = P_\varepsilon^{1/2} u(t,\cdot) \in H^1(\mathbb{R}^d)$ globally in time. In particular, since

$$\|u(t,\cdot)\|_{H^1} \leqslant \|P_\varepsilon^{1/2} u(t,\cdot)\|_{H^1},$$

we infer $u(t,\cdot) \in H^1(\mathbb{R}^d)$ for all $t \in \mathbb{R}$, provided $P_\varepsilon^{1/2} u_0 \in H^1$. This shows that for a restricted class of initial data, the solution u exhibits a sufficient amount of regularity to rule out the possibility of finite-time blow-up in the usual sense. We will see below in Section 5 that we can achieve another higher order regularity result for $\sigma > \frac{2}{d-k}$, and prove small data global existence in this regime.

4. The critical case and the case of full off-axis dispersion

In this section, we treat the two "extreme" cases and consequently prove Theorems 1.2 and 1.3.

4.1. Partial off-axis dispersion with critical nonlinearity. In the case of partial off-axis dispersion with critical nonlinearity, i.e., $\sigma = \frac{2}{d-k}$ and $0 \leqslant k \leqslant 2$, we see that the estimate obtained in Lemma 2.6 no longer yields a positive power α of T. Hence the fixed point argument employed in the subcritical case breaks down. In order to overcome this obstacle, we shall employ the same type of arguments as in [27]. To this end, we first note the particular choice of admissible pair (q,r) obtained for

$$q = r = \frac{2(d-k+2)}{d-k}$$

and introduce the following mixed space for any $I \subset \mathbb{R}_t$:

$$W(I) = L^{\frac{2(d-k+2)}{d-k}}\Big(I \times \mathbb{R}_x^{d-k}; H^{-\frac{d-k}{d-k+2}}(\mathbb{R}_y^k)\Big).$$

Then, we have the following local well-posedness result for v, which directly yields Theorem 1.2 for u via $v = P_\varepsilon^{1/2} u$.

PROPOSITION 2.10. *Let $d-k > 0$ with $k \leqslant 2$, and $\sigma = \frac{2}{d-k}$. Then for any $v_0 \in L^2(\mathbb{R}^d)$, there exist times $0 < T_{\max}, T_{\min} \leqslant \infty$ and a unique maximal solution $v \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^d)) \cap W(I)$ to (11), where I denotes any closed time interval $I \subset (-T_{\min}, T_{\max})$. Furthermore, $T_{\max} < \infty$ if and only if*

$$\|v\|_{W([0,T_{\max}))} = \infty, \tag{46}$$

and analogously for $T_{\min}$. Finally, if $\|v_0\|_{L^2_{\mathbf{x}}}$ is sufficiently small, then the solution is global.

Note that here the maximal existence time depends not only on the size of the initial datum but rather on the whole profile of the solution, or more precisely on the $W(I)$-norm of v.

PROOF. We shall only give a sketch of the proof for $t \geqslant 0$, since our arguments follow along the same lines as those in [27, Section 3]; see also [25, Chapter 4.7].

Firstly, given a $T > 0$, we claim that by choosing $\delta > 0$ sufficiently small and such that

$$\|S_\varepsilon(\cdot)v_0\|_{W([0,T])} < \delta, \tag{47}$$

we obtain a unique solution $v \in C([0,T]; L^2(\mathbb{R}^d)) \cap W([0,T])$ to (11). Indeed, under assumption (47), the operator $v \mapsto \Phi(v)$, defined by (38) with $\sigma = \frac{2}{d-k}$, admits a unique fixed point in

$$Z_{T,\delta} = \{v \in W([0,T]) \text{ s.t. } \|v\|_{W([0,T])} < 2\delta\}.$$

As in Proposition 2.7, by means of the Strichartz estimates one can then show that $v \in L_t^q L_x^r H_y^{-\delta(r)}(0,T)$ for every admissible pair (q,r). Moreover, since the solution v satisfies the integral equation (38), we also infer $v \in C([0,T]; L^2(\mathbb{R}^d))$. To see that $\Phi(v)$ has a fixed point, we use (41) with $\gamma = \rho = \frac{2(d-k+2)}{d-k}$ and (47), to obtain

$$\|\Phi(v)\|_{W([0,T])} \leqslant \delta + C_\varepsilon \|v\|_{W([0,T])}^{\frac{4+d-k}{d-k}}.$$

Since $\frac{4+d-k}{d-k} > 1$, choosing δ small enough guarantees that $\Phi : Z_{T,\delta} \to Z_{T,\delta}$. Next, Lemma 2.6 implies the estimate

$$\|\Phi(v) - \Phi(w)\|_{W([0,T])} \leqslant C_\varepsilon \left(\|v\|_{W([0,T])}^{\frac{4}{d-k}} + \|w\|_{W([0,T])}^{\frac{4}{d-k}} \right) \|v - w\|_{W([0,T])}, \tag{48}$$

where C_ε is independent of T. Here, the fact that $\frac{4}{d-k} > 0$ and $\delta > 0$ is sufficiently small (independent of v_0 and T) implies that $v \mapsto \Phi(v)$ is a contraction on $Z_{T,\delta}$. That this choice of $\delta > 0$ is always possible follows from our Strichartz estimate and from

$$\|S_\varepsilon(t)v_0\|_{W([0,T])} \xrightarrow{T\to 0} 0. \tag{49}$$

Consequently, for $T > 0$ small enough assumption (47) is satisfied, yielding a unique local-in-time solution $v(t,\cdot)$ for $t \in [0,T]$.

By a similar argument as in Proposition 2.7 (see also [25, 27]), one can prove uniqueness by letting $v = \Phi(v), w = \Phi(w) \in W([0,T])$ and having in mind that

$$\left(\|v\|_{W([0,T])}^{\frac{4}{d-k}} + \|w\|_{W([0,T])}^{\frac{4}{d-k}} \right) \xrightarrow{T\to 0} 0.$$

From (48), we thus conclude that $v = w$ for $T > 0$ sufficiently small. We can then iterate this argument to find a maximal existence time $0 < T_{\max} \leqslant \infty$ for which the unique solution exists for every admissible pair (q, r).

Next, we shall prove the blow-up alternative (46) by contradiction. Namely, let $T_{\max} < \infty$ and let us assume that $\|v\|_{W([0,T_{\max}))} < \infty$. Let $t \in [0, T_{\max})$, then for any $s \in [0, T_{\max} - t)$ we write in view of (38) that

$$S_\varepsilon(s)v(t) = v(t+s) - \mathcal{N}(v(t+\cdot))(s).$$

Applying again Lemma 2.6 we can estimate

$$\|S_\varepsilon(\cdot)v(t)\|_{W([0,T_{max}-t))} \leqslant \|v\|_{W([t,T_{max}))} + C_\varepsilon \|v\|_{W([t,T_{max}))}^{\frac{4+d-k}{d-k}}$$

and thus, for t sufficiently close to $T_{\max}$, we have

$$\|S_\varepsilon(\cdot)v(t)\|_{W([0,T_{\max}-t))} < \delta.$$

This implies we can extend the solution after the time $T_{\max}$, contradicting its maximality.

Finally, in order to conclude global existence of small solutions, we note that, by a global-in-time Strichartz estimate,

$$\|S_\varepsilon(\cdot)v_0\|_{W(\mathbb{R})} \leqslant C_1 \|v_0\|_{L^2}.$$

This implies that if $\|v_0\|_{L^2}$ is small enough depending on $\delta > 0$, we have

$$\|S_\varepsilon(\cdot)v_0\|_{W(\mathbb{R})} < \delta.$$

Hence, assumption (47) is satisfied for all $T \in \mathbb{R}$ and the same continuity argument as before allows one to repeat the contraction argument with $T = \pm\infty$, cf. [**25**, Remark 4.7.5]. In summary, this yields a unique global solution $v(t, \cdot) \in L^2(\mathbb{R}^d)$ for sufficiently small initial data. □

4.2. The case of full off-axis dispersion. We finally turn to the case of full off-axis dispersion, i.e., $d = k$. It is clear from our admissibility condition in Definition 2.1, that in this case, we cannot expect to have any Strichartz estimates (see also [**19**] for more details). We thus resort to a more basic fixed point argument to prove the following result.

LEMMA 2.11. *Let $d = k \geqslant 1$ and $\sigma \leqslant \frac{2}{(d-2)_+}$. Then, for any $v_0 \in L^2(\mathbb{R}^d)$, there exists a unique global solution $v \in C(\mathbb{R}_t, L^2(\mathbb{R}^d))$ to (11), depending continuously on the initial data and satisfying*

$$\|v(t,\cdot)\|_{L^2_{\mathbf{x}}}^2 = \|v_0\|_{L^2_{\mathbf{x}}}^2 \quad \forall\, t \in \mathbb{R}.$$

Proof. To prove this result it suffices to show that $v \mapsto \Phi(v)$ is a contraction on

$$Y_{T,M} = \{v \in L^\infty([0,T]; L^2(\mathbb{R}^d)) : \|v\|_{L^\infty_t L^2_{\mathbf{x}}} \leqslant M\}.$$

Let $v, v' \in Y_{T,M}$, and recall that $S_\varepsilon(t)$ is unitary on L^2. Using Minkowski's inequality and the scaling argument (26) then yields

$$\|\mathcal{N}(v)(t) - \mathcal{N}(v')(t)\|_{L^2_{\mathbf{x}}} \leqslant \varepsilon^{-1} \int_0^t \|g(P_\varepsilon^{-1/2} v) - g(P_\varepsilon^{-1/2} v')\|_{H^{-\frac{d\sigma}{2(\sigma+1)}}}(s)\, ds,$$

provided $\frac{d\sigma}{2(\sigma+1)} \leqslant 1$, i.e., $\sigma \leqslant \frac{2}{(d-2)_+}$.

By a similar embedding strategy as in Lemma 2.6 one finds

$$\begin{aligned}
&\|g(P_\varepsilon^{-1/2} v) - g(P_\varepsilon^{1/2} v')\|_{H^{-\frac{d\sigma}{2(\sigma+1)}}} \\
&\qquad \leqslant (\|P_\varepsilon^{-1/2} v\|^{2\sigma}_{L^\rho_{\mathbf{x}}} + \|P_\varepsilon^{-1/2} v'\|^{2\sigma}_{L^\rho_{\mathbf{x}}}) \|P_\varepsilon^{-1/2}(v - v')\|_{L^\rho_{\mathbf{x}}} \\
&\qquad \leqslant \varepsilon^{-(2\sigma+1)} (\|v\|^{2\sigma}_{L^2_{\mathbf{x}}} + \|v'\|^{2\sigma}_{L^2_{\mathbf{x}}}) \|v - v'\|_{L^2_{\mathbf{x}}},
\end{aligned}$$

which consequently implies that

$$\|\mathcal{N}(v) - \mathcal{N}(v')\|_{L^\infty_t L^2_{\mathbf{x}}} \leqslant \varepsilon^{-(2\sigma+1)} T (\|v\|^{2\sigma}_{L^\infty_t L^2_{\mathbf{x}}} + \|v'\|^{2\sigma}_{L^\infty_t L^2_{\mathbf{x}}}) \|v - v'\|_{L^\infty_t L^2_{\mathbf{x}}}.$$

Choosing $T > 0$ sufficiently small, Banach's fixed point theorem directly yields a local-in-time solution $v \in C([0,T], L^2(\mathbb{R}^d))$. The conservation property of the L^2-norm of v can then be shown analogously as in the proof of Proposition 2.7. This consequently allows us to extend the local solution v for all $t \in \mathbb{R}$. ☐

This directly yields Theorem 1.3 for u, since in the case of full-off axis dispersion $v = P_\varepsilon^{1/2} u \in L^2(\mathbb{R}^d)$ implies $u \in H^1(\mathbb{R}^d)$ for any $\varepsilon > 0$. In addition, the L^2-conservation for v directly yields (9), whereas (8) is a standard computation, and valid for any H^1-solution u. Finally, it is straightforward to extend the solution to $v(t, \cdot) \in H^s(\mathbb{R}^d)$ for any $s > 0$ provided the initial data satisfies $v_0 \in H^s(\mathbb{R}^d)$.

REMARK 2.12. *Note that* (9) *also implies a uniform-in-time bound on the H^1-norm of $u(t, \cdot)$ for any $\varepsilon > 0$. In turn, this means that both the kinetic and the nonlinear potential energy remain uniformly bounded for all $t \in \mathbb{R}$.*

5. $\mathcal{H}^1_{\mathbf{xy}}$-subcritical theory with partial off-axis variations

In this section we shall extend the well-posedness results of [5], to the case when $k = 1, 2$, where there is an apparent breakdown in the fixed point argument, see the dialogue following Lemma 2.6. In particular, we prove small data global existence and scattering for nonlinearities $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$.

5.1. Strichartz Spaces.

In order to make clear the distinction between what has been proven above, from now on we denote the space where the conserved quantity (9) is well-defined by

$$\mathcal{H}^0_{xy} := L^2_x(\mathbb{R}^{d-k}; H^1_y(\mathbb{R}^k)) \text{ with norm } \|f\|_{\mathcal{H}^0_{xy}} := \|f\|_{L^2_x H^1_y} \simeq \|P_1^{1/2} f\|_{L^2_{\mathbf{x}}}.$$

In particular, under the conditions of Theorem 1.1, we have that for $u_0 \in \mathcal{H}^0_{xy}$, there exists a unique solution $u \in C(\mathbb{R}; \mathcal{H}^0_{xy})$. By the persistence of regularity result of Proposition 2.8, if we define the space

$$\mathcal{H}^1_{xy} := \left\{ u \in \mathcal{H}^0_{xy} : P_1^{1/2} u \in H^1(\mathbb{R}^d) \right\},$$

then for $k = 1, 2$ and $\sigma < \frac{2}{d-k}$, or $k > 2$ and $\sigma \leqslant \frac{2}{d-2}$, then we have that for data $u_0 \in \mathcal{H}^1_{xy}$ there are global solutions $u \in C(\mathbb{R}; \mathcal{H}^1_{xy})$. This space can consequently be viewed as

$$\mathcal{H}^1_{xy} := \mathcal{H}^0_{xy} \cap \left(L^2_x \dot{H}^2_y \cap H^1_x H^1_y \right), \tag{50}$$

where $\dot{H}^2_y$ is the inhomogeneous Sobolev space. This is the natural space in which one can define the conserved *linear momentum*, i.e., the following quantity

$$\mathcal{P}^\varepsilon(t) := -2\mathrm{Im}\langle P_\varepsilon \overline{u} \nabla u \rangle_{L^2_{\mathbf{x}}} = -2\mathrm{Im} \int_{\mathbb{R}^d} \overline{u}_0 \nabla u_0 + \varepsilon^2 \partial_y \overline{u_0} \nabla \partial_y u_0 \, d\mathbf{x}, \tag{51}$$

which we see to be the usual conserved linear momentum when $\varepsilon = 0$. With these spaces, we may rewrite the Strichartz estimates (36) and (37) respectively in the following way:

$$\|S_\varepsilon(\cdot) f\|_{L^q_t L^r_x H^{1-\delta(r)}_y (I)} \leqslant C_1 \|f\|_{\mathcal{H}^0_{xy}}, \tag{52}$$

$$\left\| P_\varepsilon^{-1} \int_0^t S_\varepsilon(\cdot - s) F(s) \, ds \right\|_{L^q_t L^r_x H^{1-\delta(r)}_y (I)} \leqslant \varepsilon^{-2} C_2 \|F\|_{L^{\gamma'}_t L^{\rho'}_x H^{\delta(\rho)-1}_y (I)}. \tag{53}$$

This can be seen if we replace f in (36) by $P_1^{1/2} f$ and F by $P_\varepsilon^{-1} P_1^{1/2} F$ in (37) respectively, and notice from (25) and (26) the equivalence of the space-time norms.

Now given any time interval I, we define the Strichartz space $S^0_{xy}(I)$ by the norm

$$\|f\|_{S^0_{xy}} := \sup_{(q,r),\text{admissible}} \|f\|_{L^q_t L^r_x H^{1-\delta(r)}_y}.$$

We remark that since $(\infty,2)$ is admissible then $S^0_{xy}(I) \subset L^\infty_t \mathcal{H}^0_{xy}(I)$. Moreover we define the dual $N^0_{xy}(I) := (S^0_{xy}(I))^*$ by the norm

$$\|f\|_{N^0_{xy}} \leqslant \|f\|_{L^{\gamma'}_t L^{\rho'}_x H^{\delta(\rho)-1}_y},$$

where (γ,ρ) is an admissible pair to be chosen. In order to simplify certain notations in the $\mathcal{H}^1_{xy}$ well-posedness theory to follow, we introduce

$$\|u\|_{S^1_{xy}} := \|u\|_{S^0_{xy}} + \|\nabla u\|_{S^0_{xy}} \quad \text{and} \quad \|u\|_{N^1_{xy}} := \|u\|_{N^0_{xy}} + \|\nabla u\|_{N^0_{xy}},$$

where it is clear that $S^1_{xy}(I) \subset L^\infty_t \mathcal{H}^1_{xy}(I)$ as the pair $(\infty,2)$ is admissible.

5.2. Local well-posedness in $\mathcal{H}^1_{xy}$. In this subsection, we first prove a nonlinear estimate similar to (2.6), which will allow us to construct a fixed point argument and prove local existence in the space $\mathcal{H}^1_{xy}$. In order to streamline the analysis and to show a contrast in techniques, we shall *not* employ the change of unknown (10) as is done before. Instead, we shall directly employ the integral formulation for the Cauchy problem (6) and write

$$u(t) = S_\varepsilon(t)u_0 + iP_\varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(|u|^{2\sigma}u)(s)\, ds =: \Phi(u)(t). \tag{54}$$

As before, we shall shortly write the integral equation as $\Phi(u)(t) = S_\varepsilon(t)u_0 + \mathcal{N}(u)(t)$, where we denote the nonlinear term in (54) by

$$\mathcal{N}(v)(t) := iP_\varepsilon^{-1}\int_0^t S_\varepsilon(t-s)(|u|^{2\sigma}u)(s)\, ds. \tag{55}$$

LEMMA 2.13. *Let $6 \geqslant d > k$. Fix an interval $0 \in I$ and choose the admissible pair*

$$(\gamma,\rho) = \left(\frac{4(\sigma+1)}{(d-k)\sigma}, 2(\sigma+1)\right).$$

Then, in the space-time slab $I \times \mathbb{R}^d$ the following inequality

$$\|\mathcal{N}(u) - \mathcal{N}(u')\|_{S^1_{xy}} \lesssim \varepsilon^{-2}|I|^{1-\delta(\rho)}\Big(\|u\|_{S^1_{xy}} + \|u'\|_{S^1_{xy}}\Big)^{2\sigma}\|u-u'\|_{S^1_{xy}},$$

holds provided that $\frac{1}{2} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$.

PROOF. We first note that for our pair (γ, ρ) to be (nonendpoint) admissible for $d-k \geqslant 2$, we require that $\gamma > 2$, which in turn is equivalent to $\sigma < \frac{2}{(d-k-2)_+}$. However, this condition will always be fulfilled since

$$\sigma \leqslant \frac{2}{(d-2)_+} < \frac{2}{(d-k-2)_+}.$$

As a consequence, we also have that $\delta(\rho) = \frac{(d-k)\sigma}{2(\sigma+1)} < 1$. Furthermore, if we impose that $\sigma \leqslant \frac{2}{(d-2)_+}$ then it is clear that $\frac{k\sigma}{2(\sigma+1)} + \delta(\rho) \leqslant 1$ where $\rho = 2(\sigma+1)$. Hence for estimates involving $y \in \mathbb{R}^k$, we have by inclusion and Sobolev embedding the following

$$H^{1-\delta(\rho)}(\mathbb{R}^k) \subseteq H^{\frac{k\sigma}{2(\sigma+1)}}(\mathbb{R}^k) \hookrightarrow L^\rho(\mathbb{R}^k), \tag{56}$$

and moreover we have the dual of (56) given by

$$L^{\rho'}(\mathbb{R}^k) \hookrightarrow H^{\frac{-k\sigma}{2(\sigma+1)}}(\mathbb{R}^k) \subseteq H^{\delta(\rho)-1}(\mathbb{R}^k), \tag{57}$$

For $\mathbf{x} \in \mathbb{R}^d$ we will take advantage of the following inclusion and Sobolev embedding

$$H^1(\mathbb{R}^d) \subseteq H^{\frac{d\sigma}{2(\sigma+1)}}(\mathbb{R}^d) \hookrightarrow L^\rho(\mathbb{R}^d), \tag{58}$$

which holds again provided that $\sigma \leqslant \frac{2}{(d-2)_+}$.

We note from the Strichartz estimate (53) applied to the difference of $\mathcal{N}(u)$ and $\mathcal{N}(u')$ that

$$\|\mathcal{N}(u) - \mathcal{N}(u')\|_{S^1_{xy}} \leqslant \varepsilon^{-2} C_2 \left\| |u|^{2\sigma} u - |u'|^{2\sigma} u' \right\|_{N^1_{xy}} := \varepsilon^{-2} C_2 \big(\mathcal{Q}_1 + \mathcal{Q}_2\big)$$

In order to control the N^1_{xy}-norm we perform the respective computations on the *non-gradient* term $\mathcal{Q}_1$ and *gradient* term $\mathcal{Q}_2$ in N^0_{xy}. In particular, we deduce from (56) in the y-variables that

$$\|f\|_{N^0_{xy}} \leqslant \|f\|_{L_t^{\gamma'} L_x^{\rho'} H_y^{\delta(\rho)-1}} \leqslant \|f\|_{L_t^{\gamma'} L_x^{\rho'} L_y^{\rho'}}. \tag{59}$$

The term in N^0_{xy} *without* the gradient follows from (59) and Hölder's inequality in x and y with $\frac{1}{\rho'} = \frac{2\sigma}{\rho} + \frac{1}{\rho}$, which by the above program yields

$$\begin{aligned} \||u|^{2\sigma} u - |u'|^{2\sigma} u'\|_{L_x^{\rho'} L_y^{\rho'}} &\leqslant \left(\|u\|_{L^\rho}^{2\sigma} + \|u'\|_{L^\rho}^{2\sigma}\right) \|(u-u')\|_{L_x^\rho L_y^\rho} \\ &\leqslant \left(\|u\|_{H^1}^{2\sigma} + \|u'\|_{H^1}^{2\sigma}\right) \|u-u'\|_{L_x^\rho H_y^{1-\delta(\rho)}}, \end{aligned} \tag{60}$$

where in the last inequality we use (56) and (57) on the difference term.

Consequently, using the above estimates and Hölder's inequality in t with

$$\frac{1}{\gamma'} = 1 - \frac{2}{\gamma} + \frac{1}{\gamma} = 1 - \delta(\rho) + \frac{2\sigma}{\infty} + \frac{1}{\gamma}$$

we have our first notable estimate, namely

$$\mathcal{Q}_1 := \big\| |u|^{2\sigma} u - |u'|^{2\sigma} u' \big\|_{N^0_{xy}} \leqslant |I|^{1-\delta(\rho)} \left(\|u\|^{2\sigma}_{S^1_{xy}} + \|u'\|^{2\sigma}_{S^1_{xy}} \right) \|u - u'\|_{S^1_{xy}}. \tag{61}$$

Secondly, we treat the $\mathcal{Q}_2$ term, which requires a bit more care. We note by a direct computation that

$$\begin{aligned} \big|\nabla\big(|u|^{2\sigma}u\big) - \nabla\big(|u|^{2\sigma}u\big)\big| &\leqslant C_\sigma \Big(|u|^{2\sigma} |\nabla(u-u')| + \big(|u|^{2\sigma-1} + |u'|^{2\sigma-1}\big) |\nabla u'| |u - u'| \Big) \\ &= G_1(u,u') + G_2(u,u'), \end{aligned}$$

whose terms of which are integrable in every $L^q_x L^p_y$-space provided we impose that $\sigma \geqslant \frac{1}{2}$. For the first term we use the same analysis as in (61) with no gradient to give

$$\|G_1(u,u')\|_{N_0} \leqslant |I|^{1-\frac{2}{\gamma}} \|u\|^{2\sigma}_{L^\infty_t H^1} \|\nabla(u-u')\|_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y} \leqslant |I|^{1-\delta(\rho)} \|u\|^{2\sigma}_{S^1_{xy}} \|u - u'\|_{S^1_{xy}}. \tag{62}$$

Next for the term involving G_2 we note that $\dfrac{1}{\rho'} = \dfrac{2\sigma - 1}{\rho} + \dfrac{1}{\rho} + \dfrac{1}{\rho}$ which we use to apply Hölder's inequality in x and y and estimate as before to give

$$\|G_2(u,u')\|_{L^{\rho'}_x L^{\rho'}_y} \leqslant \big(\|u\|^{2\sigma-1}_{H^1} + \|u'\|^{2\sigma-1}_{H^1} \big) \|\nabla u'\|_{L^\rho_x H^{1-\delta(\rho)}_y} \|u - u'\|_{H^1}.$$

To finish off the space-time estimate for G_2 we apply Hölder in t, noting the numerology

$$\frac{1}{\gamma'} = 1 - \frac{2}{\gamma} + \frac{1}{\gamma} = 1 - \delta(\rho) + \frac{2\sigma - 1}{\infty} + \frac{1}{\gamma} + \frac{1}{\infty},$$

which in turn gives the space-time estimate for the G_2 term as follows

$$\|G_2(u,u')\|_{N^0_{xy}} \leqslant |I|^{1-\delta(\rho)} \left(\|u\|^{2\sigma-1}_{S^1_{xy}} + \|u'\|^{2\sigma-1}_{S^1_{xy}} \right) \|u'\|_{S^1_{xy}} \|u - u'\|_{S^1_{xy}}. \tag{63}$$

Collecting both estimates (62) and (63), we then obtain

$$\begin{aligned} \mathcal{Q}_2 &:= \big\| \nabla\big(|u|^{2\sigma}u\big) - \nabla\big(|u|^{2\sigma}u\big) \big\|_{N^0_{xy}} \\ &\lesssim |I|^{1-\delta(\rho)} \Big(\|u\|^{2\sigma-1}_{S^1_{xy}} + \|u'\|^{2\sigma-1}_{S^1_{xy}} \Big) \Big(\|u\|_{S^1_{xy}} + \|u'\|_{S^1_{xy}} \Big) \|u - u'\|_{S^1_{xy}}. \end{aligned} \tag{64}$$

Combining the estimates (61) and (64) gives the desired result. $\square$

Using Lemma 2.13, we are now able to prove local well-posedness for (6) in the H^1-subcritical cases for v, i.e. the cases not covered by Propositions 2.7 and 2.8. In doing so, we will require a positive exponent

$$\alpha \equiv 1 - \delta(\rho) > 0$$

of $|I|^\alpha$ in the estimate obtained in Lemma 2.13, which always holds. Hence there is only the restriction on the nonlinearity $\frac{1}{2} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$.

PROPOSITION 2.14. *Let $d > k > 0$ and $\frac{1}{2} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$. Then for any $u_0 \in \mathcal{H}^1_{xy}$, there exists a unique maximal solution to (11) such that $u \in C(I_{max}; \mathcal{H}^1_{xy}) \cap S^1_{xy}(I_{max})$, where $0 \in I_{max}(\|u_0\|_{\mathcal{H}^1_{xy}})$. Moreover, u depends continuously on the initial data, and satisfies the blow-up alternative: If $|I_{max}| < \infty$ then*

$$\lim_{t \to |I_{\max}|} \Big(\|\Delta_y u(t,\cdot)\|_{L^2_{\mathbf{x}}} + \|\nabla_x u(t,\cdot)\|_{\mathcal{H}^0_{xy}} \Big) = +\infty.$$

PROOF. (Existence): Let $M, T > 0$ and denote $I = [0, T]$ to be determined later, and set

$$B_{T,M} = \{u \in S^1_{xy}(I) : \|u\|_{S^1_{xy}(I)} \leqslant M\},$$

equipped with the metric $d(u, w) = \|u - w\|_{S^1_{xy}}$.

Let $u \in B_{T,M}$. From the nonlinear estimate of Lemma 2.13 with $u' = 0$, one finds after applying the Strichartz estimates that

$$\|\Phi(u)\|_{S^1_{xy}} \leqslant C_1 \big(\|u_0\|_{\mathcal{H}^0_{xy}} + \|\nabla u_0\|_{\mathcal{H}^0_{xy}} \big) + 2\varepsilon^{-2} C_{\sigma,2} |I|^{1-\delta(\rho)} M^{2\sigma+1},$$

Given $u_0 \in \mathcal{H}^1_{xy}$, we make way for the contraction by setting $M = 4C_1 \|u_0\|_{\mathcal{H}^1_{xy}}$ and choose $T > 0$ such that

$$C_{\sigma,2} |I|^{1-\delta(\rho)} M^{2\sigma+1} \leqslant \frac{M\varepsilon^2}{4}.$$

From this choice, it follows that $\Phi(u) \in B_{T,M}$ for all $u \in B_{T,M}$ so that $\Phi(B_{T,M}) \subset B_{T,M}$. Now, let $u, u' \in X_{T,M}$, then directly from Lemma 2.13 follows the continuity

$$\|\Phi(u) - \Phi(u')\|_{S^1_{xy}} \leqslant \frac{1}{4} \|u - u'\|_{S^1_{xy}}, \quad \text{for all } u, u' \in B_{T,M}. \tag{65}$$

Thus Φ is a contraction map on $B_{T,M}$ and Banach's fixed point theorem yields the existence of a unique fixed point $u = \Phi(u) \in B_{T,M}$.

We remark that since $u \in B_{T,M}$ then $u, \nabla u \in L_t^q L_x^r H^{1-\delta(r)}(I)$ for any admissible pair (q,r), since

$$\|u\|_{L_t^q L_x^r H_y^{1-\delta(r)}} + \|\nabla u\|_{L_t^q L_x^r H_y^{1-\delta(r)}} \leqslant \|\Phi(u)\|_{S^1_{xy}} \leqslant M.$$

Moreover we can show the solution $u \in C(I; \mathcal{H}^1_{xy})$. We first note that $u \in S^1_{xy}(I) \subset L_t^\infty(I; \mathcal{H}^1_{xy})$ and the solution satisfies the integral equation (54). From Stone's theorem $S_\varepsilon(t)$ is a strongly continuous unitary group in $\mathcal{H}^1_{xy}$, so it is clear that the linear term $S_\varepsilon(\cdot)u_0 \in C(I; \mathcal{H}^1_{xy})$. For the nonlinear part of (54), we note from Lemma 2.13 that $\mathcal{N}(u) \in L^\infty(I; \mathcal{H}^1_{xy})$ since we have for each $t \in I$ that

$$\|\mathcal{N}(u)(t)\|_{\mathcal{H}^1_{xy}} \leqslant \|\mathcal{N}(u)(t)\|_{S^1_{xy}} \leqslant 2C_{\sigma,\varepsilon}|I|^{1-\delta(\rho)}\|u(t)\|^{2\sigma+1}_{S^1_{xy}} < \infty.$$

The continuity of this term then follows for fixed $0 \leqslant t \leqslant |I|$ by writing

$$\mathcal{N}(u)(t+h) - \mathcal{N}(u)(t) = i\int_t^{t+h} S_\varepsilon(t+h-s)F(s)\,ds + i\big(S_\varepsilon(h) - 1\big)\int_0^t S_\varepsilon(t-s)F(s)\,ds,$$

where $F(\cdot) = (|u|^{2\sigma}u)(\cdot)$. Moreover it is clear that the continuity follows, since the above goes to zero in norm as $h \to 0$, after employing the same estimates as before.

Step 2: (Uniqueness and Blow-up alternative) Let $I = [0,T]$ and suppose there exists another solution $w \in S^1_{xy}(I)$ to (11) with $w_0 = u_0 = \varphi$. From the integral equation (54), we write

$$(u-w)(t) = \Phi(u)(t) - \Phi(w)(t) = \big(\mathcal{N}(u) - \mathcal{N}(w)\big)(t).$$

On an interval $I_\mu = [0,\mu] \subset I$ where $\mu > 0$ to be determined later, we note by Lemma 2.13 that

$$\begin{aligned}\|u-w\|_{S^1(I_\mu)} &= \|\mathcal{N}(u) - \mathcal{N}(w)\|_{S^1_{xy}(I_\mu)}\\ &\leqslant C_{\sigma,\varepsilon}\mu^{1-\delta(\rho)}\Big(\|u\|_{S^1_{xy}(I_\mu)} + \|w\|_{S^1_{xy}(I_\mu)}\Big)^{2\sigma}\|u-w\|_{S^1_{xy}(I_\mu)}.\end{aligned}$$

Since we have fixed the solutions $u, w \in B_{T,M}$ with $M = 4C_1\|\varphi\|_{\mathcal{H}^1_{xy}}$, then we obtain

$$\|u-w\|_{S^1_{xy}(I_\mu)} \leqslant \frac{1}{2}\|u-w\|_{S^1_{xy}(I_\mu)},$$

where we choose μ such that $C_{\sigma,\varepsilon}\mu^{1-\delta(\rho)}\|\varphi\|^{2\sigma}_{\mathcal{H}^1_{xy}} \lesssim \frac{1}{2}$. To extend uniqueness to the whole interval I, one merely repeats this argument $\frac{T}{\mu}$ times using the continuity of the integral equation. From the uniqueness we can consequently construct a maximal solution on an interval $I_{\max}$.

Next we deduce the blow-up alternative by a similar contradiction argument as in Proposition 2.7. However we note from (50) that part of the $\mathcal{H}^1_{xy}$-norm is conserved. Namely, the $\mathcal{H}^0_{xy}$-norm which corresponds to the conserved quantity (9). Hence we see that if $|I_{\max}| < +\infty$, then

$$\lim_{t\to|I_{\max}|} \|u(t)\|_{\mathcal{H}^1_{xy}} = \|u_0\|_{\mathcal{H}^0_{xy}} + \lim_{t\to|I_{\max}|} \Big(\|\Delta_y u(t)\|_{L^2_{\mathbf{x}}} + \|\nabla_x u(t)\|_{\mathcal{H}^0_{xy}} \Big) = \infty.$$

We remark that this statement of the blow-up alternative does not preclude blow-up in infinite time.

Step 3:(Continuous dependence on the initial data) Let u, u' be two such solutions. The continuous dependence follows directly from the estimate (65), where we set $u = \Phi(u), u' = \Phi(u')$ so that

$$\|u - u'\|_{S^1_{xy}} \leqslant C_1 \|u_0 - u_0'\|_{\mathcal{H}^1_{xy}} + \frac{1}{4}\|u - u'\|_{S^I_{xy}}.$$

This concludes the local well-posedness in $\mathcal{H}^1_{xy}$. $\square$

5.3. Small data scattering in $\mathcal{H}^1_{\mathbf{xy}}$. In this subsection we shall focus on the asymptotic behavior of solutions to (6). Firstly, we prove the existence of small data global solutions in the case $k = 1, 2$ and $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{d-2}$. Secondly, we show that these solutions evolve to the associated free solution for large times. More precisely, if the limit

$$u_\pm = \lim_{t\to\pm\infty} S_\varepsilon(-t)u(t)$$

exists in $\mathcal{H}^1_{xy}$, then we say that the global solution u *scatters* and we shall call $u_\pm$ the *scattering states* of u_0 at $\pm\infty$.

PROPOSITION 2.15 (Small data global existence). *Let $k + 4 \geqslant d > k$ with $k = 1, 2$, and suppose that $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$. There exists $\lambda_\varepsilon > 0$ such that for every initial data $u_0 \in \mathcal{H}^1_{xy}$ with $\|u_0\|_{\mathcal{H}^1_{xy}} \leqslant \lambda_\varepsilon$, the corresponding maximal solution u of Proposition 2.14 is global and satisfies*

$$u \in C(\mathbb{R}_t; \mathcal{H}^1_{xy}) \cap L^q_t L^r_x H^{2-\delta(r)}_y(\mathbb{R}) \quad \textit{and} \quad \nabla_x u \in L^q_t L^r_x H^{1-\delta(r)}_y(\mathbb{R})$$

for any (non-endpoint) admissible pair (q, r).

Proof. We first note that in the case $d - k \geqslant 4$ that $\frac{2}{d-k} \leqslant \frac{1}{2} \leqslant \sigma$. Hence, we assume that $d \leqslant k + 4$ and that $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{d-2}$ for the duration of the proof. Given the maximal solution of Proposition 2.14, we define the quantity

$$H(T) = \|u\|_{L^\infty_t \mathcal{H}^1_{xy}(0,T)} + \|u\|_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y(0,T)} + \|\nabla u\|_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y(0,T)},$$

for $0 < T < T_{\max}$.

