Paschke Category, K-homology, and the Riemann-Roch Transformation

by

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To

my wife,

Mozhgan

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SUMMARY

For a separable C^*-algebra A, We define an exact C^*-category called *the Paschke Category* of A, and show that its topological K-theory groups are equal to topological K-homology groups of the C^*-algebra A. Then we use the Dolbeault complex and ideas from the classical methods in Kasparov K-theory to construct an acyclic chain complex in this category, which in turn, induces a Riemann-Roch transformation from algebraic K-theory spectra of a complex manifold X, to its topological K-homology spectra. We examine the question of whether this map commutes with push-forward with respect to a proper map of complex manifolds, and how we can extend it to complex spaces.

CHAPTER 1

INTRODUCTION

This thesis is an effort to better understand the relations between the algebraic K-theory and topological K-homology of a complex manifold.

Grothendieck commenced the study of such relations concurrently with introducing the (algebraic) K_0-group (BS58) to formulate his generalization of the Hirzebruch Riemann-Roch theorem as a relative statement about proper morphisms of regular algebraic varieties. He constructed a Riemann-Roch transformation from the algebraic K_0-group of a regular algebraic variety to its Chow group. Atiyah and Hirzebruch adapted Grothendieck's ideas in the topological settings to define the topological K_0-group, and proved versions of the Grothendieck-Riemann-Roch theorem which involved manifolds, instead of regular algebraic varieties (AH59; AH62). Baum, Fulton, and MacPherson (BFM75) proved a Riemann-Roch theorem for the K_0-group of singular varieties as a natural transformation of homology theories, and later Gillet (Gil81) generalized their methods to obtain a natural transformation on K-theory spectra.

Since push-forward of a locally free sheaf with respect to a proper morphism no longer has to be locally free, it is essential to work with the category of coherent sheaves instead, which is invariant under proper push-forwards. A central fact used in Grothendieck's proof is that any coherent sheaf on a regular algebraic variety has a global resolution by locally free sheaves. This is no longer the case for complex manifolds even though there are still local resolutions. O'Brian, Toledo, and Tong (OTT81) proved a Riemann-Roch theorem for the K_0-group of com-

plex manifolds, through studying twisted products of twisting cochains associated to coherent analytic sheaves. Later, Levy proved a more general Riemann-Roch theorem for singular complex spaces, by working with the resolutions of coherent analytic sheaves with Frechet spaces (instead of Banach spaces), and analyzing how complexes of Frechet spaces are related to K-homology through Spanier-Whitehead duality (Lev87; Lev08).

In this thesis, we take a different approach towards defining the Riemann-Roch transformation on the K-theory spectra of complex manifolds, which could give us better insight into relations between the algebraic and topological K-theory spectra.

In this thesis, we are only considering *separable* Hilbert spaces and C^*-algebras.

1.1 K-theory Spectra and Grayson's Map

Quillen defined exact categories as additive categories with a suitable class of short exact sequences, and defined their K-theory (Qui73). For example for a complex manifold X, the category of holomorphic vector bundles on X, or the category of coherent analytic sheaves on X are both exact categories, and algebraic K-theory spectrum $\mathbb{K}^{alg}(X)$ is defined as the K-theory spectrum of the category of coherent analytic sheaves on X. As a matter of fact, this defines a covariant functor with respect to proper morphisms of complex analytic spaces. Waldhausen generalized Quillen's construction to categories with two classes of morphisms, called the cofibrations (admissible monomorphisms) and the weak equivalences (Wal85). These categories are now called Waldhausen categories, and he defined the K-theory spectrum of a

Waldhausen category $\mathcal{A}$ by defining the simplicial category $S.\mathcal{A}$, then constructing a topological space by restricting to weak equivalences $wS.\mathcal{A}$ of this category, and then considering the loop space $\Omega|wS.\mathcal{A}|$ of its geometric realization.

Let $\mathcal{A}$ be an exact category. Let $Ac^b(\mathcal{A})$ denote the category of bounded acyclic chain complexes in $\mathcal{A}$, and let $Bi^b(\mathcal{A})$ denote the category of bounded acyclic *double* chain complexes in $\mathcal{A}$. For example, start with a chain complex, and copy each differential to obtain a double chain complex. This induces an exact functor $\Delta : Ac^b(\mathcal{A}) \to Bi^b(\mathcal{A})$. Grayson showed that the relative K-theory spectrum $K(Ac^b(\mathcal{A}) \xrightarrow{\Delta} Bi^b(\mathcal{A}))$ is naturally homotopy equivalent to the loop space of the K-theory spectrum $K(\mathcal{A})$ (Gra12).

1.2 Analytic K-homology and Functional Calculus

In (Ati70), Atiyah brought forward ideas from theory of operator algebras to develop a K-homology theory for spaces such as complex analytic spaces, which are not smooth manifolds. Later Brown, Douglas, and Fillmore (BDF73; BDF77) gave a description of the first topological K-homology group of compact, separable, and Hausdorf topological spaces using the extensions of C^*-algebras (BDF73; BDF77). Paschke defined the dual of a unital C^*-algebra A (Pas81), whose topological K-theory groups are isomorphic to the topological K-homology groups of A (Hig95; Hig90), however this construction is not functorial.

For a pair of C^*-algebras A, B, Kasparov introduced the bivariant K-theory $KK(A, B)$ as the abelian group generated by objects called Fredholm modules, with relations induced by direct sums, homotopies, and unitary equivalences (Kas80). A Fredholm module is a triple (H, ρ, F) where H is a Hilbert B-module, (i.e. the inner product takes values in the C^*-algebra

B instead of the complex numbers, and there is a continuous action of B on H), ρ is a representation of A into B-linear bounded operators on H, and F is a pseudo-local B-linear bounded operator on H, so that $F - F^*$ and $F^2 - Id_H$ are locally compact. This construction unifies K-theory and K-homology, i.e. when $B = \mathbb{C}$ this gives the K-homology of the C^*-algebra A, and when $A = \mathbb{C}$, this gives the K-theory of the C^*-algebra B, and is functorial in both terms. However, this does not produce a concrete category whose K-theory spectrum is related to K-homology of a C^*-algebra.[1]

When A is the C^*-algebra $C_0(X)$ of complex valued continuous functions on a smooth manifold X that vanish at infinity, and D is a symmetric elliptic differential operator on X, then by applying functional calculus with respect to $\chi(t) = \frac{t}{\sqrt{1+t^2}}$, we obtain a pseudo-local bounded operator $\chi(D)$ acting on the Hilbert space $L^2(X)$ (CGT82). This process induces the Fredholm module $(L^2(X), \rho, \chi(D))$ where ρ is the representation of A given by pointwise multiplication, and hence we get a class in K-homology of A.

This suggests that when working with a holomorphic vector bundle E on the complex manifold X, one could try to use the Dolbeault operator in a proper categorical framework to obtain a map from the algebraic K-theory spectrum of X to its topological K-homology spectrum.

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[1] In (Hig88), Higson proves that through Kasparov product, KK is the hom-functor of an additive category; this category is the universal category with an additive functor from the category of C^*-algebras that respects split short exact sequences, homotopies, and inverts the tensor product with the C^*-algebra of compact operators (which is essentially the Morita equivalence). However, this description does not seem to help us with the specific needs of this thesis.

1.3 Statement of the Results

For a C^*-algebra A we define the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ of A to be the C^*-category whose objects are representations of A, and morphisms between two representations are pseudo-local operators modulo locally compact operators. We define an exact structure on this category by simply saying that a chain complex is exact, if there is a contracting homotopy. Precomposing a representation of a C^*-algebra B with a $*$-morphism $f : A \to B$, induces an exact pull-back functor $f^* : (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_A$, and this process is functorial.

Much of the machinery used by Waldhausen also works in the setting of topological categories (Mit01), and in particular for C^*-categories. Hence, we can apply Waldhausens construction on the exact C^*-category $(\mathfrak{D}/\mathfrak{C})_A$, and then apply fat geometric realization (Seg74) to obtain a topological space $\Omega\|wS_\cdot(\mathfrak{D}/\mathfrak{C})_A\|$ whose homotopy groups are the K-theory groups of the Paschke category.

Theorem 1.3.1. *Let A be a separable C^*-algebra and let $i \geq 1$. Then the topological K-theory groups $K_i((\mathfrak{D}/\mathfrak{C})_A)$ of the Paschke category of A are isomorphic to topological K-homology groups $K^{1-i}(A)$ of A.*

This means that $A \mapsto \Omega\|wS_\cdot(\mathfrak{D}/\mathfrak{C})_A\|$ is a contravariant functor that represents the topological K-homology of A.[1]

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[1]Since $X \mapsto C_0(X)$ is a contravariant functor from topological spaces and proper maps to C^*-algebras, then topological K-homology is covariant in terms of the topological space, but contravariant in terms of the C^*-algebra.

Let E be a holomorphic vector bundle on the complex manifold X, and choose Hermitian metrics on X, E. Let $D_E = \bar{\partial}_E + \bar{\partial}_E^*$ be the corresponding Dolbeault operator, and denote the sequence of the the L^2-completions of the Dolbeault complex together with $\chi(D_E)$ as morphisms between them by $\tau_X^D(E)$.[1] Similar to the case of working with the Fredholm modules, we can show that $\tau_X^D(E)$ induces an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$. Through a careful analysis of functional calculus applied to the Dolbeault operator, we prove that we get an induced exact functor

$$\tau_X^D : \mathcal{P}_{\mathrm{m,b}}(X) \to Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$$

where $\mathcal{P}_{\mathrm{m,b}}(X)$ is an exact category of holomorphic vector bundles on X with choices of metrics, and $Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ is the exact category of acyclic chain complexes in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ whose exact structure is induced by that of $\mathfrak{D}_{C_0(X)}$.

We also generalize a process used in (HR00), to obtain a natural exact functor $Ac'(\mathfrak{D}/\mathfrak{C})_A \to Bi'(\mathfrak{D}/\mathfrak{C})_A$. Then for a complex manifold, we precompose the induced map on the K-theory spectrum with τ_X^D and then compose it with the natural map induced by the Grayson's homotopy equivalence, to obtain a map in the homotopy category

$$\tau_{X,V} : K(\mathcal{P}(X)) \to \Omega K(\mathfrak{D}/\mathfrak{C})_{C_0(V)}$$

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[1] We supress the dependence on the choice of metric here for an easier notation.

for any relatively compact open subset V of X, where $\mathcal{P}(X)$ is the category of locally free sheaves on X. The map above respects restriction to further open subsets. Hence using descent, we obtain a Riemann-Roch transformation

$$\tau_X : K(\mathcal{P}(X)) \to \Omega K(\mathfrak{D}/\mathfrak{C})_{C_0(X)} \cong K^{top}(X), \tag{1.1}$$

which commutes with restriction to open subsets, and with the push-forward maps with respect to $f : X \to *$, where X is a compact complex manifold.

When the C^*-algebra A is unital (or X is compact), we define further structure on the Paschke category.

Proposition 1.3.2. *Let $\mathcal{P}_m(A)$ denote the category of finitely generated projective modules over A with a norm, then taking tensor product of a module with a representation induces a biexact functor*

$$\mathcal{P}_m(A) \times (\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})_A$$

which in turn induces a cap product on the K-theory spectra

$$K(\mathcal{P}_m(A)) \wedge K(\mathfrak{D}/\mathfrak{C})_A \to K(\mathfrak{D}/\mathfrak{C})_A$$

which is natural with respect to pull-backs and push-forwards.

When X is a compact complex manifold and $A = C_0(X)$, then taking the cap product with the Dolbeault complex $\tau_X^D(X \times \mathbb{C})$ of the trivial rank one bundle, induces a map that extends

the Riemann-Roch transformation defined in Equation 1.1 to the category of *topological* vector bundles over X.

A preliminary version of this thesis was posted on Arxiv at (Sar18).

CHAPTER 2

C^*-CATEGORIES AND THE PASCHKE CATEGORY

2.1 Definitions and Basic Properties

Let us start with giving a brief history and basic definitions of C^* *categories*. Karoubi first defined *Banach Categories* in (Kar68). A good source for this material is (Kar10). Later C^**-categories* were defined in (GLR85). Another good source for C^*-categories is (Mit02).

Definition 2.1.1. The category $\mathfrak{A}$ is called a *complex* $*$*-category* if:

A1 For each two objects A, B of $\mathfrak{A}$, $\mathfrak{A}(A, B)$ is a complex vector space and composition of arrows is bilinear.

A2 There is an involutive antilinear contravariant endofunctor $*$ of $\mathfrak{A}$ which preserves objects. The image of x under $*$ will be denoted by x^*. It follows that each $\mathfrak{A}(A, A)$ is a $*$-algebra with identity.

A3 For each $x \in \mathfrak{A}(A, B)$, x^*x is a positive element of the $*$-algebra $\mathfrak{A}(A, A)$, i.e. $x^*x = y^*y$ for some $y \in \mathfrak{A}(A, A)$. Furthermore, $x^*x = 0$ implies $x = 0$.

It follows that the mapping $\mathfrak{A}(A, B) \times \mathfrak{A}(A, B) \to \mathfrak{A}(A, A)$ defined by $(x, y) \mapsto x^*y$ is a $\mathfrak{A}(A, A)$-valued inner product on the right $\mathfrak{A}(A, A)$-module $\mathfrak{A}(A, B)$, where $\mathfrak{A}(A, A)$ acts on $\mathfrak{A}(A, B)$ by composition of arrows.

A $*$-category $\mathfrak{A}$ is called a *normed* $*$*-category* if:

A4 Each $\mathfrak{A}(A,B)$ is a normed space and $\|xy\| \leq \|x\|\|y\|$.

A normed $*$-category is called a *Banach $*$-category* if:

A5 Each $\mathfrak{A}(A,B)$ is a Banach space.

A Banach $*$-category is called a *C^*-category* if:

A6 For each arrow x of $\mathfrak{A}$, $\|x\|^2 = \|x^*x\|$.

It follows that each $\mathfrak{A}(A,A)$ is a C^*-algebra with identity. A6 shows that the norm on a C^*-category is uniquely determined by the norms on the C^*-algebra $\mathfrak{A}(A,A)$. In fact, we can say more: Let $\mathfrak{A}$ be a $*$-category where each $\mathfrak{A}(A,A)$ is a C^*-algebra, then $\mathfrak{A}$ can be made into a normed $*$-category satisfying A6 (but not A5 in general) in a unique way by setting $\|x\| = \|x^*x\|^{1/2}$.

Of course any C^*-algebra with identity can be considered as a C^*-category with a single object.

Notation 2.1.2. From now on, for a C^*-category $\mathfrak{A}$, we will denote the C^*-algebra of the endomorphisms of an object A of $\mathfrak{A}$, with $\mathfrak{A}(A)$ instead of $\mathfrak{A}(A,A)$.

Definition 2.1.3. Let $\mathfrak{A}, \mathfrak{A}'$ be C^*-categories. Then a functor $F : \mathfrak{A} \to \mathfrak{A}'$ is called a *$*$-functor* if it is a linear functor (i.e. $F : \mathfrak{A}(A,B) \to \mathfrak{A}'(F(A),F(B))$ is linear for all objects A, B of $\mathfrak{A}$.) and also $F(x)^* = F(x^*)$ for all morphisms x in $\mathfrak{A}$.