We first note the latter two terms of $H(T)$ go to zero as $T \to 0^+$ and because $u \in C([0,T];\mathcal{H}^1_{xy})$ then we have

$$H(T) \to \|u_0\|_{\mathcal{H}^1_{xy}} \quad \text{as} \quad T \to 0^+. \tag{66}$$

By way of the Strichartz estimates and the integral formulation, we deduce

$$H(T) \leqslant 2C_1\|u_0\|_{\mathcal{H}^1_{xy}} + 2\varepsilon^{-2}C_2\||u|^{2\sigma}u\|_{N^1}. \tag{67}$$

The latter term is computed with a similar strategy as in Lemma 2.13, and so using (59) we obtain

$$\||u|^{2\sigma}u\|_{N^1} \leqslant \Big(\||u|^{2\sigma}u\|_{L^{\gamma'}_t L^{\rho'}_x L^{\rho'}_y} + \|\nabla(|u|^{2\sigma}u)\|_{L^{\gamma'}_t L^{\rho'}_x L^{\rho'}_y}\Big),$$

where (γ,ρ) is the admissible pair with $\rho = 2(\sigma+1)$. Lets compute first in x and y and use $|\nabla g(w)| \leqslant C_\sigma |w|^{2\sigma}|\nabla w|$ and Remark 1.3.1 in [25] followed by Hölder's inequality and (56) to give

$$\||u|^{2\sigma}u\|_{L^{\rho'}_x L^{\rho'}_y} + \|\nabla(|u|^{2\sigma}u)\|_{L^{\rho'}_x L^{\rho'}_y} \lesssim \|u\|^{2\sigma}_{L^\rho_x L^\rho_y}\Big(\|u\|_{L^\rho_x H^{1-\delta(\rho)}_y} + \|\nabla u\|_{L^\rho_x H^{1-\delta(\rho)}_y}\Big).$$

We now perform a similar analysis as in Lemma 2.13. However we shall eliminate the factor $|I|^\alpha$ arising when employing Hölder's inequality in t. To do so, we choose the t-exponents such that

$$\frac{1}{\gamma'} = \frac{\gamma-2}{\gamma} + \frac{1}{\gamma} = \frac{2\sigma}{s} + \frac{1}{\gamma},$$

then in combining the above estimates we have in total the following

$$\||u|^{2\sigma}u\|_{N^1} \leqslant \varepsilon^{-2}\|u\|^{2\sigma}_{L^s_t L^\rho_x L^\rho_y}\Big(\|v\|_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y} + \|\nabla v\|_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y}\Big) \leqslant \varepsilon^{-2}H(T)\|u\|^{2\sigma}_{L^s_t L^\rho_x L^\rho_y}. \tag{68}$$

We now notice that $\gamma \leqslant s = \frac{2\sigma\gamma}{\gamma-2} < \infty$ provided that $\frac{2}{d-k} \leqslant \sigma$. From this, we see immediately that if $\sigma = \frac{2}{d-k}$ (in particular for the case when $k=2$) that $s=\gamma=\rho$ and one yields the same estimate (70) to follow. Then for $k=1$, we suppose $\sigma > \frac{2}{d-1}$ and interpolate in time between the exponents γ and ∞ with $\theta = 1 - \frac{\gamma}{s} \in (0,1)$ and use the Sobolev embedding (50) and (58) so that

$$\|u\|_{L^s_t L^\rho_x L^\rho_y} \leqslant \|u\|^\theta_{L^\infty_t L^\rho_{\mathbf{x}}}\|u\|^{1-\theta}_{L^\gamma_t L^\rho_x L^\rho_y} \leqslant \|u\|^\theta_{L^\infty_t H^1}\|u\|^{1-\theta}_{L^\gamma_t L^\rho_x H^{1-\delta(\rho)}_y} \leqslant H(T). \tag{69}$$

This consequently implies from (68) that inequality (67) may be estimated further to yield

$$H(T) \leqslant 2C_1\|u_0\|_{\mathcal{H}^1_{xy}} + 2\varepsilon^{-2}C_2 H(T)^{2\sigma+1} \tag{70}$$

for all $0 < T < T_{\max}$ and $\sigma \geqslant \frac{2}{d-k}$.

With this inequality in hand, we take the limit as $T \to 0^+$ and employ (66) to give

$$H(0) \leqslant 2\|u_0\|_{\mathcal{H}^1_{xy}}\Big(C_1 + \varepsilon^{-2}C_2\|u_0\|^{2\sigma}_{\mathcal{H}^1_{xy}}\Big).$$

Hence there exists $\lambda_\varepsilon > 0$ such that for $\|u_0\|_{\mathcal{H}^1_{xy}} \leqslant \lambda_\varepsilon$ with $C_2\lambda_\varepsilon^{2\sigma} \leqslant C_1\varepsilon^2$ it follows that

$$H(0) \leqslant 4C_1\lambda_\varepsilon.$$

However, in general we note that $H(0) \leqslant H(T)$ for $T < T_{\max}$. Since the interval $(0, T_{\max})$ is connected, then either $H(T) > 4C_1\lambda_\varepsilon$ or $H(T) \leqslant 4C_1\lambda_\varepsilon$. Suppose, by way of contradiction, that $H(T) > 4C_1\lambda_\varepsilon$ for $T > 0$, then from (70) we obtain

$$2C_1\lambda_\varepsilon < C_1\lambda_\varepsilon + \varepsilon^{-2}C_2H(T)^{2\sigma+1}$$

which is equivalent to

$$C_2\lambda_\varepsilon^{2\sigma+1} \leqslant C_1\varepsilon^2\lambda_\varepsilon < C_2H(T)^{2\sigma+1},$$

i.e. $\lambda_\varepsilon < H(T)$ for $T > 0$. However from (66) and the continuity of $H(T)$ we deduce that

$$\lambda_\varepsilon < \lim_{T\to 0^+} H(T) = \|u_0\|_{\mathcal{H}^1_{xy}},$$

which yields a contradiction on our choice of λ_ε.

This consequently implies that

$$\|u\|_{L^\infty_t\mathcal{H}^1_{xy}((0,T))} + \|u\|_{S^1_{xy}((0,T))} \leqslant 4C_1\|u_0\|_{\mathcal{H}^1_{xy}} \quad \text{for all} \quad 0 < T < T_{\max}.$$

In particular, we see from the continuity of $H(T)$ that

$$\|u\|_{L^\infty_t\mathcal{H}^1_{xy}((0,T_{\max}))} = \lim_{T\to T_{\max}} \|u\|_{L^\infty_t\mathcal{H}^1_{xy}((0,T))} < \infty,$$

which implies by the blow-up alternative of Proposition 2.14 that $T_{\max} = \infty$ and so it follows we have a global solution $u \in C(\mathbb{R}; \mathcal{H}^1_{xy})$. Moreover it follows from the Strichartz estimates that $u \in S^1_{xy}(\mathbb{R})$ since

$$\|u\|_{L^q_tL^r_xH^{1-\delta(r)}_y(\mathbb{R})} + \|\nabla u\|_{L^q_tL^r_xH^{1-\delta(r)}_y(\mathbb{R})} \leqslant \|u\|_{S^1_{xy}(\mathbb{R})} \lesssim \lambda_\varepsilon, \tag{71}$$

holds for every admissible pair (q, r).

We note that $u \in L_t^q L_x^r H_y^{1-\delta(r)}([T,\infty))$ for any $T>0$, and by monotone convergence we have

$$\|u\|_{L_t^q L_x^r H_y^{1-\delta(r)}([T,\infty))} + \|\nabla u\|_{L_t^q L_x^r H_y^{1-\delta(r)}([T,\infty))} \to 0 \quad \text{as} \quad T \to \infty, \tag{72}$$

for every admissible pairs $(q,r) \neq (\infty, 2)$, a fact which will be important in the scattering result below. $\square$

Now that we have proven small data global existence, we can proceed to prove the existence of scattering states. To this end, we first rewrite the integral equation (54) as

$$S_\varepsilon(-t)u(t) = u_0 + S_\varepsilon(-t)\mathcal{N}(u)(t) \tag{73}$$

and note that in this case the scattering states (if these exist) are informally given by

$$u_\pm = u_0 + iP_\varepsilon^{-1} \int_0^{\pm\infty} S_\varepsilon(-s)(|u|^{2\sigma}u)(s)\, ds. \tag{74}$$

In this way, scattering reduces to showing the convergence of the improper integral above. The key feature we use to prove scattering comes from the global Strichartz bounds i.e. $u \in S^1_{xy}(\mathbb{R})$.

PROPOSITION 2.16 (Existence of Scattering States). *Suppose u is a global solution satisfying the hypothesis of Proposition 2.15 with initial data u_0 such that $\|u_0\|_{\mathcal{H}^1_{xy}} \leqslant \lambda_\varepsilon$. Then there exists scattering states $u_\pm \in \mathcal{H}^1_{xy}$ such that*

$$\lim_{t\to\pm\infty} \|S_\varepsilon(-t)u(t) - u_\pm\|_{\mathcal{H}^1_{xy}} = 0.$$

PROOF. Let us write $w(t) = S_\varepsilon(-t)u(t)$ as in (73) then for $0 < t < \tau$, we consider the difference

$$w(\tau) - w(t) = iP_\varepsilon^{-1} \int_t^\tau S_\varepsilon(-s)(|u|^{2\sigma}u)(s)\, ds.$$

Hence using that the propagator $S_\varepsilon(t)$ is an isometry on $H^1(\mathbb{R}^d)$ and by the above we have

$$\|w(\tau) - w(t)\|_{\mathcal{H}^1_{xy}} = \|S_\varepsilon(t)P_\varepsilon^{1/2}(w(t) - w(\tau))\|_{H^1} \leqslant \|\mathcal{N}(u)\|_{L_t^\infty \mathcal{H}^1_{xy}((t,\tau))}.$$

In the case $k=2$ and $\sigma = \frac{2}{d-2}$, i.e. for $\gamma = \rho = 2(\sigma+1)$, by similar analysis as in Lemma 2.6, we obtain by employing (72) that

$$\begin{aligned} &\|\mathcal{N}(u)\|_{L_t^\infty \mathcal{H}^1_{xy}((t,\tau))} \\ &\quad \lesssim \varepsilon^{-2} \|u\|^{2\sigma}_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)} \Big(\|u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)} + \|\nabla u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)} \Big) \to 0, \quad \text{as} \quad t,\tau \to \infty. \end{aligned}$$

Lastly, in the case $k = 1$ and $\sigma \geqslant \frac{2}{d-1}$, we have from the same estimates of (68) and (69) that

$$\begin{aligned}
&\|\mathcal{N}(u)\|_{L_t^\infty H^1((t,\tau))} \\
&\quad \lesssim \varepsilon^{-2}\|u\|^{2\sigma\theta}_{L_t^\infty \mathcal{H}^1_{xy}(t,\tau)}\|u\|^{2\sigma(1-\theta)}_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)}\Big(\|u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)} + \|\nabla u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)}\Big).
\end{aligned}$$

Now, since $\|v\|_{L_t^\infty H^1(t,\tau)} \leqslant \|v\|_{L_t^\infty H^1(\mathbb{R})} = M$, then it follows from (72) that

$$\begin{aligned}
&\|w(\tau) - w(t)\|_{\mathcal{H}^1_{xy}} \\
&\leqslant C_\varepsilon M^{2\sigma\theta}\|u\|^{2\sigma(1-\theta)}_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)}\Big(\|u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)} + \|\nabla u\|_{L_t^\gamma L_x^\rho H_y^{1-\delta(\rho)}(t,\tau)}\Big) \to 0 \quad \text{as} \quad t,\tau \to \infty.
\end{aligned}$$

This is enough to conclude the convergence of the integral term in (74). Moreover from (71) it is clear by the above estimates that we obtain

$$\|u_+\|_{\mathcal{H}^1_{xy}} \lesssim C_1\lambda_\varepsilon + C_2\varepsilon^{-2}\lambda_\varepsilon^{2\sigma+1} \leqslant 2C_1\lambda_\varepsilon,$$

which implies $u_+ \in \mathcal{H}^1_{xy}$. Thus, for $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{d-2}$, there exists a scattering state $u_+ \in \mathcal{H}^1_{xy}$ such that

$$w(t) = S_\varepsilon(-t)u(t) \to u_+ \quad \text{as} \quad t \to +\infty.$$

By the same analysis we deduce the existence of a scattering state $u_- \in \mathcal{H}^1_{xy}$ as $t \to -\infty$. $\square$

In summary this yields Theorem 1.4.

6. Approximation result

This final section is dedicated to the limiting behavior of solutions to (11), in either the full or partial off-axis case. It is clear that at least formally P_ε converges to the identity operator as $\varepsilon \to 0$, and so in some sense solutions to (11) should converge to the solution to (1) the focusing NLS as $\varepsilon \to 0$.

From the persistence of regularity result, Proposition 2.8, one can construct global solutions $v^\varepsilon \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ for $s > \frac{d}{2}$ to equation (6) with $\sigma \in \mathbb{N}$ and $\sigma \leqslant \frac{2}{(d-2)_+}$. We remark that in the case when $k = 1, 2$ and $\frac{2}{d-k} \leqslant \sigma \leqslant \frac{2}{(d-2)_+}$, we obtain global solutions for data $\|v_0^\varepsilon\|_{H^1} \leqslant \lambda_\varepsilon$. Moreover, by standard arguments in [25], solutions to the focusing NLS (1) admit local-in-time solutions $u \in C([0,T], H^s(\mathbb{R}^d))$ for some $T > 0$. The following theorem shows that smooth solutions to (6) indeed converge on the existence time of the limiting equation whenever solutions are sufficiently smooth.

THEOREM 2.1. *Let $\sigma \in \mathbb{N}$. Denote the solution v^ε to (11) and u the local solution to (1) such that $v^\varepsilon|_{t=0} = u|_{t=0}$. Further suppose $u \in L^\infty([0,T]; H^{s+4}(\mathbb{R}^d))$, for some $s > \frac{d}{2}$. Then*

$$\|u - v^\varepsilon\|_{L^\infty([0,T];H^s(\mathbb{R}^d))} \lesssim \varepsilon^2$$

PROOF. Fix $s > \frac{d}{2}$ and choose $M > 0$ such that the solutions satisfy $\|v^\varepsilon\|_{H^s}, \|u\|_{H^{s+4}} \leqslant M$ for all $t \in [0,T]$. Let us denote the difference $w^\varepsilon \equiv u - v^\varepsilon$ with $w^\varepsilon(0,\mathbf{x}) = 0$, where upon taking the difference of (1) and (11) gives the equation

$$i\partial_t w^\varepsilon + P_\varepsilon^{-1}\Delta w^\varepsilon = (P_\varepsilon^{-1} - 1)\Delta u + P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v^\varepsilon) - g(u),$$

where again we denote the nonlinearity $g(z) = |z|^{2\sigma} z$.

By Duhamel's formulation this yields the integral equation

$$w^\varepsilon(t) = i\int_0^t S_\varepsilon(t-s)\Big((1 - P_\varepsilon^{-1})\Delta u + \big(g(u) - P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v^\varepsilon)\big)\Big)\, ds. \tag{75}$$

We first perform a few preliminary estimates on the integral formulation above. The estimates are identical whether we are in the case of full or partial off-axis variation. We recall the use of the Fourier variables $\mathbf{x} = (x,y) \mapsto (\xi,\eta) = \boldsymbol{\xi}$, where in the former case there is no distinction between η and ξ. We first note that we can rewrite the symbol $(\widehat{1 - P_\varepsilon^{-1}})(\eta) = \dfrac{\varepsilon^2|\eta|^2}{1 + \varepsilon^2|\eta|^2}$, and estimate the first term in H^s as follows

$$\|(1 - P_\varepsilon^{-1})\Delta u\|_{H^s} \leqslant \varepsilon^2 \big\|(1 + \varepsilon^2|\eta|^2)^{-2}\big\|^{1/2}_{L^\infty_{\xi,\eta}} \|\langle \boldsymbol{\xi}\rangle^{s+2}|\eta|^2 \widehat{u}\|_{L^2_{\xi,\eta}} \leqslant \varepsilon^2 \|u\|_{H^{s+4}}.$$

For the second term one estimates by first adding and subtracting $P_\varepsilon^{-1/2} g(u)$, and then applying the triangle inequality to give

$$\begin{aligned}\|g(u) - P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v^\varepsilon)\|_{H^s} &\leqslant \|P_\varepsilon^{-1/2} g(u) - P_\varepsilon^{-1/2} g(P_\varepsilon^{-1/2} v^\varepsilon)\|_{H^s} + \|g(u) - P_\varepsilon^{-1/2} g(u)\|_{H^s} \\ &\leqslant \|g(u) - g(P_\varepsilon^{-1/2} v^\varepsilon)\|_{H^s} + \|\big(1 - P_\varepsilon^{-1/2}\big)\big(|u|^{2\sigma} u\big)\|_{H^s} = \mathcal{Q}_1 + \mathcal{Q}_2,\end{aligned}$$

where in the first term we used the boundedness of $P_\varepsilon^{-1/2}$ i.e. $\|P_\varepsilon^{-1/2} f\|_{L^2} \leqslant \|f\|_{L^2}$.

The bound for $\mathcal{Q}_1$ begins by repeated application of the normed algebra property followed by the boundedness of $P_\varepsilon^{-1/2}$ to give

$$\mathcal{Q}_1 \leqslant C_\sigma\Big(\|P_\varepsilon^{-1/2} v^\varepsilon\|^{2\sigma}_{H^s} + \|u\|^{2\sigma}_{H^s}\Big)\|u - P_\varepsilon^{-1/2} v^\varepsilon\|_{H^s} \leqslant 2C_\sigma M^{2\sigma}\|u - P_\varepsilon^{-1/2} v^\varepsilon\|_{H^s}.$$

To finish the estimate $\mathcal{Q}_1$, we add and substract $P_\varepsilon^{-1/2}u$ and apply the triangle inequality so that

$$\begin{aligned}\|u - P_\varepsilon^{-1/2}v^\varepsilon\|_{H^s} &\leqslant \|P_\varepsilon^{-1/2}(u - v^\varepsilon)\|_{H^s} + \|(1 - P_\varepsilon^{-1/2})u\|_{H^s} \\ &\leqslant \|u - v^\varepsilon\|_{H^s} + \varepsilon^2 \big\|(1+\varepsilon^2|\eta|^2)^{-1}(1+\sqrt{1+\varepsilon^2|\eta|^2})^{-1}\big\|_{L^\infty_{\xi,\eta}} \||\eta|^2 u\|_{H^s}.\end{aligned}$$

Hence, we see in total that we have the estimate

$$\mathcal{Q}_1 \lesssim \|w^\varepsilon\|_{H^s} + \varepsilon^2 M^{2\sigma}\|u\|_{H^{s+2}}.$$

The second estimate for $\mathcal{Q}_2$ follows similarly as above

$$\mathcal{Q}_2 \leqslant \varepsilon^2 \big\|\widehat{P}_\varepsilon^{-1}(1+\widehat{P}_\varepsilon^{1/2})^{-1}\big\|_{L^\infty_{\xi,\eta}} \|\langle \boldsymbol{\xi}\rangle^s|\eta|^2\widehat{g(u)}\|_{L^2_{\xi,\eta}} \leqslant \varepsilon^2\|g(u)\|_{H^{s+2}} \leqslant \varepsilon^2\|u\|_{H^{s+2}}^{2\sigma+1},$$

where the last inequality follows by the normed algebra property for H^s.

After applying the H^s-norm to the integral equation, one applies Minkowski's inequality and uses that $S_\varepsilon(\cdot)$ is an isometry on H^s and the estimates for $\mathcal{Q}_1$ and $\mathcal{Q}_2$ above to obtain for $t \in [0,T]$ that

$$\|w^\varepsilon(t)\|_{H^s} \lesssim \varepsilon^2 T\|u\|_{L^\infty H^{s+4}}\Big(1 + \|u\|_{L^\infty H^{s+4}}^{2\sigma}\Big) + \int_0^t \|w^\varepsilon(s)\|_{H^s}\, ds.$$

After an application of Gronwall's lemma this implies the main result

$$\|w^\varepsilon(t)\|_{H^s} \lesssim \varepsilon^2 T e^T\|u\|_{L^\infty H^{s+4}}\Big(1 + \|u\|_{L^\infty H^{s+4}}^{2\sigma}\Big) \quad \text{for} \quad t \in [0,T].$$

$\square$

CHAPTER 3

On a class of derivative Nonlinear Schrödinger-type equations in two spatial dimensions [1]

The outline of Chapter 3 is organized as follows. In Section 1, we shall numerically construct nonlinear stationary states to

$$iP_\varepsilon \partial_t u + \Delta u + (1 + i\boldsymbol{\delta} \cdot \nabla)\big(|u|^{2\sigma} u\big) = 0, \quad u_{|t=0} = u_0(\mathbf{x}), \tag{76}$$

or equivalently the evolution equation

$$i\partial_t u + P_\varepsilon^{-1} \Delta u + P_\varepsilon^{-1}(1 + i\boldsymbol{\delta} \cdot \nabla)\big(|u|^{2\sigma} u\big) = 0, \quad u_{|t=0} = u_0(x_1, x_2). \tag{77}$$

These simulations also include the well-known ground states for the classical NLS. For the sake of illustration, we shall also derive explicit formulas for the one dimensional case and compare them with the well-known formulas for the classical (derivative) NLS. Certain perturbations of these stationary states will form the class of initial data considered in the numerical time-integration of (77). The numerical algorithm used to perform the respective simulations is detailed in Section 2. Here, we also include several basic numerical tests which compare the new model (77) to the classical (derivative) NLS. Analytical results yielding global well-posedness of (77) with either full or partial off-axis variations are given in Sections 3 and 5, respectively. In the former case, the picture is much more complete, which allows us to perform a numerical study of the (in-)stability properties of the corresponding stationary states, see Section 4. In the case with only partial off-axis variations, the problem of global existence is more complicated and one needs to distinguish between the cases where the action of P_ε is either parallel or orthogonal to the self-steepening. Analytically, only the former case can be treated at the present moment (see Section 5). Numerically, however, we shall present simulations for both of these cases in Section 6.

[1]This Chapter is excerpted from the following work: [7] J. Arbunich, C. Klein and C. Sparber. *On a class of derivative nonlinear Schrödinger equations in two spatial dimensions.* To appear in ESAIM Math. Model. Numer. Anal.

1. Stationary states

In this section, we focus on *stationary states*, i.e., time-periodic solutions to (76) given in the following form:

$$u(t, x_1, x_2) = e^{it} Q(x_1, x_2). \tag{78}$$

The profile Q then solves

$$P_\varepsilon Q = \Delta Q + (1 + i\boldsymbol{\delta} \cdot \nabla)(|Q|^{2\sigma} Q), \tag{79}$$

subject to the requirement that $Q(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Every non-zero solution $Q(\mathbf{x}) \in \mathbb{C}$ gives rise to a solitary wave solution (with speed zero) to (76). These solitary waves will be an important benchmark for our numerical simulations to follow. Note that in (78) we only allow for a simple time-dependence $\exp(i\omega t)$ with $\omega = 1$ in (78). This is not a restriction for the usual 2D NLS, given its scaling invariance, but it is a restriction for our model in which this invariance is broken (see also [**57, 81**] for the connection between ω and the speed of stable solitary waves).

For the classical NLS, when $\varepsilon = 0$ and $|\,\boldsymbol{\delta}\,| = 0$, there exists a particular solution Q, called *the nonlinear ground state*, which is the unique radial and positive solution to (79), cf. [**48, 101**]. Recall that in dimensions $d = 2$ the NLS is already L^2-critical and thus ground states in general cannot be obtained as minimizers of the associated energy functional (which is the same for both $\varepsilon = 0$ and $\varepsilon > 0$, see [**39**]). As we shall see below for $\varepsilon > 0$, the regularization via P_ε yields a natural modification of the ground state Q by smoothly widening its profile (while conserving positivity). We shall thus also refer to these solutions Q as the ground states for (79) with $|\,\boldsymbol{\delta}\,| = 0$ and $\varepsilon > 0$. At present, there are unfortunately no analytical results on the existence and uniqueness of such modified ground states available. However, our numerical algorithm indicates that they exist and are indeed unique (although, in general no longer radially symmetric).

The situation with derivative nonlinearity $|\,\boldsymbol{\delta}\,| \neq 0$ is somewhat more complicated, since in this case the profiles Q to (79) are always complex-valued and hence the notion of a ground state does not directly extend to this case (recall that uniqueness is only known for positive solutions). At least in $d = 1$, however, explicit calculations (see below) show, that there is a class of smooth $\boldsymbol{\delta}$-dependent *stationary solutions* to (79), which for $|\,\boldsymbol{\delta}\,| = 0$ yield the family of ε-depedent ground states.

1.1. Explicit solutions in 1D. In one spatial dimension, equation (79) allows for explicit formulas, which will serve as a basic illustration for the combined effects of self-steepening and off-axis variations. Indeed, in one spatial dimension, equation (76) simplifies to

$$i(1-\varepsilon^2\partial_x^2)\partial_t u + \partial_x^2 u + (1+i\delta\partial_x)(|u|^{2\sigma}u) = 0. \tag{80}$$

Seeking a solution of the form (78) thus yields the following ordinary differential equation:

$$(1+\varepsilon^2)Q'' + (|Q|^{2\sigma}-1)Q + i\delta(|Q|^{2\sigma}Q)' = 0. \tag{81}$$

To solve this equation, we shall use the polar representation for $Q(x) \in \mathbb{C}$

$$Q(x) = A(x)e^{i\theta(x)}, \quad A(x), \theta(x) \in \mathbb{R}$$

where we impose the requirement that $A(x) \geqslant 0$ and $\lim_{x\to\pm\infty} A(x) = 0$. Plugging this ansatz into (81), factoring $e^{i\theta}$ out and isolating the real and imaginary part yields the following coupled system:

$$\begin{aligned}\big(1+\varepsilon^2\big)A'' + (A^{2\sigma}-1)A - A\theta'\Big((1+\varepsilon^2)\theta' + \delta A^{2\sigma}\Big) &= 0,\\ \big(1+\varepsilon^2\big)\big(A\theta'' + 2\theta' A'\big) + (2\sigma+1)\delta A^{2\sigma}A' &= 0.\end{aligned}$$

Multiplying the second equation by A and integrating from $-\infty$ to x gives

$$(1+\varepsilon^2)\theta' = -\frac{(2\sigma+1)\delta A^{2\sigma}}{2(\sigma+1)},$$

where here we implicitly assume that $A^2\theta'$ vanishes at infinity. Using the above, we infer that the amplitude solves

$$\big(1+\varepsilon^2\big)A'' + (A^{2\sigma}-1)A + \frac{(2\sigma+1)\delta^2}{4(1+\varepsilon^2)(\sigma+1)^2}A^{4\sigma+1} = 0, \tag{82}$$

while the phase is given a-posteriori through

$$\theta(x) = -\frac{(2\sigma+1)\delta}{2(1+\varepsilon^2)(\sigma+1)}\int_{-\infty}^{x} A^{2\sigma}(y)\,dy. \tag{83}$$

After some lengthy computation, similar to what is done for the usual NLS, cf. [48], the solution to (82) can be written in the form

$$A(x) = \left(\frac{2(\sigma+1)}{1+K_{\varepsilon,\delta}\cosh\left(\frac{2\sigma x}{\sqrt{1+\varepsilon^2}}\right)}\right)^{1/(2\sigma)}, \tag{84}$$

where $K_{\varepsilon,\delta} = \sqrt{1 + \frac{\delta^2}{1+\varepsilon^2}} > 0$. In view of (83), this implies that the phase function θ is given by

$$\theta(x) = -\operatorname{sgn}(\delta)(2\sigma+1)\arctan\left(\frac{\sqrt{1+\varepsilon^2}}{|\delta|}\left(1 + K_{\varepsilon,\delta} e^{\frac{2\sigma x}{\sqrt{1+\varepsilon^2}}}\right)\right), \tag{85}$$

where we omitted a physically irrelevant constant in the phase (clearly, Q is only unique up to multiplication by a constant phase).

Note that in the case with no self-steepening $\delta = 0$, the phase θ is zero. Thus, $Q(x) \equiv A(x)$ and we find

$$Q(x) = (\sigma+1)^{1/(2\sigma)} \operatorname{sech}^{1/\sigma}\left(\frac{\sigma x}{\sqrt{1+\varepsilon^2}}\right).$$

For $\varepsilon = 0$, this is the well-known ground state solution to (150) in one spatial dimension, cf. [**48, 101**]. We notice that adding the off-axis dispersion ($\varepsilon > 0$) widens the profile, causing it to decay more slowly as $x \to \pm\infty$ as can be seen in Fig. 1 on the left. On the right of Fig. 1, it is shown that the maximum of the ground state decreases with σ but that the peak becomes more compressed.

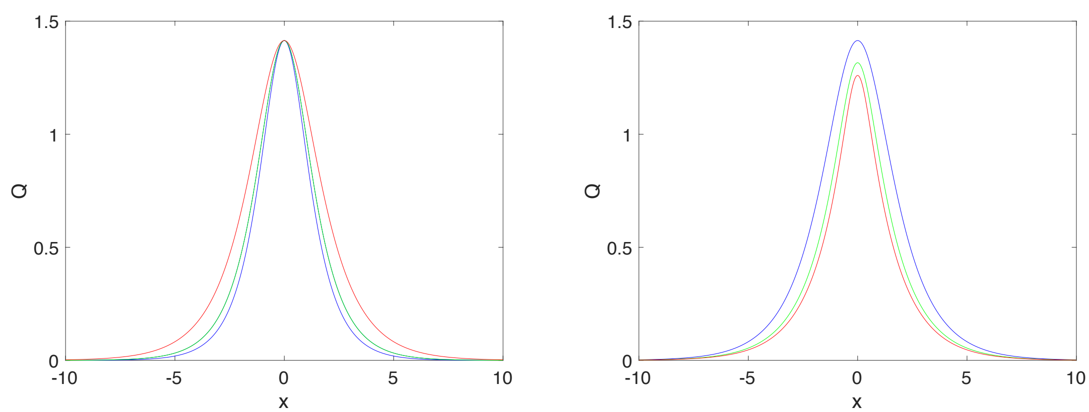

FIGURE 1. Ground state solution to (84) with $\delta = 0$: On the left for $\sigma = 1$ and $\varepsilon = 0$ (blue), $\varepsilon = 0.5$ (green) and $\varepsilon = 1$ (red). On the right for $\varepsilon = 1$ and $\sigma = 1$ (blue), $\sigma = 2$ (green) and $\sigma = 3$ (red).

REMARK 3.1. *The (σ-generalized) one-dimensional derivative NLS can be obtained from (80) by putting $\varepsilon = 0$, rescaling $u(t,x) = \delta^{-1/(2\sigma)}\tilde{u}(t,x)$, and letting $\delta \to \infty$. Note that $\tilde{u}$ solves*

$$i\partial_t \tilde{u} + \partial_x^2 \tilde{u} + (\delta^{-1} + i\partial_x)(|\tilde{u}|^{2\sigma}\tilde{u}) = 0.$$

Denoting $\tilde{Q} = \tilde{A}e^{i\tilde{\theta}(x)}$, we get from (84) and (85) the well-known zero-speed solitary wave solution of the derivative NLS, i.e.,

$$\tilde{A}(x) = \big(2(\sigma+1)\operatorname{sech}(2\sigma x)\big)^{1/(2\sigma)}, \quad \tilde{\theta}(x) = -(2\sigma+1)\arctan(e^{2\sigma x}).$$

The stability of these states has been studied in the works [**30**, **57**, **81**].

1.2. Numerical construction of stationary states. In more than one spatial dimension, no explicit formula is known for Q. Instead, we shall numerically construct Q by following an approach similar to those in [**74**, **76**]. Since we can expect Q to be rapidly decreasing, we use a Fourier spectral method and approximate

$$\mathcal{F}(Q) \equiv \widehat{Q}(\xi_1, \xi_2) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} Q(x_1, x_2) e^{-ix_1\xi_1} e^{-ix_2\xi_2} \, dx_1 dx_2$$

by a discrete Fourier transform which can be efficiently computed via the Fast Fourier Transform (FFT). In an abuse of notation, we shall in the following use the same symbols for the discrete and continuous Fourier transform. To apply FFTs, we will use a *computational domain* of the form

$$\Omega = [-\pi, \pi] L_{x_1} \times [-\pi, \pi] L_{x_2}, \tag{86}$$

and choose L_{x_1}, $L_{x_2} > 0$ sufficiently large so that the obtained Fourier coefficients of Q decrease to machine precision, roughly 10^{-16}, which in practice is slightly larger due to unavoidable rounding errors.

Now, recall that for a solution of the form (78) to satisfy (76), the function Q needs to solve (79). In Fourier space, this equation takes the simple form

$$\widehat{Q}(\xi_1, \xi_2) = \widehat{\Gamma}_\varepsilon \mathcal{F}(|Q|^{2\sigma} Q)(\xi_1, \xi_2),$$

where

$$\widehat{\Gamma}_\varepsilon(\xi_1, \xi_2) = \frac{(1 - \delta_1\xi_1 - \delta_2\xi_2)}{1 + \xi_1^2 + \xi_2^2 + \varepsilon^2 \sum_{i=1}^k \xi_i^2}.$$

For $\delta_1 = \delta_2 = 0$, the solution Q can be chosen to be real, but this will no longer be true for $\delta_{1,2} \neq 0$. In the latter situation, we will decompose

$$Q(x_1, x_2) = \alpha(x_1, x_2) + i\beta(x_1, x_2),$$

and separate (79) into its real and imaginary part, yielding a coupled nonlinear system for α, β. By using FFTs, this is equivalent to the following system for $\widehat{\alpha}$ and $\widehat{\beta}$:

$$\begin{cases} \widehat{\alpha}(\xi_1, \xi_2) - \widehat{\Gamma}_\varepsilon \mathcal{F}\Big(\big(\alpha^2 + \beta^2\big)^\sigma \alpha\Big)(\xi_1, \xi_2) = 0, \\ \widehat{\beta}(\xi_1, \xi_2) - \widehat{\Gamma}_\varepsilon \mathcal{F}\Big(\big(\alpha^2 + \beta^2\big)^\sigma \beta\Big)(\xi_1, \xi_2) = 0. \end{cases}$$

Formally, the system can be written as $M(\widehat{\mathbf{q}}) = 0$ where $\widehat{\mathbf{q}} = (\widehat{\alpha}, \widehat{\beta})^\top$ and solved via a *Newton iteration*. One thereby starts from an initial iterate $\widehat{\mathbf{q}}^{(0)}$ and computes the n-th iterate via the well known formula

$$\widehat{\mathbf{q}}^{n} = \widehat{\mathbf{q}}^{n-1} - J\left(\widehat{\mathbf{q}}^{n-1}\right)^{-1} M\left(\widehat{\mathbf{q}}^{n-1}\right) \quad n \in \mathbb{N}, \tag{87}$$

where J is the Jacobian of M with respect to $\widehat{\mathbf{q}}$. Since our required numerical resolution makes it impossible to directly compute the action of the inverse Jacobian, we instead employ a Krylov subspace approach as in [95]. Numerical experiments show that when the initial iterate $\widehat{\mathbf{q}}^{(0)}$ is sufficiently close to the final solution, then we obtain the expected quadratic convergence of our scheme and reach a precision of order 10^{-10} after only 4 to 8 iterations.

As a basic test case, we compute the ground state of the standard two-dimensional focusing NLS with $\sigma = 1$, using as an initial iterate

$$q^{(0)}(x_1, x_2) = \operatorname{sech}^2\left(\sqrt{x_1^2 + x_2^2}\right)$$

on the computational domain (86) with $L_{x_1} = L_{x_2} = 5$. By choosing $N_{x_1} = N_{x_2} = 2^9$ many Fourier modes, we have after seven iterations of (87) a residual smaller than 10^{-12}. The obtained solution is given on the left of Fig. 2. As expected, the solution is radially symmetric.

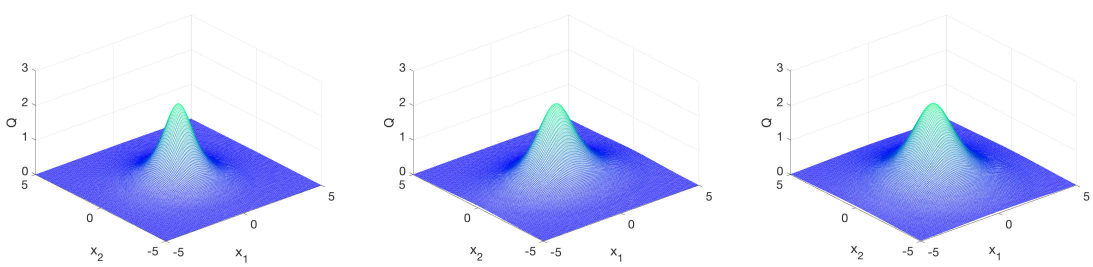

FIGURE 2. Ground state solution to equation (76) with $\sigma = 1$ and $\delta = 0$: On the left for $\varepsilon = 0$, in the middle for $\varepsilon = 1$ and $k = 1$ (partial off-axis dependence), on the right for $\varepsilon = 1$ and $k = 2$ (full off-axis dependence).

The numerical ground state solution hereby obtained will then be used as an initial iterate for the situation with non-vanishing ε and $\boldsymbol{\delta}$, as follows:

Step 1: In the case without self-steepening $\delta_1 = \delta_2 = 0$, the iteration is straightforward even for relatively large values such as $\varepsilon = 1$. It can be seen in the middle of Fig. 2, that the ground state for $\varepsilon = 1$ and $k = 1$ is no longer radially symmetric. As an effect of the partial off-axis variation, the solution is elongated in the x_1-direction. In the case of full off-axis dependence, the ground state for the same value of $\varepsilon = 1$ can be seen in Fig. 2 on the right. The solution is again radially symmetric, but as expected less localized than the ground state of the classical standard NLS. This is consistent with the explicit formulas for Q found in the one dimensional case above.

Step 2: In the case with self-steeping $\delta_1 = \delta_2 = 1$, smaller intermediate steps have to be used in the iterations: We increment $\boldsymbol{\delta}$, by first varying only δ_2 in steps of 0.2, always using the last computed value for Q as an initial iterate for the slightly larger $\boldsymbol{\delta}$. The resulting solution Q can be seen in Fig. 3. Note that the imaginary part of Q is of the same order of magnitude as the real part.

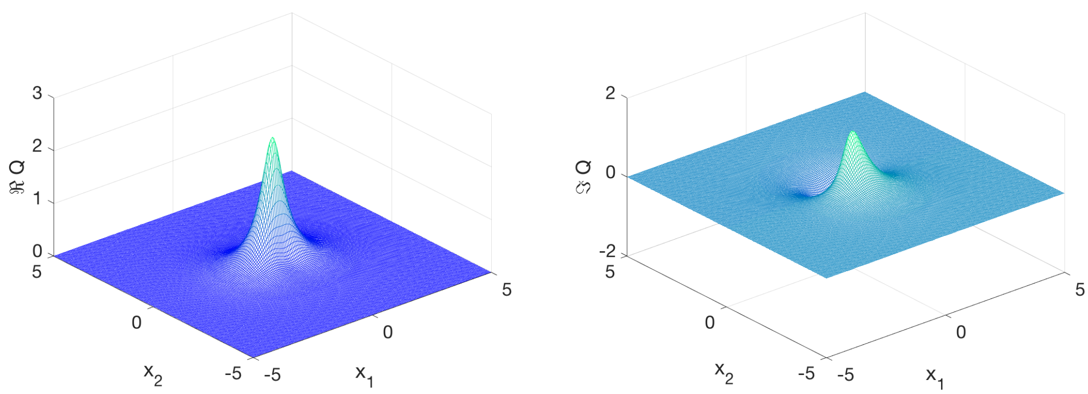

FIGURE 3. The stationary state solution Q to equation (76) with $\sigma = 1$, $\varepsilon = \delta_1 = 0$ and $\delta_2 = 1$: On the left, the real part of Q, on the right its imaginary part.

Step 3: In order to combine both effects within the same model, we shall use the (zero speed) solitary solution obtained for $\varepsilon = 0$ and $\boldsymbol{\delta} \neq \mathbf{0}$ as an initial iterate for the case of non-vanishing epsilon. In Fig. 4 we show on the left the stationary state for $\varepsilon = 1$, $k = 1$, $\delta_1 = 0$ and $\delta_2 = 1$, when the action of P_ε is orthogonal to the self-steepening. When compared to the case with $\varepsilon = 0$, the solution is seen to be elongated in the x_1-direction. Next, we simulate when P_ε acts parallel to the self-steepening, that is when $\varepsilon = 1$, $k = 1$, $\delta_1 = 1$ and $\delta_2 = 0$. The result is shown in the middle of Fig. 4. In comparison to the former case, the imaginary part of the solution is essentially rotated clockwise by 90 degrees. The elongation effect in the x_1-direction is still visible but less pronounced.

Step 4: For $\sigma > 1$ stationary states become increasingly peaked, as is seen from the 1D picture in Figure 1. Hence, to construct stationary states for higher nonlinear powers in 2D, we will consequently require more Fourier coefficients to effectively resolve these solutions. To this end, we work on the numerical domain (86) with $L_{x_1} = L_{x_2} = 3$ and use $N_{x_1} = N_{x_2} = 2^{10}$ Fourier modes. We use the ground state obtained for $\sigma = 1$ as an initial iterate for the case $\sigma = 2, 3$, and follow the same program as outlined above.

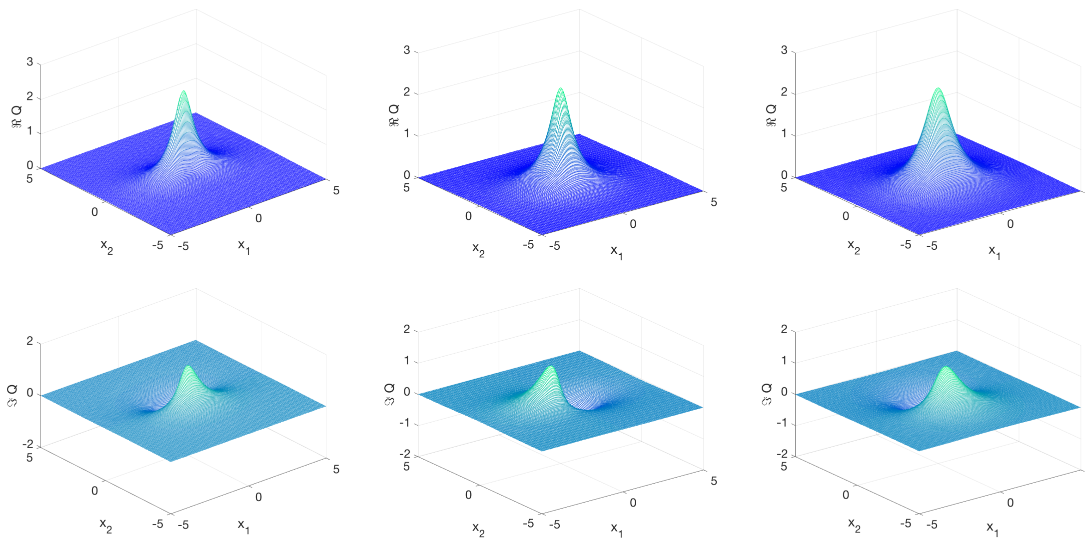

FIGURE 4. Real and imaginary parts of the stationary state Q to equation (76) with $\sigma = 1$: On the left for $\varepsilon = 1$, $k = 1$, $\delta_1 = 0$ and $\delta_2 = 1$, in the middle for $\varepsilon = 1$, $k = 1$, $\delta_1 = 1$ and $\delta_2 = 0$, and on the right for $\varepsilon = 1$, $k = 2$, $\delta_1 = 0$ and $\delta_2 = 1$.

2. Numerical method for the time evolution

2.1. A Fourier spectral method. In this section, we briefly describe the numerical algorithm used to integrate our model equation in its evolutionary form (77). After a Fourier transformation, this equation becomes

$$\partial_t \widehat{u} = -i\widehat{P}_\varepsilon^{-1}(\boldsymbol{\xi})\Big(|\boldsymbol{\xi}|^2\widehat{u} - (1 - \boldsymbol{\delta}\cdot\boldsymbol{\xi})(\widehat{|u|^{2\sigma}u})\Big), \quad \boldsymbol{\xi} \in \mathbb{R}^2.$$

Approximating the above by a discrete Fourier transform (via FFT) on a computational domain Ω given by (86), yields a finite dimensional system of ordinary differential equations, which formally reads

$$\partial_t \widehat{u} = L_\varepsilon \widehat{u} + N_\varepsilon(\widehat{u}). \tag{88}$$

Here $L_\varepsilon = -iP_\varepsilon^{-1}|\boldsymbol{\xi}|^2$ is a linear, diagonal operator in Fourier space, and $N_\varepsilon(\widehat{u})$ has a nonlinear and nonlocal dependence on $\widehat{u}$. Since $\|L_\varepsilon\|$ can be large, equation (88) belongs to a family of *stiff* ODEs, for which several efficient numerical schemes have been developed, cf. [**69, 73**] where the particular situation of semi-classical NLS is considered. Driscoll's composite Runge-Kutta (RK) method [**38**] has proven to be particularly efficient and thus will also be applied in the following simulations. This method uses a stiffly stable third order RK method for the high wave numbers of L_ε and combines it with a standard explicit fourth order RK method for the low wave numbers of L_ε and the nonlinear part $N_\varepsilon(\widehat{u})$. Despite combining a third order and a fourth order method, this approach yields fourth order in time convergence in many applications. Moreover, it provides an explicit method with much larger time steps than allowed by the usual fourth order stability conditions in stiff regimes.

REMARK 3.2. *The evolutionary form of our model* (77) *is in many aspects similar to the well-known Davey-Stewartson (DS) system, which is a non-local NLS type equation in two spatial dimensions, cf.* [**35**, **101**]. *In* [**73**, **75**, **77**]*, the possibility of self-similar blow-up in DS is studied, using a numerical approach similar to ours.*

As a first basic test of consistency, we apply our numerical code to the cubic NLS in 2D i.e. equation (1) with $\sigma = 1$. As initial data u_0 we take the ground state Q, obtained numerically as outlined in Section 1 above. We use $N_t = 1000$ time-steps for times $0 \leqslant t \leqslant 1$. In this case, we know that the exact time-dependent solution u is simply given by $u = Qe^{it}$. Comparing this to the numerical solution obtained at $t = 1$ yields an L^∞-difference of the order of 10^{-10}. This verifies both the code for the time evolution and the one for the ground state Q which in itself is obtained with an accuracy of order 10^{-10}. Thus, the time evolution algorithm evolves the ground state with the same precision as with which it is known.

For general initial data u_0, we shall control the accuracy of our code in two ways: On the one hand, the resolution in space is controlled via the decrease of the Fourier coefficients within (the finite approximation of) $\widehat{u}$. The coefficients of the highest wave-numbers thereby indicate the order of magnitude of the numerical error made in approximating the function via a truncated Fourier series. On the other hand, the quality of the time-integration is controlled via the conserved quantity defined in (9). Due to unavoidable numerical errors, the latter will numerically depend on time. For sufficient spatial resolution, the relative conservation of (9) will overestimate the accuracy in the time-integration by $1 - 2$ orders of magnitude.

2.2. Reproducing known results for the classical NLS. As already discussed in the introduction of this paper, the cubic NLS in two spatial dimensions is L^2-critical and its ground state solution Q is strongly unstable. Indeed, any perturbation of Q which lowers the L^2-norm of the initial data below that of Q itself, is known to produce purely dispersive, global-in-time solutions which behave like the free time evolution for large $|t| \gg 1$. However, perturbations that increase the L^2-norm of the initial data above that of Q are expected to generically produce a (self-similar) blow-up in finite-time. This behavior can be reproduced in our simulations.

To do so, we first take initial data of the form

$$u_0(x_1,x_2) = Q(x_1,x_2) - 0.1e^{-x_1^2-x_2^2}, \tag{89}$$

and work on the numerical domain Ω given by (86) with $L_{x_1} = L_{x_2} = 3$. We will use $N_t = 5000$ time-steps within $0 \leqslant t \leqslant 5$. We can see on the right of Fig. 5 that the L^∞-norm of the solution decreases monotonically, indicating purely dispersive behavior. The plotted absolute value of the solution at $t = 5$ confirms this behavior. In addition, the L^2-norm (2) is conserved to better than 10^{-13}, indicating that the problem is indeed well resolved in time.

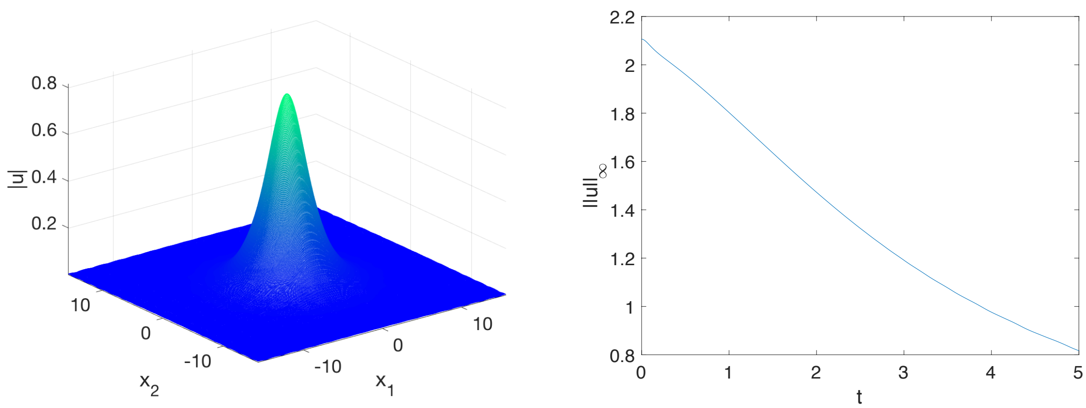

FIGURE 5. Solution to classical NLS (150) with $\sigma = 1$ and initial data (89): on the left $|u|$ at $t = 5$, and on the right the L^∞-norm of the solution as a function of t.

REMARK 3.3. *Note that we effectively run our simulations on $\Omega \simeq \mathbb{T}^2$, instead of $\mathbb{R}^2$. As a consequence, the periodicity will after some time induce radiation effects appearing on the opposite side of Ω. The treatment of (large) times $t > 5$ therefore requires a larger computational domain to suppress these unwanted effects.*

Next, for initial data of the form

$$u_0(x_1,x_2) = Q(x_1,x_2) + 0.1e^{-x_1^2-x_2^2}, \tag{90}$$

we again use $N_t = 5000$ time steps for $0 \leqslant t \leqslant 2$. As can be seen in Fig. 6 on the right, there is numerical indication for finite-time blow-up. The code is stopped at $t = 1.89$ when the relative error in the conserved L^2-norm (2) drops below 10^{-3}. The solution for $t = 1.88$ can be seen on the left of Fig. 6. This is in accordance with the self-similar blow-up established by Merle and Raphaël, cf. [**85, 87**]. In particular, we note that the result does not change notably if a higher resolution in both x and t is employed.

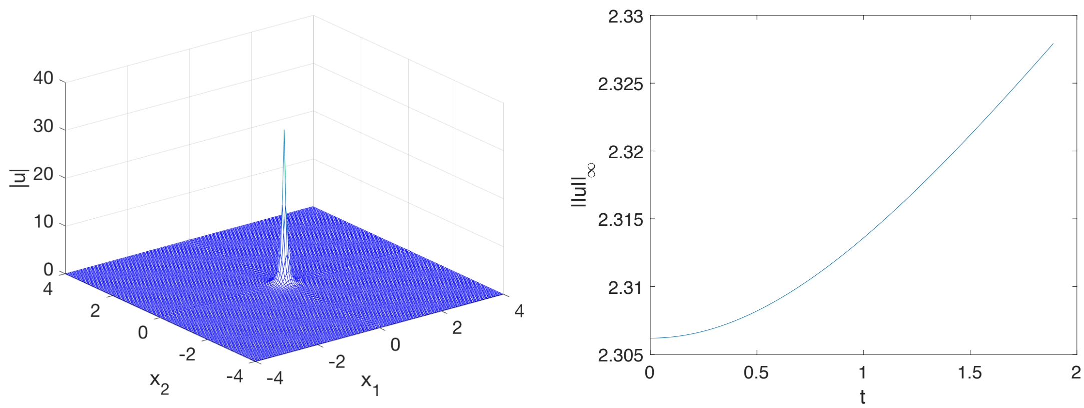

FIGURE 6. Solution to classical NLS (150) with $\sigma = 1$ and initial data (90): on the left $|u|$ at $t = 1.88$ and on the right the L^∞-norm of the solution as a function of t.

REMARK 3.4. *We want to point out that there are certainly more sophisticated methods available to numerically study self-similar blow-up, see for instance* [**76, 82, 101**] *for the case of NLS type models, as well as* [**71, 72**] *for the analogous problem in KdV type equations. However, these methods will not be useful for the present work, since as noted before, the model* (76) *does not admit a simple scaling invariance, which is the underlying reason for self-similar blow-up in NLS and KdV type models. As a result, all our numerical findings concerning finite-time blow-up have to be taken with a grain of salt. An apparent divergence of certain norms of the solution or overflow errors produced by the code can indicate a blow-up, but might also just indicate that one has run out of resolution. The results reported in this paper therefore need to be understood as being stated with respect to the given numerical resolution. However, we have checked that they remain stable under changes of the resolution within the accessible limits of the computers used to run the simulations.*

2.3. Time-dependent change of variables in the case with self-steepening. In the case of self-steepening, the ability to produce an accurate numerical time integration in the presence of a derivative nonlinearity ($\boldsymbol{\delta} \neq 0$) becomes slightly more complicated. The inclusion of such a nonlinearity can lead to localized initial data moving (relatively fast) in the direction chosen by $\boldsymbol{\delta}$. In turn, this might cause the numerical solution to "hit" the boundary of our computational domain Ω. To avoid this issue, we shall instead perform our numerical computations in a moving reference frame, chosen such that the maximum of $|u(t,x)|$ remains fixed at the origin. More precisely, we consider the transformation

$$x \mapsto x - y(t),$$

and denote $\mathbf{v}(t) = \dot{y}(t)$. The new unknown $u(t, x - y(t))$ solves

$$i\partial_t u - i\mathbf{v}\cdot\nabla u + P_\varepsilon^{-1}\Delta u + P_\varepsilon^{-1}(1 + i\boldsymbol{\delta}\cdot\nabla)\big(|u|^{2\sigma}u\big) = 0. \tag{91}$$

The quantity $\mathbf{v}(t) = (v_1(t), v_2(t))$ is then determined by the condition that the density $\rho = |u|^2$ has a maximum at $(x_1, x_2) = (0,0)$ for all $t \geqslant 0$. We get from (91) the following equation for ρ:

$$\begin{aligned}\partial_t\rho = &\,\mathbf{v}\cdot\nabla\rho + i\big(\bar{u}P_\varepsilon^{-1}\Delta u - uP_\varepsilon^{-1}\Delta\bar{u}\big) + i\big(\bar{u}P_\varepsilon^{-1}(\rho^\sigma u) - uP_\varepsilon^{-1}(\rho^\sigma\bar{u})\big)\\ &- \bar{u}P_\varepsilon^{-1}\boldsymbol{\delta}\cdot\nabla(\rho^\sigma u) - uP_\varepsilon^{-1}\boldsymbol{\delta}\cdot\nabla(\rho^\sigma\bar{u}).\end{aligned}$$

Differentiating this equation with respect to x_1 and x_2 respectively, and setting $x_1 = x_2 = 0$ yields the desired conditions for v_1 and v_2. Note that the computation of these additional derivatives appearing in this approach is expensive, since in practice it needs to be enforced in every step of the Runge-Kutta scheme. Hence, we shall restrict this approach solely to cases where the numerical results appear to be strongly affected by the boundary of Ω. In addition, we may always choose a reference frame such that one of the two components of $\boldsymbol{\delta}$ is zero, which consequently allows us to set either v_1, or v_2 equal to zero.

2.4. Basic numerical tests for a derivative NLS in 2D. As an example, we consider the case of a cubic nonlinear, two-dimensional derivative NLS of the following form

$$i\partial_t u + \Delta u + (1 + i\delta_2\partial_{x_2})\big(|u|^2 u\big) = 0, \quad u_{|t=0} = u_0(x_1, x_2). \tag{92}$$

which is obtained from our general model (77) for $\varepsilon = 0$ and $\delta_1 = 0$. We take initial data u_0 given by (90). Here, Q is the ground state computed earlier for this particular choice of parameters, see Fig. 3. We work on the computational domain (86) with $L_{x_1} = L_{x_2} = 3$, using $N_{x_1} = N_{x_2} = 2^{10}$ Fourier modes and 10^5 time-steps for $0 \leqslant t \leqslant 5$. We also apply a Krasny filter [78], which sets all Fourier coefficients smaller than 10^{-10} equal to zero. For $\delta_2 = 1$ the real and imaginary part of the solution u at the final time $t = 5$ can be seen in Fig. 7 below. Note that they are both much more localized and peaked when compared to the ground state Q shown in Fig. 3, indicating a self-focusing behavior within u. Moreover, the real part of u is no longer positive due to phase modulations.

Surprisingly, however, there is *no* indication of a finite-time blow-up, in contrast to the analogous situation without derivative nonlinearity (recall Fig. 6 above). Indeed, the Fourier coefficients of

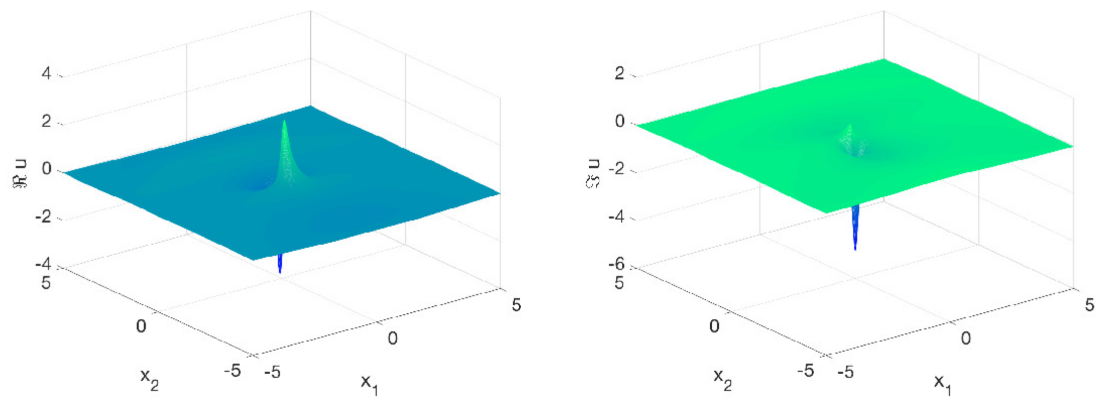

FIGURE 7. Real and imaginary part of the solution to (92) with $\delta_2 = 1$ at time $t = 5$ corresponding to $u_0 = Q + 0.1e^{-x_1^2 - x_2^2}$, where Q is the stationary state in Fig. 3.

$|u|$ at $t = 5$ are seen in Fig. 8 to decrease to the order of the Krasny filter. In addition, the L^∞-norm of the solution, plotted in the middle of the same figure, appears to exhibit a turning point shortly before $t \approx 4$. Finally, the velocity component v_2 plotted on the right in Fig. 8 seems to slowly converge to a some limiting value $v_2 \approx 2$. The latter would indicate the appearance of a stable moving soliton, but it is difficult to decide such questions numerically. All of these numerical findings are obtained with (9) conserved up to errors of the order 10^{-11}.

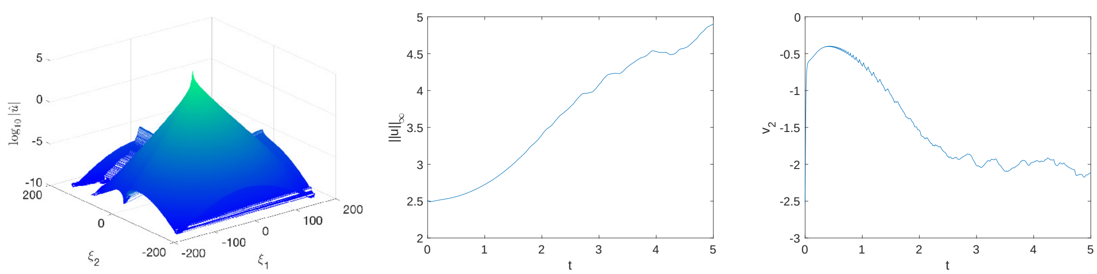

FIGURE 8. Solution to (92) with $\delta_2 = 1$ and perturbed stationary state initial data: The Fourier coefficients of $|u|$ at $t = 5$ on the left; the L^∞-norm of the solution as a function of time in the middle, and the time evolution of its velocity v_2 on the right.

It might seem extremely surprising that the addition of a derivative nonlinearity is able to suppress the appearance of finite-time blow-up. Note however, that in all the examples above we have used only (a special case of) perturbed ground states Q as initial data. For more general initial data, the situation is radically different, as can be illustrated numerically in the following example. We solve (92) with purely Gaussian initial data of the form

$$u_0(x_1, x_2) = 4e^{-(x_1^2 + x_2^2)} \tag{93}$$

on a numerical domain Ω with $L_{x_1} = L_{x_2} = 2$, using $N_{x_1} = N_{x_2} = 2^{10}$ Fourier coefficients and $N_t = 10^5$ time steps for $0 \leqslant t \leqslant 0.25$. This case appears to exhibit finite-time blow-up, as is illustrated in Fig. 9. The conservation of the numerically computed L^2-norm i.e. (2) drops below 10^{-3} at $t \approx T = 0.1955$ which indicates that plotting accuracy is no longer guaranteed. Consequently we ignore data taken for later times, but note that the code stops with an overflow error for $t \approx 0.202$.

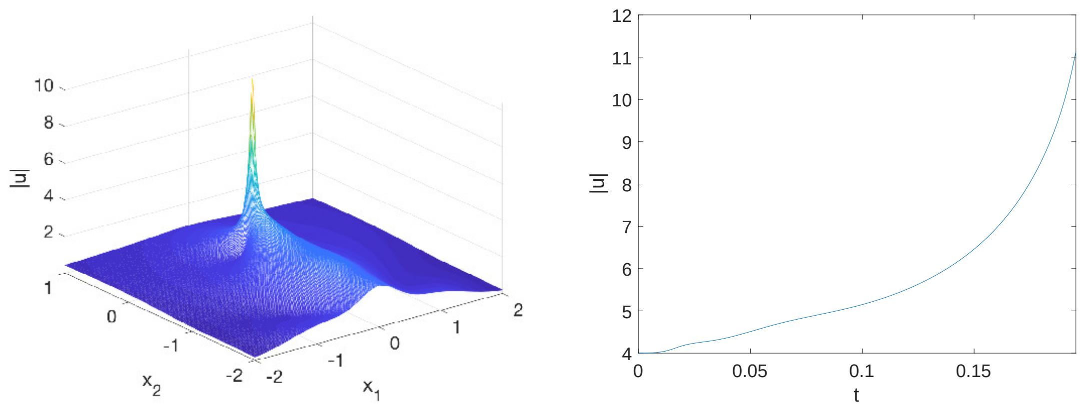

FIGURE 9. The modulus of the solution to (92) with $\delta_2 = 1$ for Gaussian initial data $u_0 = 4e^{-x_1^2 - x_2^2}$, at time $t = 0.195$. On the right, the L^∞-norm of the solution as a function of time.

REMARK 3.5. *These numerical findings are consistent with analytical results for derivative NLS in one spatial dimension. For certain values of $\sigma \geqslant 1$ and certain velocities v, the corresponding solitary wave solutions are found to be orbitally stable, see* [**30**, **57**, **81**]. *However, for general initial data and $\sigma > 1$ large enough, one expects finite-time blow-up, cf.* [**82**].

3. Global well-posedness with full off-axis variation

In this section we will analyze the Cauchy problem corresponding to (77) in the case of full off-axis dependence, i.e. $k = 2$, so that

$$P_\varepsilon = 1 - \varepsilon^2 \Delta.$$

In this context, we expect the solution u of (77) to be very well behaved due to the strong regularizing effect of the elliptic operator P_ε acting in both spatial directions.

To prove a global-in-time existence result, we rewrite (77), using Duhamel's formula,

$$u(t) = S_\varepsilon(t)u_0 + i\int_0^t S_\varepsilon(t-s)P_\varepsilon^{-1}(1 + i\boldsymbol{\delta}\cdot\nabla)\big(|u|^{2\sigma}u\big)(s)\, ds \equiv \Phi(u)(t), \tag{94}$$

where $S_\varepsilon(t) = e^{itP_\varepsilon^{-1}\Delta}$ is the corresponding linear propagator introduced in Chapter 2, which we recall is an isometry on $H^s(\mathbb{R}^2)$ for any $s \in \mathbb{R}$. It is known that in the case with full off-axis variation, $S_\varepsilon(t)$ does not allow for any Strichartz estimates, see [19]. However, the action of P_ε^{-1} allows us to "gain" two derivatives and offset the action of the gradient term in the nonlinearity of (94). By way of a fixed point argument, we can therefore prove the following result, which is similar in spirit to Lemma 2.11.

THEOREM 3.1 (Full off-axis variations). *Let $\varepsilon > 0$, $k = 2$ and $\sigma > \frac{1}{4}$. Then for any $\boldsymbol{\delta} \in \mathbb{R}^2$ and any $u_0 \in H^1(\mathbb{R}^2)$, there exists a unique global-in-time solution $u \in C(\mathbb{R}_t; H^1(\mathbb{R}^2))$ to (77), depending continuously on the initial data. Moreover, we have the uniform-in-time estimate*

$$\|u(t,\cdot)\|_{H^1} \leqslant C(\varepsilon, \|u_0\|_{H^1}), \quad \forall\, t \in \mathbb{R}.$$

PROOF. Let $T, M > 0$. We aim to show that $u \mapsto \Phi(u)$ is a contraction on the ball

$$X_{T,M} = \{u \in L^\infty([0,T); H^1(\mathbb{R}^2)) : \|u\|_{L_t^\infty H_x^1} \leqslant M\}.$$

To this end, let us shortly denote

$$\Phi(u)(t) = S_\varepsilon(t)u_0 + \mathcal{N}(u)(t), \tag{95}$$

where for $g(u) = |u|^{2\sigma}u$, we write

$$\mathcal{N}(u)(t) := i\int_0^t S_\varepsilon(t-s)P_\varepsilon^{-1}(1 + i\boldsymbol{\delta}\cdot\nabla)g(u(s))\,ds.$$

Now, let $u, u' \in X_{T,M}$. Using Minkowski's inequality and recalling that $S_\varepsilon(t)$ is an isometry on $H^1(\mathbb{R}^2)$ yields

$$\big\|\big(\mathcal{N}(u)(t) - \mathcal{N}(u')(t)\big)\big\|_{H^1} \leqslant \varepsilon^{-2}(1 + |\,\boldsymbol{\delta}\,|)\int_0^t \|g(u) - g(u')\|_{L^2_{\mathbf{x}}}(s)\,ds.$$

To bound the integrand, we first note that

$$|g(u) - g(u')| \leqslant C_\sigma(|u|^{2\sigma} + |u'|^{2\sigma})|u - u'|. \tag{96}$$

If we impose $\sigma > \frac{1}{4}$, then we have by Sobolev's embedding that

$$H^1(\mathbb{R}^2) \subset H^{\frac{(4\sigma-1)}{4\sigma}}(\mathbb{R}^2) \hookrightarrow L^{8\sigma}(\mathbb{R}^2) \quad \text{and} \quad H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2).$$

This allows us to estimate further after using (96) and Hölder's inequality in space to give

$$\begin{aligned}\|g(u)-g(u')\|_{L^2_{\mathbf{x}}} &\leqslant \big(\|u\|^{2\sigma}_{L^{8\sigma}_{\mathbf{x}}} + \|u'\|^{2\sigma}_{L^{8\sigma}_{\mathbf{x}}}\big)\|u-u'\|_{L^4_{\mathbf{x}}} \\ &\leqslant \Big(\|u\|^{2\sigma}_{H^1} + \|u'\|^{2\sigma}_{H^1}\Big)\|u-u'\|_{H^1}.\end{aligned}$$

Together with Hölder's inequality in t, we can consequently bound

$$\big\|\mathcal{N}(u)-\mathcal{N}(u')\big\|_{L^\infty_t H^1} \leqslant 2\varepsilon^{-2}(1+|\,\boldsymbol{\delta}\,|)TM^{2\sigma}\|u-u'\|_{L^\infty_t H^1}.$$

By choosing $T>0$ sufficiently small, Banach's fixed point theorem directly yields a unique local-in-time solution $u \in C([0,T],H^1(\mathbb{R}^2))$. Standard arguments (see, e.g., [25]) then allow us to extend this solution up to a maximal time of existence $T_{\max} = T_{\max}(\|u_0\|_{H^1_{\mathbf{x}}}) > 0$ and we also infer continuous dependence on the initial data.

Next, we shall prove that

$$\|P_\varepsilon^{1/2}u(t)\|_{L^2_{\mathbf{x}}} = \|P_\varepsilon^{1/2}u_0\|_{L^2_{\mathbf{x}}}, \quad \text{for all } t\in[0,T] \text{ and } T<T_{\max}. \tag{97}$$

For $\varepsilon>0$, this conservation law yields a uniform bound on the H^1-norm of u, since

$$c_\varepsilon\|P_\varepsilon^{1/2}\varphi\|_{L^2_{\mathbf{x}}} \leqslant \|\varphi\|_{H^1} \leqslant C_\varepsilon\|P_\varepsilon^{1/2}\varphi\|_{L^2_{\mathbf{x}}}, \quad C_\varepsilon, c_\varepsilon>0.$$

We consequently can re-apply the fixed point argument as many times as we wish, thereby preserving the length of the maximal interval in each iteration, to yield $T_{\max}=+\infty$. Since the equation is time-reversible modulo complex conjugation, we obtain a global H^1-solution for all $t\in\mathbb{R}$, provided (97) holds.

To prove (97), we adapt and (slightly) modify an elegant argument given in [90], used as in the proof of Theorem 2.7. The advantage of this argument resides in that it does not require an approximation procedure via a sequence of sufficiently smooth solutions (as is classically done, see e.g. [25]).

Let $t\in[0,T]$ for $T<T_{\max}$. We first rewrite Duhamel's formula (94), using the continuity of the semigroup S_ε to propagate backwards in time

$$S_\varepsilon(-t)u(t) = u_0 + S_\varepsilon(-t)\mathcal{N}(u)(t). \tag{98}$$

As $S_\varepsilon(\cdot)$ is unitary in L^2, we have $\|P_\varepsilon^{1/2}u(t)\|_{L^2_{\mathbf{x}}} = \|S_\varepsilon(-t)P_\varepsilon^{1/2}u(t)\|_{L^2_{\mathbf{x}}}$. The latter can be expressed using the above identity:

$$\|P_\varepsilon^{1/2}\,u(t)\|_{L^2_{\mathbf{x}}} = \|P_\varepsilon^{1/2}u_0\|_{L^2_{\mathbf{x}}} + 2\mathrm{Re}\left\langle S_\varepsilon(-t)P_\varepsilon^{1/2}\mathcal{N}(u)(t), P_\varepsilon^{1/2}u_0\right\rangle_{L^2_{\mathbf{x}}} + \|S_\varepsilon(-t)P_\varepsilon^{1/2}\mathcal{N}(u)(t)\|^2_{L^2_{\mathbf{x}}}$$
$$\equiv \|P_\varepsilon^{1/2}u_0\|_{L^2_{\mathbf{x}}} + \mathcal{I}_1 + \mathcal{I}_2.$$

We want to show that $\mathcal{I}_1 + \mathcal{I}_2 = 0$. In view of (95) we can rewrite

$$\begin{aligned}\mathcal{I}_1 &= -2\mathrm{Im}\left\langle \int_0^t S_\varepsilon(-s)P_\varepsilon^{-1/2}(1+i\boldsymbol{\delta}\cdot\nabla)g(u)(s)\,ds, P_\varepsilon^{1/2}u_0\right\rangle_{L^2_{\mathbf{x}}} ds\\ &= -2\mathrm{Im}\int_0^t \left\langle (1+i\boldsymbol{\delta}\cdot\nabla)g(u)(s), S_\varepsilon(s)u_0\right\rangle_{L^2_{\mathbf{x}}} ds.\end{aligned}$$

By the Cauchy-Schwarz inequality we find that this quantity is indeed finite, since

$$|\mathcal{I}_1| \leqslant 2T\|(1+i\boldsymbol{\delta}\cdot\nabla)g(u)\|_{L^\infty_t L^2_{\mathbf{x}}}\|S_\varepsilon(\cdot)u_0\|_{L^\infty_t L^2_{\mathbf{x}}} < \infty.$$

Denoting for simplicity $G_\varepsilon(\cdot) = P_\varepsilon^{-1}(1+i\boldsymbol{\delta}\cdot\nabla)g(u)(\cdot)$, we find after a lengthy computation (see Proposition 2.7 for more details) that the integral

$$\mathcal{I}_2 = 2\mathrm{Re}\int_0^t \left\langle P_\varepsilon G_\varepsilon(s), -i\mathcal{N}(u)(s)\right\rangle_{L^2_{\mathbf{x}}} ds.$$

We can express $-i\mathcal{N}(u)(s)$ using the integral formulation (98) and write

$$\mathcal{I}_2 = 2\mathrm{Re}\Big(\int_0^t \left\langle P_\varepsilon G_\varepsilon(s), iS_\varepsilon(s)u_0\right\rangle_{L^2_{\mathbf{x}}} ds + \int_0^t \left\langle P_\varepsilon G_\varepsilon(s), -iu(s)\right\rangle_{L^2_{\mathbf{x}}} ds\Big). \tag{99}$$

Next, we note that the particular form of our nonlinearity implies

$$\begin{aligned}\mathrm{Re}\left\langle P_\varepsilon G_\varepsilon, -iu\right\rangle_{L^2_{\mathbf{x}}} &= \mathrm{Im}\left\langle (1+i\boldsymbol{\delta}\cdot\nabla)g(u), u\right\rangle_{L^2_{\mathbf{x}}}\\ &= \mathrm{Im}\,\|u\|^{2\sigma+2}_{L^{2\sigma+2}_{\mathbf{x}}} - \mathrm{Re}\left\langle g(u), (\boldsymbol{\delta}\cdot\nabla)u\right\rangle_{L^2_{\mathbf{x}}}.\end{aligned}$$

Here, the first expression in the last line is obviously zero, whereas for the second term we compute

$$\mathrm{Re}\left\langle g(u), (\boldsymbol{\delta}\cdot\nabla)u\right\rangle_{L^2_{\mathbf{x}}} = \int_{\mathbb{R}^2}|u|^{2\sigma}\mathrm{Re}\big(u(\boldsymbol{\delta}\cdot\nabla)\overline{u}\big)\,d\mathbf{x} = \frac{1}{2(\sigma+1)}\int_{\mathbb{R}^2}(\boldsymbol{\delta}\cdot\nabla)(|u|^{2\sigma+2})\,d\mathbf{x} = 0,$$

for H^1-solutions u. In summary, the second term on the right-hand side of (99) simply vanishes and we find

$$\mathcal{I}_2 = 2\mathrm{Im}\int_0^t \left\langle (1+i\boldsymbol{\delta}\cdot\nabla)g(u)(s), S_\varepsilon(s)u_0\right\rangle_{L^2_{\mathbf{x}}} ds = -\mathcal{I}_1.$$

This finishes the proof of (97). $\square$

4. (In-)stability properties of stationary states with full off-axis variation

In this section, we shall perform numerical simulations to study the orbital stability or instability properties of the (zero speed) solitary wave Qe^{it} in the case with self-steepening $|\boldsymbol{\delta}| \neq 0$ and full off-axis variation $k = 2$. In view of Theorem 3.1, we know that there cannot be any strong instability, i.e., instability due to finite-time blow-up. Nevertheless, we shall see that there is a wealth of possible scenarios, depending on the precise choice of parameters, σ, $\boldsymbol{\delta}$, and on the way we perturb the initial data.

To be more precise, we shall consider initial data to equation (77) with $k = 2$, given by

$$u_0(x_1, x_2) = Q(x_1, x_2) \pm 0.1e^{-x_1^2 - x_2^2}, \tag{100}$$

where Q is again the stationary state constructed numerically as described in Section 1. We will use $N_{x_1} = N_{x_2} = 2^{10}$ Fourier modes, a numerical domain Ω of the form (86) with $L_{x_1} = L_{x_2} = 3$, and a time step of $\Delta t = 10^{-2}$.

Recall that in a stable regime, the time-dependent solution u typically oscillates around some time-periodic state plus a (small) remainder which radiates away as $t \to \pm\infty$ (see, e.g., [**?**, Section 4.5.1] for more details). In our simulations, however, we work on $\mathbb{T}^2$ instead of $\mathbb{R}^2$ which implies that radiation cannot escape to infinity. Thus, we will not be able to numerically verify the precise behavior of u for large times. Having this in mind, we take it as numerical evidence for (orbital) stability, if both perturbations (100) of Q generate stable oscillations of $\|u(t, \cdot)\|_{L^\infty}$, see also [**74, 76**] for similar studies.

4.1. The case without self-steepening.

Let us first address the case $\delta_1 = \delta_2 = 0$ for non-linear strengths $\sigma = 1, 2, 3$. This subsection is then dedicated to the two dimensional simulations of the previous chapter in the case of partial off-axis dependence.

For $\sigma = 1$, we find that the perturbed ground state is unstable, and that the initial pulse disperses towards infinity as can be seen in Fig. 10. The modulus of the solution at $t = 10$ in the same figure on the right shows that the initial pulse disperses with an annular profile. A " $-$ " perturbation in (100) leads to the same qualitative behavior and a corresponding figure is omitted.

The situation is found to be different for $\sigma = 2$, where Q appears to be stable, see Fig. 11. The L^∞-norm of the solution thereby oscillates for both signs of the perturbation.