Definition 2.1.4. (Mit02, 3.1.) A *non-unital category*, is a category of objects and morphisms similar to a category, except that there need not exist an identity morphism $1 \in Hom(A,A)$ for each object A. A *non-unital functor* $F : \mathfrak{A} \to \mathfrak{B}$ between (possibly non-unital) categories

$\mathfrak{A}, \mathfrak{B}$ is a transformation similar to a functor, except that there is no condition on the identity morphisms of the category $\mathfrak{A}$. Similarly, we can define non-unital C^*-categories, and $*$-functors between them.

Definition 2.1.5. (Mit02, 4.2.) Let $\mathfrak{A}$ be a C^*-category, then a *C^*-ideal* $\mathfrak{I}$ in the category $\mathfrak{A}$ is (a probably non-unital) subcategory of $\mathfrak{A}$ so that:

- The subcategory $\mathfrak{I}$ has the same objects as the category $\mathfrak{A}$.
- Each morphism set $\mathfrak{I}(A, B)$ is a norm closed subspace of the space $\mathfrak{A}(A, B)$.
- The composition of an arrow in the category $\mathfrak{A}$ with an arrow in the subcategory $\mathfrak{I}$ is an arrow in the subcategory $\mathfrak{I}$.

As a result of the definition above we have:

Proposition 2.1.6. *(Mit02, 4.7.) Let $j \in \mathfrak{I}(A, B)$ be a morphism in the C^*-ideal $\mathfrak{I}$ of the C^*-category $\mathfrak{A}$. Then the adjoint morphism j^* is also a morphism in the ideal $\mathfrak{I}$.*

Also, we can define the quotient $\mathfrak{A}/\mathfrak{I}$ to be the category with the same objects as $\mathfrak{A}$ and with morphism sets the quotient Banach spaces

$$(\mathfrak{A}/\mathfrak{I})(A, B) = \frac{\mathfrak{A}(A, B)}{\mathfrak{I}(A, B)},$$

for objects A, B in $\mathfrak{A}$. This is also a C^*-category.

Example 2.1.7. We will use $\mathfrak{B}$ to denote the category of Hilbert spaces with bounded operators between them, which is an additive C^*-category as products and coproducts of a finite number of

Hilbert spaces is just their direct sum, and also the set of bounded operators between two Hilbert spaces forms an abelian group, as we can add the operators with each other and composition on both sides is linear.

We will denote the C^*-ideal subcategory of compact operators by $\mathfrak{K}$.

Example 2.1.8. Let A be a C^*-algebra, and let $\mathfrak{A}$ be a C^*-category. Let $\mathcal{R}ep_{\mathfrak{A}}(A)$ denote the category of *representations of* A, i.e. a category whose objects are representations $\rho : A \to \mathfrak{A}(H)$, where H is an object in $\mathfrak{A}$, and whose morphisms between two representations $\rho_1 : A \to \mathfrak{A}(H_1)$ and $\rho_2 : A \to \mathfrak{A}(H_2)$ are morphisms $T : H_1 \to H_2$ in $\mathfrak{A}(H_1, H_2)$ such that for each $a \in A$, $T\rho_1(a) = \rho_2(a)T$. Notice that we are *not* restricting our attention to unital representations, i.e. we also consider the zero representation, and other non-unital representations.

It is easy to check that $\mathcal{R}ep_{\mathfrak{A}}(A)$ is a C^*-category, and if $\mathfrak{A}$ is additive, then it is easy to check that $\mathcal{R}ep_{\mathfrak{A}}(A)$ is an additive C^*-category as well.

Definition 2.1.9. Let A be a C^*-algebra, and let $\rho_i : A \to \mathfrak{B}(H_i)$, be representations of A for $i = 1, 2$. A bounded operator $T : H_1 \to H_2$ is called *pseudo-local*, if $\rho_2(a)T - T\rho_1(a) \in \mathfrak{K}(H_1, H_2)$, for all $a \in A$, and T is *locally compact*, if both $\rho_2(a)T, T\rho_1(a)$ are in $\mathfrak{K}(H_1, H_2)$ for all $a \in A$.

Definition 2.1.10. Let A be a C^*-algebra. Then we define the *Paschke category* of A to be the quotient category $(\mathfrak{D}/\mathfrak{C})_A \coloneqq \mathfrak{D}_A/\mathfrak{C}_A$, where $\mathfrak{D}_A$ is the category with objects the representations $\rho : A \to \mathfrak{B}(H)$ of A, and morphisms between two representations are the pseduo-local operators

between them, and the C^*-ideal $\mathfrak{C}_A$ in $\mathfrak{D}_A$ has the same objects, but the morphisms are locally compact operators.

We define the *Calkin-Paschke category* of A to be the quotient category $(\mathfrak{D}/\mathfrak{C})'_A \coloneqq \mathfrak{D}'_A/\mathfrak{C}'_A$, where $\mathfrak{D}'_A = \mathcal{R}ep_{\mathfrak{B}/\mathfrak{K}}(A)$ is the category of representations of A into the Calkin algebra $(\mathfrak{B}/\mathfrak{K})(H)$ of some Hilbert space H (cf. example 2.1.8), and the C^*-ideal $\mathfrak{C}'_A$ in $\mathfrak{D}'_A$ has the same objects, but a morphism $T \in \mathfrak{D}'_A(\rho'_1, \rho'_2)$ is in $\mathfrak{C}'_A(\rho'_1, \rho'_2)$ iff $T\rho'_1(a) = 0 = \rho'_2(a)T$ for all $a \in A$.

Note that there is a natural functor $(\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})'_A$, which sends a representation $\rho : A \to \mathfrak{B}(H)$ to $\rho' : A \to \mathfrak{B}(H) \to (\mathfrak{B}/\mathfrak{K})(H)$. Therefore $(\mathfrak{D}/\mathfrak{C})_A$ is equivalent to the full subcategory of $(\mathfrak{D}/\mathfrak{C})'_A$ consisting of representations of A to some Calkin algebra $(\mathfrak{B}/\mathfrak{K})(H)$ that lift to the algebra $\mathfrak{B}(H)$ of bounded operators.

Notation 2.1.11. From now on, we will use letters such as ρ, ν to refer to objects of the Paschke category, and use similar letters with "primes", i.e. ρ', ν' to refer to objects of the Calkin-Paschke categories.

In case no confusion should arise, instead of writing $T\rho(a)$ is compact for all $a \in A$, we will simply say $T\rho$ is compact, and similarly for ρ'.

Example 2.1.12 (See (HR00, 5.3.2.))**.** One can generalize the definition above, by introducing a "relative" version. Let A be a C^*-algebra and $I \subset A$ a C^*-ideal. Then for representations $\rho_i : A \to \mathfrak{B}(H_i)$ for $i = 1, 2$, define $\mathfrak{D}_A(\rho_1, \rho_2)$ to be the same as the above example, and let

$$\mathfrak{C}_{I,A}(\rho_1, \rho_2) = \{T \in \mathfrak{D}_A(\rho_1, \rho_2) | T\rho(a), \rho(a)T \in \mathfrak{K}(H_1, H_2),\ \forall a \in I\}.$$

Note that when $I \subset J$ then $\mathfrak{C}_A \subset \mathfrak{C}_{J,A} \subset \mathfrak{C}_{I,A}$, and if $I = A$, then we recover the definition above. All of the results on the Paschke category also holds for this relative version, however, by Theorem 5.1.1 and excision for K-homology (cf. (HR00, 5.4.5.)), this does not provide any new information.

Definition 2.1.13. (Kar10, 1.1.6.7.) Let $\mathfrak{A}$ be an additive category. Then $\mathfrak{A}$ is called *pseudo-abelian* if for each object H of $\mathfrak{A}$ and every morphism $p : H \to H$ so that $p^2 = p$, the kernel of p exists.

In the case when $\mathfrak{A}$ is an additive C^*-category, and each self-adjoint projection has a kernel, then we say $\mathfrak{A}$ is *weakly pseudo-abelian*. (Kan00, Def 8.)

Proposition 2.1.14. *(Kar10, 1.1.6.9.) Let $\mathfrak{A}$ be a (weakly) pseudo-abelian category, let H be an object of $\mathfrak{A}$ and let $p : H \to H$ be such that $p^2 = p$ (and also $p = p^*$). Then the object H splits into the direct sum $H = \ker(p) \oplus \ker(1 - p)$.*

Proposition 2.1.15. *(Kar10, 1.1.6.10.) Let $\mathfrak{A}$ be an additive category. Then there exists a pseudo-abelian category $\tilde{\mathfrak{A}}$, and an additive functor $\phi : \mathfrak{A} \to \tilde{\mathfrak{A}}$ which is fully faithful and is universal among additive functors from $\mathfrak{A}$ to a pseudo-abelian category. The pair $(\phi, \tilde{\mathfrak{A}})$ is unique up to equivalence of categories.*

$\tilde{\mathfrak{A}}$ is equivalent to the category where objects are pairs (H, p) where H is an object in $\mathfrak{A}$ and $p : H \to H$ is a projector (i.e. $p^2 = p$), and morphisms between (H_1, p_1) and (H_2, p_2) are morphisms $f : H_1 \to H_2$ in $\mathfrak{A}$ such that $fp_1 = p_2 f = f$ in $\mathfrak{A}$. This category is called the pseudo-abelianization *of $\mathfrak{A}$.*

The same statement is true for a C^*-category and its weakly pseudo-abelian counterpart. (Kan00, Thm 9.)

Proposition 2.1.16. *The weak pseudo-abelianization of the C^*-category $(\mathfrak{B}/\mathfrak{K})$ is isomorphic to the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_{\mathbb{C}}$.*

Proof. The objects in the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_{\mathbb{C}}$ can be considered as pairs $(H, \rho'(1))$ of a Hilbert space H and a self-adjoint projection $p = \rho'(1) \in (\mathfrak{B}/\mathfrak{K})(H)$, and morphisms $\rho'_1 \to \rho'_2$ in $(\mathfrak{D}/\mathfrak{C})'_{\mathbb{C}}$ are the pseudo-local operators modulo locally compact ones, i.e. the operators $F \in (\mathfrak{B}/\mathfrak{K})(H_1, H_2)$ so that $Fp_1 = p_2F$, modulo the ones that $Fp_1 = 0 = p_2F$. In other words since $F(1 - p_1)$ is locally compact, hence $F = Fp_1 = p_2F$ in the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_{\mathbb{C}}$. Therefore we have a natural functor $(\mathfrak{D}/\mathfrak{C})'_{\mathbb{C}} \to \widetilde{(\mathfrak{B}/\mathfrak{K})}$.

This functor is faithful, because $F = Fp_1 = p_2F$ are all zero in the category $(\mathfrak{B}/\mathfrak{K})$ iff F is locally compact in the Calkin-Paschke category. The functor is also full, because any $F \in (\mathfrak{B}/\mathfrak{K})(H_1, H_2)$ that satisfies $Fp_1 = p_2F$ is pseudo-local. □

Proposition 2.1.17. *The Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_A$ is a weakly pseudo-abelian category.*

We need the following two lemmas to prove the proposition above.

Lemma 2.1.18. *Each self-adjoint projection in $(\mathfrak{D}/\mathfrak{C})'_A$ has a representative in $\mathfrak{D}_A$ which is a self-adjoint projection.*

Proof. Let $P \in \mathfrak{D}_A(\rho')$ be a representative for a self-adjoint projection in $(\mathfrak{D}/\mathfrak{C})'_A$. Hence $\rho'(P - P^*)$ and $(P - P^*)\rho'$ are compact operators. Set $P' = (P + P^*)/2$. Then P' is a self-adjoint operator, hence by Weyl-Von Neumann theorem (HR00, 2.2.5.) there is a diagonal

compact perturbation of P', i.e. there exists an operator P_1 with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of eigenvectors of P_1 for H with complex numbers λ_i as eigenvalues so that and $P_1 - P'$ is a compact operator. Therefore $P - P_1$ is in $\mathfrak{C}_A(\rho')$.

Let $I \subset \mathbb{N}$ be the set of indices i such that $|\lambda_i| < 1/2$. Now define the bounded operator Q by $Q(e_i) = e_i$ if $i \notin I$, and set $Q(e_i) = 0$ otherwise. Evidently, Q is a self-adjoint projection in the category $\mathfrak{B}$. We want to show that $P_1 - Q$ is in $\mathfrak{C}_A(\rho')$.

Define $D(e_i) = \frac{1}{1-\lambda_i} e_i$ if $i \in I$, and $D(e_i) = \frac{1}{\lambda_i} e_i$ otherwise. Notice that D is a bounded diagonal operator (of norm at most 2). Also $(P_1 - Q)e_i = \lambda_i e_i$ when $i \in I$, and $(P_1 - Q)e_i = (\lambda_i - 1)e_i$ otherwise. Furthermore $(P_1 - P_1^2)e_i = (\lambda_i - \lambda_i^2)e_i$ for all $i \in \mathbb{N}$. Therefore $(P_1 - Q)(e_i) = D(P_1 - P_1^2)(e_i) = (P_1 - P_1^2)D(e_i)$ for all i, hence $P_1 - Q = D(P_1 - P_1^2) = (P_1 - P_1^2)D$. But since $P_1 - P_1^2 \in \mathfrak{C}_A(\rho')$, then $(P_1 - P_1^2)\rho', \rho'(P_1 - P_1^2)$ are compact operators. Therefore $(P_1 - Q)\rho', \rho'(P_1 - Q)$ are also compact, which proves that $P_1 - Q \in \mathfrak{C}_A$ and hence $Q \in \mathfrak{D}_A(\rho')$. □

Lemma 2.1.19. *Let $T \in \mathfrak{D}_A(\rho_1', \rho_2')$ be a pseudo-local operator with closed image $V_2 \subset H_2$. Let $V_1 \subset H_1$ be the orthogonal complement of* $\ker(T)$. *Then for $i = 1, 2$, the projections $\pi_i : H_2 \to V_i$ and the inclusions $\iota_i : V_i \to H_i$ are pseudo-local operators.*

Proof. Since T has a closed image, it induces an isomorphism of Hilbert spaces from V_1 to V_2. To simplify the notation, denote $T' = \pi_2 T \iota_1 : V_1 \to V_2$. Let $S' \in \mathfrak{B}(V_2, V_1)$ be the inverse to T' and set $S = S' \oplus 0 : V_2 \oplus V_2^{\perp} = H_2 \to V_1 \oplus V_1^{\perp} = H_1$. We have $ST = \iota_1 \pi_1$ and $TS = \iota_2 \pi_2$. First we show that S is also pseudo-local. This will show that ι_i, π_i are pseudo-local.

Let $\rho'_i = \begin{pmatrix} \rho'^{11}_i & \rho'^{12}_i \\ \rho'^{21}_i & \rho'^{22}_i \end{pmatrix} \in (\mathfrak{B}/\mathfrak{K})(V_i \oplus V_i^{\perp})$ for $i = 1, 2$. Since T is pseudo-local, we have $T'\rho'^{11}_1 - \rho'^{11}_2 T'$ and $T'\rho'^{12}_2$ and $\rho'^{21}_2 T'$ are compact[1]. Therefore $\rho'^{12}_1 = S'T'\rho'^{12}_1 = 0$ and $\rho'^{21}_2 = \rho'^{21}_2 T'S' = 0$. Also, since $\rho'^{12}_i(a)^* = \rho'^{21}_i(a^*)$, then we can say that $\rho'^{12}_i, \rho'^{21}_i$ are zero, for $i = 1, 2$. Therefore $\rho'_1 S - S\rho'_2 = (\rho'^{11}_1 S' - S'\rho'^{11}_2) \oplus 0$. But $(\rho'^{11}_1 S' - S'\rho'^{11}_2) = S'T'(\rho'^{11}_1 S' - S'\rho'^{11}_2) = S'\rho'^{11}_2 T'S' - S'\rho'^{11}_2 = 0$. Hence S is pseudo-local.

We have $\iota_i\rho'^{11}_i - \rho'_i\iota_i = (\rho'^{11}_i, 0) - (\rho'^{11}_i, \rho'^{12}_i) = 0$. This proves that ι_i is pseudo-local, for $i = 1, 2$. It only remains to show that $\rho'^{11}_i = \pi_i\rho'_i\iota_i : A \to (\mathfrak{B}/\mathfrak{K})(V_i)$ is an object of the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_A$. This follows from adjointness of ι_i and π_i and[2]

$$\rho'^{11}_i(ab^*) = \pi_i\rho'_i(a)\rho'_i(b^*)\iota_i = \pi_i\iota_i\pi_i\rho'_i(a)\rho'_i(b^*)\iota_i = \pi_i\rho'_i(a)\iota_i(\pi_i\rho'_i(b)\iota_i)^*.$$

□

Proof of Proposition. Let $P \in \mathfrak{D}_A(\rho')$ be a representative for a self-adjoint projection in $(\mathfrak{D}/\mathfrak{C})'_A$, and let Q be the projection as in proof of Lemma 2.1.18 (we will only use the fact that Q has a closed image). Let ι be the inclusion of $\ker(Q)$ in H, and let π be the projection onto the kernel. Then we want to show that ι is a kernel for Q. We clearly have $Q\iota = 0$. Also since Q is bounded from below on the orthogonal complement of its kernel, then it has closed image. It follows from Lemma 2.1.19 that ι is pseudo-local. Let Q' denote "inverse" of Q restricted to

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[1]Where in here, we are denoting the operator induced by T' in $(\mathfrak{B}/\mathfrak{K})(V_1, V_2)$ again by T'. We abuse notation in a similar way for S', S, T, π_i, ι_i.

[2]Note that this part of the argument only works in the Calkin-Paschke category. In fact, this is the only part of the proof of pseudo-abelianness of the Calkin-Paschke category that does not apply to the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$.

orthogonal complement of $\ker(Q)$, i.e. Q' sends image of the projection Q isometrically to the orthogonal complement of $\ker(Q)$.

Now let $F \in \mathfrak{D}_A(\rho_0', \rho')$ be an operator so that $QF \in \mathfrak{C}_A(\rho_0', \rho')$. Then we want to show that F factors through ι up to locally compact operators. We have $\iota\pi F = F$ modulo compact operators because $(Id_H - \iota\pi)F = Q'QF = 0$. Also, if we have $\rho'(\iota G - F) = 0$, then $\rho'\iota(G - \pi F) = 0$. Therefore $\rho'(G - \pi F) = 0$. This completes the proof. □

Definition 2.1.20. Let $\mathfrak{A}$ be an additive category. Then we say that a chain complex

$$\ldots \xrightarrow{T_{i-1}} \rho_i \xrightarrow{T_i} \rho_{i+1} \xrightarrow{T_{i+1}} \rho_{i+2} \xrightarrow{T_{i+2}} \ldots$$

is *exact* if there is a *contracting homotopy*, i.e. if there are morphisms S_i in $\mathfrak{A}$ from ρ_{i+1} to ρ_i so that $T_{i-1}S_{i-1} + S_iT_i = Id_{\rho_i}$ in $\mathfrak{A}$.

As a result of this definition, every short exact sequence in $\mathfrak{A}$ is split, hence $\mathfrak{A}$ is an *exact category* in the sence of Quillen (Qui73, Sec 2.). Note that this does not necessarily mean all exact sequences are split.

In particular, (using the definition above) the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$, the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_A$, and also $\mathfrak{D}_A, \mathfrak{B}, (\mathfrak{B}/\mathfrak{K})$ are all exact C^*-categories.

Notice that a map $f : A \to B$ of C^*-algebras, induces pull-back maps $f^* : (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_A$ and also $f^* : (\mathfrak{D}/\mathfrak{C})'_B \to (\mathfrak{D}/\mathfrak{C})'_A$ of categories, by precomposing with the representation. This map preserves exact sequences, hence the pull-back functor is exact, and this process is functorial.

2.2 Subcategories of the Paschke Category

We start this section by giving a definition similar to (Wal85)[1].

Definition 2.2.1. Let $\mathcal{B}$ be an additive category. Then a full additive subcategory $\mathcal{A}$ is called *cofinal* if for every object B of the category $\mathcal{B}$, there is an object B' in $\mathcal{B}$ so that $B \oplus B'$ is isomorphic to an object in $\mathcal{A}$. If we can always take B' to be an object in $\mathcal{A}$, then $\mathcal{A}$ is called *strictly cofinal*.

In case the category $\mathcal{B}$ is exact, we require the subcategory $\mathcal{A}$ to be exact as well.

Let us recall some definitions and useful properties of representations.

Definition 2.2.2. A representation $\rho : A \to \mathfrak{B}(H)$ of a C^*-algebra is called *non-degenerate* if $\rho(A)H$ is a dense subset of H (or equivalently, it is the whole H, cf. (HR00, 1.9.17.).). Another equivalent definition is that for each $h \in H, h \neq 0$, there exists an $a \in A$ so that $\rho(a)h \neq 0$.

A representation $\rho : A \to \mathfrak{B}(H)$ is called *ample* if it is non-degenerate, and also for each $a \in A, a \neq 0$, $\rho(a)$ is not a compact operator.

Proposition 2.2.3. *Let $\mathfrak{Q}_A$ denote the full subcategory of $(\mathfrak{D}/\mathfrak{C})_A$ whose objects are ample representations, together with the zero representation $A \to 0$. This is an exact strictly cofinal subcategory of $(\mathfrak{D}/\mathfrak{C})_A$.*

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[1]This was originally defined for Waldhausen categories by considering coproduct instead of the direct sum. In this thesis, we will only apply the definition to certain Waldhausen categories.

Let A be a unital C^*-algebra and let $\mathfrak{Q}'_A$ denote the full subcategory of $(\mathfrak{D}/\mathfrak{C})'_A$ whose objects are unital injective representations, together with the zero representation $A \to 0$. This is an exact strictly cofinal subcategory of $(\mathfrak{D}/\mathfrak{C})'_A$.

Proof. Note that direct sum of two non-degenerate representations is non-degenerate, and the direct sum of a non-degenerate and an ample representation is ample. Given some representation $\rho : A \to \mathfrak{B}(H)$, let $H_1 = \overline{\rho(A)H} \subset H$, $\pi : H \to H_1$ be the orthogonal projection onto the closed subspace, and let $\iota : H_1 \to H$ be the inclusion. Then we can define $\rho_1 : A \to \mathfrak{B}(H_1)$ by $\rho_1 = \pi\rho\iota$. Since π, ι are adjoints to each other, and $\iota\pi\rho = \rho = \rho\iota\pi$ then ρ_1 is indeed a representation. Also these two representations are isomorphic as objects of $(\mathfrak{D}/\mathfrak{C})_A$, as π, ι are pseudo-local and induce the isomorphism. But ρ_1 is a non-degenerate representation. Hence for any object ρ of $(\mathfrak{D}/\mathfrak{C})_A$, and any ample representation ρ_0 of A, $\rho \oplus \rho_0$ is isomorphic to an ample representation in $(\mathfrak{D}/\mathfrak{C})_A$.

If the C^*-algebra A is unital and $\rho' : A \to (\mathfrak{B}/\mathfrak{K})(H)$ is an object of $(\mathfrak{D}/\mathfrak{C})'_A$, then by Lemma 2.1.18 $\rho'(1)$ has a representative $\pi \in \mathfrak{B}(H)$ which is a self-adjoint projection. By repeating the argument above, ρ' is isomorphic to the unital representation $\rho'_1 : A \to (\mathfrak{B}/\mathfrak{K})(H_1)$, where $H_1 \subset H$ is image of π. Also, direct sum of an injective representation ρ'_0 with any representation ρ' is injective.

□

Another important property of ample representations, is the following corollary of Voiculescu's theorem (Voi76), which we mention similar to as stated in (HR00, 3.4.2.).

Theorem 2.2.4 (Voiculescu)**.** *Let A be a unital C^*-algebra, let $\rho : A \to \mathfrak{B}(H)$ be a non-degenerate representation, and let $\nu' : A \to (\mathfrak{B}/\mathfrak{K})(H')$ be an object in $(\mathfrak{D}/\mathfrak{C})'_A$. Assume that for each $a \in A$ with $\rho(a) \in \mathfrak{K}(H)$, we have $\nu'(a) = 0$. Then there exists an isometry $V : H' \to H$ so that $V^*\rho(a)V - \nu'(a) = 0$ for all $a \in A$*[1].

An important corollary is the following:

Corollary 2.2.5. *Let ρ_1, ρ_2 be two ample representations of the C^*-algebra A. Then there is a unitary operator $U : H_1 \to H_2$ so that $U\rho_1(a)U^* - \rho_2(a)$ is compact for all a.*

In other words, any two ample representations in the Paschke category are isomorphic. Hence we can denote the isomorphism class of endomorphisms of an ample representation ρ of A by $\mathfrak{Q}(A)$, which is also known as the *Paschke dual*.

Remark 2.2.6. The natural map $(\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})'_A$ is fully faithful, and by Voiculescu's theorem, each object ν' of $(\mathfrak{D}/\mathfrak{C})'_A$ has an admissible monomorphism to an object which lifts to a non-degenerate representation of A. Since $(\mathfrak{D}/\mathfrak{C})'_A$ is weakly pseudo-abelian, therefore by Voiculescu's theorem, the full subcategory of the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_A$ consisting of objects which lift to the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ is cofinal.

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[1] *We are abusing the notation for the class of $V^*\rho(a)V$ in $(\mathfrak{B}/\mathfrak{K})(H')$.*

CHAPTER 3

K-THEORY

3.1 Waldhausen's Cofinality

In this section, we recall some standard facts about K-theory and fix our notation for the rest of the section. Aside from Waldhausen's original paper (Wal85), a good source for more information is (Wei13).

Definition 3.1.1. (Seg74, A.) Let $X_\cdot$ be a simplicial space. Then define the topological space $\|X_\cdot\|$ called the *fat geometric realization* of $X_\cdot$, as the quotient

$$\coprod_n X_n \times \Delta^n_{top} / \sim_+$$

where the relation $\sim_+$ is generated by $(x, f_* p) \sim_+ (f^* x, p)$ for $x \in X_n, p \in \Delta^m_{top}$, whenever $f : [m] \to [n]$ in the simplex category Δ is a *face* (injective) map.

Let $X_\cdot$ be a simplicial set (or a *discrete* simplicial space). Then define the topological space $|X_\cdot|$ called the *geometric realization* of $X_\cdot$, as the quotient

$$\coprod_n X_n \times \Delta^n_{top} / \sim$$

where the relation $\sim$ is generated by $(x, f_* p) \sim_+ (f^* x, p)$ for $x \in X_n, p \in \Delta^m_{top}$, for *any* morphism $f : [m] \to [n]$ in the simplex category Δ.

Remark 3.1.2. For a simplicial space $X_\cdot$, there is a natural quotient map $\|X_\cdot\| \to |X_\cdot|$. This map is a homotopy equivalence if $X_\cdot$ is a discrete simplicial space.

In (Seg74, A.1.), Segal shows that this quotient map is a homotopy equivalence if for all n and all degeneracy (surjective) maps $[n] \to [n-1]$ in the simplex category Δ, the corresponding degeneracy map $A_{n-1} \to A_n$ is a closed cofibration.

Proposition 3.1.3. *:(Seg74, A.1.) Let $X_\cdot, Y_\cdot$ be simplicial topological spaces.*

1. *If each X_n has the homotopy type of a CW-complex, then so does $\|X_\cdot\|$.*
2. *If $X_\cdot \to Y_\cdot$ is a simplicial map such that $X_n \to Y_n$ is a weak homotopy equivalence for each n, then $\|X_\cdot\| \to \|Y_\cdot\|$ is also a weak homotopy equivalence.*
3. *$\|X_\cdot \times Y_\cdot\|$ is weakly homotopy equivalent to $\|X_\cdot\| \times \|Y_\cdot\|$.*

The advantage of the notion of fat geometric realization to the usual one is that the usual geometric realization of simplicial spaces does not satisfy the first two properties.

Let us recall the general process of defining the algebraic K-theory spectrum of a small *Waldhausen category* $(\mathcal{A}, w)$ (for more details, see(Wal85)).

Definition 3.1.4. Let $\mathcal{A}$ be a Waldhausen category. Define the simplicial category $S_\cdot\mathcal{A}$ as follows. First, consider the category of ordered pairs of integers (j,k) with $0 \le j \le k \le n$ that has a unique morphism from (j,k) to (j',k') iff $j \le j'$ and $k \le k'$. Then the objects in $S_n\mathcal{A}$ are the functors A from this category of pairs to the category $\mathcal{A}$, so that $A(j,j) = 0$ and $A(j,k) \rightarrowtail A(j,l) \twoheadrightarrow A(k,l)$ is a cofibration sequence in $\mathcal{A}$ whenever $0 \le j \le k \le l \le n$. The morphisms in $S_n\mathcal{A}$ are the natural transformations $A \to A'$, and the weak equivalences are the

morphisms that $A(j,k) \to A'(j,k)$ are all weak equivalences in $\mathcal{A}$. The cofibrations are the morphisms that $A(j,k) \to A'(j,k)$ are all cofibrations, and $A(j,l) \coprod_{A(j,k)} A'(j,k) \to A'(j,l)$ are also cofibrations in $\mathcal{A}$ whenever $0 \leq j \leq k \leq l \leq n$. Note that a morphism $f : [n] \to [m]$ in the opposite simplex category Δ^{op} induces a functor $S_n\mathcal{A} \to S_m\mathcal{A}$, which sends the object $(j,k) \mapsto A(j,k)$ of $S_n\mathcal{A}$ to the object $(r,s) \mapsto A(f(r), f(s))$ in $S_m\mathcal{A}$. This defines a simplicial structure on $S_\bullet\mathcal{A}$ [1].