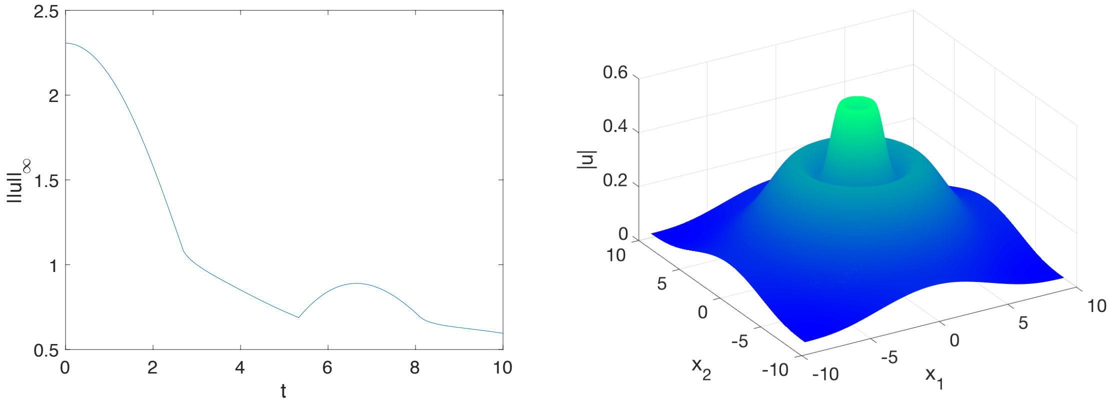

FIGURE 10. Solution to equation (77) with $\sigma = 1$, $\varepsilon = 1$, $k = 2$, $\boldsymbol{\delta} = 0$, and initial data (100) with the "+" sign: On the left the L^∞-norm of the solution as a function of t, and on the right the modulus of the solution for $t = 10$.

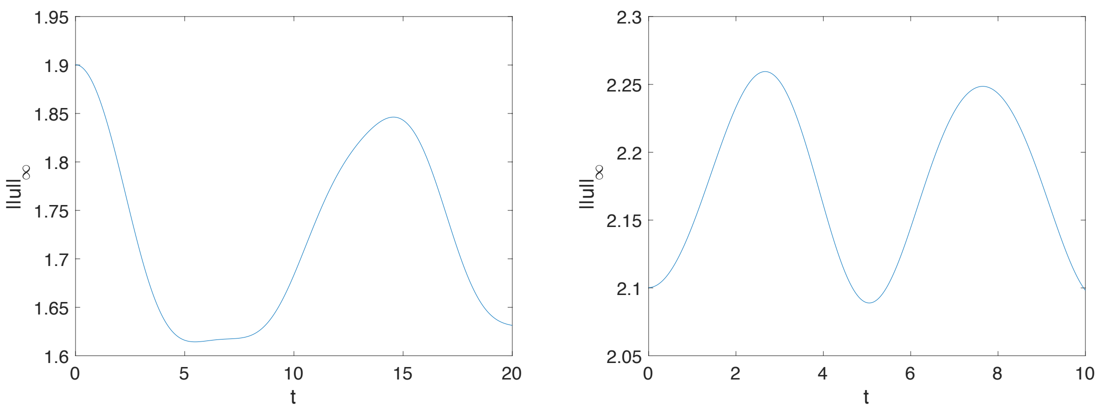

FIGURE 11. L^∞-norm of the solution to equation (77) with $\sigma = 2$, $\varepsilon = 1$, $k = 2$, $\boldsymbol{\delta} = 0$, and initial data (100): On the left for the " $-$ " sign, and on the right for the " $+$ " sign.

Finally, for $\sigma = 3$ we find that the behavior depends on how we perturb the initial ground state Q. Perturbations with a " $+$ " sign in (100) again exhibit an oscillatory behavior of the L^∞-norm, see the right of Fig. 12. However, a " $-$ " perturbation yields a monotonically decreasing L^∞-norm of the solution. The latter is again dispersed with an annular profile.

4.2. The case with self-steepening. In this subsection, we shall perform the same numerical study, but in the case with self-steepening, i.e. $\delta_{1,2} \neq 0$. For $\sigma = 1$, the corresponding stationary state Q seems to remain stable, since both types of perturbations yield an oscillatory behavior of the L^∞-norm in time, see Fig. 13. This is in sharp contrast to the $\sigma = 1$ case without self-steepening depicted in Fig. 10 above. In addition, we see that the solution no longer displays an annular profile. This stable behavior is lost in the case of higher nonlinearities. More precisely, for both

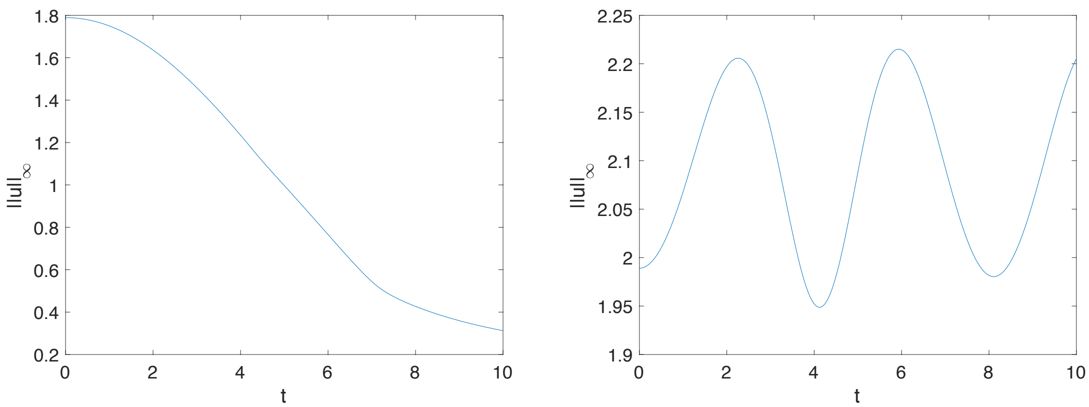

FIGURE 12. L^∞-norm of the solution to equation (77) with $\varepsilon = 1$, $k = 2$, $\sigma = 3$, $\boldsymbol{\delta} = 0$ and initial data (100): On the left for the "$-$" sign, on the right for the "$+$" sign.

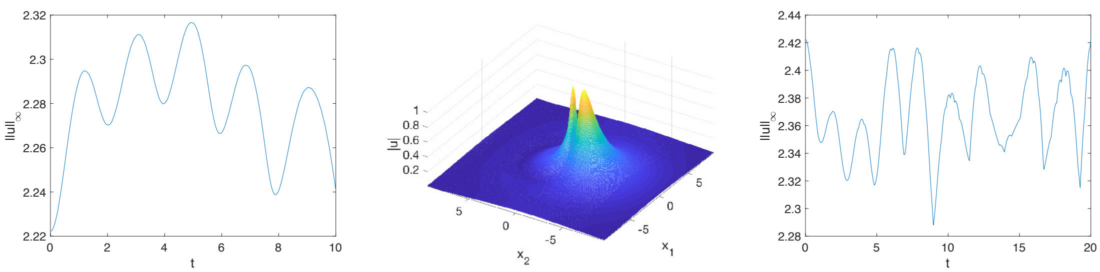

FIGURE 13. Solution to equation (77) with $\varepsilon = 1$, $k = 2$, $\sigma = 1$, $\delta_1 = 0$, $\delta_2 = 1$, and initial data (100): On the left the L^∞-norm for the "$-$" sign, in the middle $|u|$ plotted at the final time, and on the right the L^∞-norm for the solution with the "$+$" sign.

$\sigma = 2$ and 3 we find that the behavior of the solution u depends on the sign of the considered Gaussian perturbation. On the one hand, for the "$+$" perturbation in (100), both $\sigma = 2$ and $\sigma = 3$ yield an oscillatory behavior of the L^∞-norm, see Fig. 14. On the other hand, the "$-$" perturbation for both nonlinearities produce a solution with decreasing L^∞-norm in time. Although for $\sigma = 2$ this decrease is no longer monotonic.

REMARK 3.6. *Our numerical findings are reminiscent of recent results for the (generalized) BBM equation, see* [18]*. In there, it is found that for* $p \geqslant 5$*, the regime where the underlying KdV equation is expected to exhibit blow-up, solitary waves can be both stable and unstable and are sensitive to the type of perturbation considered. The main difference to our case is of course that these earlier studies are done in only one spatial dimension.*

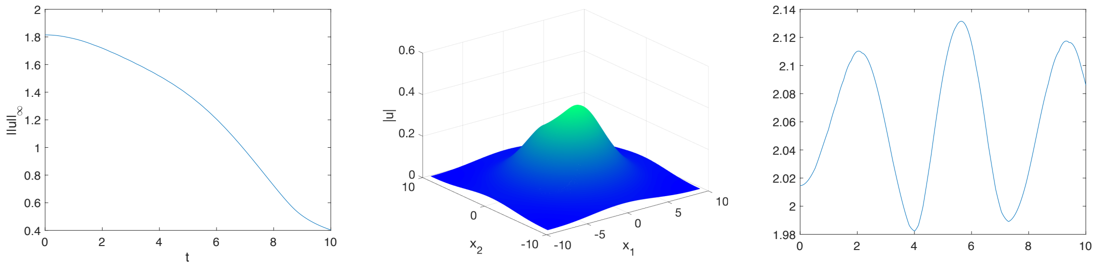

FIGURE 14. Solution to equation (77) with $\varepsilon = 1$, $k = 2$, $\sigma = 3$, $\delta_1 = 0$, $\delta_2 = 0.1$, and initial data (100): On the left, the L^∞-norm for the " $-$ " perturbation, in the middle $|u|$ plotted at the final time, and on the right the L^∞-norm for the solution with the " $+$ " sign.

5. Well-posedness results for the case with partial off-axis variation

From a mathematical point of view, the most interesting situation arises in the case where there is only a *partial* off-axis variation. To study such a situation, we shall without loss of generality assume that P_ε acts only in the x_1-direction, i.e.

$$P_\varepsilon = 1 - \varepsilon^2 \partial_{x_1}^2.$$

In this case (76) becomes

$$i(1 - \varepsilon^2 \partial_{x_1}^2)\partial_t u + \Delta u + (1 + i\boldsymbol{\delta} \cdot \nabla)(|u|^{2\sigma} u) = 0, \quad u_{|t=0} = u_0(x_1, x_2). \tag{101}$$

When $\boldsymbol{\delta} = (\delta, 0)^\top$ and $\sigma = 1$, this is precisely the model proposed in [39, Section 4.3]. Motivated by this, we shall in our analysis only consider the case where the regularization P_ε and the derivative nonlinearity act in the same direction. Numerically, however, we shall also treat the orthogonal case where, instead, $\boldsymbol{\delta} = (0, \delta)^\top$, see below.

5.1. Change of unknown and Strichartz estimates.

In [5], which treats the case without self-steepening, the following change of unknown is proposed in order to streamline the analysis:

$$v(t, x_1, x_2) := P_\varepsilon^{1/2} u(t, x_1, x_2). \tag{102}$$

Rewriting the evolutionary form of (101) with $\boldsymbol{\delta} = (\delta, 0)^\top$ in terms of v yields

$$i\partial_t v + P_\varepsilon^{-1}\Delta v + (1 + i\delta\partial_{x_1}) P_\varepsilon^{-1/2}(|P_\varepsilon^{-1/2} v|^{2\sigma} P_\varepsilon^{-1/2} v) = 0, \tag{103}$$

subject to initial data

$$v_{|t=0} = v_0(x_1, x_2) \equiv P_\varepsilon^{1/2} u_0(x_1, x_2).$$

Instead of (9), as before one finds the new conservation law

$$\|v(t,\cdot)\|_{L^2_{\mathbf{x}}}^2 = \|P_\varepsilon^{1/2}u(t,\cdot),\|_{L^2_{\mathbf{x}}}^2 = \|P_\varepsilon^{1/2}u_0\|_{L^2_{\mathbf{x}}}^2 = \|v_0\|_{L^2_{\mathbf{x}}}^2, \tag{104}$$

where we recall that $P_\varepsilon^{1/2}$ only acts in the x_1-direction, via its Fourier symbol

$$\widehat{P}_1^{s/2}(\boldsymbol{\xi}) = (1+\xi_1^2)^{s/2}, \quad \xi_1 \in \mathbb{R}.$$

This suggests to work in the two-dimensional mixed Sobolev-type spaces $L^p(\mathbb{R}_{x_2}; H^s(\mathbb{R}_{x_1}))$, which for any $s \in \mathbb{R}$ are defined through the following norm:

$$\|f\|_{L^p_{x_2}H^s_{x_1}} := \left\|P_1^{s/2}f\right\|_{L^p_{x_2}L^2_{x_1}} := \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |P_1^{s/2}f(x_1,x_2)|^2\,dx_1\right)^{\frac{p}{2}}\,dx_2\right)^{\frac{1}{p}}.$$

We will also make use of the mixed space-time spaces $L^q_tL^p_{x_2}H^s_{x_1}(I)$ for some time interval I, or simply $L^q_tL^p_{x_2}H^s_{x_1}$ when the interval is clear from context, see Section 1 of Chapter 2 for more notation.

We recall Strichartz admissibility and the corresponding Strichartz estimates from Section 2 of Chapter 2 in the case of $(d,k)=(2,1)$. To this end, we say that a pair (q,r) is Strichartz *admissible*, if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}, \quad \text{for } 2 \leqslant r \leqslant \infty, 4 \leqslant q \leqslant \infty. \tag{105}$$

Now, let $(q,r),(\gamma,\rho)$ be two arbitrary admissible pairs. It is proved in Proposition 2.4 that there exist constants $C_1, C_2 > 0$ independent of ε, such that

$$\|S_\varepsilon(\cdot)f\|_{L^q_tL^r_{x_2}H^{-\frac{2}{\gamma}}_{x_1}} \leqslant C_1\|f\|_{L^2_{\mathbf{x}}}, \tag{106}$$

as well as

$$\left\|\int_0^t S_\varepsilon(\cdot - s)F(s)\,ds\right\|_{L^q_tL^r_{x_2}H^{-\frac{2}{q}}_{x_1}} \leqslant C_2\|F\|_{L^{\gamma'}_tL^{\rho'}_{x_2}H^{\frac{2}{\gamma}}_{x_1}}. \tag{107}$$

Here, one should note the loss of derivatives in the x_1-direction.

5.2. Global existence results. Using the Strichartz estimates stated above, we shall now prove some L^2-based global existence results for the solution v to (103). In turn, this will yield global existence results (in mixed spaces) for the original equation (101) via the transformation $v = P_\varepsilon^{1/2}u$. To this end, we first recall Theorem 1.1 in dimension two, in the case without self-steepening $\delta = 0$. In particular, for $\sigma < 2$ and $u_0 \in L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1}))$, there exists a unique global-in-time solution $u \in C(\mathbb{R}_t; L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1})))$ to

$$i(1-\varepsilon^2\partial_{x_1}^2)\partial_t u + \Delta u + |u|^{2\sigma}u = 0, \quad u_{|t=0} = u_0(x_1, x_2). \tag{108}$$

We shall see that our numerical findings in the next section indicate that this result is indeed sharp, that is for $\sigma \geqslant 2$ global existence in general no longer holds.

In the next result, we shall take into account the effect of self-steepening, and rewrite (103) using Duhamel's formula:

$$v(t) = S_\varepsilon(t)v_0 + i\int_0^t S_\varepsilon(t-s)P_\varepsilon^{-1/2}(1+i\delta\partial_{x_1})(|P_\varepsilon^{-1/2}v|^{2\sigma}P_\varepsilon^{-1/2}v)(s)\,ds \equiv \Phi(v)(t). \tag{109}$$

To prove that Φ is a contraction mapping, the following lemma is key.

LEMMA 3.7. *Let $g(z) = |z|^{2\sigma}z$ with $\sigma \in \mathbb{N}$. For $t \in [0,T]$ denote*

$$\mathcal{N}(v)(t) := i\int_0^t S_\varepsilon(t-s)P_\varepsilon^{-1/2}(1+i\delta\partial_{x_1})g(P_\varepsilon^{-1/2}v(s))\,ds, \tag{110}$$

and choose the admissible pair $(\gamma,\rho) = \big(\frac{4(\sigma+1)}{\sigma}, 2(\sigma+1)\big)$. Then for $\varepsilon, \delta > 0$, it holds:

$$\begin{aligned}
&\big\|\mathcal{N}(v) - \mathcal{N}(v')\big\|_{L_t^\gamma L_{x_2}^\rho H_{x_1}^{-\frac{2}{\gamma}}} \\
&\qquad \lesssim \varepsilon^{-2(\sigma+1)}(1+\delta)T^{1-\frac{\sigma}{2}}\Big(\|v\|^{2\sigma}_{L_t^\gamma L_{x_2}^\rho H_{x_1}^{-\frac{2}{\gamma}}} + \|v'\|^{2\sigma}_{L_t^\gamma L_{x_2}^\rho H_{x_1}^{-\frac{2}{\gamma}}}\Big)\|v-v'\|_{L_t^\gamma L_{x_2}^\rho H_{x_1}^{-\frac{2}{\gamma}}}.
\end{aligned}$$

PROOF. First it is easy to check that

$$(\gamma,\rho) = \Big(\frac{4(\sigma+1)}{\sigma}, 2(\sigma+1)\Big)$$

is admissible in the sense of (105). Moreover, since $\gamma > 4$ we have $\frac{2}{\gamma} < \frac{1}{2}$, from which we infer that $H^{1-\frac{2}{\gamma}}(\mathbb{R})$ is indeed a normed Banach algebra, a fact to be used below. Using the Strichartz estimate (107) we have

$$\big\|\mathcal{N}(v) - \mathcal{N}(v')\big\|_{L_t^\gamma L_{x_2}^\rho H_{x_1}^{-\frac{2}{\gamma}}} \leqslant C_2\big\|P_\varepsilon^{-1/2}\big(1+i\delta\partial_{x_1}\big)\big(g(P_\varepsilon^{-1/2}v) - g(P_\varepsilon^{-1/2}v')\big)\big\|_{L_t^{\gamma'} L_{x_2}^{\rho'} H_{x_1}^{\frac{2}{\gamma}}}.$$

For simplicity we shall in the following denote $u = P_\varepsilon^{-1/2}v, u' = P_\varepsilon^{-1/2}v'$ in view of (102). Keeping t and x_2 fixed we can estimate

$$\begin{aligned}
\big\|P_\varepsilon^{-1/2}\big(1+i\delta\partial_{x_1}\big)\big(g(P_\varepsilon^{-1/2}v) - g(P_\varepsilon^{-1/2}v')\big)\big\|_{H_{x_1}^{\frac{2}{\gamma}}} &\leqslant \varepsilon^{-1}\big\|\big(1+i\delta\partial_{x_1}\big)\big(g(u) - g(u')\big)\big\|_{H_{x_1}^{\frac{2}{\gamma}-1}} \\
&\leqslant \varepsilon^{-1}(1+\delta)\|g(u) - g(u')\|_{H_{x_1}^{1-\frac{2}{\gamma}}},
\end{aligned}$$

where in the last inequality we have used the fact that $H^{1-\frac{2}{\gamma}}(\mathbb{R}) \subset H^{\frac{2}{\gamma}}(\mathbb{R})$. Next, use again (96) which together with the algebra property of $H^{1-\frac{2}{\gamma}}(\mathbb{R})$ for $\sigma \in \mathbb{N}$ implies

$$\begin{aligned}\|g(u)-g(u')\|_{H^{1-\frac{2}{\gamma}}_{x_1}} &\lesssim \Big(\|u\|^{2\sigma}_{H^{1-\frac{2}{\gamma}}_{x_1}} + \|u'\|^{2\sigma}_{H^{1-\frac{2}{\gamma}}_{x_1}}\Big)\|u-u'\|_{H^{1-\frac{2}{\gamma}}_{x_1}} \\ &\lesssim \varepsilon^{-2(\sigma+1)}\Big(\|v\|^{2\sigma}_{H^{-\frac{2}{\gamma}}_{x_1}} + \|v'\|^{2\sigma}_{H^{-\frac{2}{\gamma}}_{x_1}}\Big)\|v-v'\|_{H^{-\frac{2}{\gamma}}_{x_1}}.\end{aligned}$$

It consequently follows after Hölder's inequality in x_2, that we obtain

$$\big\|\mathcal{N}(v)-\mathcal{N}(v')\big\|_{L^\rho_{x_2}H^{-\frac{2}{\gamma}}_{x_1}} \lesssim \varepsilon^{-(2\sigma+1)}(1+\delta)\Big(\|v\|^{2\sigma}_{L^\rho_{x_2}H^{-\frac{2}{\gamma}}_{x_1}} + \|v'\|^{2\sigma}_{L^\rho_{x_2}H^{-\frac{2}{\gamma}}_{x_1}}\Big)\|v-v'\|_{L^\rho_{x_2}H^{-\frac{2}{\gamma}}_{x_1}}.$$

The result then follows after applying yet another Hölder's inequality in t. □

This lemma allows us to prove the following global existence result for (101).

THEOREM 3.2 (Partial off-axis variation with parallel self-steepening). *Let $\sigma = 1$ and $\boldsymbol{\delta} = (\delta, 0)^\top$ for $\delta \in \mathbb{R}$. Then for any $u_0 \in L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1}))$ there exists a unique global solution $u \in C(\mathbb{R}_t; L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1})))$ to (101).*

Here, the restriction $\sigma = 1$ is due to the fact that this is the only $\sigma \in \mathbb{N}$ (required for the normed algebra property above) for which the problem is subcritical. Indeed, in view of the estimate in Lemma 3.7, the exponent $1 - \frac{\sigma}{2} > 0$ yields a contraction for small times.

PROOF. We seek to show that $v \mapsto \Phi(v)$ is a contraction mapping in a suitable space. To this end, we denote as before

$$\Phi(v)(t) = S_\varepsilon(t)v_0 + \mathcal{N}(v)(t),$$

where $\mathcal{N}(v)$ is given by (110). Let $T, M > 0$ and denote

$$Y_{T,M} = \{v \in L^\infty([0,T); L^2(\mathbb{R}^2)) \cap L^8([0,T); L^4(\mathbb{R}_{x_2}; H^{-\frac{1}{4}}(\mathbb{R}_{x_1}))) : \ \|v\|_{L^\infty_t L^2_{\mathbf{x}}} + \|v\|_{L^8_t L^4_{x_2} H^{-\frac{1}{4}}_{x_1}} \leqslant M\}.$$

The Strichartz estimates (106) and (107) together with Lemma 3.7 imply that for any admissible pair (q, r) and solutions $v, v' \in Y_{T,M}$ that

$$\begin{aligned}\|\Phi(v)-\Phi(v')\|_{L^q_t L^r_{x_2} H^{-\frac{2}{q}}_{x_1}} &\leqslant \|S_\varepsilon(t)(v_0-v_0')\|_{L^q_t L^r_{x_2} H^{-\frac{2}{q}}_{x_1}} + \|\mathcal{N}(v)-\mathcal{N}(v')\|_{L^q_t L^r_{x_2} H^{-\frac{2}{q}}_{x_1}} \\ &\leqslant C_1\|v_0-v_0'\|_{L^2_{\mathbf{x}}} + C_2\big\|P_\varepsilon^{-1/2}\big(1+i\delta\partial_{x_1}\big)\big(g(P_\varepsilon^{-1/2}v)-g(P_\varepsilon^{-1/2}v')\big)\big\|_{L^{\frac{8}{7}}_t L^{\frac{4}{3}}_{x_2} H^{\frac{1}{4}}_{x_1}} \\ &\leqslant C_{\sigma,\varepsilon}\Big(\|v_0-v_0'\|_{L^2_{\mathbf{x}}} + T^{1/2}M^2\|v-v'\|_{L^8_t L^4_{x_2} H^{-\frac{1}{4}}_{x_1}}\Big).\end{aligned}$$

Choosing $M = M(\|v_0\|_{L^2_{\mathbf{x}}})$ and T sufficiently small, it is clear that Φ is a contraction on $Y_{T,M}$. Banach's fixed point theorem and a standard continuity argument thus yield the existence of a unique maximal solution $v \in C([0, T_{\max}), L^2(\mathbb{R}^2))$ where $T_{\max} = T_{\max}(\|v_0\|_{L^2_{\mathbf{x}}})$. Continuous dependence on the initial data follows by classical arguments.

The conservation property (104) for v follows similarly as in the proof of Proposition 2.7 above and we shall therefore only sketch its main steps below. By the unitary of $S_\varepsilon(\cdot)$ in L^2 we obtain

$$\|v(t)\|_{L^2_{\mathbf{x}}} = \|v_0\|_{L^2_{\mathbf{x}}} + 2\mathrm{Re}\left\langle S_\varepsilon(-t)\mathcal{N}(v)(t), v_0\right\rangle_{L^2_{\mathbf{x}}} + \|S_\varepsilon(-t)\mathcal{N}(v)(t)\|^2_{L^2_x} =: \|v_0\|_{L^2_{\mathbf{x}}} + \mathcal{I}_1 + \mathcal{I}_2.$$

To show that $\mathcal{I}_1 + \mathcal{I}_2 = 0$, we use (95) and rewrite

$$\mathcal{I}_1 = -2\mathrm{Im}\int_0^t \left\langle P_\varepsilon^{-1/2}(1 + i\delta\partial_{x_1})g(P_\varepsilon^{-1/2}v)(s), S_\varepsilon(s)v_0\right\rangle_{L^2_x} ds.$$

By duality in x_1 and Hölder's inequality in t and x_2 we find that this quantity is indeed finite, since

$$|\mathcal{I}_1| \leqslant 2\|P_\varepsilon^{-1/2}(1 + i\delta\partial_{x_1})g(P_\varepsilon^{-1/2}v)\|_{L_t^{\gamma'}L_{x_2}^{\rho'}H_{x_1}^{\frac{2}{\gamma}}}\|S_\varepsilon(\cdot)v_0\|_{L_t^{\gamma}L_{x_2}^{\rho}H_{x_1}^{-\frac{2}{\gamma}}} < \infty.$$

Once again we find, after a lengthy computation (see Proposition 2.7 for more details), that

$$\mathcal{I}_2 = 2\mathrm{Re}\int_0^t \left\langle P_\varepsilon^{-1/2}(1 + i\delta\partial_{x_1})g(P_\varepsilon^{-1/2}v)(s), -i\mathcal{N}(u)(s)\right\rangle_{L^2_{\mathbf{x}}} ds.$$

We express $-i\mathcal{N}(u)(s)$ using the integral formulation (109) and write

$$\begin{aligned}\mathcal{I}_2 = 2\mathrm{Re}\int_0^t &\left\langle P_\varepsilon^{-1/2}(1 + i\delta\partial_{x_1})g(P_\varepsilon^{-1/2}v)(s), iS_\varepsilon(s)u_0\right\rangle_{L^2_{\mathbf{x}}} ds \\ &+ \int_0^t \mathrm{Im}\,\|P_\varepsilon^{-1/2}v(s)\|^{2\sigma+2}_{L_x^{2\sigma+2}} - \delta\,\mathrm{Re}\left\langle g(P_\varepsilon^{-1/2}v), \partial_{x_1}P_\varepsilon^{-1/2}v\right\rangle_{L^2_{\mathbf{x}}}(s)\, ds.\end{aligned}$$

Here the second time integral vanishes entirely, and as in the full off-axis case, the latter term in the integrand vanishes due to

$$\mathrm{Re}\left\langle g(P_\varepsilon^{-1/2}v), \partial_{x_1}P_\varepsilon^{-1/2}v\right\rangle_{L^2_{\mathbf{x}}} = \frac{2}{(\sigma+1)}\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_{x_1}(|P_\varepsilon^{-1/2}v|^{2\sigma+2})\, dx_1 dx_2 = 0.$$

In summary, we find that

$$\mathcal{I}_2 = 2\mathrm{Im}\int_0^t \left\langle P_\varepsilon^{-1/2}g(P_\varepsilon^{-1/2}v)(s), S_\varepsilon(s)u_0\right\rangle_{L^2_{\mathbf{x}}} ds = -\mathcal{I}_1,$$

which finishes the proof of (104). We can thus extend v to become a global solution by repeated iterations to conclude $T_{\max} = +\infty$. Finally, we use that $v = P_\varepsilon^{1/2}u$ to obtain a unique global-in-time solution $u \in C(\mathbb{R}_t; L^2(\mathbb{R}_{x_2}; H^1(\mathbb{R}_{x_1})))$ which finishes the proof. $\square$

REMARK 3.8. *It is possible to treat the critical case $\sigma = 2$ using the same type of arguments as in* [27] *(see also Section 4). Unfortunately, this will only yield local-in-time solutions up to some time $T = T(u_0) > 0$, which depends on the initial profile u_0 (and not only its norm). Only for sufficiently small initial data $\|u_0\|_{L^2_{x_2}H^1_{x_1}} < 1$, does one obtain a global-in-time solution. But since it is hard to detect small nonlinear effects numerically, we won't be concerned with this case in the following. We also mention the possibility of obtaining (not necessarily unique) global weak solutions for derivative NLS, which has been done in* [4] *in one spatial dimension.*

We note that Theorem 3.2 covers the situation in which a partial off-axis regularization acts parallel to the self-steepening. At present, no analytical result for the case where the two effects act orthogonal to each other is available. However, numerically it is possible to study such a scenario. To this end, we recall that from the physics point of view, both ε and $|\boldsymbol{\delta}|$ have to be considered as (very) small parameters. With this in mind, we study the time-evolution of (101) with $\sigma = 1$, Gaussian initial data of the form (93), and a relatively small self-steepening, furnished by $\delta_1 = 0$ and $\delta_2 = 0.1$. In the case where $\varepsilon = 0$, it can be seen on the left of Fig. 15 that the L^∞-norm of the solution indicates a finite-time blow-up at $t \approx T = 0.1445$. In the same situation with a small, but nonzero $\varepsilon = 0.1$, one can see that, instead, oscillations appear within the L^∞-norm of the solution for $t \geqslant T$.

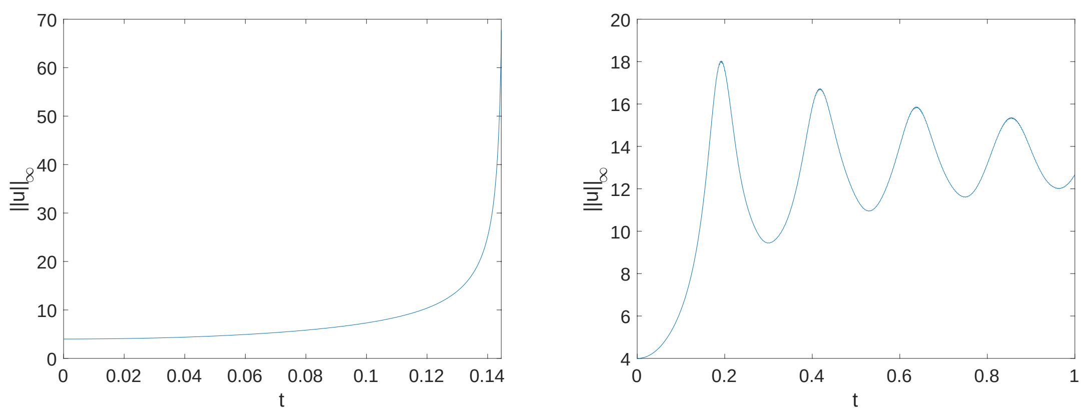

FIGURE 15. L^∞-norm of the solution to (101) with $\sigma = 1$, $\boldsymbol{\delta} = (0, 0.1)$, and initial data $u_0 = 4\exp^{-x_1^2 - x_2^2}$: On the left for $\varepsilon = 0$, on the right for $\varepsilon = 0.1$.

Note that these oscillations appear to decrease in amplitude, which indicates the possibility of an asymptotically stable final state as $t \to +\infty$. A similar behavior can be seen for different choices of parameters and also for a full, two-dimensional off-axis variation (not shown here).

6. Numerical studies for the case with partial off-axis variation

In this section we present numerical studies for the model (101) with $\varepsilon = 1$ and different values of the self-steepening parameter $\boldsymbol{\delta}$, as well as $\sigma > 0$. We will always use $N_{x_1} = N_{x_2} = 2^{10}$ Fourier coefficients on the numerical domain Ω given by (86) with $L_{x_1} = L_{x_2} = 3$. The time step is $\Delta t = 10^{-2}$ unless otherwise noted. The initial data is the same as in (100), i.e. a numerically constructed stationary state Q perturbed by respectively adding and subtracting small Gaussians.

6.1. The case without self-steepening. We shall first study the particular situation furnished by equation (108) with $\varepsilon = 1$. It is obtained from the general model (76) in the case without self-steepening $\delta_1 = \delta_2 = 0$, i.e. equation (6) from Chapter 2.

In the case $\sigma = 1$, the ground state perturbation in (100) with a " + " sign is unstable and results in a purely dispersive solution with monotonically decreasing L^∞-norm, see Fig. 16. The modulus of the solution at time $t = 2.5$ is shown on the right of the same figure. Interestingly, the initial hump appears to separate into four smaller humps and we thus lose radial symmetry of the solution. The situation is qualitatively similar for perturbations corresponding to the " $-$ " sign in (100) and we thus omit a corresponding figure.

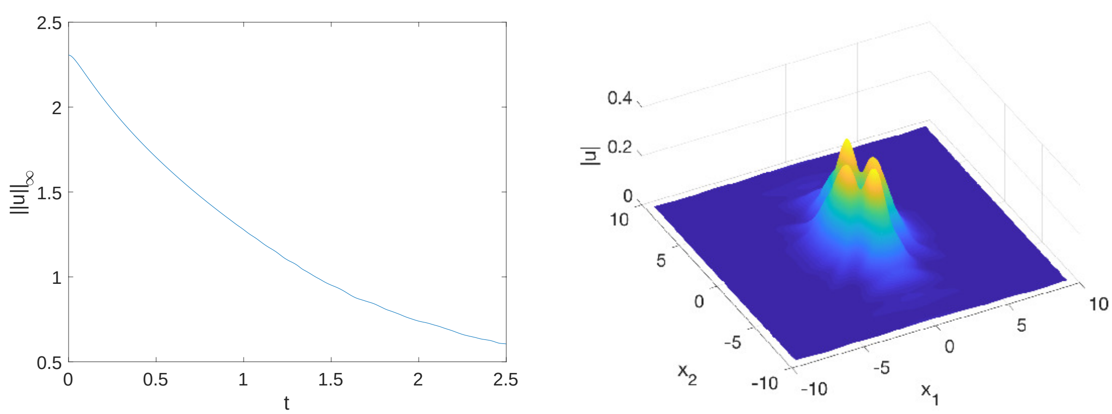

FIGURE 16. Solution to (108) with $\varepsilon = 1$, $\sigma = 1$, and initial data (100) with a " + " sign: On the left the L^∞-norm in dependence of time, on the right the modulus of u at $t = 2.5$.

The situation changes significantly for $\sigma = 2$, as can be seen in Fig. 17. While the L^∞-norm of the solution obtained from initial data (100) with the " $-$ " sign is again decreasing, the " + " sign yields a monotonically increasing L^∞-norm indicating a blow-up at $t \approx 0.64$. The modulus of the solution at the last recorded time $t = 0.6405$ is shown in Fig. 18 on the left. It can be seen that it is strongly

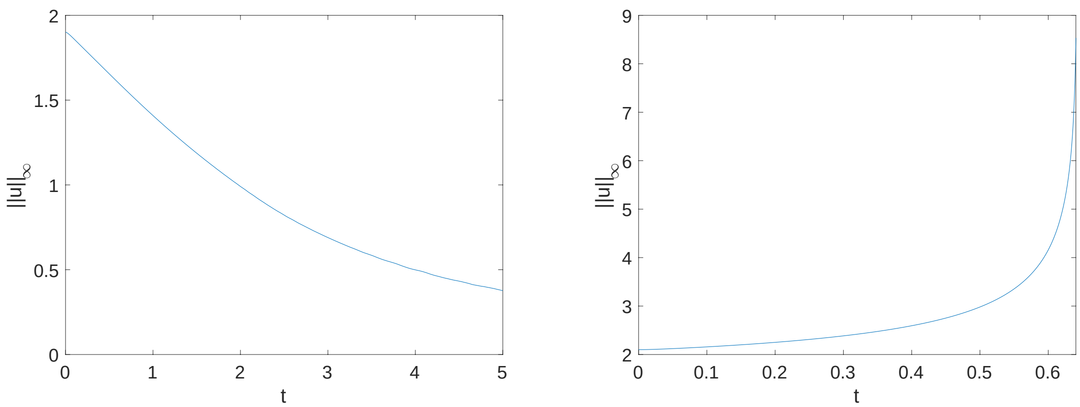

FIGURE 17. Time-dependence of the L^∞-norm of the solution to (108) with $\varepsilon = 1$, $\sigma = 2$, and initial data (100): On the left, the case with a " $-$ " perturbation; on the right the case with " $+$ " sign.

compressed in the x_2-direction. The corresponding Fourier coefficients are shown on the right of the same figure. They also indicate the appearance of a singularity in the x_2-direction.

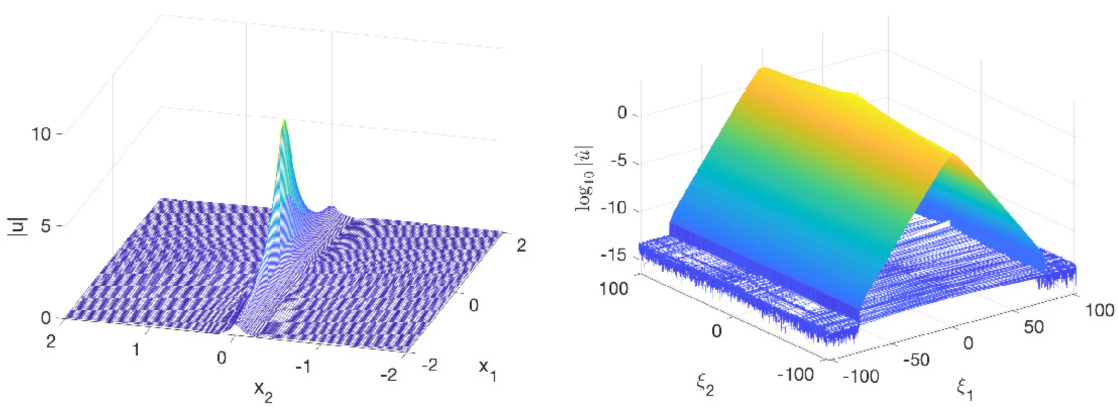

FIGURE 18. Solution to (108) for $\varepsilon = 1$, $\sigma = 2$ and initial data (100) with the " $+$ " sign: On the left the modulus of the solution at the last recorded time $t = 0.6045$; on the right the corresponding Fourier coefficients of u given by $\hat{u}$.

These numerical findings indicate that the global existence result stated in Theorem 1.1 is indeed sharp. It also shows that the two-dimensional model with partial off-axis variation essentially behaves like the classical one-dimensional focusing NLS in the unmodified x_2-direction, that is the direction in which P_ε does not act. Recall that for the classical one-dimensional (focusing) NLS, finite-time blow-up is known to appear as soon as $\sigma \geqslant 2$.

6.2. The case with self-steepening parallel to the off-axis variation. In this subsection, we include the effect of self-steepening and consider equation (101) with $\varepsilon = 1$, $\delta_2 = 0$, and $\delta_1 > 0$.

For $\sigma = 1$, the stationary state Qe^{it} appears to be stable against all studied perturbations. Indeed, the situation is found to be qualitatively similar to the case with full off-axis perturbations (except for a loss of radial symmetry) and we therefore omit a corresponding figure.

When $\sigma = 2$, the stationary state no longer appears to be stable. However, we also do not have any indication of finite-time blow-up in this case. Indeed, given a " $-$ " perturbation in the initial data (100), it can be seen on the left of Fig. 19 that the L^∞-norm of the solution simply decreases monotonically in time.

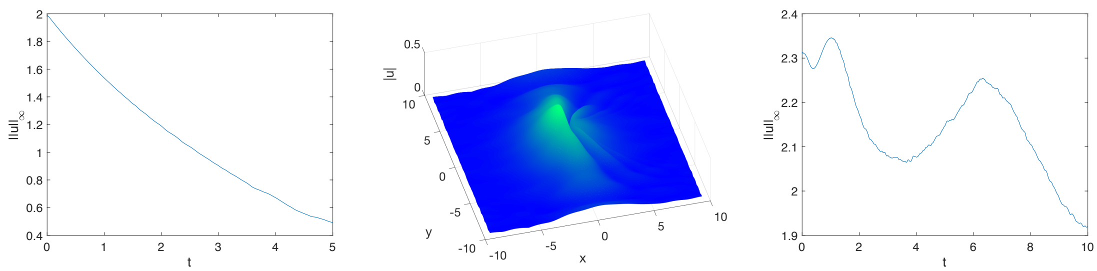

FIGURE 19. Solution to (101) with $\varepsilon = 1$, $\sigma = 2$, and $\boldsymbol{\delta} = (0.3, 0)^\top$: On the left, the L^∞-norm of the solution obtained for initial data (100) with the " $-$ " sign, on the right for the " $+$ ", and in the middle $|u|$ at $t = 5$ for the " $-$ " sign perturbation.

Notice, that there is still an effect of self-steepening visible in the modulus of the solution $|u|$, depicted in the middle of the same figure. The behavior of the L^∞-norm in the case of a " $+$ " perturbation is shown on the right of Fig. 19. It is no longer monotonically decreasing but still converges to zero.

For $\sigma = 3$, a " $-$ " perturbation of (100) is found to be qualitatively similar to the case $\sigma = 2$ and we therefore omit a figure illustrating this behavior. However, the situation radically changes if we consider a perturbation with the " $+$ " sign, see Fig. 20. The L^∞-norm of the solution indicates a blow-up for $t \approx 0.1555$, where the code stops with an overflow error. In this particular simulation we have used 10^4 time steps for $t \in [0, 0.17]$ and $N_{x_1} = 2^{10}$, $N_{x_2} = 2^{11}$ Fourier modes (since the maximum of the solution hardly moved, it was not necessary to use a co-moving frame). The solution is still well-resolved in time at $t = 0.155$ since (9) remains numerically conserved up to the order of 10^{-11}. But despite the higher resolution in x_2 used for this simulation, the Fourier coefficients indicate a loss of resolution in the x_2-direction. The modulus of the solution at the last recorded time is plotted in Fig. 21. Note that $|u|$ is still regular in the x_1-direction in which P_ε^{-1} acts, but it has become strongly compressed in the x_2-direction.

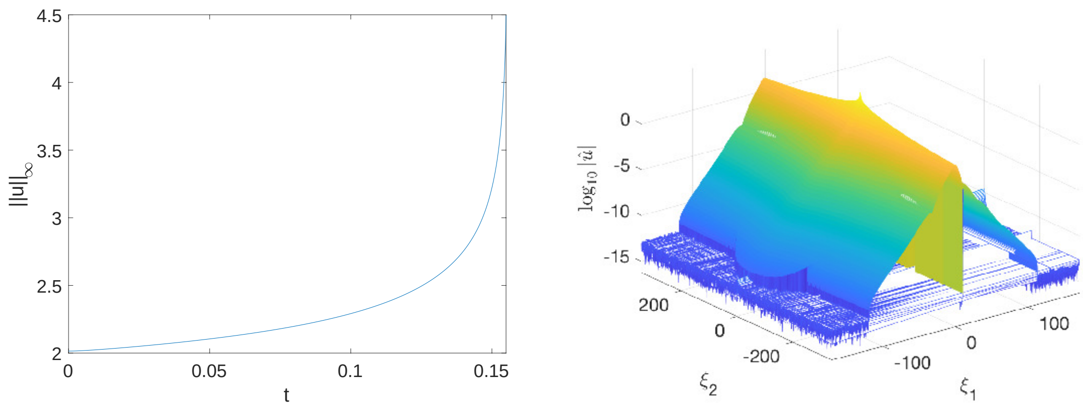

FIGURE 20. L^∞-norm of the solution to (101) with $\varepsilon = 1$, $\sigma = 3$, $\boldsymbol{\delta}_1 = (0.1, 0)^\top$, and initial data (100) with the " + " sign. On the right the modulus of the Fourier coefficients of the solution at time $t = 0.155$.

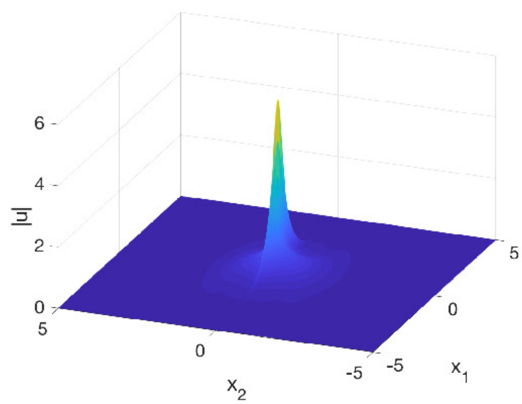

FIGURE 21. The modulus of the solution to equation (101) with $\varepsilon = 1$, $\sigma = 3$, $\boldsymbol{\delta}_1 = (0.1, 0)$, and initial data (100) with the " + " sign, plotted at time $t = 0.155$.

6.3. The case with self-steepening orthogonal to the off-axis variation. Finally, we shall consider the same model equation (101) with $\varepsilon = 1$, but this time we let $\delta_1 = 0$ for non-vanishing $\delta_2 > 0$. This is the only case, for which we do *not* have any analytical existence results at present. For $\sigma = 1$, it can be seen that a " $-$ " sign in the initial data (100) yields a purely dispersive solution with monotonically decreasing L^∞-norm, see Fig. 22 which also shows a picture of $|u|$ at $t = 20$. The "+" sign again leads to oscillations of the L^∞-norm in time, indicating stability of the ground state. The situation for $\sigma = 2$ is qualitatively very similar and hence we omit the corresponding figure. For $\sigma = 3$ and a " $-$ " sign in the initial data (100), we again find a purely dispersive solution. However, the behavior of the solution obtained from a perturbation of Q with the " + " sign is less clear. As one can see in Fig. 23, the solution is initially focused up to a certain point after which its L^∞-norm decreases again.

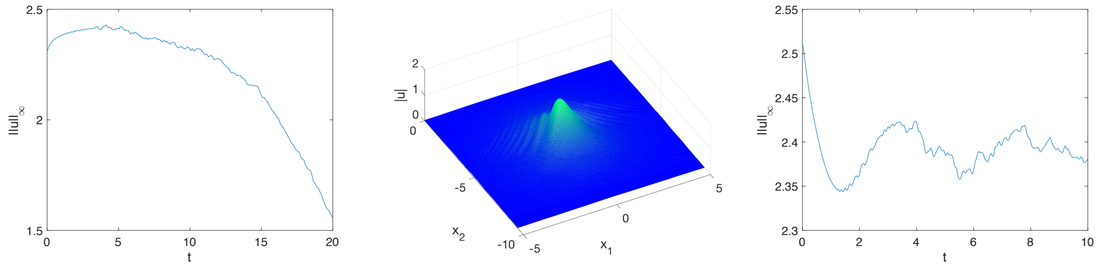

FIGURE 22. Solution to (101) with $\varepsilon = 1$, $\sigma = 1$, and $\boldsymbol{\delta}_1 = (0,1)^\top$. On the left the L^∞-norm of the solution for initial data (100) with the " $-$ " sign, on the right for the one with " $+$ " sign, and in the middle $|u|$ at time $t = 20$ for the " $-$ " sign.

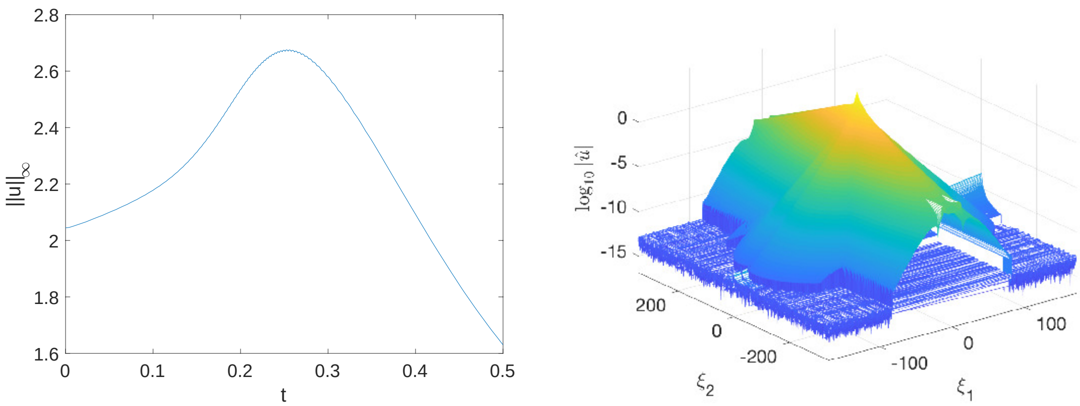

FIGURE 23. Solution to (101) with $\varepsilon = 1$, $\sigma = 3$, $\boldsymbol{\delta}_1 = (0, 0.1)^\top$, and initial data (100) with the " $+$ " sign: On the left the L^∞-norm of u as a function of time, on the right the Fourier coefficients $\widehat{u}$ at $t = 0.25$.

This simulation is done with $N_{x_1} = 2^{10}$, $N_{x_2} = 2^{11}$ Fourier modes and $N_t = 10^4$ time steps for $t \in [0, 0.5]$. The relative conservation of the numerically computed quantity (9) is better than 10^{-10} during the whole computation indicating an excellent resolution in time. The spatial resolution is indicated by the Fourier coefficients of the solution near the maximum of the L^∞-norm as shown on the right of Fig. 23. Obviously, a much higher resolution is needed in the x_2-direction, but even near the maximum of the L^∞-norm the modulus of the Fourier coefficients decreases to the order of 10^{-5}. The modulus of the solution at time $t = 0.5$ can be seen in Fig. 24. It shows a strong compression in the x_2-direction but nevertheless remains regular for all times. This is in stark contrast to the analogous situation with parallel self-steepening and off-axis variations, cf. Figures 20 and 21 above.

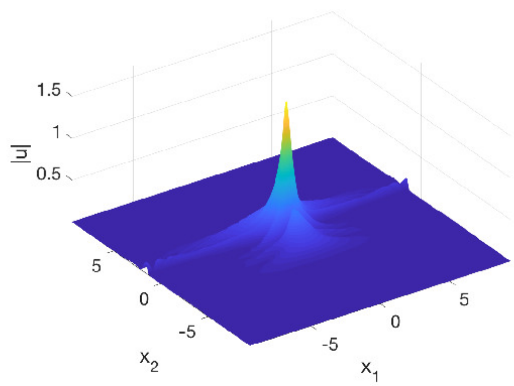

FIGURE 24. The modulus of the solution to (101) with $\varepsilon = 1$, $\sigma = 3$, $\boldsymbol{\delta}_1 = (0, 0.1)^\top$, and initial data (100) with the " + " sign, plotted at $t = 0.5$.

CHAPTER 4

Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures [1]

The outline of Chapter 4 is as follows. In Section 1 we recall some basic properties of honeycomb lattice structures, the associated spectrum of Λ-periodic Schrödinger operators and Dirac Points. Then, we shall perform a formal multi-scale expansion in Section 2, for which we will set up a rigorous framework in Section 3. As a last step, we shall prove in Section 2 that the approximate solutions thereby obtained are stable under the nonlinear time evolution of (19). Finally, we shall briefly discuss the case of Hartree nonlinearities in Section 5.

1. Dynamics in Honeycomb Structures

1.1. Honeycomb lattice structures. In the context of electron wave dynamics in graphene, we consider a physical domain where we fix identical carbon atoms in a periodic, two dimensional hexagonal or honeycomb structure. In particular, a honeycomb structure H is the union of two triangular sub-lattices $\Lambda_{\mathbf{A}} = \mathbf{A} + \Lambda$ and $\Lambda_{\mathbf{B}} = \mathbf{B} + \Lambda$, cf. [**43, 44**], where $\mathbf{A}$ and $\mathbf{B}$ preserve the rotational symmetry of the lattice. We recall the basic geometry of a triangular lattice $\Lambda = \mathbb{Z}\mathbf{v}_1 \oplus \mathbb{Z}\mathbf{v}_2$ spanned by the basis vectors

$$\mathbf{v}_1 = a \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \\ \frac{1}{2} \end{pmatrix}, \ \mathbf{v}_2 = a \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \\ -\frac{1}{2} \end{pmatrix}, \quad a > 0.$$

Due the symmetry, it is enough to focus on simply one such triangular lattice Λ. Also, since the lattice Λ is a periodic structure which can be reproduced by proper translations along the basis vectors and rotations by $2\pi/3$, it makes sense to define a unit cell which defines the characteristic symmetry of the lattice. We define the fundamental cell by the following region

$$Y = \{\theta_1 v_1 + \theta_2 v_2 : 0 \leqslant \theta_j \leqslant 1, j = 1, 2\}.$$

[1]This Chapter has been excerpted from the following work:
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The corresponding dual lattice $\Lambda^* = \mathbb{Z}\mathbf{k}_1 \oplus \mathbb{Z}\mathbf{k}_2$ is spanned by the dual basis vectors

$$\mathbf{k}_1 = q \begin{pmatrix} \frac{1}{2} \\ \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \ \mathbf{k}_2 = q \begin{pmatrix} \frac{1}{2} \\ \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \ q \equiv \frac{4\pi}{a\sqrt{3}},$$

such that $\mathbf{v}_j \cdot \mathbf{k}_j = 2\pi$. The Brillouin zone, Y^*, is a choice of fundamental cell in the dual lattice which we choose to be a regular hexagon centered at the origin. Due to symmetry, the vertices of Y^* fall into two equivalence classes of points, $\mathbf{K} \equiv \frac{1}{3}(\mathbf{k}_1 + \mathbf{k}_2)$ and $\mathbf{K}' \equiv -\mathbf{K} = \frac{1}{3}(\mathbf{k}_2 - \mathbf{k}_1)$. The other vertices of Y^* are generated by the action of $2\pi/3$ rotation matrix R, given by

$$R = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

so that

$$R\mathbf{K} = \mathbf{K} + \mathbf{k}_2, \quad R^2\mathbf{K} = \mathbf{K} - \mathbf{k}_1.$$

$$R\mathbf{K}' = \mathbf{K}' - \mathbf{k}_2, \quad R^2\mathbf{K}' = \mathbf{K}' + \mathbf{k}_1.$$

Lastly, we make clear the following functions of interest that live on our our lattice Λ, and the following rotationally invariant subspaces that will be of particular importance in the analysis to follow. Denote $L^2_{\mathrm{per}} \equiv L^2(\mathbb{R}^2/\Lambda)$, and say an element $f \in L^2_{\mathrm{per}}$ is called Λ-periodic if it satisfies

$$f(\mathbf{y} + \mathbf{v}) = f(\mathbf{y}), \quad \text{for all } \mathbf{y} \in \mathbb{R}^2 \text{ and } \mathbf{v} \in \Lambda.$$

We shall call a function $\mathbf{k}-$pseudo periodic and denote $f \in L^2_{\mathbf{k}}$ if it satisfies

$$f(\mathbf{y} + \mathbf{v}) = f(\mathbf{y})e^{i\mathbf{k}\cdot\mathbf{v}} \quad \text{for all } \mathbf{y} \in \mathbb{R}^2 \text{ and } \mathbf{v} \in \Lambda.$$

Now, let $\mathbf{K}_*$ be a vertex of $\mathbf{K}$ or $\mathbf{K}'$ type. For any function $f \in L^2_{\mathbf{K}_*}$, we introduce the unitary operator

$$\mathcal{R} : f \mapsto \mathcal{R}[f] = f(R^*\mathbf{y}) = f(R^{-1}\mathbf{y}). \tag{111}$$

One checks that $\mathcal{R}$ has eigenvalues $1, \tau, \overline{\tau}$, where $\tau = e^{2\pi i/3}$.

This consequently yields the following pairwise orthogonal subspaces to be used later on:

$$L^2_{\mathbf{K}_*,1} \equiv \{f \in L^2_{\mathbf{K}_*} : \mathcal{R}[f] = f\}$$
$$L^2_{\mathbf{K}_*,\tau} \equiv \{f \in L^2_{\mathbf{K}_*} : \mathcal{R}[f] = \tau f\}$$
$$L^2_{\mathbf{K}_*,\overline{\tau}} \equiv \{f \in L^2_{\mathbf{K}_*} : \mathcal{R}[f] = \overline{\tau} f\}.$$

Moreover to follow, for functions living on the fundamental cell Y, we define the associated $L^2(Y)$-inner product by

$$\langle f, g\rangle_{L^2(Y)} = \frac{1}{|Y|}\int_Y \overline{f(\mathbf{y})} g(\mathbf{y})\, d\mathbf{y}.$$

1.2. Periodic Schrödinger operators. We consider the Hamiltonian given by the following periodic Schrödinger operator

$$H = -\Delta + V_{\rm per}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2. \tag{112}$$

Physically, we can think of the $-\Delta$ as representing the kinetic energy, which classically is proportional to the square of the momentum operator $i\nabla$. The $V_{\rm per}$ term represents the potential energy that governs the electron's interaction with the atoms that constitute the lattice. As in [**43**], we classify this particular interaction between the particle and the honeycomb lattice by the following class of smooth periodic potentials.

DEFINITION 4.1. *Let $V_{\rm per} \in C^\infty(\mathbb{R}^2;\mathbb{R})$. Then $V_{\rm per}$ is called a **honeycomb lattice potential** if there exists an $\mathbf{y}_0 \in \mathbb{R}^2$ such that $\tilde{V} = V_{\rm per}(\mathbf{y} - \mathbf{y}_0)$ has the following properties :*

(1) *$\tilde{V}$ is Λ-periodic.*
(2) *$\tilde{V}$ is inversion symmetric, i.e., $\tilde{V}(-\mathbf{y}) = \tilde{V}(\mathbf{y})$.*
(3) *$\tilde{V}$ is $\mathcal{R}-$invariant, i.e., $\tilde{V}(R^*\mathbf{y}) = \tilde{V}(\mathbf{y})$.*

REMARK 4.2. *In physics experiments, honeycomb lattices are typically generated by the interference of three laser beams. For a concrete example of such a potential, see, e.g.,* [**1, 11**].

Bloch-Floquet theory consequently asserts that the spectrum of H is given by (see [**105**]):

$$\sigma(H) = \bigcup_{m\in\mathbb{N}} E_m,$$

where the graph $E_m = \{\mu_m(\mathbf{k}) : \mathbf{k} \in Y^*\}$ is called the m-th energy band, or Bloch band.

These bands are surfaces that describe the effective dispersion relation within the periodic structure. The eigenvalues $\mu_m(\mathbf{k})$ are thereby obtained through (17), the $\mathbf{k}-$pseudo periodic eigenvalue problem.

More directly, for fixed $k \in Y^*$, we make the change of variables $\chi(\mathbf{y};\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{y}}\Phi(\mathbf{y},\mathbf{k})$. Then (17) reduces to the periodic eigenvalue problem

$$
\begin{cases} \Big((i\nabla - \mathbf{k})^2 + V_{\text{per}}(y)\Big)\chi(\mathbf{y};\mathbf{k}) = \mu(\mathbf{k})\chi(\mathbf{y};\mathbf{k}), \ \mathbf{y} \in Y, \\ \chi(\mathbf{y}+\mathbf{v};\mathbf{k}) = \chi(\mathbf{y};\mathbf{k}), \ \mathbf{v} \in \Lambda. \end{cases} \tag{113}
$$

Hence in order to determine the spectrum of H, one must analyze the spectrum of the operator given by

$$
H_{\mathbf{k}} = (i\nabla - \mathbf{k})^2 + V_{\text{per}}(y) \quad \text{with domain} \quad D(H_{\mathbf{k}}) = H^2_{\text{per}}.
$$

For generic smooth potentials V_{per}, one can show rather rigorously see [46], that $H_{\mathbf{k}}$ is self-adjoint on H^2_{per} with spectrum $\sigma(H_{\mathbf{k}}) \subseteq \big[-\sup_{\mathbf{y}\in\mathbb{R}^2} V_{\text{per}}(\mathbf{y}), \infty\big)$. Furthermore, by the resolvent identity one can show that

$$
\big(H_{\mathbf{k}} - \lambda\mathbb{I}\big)^{-1} = \Big(\mathbb{I} - \big(H_{\mathbf{k}} - \lambda\mathbb{I}\big)^{-1}\big(V_{\text{per}} - (1+\lambda)\mathbb{I}\big)\Big)\Big((i\nabla - \mathbf{k})^2 + \mathbb{I}\Big)^{-1}
$$

is a compact operator (in fact Hilbert-Schmidt in $\mathbb{R}^2$) when $\lambda \notin \sigma(H_{\mathbf{k}})$. This follows from [46] since the operator acting on the outside is a bounded operator and the operator $\big((i\nabla - \mathbf{k})^2 + \mathbb{I}\big)^{-1}$ is compact (and Hilbert-Schmidt in $\mathbb{R}^2$). Then by the spectral theorem for compact operators it follows that the spectrum of the resolvent $\big(H_{\mathbf{k}} - \lambda\mathbb{I}\big)^{-1}$ consists of a sequence of countable eigenvalues converging to zero. Moreover, this ultimately gives that the eigenvalues of $H_{\mathbf{k}}$ (and hence H) satisfy $\mu_m(\mathbf{k}) \to \infty$ as $m \to \infty$. It should also be noted that the eigenvalues are continuous and periodic with respect to $\mathbf{k} \in Y^*$.

The associated eigenfunctions of (17), given by $\Phi_m(\cdot;\mathbf{k}) \in H^2_{\text{per}}$ are $\mathbf{k}-$pseudo periodic, and are usually referred to as Bloch waves. For every fixed $\mathbf{k} \in Y^*$, the eigenfunctions form a complete orthonormal basis on $L^2(Y)$. This consequently allows one to write the linear time evolution associated to (19) as

$$
\Psi(t,\mathbf{x}) = e^{-iHt/\varepsilon}\Psi_0(\mathbf{x}) = \sum_{m\geqslant 1} \int_{Y^*} e^{-i\mu_m(\mathbf{k})t/\varepsilon} \langle \Phi_m(\cdot\,;\mathbf{k}), \Psi_0 \rangle_{L^2(\mathbb{R}^2)} \Phi_m(\mathbf{x};\mathbf{k})\, d\mathbf{k},
$$

for any initial data $\Psi_0 \in L^2(\mathbb{R}^2)$.

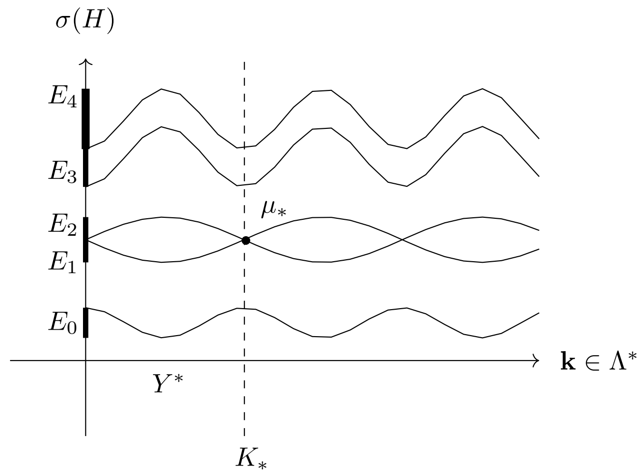

FIGURE 25. Characteristic band structure for Periodic Schrödinger operators in the case of Honeycomb lattice potentials. The Dirac point labeled μ^* at the edge of the Brillouin zone $K_* \in Y^*$, note that the intersection should appear conical.

1.3. Dirac Points. Recall that electrons in an isolated (carbon) atom occupy particular atomic orbitals. Each orbital has associated to it a discrete energy level. However, when many carbon atoms coalesce into a crystalline structure like graphene, the number of orbitals increases and the associated energy levels become very closely spaced, creating essentially a continuous energy band E_m, see Figure 25 above. Mathematically, we note that for any $m \in \mathbb{N}$ (on any band) there exists a closed subset $I \subset Y^*$ such that the functions $\mu_m(\mathbf{k})$ are real analytic functions for all $\mathbf{k} \in Y^*/I$, and we have the following condition

$$\mu_{m-1}(\mathbf{k}) < \mu_m(\mathbf{k}) < \mu_{m+1}(\mathbf{k}),\ \ \mathbf{k} \in Y^*/I.$$

We call E_m an isolated Bloch band if the condition above holds for all $\mathbf{k} \in Y^*$. Physically, such an isolated band ensures the existence of a band gap, whose width corresponds to how the given orbitals overlap. Lastly, it is known, that intersections of two adjacent bands only occur on a set of measure zero, namely

$$|I| = |\{\mathbf{k} \in Y^* : \mu_m(\mathbf{k}) = \mu_{m+1}(\mathbf{k})\}| = 0.$$

It should be noted, that at such band crossings the eigenvalues and eigenfunctions are not differentiable in $\mathbf{k}$.

An important mathematical feature of honeycomb lattice potentials is the presence of such crossings called Dirac points. These particular band crossings intersect conically and occur only at the vertices of the Brillouin zone. The following definition is taken from [43]:

DEFINITION 4.3. *Let V_{per} be a smooth honeycomb lattice potential. Then a vertex $\boldsymbol{K} = \mathbf{K}_* \in Y^*$ is called a **Dirac point** if the following holds: There exists $m_1 \in \mathbb{N}$, a real number μ_*, and strictly positive numbers, λ and δ, such that:*

1. *μ_* is a degenerate eigenvalue of H with associated $\boldsymbol{K}_*$-pseudo-periodic eigenfunctions.*
2. *dim Nullspace$(H - \mu_*) = 2$.*
3. *Nullspace$(H - \mu_*) = $ span$\{\Phi_1, \Phi_2\}$, where $\Phi_1 \in L^2_{\boldsymbol{K},\tau}$ and $\Phi_2 \in L^2_{\boldsymbol{K},\overline{\tau}}$.*
4. *There exists Lipschitz functions $\mu_\pm(\boldsymbol{k})$,*

$$\mu_{m_1}(\boldsymbol{k}) = \mu_-(\boldsymbol{k}), \quad \mu_{m_1+1}(\boldsymbol{k}) = \mu_+(\boldsymbol{k}), \quad \mu_{m_1}(\boldsymbol{K}_*) = \mu_*,$$

and $E_\pm(\boldsymbol{k})$, defined for $|\boldsymbol{k} - \boldsymbol{K}_*| < \delta$, such that

$$\begin{aligned} \mu_+(\boldsymbol{k}) - \mu_* &= +\lambda|\boldsymbol{k} - \boldsymbol{K}_*|(1 + E_+(\boldsymbol{k})) \\ \mu_-(\boldsymbol{k}) - \mu_* &= -\lambda|\boldsymbol{k} - \boldsymbol{K}_*|(1 + E_-(\boldsymbol{k})), \end{aligned}$$

where $|E_\pm(\boldsymbol{k})| < C|\boldsymbol{k} - \boldsymbol{K}_*|$ for some $C > 0$.

For later purposes we also recall the following result from [43] which is computed using the Fourier series expansion of $\Phi_{1,2}$, spanning the two-dimensional eigenspace associated to the Dirac point μ_*.

PROPOSITION 4.4. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ some vector. Then it holds*

$$\begin{aligned} &\langle \Phi_n, \zeta \cdot \nabla \Phi_n \rangle_{L^2(Y)} = 0 \ , \ n = 1, 2, \\ &2i\langle \Phi_1, \zeta \cdot \nabla \Phi_2 \rangle_{L^2(Y)} = \overline{2i\langle \Phi_2, \zeta \cdot \nabla \Phi_1 \rangle}_{L^2(Y)} = -\overline{\lambda_\#}(\zeta_1 + i\zeta_2), \\ &2i\langle \Phi_2, \zeta \cdot \nabla \Phi_1 \rangle_{L^2(Y)} = -\lambda_\#(\zeta_1 - i\zeta_2), \end{aligned}$$

where $\lambda_\# \in \mathbb{C}$ is defined by $\lambda_\# = 3|Y| \sum_{\mathbf{m} \in S} c(\mathbf{m})^2 (1, i)^T \cdot \mathbf{K}^*_{\mathbf{m}}$.

Here $\{c(\mathbf{m})\}_{\mathbf{m} \in S \subset \mathbb{Z}^2}$ denotes the sequence of $L^2_{\mathbf{K}_*,\tau}$ Fourier coefficients of the normalized eigenstate $\Phi_1(\mathbf{x})$ and $\mathbf{K}^*_{\mathbf{m}} = \mathbf{K}_* + m_1 \mathbf{k}_1 + m_2 \mathbf{k}_2$.

If $\lambda_\# \neq 0$, then (4) in Definition 4.3 above holds with $\lambda = |\lambda_\#|$. For the present work, we shall thus make the following assumption that is proved in [43] to be generically satisfied.

ASSUMPTION 2. *$V_{\text{per}} \in C^\infty(\mathbb{R}^2; \mathbb{R})$ is a smooth honeycomb lattice potential, which admits Dirac points such that $\lambda_\# \neq 0$.*

2. Multi-scale asymptotic expansions

2.1. Formal derivation of the Dirac system. In this section, we shall follow the ideas in [20, 53], and perform a formal multi-scale expansion of the solution to (19) under the Assumption 2. To this end, we seek a solution of the form

$$\Psi^\varepsilon(t,\mathbf{x}) \underset{\varepsilon\to 0}{\sim} \Psi^\varepsilon_N(t,\mathbf{x}) := e^{-i\lambda t/\varepsilon}\sum_{n=0}^{N}\varepsilon^n u_n\Big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big), \quad \lambda\in\mathbb{R},$$

where each $u_n(t,\mathbf{x},\mathbf{y})$ is supposed to be $\mathbf{k}-$pseudo periodic with respect to the fast variable $\mathbf{y}=\frac{\mathbf{x}}{\varepsilon}$. From now on, we denote the linear Hamiltonian by

$$H^\varepsilon = -\varepsilon^2\Delta + V_{\text{per}}\Big(\frac{\mathbf{x}}{\varepsilon}\Big) \tag{114}$$

and formally plug the ansatz Ψ^ε_N into (19). This yields

$$i\varepsilon\partial_t\Psi^\varepsilon_N - H^\varepsilon\Psi^\varepsilon_N - \varepsilon\kappa|\Psi^\varepsilon_N|^2\Psi^\varepsilon_N = e^{-i\lambda(t/\varepsilon)}\sum_{n=0}^{N}\varepsilon^n X_n + \rho(\Psi^\varepsilon_N),$$

where the remainder is

$$\rho\left(\Psi^\varepsilon_N\right) = e^{-i\lambda(t/\varepsilon)}\sum_{n=N+1}^{3N+1}\varepsilon^n X_n. \tag{115}$$

Introducing the operators

$$L_0 = \lambda - H, \quad L_1 = i\partial_t + 2\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}, \quad L_2 = \Delta_{\mathbf{x}},$$

with H given by (112), we find (after some lengthy calculations) that

$$X_n = L_0 u_n + L_1 u_{n-1} + L_2 u_{n-2} - \kappa\sum_{\substack{j+k+l+1=n\\ 0\leqslant j,k,l<n}} u_j u_k \overline{u}_l.$$

We can then proceed by solving $X_n=0$ for all $n\leqslant N$, to obtain an approximate solution Ψ^ε_N, which formally solves (19) up to an error of order $\mathcal{O}(\varepsilon^{N+1})$. To this end, the leading order equation is $L_0u_0=0$, which means

$$Hu_0 \equiv (-\Delta_{\mathbf{y}} + V_{\text{per}}(\mathbf{y}))u_0 = \lambda u_0.$$

In view of the assumption that u_0 is $\mathbf{k}-$pseudo periodic, this implies that $\lambda=\mu(\mathbf{k})$ is a Bloch eigenvalue. We shall from now on fix $\mathbf{k}=\mathbf{K}_*$ to be a Dirac point satisfying Assumption 2, and denote the associated eigenvalue by $\lambda = \mu_*$. The leading order amplitude u_0 can thus be written as

$$u_0(t,\mathbf{x},\mathbf{y}) = \sum_{j=1}^{2} \alpha_j(t,\mathbf{x})\Phi_j(\mathbf{y};\mathbf{K}_*), \tag{116}$$

where $\Phi_{1,2}$ span the two-dimensional eigenspace of μ_*, as in Definition 4.3.

To determine the leading order amplitudes α_1, α_2 we set $X_1 = 0$ to obtain

$$L_0 u_1 = \kappa |u_0|^2 u_0 - L_1 u_0. \tag{117}$$

Explicitly, the right hand side reads

$$\kappa|u_0|^2 u_0 - L_1 u_0 = -i \sum_{j,k,l=1}^{2} \Big(\Phi_j \partial_t \alpha_j - 2i \nabla_{\mathbf{x}} \alpha_j \cdot \nabla_{\mathbf{y}} \Phi_j + i\kappa \alpha_j \overline{\alpha}_k \alpha_l \Phi_j \overline{\Phi}_k \Phi_l \Big). \tag{118}$$

By Fredholm's alternative, a necessary condition for the solvability of (117) is that the right hand side $\notin \ker(L_0)$. Denoting by $\mathbb{P}_* = \mathbb{P}_*^2$ the $L^2(Y)-$projection on the spectral subspace corresponding to μ_*, we consequently require $\mathbb{P}_*(\kappa|u_0|^2 u_0 - L_1 u_0) = 0$, or, more explicitly:

$$\sum_{j,k,l=1}^{2} \big\langle \Phi_m, \Phi_j \partial_t \alpha_j - 2i\nabla_{\mathbf{x}} \alpha_j \cdot \nabla_{\mathbf{y}} \Phi_j + i\kappa \alpha_j \overline{\alpha}_k \alpha_l \Phi_j \overline{\Phi}_k \Phi_l \big\rangle_{L^2(Y)} = 0, \text{ for } m = 1,2.$$

Applying Proposition 4.4 with $\zeta = \nabla_{\mathbf{x}} \alpha_j \in \mathbb{C}^2$ for $j = 1,2$, respectively, we obtain the following system of equations

$$\partial_t \alpha_1 + \overline{\lambda}_{\#} \big(\partial_{x_1} + i\partial_{x_2}\big)\alpha_2 + i \sum_{j,k,l=1}^{2} \kappa_{(j,k,l,1)} \alpha_j \overline{\alpha}_k \alpha_l = 0$$

$$\partial_t \alpha_2 + \lambda_{\#} \big(\partial_{x_1} - i\partial_{x_2}\big)\alpha_1 + i \sum_{j,k,l=1}^{2} \kappa_{(j,k,l,2)} \alpha_j \overline{\alpha}_k \alpha_l = 0,$$

where $\lambda_{\#} \neq 0$ is as above, and

$$\kappa_{(j,k,l,m)} = \kappa \big\langle \Phi_m, \Phi_j \overline{\Phi}_k \Phi_l \big\rangle_{L^2(Y)}.$$

The latter can now be evaluated, using symmetry considerations of the action of the operator $\mathcal{R}$, defined in (111), when acting onto Bloch eigenfunctions. We recall that $\Phi_1 \in L^2_{\mathbf{K},\tau}, \Phi_2 \in L^2_{\mathbf{K},\overline{\tau}}$ are eigenfunctions for $\mathcal{R}$ such that

$$\mathcal{R}[\Phi_1(\mathbf{y})] = \Phi_1(R^*\mathbf{y}) = \tau \Phi_1(\mathbf{y}), \quad \mathcal{R}[\Phi_2(\mathbf{y})] = \overline{\Phi}_1(-R^*\mathbf{y}) = \overline{\tau}\Phi_2(\mathbf{y}),$$

where $\tau = e^{2\pi i/3}$.