Let $wS_\bullet\mathcal{A}$ be the simplicial category obtained by only considering the weak equivalences in $S_\bullet\mathcal{A}$, and form the nerves in each degree, which yeilds a bisimplicial set $N_\bullet wS_\bullet\mathcal{A}$. Define the *algebraic K-theory spectrum* $K^{alg}(\mathcal{A})$ of the discrete Waldhausen category $\mathcal{A}$ as the spectrum whose n'th space is the goemetric realization $|N_\bullet w \underbrace{S_\bullet S_\bullet \dots S_\bullet}_{\text{n } times} \mathcal{A}|$. (Or we could have defined the algebraic K-theory *space* to be the loop space $\Omega|N_\bullet wS_\bullet\mathcal{A}|$. In fact, they have the same (stable) homotopy groups, hence we may sometimes use the space instead of the spectrum.)

By (Mit01), we can define the *toplogical K-theory spectrum* $K^{top}(\mathcal{A})$ of the topological Waldhausen category $\mathcal{A}$ in a similar fashion, i.e. as a spectrum whose n'th space is the fat geometric realization $\|N_\bullet w \underbrace{S_\bullet S_\bullet \dots S_\bullet}_{\text{n } times} \mathcal{A}\|$.

Definition 3.1.5. Let $\mathcal{A}$ be a topological Waldhausen category. Let $\mathcal{A}^\delta$ denote the discrete Waldhausen category obtained by forgetting the topological structure. There is a natural exact functor $\mathcal{A}^\delta \to \mathcal{A}$ which induces a natural map $K^{top}(\mathcal{A}^\delta) \to K^{top}(\mathcal{A})$. By remark 3.1.2, there is a natural equivalence of K-theory spectra $K^{alg}(\mathcal{A}^\delta) \cong K^{top}(\mathcal{A}^\delta)$.

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[1] This description is taken from (TT90, 1.5.1.).

Hence there is a natural *comparison map* $c : K^{alg}(\mathcal{A}^{\delta}) \to K^{top}(\mathcal{A})$.

Notation 3.1.6. If $\mathcal{A}$ is a topological Waldhausen category, we will simply write $K^{alg}(\mathcal{A})$ instead of $K^{alg}(\mathcal{A}^{\delta})$.

Recall from (Wal85, 1.3.):

Definition 3.1.7. Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories. Then we say that a sequence $F_0 \to F_1 \to F_2$ of exact functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them is a *short exact sequence of functors*, if for each object A of $\mathcal{A}$ we have a cofibration sequence $F_0(A) \rightarrowtail F_1(A) \twoheadrightarrow F_2(A)$ in $\mathcal{B}$.

Theorem 3.1.8 (Additivity Theorem)**.** *(Qui73, Sec 3.) (Wal85, 1.3.2.) Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories, and let $F_0 \rightarrowtail F_1 \twoheadrightarrow F_2$ be a short exact sequence of functors from $\mathcal{A}$ to $\mathcal{B}$. Then $F_{1*}, F_{0*} + F_{2*} : K(\mathcal{A}) \to K(\mathcal{B})$ are homotopic to each other. By (Mit01, 4.2.), the same holds for the topological Waldhausen categories.*

Definition 3.1.9 (Relative K-theory Space)**.** (Wal85, 1.5.) Let $\mathcal{A}, \mathcal{B}$ be Waldhausen categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then define the category $[\mathcal{A} \xrightarrow{F} \mathcal{B}].$ by $[\mathcal{A} \xrightarrow{F} \mathcal{B}]_n = S_n\mathcal{A} \times_{S_n\mathcal{B}} S_{n+1}\mathcal{B}$. There is a natural simplicial structure on $[\mathcal{A} \xrightarrow{F} \mathcal{B}].$, and the Waldhausen category structures of $S.\mathcal{A}, S.\mathcal{B}$ induce one on $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$ in a natural way.

Proposition 3.1.10. *(Wal85, 1.5.5.) There are natural functors of Waldhausen categories $\mathcal{B} \to [\mathcal{A} \xrightarrow{F} \mathcal{B}]. \to S.\mathcal{A}$ which in turn induce the homotopy fibration sequence*

$$wS.\mathcal{B} \to wS.[\mathcal{A} \xrightarrow{F} \mathcal{B}]. \to wS.S.\mathcal{A}.$$

By (Mit01, 4.4.), the same holds for topological Waldhausen categories.

Definition 3.1.11. (Wal85, After 1.5.3.) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Waldhausen categories, then a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *biexact* if for each object A of $\mathcal{A}$ and B of $\mathcal{B}$, the functors $F(A, -)$ and $F(-, B)$ are exact, and also for each cofibration $A \rightarrowtail A'$ in $\mathcal{A}$ and $B \rightarrowtail B'$ in $\mathcal{B}$, the map below is a cofibration in $\mathcal{C}$.

$$F(A, B') \cup_{F(A,B)} F(A', B) \to F(A', B').$$

Proposition 3.1.12. *(Wal85, After 1.5.3.) A biexact functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ of Waldhausen categories, induces a map of bisimplicial categories $wS.\mathcal{A} \wedge wS.\mathcal{B} \to wwS.S.\mathcal{C}$ which in turn induces a map of K-theory spectra*

$$K(\mathcal{A}) \wedge K(\mathcal{B}) \to K(\mathcal{C}).$$

The same holds for the topological categories (Mit01, 2.8.).

Definition 3.1.13. We say that a (topological) Waldhausen subcategory $\mathcal{A}$ of $\mathcal{B}$ is *closed under extensions* if for each cofibration sequence in $\mathcal{B}$ where the source and the quotient are in $\mathcal{A}$, then the target is isomorphic to an object in $\mathcal{A}$.

Proposition 3.1.14. *(Wal85, 1.5.9.) If $\mathcal{A}$ is a strictly cofinal (topological) Waldhausen subcategory of $\mathcal{B}$, then the natural map $K(\mathcal{A}) \to K(\mathcal{B})$ is a homotopy equivalence. If $\mathcal{A}$ is only a cofinal (topological) Waldhausen subcategory of $\mathcal{B}$ which is also closed under extensions, then the*

natural map of K-theory spectra $K(\mathcal{A}) \to K(\mathcal{B})$ induces an isomorphism on the i'th homotopy group when $i \geq 1$.

The above statement was originally proved for discrete categories, however, in here we will need to apply it to a certain cofinal subcategory of the Paschke category. The proof goes through for topological categories with no change, but for the sake of completeness, we repeat the argument here.

Notation 3.1.15. Let $\mathcal{A}, \mathcal{B}$ denote topological Waldhausen categories and let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then we denote the space of objects in $S.\mathcal{A}$ by $s.\mathcal{A}$, and denote the space of objects in $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$ by $[s(\mathcal{A} \xrightarrow{F} \mathcal{B})]$.. Beware that the second notation is not standard.

Lemma 3.1.16. *(Wal85, 1.4.1.) Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of topological Waldhausen categories. Then there is an induced map $s.F : s.\mathcal{A} \to s.\mathcal{B}$. An isomorphism between two such functors F_0, F_1 induces a homotopy between $s.F_0, s.F_1$.*

Proof. The first statement is clear (cf. (Mit01, Page 6.)). To prove the second part, we will explicitly write down a simplicial homotopy. Simplicial objects in a category $\mathcal{C}$ can be considered as functors $X : \Delta^{op} \to \mathcal{C}$, and maps of simplicial objects are natural transformations of such functors. Simplicial homotopies can be described similarly; namely let $\Delta/[1]$ be denote the category of objects over [1] in the simplex category, i.e. objects are maps $[n] \to [1]$. For any $X : \Delta^{op} \to \mathcal{C}$, let X^* denote the composited functor

$$(\Delta/[1])^{op} \to \Delta^{op} \xrightarrow{X} \mathcal{C}$$

$$([n] \to [1]) \mapsto [n] \mapsto X[n]$$

Then a simplicial homotopy of maps may be identified with a natural transformation $X^* \to Y^*$.

Now, suppose there is a functor isomorphism from F_0 to F_1 given by $F : \mathcal{A} \times [1] \to \mathcal{B}$. The required simplicial homotopy then is a map from $([n] \to [1]) \mapsto s_n\mathcal{A}$ to $([n] \to [1]) \mapsto s_n\mathcal{B}$ given by

$$(a : [n] \to [1]) \mapsto ((A : Ar[n] \to \mathcal{A}) \mapsto (B : Ar[n] \to \mathcal{B}))$$

where B is defined as the composition

$$Ar[n] \xrightarrow{(A,a_*)} \mathcal{A} \times Ar[1] \xrightarrow{Id \times p} \mathcal{A} \times [1] \xrightarrow{F} \mathcal{B}$$

and $p : Ar[1] \to [1]$ is given by $(0,0) \mapsto 0$, $(0,1) \mapsto 1$, $(1,1) \mapsto 1$. □

Corollary 3.1.17. *(Wal85) An equivalence of Waldhausen topological categories $\mathcal{A} \to \mathcal{B}$ induces a homotopy equivalence $s.\mathcal{A} \to s.\mathcal{B}$. Therefore if weak equivalences of $\mathcal{A}$ are the isomorphisms, denoted by i, then $s.\mathcal{A} \to iS.\mathcal{A}$ is a homotopy equivalence.*

The first part of this corollary is clear consequence of the lemma. The second part is a result of considering the simplicial object $[m] \to i_m S.\mathcal{A}$, the nerve of $iS.\mathcal{A}$ in the i-direction, and noting that $i_0 S.\mathcal{A} = s.\mathcal{A}$ and that face and degeneracy maps are homotopy equivalences by the first part of the corollary.

Proof of Propositoin 3.1.14. To prove that a strictly cofinal topological Waldhausen subcategory $\mathcal{A}$ of $\mathcal{B}$ and $\mathcal{B}$ have homotopy equivalent K-theory spaces (and similarly spectra), it suffices

to show that the relative K-theory category $wS_\cdot[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot$ is contractible. By property 2 of fat geometric realization, it suffices to show that each $wS_n[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot$ is contractible. Consider the inclusions $S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}$. Then $wS_n[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot$ is equivalent to $w[S_n\mathcal{A} \hookrightarrow S_n\mathcal{B}]$. But it is easy to show that $S_n\mathcal{A}$ is a strictly cofinal subcategory of $S_n\mathcal{B}$: take an object $\{B_{jk}\}_{0\leq j<k\leq n}$ in $S_n\mathcal{B}$. Then for each B_{jk} in $\mathcal{B}$, there exists an object A_{jk} in $\mathcal{A}$ so that $B_{jk} \amalg A_{jk}$ is isomorphic to an object in $\mathcal{A}$. Hence if $A = \amalg_{j,k} A_{jk}$, then for all j, k, $B_{jk} \amalg (\amalg_{l=1}^{k-j} A)$ is isomorphic to an object in $\mathcal{A}$, as $\mathcal{A}$ is closed under finite coproduct. Therefore $\{B_{jk} \amalg (\amalg_{l=1}^{k-j} A)\}_{0\leq j<k\leq n}$ is isomorphic to an object in $S_n\mathcal{A}$.

To show $w[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot$ is contractible, again, by property 2 of fat geometric realization, it suffices to show that $w_m[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot$ (the m-degree part in the w-direction) is contractible for all m. Let $\mathcal{A}(m, w)$ denote a sequence $A_0 \xrightarrow{\simeq} A_1 \xrightarrow{\simeq} \ldots \xrightarrow{\simeq} A_m$ of m-weak equivalences in $\mathcal{A}$, and similarly define $\mathcal{B}(m, w)$, and consider the inclusion $\mathcal{A}(m, w) \hookrightarrow \mathcal{B}(m, w)$. Similar to before, $\mathcal{A}(m, w)$ is strictly cofinal in $\mathcal{B}(m, w)$. It is easy to see that $w_m[\mathcal{A} \hookrightarrow \mathcal{B}]_\cdot \simeq [s(\mathcal{A}(m, w) \hookrightarrow \mathcal{B}(m, w))]_\cdot$. Hence it suffices to prove that when $\mathcal{A} \hookrightarrow \mathcal{B}$ is an inclusion of a strictly cofinal topological Waldhausen subcategory, then $[s(\mathcal{A} \hookrightarrow \mathcal{B})]_\cdot$ is contractible (cf. 3.1.15).

First note that the simplicial space $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]_\cdot$ is the nerve of the (topological) category of cofibrations in $\mathcal{A}$ which has an initial object, and hence is contractible[1]. Now we want to

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[1]The point and the 1-simplex [1] are both *good* simplicial spaces (Seg74, A.4.), hence their fat geometric realizations are homotopy equivalent to their (usual) geometric realizations (Seg74, A.5.) which are both contractible. Hence, by property 3 of the fat geometric realization, the argument in (Seg68, 2.1.) goes through to show that a natural transformation between two functors between topological categories induces a homotopy between the induced maps on the fat geometric realizations. If the topological category $\mathcal{C}$ has an initial object, then there is an induced homotopy between the fat geometric realization of the nerve of the category $\mathcal{C}$, and the fat geometric realization of a point, which is contractible.

show that the inclusion $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]. \to [s(\mathcal{A} \hookrightarrow \mathcal{B})].$ is a homotopy equivalence. Consider the category of cofibrations in $\mathcal{B}$. Then $[s(\mathcal{A} \hookrightarrow \mathcal{B})].$ is homotopic to a simplicial subset of the nerve of this category (through forgetting the choices of quotients $B_{jk} \simeq B_{0k}/B_{0j}$), and taking a pushout with a fixed object is a natural transformation of the identity functor on this category to the pushout functor. In other words, there is a homotopy from the identity functor on $[s(\mathcal{A} \hookrightarrow \mathcal{B})].$ to the functor κ_A, where $\kappa_A(A_{jk}, B_{jk}) = (A_{jk}, B'_{jk})$, where $B'_{jk} = B_{jk} \amalg A$ when $j = 0$ and $B'_{jk} = B_{jk}$ otherwise. But for any simplicial set L in $[s(\mathcal{A} \hookrightarrow \mathcal{B})].$ with *finitely* many non-degenerate simplicies, as we argued before there exists an object A in $\mathcal{A}$ so that κ_A applied to L, would send each of the non-degenarate simplicies to simplicies (weakly equivalent to simplicies) in $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]..$ But then κ_A sends all of L to simplicies (weakly equivalent to simplicies) in $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]..$ Therefore there is a homotopy from the inclusion of L in $[s(\mathcal{A} \hookrightarrow \mathcal{B})].$ to a map from L to $[s(\mathcal{A} \xrightarrow{Id} \mathcal{A})]..$

The proof for when $\mathcal{A}$ in $\mathcal{B}$ is only cofinal, goes through similarly. To show that the connected component of the zero object in $[s(\mathcal{A} \hookrightarrow \mathcal{B})].$ is contractible, one needs to use the assumption that $\mathcal{A}$ is closed under extensions, which in turn shows that the object A (that was obtained by using the cofinality assumption applied to the objects B_{jk}) used in the paragraph above, is in fact isomorphic to an object of $\mathcal{A}$.

□

Remark 3.1.18. We will only use cofinality in the case when there exists an object A_0 of $\mathcal{A}$ so that for each object B of $\mathcal{B}$, $A_0 \oplus B$ is isomorphic to an object in $\mathcal{A}$. The proof of the lemma above for this special case is slightly simpler and easier to understand.

3.2 Grayson's Map

We start this section by going through an unfortunately rather long list of notations and definitions, and then we will use a construction of Grayson to give a natural map in the homotopy category of spectra[1], from the K-theory spectrum of the category of acyclic *binary* chain complexes in an exact category $\mathcal{A}$, to the loop space of the K-theory spectrum of $\mathcal{A}$. We will closely follow the construction to see the extent to which it can be applied to topological exact categories.

Notation 3.2.1. Let $X_\cdot$ be a spectrum. For $n \in \mathbb{Z}^{\geq 0}$, let $V^n X_\cdot$ denote the n'th stage of the postnikov filtration of $X_\cdot$, obtained by killing the stable homotopy groups $\pi_m(X_\cdot)$ for $m < n$. In particular, $V^0 X_\cdot$ is the connective part of $X_\cdot$.

Following (Wal85, Sec 1.6.), let $(\mathcal{A}, w)$ be a (topological) Waldhausen category with the subcategory $w\mathcal{A}$ (sometimes abbreviated to just w) of weak equivalences. If $(\mathcal{A}, v)$ is also a (topological) Waldhausen category with weak equivalences so that w is a subcategory of v, then let $\mathcal{A}^v$ denote the full (topological) Waldhausen subcategory of $(\mathcal{A}, w)$ whose objects A are the ones with the property that $* \to A$ is in v. Recall that for a (topological) category $\mathcal{A}$ with cofibrations, if $i\mathcal{A}$ denotes the subcategory of isomorphisms, then $(\mathcal{A}, i)$ is a (topological) Waldhausen category.

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[1]The *stable homotopy category* (cf. (Ada95)) can be considered as the localization of the category of spectra at the weak homotopy equivalences. In particular, all homotopic maps are equivalent to each other in the homotopy category, and homotopy equivalences are invertible.

For a (topological) exact category $\mathcal{A}$, let $C\mathcal{A}$ denote the category of *chain complexes* in $\mathcal{A}$ and let $Ac\mathcal{A}$ be the category of *acyclic chain complexes* in $\mathcal{A}$, both of which have chain maps as morphisms. The categories $C\mathcal{A}, Ac\mathcal{A}$ have a natural (topological) exact structure; a sequence is called exact iff it is exact degreewise. This means the cofibrations are the morphisms which are degree-wise cofibrations (admissible monomorphisms) and the weak equivalences i are the degree-wise *isomorphisms*[1]. We introduce a different structure of a (topological) Waldhausen category on $C\mathcal{A}$ by defining the cofibrations to be the degree-wise cofibrations again, and define the weak equivalences to be the *quasi-isomorphisms*, which we denote by q. Note that quasi-isomorphisms are considered with respect to embedding the exact category $\mathcal{A}$ into an abelian category. This definition does not depend on the choice of the embedding if $\mathcal{A}$ either *supports long exact sequences* (Gra12, 1.4.), or if $\mathcal{A}$ satisfies the condition in (TT90, 1.11.3.). These conditions are both satisfied if $\mathcal{A}$ is a (topological) pseudo-abelian category, cf (TT90, 1.11.5.) and (Gra12, 4.). Evidently, $(C\mathcal{A}, q)$ is a (topological) Waldhausen category, and $Ac(\mathcal{A})$ is equal to $(C\mathcal{A})^q$. Furthermore, denote the full subcategories of *bounded chain complexes* and *bounded acyclic chain complexes* by $C^b\mathcal{A}$ and $Ac^b\mathcal{A}$ respectively. Again we have $(C^b\mathcal{A})^q = Ac^b\mathcal{A}$.

Definition 3.2.2. (Gra12, 3.1.) Let $\mathcal{A}$ be a (topological) exact category. We define a *binary chain complex* in $\mathcal{A}$ to be a chain complex in $\mathcal{A}$ with *two* differentials instead of one, i.e. a pair of chain complexes with the same objects but possibly different differentials, called the *top differential* of the top chain complex, and the *bottom differential* of the bottom chain complex.

---

[1]We will abuse notation and denote the class of isomorphisms of different categories by i.

A binary chain complex is *acyclic* if both the top and the bottom chain complexes are acyclic. If we denote a binary chain complex by $(A^{\cdot}, d_1^{\cdot}, d_2^{\cdot})$, then $A^{\cdot}$ are the objects of the complex, $d_1^{\cdot}$ are the top differentials, and $d_2^{\cdot}$ are the bottom differentials. Let $B\mathcal{A}$ and $Bi\mathcal{A}$ be the (topological) category of binary chain complexes in $\mathcal{A}$ and acyclic binary chain complexes in $\mathcal{A}$. Also denote the (topological) category of *bounded binary chain complexes* and the category of *bounded acyclic binary chain complexes* in $\mathcal{A}$ by $B^b\mathcal{A}$ and $Bi^b\mathcal{A}$, respectively. A morphism between two (respectively, acyclic) binary chain complexes is a map between the underlying objects which is a chain map with respect to both chain complexes; in other words, a chain map when we consider only the top chain complexes, and also a chain map when we consider only the bottom chain complexes.

Similar to before, the categories $B\mathcal{A}, Bi\mathcal{A}, B^b\mathcal{A}, Bi^b\mathcal{A}$ have a natural (topological) exact structure given by exactness at each degree. This means the cofibrations are degree-wise cofibrations, and weak equivalences i are the degree-wise isomorphisms. We can define another structure of a (topological) Waldhausen category on $B\mathcal{A}, B^b\mathcal{A}$ with the cofibrations being the degree-wise cofibrations and the weak equivalences being the *quasi-isomorphisms* which we again denote by q. This again may depend on the choice of embedding $\mathcal{A}$ in an abelian category, but does not depend on that choice if $\mathcal{A}$ is a (topological) pseudo-abelian category. Hence we have a (topological) Waldhausen category $(B\mathcal{A}, q)$, and again $(B\mathcal{A})^q, (B^b\mathcal{A})^q$ are the categories $Bi\mathcal{A}, Bi^b\mathcal{A}$, respectively.

Let us denote the morphism that sends a chain complex $(A^{\cdot}, d^{\cdot})$ to the binary chain complex $(A^{\cdot}, d^{\cdot}, d^{\cdot})$ by $\Delta : C\mathcal{A} \to B\mathcal{A}$, and denote the morphisms that send a binary chain complex to

respectively the top and the bottom chain complex by $\top, \perp : B\mathcal{A} \to C\mathcal{A}$. These are exact functors, and we use the same notation for their restriction to $C^b\mathcal{A} \to B^b\mathcal{A}$, $Ac\mathcal{A} \to Bi\mathcal{A}$, $B^b\mathcal{A} \to C^b\mathcal{A}$ and $Bi^b\mathcal{A} \to Ac^b\mathcal{A}$. Let τ, τ^b denote the category of maps f in $B\mathcal{A}$ and $B^b\mathcal{A}$ respectively, such that $\top f$ is in $qC\mathcal{A}, qC^b\mathcal{A}$, respectively, and let β, β^b denote the category of maps f in $B\mathcal{A}$ and $B^b\mathcal{A}$, such that $\perp f$ is in $qC\mathcal{A}, qC^b\mathcal{A}$, respectively. Define $F : (C\mathcal{A}, q) \to (B\mathcal{A}, \tau)$ by $F(A^{\cdot}, d^{\cdot}) = (A^{\cdot}, d^{\cdot}, 0)$. Then the composition $\top \circ F$ is the identity functor on $C\mathcal{A}$, and $F \circ \top$ is an exact endofunctor of $(B\mathcal{A}, \tau)$.

Recall from definition 3.1.9 that for an exact functor $F : \mathcal{A} \to \mathcal{B}$ between (topological) Waldhausen categories, we have the relative K-theory space denoted by $[\mathcal{A} \xrightarrow{F} \mathcal{B}]$. We have the following proposition from Grayson (Gra12, Sec 7.):

Proposition 3.2.3. *Let $\mathcal{A}$ be a discrete exact category. Then there is a natural homotopy equivalence of spectra*

$$K[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \simeq V^0\Omega K(\mathcal{A})$$

In particular, there is a natural isomorphism of K-theory groups $K_{n-1}[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \cong K_n(\mathcal{A})$ when $n \geq 1$, and there is a natural map in the homotopy category of spectra

$$\tau^G_{\mathcal{A}} : K(Bi^b\mathcal{A}, i) \to \Omega K(\mathcal{A}). \tag{3.1}$$

The proposition above uses ingredients such as Waldhausen's fibration and approximation theorems (Wal85, 1.6.4, 1.6.7.), the Gillet-Waldhausen theorem (Gil81, 6.2.), and Thomason's cofinality theorem (TT90, 1.10.1.).

Let $\mathcal{A}$ be (a topological) an exact category. Recall that by definition of the relative K-theory space, there is an exact functor $Bi^b(\mathcal{A}) \to [Ac^b\mathcal{A} \xrightarrow{\Delta} Bi^b\mathcal{A}]$ for a (topological) Waldhausen category $\mathcal{A}$. Now, assuming that the (topological) category $\mathcal{A}$ "supports long exact sequences", we give a series of maps in the homotopy category of spectra as follows. Following the proof of (Gra12, 4.3.), we first give a map in the homotopy category of spectra

$$G_1 : K[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \to \Omega K[(C\mathcal{A}, i) \xrightarrow{\Delta} (B\mathcal{A}, i)].$$

We have the following commutative diagrams:

$$\begin{array}{ccc} K(Ac^b\mathcal{A}, i) & \longrightarrow & K(C^b\mathcal{A}, i) \\ \downarrow & & \downarrow \\ K(Ac^b\mathcal{A}, q) & \longrightarrow & K(C^b\mathcal{A}, q) \end{array} \qquad \begin{array}{ccc} K(Bi^b\mathcal{A}, i) & \longrightarrow & K(B^b\mathcal{A}, i) \\ \downarrow & & \downarrow \\ K(Bi^b\mathcal{A}, q) & \longrightarrow & K(B^b\mathcal{A}, q) \end{array}$$

By Waldhausen's fibration theorem (Wal85, 1.6.4.), the squares above are cartesian when the category $\mathcal{A}$ is discrete. Therefore we get the cartesian square below.

$$\begin{array}{ccc} K[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] & \longrightarrow & K[(C^b\mathcal{A}, i) \xrightarrow{\Delta} (B^b\mathcal{A}, i)] \\ \downarrow & & \downarrow \\ K[(Ac^b\mathcal{A}, q) \xrightarrow{\Delta} (Bi^b\mathcal{A}, q)] & \longrightarrow & K[(C^b\mathcal{A}, q) \xrightarrow{\Delta} (B^b\mathcal{A}, q)] \end{array}$$

In the case of topological exact categories, the square above is still commutative (but not necessarily cartesian). Also the argument below works for topological exact categories as well.

The lower left hand corner of the diagram is contractible as each of the two categories in the relative K-theory space are contractible. The map $K(Ac^b\mathcal{A}, i) \xrightarrow{K\Delta} K(Bi^b\mathcal{A}, i)$ is a homotopy equivalence, because the functor $P^j : (C^b\mathcal{A}, i) \to \mathcal{A}$ which sends a chain complex to the term in degree j is exact, and by induction and the additivity theorem, induces an isomorphism

$K(C^b\mathcal{A}, i) \cong K(\coprod_{\mathbb{Z}} \mathcal{A})$ (cf. (Gil81, 6.2.)). Similarly we can say the same for $K(B^b\mathcal{A}, i)$, and note that Δ commutes with these isomorphisms and the identity map on $K(\coprod_{\mathbb{Z}} \mathcal{A})$. Thus the upper right hand corner of the diagram above is also contractible. Therefore we have the following sequence of natural maps:

$$K[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \cong K\left[[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \to 0\right] \xleftarrow{\sim}$$
$$K\left[[(Ac^b\mathcal{A}, i) \xrightarrow{\Delta} (Bi^b\mathcal{A}, i)] \to [(Ac^b\mathcal{A}, q) \xrightarrow{\Delta} (Bi^b\mathcal{A}, q)]\right] \xrightarrow{*}$$
$$\Omega K\left[[(C^b\mathcal{A}, i) \xrightarrow{\Delta} (B^b\mathcal{A}, i)] \to [(C^b\mathcal{A}, q) \xrightarrow{\Delta} (B^b\mathcal{A}, q)]\right] \xleftarrow{\sim} \Omega K[(C^b\mathcal{A}, q) \xrightarrow{\Delta} (B^b\mathcal{A}, q)].$$

When $\mathcal{A}$ is a discrete category, all of the maps above are homotopy equivalences, hence G_1 is a homotopy equivalence. Note that if the fibration theorem holds for topological exact categories, then $*$ (and therefore G_1) is a homotopy equivalence for topological categories as well.

The next step is to define the homotopy equivalence

$$G_2 : \Omega K[(B^b\mathcal{A}, q) \xrightarrow{\top} (C^b\mathcal{A}, q)] \to K[(C^b\mathcal{A}, q) \xrightarrow{\Delta} (B^b\mathcal{A}, q)].$$

This map is induced by the commutative diagram of (Gra12, 4.5.). To be more precise, G_2 is the composition of the following sequence of maps:

$$\Omega K\left[(B^b\mathcal{A}, q) \xrightarrow{\top} (C^b\mathcal{A}, q)\right] \cong \Omega K\begin{bmatrix} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ (B^b\mathcal{A}, q) & \xrightarrow{\top} & (C^b\mathcal{A}, q) \end{bmatrix} \xrightarrow{\sim}$$
$$\Omega K\begin{bmatrix} (C^b\mathcal{A}, q) & \xrightarrow{1} & (C^b\mathcal{A}, q) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle 1} \\ (B^b\mathcal{A}, q) & \xrightarrow{\top} & (C^b\mathcal{A}, q) \end{bmatrix} \xrightarrow{\sim} \Omega K\begin{bmatrix} (C^b\mathcal{A}, q) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Delta} & & \downarrow \\ (B^b\mathcal{A}, q) & \longrightarrow & 0 \end{bmatrix} \cong K\left[(C^b\mathcal{A}, q) \xrightarrow{\Delta} (B^b\mathcal{A}, q)\right].$$

Where we used the fact that $\top \circ \Delta = 1$ and K-theory of squares is a generalization of relative K-theory which was defined in (Gra92, Sec 4.)[1].