To compute the integrals $\left\langle \Phi_m, \Phi_j\overline{\Phi}_k\Phi_l\right\rangle_{L^2(Y)}$ we use the change of coordinates $\mathbf{y} = R^*\mathbf{w}$ and apply the relations above, to obtain:

$$\begin{aligned}\kappa_{(j,k,l,m)} &= \kappa \int_{R(Y)} \overline{\Phi}_m(R^*\mathbf{w})\Phi_j(R^*\mathbf{w})\overline{\Phi}_k(R^*\mathbf{w})\Phi_l(R^*\mathbf{w})\, d\mathbf{w} \\ &= \kappa \int_Y \overline{\tau}_m\overline{\Phi}_m(\mathbf{w})\tau_j\Phi_j(\mathbf{w})\overline{\tau}_k\overline{\Phi}_k(\mathbf{w})\tau_l\Phi_l(\mathbf{w})\, d\mathbf{w} = \tau_j\overline{\tau}_k\tau_l\overline{\tau}_m\kappa_{(j,k,l,m)},\end{aligned}$$

so that

$$\kappa_{(j,k,l,m)}\Big(1 - C_{(j,k,l,m)}\Big) = 0, \quad C_{(j,k,l,m)} = \tau_j\overline{\tau}_k\tau_l\overline{\tau}_m.$$

We consequently find that $\kappa_{(j,k,l,m)}$ vanishes, whenever $C_{(j,k,l,m)} \neq 1$. A computation shows

$$\begin{aligned}&C_{(j,j,j,m)} = \tau_j\overline{\tau}_j\tau_j\overline{\tau}_m = \tau_j\overline{\tau}_m \neq 1 \text{ for } m \neq j, \\ &C_{(j,k,j,m)} = C_{(k,j,m,j)} = \tau_j^2\overline{\tau}_k\overline{\tau}_m = \tau_k\overline{\tau}_k\overline{\tau}_m = \overline{\tau}_m \neq 1 \text{ for } m = 1, 2, \\ &C_{(j,j,k,m)} = \tau_j\overline{\tau}_j\tau_k\overline{\tau}_m = \tau_k\overline{\tau}_m \neq 1 \text{ for } k \neq m = j,\end{aligned}$$

and we consequently deduce that the only non-vanishing coefficients are

$$\begin{aligned}b_1 &:= \kappa_{(1,1,1,1)} = \kappa_{(2,2,2,2)} = \int_Y |\Phi_1(\mathbf{y})|^4\, d\mathbf{y} = \int_Y |\Phi_2(\mathbf{y})|^4\, d\mathbf{y}, \\ b_2 &:= \kappa_{(1,1,2,2)} = \kappa_{(1,2,2,1)} = \kappa_{(2,2,1,1)} = \kappa_{(2,1,1,2)} = \int_Y |\Phi_1(\mathbf{y})|^2|\Phi_2(\mathbf{y})|^2\, d\mathbf{y}.\end{aligned}$$

In summary, we find the nonlinear Dirac system as announced in the Introduction:

$$\begin{cases}\partial_t\alpha_1 + \overline{\lambda}_\#(\partial_{x_1} + i\partial_{x_2})\alpha_2 + i\kappa\Big(b_1|\alpha_1|^2 + 2b_2|\alpha_2|^2\Big)\alpha_1 = 0, \\ \partial_t\alpha_2 + \lambda_\#(\partial_{x_1} - i\partial_{x_2})\alpha_1 + i\kappa\Big(b_1|\alpha_2|^2 + 2b_2|\alpha_1|^2\Big)\alpha_2 = 0.\end{cases}$$

Obviously, this system needs to be supplemented with initial data $\alpha_{1,0}$, $\alpha_{2,0}$, which for simplicity we assume to live in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

2.2. Higher order corrections. Assuming that the leading order amplitudes α_1, α_2 satisfy the nonlinear Dirac system allows us to proceed with our expansion and solve (117) for u_1. We obtain a unique solution in the form

$$u_1(t, \mathbf{x}, \mathbf{y}) = \tilde{u}_1(t, \mathbf{x}, \mathbf{y}) + u_1^{\perp}(t, \mathbf{x}, \mathbf{y}), \tag{119}$$

where $\tilde{u}_1 \in (\ker L_0)$ so that

$$\tilde{u}_1(t, \mathbf{x}, \mathbf{y}) = \sum_{j=1}^{2} \beta_j(t, \mathbf{x})\Phi_j(\mathbf{y}, \mathbf{K}_*) \tag{120}$$

for some yet to be determined amplitudes $\beta_{1,2}$, and

$$u_1^{\perp}(t,\mathbf{x},\mathbf{y}) = -L_0^{-1}\big(L_1 u_0 - \kappa |u_0|^2 u_0\big) \in (\ker L_0)^{\perp}. \tag{121}$$

Here we denote by L_0^{-1} the partial inverse (or partial resolvent) of L_0, i.e.

$$L_0^{-1} = (1-\mathbb{P}_*)(\mu_* - H)^{-1}(1-\mathbb{P}_*).$$

Note that at $t=0$, the function $u_1^{\perp}(0,\mathbf{x},\mathbf{y})$ cannot be chosen, but rather has to be determined from the initial data $\alpha_{0,1}$, $\alpha_{0,2}$ according to the formula above.

Proceeding further, we determine the amplitudes β_1, β_2 by setting $X_2 = 0$ so that

$$L_0 u_2 = -L_1 u_1 - L_2 u_0 + \kappa\big(u_0^2 \overline{u}_1 + 2|u_0|^2 u_1\big).$$

By the same arguments as before, we obtain the following system of linear, inhomogeneous Dirac equations for $\beta_{1,2}$ as the corresponding solvability condition:

$$\begin{cases} \partial_t \beta_1 + \overline{\lambda}_\#\big(\partial_{x_1} + i\partial_{x_2}\big)\beta_2 + i \displaystyle\sum_{j,k,l=1}^{2} \kappa_{(j,k,l,1)}\Big(\alpha_j \overline{\beta}_k \alpha_l + 2\beta_j \overline{\alpha}_k \alpha_l\Big) = i\Theta_1 \\ \partial_t \beta_2 + \lambda_\#\big(\partial_{x_1} - i\partial_{x_2}\big)\beta_1 + i \displaystyle\sum_{j,k,l=1}^{2} \kappa_{(j,k,l,2)}\Big(\alpha_j \overline{\beta}_k \alpha_l + 2\beta_j \overline{\alpha}_k \alpha_l\Big) = i\Theta_2, \end{cases} \tag{122}$$

where the right hand side source terms can be written as

$$\Theta_n = \Delta \alpha_n + \langle \Phi_n, L_1 u_1^{\perp}\rangle_{L^2(Y)} - \kappa \langle \Phi_n, (u_0 \overline{u}_1^{\perp} + 2\overline{u}_0 u_1^{\perp}) u_0\rangle_{L^2(Y)}. \tag{123}$$

It should be noted here that we have the freedom to choose vanishing initial data $\beta_{1,2}(0,\mathbf{x}) = 0$ for the system (122), which nevertheless will have a non-vanishing solution due to the source terms. In summary, this allows us to write $u_2(t,\mathbf{x},\mathbf{y}) = \tilde{u}_2(t,\mathbf{x},\mathbf{y}) + u_2^{\perp}(t,\mathbf{x},\mathbf{y})$, where $\tilde{u}_2 \in (\ker L_0)$ and $u_2^{\perp} \in (\ker L_0)^{\perp}$. The latter is obtained by elliptic inversion on $(\ker L_0)^{\perp}$, such that

$$u_2^{\perp}(t,\mathbf{x},\mathbf{y}) = -L_0^{-1}\Big(L_1 u_1 + L_2 u_0 - \kappa\big(u_0^2 \overline{u}_1 + 2|u_0|^2 u_1\big)\Big). \tag{124}$$

All higher order terms u_n can then be determined analogously. However, since we are mainly interested in deriving the nonlinear Dirac system for the leading order amplitudes $\alpha_{1,2}$, we shall see that it is sufficient to stop our expansion at $N = 2$. Note that in order to satisfy $X_2 = 0$, one does not need to determine the amplitudes within $\tilde{u}_2 \in (\ker L_0)$, which will simplify our treatment below.

3. Mathematical framework for the approximate solution

3.1. Local well-posedness of the Dirac equations. Our aim in this subsection is to make the formal multi-scale expansion of the foregoing section mathematically rigorous. In a first step, we shall construct a unique local-in-time solution to the nonlinear Dirac model (21). We shall choose data in the Sobolev space $H^s(\mathbb{R}^2)$ with $s > 1$, with the understanding that for complex vector-valued functions $\mathbf{u} = (u_1, u_2)(\mathbf{x}) \in \mathbb{C}^2$ we can define the H^s-norm via the Fourier transform and the bracket $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$ such that

$$\|\mathbf{u}\|_{H^s}^2 = \int_{\mathbb{R}^2} \langle\boldsymbol{\xi}\rangle^s \big(\widehat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \overline{\widehat{\mathbf{u}}}(\boldsymbol{\xi})\big)\, d\boldsymbol{\xi} = \|u_1\|_{H^s}^2 + \|u_2\|_{H^s}^2,$$

where the " $\cdot$ " represents the usual Euclidean inner product. To this end, we will work in the Banach space $X = C([0,T); H^s(\mathbb{R}^2))$ for $s > 1$, endowed with the norm

$$\|\mathbf{u}\|_X \equiv \sup_{0 \leqslant t \leqslant T} \Big(\|u_1(t)\|_{H^s}^2 + \|u_2(t)\|_{H^s}^2\Big)^{1/2}.$$

Let us first rewrite the system (21) as a vectorial equation. Namely, let $\boldsymbol{\alpha}(t,\mathbf{x}) = (\alpha_1, \alpha_2)(t,\mathbf{x})$ and define the matrix operator by

$$D(\nabla) = \begin{pmatrix} 0 & \overline{\lambda}_{\#}\big(\partial_{x_1} + i\partial_{x_2}\big) \\ \lambda_{\#}\big(\partial_{x_1} - i\partial_{x_2}\big) & 0 \end{pmatrix}.$$

Then we can write (21) in the following form

$$\partial_t \boldsymbol{\alpha} + D(\nabla)\boldsymbol{\alpha} = -i\kappa \mathbf{F}(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha}_{|t=0} = \boldsymbol{\alpha}_0(\mathbf{x}), \tag{125}$$

where $\mathbf{F}(\boldsymbol{\alpha})$ is the nonlinearity discussed in the following lemma.

LEMMA 4.5. *Consider the function*

$$G(\alpha_1, \overline{\alpha}_1, \alpha_2, \overline{\alpha}_2) = \frac{b_1}{2}\big(|\alpha_1|^4 + |\alpha_2|^4\big) + 2b_2|\alpha_1|^2|\alpha_2|^2,$$

with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in H^s(\mathbb{R}^2)^2$. *Then, the nonlinear vector field* $\mathbf{F}(\boldsymbol{\alpha}) = \begin{pmatrix} \partial_{\overline{\alpha}_1} G \\ \partial_{\overline{\alpha}_2} G \end{pmatrix}$ *is a map from* $H^s(\mathbb{R}^2)^2 \to H^s(\mathbb{R}^2)^2$ *for any* $s > 1$.

PROOF. We note that for $s > 1$, $H^s(\mathbb{R}^2)$ is a Banach algebra so that we have

$$\|fg\|_{H^s} \leqslant C_s \|f\|_{H^s} \|g\|_{H^s}, \ \forall f, g \in H^s(\mathbb{R}^2),$$

which implies the trilinear estimate

$$\|fgh\|_{H^s} \leqslant C_s^2 \|f\|_{H^s}\|g\|_{H^s}\|h\|_{H^s},\ \forall f,g,h \in H^s(\mathbb{R}^2).$$

One notices that for $i \neq j$ that

$$\partial_{\overline{\alpha}_i} G = b_1|\alpha_i|^2\alpha_i + 2b_2|\alpha_j|^2\alpha_i = b_1\alpha_i\alpha_i\overline{\alpha}_i + 2b_2\alpha_j\overline{\alpha}_j\alpha_i.$$

After use of the trilinear estimate gives for $\boldsymbol{\alpha} \in H^s(\mathbb{R}^2)$ that

$$\|\mathbf{F}(\boldsymbol{\alpha})\|_{H^s} \leqslant \big(b_1 + 2b_2\big)C_s^2\|\boldsymbol{\alpha}\|_{H^s}^3 < \infty.$$

$\square$

Now that we can control the nonlinear term in $H^s(\mathbb{R}^2)$, we shall determine the linear time evolution governed by the continuous matrix semi-group acting on $\mathbf{f} \in \mathcal{S}(\mathbb{R}^2)$, namely

$$U(t)\mathbf{f}(\mathbf{x}) = e^{-tD(\nabla)}\mathbf{f}(\mathbf{x}) \equiv \mathcal{F}^{-1}\Big(e^{-tD(\boldsymbol{\xi})}\widehat{\mathbf{f}}(\boldsymbol{\xi})\Big)(\mathbf{x}), \tag{126}$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2)$ represent the Fourier variables associated to $\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2$. By Fourier transforming the linear part of (125), we obtain the following system of ODE

$$\partial_t\widehat{\boldsymbol{\alpha}} = -D(\boldsymbol{\xi})\widehat{\boldsymbol{\alpha}}, \tag{127}$$

with the Fourier transformed matrix given by

$$D(\boldsymbol{\xi}) = \begin{pmatrix} 0 & -\overline{\lambda_\#\big(i\xi_1+\xi_2\big)} \\ \lambda_\#\big(i\xi_1+\xi_2\big) & 0 \end{pmatrix}.$$

For each fixed $\boldsymbol{\xi} \in \mathbb{R}^2$, one notices that $D(\boldsymbol{\xi})$ is a *skew-Hermitian* matrix such that the conjugate transpose $D(\boldsymbol{\xi})^\dagger := \overline{D(\boldsymbol{\xi})}^T = -D(\boldsymbol{\xi})$. Any skew-Hermitian matrix is diagonalizable with purely imaginary eigenvalues, in this case, given by $\pm i|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2}$. Further, it is clear from standard ODE theory that the solution of (152) is given by the matrix exponential

$$\widehat{\boldsymbol{\alpha}}(\boldsymbol{\xi},t) = e^{-tD(\boldsymbol{\xi})}\widehat{\boldsymbol{\alpha}}_0(\boldsymbol{\xi}).$$

We shall explicitly determine the exponential matrix below, but the fact that $-D(\boldsymbol{\xi})$ is skew-Hermitian implies that the exponential map $e^{-tD(\boldsymbol{\xi})}$ is a unitary matrix for $t \in \mathbb{R}$. Furthermore $U(t) = e^{-tD(\nabla)}$ is a strongly continuous unitary group on $H^s(\mathbb{R}^2)$ for $s > 1$.

To see this we compute explicitly

$$\|U(t)\mathbf{u}\|_{H^s}^2 = \|\langle\boldsymbol{\xi}\rangle^{s/2}e^{-tD(\boldsymbol{\xi})}\widehat{\mathbf{u}}\|_{L^2}^2 = \int_{\mathbb{R}^2}\langle\boldsymbol{\xi}\rangle^s\Big(e^{-tD(\boldsymbol{\xi})}\widehat{\mathbf{u}}\cdot \overline{e^{-tD(\boldsymbol{\xi})}\widehat{\mathbf{u}}}\Big)\,d\boldsymbol{\xi}$$
$$= \int_{\mathbb{R}^2}\langle\boldsymbol{\xi}\rangle^s\Big(e^{-t\big(D(\boldsymbol{\xi})^\dagger + D(\boldsymbol{\xi})\big)}\widehat{\mathbf{u}}\cdot\overline{\widehat{\mathbf{u}}}\Big)\,d\boldsymbol{\xi} = \|\mathbf{u}\|_{H^s}^2.$$

Lastly, in order to determine the propagator explicitly, we must diagonalize $D(\xi)$. With the eigenvalues above, one can construct the matrix of eigenvectors so that

$$E(\boldsymbol{\xi}) = \begin{pmatrix} i|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2} & i|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2} \\ \lambda_\#\big(i\xi_1+\xi_2\big) & -\lambda_\#\big(i\xi_1+\xi_2\big)\end{pmatrix},$$

and the respective inverse for $\boldsymbol{\xi}\neq\mathbf{0}$ given by

$$E^{-1}(\boldsymbol{\xi}) = \begin{pmatrix} \frac{1}{2i|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2}} & \frac{-\lambda_\#(i\xi_1-\xi_2)}{2|\lambda_\#|^2(\xi_1^2+\xi_2^2)} \\ \frac{1}{2i|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2}} & \frac{\lambda_\#(i\xi_1-\xi_2)}{2|\lambda_\#|^2(\xi_1^2+\xi_2^2)}\end{pmatrix}.$$

In this regard we note that we can formally compute the exponential matrix as follows

$$e^{-tD(\boldsymbol{\xi})} = E(\boldsymbol{\xi})\begin{pmatrix} e^{it|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2}} & 0 \\ 0 & e^{-it|\lambda_\#|\sqrt{\xi_1^2+\xi_2^2}}\end{pmatrix}E^{-1}(\boldsymbol{\xi})$$
$$= \begin{pmatrix} \cos\left(|\lambda_\#\boldsymbol{\xi}|t\right) & \frac{\lambda_\#(i\xi_1+\xi_2)\sin\big(|\lambda_\#\boldsymbol{\xi}|t\big)}{|\lambda_\#\boldsymbol{\xi}|} \\ \frac{\lambda_\#(i\xi_1+\xi_2)\sin\big(|\lambda_\#\boldsymbol{\xi}|t\big)}{|\lambda_\#\boldsymbol{\xi}|} & \frac{\lambda_\#^2}{|\lambda_\#|^2}\cos\left(|\lambda_\#\boldsymbol{\xi}|t\right)\end{pmatrix}.$$

In this way we can define the linear propagator exactly via (126) and the above expression. Together with the foregoing lemma, we can use the above to prove the following local well-posedness result.

PROPOSITION 4.6. *For any $\boldsymbol{\alpha}_0\in\mathcal{S}(\mathbb{R}^2)^2$ there exists a time $T>0$ and a unique maximal solution $\boldsymbol{\alpha}\in C\big([0,T);H^s(\mathbb{R}^2)\big)\cap C^1\big((0,T);H^{s-1}(\mathbb{R}^2)\big)$ for all $s>1$ to (125). Moreover, it holds*

$$\|\alpha_1(t,\cdot)\|_{L^2} + \|\alpha_2(t,\cdot)\|_{L^2} = \text{ const. } \forall t\in[0,T).$$

PROOF. This follows by a fixed point argument applied to Duhamel's integral formulation of (125), i.e.

$$\boldsymbol{\alpha}(t) = U(t)\boldsymbol{\alpha}_0 - i\kappa\int_0^t U(t-\tau)\mathbf{F}(\mathbf{u}(\tau))d\tau =: \Phi(\boldsymbol{\alpha})(t). \tag{128}$$

Indeed, using Lemma 4.5 and the unitarity of $U(t)$ on H^s for $s>1$, one easily sees that for any $\boldsymbol{\alpha}_0\in H^s(\mathbb{R}^2)$ such that $\|\boldsymbol{\alpha}_0\|_{H^s}\leqslant R$, the functional $\Phi: X\to X$ is a contraction mapping on

$K = \{\boldsymbol{\alpha} \in X : \|\boldsymbol{\alpha}\|_X \leqslant 2R\}$, provided $T = \dfrac{1}{8(b_1 + 2b_2)R^2 C_s^2} > 0$. The asserted regularity in time then follows by differentiating the equation. Finally, the identity for mass conservation is obtained by multiplying the equations by $\overline{\alpha}_j$, integrating and taking the imaginary part. □

With this result in hand, we can also obtain local well-posedness for the inhomogeneous linear Dirac equation (122) on the same time interval, namely we have

$$\boldsymbol{\beta} \in C\big([0, T); H^{s-2}(\mathbb{R}^2)\big). \tag{129}$$

This follows in view of the fact that the coefficients $\boldsymbol{\alpha} \in C([0, T); L^\infty(\mathbb{R}^2))$ and the source terms satisfy $\boldsymbol{\theta} \in C\big([0, T); H^{s-2}(\mathbb{R}^2)\big)$. The latter can be seen from (123) which involves the Laplacian of u_0 and thus we lose two derivatives.

REMARK 4.7. *The main obstacle in obtaining global well-posedness of the nonlinear Dirac equation* (125) *is the fact that the corresponding energy does not have a definite sign, i.e.,*

$$E(t) = \operatorname{Im}\left(\overline{\lambda_\#} \int_{\mathbb{R}^2} \alpha_2(\partial_{x_1} + i\partial_{x_2})\overline{\alpha}_1 \, d\mathbf{x}\right) - \frac{\kappa}{4} \int_{\mathbb{R}^2} b_1 |\boldsymbol{\alpha}|^2 + 4b_2 |\alpha_1|^2 |\alpha_2|^2 \, d\mathbf{x}.$$

So far, the existence of global in time solutions is thus restricted to small initial data cases, see, e.g., [**14, 40, 41, 83**] *and the references therein. Also due to the fact that* (125) *is also a massless equation proves to be an additional complication. Since the spectral subspaces for the corresponding free Dirac operator are no longer separated, this requires considerable more care than the case with nonzero mass.*

3.2. Estimates on the approximate solution and the remainder. Formally, the approximate solution Ψ_N^ε derived in Section 2 solves (19) up to errors of order $\mathcal{O}(\varepsilon^{N+1})$. To make this error estimate rigorous on time-scales of order $\mathcal{O}(1)$, we shall prove a nonlinear stability result in Section 2 below. The latter will require us to work with an approximate solution of order $N > 1$. We consequently need to work (at least) with Ψ_2^ε. Although as was already remarked above, in order to solve (19) up to reminders of $\mathcal{O}(\varepsilon^2)$, one does not need to determine the highest order amplitudes within $\tilde{u}_2 \in (\ker L_0)$. We shall thus set them identically equal to zero and work with an approximate solution of the following form

$$\begin{aligned} \Psi_{\mathrm{app}}^\varepsilon(t, \mathbf{x}) &= e^{-i\mu_* t/\varepsilon}\Big(u_0 + \varepsilon u_1 + \varepsilon^2 u_2\Big)\Big(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big) \\ &= e^{-i\mu_* t/\varepsilon} \sum_{j=1}^{2} \big(\alpha_j + \varepsilon\beta_j\big)(t, \mathbf{x})\Phi_j\Big(\frac{\mathbf{x}}{\varepsilon}; \mathbf{K}_*\Big) + \varepsilon^j u_j^\perp\Big(t, \mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\Big), \end{aligned} \tag{130}$$

where $\alpha_{1,2} \in C\big([0,T); H^s(\mathbb{R}^2)\big)$ and $\beta_{1,2} \in C\big([0,T); H^{s-2}(\mathbb{R}^2)\big)$ for $s > 1$, are the leading and first order amplitudes guaranteed to exist by the results of the previous subsection. The term of order $\mathcal{O}(\varepsilon^2)$ within this approximation is solely determined by elliptic inversion and thereby depends on the two lower order terms.

REMARK 4.8. *The nonlinear Dirac system* (125) *has been formally derived in* [**44**] *and plays the same role as the coupled mode equations derived in* [**53**], *as well as the semi-classical transport equations derived in* [**16, 20**]. *This becomes even more apparent when we recall that the Bloch waves* $\Phi_{1,2}$ *can be written as*

$$\Phi_{1,2}\left(\frac{\mathbf{x}}{\varepsilon}\right) = \chi_{1,2}\left(\frac{\mathbf{x}}{\varepsilon}\right) e^{i\mathbf{K}_*\cdot\mathbf{x}/\varepsilon},$$

where $\chi_{1,2}(\cdot)$ *is purely* Λ*-periodic. This shows that* (20) *is of the form of a two-scale Wentzel-Kramers-Brillouin (WKB) ansatz, first introduced in* [**16**] *involving a highly oscillatory phase function*

$$S(t,x) = \mathbf{K}_* \cdot \mathbf{x} - \mu_* t.$$

The latter is seen to be the unique, global-in-time, smooth solution (i.e., no caustics) of the semi-classical Hamilton-Jacobi equation

$$\partial_t S + \mu(\nabla S) = 0, \quad S(0,\mathbf{x}) = \mathbf{K}_* \cdot \mathbf{x}.$$

The fact that the phase function S *does not suffer from caustics, allows us to prove that our approximation* (130) *holds for (finite) time-intervals of order* $\mathcal{O}(1)$*, bounded by the existence time of* (21). *In contrast to that, it is known from earlier results, cf.* [**16, 20**], *that caustics within* S *lead to the breakdown of a single-phase WKB approximation. Also note that the group velocity* $\nabla S = \mathbf{K}_*$ *being constant allows us to localize around Dirac points.*

Now, since our approximate solution $\Psi^\varepsilon_{\mathrm{app}}$ involves highly oscillatory Bloch eigenfuntions, we cannot expect to obtain uniform-in-epsilon estimates in the usual Sobolev spaces $H^s(\mathbb{R}^2)$. We shall therefore work in the following ε-scaled spaces, as used in [**20, 53**].

DEFINITION 4.9. *Let* $s \in \mathbb{Z}$, $0 < \varepsilon \leqslant 1$, *and* $f \in H^s_\varepsilon(\mathbb{R}^2)$ *and define the* $H^s_\varepsilon(\mathbb{R}^2)-$*norm by*

$$\|f\|^2_{H^s_\varepsilon} := \sum_{|\gamma|\leqslant s} \|(\varepsilon\partial_x)^\gamma f\|^2_{L^2}.$$

A family f^ε *is bounded in* $H^s_\varepsilon(\mathbb{R}^2)$ *whenever*

$$\sup_{0<\varepsilon\leqslant 1} \|f^\varepsilon\|_{H^s_\varepsilon} < \infty.$$

We note that in $H^s_\varepsilon(\mathbb{R}^2)$ the following Gagliardo-Nirenberg inequality holds

$$\forall s > 1 \ \exists C_\infty > 0 : \ \|f\|_{L^\infty} \leqslant C_\infty \varepsilon^{-1} \|f\|_{H^s_\varepsilon}, \tag{131}$$

where the "bad" factor ε^{-1} is obtained by scaling.

The following proposition then collects the necessary regularity estimates for our approximate solution and its corresponding remainder.

PROPOSITION 4.10. *Let $V_{\rm per}$ satisfy Assumption 2 and choose $S \in (5, \infty)$ such that $S > s + 4$ for any $s > 1$. Let*

$$\boldsymbol{\alpha} \in C\Big([0,T), H^S(\mathbb{R}^2)\Big), \quad \boldsymbol{\beta} \in C\Big([0,T), H^{S-2}(\mathbb{R}^2)\Big)$$

be the solutions to (non-)linear Dirac systems (125) and (122), respectively.

Then, the approximate solution $\Psi^\varepsilon_{\rm app}(t,\cdot)$, given in (130), satisfies the following estimates for all $t \in [0,T)$ and for any $|\gamma| \leqslant s$:

$$\|(\varepsilon\partial)^\gamma \Psi^\varepsilon_{\rm app}(t,\cdot)\|_{L^\infty} \leqslant C_a, \quad \|\Psi^\varepsilon_{\rm app}(t,\cdot)\|_{H^s_\varepsilon} \leqslant C_b, \quad \|\rho(\Psi^\varepsilon_{\rm app})(t,\cdot)\|_{H^s_\varepsilon} \leqslant C_r \varepsilon^3.$$

with $C_a, C_b, C_r > 0$ independent of ε.

PROOF. To prove the estimates of the lemma, we need to first establish the regularity of $u_n(t) = u_n\big(t, \mathbf{x}, \mathbf{y}\big)$ for $n = 0, 1, 2$. We note first that Assumption 2 implies that the eigenfunctions $\Phi_j(\cdot\,; \mathbf{K}^*) \in C^\infty(\overline{Y})$ for $j = 1, 2$, cf. [105]. We shall then prove the following, preliminary estimate for $t \in [0,T)$

$$u_n(t) \in H^{S-3}(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}), \quad n = 0, 1, 2, \tag{132}$$

where $C^\infty_{\mathbf{K}_*}(\overline{Y})$ is the space of smooth $\mathbf{K}_*-$pseudo-periodic functions on $\overline{Y}$. To this end, we have for $t \in [0,T)$ that

$$u_0(t) \in H^S(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}),$$

due to (116) and Proposition 4.6. Next, recall that u_1 is of the form (119) with $\tilde{u}_1$ and $u_1^\perp$ given by (120) and (121), respectively. In view of (120) and (129), we directly infer

$$\tilde{u}_1(t) \in H^{S-2}(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}).$$

Moreover, since $L_0^{-1} : L^2(Y) \to H^2_{\rm per}(Y)$ and $L_1 = i\partial_t + 2\nabla x \cdot \nabla_y$ it follows from (117) that

$$u_1^\perp(t) \in H^{S-1}(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}),$$

and thus

$$u_1(t) \in H^{S-2}(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}),$$

since $H^s(\mathbb{R}^2) \subset H^{s-1}(\mathbb{R}^2)$ for all $s > 0$. Lastly, we recall that u_2 has the same type of structure with $u_2^\perp$ given by (124). Similarly, as before it then follows that

$$u_2(t) \in H^{S-3}(\mathbb{R}^2) \times C^\infty_{\mathbf{K}_*}(\overline{Y}).$$

In summary, this yields (132) which implies $u_n\Big(t, \cdot, \frac{\cdot}{\varepsilon}\Big) \in H_\varepsilon^{S-3}(\mathbb{R}^2)$ for $n = 0, 1, 2$. Hence it follows that for any $s > 1$

$$u_n\Big(t, \cdot, \frac{\cdot}{\varepsilon}\Big) \in H^s_\varepsilon(\mathbb{R}^2), \quad n = 0, 1, 2,$$

whenever $S - 3 \geqslant s$. Moreover it follows that

$$\|\Psi_{\text{app}}(t, \cdot)\|_{H^s_\varepsilon} \leqslant \sum_{n=0}^{2} \varepsilon^n \|u_n\|_{H^s_\varepsilon} \leqslant \sum_{n=0}^{2} \|u_n\|_{H_\varepsilon^{S-3}} = C_b.$$

Having established this, our next step is to prove the first inequality of the lemma. It suffices to show that there exists a constant $C_0 > 0$ such that

$$\Big\|(\varepsilon\partial)^\gamma u_0\Big(t, \cdot, \frac{\cdot}{\varepsilon}\Big)\Big\|_{L^\infty} \leqslant C_0 \tag{133}$$

holds for all $\varepsilon \in (0, 1), |\gamma| \leqslant s$, and $t \in [0, T)$. Since u_0 is of the form (116), we need only to show that

$$\Big\|(\varepsilon\partial)^\gamma\Big(\alpha_{1,2}(t, \cdot)\Phi_{1,2}\Big(\frac{\cdot}{\varepsilon}; \mathbf{K}_*\Big)\Big)\Big\|_{L^\infty} \leqslant C,$$

where C is a constant independent of epsilon. By the Leibniz rule one has

$$(\varepsilon\partial)^\gamma\Big(\alpha_{1,2}(t, \mathbf{x})\Phi_{1,2}\Big(\frac{\mathbf{x}}{\varepsilon}; \mathbf{K}_*\Big)\Big) = \sum_{|\sigma| \leqslant |\gamma|} \begin{pmatrix} |\gamma| \\ \sigma \end{pmatrix} (\varepsilon\partial)^\sigma \alpha_{1,2}(t, \mathbf{x}) \cdot (\varepsilon\partial)^{\sigma-\gamma}\Phi_{1,2}\Big(\frac{\mathbf{x}}{\varepsilon}; \mathbf{K}_*\Big),$$

and we can thus estimate

$$\begin{aligned} \Big\|(\varepsilon\partial)^\gamma\Big(\alpha_{1,2}(t, \cdot)\Phi_{1,2}\Big(\frac{\cdot}{\varepsilon}; \mathbf{K}_*\Big)\Big)\Big\|_{L^\infty} &\leqslant C_1 \sum_{|\sigma| \leqslant |\gamma|} \Big\|(\varepsilon\partial)^\sigma \alpha_{1,2}(t, \cdot)\Big\|_{L^\infty} \Big\|(\varepsilon\partial)^{\sigma-\gamma}\Phi_{1,2}\Big(\frac{\cdot}{\varepsilon}; \mathbf{K}_*\Big)\Big\|_{L^\infty} \\ &\leqslant C_2 \Big\|\Phi_{1,2}\Big(\cdot; \mathbf{K}_*\Big)\Big\|_{C^{|\gamma|}_{\mathbf{K}_*}} \sum_{|\sigma| \leqslant |\gamma|} \Big\|\partial^\sigma \alpha_{1,2}(t, \cdot)\Big\|_{L^\infty} \leqslant C_3 \Big\|\alpha_{1,2}(t, \cdot)\Big\|_{H^{2+|\gamma|}} \leqslant C, \end{aligned}$$

where the second to last inequality follows by Sobolev embedding. Hence (133) follows by the triangle inequality. For $n = 1, 2$, one invokes (131) directly to obtain for $|\gamma| \leqslant s$ that

$$\|(\varepsilon\partial)^\gamma u_n\Big(t, \cdot, \frac{\cdot}{\varepsilon}\Big)\|_{L^\infty} \leqslant C_\infty \varepsilon^{-1} \|u_n\|_{H_\varepsilon^{m+|\gamma|}} \leqslant \varepsilon^{-1} \|u_n\|_{H^{m+s}} \leqslant \varepsilon^{-1} \|u_n\|_{H^{S-3}} \leqslant \varepsilon^{-1} C_n,$$

provided that $m > 1$ and $1 + s < S - 3$, which implies $S > s + 4$ which gives the condition stated above. The desired inequality then follows, since

$$\|(\varepsilon\partial)^\gamma \Psi^\varepsilon_{\text{app}}(t,\cdot)\|_{L^\infty} \leqslant C_0 + \sum_{n=1}^{2} \varepsilon^{n-1} \Big\|(\varepsilon\partial)^\gamma u_n\Big(t,\cdot,\frac{\cdot}{\varepsilon}\Big)\Big\|_{L^\infty} \leqslant C_a,$$

where C_a is a constant independent of epsilon. One notices that the last inequality follows from the fact that for $n = 3, \dots, 7$: $X_n \in H^s_\varepsilon(\mathbb{R}^2)$ and so

$$\|\rho(\Psi_{\text{app}})(t,\cdot)\|_{H^s_\varepsilon} \leqslant \sum_{n=3}^{7} \varepsilon^n \|X_n\|_{H^s_\varepsilon} \leqslant C_r \varepsilon^3.$$

□

With these estimates at hand, we can now prove the stability of our approximation.

4. Nonlinear stability of the approximation

Before we prove the nonlinear stability of our approximation, we recall that it was proved in [53], that the linear Schrödinger group

$$S^\varepsilon(t) := e^{-iH^\varepsilon t/\varepsilon} \tag{134}$$

generated by the periodic Hamiltonian H^ε defined in (114) satisfies

$$\|S^\varepsilon(t) f\|_{H^s_\varepsilon} \leqslant C_s \|f\|_{H^s_\varepsilon} \text{ for all } t \in \mathbb{R} \text{ and all } \varepsilon \in (0,1), \tag{135}$$

where $C_s > 0$ is independent of ε. To this end, one uses the fact that V_{per} is assumed to be smooth and periodic (as guaranteed by Assumption 2). Using this, it is straightforward to obtain the following basic well-posedness result for nonlinear Schrödinger equations.

LEMMA 4.11. *Let V_{per} satisfy Assumption 2 and $\Psi^\varepsilon_0 \in H^s_\varepsilon(\mathbb{R}^2)$, for $s > 1$. Then there exists a $T^\varepsilon > 0$ and a unique solution $\Psi^\varepsilon \in C([0, T^\varepsilon); H^s_\varepsilon(\mathbb{R}^2))$ to the initial value problem (19).*

PROOF. In view of (135), the proof is a straightforward extension of the one given in, e.g., [102, Proposition 3.8] for the case without periodic potential. □

REMARK 4.12. *Note that this result does not preclude the possibility that $T^\varepsilon \to 0_+$, as $\varepsilon \to 0_+$. However, it will be a by-product of our main theorem below that this is not the case (at least not for the class of initial data considered in this work).*

As a final preparatory step, we recall the following Moser-type lemma proved in [93, Lemma 8.1.1].

LEMMA 4.13. *Let $R > 0$, $s \in [0, \infty)$, and $\mathcal{N}(z) = \kappa|z|^2 z$ with $\kappa \in \mathbb{R}$. Then there exists a $C_s = C_s(R, s, d, \kappa)$ such that if f satisfies*

$$\|(\varepsilon\partial)^\gamma f\|_{L^\infty} \leqslant R \quad \forall|\gamma| \leqslant s,$$

and g satisfies

$$\|g\|_{L^\infty} \leqslant R,$$

then

$$\|\mathcal{N}(f+g) - \mathcal{N}(f)\|_{H^s_\varepsilon} \leqslant C_s \|g\|_{H^s_\varepsilon}$$

In contrast to other estimates (e.g., Schauder etc.), the above result has the advantage that we obtain a linear bound on the nonlinearity, a fact we shall use in the proof below. Indeed, we are now in the position to prove our main result.

THEOREM 4.1. *Let V_{per} satisfy Assumption 2, let $\boldsymbol{\alpha} \in C([0,T), H^S(\mathbb{R}^2))$ be a solution to (21) and $\boldsymbol{\beta} \in C([0,T), H^{S-2}(\mathbb{R}^2))$ be a solution to (122) for some $S > s+4$ with $s > 1$. Finally, assume that there is a $c > 0$ such that the initial data Ψ_0^ε of (19) satisfies*

$$\left\| \Psi_0^\varepsilon - \big(u_0 + \varepsilon u_1\big)\left(0, \cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^s_\varepsilon} \leqslant c\varepsilon^2.$$

Then, for any $T_* \in [0, T)$ there exists an $\varepsilon_0 = \varepsilon_0(T_*) \in (0,1)$ and a constant $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the solution $\Psi^\varepsilon \in C([0,T_*); H^s_\varepsilon(\mathbb{R}^2))$ of (19) exists and moreover

$$\sup_{0 \leqslant t \leqslant T_*} \left\| \Psi^\varepsilon(t, \cdot) - e^{-i\mu_* t/\varepsilon}\big(u_0 + \varepsilon u_1\big)\left(t, \cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^s_\varepsilon} \leqslant C\varepsilon^2.$$

PROOF. Consider the difference in the exact and approximate solution $\varphi^\varepsilon = \Psi^\varepsilon - \Psi^\varepsilon_{\mathrm{app}}$, then the difference $\varphi^\varepsilon(t, \mathbf{x})$ satisfies

$$i\partial_t \varphi^\varepsilon = \frac{1}{\varepsilon} H^\varepsilon \varphi^\varepsilon + \Big(g(\Psi^\varepsilon_{\mathrm{app}} + \varphi^\varepsilon) - g(\Psi^\varepsilon_{\mathrm{app}})\Big) - \frac{1}{\varepsilon}\rho(\Psi^\varepsilon_{\mathrm{app}}), \quad \varphi^\varepsilon_{|t=0} = \varphi^\varepsilon_0,$$

where H^ε is defined in (114), the nonlinearity is $g(z) = \kappa|z|^2 z$, and $\rho(\Psi^\varepsilon_{\mathrm{app}})$ is the remainder obtained in (115) for $N = 2$. We denote $w^\varepsilon(t) := \|\varphi^\varepsilon(t)\|_{H^s_\varepsilon}$ and first note by assumption that

$$w^\varepsilon(0) \equiv \|\varphi^\varepsilon_0\|_{H^s_\varepsilon} \leqslant \left\| \Psi^\varepsilon(0, \cdot) - \big(u_0 + \varepsilon u_1\big)\left(0, \cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{H^s_\varepsilon} + \varepsilon^2 \|u_2^\perp(0, \cdot)\|_{H^s_\varepsilon} \leqslant \tilde{c}\varepsilon^2,$$

for some constant $\tilde{c} \in \mathbb{R}$. We shall prove that there exists $\tilde{C} > 0$, $\varepsilon_0 \in (0,1)$, such that for all $\varepsilon \in (0, \varepsilon_0]$ that $w^\varepsilon(t) \leqslant \tilde{C}\varepsilon^2$ for $t \in [0, T_*]$.