For the next step (Gra12, 4.9.), Grayson defines a map $K\left((B^b\mathcal{A})^\tau, q\right) \to \Omega K[(B^b\mathcal{A}, q) \xrightarrow{\top} (C^b\mathcal{A}, q)]$, which is a homotopy equivalence in the case of discrete categories. Instead we define the homotopy equivalence below for the (topological) exact category $\mathcal{A}$.

$$G_3 : K[(B^b\mathcal{A}, q) \xrightarrow{\top} (C^b\mathcal{A}, q)] \to K[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)].$$

Notice that we have the following commutative diagram, where each row is a cofiber sequence[2], and $F : (C^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)$ is defined by $F(A^{\cdot}, d^{\cdot}) = (A^{\cdot}, d^{\cdot}, 0)$.

$$\begin{array}{ccccc} K(B^b\mathcal{A}, q) & \xrightarrow{K\top} & K(C^b\mathcal{A}, q) & \longrightarrow & K[(B^b\mathcal{A}, q) \xrightarrow{\top} (C^b\mathcal{A}, q)] \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle KF} & & \downarrow{\scriptstyle \exists G_3} \\ K(B^b\mathcal{A}, q) & \longrightarrow & K(B^b\mathcal{A}, \tau) & \longrightarrow & K\left[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)\right] \end{array}$$

This induces the desired map G_3. Similar to (Gra12, 4.8.), we can argue that for the (topological) exact category $\mathcal{A}$, the maps $KF, K\top$ are inverses to each other up to homotopy. (The argument relies on the fact that *weak equivalence* between two functors induces a homotopy between the corresponding maps of K-theory (Wal85, 1.3.1.), which also holds for topological categories; see (Seg68, 2.1.).) Hence KF is a homotopy equivalence for the (topological) exact category $\mathcal{A}$, which means that G_3 is also a homotopy equivalence.

---

[1]The proofs only rely on the additivity theorem, which holds for topological categories.

[2]Recall that fibration and cofiber sequences are the same in the category of spectra.

Let $(C^b\mathcal{A})^x$ be the subcategory of chain complexes $(A^{\cdot}, d^{\cdot})$ in $C^b\mathcal{A}$, whose euler characteristic $\chi(A^{\cdot}) = \sum_n (-1)^n A^n$ is equal to zero. When $\mathcal{A}$ is a discrete category, according to (Gra12, 5.8.), as a corollary of Thomason's cofinality theorem (TT90, 1.10.1.), we have a homotopy equivalence $K\left((C^b\mathcal{A})^x, q\right) \xrightarrow{\sim} V^1K(C^b\mathcal{A}, q)$. Then for the discrete exact category $\mathcal{A}$, one has the following sequence of homotopy equivalences:

$$\begin{aligned} \Omega K\left((B^b\mathcal{A})^\tau, q\right) \cong \Omega K\left((B^b\mathcal{A})^\beta, q\right) = \Omega K\left((B^b\mathcal{A})^\beta, \tau\right) \xrightarrow{\sim} \Omega K\left((C^b\mathcal{A})^x, q\right) \\ \xrightarrow{\sim} \Omega V^1K(C^b\mathcal{A}, q) \xleftarrow{\sim} \Omega V^1K(\mathcal{A}) \cong V^0\Omega K(\mathcal{A}) \end{aligned} \tag{3.2}$$

where the first homotopy equivalence is given by interchanging the top and the bottom differentials; the second is done by observing that $((B^b\mathcal{A})^\beta, q) = ((B^b\mathcal{A})^\beta, \tau)$; the third map is induced by the functor $\top : ((B^b\mathcal{A})^\beta, \tau) \to ((C^b\mathcal{A})^x, q)$ (note that this is well-defined since $(A^{\cdot}, d_1^{\cdot}, d_2^{\cdot})$ in $((B^b\mathcal{A})^\beta, \tau)$ is sent to $(A^{\cdot}, d_1^{\cdot})$, but acyclicity of $(A^{\cdot}, d_2^{\cdot})$ shows that the euler characteristic is zero.), which by (Gra12, 5.9.) is a homotopy equivalence for discrete categories[1]; the fourth one by (Gra12, 5.8.), is a corollary of Thomason's cofinality theorem (TT90, 1.10.1.); the last one is true for any spectrum; and finally, the fifth map is induced by the inclusion $\mathcal{A} \to C^b\mathcal{A}$ as the chain complex concentrated in degree zero, which is a homotopy equivalence is by (Gil81, 6.2.) (Also see (TT90, 1.11.7.))[2]. However, since the map is going in the opposite direction, the homotopy equivalence does not necessarily induce a map for topological exact categories.

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[1]The proof relies on Waldhausen's approximation theorem (Wal85, 1.6.7.) whose proof is quite long!

[2]We need the extra assumption (TT90, 1.11.3.) for this theorem to hold, however we *are* assuming that $\mathcal{A}$ "supports long exact sequences", which ensures that there is no problem.

Lemma 3.2.4. *Let $\mathcal{A}$ be a topological exact category, that "supports long exact sequences", and assume that $K(\mathcal{A}) \to K(C^b\mathcal{A}, q)$ is a homotopy equivalence, where the map is induced by inclusion as the chain complex concentrated in degree zero. Then the sequence of maps in Equation 3.2, composed with the natural map $V^0\Omega K(\mathcal{A}) \to \Omega K(\mathcal{A})$ (given by definition of Postnikov tower) factors through $K[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)]$.*

Before proving the lemma above, let us summarize (Gra12, Sec 6.) on what happens when the (topological) exact category $\mathcal{A}$ does not "support long exact sequences".

The pseudo-abelianization (cf. Proposition 2.1.15) $\tilde{\mathcal{A}}$ of the (topological) exact category $\mathcal{A}$ inherits (both the topological and) the exact structure of $\mathcal{A}$. Also $\mathcal{A}$ embeds in $\tilde{\mathcal{A}}$ as a cofinal subcategory. Hence $K_n(\mathcal{A}) \to K_n(\tilde{\mathcal{A}})$ is an isomorphism when $n > 0$ and is injective for $n = 0$. Similar to (Gra12, 6.3.), the induced inclusions $Ac^b(\mathcal{A}) \hookrightarrow Ac^b(\tilde{\mathcal{A}})$ and $Bi^b(\mathcal{A}) \hookrightarrow Bi^b(\tilde{\mathcal{A}})$ are also cofinal, and by repeating the argument in (Gra12, 6.2.), the natural map from the cofiber of $K(Ac^b(\mathcal{A})) \to K(Bi^b(\mathcal{A}))$ to the cofiber of $K(Ac^b(\tilde{\mathcal{A}})) \to K(Bi^b(\tilde{\mathcal{A}}))$ is a homotopy equivalence.[1] Again by cofinality, $V^0\Omega K(\mathcal{A}) \to V^0\Omega K(\tilde{\mathcal{A}})$ is a homotopy equivalence.

Since $\tilde{\mathcal{A}}$ is pseudo-abelian hence as explained before, $\tilde{\mathcal{A}}$ "supports exact sequences", and when $\mathcal{A}$ is a discrete category, then there is an induced natural map in the homotopy category of spectra

$$\tau^G_{\mathcal{A}}: \quad K(Bi^b(\mathcal{A})) \to K(Bi^b(\tilde{\mathcal{A}})) \xrightarrow{\tau^G_{\tilde{\mathcal{A}}}} V^0\Omega K(\tilde{\mathcal{A}}) \xleftarrow{\sim} V^0\Omega K(\mathcal{A}).$$

---

[1] The reason why the n'th homotopy groups are isomorphic follows from cofinality when $n > 0$. But for $n = 0$ an extra argument is needed.

Proof of Lemma. Let $(A^{\cdot}, d_1^{\cdot}, d_2^{\cdot})$ be an object of $(B^b\mathcal{A}, q)^\tau$. By definition, the top chain complex $(A^{\cdot}, d_1^{\cdot})$ is acyclic. This goes to $(A^{\cdot}, d_2^{\cdot}, d_1^{\cdot})$ through the first map in Equation 3.2, and the second map is the identity. The third map sends it to the top chain complex $(A^{\cdot}, d_2^{\cdot})$ in $((C^b\mathcal{A})^x, q)$. The composition

$$\Omega K\left((C^b\mathcal{A})^x, q\right) \sim \Omega V^1 K(C^b\mathcal{A}, q) \sim \Omega V^1 K(\mathcal{A}) \sim V^0 \Omega K(\mathcal{A}) \to \Omega K(\mathcal{A})$$

is equal to the composition

$$G_5 : \quad \Omega K\left((C^b\mathcal{A})^x, q\right) \to \Omega K(C^b\mathcal{A}, q) \sim \Omega K(\mathcal{A}),$$

where the first map is induced by inclusion of categories, and the second is given by the hypothesis of the lemma.

The natural map $\Omega K((B^b\mathcal{A})^\tau, q) \to K[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)]$ is induced by inclusion. This sends the object $(A_{jk}^{\cdot}, d_{1,jk}^{\cdot}, d_{2,jk}^{\cdot})_{0\le j\le k\le n}$ of $S_n(B^b\mathcal{A})^\tau$ to the pair

$$\left((A_{jk}^{\cdot}, d_{1,jk}^{\cdot}, d_{2,jk}^{\cdot})_{0\le j\le k\le n}, (0)_{0\le j\le k\le n+1}\right).$$

Now, define the map $G_4 : K[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)] \to \Omega K(C^b\mathcal{A}, q)$ by

$$G_4\left((A_{jk}^{\cdot}, d_{1,jk}^{\cdot}, d_{2,jk}^{\cdot})_{0\le j\le k\le n}, (A', d_1', d_2')_{n+1}\right) = (A_{jk}^{\cdot}, d_{2,jk}^{\cdot})_{0\le j\le k\le n},$$

where $(A', d'_1, d'_2)_{n+1}$ is an object of $S_{n+1}(B^b\mathcal{A}, \tau)$, the first term is an object of $\big[(B^b\mathcal{A}, q) \to (B^b\mathcal{A}, \tau)\big]$ and the second term is an object of $S_n(C^b\mathcal{A}, q)$. Then use the natural homotopy equivalence $\|wS.S.\mathcal{E}\| \cong \Omega\|wS.\mathcal{E}\|$ for the topological Waldhausen category $\mathcal{E} = (C^b\mathcal{A}, q)$.

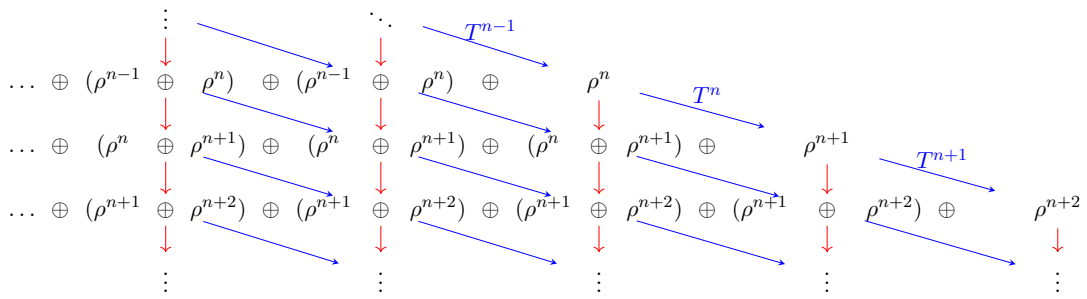
□

3.3 Higson's Functor

Generalizing a construction given by Higson in (Hig95, Page 6.), for a C^*-algebra A we define the functor $\tau_A^H : C(\mathfrak{D}/\mathfrak{C})_A \to B(\mathfrak{D}/\mathfrak{C})_A$ below.

Definition 3.3.1. Let $(\rho^{\cdot}, T^{\cdot})$ be a chain complex in $C(\mathfrak{D}/\mathfrak{C})_A$. Define $\tau_A^H(\rho^{\cdot}, T^{\cdot})$ to be the binary chain complex whose n'th term is the graded object $\nu^n = \big(\oplus_{i=-\infty}^{n-1}(\rho^{n-1} \oplus \rho^n)\big) \oplus \rho^n$ in $(\mathfrak{D}/\mathfrak{C})_A$, where the index i in the infinite direct sum of $\rho^{n-1} \oplus \rho^n$ refers to the degree of the term, and the separate copy of ρ^n is of degree n. The top differential (temporarily denoted by) $\top^n$ from ν^n to ν^{n+1} is a degree 1 map, where its i'th degree piece from $\rho^{n-1} \oplus \rho^n$ (of degree i) to $\rho^n \oplus \rho^{n+1}$ (of degree $i+1$) is the trivial one (i.e. is identity on ρ^n and zero on ρ^{n-1}.) for $i \leq n-1$, and its n'th degree piece is equal to $\rho^n \xrightarrow{T^n} \rho^{n+1}$. The bottom differential (temporarily denoted by) $\perp^n$ from ν^n to ν^{n+1} is a degree 0 map, where its i'th degree piece from $\rho^{n-1} \oplus \rho^n$ to $\rho^n \oplus \rho^{n+1}$ is again the trivial one (i.e. is identity on ρ^n and zero on ρ^{n-1}.) for $i \leq n-1$, and its n'th degree piece is the trivial inclusion $\rho^n \xrightarrow{(Id,0)} \rho^n \oplus \rho^{n+1}$.

The rows of the diagram below display the terms of the binary chain complex $\tau_A^H(\rho^{\cdot}, T^{\cdot})$, the diagonal arrows are the top differentials, and the vertical arrows are the bottom differentials.



It is easy to see that the bottom chain complex is split exact, and the top chain complex is exact iff the original chain complex $(\rho^{\cdot}, T^{\cdot})$ is exact. This process is functorial with respect to chain maps in a trivial way. Finally, note that if we start with a chain complex of length n, then we will get a binary chain complex of length $n+1$. Hence we also have a natural functor $\tau_A^H : Ac^b(\mathfrak{D}/\mathfrak{C})_A \to Bi^b(\mathfrak{D}/\mathfrak{C})_A$. This functor is not exact; however we can tweak the structures of the categories to obtain an exact functor.

Definition 3.3.2. Let A be a C^*-algebra and let $Ac'(\mathfrak{D}/\mathfrak{C})_A, Bi'(\mathfrak{D}/\mathfrak{C})_A$ denote the categories with the same objects as $Ac^b(\mathfrak{D}/\mathfrak{C})_A, Bi^b(\mathfrak{D}/\mathfrak{C})_A$ respectively, but with morphisms and exact structure coming from the category $\mathfrak{D}_A$. To be precise, a morphism in $Ac'(\mathfrak{D}/\mathfrak{C})_A$ from the chain complex $(\rho^{\cdot}, T^{\cdot})$ to $(\nu^{\cdot}, S^{\cdot})$ is given by a chain map $f^n : \rho^n \to \nu^n$ in the category $\mathfrak{D}_A$, and morphisms in $Bi'(\mathfrak{D}/\mathfrak{C})_A$ are defined similarly as a chain map in $\mathfrak{D}_A$ with respect to both the top and the bottom chain complex. We say a sequence of chain complexes in $Ac'(\mathfrak{D}/\mathfrak{C})_A$ is exact, iff the sequence is exact at each degree in $\mathfrak{D}_A$, and similarly define the exact structure on $Bi'(\mathfrak{D}/\mathfrak{C})_A$.

There are natural functors $Ac'(\mathfrak{D}/\mathfrak{C})_A \to Ac^b(\mathfrak{D}/\mathfrak{C})_A$ and $Bi'(\mathfrak{D}/\mathfrak{C})_A \to Bi^b(\mathfrak{D}/\mathfrak{C})_A$. These functors are exact, since exactness in $\mathfrak{D}_A$ guarantees exactness in $(\mathfrak{D}/\mathfrak{C})_A$.