To this end, we rewrite the equation above using Duhamel's principle, i.e.

$$\varphi^\varepsilon(t) = S^\varepsilon(t)\varphi_0^\varepsilon - i \int_0^t S^\varepsilon(t-\tau)\big(g\big(\Psi^\varepsilon_{\mathrm{app}}(\tau) + \varphi^\varepsilon(\tau)\big) - g(\Psi^\varepsilon_{\mathrm{app}}(\tau))\big)\, d\tau$$
$$+ \frac{i}{\varepsilon} \int_0^t S^\varepsilon(t-\tau)\rho(\Psi^\varepsilon_{\mathrm{app}}(\tau))\, d\tau,$$

where $S^\varepsilon(t)$ is the Schrödinger group (134). Now, using the propagation estimate (135) together with the estimate on the remainder stated in Proposition 4.10, we obtain

$$w^\varepsilon(t) \leqslant C_l(\tilde{c} + C_r T_*)\varepsilon^2 + C_l \int_0^t \big\| g\big(\Psi^\varepsilon_{\mathrm{app}}(\tau) + \varphi^\varepsilon(\tau)\big) - g\big(\Psi^\varepsilon_{\mathrm{app}}(\tau)\big)\big\|_{H^s_\varepsilon}\, d\tau.$$

for all $t \leqslant T_*$.

Set $\tilde{C} := C_l(\tilde{c} + C_r T_*)e^{C_l C_s T_*}$ and choose $M > \max\{\tilde{c}, \tilde{C}\}$ such that $M\varepsilon^2 > \tilde{c}\varepsilon^2 \geqslant w^\varepsilon(0)$. Since $w^\varepsilon(t)$ is continuous in time, there exists, for every $\varepsilon \in (0,1)$, a positive time $t^\varepsilon_M > 0$, such that $w^\varepsilon(t) \leqslant \varepsilon^2 M$ for $t \leqslant t^\varepsilon_M$. The Gagliardo-Nirenberg inequality (131) then yields for $s > 1$

$$\|\varphi^\varepsilon(t)\|_{L^\infty} \leqslant \varepsilon^{-1} C_\infty \|\varphi^\varepsilon(t)\|_{H^s_\varepsilon} = \varepsilon^{-1} C_\infty w^\varepsilon(t) \leqslant \varepsilon C_\infty M$$

for $t \leqslant t^\varepsilon_M$. Hence there exists an $\varepsilon_0 \in (0,1)$, such that

$$\|\varphi^\varepsilon(t)\|_{L^\infty} \leqslant C_a,$$

for $\varepsilon \in (0, \varepsilon_0]$ and $t \leqslant t^\varepsilon_M$. By Proposition 4.10 we also have

$$\|(\varepsilon\partial)^\gamma \Psi_{\mathrm{app}}(t, \cdot)\|_{L^\infty} \leqslant C_a$$

for $|\gamma| \leqslant s, \varepsilon \in (0,1)$, and $t < T$. Thus we are in a position to apply the Moser-type Lemma 4.13 with $R = C_a$ to obtain

$$w^\varepsilon(t) \leqslant C_l(c + C_r T_*)\varepsilon^2 + C_l C_s \int_0^t w^\varepsilon(\tau)\, d\tau, \text{ for } \varepsilon \in (0, \varepsilon_0] \text{ and } t \leqslant t^\varepsilon_M.$$

Gronwall's lemma then yields

$$w^\varepsilon(t) \leqslant C_l(\tilde{c} + C_r T_*)\varepsilon^2 e^{C_l C_s t} \leqslant \tilde{C}\varepsilon^2 < M \text{ for all } \varepsilon \in (0, \varepsilon_0] \text{ and } t \leqslant T_*.$$

Hence for any choice of T_*, by continuity, we may extend further in time in that $t^\varepsilon_M \geq T_*$, and so we have proved that for any $T_* < T$:

$$\sup_{0 \leqslant t \leqslant T_*} w^\varepsilon(t) \equiv \sup_{0 \leqslant t \leqslant T_*} \|\Psi^\varepsilon(t,\cdot) - \Psi^\varepsilon_{\text{app}}(t,\cdot)\|_{H^s_\varepsilon} \leqslant \tilde{C}\varepsilon^2.$$

Another triangle inequality, then yields the desired result. □

This theorem implies the approximation result announced in (20), uniformly on finite-time intervals bounded by the local existence time of (125), provided the initial data is sufficiently well prepared, i.e. up to errors of order $\mathcal{O}(\varepsilon^2)$.

REMARK 4.14. *Unfortunately, our proof requires us to work with an approximate solution of order $\mathcal{O}(\varepsilon^2)$, in which we need to control also the first order corrector $\propto u_1$. In turn, this requires the initial data to be somewhat better than one would like it to be. The reason for this is the scaling factor ε^{-1} appearing in the Gagliardo-Nirenberg inequality* (131)*. It is conceivable that this is merely a technical issue which can be overcome by the use of a different functional framework. For example, the authors in* [**36, 37**] *work in a Wiener-type space for the Bloch-transformed solution and its approximation.*

5. The case of Hartree nonlinearities

In this section, we shall briefly discuss the case of semiclassical NLS with Hartree nonlinearity, i.e., instead of (19), we consider

$$i\varepsilon\partial_t\Psi^\varepsilon = -\varepsilon^2\Delta\Psi^\varepsilon + V_{\text{per}}\left(\frac{\mathbf{x}}{\varepsilon}\right)\Psi^\varepsilon + \varepsilon\kappa\left(\frac{1}{|\cdot|} * |\Psi^\varepsilon|^2\right)\Psi^\varepsilon, \quad \Psi^\varepsilon_{|t=0} = \Psi^\varepsilon_0(\mathbf{x}). \tag{136}$$

This model describes the (semi-classical) dynamics of electrons inside a graphene layer, under the influence of a self-consistent electric field. We again seek an asymptotic expansion of the form

$$\Psi^\varepsilon(t,\mathbf{x}) \underset{\varepsilon\to 0}{\sim} e^{-it\mu_*/\varepsilon}\sum_{n=0}^{2}\varepsilon^n u_n\Big(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon}\Big),$$

where as before the u_n are assumed to be $\mathbf{k}-$pseudo periodic with respect to the fast variable. By the same procedure as in Section 2, we obtain that

$$u_0(t,\mathbf{x},\mathbf{y}) = \sum_{j=1}^{2}\alpha_j(t,\mathbf{x})\Phi_j(\mathbf{y};\mathbf{K}_*).$$

Plugging this into the Hartree nonlinearity, yields the following nonlinear potential

$$V^\varepsilon(t,\mathbf{x}) = \frac{1}{|\cdot|} * \sum_{j,k=1}^{2}\big(\alpha_j\overline{\alpha}_k\Phi_j\overline{\Phi}_k\big)\Big(t,\cdot,\frac{\cdot}{\varepsilon}\Big),$$

which unfortunately does not directly exhibit the required two-scale structure. However, we shall prove the following averaging result.

LEMMA 4.15. *Let $\alpha_j(t,\mathbf{x}) \in H^s(\mathbb{R}^2)$, for $s>1$, then*

$$\lim_{\varepsilon\to 0} V^\varepsilon(t,\mathbf{x}) = \Big(\frac{1}{|\cdot|} * \big(|\alpha_1|^2 + |\alpha_2|^2\big)\Big)(t,\mathbf{x}).$$

PROOF. We recall that Bloch eigenfunctions concentrated at a Dirac point have a Fourier series expansion of the following form, cf. [43]:

$$\Phi_1(\mathbf{y},\mathbf{K}^*) = \sum_{\mathbf{m}\in\mathbb{Z}^2} c(\mathbf{m}) e^{i\mathbf{K}^*_{\mathbf{m}}\cdot\mathbf{x}} \quad \text{and} \quad \Phi_2(\mathbf{x},\mathbf{K}^*) = \overline{\Phi}_1(-\mathbf{x},\mathbf{K}^*) = \sum_{\mathbf{m}\in\mathbb{Z}^2} \overline{c}(\mathbf{m}) e^{i\mathbf{K}^*_{\mathbf{m}}\cdot\mathbf{x}},$$

where $\mathbf{K}^*_{\mathbf{m}} = \mathbf{K}_* + m_1\mathbf{k}_1 + m_2\mathbf{k}_2 = \mathbf{K}_* + \mathbf{k}_{\mathbf{m}}$.

By orthogonality and Parseval's identity one finds the following relations to be used below

$$\sum_{\mathbf{m}\in\mathbb{Z}^2} |c(\mathbf{m})|^2 = \int_Y |\Phi_1(\mathbf{x})|^2 \, d\mathbf{x} = 1,$$
$$\sum_{\mathbf{m}\in\mathbb{Z}^2} c(\mathbf{m})^2 = \langle c(\mathbf{m}), \overline{c}(\mathbf{m})\rangle_{\ell_2(\mathbb{Z}^2)} = \int_Y \Phi_1(\mathbf{x})\overline{\Phi}_2(\mathbf{x}) \, d\mathbf{x} = 0,$$
$$\sum_{\mathbf{m}\in\mathbb{Z}^2} \overline{c}(\mathbf{m})^2 = \langle \overline{c}(\mathbf{m}), c(\mathbf{m})\rangle_{\ell_2(\mathbb{Z}^2)} = \int_Y \Phi_2(\mathbf{x})\overline{\Phi}_1(\mathbf{x}) \, d\mathbf{x} = 0.$$

Next, we decompose the nonlinear potential term as follows

$$V^\varepsilon = V_1^\varepsilon + V_2^\varepsilon \equiv \frac{1}{|\cdot|} * \sum_{j=1}^2 |\alpha_j|^2 |\Phi_j|^2 + \frac{1}{|\cdot|} * \sum_{j\neq k} \alpha_j \overline{\alpha}_k \Phi_j \overline{\Phi}_k.$$

Explicitly, we find the former term

$$\begin{aligned} V_1^\varepsilon(t,\mathbf{x}) &= \int_{\mathbb{R}^2} \Big(\big|\alpha_1(t,\boldsymbol{\eta})\big|^2 \big|\Phi_1\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big)\big|^2 + \big|\alpha_2(t,\boldsymbol{\eta})\big|^2 \big|\Phi_2\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big)\big|^2 \Big) \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} \\ &= \sum_{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^2} c(\mathbf{m})\overline{c}(\mathbf{m}') \int_{\mathbb{R}^2} \Big(\big|\alpha_1(t,\boldsymbol{\eta})\big|^2 + \big|\alpha_2(t,\boldsymbol{\eta})\big|^2 \Big) e^{i(\mathbf{k}_{\mathbf{m}-\mathbf{m}'}\cdot\boldsymbol{\eta})/\varepsilon} \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|}, \end{aligned}$$

and similarly for the latter term that

$$\begin{aligned} V_2^\varepsilon(t,\mathbf{x}) &= \int_{\mathbb{R}^2} \Big(\alpha_1(t,\boldsymbol{\eta})\overline{\alpha}_2(t,\boldsymbol{\eta})\Phi_1\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big)\overline{\Phi}_2\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big) + \overline{\alpha}_1(t,\boldsymbol{\eta})\alpha_2(t,\boldsymbol{\eta})\overline{\Phi}_1\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big)\Phi_2\Big(\frac{\boldsymbol{\eta}}{\varepsilon}\Big) \Big) \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} \\ &= \sum_{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^2} c(\mathbf{m})c(\mathbf{m}') \int_{\mathbb{R}^2} \alpha_1(t,\boldsymbol{\eta})\overline{\alpha}_2(t,\boldsymbol{\eta}) e^{i(\mathbf{k}_{\mathbf{m}-\mathbf{m}'}\cdot\boldsymbol{\eta})/\varepsilon} \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} \\ &\quad + \sum_{\mathbf{m},\mathbf{m}'\in\mathbb{Z}^2} \overline{c}(\mathbf{m})\overline{c}(\mathbf{m}') \int_{\mathbb{R}^2} \overline{\alpha}_1(t,\boldsymbol{\eta})\alpha_2(t,\boldsymbol{\eta}) e^{-i(\mathbf{k}_{\mathbf{m}-\mathbf{m}'}\cdot\boldsymbol{\eta})/\varepsilon} \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|}. \end{aligned}$$

Now, since the kernel $\frac{1}{|\cdot|}$ in two spatial dimensions is integrable at the origin and since $\alpha_{1,2} \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the Riemann-Lebesgue lemma implies that as $\varepsilon \to 0$, all $\varepsilon-$oscillatory terms vanish, i.e. all terms for which $\mathbf{m} \neq \mathbf{m}'$. In view of the above identities for the Fourier coefficients we thus find

$$\lim_{\varepsilon \to 0} V_1^\varepsilon(t,\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} |c(\mathbf{m})|^2 \int_{\mathbb{R}^2} \left(\left|\alpha_1(t,\boldsymbol{\eta})\right|^2 + \left|\alpha_2(t,\boldsymbol{\eta})\right|^2\right) \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} = \left(\frac{1}{|\cdot|} * \left(\left|\alpha_1\right|^2 + \left|\alpha_2\right|^2\right)\right)(t,\mathbf{x}),$$

whereas

$$\lim_{\varepsilon \to 0} V_2^\varepsilon(t,\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^2} c(\mathbf{m})^2 \int_{\mathbb{R}^2} \alpha_1(t,\boldsymbol{\eta})\overline{\alpha}_2(t,\boldsymbol{\eta}) \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} + \sum_{\mathbf{m}\in\mathbb{Z}^2} \overline{c}(\mathbf{m})^2 \int_{\mathbb{R}^2} \overline{\alpha}_1(t,\boldsymbol{\eta})\alpha_2(t,\boldsymbol{\eta}) \frac{d\boldsymbol{\eta}}{|\mathbf{x}-\boldsymbol{\eta}|} = 0,$$

since both sums vanish individually by the above considerations. □

We therefore expect that as $\varepsilon \to 0$, the dynamics of WKB waves spectrally localized around $\mathbf{K}_*$ are governed by the following Dirac-Hartree system:

$$\begin{cases} \partial_t \alpha_1 + \overline{\lambda}_\#(\partial_{x_1} + i\partial_{x_2})\alpha_2 = -i\kappa\left(\frac{1}{|\cdot|} * \left(\left|\alpha_1\right|^2 + \left|\alpha_2\right|^2\right)\right)\alpha_1, \\ \partial_t \alpha_2 + \lambda_\#(\partial_{x_1} - i\partial_{x_2})\alpha_1 = -i\kappa\left(\frac{1}{|\cdot|} * \left(\left|\alpha_1\right|^2 + \left|\alpha_2\right|^2\right)\right)\alpha_2, \end{cases}$$

The Cauchy problem for this system has been rigorously studied in [60] as an ad-hoc model for electrons in graphene (see also [62, 88] for related results). Our analysis above indicates that this is indeed the correct model and we believe that a fully rigorous proof can be achieved along the same lines as in the case of a cubic nonlinearity. However, a rigorous averaging argument for the required second order approximate solution would require considerably more work, a direction we do not want to pursue here.

REMARK 4.16. *Let us finally note that while the Hartree nonlinearity is equivalent to a coupling with a Poisson equation* $-\Delta V = |\Psi^\varepsilon|^2$ *in three spatial dimensions, this is no longer the case in 2D. If one were to pursue the coupled system instead of* (136)*, the effective model obtained would be a Dirac-Poisson system as studied in* [28].

CHAPTER 5

Stability and Instability of rotating Bose-Einstein condensates [1]

From the dialogue in the Introduction, we recall briefly the Cauchy problem for the Gross–Pitaevskii (GP) equation of interest which models rotating trapped BEC, with $\psi = \psi(t, \mathbf{x})$ the wave function of the condensate, given by

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(\mathbf{x})\psi + a|\psi|^{2\sigma}\psi - (\mathbf{\Omega}\cdot\mathbf{L})\psi\,, \quad \psi_{|t=0} = \psi_0(\mathbf{x}), \tag{137}$$

where $V(\mathbf{x})$ is the harmonic potential given in (23) and $\omega \equiv \min_{j=1,\ldots,d}\{\omega_j\}$ is the smallest trapping frequency. Here we also recall our underlying criticality assumption for well-posedness such that $a > 0$ (defocusing) and $0 < \sigma < \frac{2}{(d-2)_+}$, or $a < 0$ (focusing) and $0 < \sigma < \frac{2}{d}$. This yields global existence of solutions in the space

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \,:\, |\mathbf{x}|u \in L^2(\mathbb{R}^d)\},$$

which conserves the L^2-norm

$$N(\psi(t,\cdot)) = \int_{\mathbb{R}^d} |\psi(t,\mathbf{x})|^2\, d\mathbf{x} = N(\psi_0) \quad \forall\, t \in \mathbb{R}, \tag{138}$$

and the corresponding GP energy functional

$$E_\Omega(\psi(t,\cdot)) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{a}{\sigma+1}|\psi|^{2\sigma+2} - \overline{\psi}(\mathbf{\Omega}\cdot\mathbf{L})\psi\, d\mathbf{x} = E_\Omega(\psi_0) \quad \forall\, t \in \mathbb{R}. \tag{139}$$

The outline of Chapter 5 is organized as follows. In Section 1 below we shall prove the existence of nonlinear ground states. Their orbital stability (and several further consequences) is proved in Section 2. Finally, we turn to the analysis of possible resonances in Section 3.

[1] This Chapter is excerpted from the following work:
[8] J. Arbunich, I. Nenciu and C. Sparber *Stability and instability properties of rotating Bose-Einstein condensates.* Lett. Math. Phys (2019) 109: 1415.

1. Existence of ground states

In this section we shall prove the existence of time-periodic solutions $\psi(t,\mathbf{x}) = e^{-i\mu t}\varphi(\mathbf{x})$ to (137), whose profiles φ satisfy the following nonlinear elliptic equation

$$\mu\varphi = \Big(-\frac{1}{2}\Delta + V(\mathbf{x}) - (\mathbf{\Omega}\cdot\mathbf{L})\Big)\varphi + a|\varphi|^{2\sigma}\varphi. \tag{140}$$

Note that if φ solves this equation, then so does $\varphi e^{i\theta}$ with $\theta\in\mathbb{R}$, and so we have symmetry under gauge transformations. For any given total mass $N>0$, a particular class of solutions $\varphi\in\Sigma$ to (140), called ground states, are obtained by considering the following constrained minimization problem:

$$e(N,\mathbf{\Omega}) := \inf\{E_\Omega(\varphi) : \varphi\in\Sigma,\ N(\varphi)=N\}, \tag{141}$$

where the infimum can be replaced by a minimum whenever the energy functional (139) is bounded from below. In this case $e(N,\mathbf{\Omega}) > -\infty$ denotes the ground state energy. Note that $E_\Omega(\varphi)$ is well-defined for any $\varphi\in\Sigma$, since Assumption 1 and Sobolev's embedding imply $\Sigma\hookrightarrow L^{2\sigma+2}$ provided $\sigma<\frac{2}{(d-2)_+}$. Moreover, for any $\gamma>0$ we have

$$|\langle\psi,(\mathbf{\Omega}\cdot\mathbf{L})\psi| \leqslant \|(\mathbf{\Omega}\wedge\mathbf{x})\psi\|_{L^2}\|\nabla\psi\|_{L^2} \leqslant \frac{1}{2\gamma}|\mathbf{\Omega}|^2\|\mathbf{x}\psi\|^2_{L^2} + \frac{\gamma}{2}\|\nabla\psi\|^2_{L^2}, \tag{142}$$

which in itself follows by rewriting $\mathbf{\Omega}\cdot\mathbf{L} = (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla$ and employing Young's inequality.

The existence and orbital stability of ground state solutions will be proved by the same method as in [26]. To this end, we shall first show that the energy functional (139) is coercive, provided the angular velocity $|\mathbf{\Omega}|$ is less than the smallest trapping frequency:

PROPOSITION 5.1. *Let $|\mathbf{\Omega}|<\omega$ and Assumption 1 hold. Then for any $\varphi\in\Sigma$ with $\|\varphi\|^2_{L^2}=N$, there is a $\delta>0$ such that*

$$E_\Omega(\varphi) \geqslant \delta\|\varphi\|^2_\Sigma - C_N \geqslant 0, \tag{143}$$

Moreover, $\varphi\mapsto E_\Omega(\varphi)$ is weakly lower semicontinuous in Σ, i.e. for $\{\varphi_k\}_{k=1}^\infty\subset\Sigma$ such that $\varphi_k\rightharpoonup\varphi\in\Sigma$, then we have $E_\Omega(\varphi)\leqslant\liminf\limits_{k\to\infty}E_\Omega(\varphi_k)$.

PROOF. The coercivity follows from (142) and the fact that $V(x)\geqslant\frac{1}{2}\omega^2|\mathbf{x}|^2$ where $\omega>0$ is defined above. Thus one finds, for $0<\gamma<1$:

$$E_\Omega(\varphi) \geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|^2_{L^2} + \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\|\mathbf{x}\varphi\|^2_{L^2} + \frac{a}{\sigma+1}\|\varphi\|^{2\sigma+2}_{L^{2\sigma+2}}. \tag{144}$$

In the case $a > 0$, we directly obtain

$$E_\Omega(\varphi) \geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\|\mathbf{x}\varphi\|_{L^2}^2 \geqslant \delta\|\varphi\|_\Sigma^2 - \frac{N}{2}\,,$$

where we choose $\gamma \in (0,1)$ such that $\frac{|\Omega|^2}{\omega^2} < \gamma < 1$, and we set

$$\delta = \min\left\{\frac{1-\gamma}{2}, \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\right\} > 0\,.$$

In the case $a < 0$, we first note from the Gagliardo-Nirenberg inequality that

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{\sigma,d}\|\nabla u\|_{L^2}^{d\sigma}\|u\|_{L^2}^{2-\sigma(d-2)}, \tag{145}$$

with the optimal constant $C_{\sigma,d} = \frac{\sigma+1}{\|Q\|_{L^2}^{2\sigma}}$ obtained in [104], where Q satisfies

$$\frac{d\sigma}{2}\Delta Q - \Big(1 + \frac{\sigma(d-2)}{2}\Big)Q + Q^{2\sigma+1} = 0.$$

Then applying (145) to (144) and employing Young's inequality with

$$(p,q) = \Big(\frac{2}{d\sigma}, \frac{1}{1-d\sigma/2}\Big)$$

yields the following lower bound for any $\varepsilon > 0$:

$$\begin{aligned} E_\Omega(\varphi) &\geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|_{L^2}^2 - \frac{|a|}{\|Q\|_{L^2}^{2\sigma}}\|\nabla\varphi\|_{L^2}^{d\sigma}\|\varphi\|_{L^2}^{2+\sigma(d-2)} + \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\|\mathbf{x}\varphi\|_{L^2}^2 \\ &\geqslant \Big(\frac{1-\gamma}{2} - \frac{d\sigma|a|\varepsilon^p}{2\|Q\|_{L^2}^{2\sigma}}\Big)\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\|\mathbf{x}\varphi\|_{L^2}^2 - \frac{|a|(1-\frac{d\sigma}{2})}{\|Q\|_{L^2}^{2\sigma}\varepsilon^q}\|\varphi\|_{L^2}^{\frac{2+\sigma(d-2)}{1-\frac{d\sigma}{2}}}\,. \end{aligned}$$

Now recall $p = \frac{2}{d\sigma} > 1$ and choose $\gamma \in (0,1)$, as above, such that $\dfrac{|\mathbf{\Omega}|^2}{\omega^2} < \gamma < 1$, and then $\varepsilon > 0$ such that

$$\frac{d\sigma|a|\varepsilon^p}{2\|Q\|_{L^2}^{2\sigma}} = \frac{1-\gamma}{4}\,.$$

After recalling that $\|\varphi\|_{L^2}^2 = N$, we find

$$\begin{aligned} E_\Omega(\varphi) &= \frac{1-\gamma}{4}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\|\mathbf{x}\varphi\|_{L^2}^2 - \frac{|a|(1-\frac{d\sigma}{2})}{\|Q\|_{L^2}^{2\sigma}\varepsilon^q}N^{\frac{2+\sigma(d-2)}{2-d\sigma}} \\ &\geqslant \tilde{\delta}\|\varphi\|_\Sigma^2 - C(a,d,\sigma,N,\|Q\|_{L^2}^{2\sigma}) \end{aligned}$$

where

$$\tilde{\delta} = \min\left\{\frac{1-\gamma}{4}, \frac{1}{2}\left(\omega^2 - \frac{|\mathbf{\Omega}|^2}{\gamma}\right)\right\}\,.$$

Moreover, since the Σ-norm is weakly lower semicontinuous, then estimate (143) directly implies the same holds for E_Ω, since its quadratic part together with a multiple of the L^2-norm forms a norm on Σ equivalent to the usual one. $\square$

To proceed further, we recall the following compactness result, see [**58, 107**].

LEMMA 5.2. *For $2 \leqslant q < \frac{2d}{(d-2)_+}$, the embedding $\Sigma \hookrightarrow L^q$ is compact.*

Using this we can prove existence of a (constrained) minimizer.

PROPOSITION 5.3. *Let $|\mathbf{\Omega}| < \omega$ and Assumption 1 hold. Then for a given $N > 0$, there exists a $\varphi_\infty \in \Sigma$ such that $\|\varphi_\infty\|_{L^2}^2 = N$ and*

$$E_\Omega(\varphi_\infty) = \min_{\varphi \in \Sigma} E_\Omega(\varphi) = e(N, \mathbf{\Omega}).$$

In addition, φ_∞ is a weak solution to (140) with $\mu \in \mathbb{R}$ a Lagrange multiplier associated to the mass constraint.

PROOF. Choose a minimizing sequence $\{\varphi_k\}_{k=1}^\infty \subset \Sigma$ such that $\|\varphi_k\|_{L^2}^2 = N$. First we show $\{\varphi_k\}_{k=1}^\infty$ is a bounded sequence in Σ. From Proposition 5.1 we know that $0 < E_\Omega(\varphi_k) < \infty$ and the coercivity implies that any minimizing sequence $\{\varphi_k\}_{k=1}^\infty$ is a bounded sequence in Σ. By Banach-Alaoglu, there exists a weakly convergent subsequence $\{\varphi_{k_j}\}_{j=1}^\infty \subset \{\varphi_k\}_{k=1}^\infty$ such that

$$\varphi_{k_j} \rightharpoonup \varphi_\infty \quad \text{as } j \to \infty,$$

for some $\varphi_\infty \in \Sigma$. The compact embedding of Lemma 5.2 implies that $\varphi_{k_j} \to \varphi_\infty$ strongly (and hence in norm) in L^2 and in $L^{2\sigma+2}$, provided $\sigma < \frac{2}{(d-2)_+}$. In particular

$$\|\varphi_\infty\|_{L^2}^2 = \lim_{j \to \infty} \|\varphi_{k_j}\|_{L^2}^2 = N. \tag{146}$$

By the lower semicontinuity of the functional E_Ω we have

$$E_N := \inf_{\varphi \in \Sigma, \|\varphi\|_2^2 = N} E_\Omega(\varphi) \leqslant E_\Omega(\varphi_\infty) \leqslant \lim_{j \to \infty} \inf E_\Omega(\varphi_{k_j}) = E_N.$$

Furthermore, since $e(N, \mathbf{\Omega}) \equiv E_\Omega(\varphi_\infty) = \lim_{j \to \infty} E_\Omega(\varphi_{k_j})$, we see that $\|\varphi_{k_j}\|_\Sigma \to \|\varphi_\infty\|_\Sigma$, as $j \to \infty$. Together with the weak convergence of the minimizing sequence this implies strong convergence to some $\varphi_\infty \in \Sigma$.

It is then straightforward to compute the first variation $\langle \frac{\delta E_\Omega}{\delta \varphi}, \chi \rangle = 0$ to see that a minimizer $\varphi_\infty \in \Sigma$ indeed solves (140) in the weak sense, i.e.

$$\mu \int_{\mathbb{R}^d} \overline{\varphi}_\infty \chi \, d\mathbf{x} = \frac{1}{2} \int_{\mathbb{R}^d} \nabla \overline{\varphi}_\infty \cdot \nabla \chi + V(\mathbf{x}) \overline{\varphi}_\infty \chi - \overline{\varphi}_\infty (\mathbf{\Omega} \cdot \mathbf{L}) \chi + a |\varphi_\infty|^{2\sigma} \overline{\varphi}_\infty \chi \, d\mathbf{x},$$

for all $\chi \in \Sigma$. $\square$

REMARK 5.4. *It is straightforward to generalize all of the results in this section to GP equations with general confinement potentials $V(\mathbf{x}) \to +\infty$, as $|\mathbf{x}| \to \infty$, provided an appropriate energy space Σ is chosen.*

2. Orbital stability

The set of all ground states with a given mass N will be denoted by

(147) $$\mathcal{G}_\Omega = \Big\{ \varphi \in \Sigma : E_\Omega(\varphi) = e(N, \mathbf{\Omega}) \text{ and } N(\varphi) = N \Big\} \neq \emptyset.$$

Recall that by gauge symmetry $\varphi \in \mathcal{G}_\Omega$ if and only if $e^{i\theta}\varphi \in \mathcal{G}_\Omega$, for some $\theta \in \mathbb{R}$. In the case without rotation ($\mathbf{\Omega} \equiv 0$) and for radially symmetric potentials V with $\omega_1 = \omega_2 = \omega_3$, one can show that the energy minimizer is indeed radially symmetric and positive on all of $\mathbb{R}^d$, see [**58, 63**] and the references therein. In other words, in this case

(148) $$\mathcal{G}_0 = \{ u e^{i\theta}, \; u \equiv u(|\mathbf{x}|) > 0, \, \theta \in \mathbb{R} \}.$$

Moreover, since the action of $\mathbf{\Omega} \cdot \mathbf{L}$ vanishes on radially symmetric functions, then any radially symmetric $\varphi \in \mathcal{G}_\Omega$ is also in $\mathcal{G}_0$, and hence of the form above. However, the symmetry breaking results in [**97, 98**] imply that for $|\mathbf{\Omega}| \neq 0$, a minimizer $\varphi_\infty \in \mathcal{G}_\Omega$ is in general *not radially symmetric*. More precisely, it is proved that for $|\mathbf{\Omega}| > \mathbf{\Omega}_{\text{crit}} > 0$ no eigenfunction of the angular momentum operator $\mathbf{L}$ can be a minimizer (and a radial function u is an eigenfunction with zero eigenvalue), even if the GP functional is invariant under rotations around the $\mathbf{\Omega}$-axis. This implies that φ_∞ in the case *with* rotation cannot be unique (up to gauge transforms), since by rotating a minimizer one obtains another minimizer. In this context, an estimate for the critical rotation speed $\mathbf{\Omega}_{\text{crit}}$ in $d = 2$ can be found in [**64**]. In summary, these results show that $\mathcal{G}_\Omega$, in general, will be a more complicated set than $\mathcal{G}_0$. Moreover, $\mathcal{G}_\Omega$ should also be distinguished from the set of rotationally symmetric vortex solutions studied in [**52**].

Our first main result is as follows:

THEOREM 5.1 (Orbital stability of ground states). *Let $|\mathbf{\Omega}| < \omega$ and Assumption 1 hold. Then the set of ground states $\mathcal{G}_\Omega \neq \emptyset$ is orbitally stable in Σ. That is, for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $\psi_0 \in \Sigma$ satisfies*

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi_0 - \varphi\|_\Sigma < \delta,$$

then the solution $\psi \in C(\mathbb{R}_t, \Sigma)$ to (137) with $\psi(0, x) = \psi_0 \in \Sigma$ satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\varphi \in \mathcal{G}_\Omega} \|\psi(t, \cdot) - \varphi\|_\Sigma < \varepsilon.$$

This theorem generalizes earlier results on the orbital stability of standing waves in nonlinear Schrödinger equations with (unbounded) potential (see, e.g., [**26, 49, 51, 58, 107, 108**] and the references therein) to the case with harmonic potential and additional rotation. Note that for $\Omega = 0$, the simple structure of $\mathcal{G}_0$, given in (148), allows one to rephrase the infimum over $\mathcal{G}_0$ as an infimum over $\theta \in \mathbb{R}$. Also note Theorem 5.1 holds for defocusing and focusing nonlinearities satisfying Assumption 1 (see also Remark 5.5 below). In this context, we also mention the papers [**49, 50**], in which the authors study various instability properties of standing wave solutions to focusing nonlinear Schrödinger equations with potentials.

PROOF. By way of contradiction, assume that the set of ground states $\mathcal{G}_\Omega \neq \emptyset$ is unstable. Then there exist $\varepsilon_0 > 0$, $\varphi_0 \in \mathcal{G}_\Omega$, a sequence of initial data $\{\psi_0^k\}_{k \in \mathbb{N}} \subset \Sigma$ satisfying

$$\|\psi_0^k - \varphi_0\|_\Sigma \to 0 \quad \text{as} \quad k \to \infty,$$

and a sequence of times $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$, such that

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi^k(t_k, \cdot) - \varphi\|_\Sigma > \varepsilon_0.$$

Here $\psi^k(t, \mathbf{x}) \in C(\mathbb{R}, \Sigma)$ is the unique global solution to (137) with initial data ψ_0^k. For simplicity set $u_k(\mathbf{x}) := \psi^k(t_k, \mathbf{x})$. From mass conservation (138) we have, as $k \to \infty$:

$$\|u_k\|_{L^2}^2 \equiv \|\psi^k(t_k, \cdot)\|_{L^2}^2 = \|\psi_0^k\|_{L^2}^2 \xrightarrow{k \to \infty} \|\varphi_0\|_{L^2}^2 = N.$$

Moreover, by energy conservation (2.2) it also follows that

$$E_\Omega(u_k) \equiv E_\Omega(\psi^k(t_k, \cdot)) = E_\Omega(\psi_0^k) \xrightarrow{k \to \infty} E_\Omega(\varphi_0) = e(N, \mathbf{\Omega}).$$

Consequently, the continuity in time implies that u_k is a minimizing sequence in Σ. By the proof of Proposition 5.3, there exists a subsequence such that $u_{k_j} \to \varphi_\infty \in \Sigma$ strongly, as $j \to \infty$. Thus

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi^{k_j}(t_{k_j}, \cdot) - \varphi\|_\Sigma \leqslant \|u_{k_j} - \varphi_\infty\|_\Sigma \xrightarrow{j\to\infty} 0,$$

which contradicts our assumption. □

REMARK 5.5. *It is possible to generalize this result to the case of an attractive ($a < 0$) mass-critical nonlinearity $\sigma = \frac{2}{d}$, under the assumption that $N < \|Q\|_{L^2}^2$, see, e.g., [108, 109] for analogous results in the case without rotation. We shall not go into further details here, but note that the associated question of a blow-up profile as $N \to \|Q\|_{L^2}^2$ in the case with rotation has recently been studied in* [79].

Theorem 5.1 has the following interesting consequence: Recall that $\mathbf{\Omega} \cdot \mathbf{L}$ is the generator of rotations around the $\mathbf{\Omega}$-axis, in the sense that

$$e^{t\Omega \cdot L} u(\mathbf{x}) = u\left(e^{t\Theta}\mathbf{x}\right), \quad \forall\, u \in L^2(\mathbb{R}^d),$$

where Θ is the skew symmetric matrix given by

$$\Theta = \begin{pmatrix} 0 & |\mathbf{\Omega}| \\ -|\mathbf{\Omega}| & 0 \end{pmatrix} \text{ for } d = 2, \text{ and } \Theta = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \text{ for } d = 3.$$

Clearly, this is a unitary operator on both $L^2(\mathbb{R}^d)$ and Σ. It was shown in [6] that if $\psi(t, \mathbf{x})$ solves (137), i.e., the GP equation with rotation, then

$$\Psi(t, \mathbf{x}) := \left(e^{t\Omega \cdot L}\psi(t, \cdot)\right)(\mathbf{x}), \tag{149}$$

solves the following nonlinear Schrödinger equation with time-dependent potential:

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi + W_\Omega(t, \mathbf{x})\Psi + a|\Psi|^{2\sigma}\Psi, \quad \Psi_{|t=0} = \psi_0(\mathbf{x})\,. \tag{150}$$

Here, the new potential W_Ω is given by

$$W_\Omega(t, \mathbf{x}) := e^{t\Omega \cdot L} V(\mathbf{x}) \equiv V\left(e^{t\Theta}\mathbf{x}\right).$$

The global existence result for (137) then directly translates to the existence of a unique global solution $\Psi \in C(\mathbb{R}_t; \Sigma)$ to (150) (see also [19] for related results). Moreover, we have that (150) conserves the total mass, i.e., $N(\Psi(t,\cdot)) = N(\psi_0)$ for all $t \in \mathbb{R}$. The associated energy, however, is *no longer conserved* unless $V(\mathbf{x})$ is rotationally or at least axisymmetric with respect to $\mathbf{\Omega}$, cf. [6] for more details.

COROLLARY 5.6. *Under the same assumptions as in Theorem 5.1 it holds: For all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that if* $\psi_0 \in \Sigma$ *satisfies* $\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi_0 - \varphi\|_\Sigma < \delta$, *then the solution* $\Psi \in C(\mathbb{R}_t, \Sigma)$ *to* (150) *with* $\psi(0,x) = \psi_0 \in \Sigma$ *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{\varphi\in\mathcal{G}_\Omega} \|\Psi(t,\cdot) - e^{t\Omega\cdot L}\varphi(\cdot)\|_\Sigma < \varepsilon.$$

In other words, we have orbital stability of the set $e^{t\Omega\cdot L}\mathcal{G}_\Omega$ under the dynamics of (150). To the best of our knowledge, this is the only orbital stability result for nonlinear Schrödinger equations with a time-dependent potential available to date.

In the particular situation where V is *rotationally symmetric*, i.e., $V(\mathbf{x}) = \frac{1}{2}\omega^2|\mathbf{x}|^2$, one finds

$$W_\Omega(t,\mathbf{x}) = V(\mathbf{x}), \quad \text{for any } \mathbf{\Omega} \in \mathbb{R}^d,$$

yielding the usual Gross–Pitaevskii equation for (harmonically) trapped Bose gases

$$(151) \qquad i\partial_t\Psi = -\frac{1}{2}\Delta\Psi + \frac{1}{2}\omega^2|\mathbf{x}|^2 + a|\Psi|^{2\sigma}\Psi, \quad \Psi_{|t=0} = \psi_0(\mathbf{x}),$$

In contrast to (150), this equation does conserve the associated Gross–Pitaevskii energy, $E_0(\Psi(t,\cdot)) = E_0(\psi_0)$, for all $t \in \mathbb{R}$. The orbital stability result proved above then has the following consequence:

COROLLARY 5.7. *Let Assumption 1 hold and* V *be rotationally symmetric. Then*

$$\mathcal{O} = \cup_{(\Omega\in\mathbb{R}^d;|\Omega|<\omega)} \left(e^{t\Omega\cdot L}\mathcal{G}_\Omega\right),$$

is an orbitally stable set of solutions to (151).