Lemma 3.3.3. *The functor τ_A^H defined in 3.3.1 induces an exact functor*

$$\tau_A^H : Ac'(\mathfrak{D}/\mathfrak{C})_A \to Bi'(\mathfrak{D}/\mathfrak{C})_A. \tag{3.3}$$

Therefore, we have a natural map of K-theory spectra

$$\tau_A^H : K(Ac'(\mathfrak{D}/\mathfrak{C})_A) \to K(Bi'(\mathfrak{D}/\mathfrak{C})_A).$$

This is proved by observing that infinite direct sum of identities is equal to identity in $\mathfrak{D}_A$. Note that this is *not* true in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$.

Proof. Let $(\rho_i^{\cdot}, T_i^{\cdot})$ denote objects in $Ac'(\mathfrak{D}/\mathfrak{C})_A$ for $i \in \mathbb{Z}$, and let $f_i^{\cdot} : (\rho_i^{\cdot}, T_i^{\cdot}) \to (\rho_{i+1}^{\cdot}, T_{i+1}^{\cdot})$ be morphisms that give an exact sequence in $Ac'(\mathfrak{D}/\mathfrak{C})_A$, with the degree-wise contracting homotopy given by $g_i^{\cdot} : (\rho_{i+1}^{\cdot}, T_{i+1}^{\cdot}) \to (\rho_i^{\cdot}, T_i^{\cdot})$. Then we need to show that $\tau_A^H(g_i^{\cdot})\tau_A^H(f_i^{\cdot}) + \tau_A^H(f_{i-1}^{\cdot})\tau_A^H(g_{i-1}^{\cdot})$ is equal to identity at each degree of $\tau_A^H(\rho_i^{\cdot}, T_i^{\cdot})$. This is true since at degree n this is given by infinite direct sum $g_i^n f_i^n + f_{i-1}^n g_{i-1}^n$ and $g_i^{n-1} f_i^{n-1} + f_{i-1}^{n-1} g_{i-1}^{n-1}$. But by assumption, each term is equal to identity in $\mathfrak{D}_A$, hence their infinite direct sum is also equal to identity. □

None of this process works in the Calkin-Paschke category $(\mathfrak{D}/\mathfrak{C})'_A$, as infinite direct sums of $\rho' : A \to (\mathfrak{B}/\mathfrak{K})(H)$ is not necessarily defined, since infinite direct sum of compact operators

does not have to be compact. In fact if the infinite direct sum of ρ' is well-defined, then by an *Eilenberg swindle* argument we can show that the class corresponding to ρ' in $K_1^{top}(A) = Ext(A)$ defined in (BDF77) is zero.

CHAPTER 4

COMPLEX MANIFOLDS AND THE DOLBEAULT COMPLEX

This section will contain a great deal of computations, and to ease the readability, we will fix some of our notations.

Notation 4.0.1. Fix $\chi(t) = \frac{t}{\sqrt{1+t^2}}$. The functions χ, ϕ will be used for functional calculus. The letter X denotes manifolds, U, V are used for open subsets of the manifold, λ for a partition of unity, and γ for cutoff functions. The letters D, d will be used for differential operators, and $\bar{\partial}$ will denote the Dolbeault operator.

We will use E for vector bundles, g, h will be reserved for a metric on the manifold, and on the bundle respectively. The letters α, β will be isomorphisms of vector bundles, φ, σ, ψ will be maps of vector bundles.

The letter I will be used as a map of Hilbert spaces induced by identity map on a bundle (with different choices of metrics), π will refer to projection onto the L^2-integrable functions on an open subset, and ι will denote extension by zero of L^2 sections on an open subset to the whole space.

4.1 The Dolbeault Functor

Lemma 4.1.1. *Let X be a differentiable manifold, E a differentiable vector bundle over X, and let $D \in Diff_1(E, E)$ be a differential operator of order 1. Let $\rho : C_0(X) \to \mathfrak{B}(L^2(X, E))$ be the representation of $C_0(X)$ given by pointwise multiplication of a function on X with an*

L^2-section. Recall from definition 2.1.9 that with respect to a given representation, pseudo-local operators commute with the representation up to compact operators, and locally compact operators composed with the representation give a compact operator.

1. *Let ϕ be a bounded Borel function on $\mathbb{R}$, whose Fourier transform is compactly supported. Then $\phi(D)$ is a well-defined bounded operator acting on $L^2(X, E)$, which is in fact, pseudo-local. (HR00, 10.3.5, 10.6.3.)*

2. *Assume in addition that D is an elliptic operator. Let $\phi \in C_0(\mathbb{R})$, then $\phi(D) : L^2(X, E) \to L^2(X, E)$ is a locally compact operator. (HR00, 10.5.2.)*

Now we are ready to define the functor $\hat{\tau}_X^D$.

Definition 4.1.2. Let X be a complex manifold of dimension n, and let E be a holomorphic vector bundle on X. We will use the Dolbeault complex (see Equation A.1) to define an exact sequence in the Paschke category of $C_0(X)$.

Fix some Hermitian metric g on X and a Hermitian metric h on E and let H^i be the space of L^2-integrable sections of the bundle $\wedge^{0,i}T^*X \otimes E$ over X. There are natural representations $\rho^i : C_0(X) \to \mathfrak{B}(H^i)$ given by pointwise multiplication. Let $\bar{\partial}_E^*$ be the formal adjoint of the Dolbeault operator $\bar{\partial}_E$ (with respect to the metrics g and h), and consider the essentially self-adjoint differential operator $D_E = \bar{\partial}_E + \bar{\partial}_E^*$ of order 1 (HR00, 11.8.1.). Therefore we can apply functional calculus to D_E with respect to the function $\chi(t) = \frac{t}{\sqrt{1+t^2}}$, to obtain a bounded operator $\frac{D_E}{\sqrt{1+D_E^2}} = \chi(D_E) \in \mathfrak{B}(\oplus_i H^i)$. By Lemma 4.1.1, this is a pseudo-local operator with

respect to the ρ^i's, so if $\chi_i(D_E) = \frac{\bar{\partial}_i}{\sqrt{1+(D_E)^2}}$ denotes the restriction of $\chi(D_E)$ to $\mathfrak{B}(H^i, H^{i+1})$, then we have the following chain complex in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

$$\hat{\tau}^D_{X,g}(E,h): \quad 0 \to \rho^0 \xrightarrow{\chi_0(D_E)} \rho^1 \xrightarrow{\chi_1(D_E)} \dots \xrightarrow{\chi_{n-1}(D_E)} \rho^n \to 0. \tag{4.1}$$

To show that this is in fact an exact sequence in the Paschke category, we need to find pseudo-local operators $P_i : H^{i+1} \to H^i$ which give a contracting homotopy[1], i.e. $P_i\chi_i(D_E) + \chi_{i-1}(D_E)P_{i-1} - Id_{H^i}$ is a locally compact operator. It is easy to see if $P_i = \frac{\bar{\partial}_i^*}{\sqrt{1+D_E^2}} : H^{i+1} \to H^i$, then $P_i\chi_i(D_E)+\chi_{i-1}(D_E)P_{i-1} - Id_{H^i} = \frac{1}{1+D_E^2}$, which is locally compact by Lemma 4.1.1. This shows that the complex in Equation 4.1 is an exact sequence.

Proposition 4.1.3. *Let X be a compact complex manifold and E a holomorphic vector bundle. Then considered as complexes of vector spaces, the chain complex in Equation 4.1 is quasi-isomorphic to the Dolbeault complex (see Equation A.1) with coefficients in E.*

Proof. It is easy to see that the diagram below commutes:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathscr{A}_X^{0,0}(X,E) & \xrightarrow{\bar{\partial}_0} & \mathscr{A}_X^{0,1}(X,E) & \xrightarrow{\bar{\partial}_1} & \dots & \xrightarrow{\bar{\partial}_{n-1}} & \mathscr{A}_X^{0,n}(X,E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (1+D_E^2)^{-1/2}} & & & & \downarrow{\scriptstyle (1+D_E^2)^{-n/2}} & & \\
0 & \longrightarrow & H^0 & \xrightarrow{\chi_0(D_E)} & H^1 & \xrightarrow{\chi_1(D_E)} & \dots & \xrightarrow{\chi_{n-1}(D_E)} & H^n & \longrightarrow & 0
\end{array}$$

Since the image of the vertical maps are dense, then by the Hodge decomposition A.0.5, we can see this sends harmonic forms isomorphically to the cohomology of the complex below. □

---

[1]The operators P_i are also called the *parametrices*

The definition 4.1.2, is not very easy to work with when we restrict to open subsets, because restriction of an essentially self-adjoint operator to an open subset is not necessarily essentially self-adjoint. We will give an equivalent definition in 4.1.6 for any symmetric elliptic operator.

Lemma 4.1.4. *(HR00, 10.8.4.) Let X be a differentiable manifold, E a differentiable vector bundle on it, $U \subset X$ an open subset, and suppose that $D_1, D_2 \in Diff_1(E,E)$ are order one essentially self-adjoint differential operators, so that $D_1|_U = D_2|_U$. Then if $f \in C_0(U)$, we have $\rho(f)\chi(D_1) - \rho(f)\chi(D_2)$ is a compact operator, where $\rho : C_0(U) \to \mathfrak{B}(L^2(X,E))$ is given by pointwise multiplication, and $\chi(t) = \frac{t}{\sqrt{1+t^2}}$.*

Definition 4.1.5. Let X be a locally compact, and Hausdorf topological space. We say that the open cover $\{U_j\}_j$ is a *good cover*, if it is countable, locally finite, and each open set U_j is relatively compact.

Definition 4.1.6. (HR00, 10.8.) Let X be a (non-compact) differentiable manifold, E a differentiable vector bundle on X, and let $D \in \mathrm{Diff}_1(E,E)$ be a symmetric elliptic differential operator of order 1. Let $\{U_j\}_j$ be a good cover (definition 4.1.5), and let $\{\lambda_j\}_j$ be a partition of unity subbordinate to the cover, and let $\{\gamma_j\}_j$ be compactly supported non-negative continuous functions, so that $\gamma_j|_{U_j}$ is equal to (the constant function) one. Then the symmetric differential operator $D_j = \gamma_j D \gamma_j$ is supported on a compact set, hence by Lemma B.0.5, is essentially self-adjoint. Therefore if the representation $\rho : C_0(X) \to \mathfrak{B}(L^2(X,E))$ (where the L^2-completion

is of course defined with respect to choices of metrics on X and on E) is given by pointwise multiplication, then we can define

$$\chi_D \coloneqq \sum_j \rho(\lambda_j^{1/2})\chi(D_j)\rho(\lambda_j^{1/2}), \tag{4.2}$$

as the partial sums are bounded in norm and the series converges in the strong operator topology.

One can see that χ_D is self-adjoint. χ_D depends on the choice of the open cover, the partition of unity, and the cut-off functions γ_j's, but if $f \in C_0(X)$ is compactly supported, then $\rho(f)\chi_D$ has only finitely many terms, hence by Lemma 4.1.4, if $D_1 \in \mathrm{Diff}_1(E,E)$ is any essentially self-adjoint differential operator which agrees with D on support of f, then $\rho(f)\chi_D - \rho(f)\chi(D_1)$ is compact, and in particular if D is itself essentially self-adjoint, then $\chi_D - \chi(D)$ is locally compact. Hence the choices do not matter up to locally compact operators. Therefore, we have a well-defined operator χ_D in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

Definition 4.1.7. Let X be a complex manifold, then denote the category of holomorphic vector bundles on X by $\mathcal{P}(X)$. This is an exact category.

It is straightforward to show that $\mathcal{P}(X)$ has a small skeletal subcategory. For each vector bundle on X, there is a *set* of metrics, hence if we denote the category of holomorphic vector bundles with a choice of metric by $\mathcal{P}_{\mathrm{m}}(X)$, (i.e. objects are pairs (E,h) of a holomorphic vector bundle with a Hermitian metric, and morphisms are maps of holomorphic bundles) then without loss of generality, we can assume that this is a small category. The category $\mathcal{P}_{\mathrm{m}}(X)$ inherits an exact structure from the equivalent category $\mathcal{P}(X)$.

Let g be a Hermitian metric on X, then denote the *bounded* category of holomorphic vector bundles with a choice of metric by $\mathcal{P}_{\mathrm{m,\,b}}(X,g)$, where objects again are pairs (E,h) of a holomorphic vector bundle with a choice of a Hermitian metric, and a morphism from (E_1,h_1) to (E_2,h_2) is a map of holomorphic bundles $E_1 \to E_2$, so that the induced map $L^2(X,\wedge^{0,*}T^*X\otimes E_1) \to L^2(X,\wedge^{0,*}T^*X\otimes E_2)$ is a bounded map of Hilbert spaces. We say a sequence $\ldots \to (E_{i-1},h_{i-1}) \to (E_i,h_i) \to (E_{i+1},h_{i+1}) \to \ldots$ is exact in $\mathcal{P}_{\mathrm{m,\,b}}(X)$, if there are smooth maps of bundles $\sigma_i : E_{i+1} \to E_i$ which are a contracting homotopy, and also the induced maps $L^2(X,\wedge^{0,*}T^*X\otimes E_{i+1}) \to L^2(X,\wedge^{0,*}T^*X\otimes E_i)$ are bounded.[1] This is again a small category.

Note that there $\mathcal{P}_{\mathrm{m,\,b}}(X)$ is a subcategory of $\mathcal{P}_{\mathrm{m}}(X)$, and there is a forgetful map $\mathcal{P}_{\mathrm{m}}(X) \to \mathcal{P}(X)$. Both of these functors are exact.

Proposition 4.1.8. *Let X be a complex manifold, and let g be a Hermitian metric on X. Recall from definition 3.3.2 that $Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ denotes the category of bounded acyclic chain complexes in $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$ where the exact structure is induced by that of $\mathfrak{D}_{C_0(X)}$.*

The map $\hat{\tau}^D_{X,g}$ defined in 4.1.2 induces an exact functor from $\mathcal{P}_{m,b}(X,g)$ to $Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

The proof of proposition above, will take the entirety of the next section, as we will need to prove a series of technical lemmas (which are probably known to the experts). We include them with great details for readers who may not have a background in the topic.

Let us investigate effect of the choice of the metric g on $\hat{\tau}^D_{X,g}$.

---

[1]Notice that a smooth contracting homotopy always exists (Ati67, 1.4.11.). The only condition here is boundedness of σ_i's.