The usual orbital stability result for ground states associated to (151) applies to $\mathcal{G}_0$, see, e.g., [26]. Note that if, for some $\mathbf{\Omega}$, all minimizers $\varphi \in \mathcal{G}_\Omega$ are rotationally symmetric, then $e^{t\Omega\cdot L}\mathcal{G}_\Omega = \mathcal{G}_\Omega = \mathcal{G}_0$. However, the results of [64, 97, 98] show that, in general, $\varphi \in \mathcal{G}_\Omega$ is *not* rotationally symmetric, in which case $e^{t\Omega\cdot L}\mathcal{G}_\Omega$, does *not* contain stationary solutions to (151) given by $\Psi(t,\mathbf{x}) = e^{i\mu t}\varphi(\mathbf{x})$. Again, to the best of our knowledge, this is the only orbital stability result for (151) based on non-stationary solutions.

3. A resonance-type phenomenon in non-isotropic potentials

All the preceding results are obtained under the condition $|\Omega| < \omega$, which is necessary for the existence of nonlinear ground states. However, one may wonder (in particular in view of Corollary 5.7) if there are any qualitative changes to the time-dependent solution of (137) for $|\mathbf{\Omega}| \geqslant \omega$. At least in the case of *non-isotropic potentials* $V(\mathbf{x})$, we will see below that this is indeed the case.

To this end, we denote for $\psi(t, \cdot) \in \Sigma$, the quantum mechanical mean position and momentum by

$$\mathbf{X}(t) := \int_{\mathbb{R}^d} \mathbf{x}|\psi(t,\mathbf{x})|^2 \, d\mathbf{x}, \quad \mathbf{P}(t) := -i \int_{\mathbb{R}^d} \overline{\psi}(t,\mathbf{x}) \nabla \psi(t,\mathbf{x}) \, d\mathbf{x}.$$

LEMMA 5.8. *Let $\psi \in C(\mathbb{R}_t; \Sigma)$ be a solution to (137), then, for all $t \in \mathbb{R}$:*

$$\begin{aligned} \mathbf{X}(t) &= \mathbf{X}(0) + \int_0^t \mathbf{P}(s) - \mathbf{\Omega} \wedge \mathbf{X}(s) \, ds \\ \mathbf{P}(t) &= \mathbf{P}(0) - \int_0^t \nabla V(\mathbf{X}(s)) + \mathbf{\Omega} \wedge \mathbf{P}(s) \, ds. \end{aligned} \tag{152}$$

This system can be regarded as a generalization of the results in [89, Section 6], obtained for $\mathbf{\Omega} = 0$. It should be noted that the nonlinearity does not enter into (152).

PROOF. We shall assume that ψ is sufficiently smooth (and decaying) such that all of our computations below are rigorous. A classical density argument, combined with the continuous dependence of ψ on its initial data, then allows us to extend the result to solutions $\psi \in C(\mathbb{R}_t; \Sigma)$.

We start by calculating the time derivative of $\mathbf{X}$:

$$\begin{aligned} \dot{\mathbf{X}} &= 2\mathrm{Re}\langle \partial_t \psi, \mathbf{x}\psi \rangle = 2\mathrm{Re}\langle i(\tfrac{1}{2}\Delta\psi - V(\mathbf{x})\psi - a|\psi|^{2\sigma}\psi + (\mathbf{\Omega}\cdot\mathbf{L})\psi), \mathbf{x}\psi\rangle \\ &= \mathrm{Re}\langle i\Delta\psi, \mathbf{x}\psi\rangle + 2\mathrm{Re}\langle i(\mathbf{\Omega}\cdot\mathbf{L})\psi, \mathbf{x}\psi\rangle + 2\mathrm{Im}\, \underbrace{\langle V(\mathbf{x})\psi + a|\psi|^{2\sigma}\psi, \mathbf{x}\psi\rangle}_{\in\mathbb{R}} \equiv J_1 + J_2. \end{aligned}$$

An integration by parts then implies

$$J_1 = \mathrm{Re}\langle -i\nabla\psi, \nabla(\mathbf{x}\psi)\rangle = \mathrm{Im}\, \underbrace{\langle \nabla\psi, \mathbf{x}\nabla\psi\rangle}_{\in\mathbb{R}} + \mathrm{Re}\langle -i\nabla\psi, \psi\nabla\mathbf{x}\rangle = \mathbf{P}.$$

The term J_2 can be rewritten using $(\mathbf{\Omega}\cdot\mathbf{L}) = -i(\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla$ and integration by parts

$$\begin{aligned} J_2 &= 2\mathrm{Re}\langle (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla\psi, \mathbf{x}\psi\rangle = 2\mathrm{Re} \sum_{\ell,j=1}^d \langle \partial_{x_j}\psi, (\mathbf{\Omega}\wedge\mathbf{x})_j x_\ell \psi\rangle \mathbf{e}_\ell \\ &= -2\sum_{j=1}^d \langle \psi, (\mathbf{\Omega}\wedge\mathbf{x})_j\psi\rangle \mathbf{e}_j - 2\mathrm{Re}\langle \mathbf{x}\psi, (\mathbf{\Omega}\wedge\mathbf{x})\cdot\nabla\psi\rangle = -2\langle\psi, (\mathbf{\Omega}\wedge\mathbf{x})\psi\rangle - J_2, \end{aligned}$$

which implies that

$$J_2 = -\langle \psi, (\mathbf{\Omega} \wedge \mathbf{x})\psi \rangle = -\mathbf{\Omega} \wedge \mathbf{X}.$$

In summary this yields the following equation of motion for X:

$$\dot{\mathbf{X}} = \mathbf{P} - \mathbf{\Omega} \wedge \mathbf{X}, \tag{153}$$

which is the time-differentiated version of the first equation in (152).

Next, we calculate the time-derivative of $\mathbf{P}$ as:

$$\dot{\mathbf{P}} = 2\mathrm{Re}\langle i\partial_t \psi, \nabla\psi \rangle = 2\mathrm{Re}\langle V(\mathbf{x})\psi + a|\psi|^{2\sigma}\psi - \tfrac{1}{2}\Delta\psi - (\mathbf{\Omega} \cdot \mathbf{L})\psi, \nabla\psi \rangle \equiv I_1 + I_2 + I_3 + I_4.$$

For the first term, a straightforward integration by parts yields

$$I_1 = -2\mathrm{Re}\langle \nabla\big(V\psi\big), \psi \rangle = -2\int_{\mathbb{R}^d} \nabla V(\mathbf{x})|\psi(t,\mathbf{x})|^2 \, d\mathbf{x} - I_1,$$

which implies

$$I_1 = -\int_{\mathbb{R}^d} \nabla V(\mathbf{x})|\psi(t,\mathbf{x})|^2 \, d\mathbf{x} = -\nabla V(\mathbf{X}),$$

since $\nabla V(\mathbf{x}) = \sum_{j=1}^d \omega_j^2 x_j$. Furthermore, I_2 vanishes, since

$$I_2 = \frac{a}{\sigma+1}\int_{\mathbb{R}^d} \nabla\big(|\psi|^{2(\sigma+1)}\big) \, d\mathbf{x} = 0,$$

and one also finds $I_3 = -\mathrm{Re}\langle \Delta\psi, \nabla\psi \rangle = 0$. Finally, using standard vector identities we compute

$$I_4 = -2\mathrm{Re}\langle (\mathbf{\Omega} \cdot \mathbf{L})\psi, \nabla\psi \rangle = -2\mathbf{\Omega} \wedge \mathbf{P} - I_4,$$

which implies that

$$\dot{\mathbf{P}} = -\nabla V(\mathbf{X}) - \mathbf{\Omega} \wedge \mathbf{P}, \tag{154}$$

which is the differential version of the second line in (152). $\square$

Given that (152) constitutes a closed system for $\mathbf{X}$ and $\mathbf{P}$, one can study its solution independently of (137). As a first step, we have the following global existence result.

LEMMA 5.9. *For any* $(\mathbf{X}_0, \mathbf{P}_0) \in \mathbb{R}^{2d}$*, the system* (152) *admits a unique global-in-time solution* $(\mathbf{X}, \mathbf{P}) \in C^\infty(\mathbb{R}_t; \mathbb{R}^{2d})$ *with* $(\mathbf{X}(0), \mathbf{P}(0)) = (\mathbf{X}_0, \mathbf{P}_0)$.

Proof. Denote $\mathbf{\Xi} = (\mathbf{X}, \mathbf{P})^\top$, then (153), (154) are equal to

$$\dot{\mathbf{\Xi}} = M_d \mathbf{\Xi}, \quad \mathbf{\Xi}(0) = \mathbf{\Xi}_0, \tag{155}$$

where $\mathbf{\Xi}_0 = (\mathbf{X}_0, \mathbf{P}_0)^\top$, and

$$M_2 = \begin{pmatrix} 0 & |\mathbf{\Omega}| & 1 & 0 \\ -|\mathbf{\Omega}| & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & |\mathbf{\Omega}| \\ 0 & -\omega_2^2 & -|\mathbf{\Omega}| & 0 \end{pmatrix} \quad \text{for } d = 2,$$

and

$$M_3 = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 & 1 & 0 & 0 \\ -\Omega_3 & 0 & \Omega_1 & 0 & 1 & 0 \\ \Omega_2 & -\Omega_1 & 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & 0 & \Omega_3 & -\Omega_2 \\ 0 & -\omega_2^2 & 0 & -\Omega_3 & 0 & \Omega_1 \\ 0 & 0 & -\omega_3^2 & \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \quad \text{for } d = 3.$$

Equation (155) is a linear matrix-valued ordinary differential equation with constant coefficients. Thus, (155) and equivalently (152), admits a unique smooth solution given by:

$$\mathbf{\Xi}(t) = e^{tM_d} \mathbf{\Xi}_0, \quad \text{for all } t \in \mathbb{R}.$$

□

To simplify the following discussion, we shall assume that $\mathbf{\Omega} \in \mathbb{R}^3$ is aligned with one of the coordinate axes, say, $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|)^\top$. In this way, (24) automatically holds and thus the two-dimensional situation is included in what follows.

PROPOSITION 5.10. *Let $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|)^\top$. Assume that*

$$\omega_1 \neq \omega_2 \text{ and } \min\{\omega_1, \omega_2\} \leqslant |\mathbf{\Omega}| \leqslant \max\{\omega_1, \omega_2\}. \tag{156}$$

Then for all $(\mathbf{X}_0, \mathbf{P}_0) \in \mathbb{R}^{2d} \setminus \mathcal{H}$, where $\mathcal{H} = \mathcal{H}(\omega_1, \ldots, \omega_d, \mathbf{\Omega})$ is a linear subspace of $\mathbb{R}^{2d}$, it holds

$$\lim_{t \to +\infty} |\mathbf{X}(t)| = \lim_{t \to +\infty} |\mathbf{P}(t)| = +\infty, \quad \text{or} \quad \lim_{t \to -\infty} |\mathbf{X}(t)| = \lim_{t \to -\infty} |\mathbf{P}(t)| = +\infty.$$

Moreover, if both inequalities in (156) are strict, this growth is exponentially fast and $\dim \mathcal{H} = 2(d-1)$. If, however $|\mathbf{\Omega}| \in \{\omega_1, \omega_2\}$, then the growth is only linear in time and $\dim \mathcal{H} = 2d - 1$.

Proof. Observe that for $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|)^\top$, the matrix M_3 decomposes as a direct sum of M_2 and the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_3^2 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of M_3 is

$$\det(\lambda - M_3) = \det(\lambda - M_2) \cdot \det(\lambda - A) = \det(\lambda - M_2) \cdot (\lambda^2 + \omega_3^2).$$

Note that $\lambda^2 + \omega_3^2$ has purely imaginary roots, leading to bounded oscillations in the solution of (152). Thus, for both $d = 2$ and $d = 3$ the characteristic polynomial of M_2 is the only possible source of growth in the solution. One finds that

$$\det(\lambda - M_2) = \lambda^4 + b\lambda^2 + c$$

with

$$b = 2|\mathbf{\Omega}|^2 + \omega_1^2 + \omega_2^2 \quad \text{and} \quad c = \big(|\mathbf{\Omega}|^2 - \omega_1^2\big)\big(|\mathbf{\Omega}|^2 - \omega_2^2\big).$$

As a quadratic polynomial in λ^2, it has discriminant

$$D = \big(\omega_1^2 - \omega_2^2\big)^2 + 8|\mathbf{\Omega}|^2\big(\omega_1^2 + \omega_2^2\big) > 0,$$

and thus $\lambda^2 \in \mathbb{R}$. This implies that a necessary condition for the fact that at least one of the two limits

$$\lim_{t\to\pm\infty} |\mathbf{\Xi}(t)| = +\infty,$$

is that $\lambda^2 \geqslant 0$. This growth occurs on $\mathbb{R}^{2d} \setminus \mathcal{H}$, where $\mathcal{H}$ is the orthogonal complement of the eigenspace corresponding to the real eigenvalue(s) λ.

Computing the roots, we find that since $b > 0$, the root

$$\lambda^2 = \frac{-b - \sqrt{b^2 - 4c}}{2} < 0.$$

In addition, the other root satisfies

$$\lambda^2 = \frac{-b + \sqrt{b^2 - 4c}}{2} \geqslant 0, \text{ if and only if } c \leqslant 0.$$

The latter is equivalent to $\min\{\omega_1, \omega_2\} \leqslant |\mathbf{\Omega}| \leqslant \max\{\omega_1, \omega_2\}$.

Now if $c < 0$ then $\lambda^2 > 0$. Hence, the system has a positive and a negative simple eigenvalue, implying exponential growth for $t \to \pm\infty$ and co-dimension of $\mathcal{H}$ equal to 2. The fact that both $\mathbf{X}$ and $\mathbf{P}$

grow individually can be seen from computing the eigenvector $V = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)^\top$ associated to λ. This can be done using the block structure of M_2 to derive a new eigenvalue equation for $(\mathbf{v}_1, \mathbf{v}_2)^\top$, given by

$$\begin{pmatrix} |\mathbf{\Omega}|^2 - \omega_1^2 & -2\lambda|\mathbf{\Omega}| \\ 2\lambda|\mathbf{\Omega}| & |\mathbf{\Omega}|^2 - \omega_2^2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}.$$

In addition, one finds that

$$\begin{pmatrix} \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} \lambda & |\mathbf{\Omega}| \\ -|\mathbf{\Omega}| & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}.$$

This yields the expression for V after which a straightforward but somewhat tedious analysis leads to the desired conclusion.

When $c = 0$ then $\lambda = 0$ is a double eigenvalue, in which case one needs to study the dimension $d_0 \in \mathbb{N}$ of the associated eigenspace. A straightforward computation shows that if $\omega_1 = \omega_2$ (the axisymmetric case), then $d_0 = 2$ is maximal and hence the solution does not grow in t. By contrast if $\omega_1 \neq \omega_2$, then $d_0 = 1$, and there exists a linearly independent solution $\propto t$, stemming from the eigenvector $V = (1, 0, 0, -|\mathbf{\Omega}|)^\top$. $\square$

REMARK 5.11. *In the case without rotation, i.e.* $|\mathbf{\Omega}| = 0$, *one finds*

$$\lambda^2 = -\frac{\omega_1^2 + \omega_2^2}{2} \pm \left|\frac{\omega_1^2 - \omega_2^2}{2}\right|,$$

which implies $\lambda = \pm i\omega_1, \pm i\omega_2$, *and thus a purely oscillatory solution.*

We are now in position to prove the second main result of this work.

THEOREM 5.2 (Resonance in non-isotropic potentials). *Let Assumption 1 hold and* $\mathbf{\Omega} = (0, 0, |\mathbf{\Omega}|)^\top$. *If condition (156) holds and if* $\psi_0 \in \Sigma$ *is such that the associated averages* $(\mathbf{X}_0, \mathbf{P}_0) \notin \mathcal{H}$, *then the solution* $\psi \in C(\mathbb{R}_t; \Sigma)$ *satisfies*

$$\lim_{t\to+\infty} \|\psi(t,\cdot)\|_\Sigma = +\infty, \ \ \textit{or} \lim_{t\to-\infty} \|\psi(t,\cdot)\|_\Sigma = +\infty.$$

PROOF. Recall that both (137) and (152) have unique solutions. Thus, if $\psi(t, \cdot)$ solves (137) with initial data $\psi_0 \in \Sigma$ and if $\mathbf{X}_0 = \langle \psi_0, \mathbf{x}\psi_0 \rangle$ and $\mathbf{P}_0 = -i\langle \psi_0, \nabla\psi_0 \rangle$ are the initial data to (152), then

$$\mathbf{X}(t) = \langle \psi(t, \cdot), \mathbf{x}\psi(t, \cdot) \rangle, \quad \mathbf{P}(t) = -i\langle \psi(t, \cdot), \nabla\psi(t, \cdot) \rangle, \quad \forall\, t \in \mathbb{R}.$$

By the Cauchy-Schwarz inequality we have

$$|\mathbf{X}| \leqslant \|\psi\|_{L^2}\|\mathbf{x}\psi\|_{L^2}, \quad |\mathbf{P}| \leqslant \|\psi\|_{L^2}\|\nabla\psi\|_{L^2},$$

which together with the results of Proposition 5.10 and the mass conservation property (138) implies the assertion of the theorem. □

REMARK 5.12. *The fact that there are nontrivial $\psi_0 \in \Sigma$ for which the associated $(\mathbf{X}_0, \mathbf{P}_0) \notin \mathcal{H}$, can be easily seen by considering initial data of the form:*

$$\psi_0(x) = e^{i\mathbf{p}_0\cdot\mathbf{x}}e^{-(\mathbf{x}-\mathbf{x}_0)^2/2}, \quad \mathbf{x}_0, \mathbf{p}_0 \in \mathbb{R}^d.$$

In this case, $\mathbf{X}_0 = \pi^{d/2}\mathbf{x}_0$ and $P_0 = \pi^{d/2}\mathbf{p}_0$ and thus one obtains a growing Σ-norm of the solution ψ provided $(\mathbf{x}_0, \mathbf{p}_0) \notin \mathcal{H}$.

Indeed, the proof of Proposition 5.10 shows that if condition (156) holds, there are solutions to (137) for which

$$\|\nabla\psi(t,\cdot)\|_{L^2}, \|\mathbf{x}\psi(t,\cdot)\|_{L^2} \to \infty,$$

if $t \to +\infty$, or $t \to -\infty$. In other words, these solutions develop frequencies which are larger than those controlled by the Σ-norm, and in addition their mass is transferred to infinity resulting in a weaker decay of ψ. This is in sharp contrast to the case $\omega_1 = \omega_2 = \omega_3$, where (137) is equivalent, up to the time-dependent change of variables (149), to the classical NLS with harmonic trapping (151). The latter conserves the energy $E_0(\Psi(t,\cdot)) = E_0(\psi_0)$, which in the defocusing case $a > 0$ directly yields the uniform bound

$$\|\Psi(t,\cdot)\|_\Sigma = \|\psi(t,\cdot)\|_\Sigma \leqslant E_0(\psi_0), \quad \forall\, t \in \mathbb{R}.$$

REMARK 5.13. *The growth of (higher order) Sobolev-norms of solutions to nonlinear Schrödinger equations with time-dependent, quadratic potentials was also studied in [19]. One can check that (150) (obtained from (137), via the change of variables) falls into the class of models for which exponentially growing upper bounds were established in [19]. Theorem 5.2 shows that, in general, such exponential growth indeed occurs, and that this is true even for linear Schrödinger equations. There exponential growth naturally occurs in the case of (even only partially) repulsive harmonic potentials. We finally mention that very recently a somewhat similar instability phenomenon for linear Schrödinger equations with quadratic time-dependent Hamiltonian has been established in [12].*

It is very likely that additional (in-)stability phenomena appear for general $\mathbf{\Omega} \in \mathbb{R}^3$, not necessarily aligned to one of the axis. However, the calculations of the roots of the associated degree 6 characteristic polynomial become extremely involved, see also [17]. Since our main goal was to establish an instability result for ψ we do not investigate the general case in full detail.

APPENDIX A

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Cited Literature

[1] M. J. Ablowitz and Y. Zhu. *Nonlinear waves in shallow honeycomb lattices.* SIAM J. Appl. Math. **72** (2012), no. 1, 240–260.

[2] M. J. Ablowitz and Y. Zhu. *Nonlinear wave packets in deformed honeycomb lattices.* SIAM J. Appl. Math. **73** (2013), no. 6, 1959–1979.

[3] A. Aftalion, *Vortices in Bose–Einstein condensates.* Progress in Nonlinear Differential Equations and their Applications vol. 67, Springer, 2006.

[4] D. M. Ambrose and G. Simpson. *Local existence theory for derivative nonlinear Schrödinger equations with non-integer power nonlinearities.* SIAM J. Math. Anal. **47** (2015), no. 3, 2241–2264.

[5] P. Antonelli, J. Arbunich and C. Sparber. *Regularizing nonlinear Schrödinger equations through partial off-axis variations.* SIAM J. Math. Anal. **51** (2019), no. 1, pp. 110–130.

[6] P. Antonelli, D. Marahrens, and C. Sparber. *On the Cauchy problem for nonlinear Schrödinger equations with rotation.* Discrete Contin. Dyn. Syst. **32** (2012), no. 3, 703–715.

[7] J. Arbunich, C. Klein and C. Sparber. *On a class of derivative nonlinear Schrödinger equations in two spatial dimensions.* To appear in ESAIM Math. Model. Numer. Anal.

[8] J. Arbunich, I. Nenciu and C. Sparber *Stability and instability properties of rotating Bose-Einstein condensates.* Lett. Math. Phys (2019) 109: 1415.

[9] J. Arbunich and C. Sparber *Rigorous derivation of nonlinear Dirac equations for wave propagation in honeycomb structures.* J. Math. Phys. **59** (2018), 011509.

[10] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations.* Grundlehren der Mathematischen Wissenschaften Vol. 343, Springer Verlag, New York, 2011.

[11] O. Bahat-Treidel, O. Peleg, and M. Segev, *Symmetry breaking in honeycomb photonic lattices.* Optics Lett. **33** (2008), no. 19, 2251–2253.

[12] D. Bambusi, B. Grébert, A. Maspero, and D. Robert, *Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time-dependent perturbation.* Anal. PDE **11** (2018), no. 3, 775–799.

[13] W. Bao, H. Wang, and P. Markowich, *Ground, symmetric and central vortex states in rotating Bose-Einstein condensates.* Comm. Math. Sci. **3** (2005), no. 1, 57–88.

[14] I. Bejenaru and S. Herr, *The cubic Dirac equation: Small initial data in* $H^{1/2}(\mathbb{R}^2)$. Comm. Math. Phys. **343** (2016), no. 2, 515–562.

[15] T. B. Benjamin, J. L. Bona, and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems.* Phil. Trans. Royal Soc. London. Series A, Math. Phys. Sci. **272** (1972), 47–78.

[16] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures.* North-Holland Pub. Co., 1978.

[17] I. Bialynicki-Birula and T. Sowiński, *Gravity-induced resonances in a rotating trap.* Phys. Rev. A **71** (2005), 043610, 8pp.

[18] J. L. Bona, W. R. McKinney, and J. M. Restrepo, *Stable and unstable solitary-wave solutions of the generalized regularized long-wave equation.* J. Nonl. Sci. **10** (2000), no. 6, 603–638.

[19] R. Carles, *Nonlinear Schrödinger equation with time dependent potential.* Comm. Math. Sci. **9** (4), 937–964.

[20] R. Carles, P. Markowich, and C. Sparber, *Semiclassical asymptotics for weakly nonlinear Bloch waves.* J. Statist. Phys. **117** (2004), no. 1-2, 343–375.

[21] R. Carles and C. Sparber, *Semiclassical wave packet dynamics in Schrödinger equations with periodic potentials.* Discrete Contin. Dyn. Syst. Ser. B. **17** (2012), no. 3, 759–774.

[22] R. Carles and C. Sparber, *Nonlinear dynamics of semiclassical coherent states in periodic potentials.* J. Phys. A **45** (2012), no. 24, 244032, 14 pp.

[23] L. D. Carr and C. W. Clark, *Vortices in attractive Bose-Einstein condensates in two dimensions.* Phys. Rev. Lett. **97** (2006), pp. 010403.

[24] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, *The electronic properties of graphene.* Rev. Mod. Phys. **81** (2009), no. 1 ,109–162.

[25] T. Cazenave, *Semilinear Schrödinger equations.* Courant Lecture Notes in Mathematics vol. 10, American Mathematical Society, 2003.

[26] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations.* Comm. Math. Phys. **85** (1982), no. 3, 549–561.

[27] T. Cazenave, F. Weissler, *Some remarks on the nonlinear Schrödinger equation in the critical case.* In: Lecture Notes Math. **1394**, pp. 18–29, Springer, 1989.

[28] J. T. Chadam and R. T. Glassey, *On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions*, J. Math. Anal. Appl. **53** (1976), no. 3, 495–507.

[29] M. Christ and A. Kiselev, *Maximal Functions Associated to Filtrations.* J. Funct. Anal. **179** (2001), 409–425.

[30] M. Colin and M. Ohta, *Stability of solitary waves for derivative nonlinear Schrödinger equation.* Ann. Inst. H. Poincaré Anal. Non Lineaire **23** (2006), no. 5, 753–764.

[31] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. *The theory of nonlinear Schrödinger equations : Part I.* `http://math.mit.edu/~gigliola/allethlectures.pdf`

[32] A. Collin, E. Lundh, and K.-A. Suominen, *Center-of-mass rotation and vortices in an attractive Bose gas*. Phys. Rev. A **71** (2005), pp. 023613.
[33] N. R. Cooper, *Rapidly rotating atomic gases.* Advances Phys. **57** (2008), 539–616.
[34] F. Dalfovo and S. Stringari, *Bosons in anisotropic traps: Ground state and vortices.* Phys. Rev. A **53** (1996), 2477–2485.
[35] A. Davey and K. Stewartson, *On three-dimensional packets of water waves.* Proc. R. Soc. Lond. Ser. A **338** (1974), no. 1613, 101–110.
[36] T. Dohnal and L. Helfmeier, *Justification of the coupled mode asymptotics for localized wavepackets in the periodic nonlinear Schrödinger equation.* J. Math. Anal. Appl. **450** (2017), no. 1, 691–726.
[37] T. Dohnal and H. Uecker, *Coupled Mode Equations and Gap Solitons for the 2D Gross-Pitaevskii equation with a non-separable periodic potential.* Phys. D **238** (2009), no. 9-10, 860–879.
[38] T. Driscoll, *A composite Runge-Kutta Method for the spectral solution of semilinear PDEs.* J. Comput. Phys. **182** (2002), 357–367.
[39] E. Dumas, D. Lannes, and J. Szeftel, *Variants of the focusing NLS equation. Derivation, justification and open problems related to filamentation.* In: CRM Series in Mathematical Physics, pp. 19–75. Springer, 2016.
[40] M.Escobedo and L.Vega, *A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$*, SIAM J. Math. Anal. **28**(1997), no. 2, 338–362.
[41] M. Esteban and E. Séré, *An overview on linear and nonlinear Dirac equations*, Discrete Contin. Dyn. Syst. **8** (2002), no. 2, 381–397.
[42] C. L. Fefferman, J. P. Lee-Thorp, M. I. Weinstein, *Honeycomb Schrödinger operators in the strong binding regime.* Commun. Pure Appl. Math. (2017)
[43] C. L. Fefferman and M. I. Weinstein, *Honeycomb lattice potentials and Dirac points.* J. Amer. Math. Soc. **25** (2012), no. 4, 1169–1220.
[44] C. L. Fefferman and M. I. Weinstein, *Waves in Honeycomb Structures.* J. Équ. dérivées part. Exp. **12** (2012), 1–12.
[45] C. L. Fefferman and M. I. Weinstein, *Wave packets in honeycomb structures and two dimensional Dirac equations.* Commun. Math. Phys. **326** (2014), no. 1, 251–286.
[46] J. Feldman, *The spectrum of periodic Schrödinger operators.* `http://www.math.ubc.ca/~feldman/m511/persch.pdf` (2009)
[47] A. Fetter, *Rotating trapped Bose-Einstein condensates*, Rev. Mod. Phys. **81** (2009), pp. 647.
[48] G. Fibich, *The nonlinear Schrödinger equation; Singular solutions and optical collapse.* Appl. Math. Sciences vol. 192, Springer Verlag, 2015.
[49] R. Fukuizumi, *Stability and instability of standing waves for the nonlinear Schrödinger equation with harmonic potential.* Discrete Contin. Dyn. Syst. **7** (2001), no. 3, 525–544.
[50] R. Fukuizumi and M. Ohta, *Instability of standing waves for nonlinear Schrödinger equations with potentials.* Diff. Integral Equ. **16** (2003), no. 6, 691–706.
[51] R. Fukuizumi and M. Ohta, *Stability of standing waves for nonlinear Schrödinger equations with potentials.* Diff. Integral Equ. **16** (2003), no. 1, 111–128.
[52] C. Garcia-Azpeitia and D. E. Pelinovsky, *Bifurcations of multi-vortex configurations in rotating Bose–Einstein condensates.* Milan J. Math. **85** (2017), no. 2, 331–367.
[53] J. Giannoulis, A. Mielke, and C. Sparber, *Interaction of modulated pulses in the nonlinear Schrödinger equation with periodic potential.* J. Differential Equ. **245** (2008), no. 4, 939–963.
[54] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **144** (1992), no. 1, 163–188.
[55] J. A. Goldstein and B.J. Wichnoski, *On the Benjamin-Bona-Mahony equation in higher dimensions*, Nonlin. Anal. **4** (1980), no. 4, 665–675.
[56] R. Goodman, P. Holmes, and M. I. Weinstein, *Nonlinear propagation of light in one-dimensional periodic structures.* J. Nonlinear Sci. **11** (2001), no. 2, 123–168.
[57] Z. Guo, C. Ning, and Y. Wu, *Instability of the solitary wave solutions for the genenalized derivative Nonlinear Schrödinger equation in the critical frequency case.* Preprint `arXiv:1803.07700`.
[58] F. Hadj Selem, H. Hajaiej, P.A. Markowich, and S. Trabelsi *Variational approach to the orbital stability of standing waves of the Gross-Pitaevskii equation* Milan J. Math. **84** (2014), no. 2, 273–295.
[59] C. Hainzl, M. Lewin, and C. Sparber, *Ground state properties of graphene in Hartree-Fock theory.* J. Math. Phys. **53** (2012), no. 9, 095220, 27 pp.
[60] R. Hajj and F. Méhats, *Analysis of Models for Quantum Transport of Electrons in Graphene Layers*, Math. Models Methods Appl. Sci. **24** (2014), no. 11, 2287–2310.
[61] N. Hayashi and T. Ozawa. *On the derivative nonlinear Schrödinger equation.* Phys. D **55** (1992), no. 1-2, 14–36.
[62] S. Herr and A. Tesfahun, *Small data scattering for semi-relativistic equations with Hartree type nonlinearity.* J. Diff. Equ. **259** (2015), no. 10, 5510–5532.
[63] M. Hirose and M. Ohta, *Uniqueness of positive solutions to scalar field equation with harmonic potential.* Funkcial Ekvac. **50** (2007), 67–100.
[64] R. Ignat and V. Millot, *The critical velocity for vortex existence in a two-dimensional rotating Bose–Einstein condensate.* J. Funct. Anal. **233** (2006), no. 1, 260–306.
[65] R. Jenkins, J. Liu, P. Perry, and C. Sulem, *Soliton resolution for the derivative nonlinear Schrödinger Equation.* Preprint `arXiv:1710.03819`.
[66] R. Jenkins, J. Liu, P. Perry, and C. Sulem, *Global well-posesedness for the derivative nonlinear Schrödinger Equation.* Preprint `arXiv:1710.03810.`
[67] M. Keel and T. Tao, *Endpoint Strichartz Estimates.* Amer. J. Math. **120** (1998), 955–980.
[68] C. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case.* Invent. math. **166** (2006), no. 3, 645–675.
[69] C. Klein, *Fourth-order time-stepping for low dispersion Korteweg-de Vries and nonlinear Schrödinger Equation.* Electronic Trans. Num. Anal. **39** (2008), 116–135.
[70] C. Klein, B. Muite, and K. Roidot, *Numerical study of blowup in the Davey-Stewartson System.* Discrete Contin. Dyn. Syst. Ser. B, **18** (2013), no. 5, 1361–1387.

[71] C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Kadomtsev-Petviashvili equations.* Discrete Contin. Dyn. Syst. Ser. B **19** (2014), no. 6, 1689–1717.

[72] C. Klein and R. Peter, *Numerical study of blow-up in solutions to generalized Korteweg-de Vries equations*. Phys. D **304** (2015), 52–78.

[73] C. Klein and K. Roidot, *Fourth order time-stepping for Kadomtsev-Petviashvili and Davey-Stewartson equations.* SIAM J. Sci. Comput. **33** (2011), no. 6, 3333–3356.

[74] C. Klein and J.-C. Saut, *A numerical approach to blow-up issues for dispersive perturbations of Burgers' equation.* Phys. D **295** (2015), 46–65.

[75] C. Klein and J.-C. Saut, *A numerical approach to Blow-up issues for Davey-Stewartson II type systems*. Comm. Pure Appl. Anal. **14** (2015), no. 4, 1443–1467.

[76] C. Klein, C. Sparber, and P. Markowich, *Numerical study of fractional Nonlinear Schrödinger equations.* Proc. R. Soc. Lond. Ser. A. **470** (2014) 20140364, 26pp.

[77] C. Klein and N. Stoilov, *A numerical study of blow-up mechanisms for Davey-Stewartson II systems*, Stud. Appl. Math. (2018), to appear.

[78] R. Krasny, *A study of singularity formation in a vortex sheet by the point-vortex approximation*, J. Fluid Mech. **167** (1986), 65–93.

[79] M. Lewin, P.T. Nam, and N. Rougerie *Blow-up profile of rotating 2D focusing Bose gases.* In: Macroscopic Limits of Quantum Systems, Springer Verlag, 2018.

[80] E. H. Lieb, R. Seiringer, and J. Yngvason, *Bosons in a trap : A rigorous derivation of the Gross-Pitaevskii energy functional.* Phys. Rev. A **61** (2000), 043602, 8pp.

[81] X. Liu, G. Simpson, and C. Sulem, *Stability of solitary waves for a generalized derivative nonlinear Schrödinger Equation.* J. Nonlin. Sci. **23** (2013), no. 4, 557–583.

[82] X. Liu, G. Simpson, and C. Sulem, *Focusing singularity in a derivative nonlinear Schrödinger equation.* Phys. D **262** (2013), 45–58.

[83] S. Machihara, M. Nakamura, T. Ozawa, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation.* Rev. Mat. Iberoamericana, **19** (2003), no. 1, 179-194.

[84] F. Méhats and C. Sparber, *Dimension reduction for rotating Bose-Einstein condensates with anisotropic confinement.* Discrete Contin. Dyn. Syst. **36** (2016), no. 9, 5097–5118.

[85] F. Merle, and P. Raphaël, *On universality of blow up profile for L^2 critical nonlinear Schrödinger equation.* Invent. Math. **156** (2004), 565–672.

[86] F. Merle, and P. Raphaël, *Sharp lower bound on the blow up rate for critical nonlinear Schrödinger equation.* J. Amer. Math. Soc. **19** (2006), no. 1, 37–90.

[87] F. Merle, and P. Raphaël, *Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation.* Comm. Math. Phys. **253** (2005), no. 3, 675–704.

[88] M. Nakamuraa and K. Tsutayab, *Scattering theory for the Dirac equation of Hartree type and the semirelativistic Hartree equation*, Nonlin. Anal. **75** (2012), no. 8, 3531–3542.

[89] Y.-G. Oh, *Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials*. J. Differential Eq. **81** (1989), no. 2, 255–274.

[90] T. Ozawa, *Remarks on proofs of conservation laws for nonlinear Schrödinger equations*, Calc. Var. Partial Differ. Equ. 25, No. 3, 403-408 (2006).

[91] D. Pelinovsky and G. Schneider, *Justification of the coupled-mode approximation for a nonlinear elliptic problem with a periodic potential.* Appl. Anal. **86** (2007), no. 8, 1017–1036.

[92] D. Pelinovsky and G. Schneider, *Moving gap solitons in periodic potentials.* Math. Methods Appl. Sci. **31** (2008), no. 14, 1739–1760.

[93] J. Rauch, *Lectures on Nonlinear Geometrical Optics.* IAS/Park City Math. Series 5, American Math. Soc., 1999.

[94] J. E. Rothenberg, *Space-time focusing: breakdown of the slowly varying envelope approximation in the self-focusing of femtosecond pulses.* Optics Lett. **17** (1992), 1340–1342.

[95] Y. Saad and M. Schultz, *GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems.* SIAM J. Sci. Comput. **7** (1986), no. 3, 856–869.

[96] H. Saito and M. Ueda, *Split-merge cycle, fragmented collapse, and vortex disintegration in rotating Bose-Einstein condensates with attractive interactions.* Phys. Rev. A **69** (2004), pp. 013604.

[97] R. Seiringer, *Gross-Pitaevskii theory of the rotating gas.* Comm. Math. Phys. **229** (2002), 491–509.

[98] R. Seiringer, *Ground state asymptotics of a dilute, rotating gas.* J. Phys. A **36** (2003), no. 37, pp. 9755.

[99] E. M. Stein, *Interpolation of linear operators.* Trans. Amer. Math. Soc. **83** (1956), 482–492.

[100] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on euclidean spaces*, Princeton University Press, 1971.

[101] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation, Self-focusing and wave collapse*. Springer-Verlag, 1999.

[102] T. Tao, *Nonlinear Dispersive Equations.* CBMS Regional Conference Series in Mathematics 106, Amer. Math. Soc., Providence 2006.

[103] M. Tsutsumi and I. Fukuda, *On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem.* Fako l'Funk. Ekvacioj Japana Mat. Societo. **23** (1980), no. 3, 259–277.

[104] M. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates.* Comm. Math. Phys. **87** (1983), 567–576.

[105] C.H. Wilcox, *Theory of Bloch Waves*, J. d' Anal. Math. **33** (1978), no. 1, 146–167.

[106] Y. Wu, *Global well-posedness on the derivative nonlinear Schrödinger equation revisited.* Anal. PDE **8** (2015), no. 5, 1101–1112.

[107] J. Zhang, *Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials*. Z. angew. Math. Phys. (2000) **51**, no. 3, 498–503.

[108] J. Zhang, *Stability of attractive Bose-Einstein condensates.* J. Stat. Phys. **101** (2000), 731–746.

[109] J. Zhang, *Sharp threshold for global existence and blowup in nonlinear Schrödinger equation with harmonic potential.* Comm. Partial Differ. Equ. **30** (2005), 1429–1443.

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"Stability and instability properties of rotating Bose-Einstein condensates."
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