Lemma 4.1.9. *Let X be a differentiable manifold, and let E be a differentiable vector bundle on X. Let $d \in Diff_1(E,E)$ be a differential operator, so that for each metric g on X and h on E, $D = d + d^*$ is an essentially self-adjoint elliptic differential operator.*[1] *Let g_0, g_1 be a pair of metrics on X, and let h_0, h_1 be metrics on E. Denote $d + d^*$ with respect to $(g_0, h_0), (g_1, h_1)$ by D_0, D_1 respectively. Then there is a unitary isomorphism $L^2(X, E; g_0, h_0) \to L^2(X, E; g_1, h_1)$ that commutes with $\chi(D)$ up to locally compact operators.*

Proof. Let $g_t = (1-t)g_0 + tg_1, h_t = (1-t)h_0 + th_1$ for $0 \leq t \leq 1$. Then both g_t, h_t are metrics (and in case both g_0, g_1 are Hermitian, then so are g_t and etc.). Denote the Hilbert space of L^2-sections of E with respect to the metric g_t, h_t by H_t. Let $\nu_t : X \to \mathbb{R}^{\geq 0}$ be the "square root of the measure" given by the Radon-Nikodym theorem so that $d\mu_t(Z) = \int_Z \nu_t^2 d\mu_0$ for each measurable subset Z of X and each t. Let $S_t : E \to E$ be the square root[2] of the positive definite map $E \xrightarrow{h_0} E^* \xrightarrow{h_t^*} E$. Then $T_t(x) = \eta(x) S_t(x)$ acts fiberwise, hence it is pseudo-local. Also for L^2 sections η, ζ in H_0,

$$\begin{aligned}\langle T_t\eta, T_t\zeta\rangle_t &= \int_X (h_t)(T_t\eta)(T_t\zeta)d\mu_t = \int_X \nu^2 h_t(S_t\eta)(S_t\zeta)d\mu_t \\ &= \int_X h_t(S_t^* S_t\eta)(\zeta)d\mu_0 = \int_X h_t h_t^* h_0(\eta)(\zeta)d\mu_0 = \langle \eta, \zeta\rangle_0.\end{aligned}$$

---

[1] *We only use this condition when $g = g_t = (1-t)g_0 + tg_1$ and $h = h_t = (1-t)h_0 + th_1$, where $0 \leq t \leq 1$.*

[2] Notice that by (Lax07, Page 150.) this exists and varies continuously.

Therefore we have unitary maps $T_t : L^2(X,E) \to L^2(X,E)$ (where the L^2-completions are with respect to $(g_0,h_0),(g_t,h_t)$, respectively.). Consider the the path $t \mapsto T_t^*\chi(D_t)T_t$ from $\chi(D_0)$ to $T_1^*\chi(D_1)T_1$. Since $\chi^2-1 \in C_0(\mathbb{R})$, therefore $T_t^*\frac{\chi(D_t)+1}{2}T_t \in (\mathfrak{D}/\mathfrak{C})(\rho_0)$ is a self-adjoint projection up to locally compact operators. Hence by Lemma 2.1.18, without loss of generality we can assume $T_t^*\frac{\chi(D_t)+1}{2}T_t \in \mathfrak{B}(L^2(X,E))$ (where the L^2-completion is with respect to (g_0,h_0).) is a self-adjoint projection, and by (HR00, 4.1.8.) this path of projections induces a unitary operator $W_1 : L^2(X,E;g_0,h_0) \to L^2(X,E;g_t,h_t)$ such that $W_1^*(T_1^*\frac{\chi(D_1)+1}{2}T_1)W_1 = \frac{\chi(D_0)+1}{2}$. Therefore, W_1T_1 is the unitary isomorphism that commutes with $\chi(D)$ up to locally compact operators. □

For the rest of this thesis, we will fix a single metric on the manifold X.

4.2 Proof of Proposition 4.1.8

Notation 4.2.1. Let X be a differentiable manifold, and let E_1, E_2 be differentiable vector bundles on X. Choose metrics g on X and h_1, h_2 on E_1, E_2. To shorten the notation, we say $(X, g; E_1, h_1; E_2, h_2)$ is a *metric pair*.

Let X be a complex manifold, and let E_1, E_2 be holomorphic vector bundles on X. Choose Hermitian metrics g on X and h_1, h_2 on E_1, E_2, and let set $D_{E_1} = \bar{\partial}_{E_1} + \bar{\partial}_{E_1}^*$ and $D_{E_2} = \bar{\partial}_{E_2} + \bar{\partial}_{E_2}^*$ be the corresponding Dolbeault operators. To shorten the notation, we say

$$(X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2})$$

is a *Hermitian pair*.

Definition 4.2.2. Let $(X, g; E_1, h_1; E_2, h_2)$ be a metric pair. We say an operator T (or a family of operators) is *locally bounded* with respect to $(X, g; E_1, h_1; E_2, h_2)$, if for each relatively compact open subset U of X, there exists an induced operator $T_U : L^2(U, E_1|_U) \to L^2(U, E_2|_U)$ so that T_U is a bounded linear operator, and if for each pair of relatively compact open subsets U_1, U_2, we have

$$\pi^{U_1}_{U_1\cap U_2} T_{U_1} \iota^{U_1}_{U_1\cap U_2} = \pi^{U_2}_{U_1\cap U_2} T_{U_2} \iota^{U_2}_{U_1\cap U_2}$$

where in here $\pi^U_V : L^2(U, E|_U) \to L^2(V, E|_V)$ is the projection defined by multiplication by the characteristic function of $U \cap V$, and $\iota^U_V : L^2(V, E|_V) \to L^2(U, E|_U)$ is extension by zero.

Beware that this definition is not exactly the same as more well-known definitions of local boundedness. Also note that there does not need to be a uniform bound on $\|T_U\|$. However, in case there is a uniform bound on T_U (say M), then we can "glue" them to obtain $T : L^2(X, E_1) \to L^2(X, E_2)$, by simply choosing a relatively compact open neighborhood U of x, and setting $T(\zeta)(x) = T_U(\pi_U \zeta \pi^*_U)(x)$. This is independent of choice of U and $\|T(\zeta)\|_2 \leq M\|\zeta\|_1$, as this holds for *almost every*[1] point x.

Example 4.2.3. Let $(X, g; E_1, h_1; E_2, h_2)$ be a metric pair (definition 4.2.1), and let $\varphi : E_1 \to E_2$ be a continuous bundle map. Then φ is locally bounded (definition 4.2.2).

Example 4.2.4. Let $(X, g; E, h_1, E, h_2)$ be a metric pair (definition 4.2.1), and let $L^2(X, E; h_i) = L^2(X, E; g, h_i)$ denote the space of L^2-sections of E on X with respect to the metric h_i on E

---

[1]Recall that evaluating $\zeta \in L^2(X, E_1)$ at a point $x \in X$ only makes sense up to subsets of measure zero in X.

(and g on X). Then the identity map $Id : E \to E$ induces a locally bounded map (definition 4.2.2) from $L^2(X, E; h_1)$ to $L^2(X, E; h_2)$ which we denote by $I(h_2, h_1)$ throughout this section.

Lemma 4.2.5. *Let $(X, g; E, h_1, D_{E,1}; E, h_2, D_{E,2})$ be a Hermitian pair (definition 4.2.1). Then $D_{E,1} - D_{E,2}$ is locally bounded (definition 4.2.2).*

Proof. Recall from definition A.0.6 that the metric h_i can be considered as a linear map of bundles from E to the dual bundle E^*, which by abuse of notation we denote with h_i again. Let $h_i^* : E^* \to E$ denote the dual maps induced by h_i, let θ denote the composition $E \xrightarrow{h_1} E^* \xrightarrow{h_2^*} E$, and let ϑ^* denote the composition $E^* \xrightarrow{h_1^*} E \xrightarrow{h_2} E^*$.

Consider $f \otimes e \in \mathscr{C}^\infty(X, \wedge^{0,*} T^*X \otimes E)$, we have:

$$
\begin{aligned}
&D_{E,1}(f \otimes e) - D_{E,2}(f \otimes e) \\
=&\left(\bar\partial_E + (\star \otimes h_1^*)\bar\partial(\star \otimes h_1)\right)(f \otimes e) - \left(\bar\partial_E + (\star \otimes h_2^*)\bar\partial(\star \otimes h_2)\right)(f \otimes e) \\
=&(\star \otimes h_1^*)\bar\partial(\star f \otimes h_1(e)) - (\star \otimes h_2^*)\bar\partial(\star f \otimes h_2(e)) \\
=&(\star \otimes h_1^*)\left(\bar\partial(\star f) \otimes h_1(e) + \star f \otimes \bar\partial h_1(e)\right) - (\star \otimes h_2^*)\left(\bar\partial(\star f) \otimes h_2(e) + \star f \otimes \bar\partial h_2(e)\right) \\
=&\left(\star\bar\partial \star f \otimes e + \star\star f \otimes h_1^*\bar\partial h_1(e)\right) - \left(\star\bar\partial \star f \otimes e + \star\star f \otimes h_2^*\bar\partial h_2(e)\right) \\
=&\star\star f \otimes (h_1^*\bar\partial(e^*) - h_2^*\bar\partial(\vartheta^* e^*)) \\
=&\star\star f \otimes \left(h_1^*\bar\partial(e^*) - h_2^*(\vartheta^*\bar\partial(e^*) + e^*\bar\partial(\vartheta^*)\right) \\
=&\star\star f \otimes \theta(e)\bar\partial(\vartheta^*)
\end{aligned}
$$

where in here, $e^* = h_1(e)$. The term above does not have any differentials of $f \otimes e$; recall $\star\star$ is ± 1, and $\|\theta\|_i, \|\vartheta^*\|_i$ vary continuously with respect to $x \in X$, and $i = 1, 2$, hence the term $\theta(e)\bar{\partial}(\vartheta^*)$ is bounded with respect to both norms on the relatively compact set U. □

Lemma 4.2.6. *Let $(X, g; E, h_1; E, h_2)$ be a metric pair (definition 4.2.1), let and let $D_i \in Diff_1(E, E)$ be an essentially self-adjoint differential operators with respect to the metric h_i for $i = 1, 2$, so that $D_1 - D_2$ is a locally bounded operator (definition 4.2.2). Let $I(h_2, h_1)$ denote the locally bounded map induced by the identity map of E (Example 4.2.4). Then for each relatively compact open subset U of X, we have*

$$\pi_1 \chi(D_1) \iota_1 I(h_1, h_2)_U = I(h_1, h_2)_U \pi_2 \chi(D_2) \iota_2$$

in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(U)}$, where in here, $\pi_i : L^2(X, E; h_i) \to L^2(U, E|_U; h_i)$ is the projection and ι_i is extension by zero, for $i = 1, 2$.

This proof closely follows that of (HR00, 10.9.5.).[1]

Proof. Similar to (HR00, 10.3.5.) we argue that if u, v are compactly supported smooth sections of $\wedge^{0,*} T^*X \otimes E$, and ϕ is a *Schwartz function*, then since $\phi(x) = \frac{1}{2\pi} \int e^{\sqrt{-1}sx} \hat{\phi}(s) ds$, then we

---

[1] We can not directly apply this result here, even though they look similar; the problem is that in (HR00, 10.9.5.) it is required for both D_1, D_2 to be essentially self-adjoint with respect to the same given inner product, which is not the case here.

can pair $\hat{\phi}$ with the smooth function $s \mapsto \langle (I(h_2,h_1)e^{\sqrt{-1}sD_1}I(h_1,h_2))u,v\rangle_2 = \langle e^{\sqrt{-1}sD_1}u,v\rangle_2$ to obtain

$$\langle \phi(D_1)u,v\rangle_2 = \langle I(h_2,h_1)\phi(D_1)I(h_1,h_2)u,v\rangle_2 = \frac{1}{2\pi}\int \langle I(h_2,h_1)e^{\sqrt{-1}sD_1}I(h_1,h_2)u,v\rangle_2 \hat{\phi}(s)ds,$$

and then use the rest of the argument in (HR00, 10.3.5.), to generalize this for any bounded Borel function whose Fourier transform is compactly supported.

Let ϕ, u, v be as above (with the extra assumption that $s\hat{\phi}(s)$ is a smooth function, also note that D_1, D_2 both share the invariant domain of smooth compactly supported functions.), then we have

$$\begin{gathered}\langle (I(h_2,h_1)\phi(D_1)I(h_1,h_2) - \phi(D_2))u,v\rangle_2 \\ = \tfrac{1}{2\pi}\int \langle (I(h_2,h_1)e^{\sqrt{-1}sD_1}I(h_1,h_2) - e^{\sqrt{-1}sD_2})u,v\rangle_2 \hat{\phi}(s)ds.\end{gathered}$$

By the fundamental theorem of calculus we know that

$$\begin{gathered}\langle (I(h_2,h_1)e^{\sqrt{-1}sD_1}I(h_1,h_2) - e^{\sqrt{-1}sD_2})u,v\rangle_2 = \langle (e^{\sqrt{-1}sD_1} - e^{\sqrt{-1}sD_2})u,v\rangle_2 \\ = \sqrt{-1}\int_0^s \langle (I(h_2,h_1)e^{\sqrt{-1}tD_1}I(h_1,h_2)(D_1-D_2)e^{\sqrt{-1}(s-t)D_2})u,v\rangle_2,\end{gathered}$$

and by repeating the argument in (HR00, 10.3.6, 10.3.7.) we obtain that there exists a constant $C_\phi < \infty$ (which only depends on ϕ) so that $\|I(h_2,h_1)\phi(D_1)I(h_1,h_2) - \phi(D_2)\|_2 \leq C_\phi \|D_1 - D_2\|_2$.

Now, let ϕ be a *normalizing function* (i.e. $\phi - \chi \in C_0(\mathbb{R})$.) that satisfies the conditions above, and let $\phi_\epsilon(x) = \phi(\epsilon x)$. Then ϕ_ϵ is also a normalizing function, and hence $\phi_\epsilon(D_i) - \chi(D_i)$ is a locally compact operator for any $\epsilon > 0$. But as $\epsilon \to 0$, we get

$$\begin{aligned}\|I(h_2,h_1)\phi_\epsilon(D_1)I(h_1,h_2) - \phi_\epsilon(D_2)\|_2 &= \|I(h_2,h_1)\phi(\epsilon D_1)I(h_1,h_2) - \phi(\epsilon D_2)\|_2 \\ &\leq \epsilon C_\phi \|D_1 - D_2\|_2 \to 0.\end{aligned}$$

In other words, the operators which are equivalent to $I(h_2,h_1)\chi(D_1)I(h_1,h_2)$ modulo locally compact operators, can be arbitrarily close in norm to the ones which are equivalent to $\chi(D_2)$ modulo locally compact operators. But these equivalency classes form linear subspaces of pseudo-local operators, hence these subspaces have to be the same, i.e. $I(h_2,h_1)\chi(D_1)I(h_1,h_2) - \chi(D_2)$ is locally compact. This finishes the proof.

□

Corollary 4.2.7. *Let $(X, g; E, h_1, D_{E,1}; E, h_2, D_{E,2})$ be a Hermitian pair (definition 4.2.1), let $I(h_2,h_1)$ be the locally bounded map induced by the identity map of E (Example 4.2.4). Let U be a relatively compact open subset of X, and let $\pi_i : L^2(X, \wedge^{0,*}T^*X \otimes E; h_i) \to L^2(U, \wedge^{0,*}T^*X \otimes E|_U; h_i)$ be the projection and let ι_i be its adjoint. Then $\pi_1\chi_{D_{E_1}}\iota_1 I(h_1,h_2)_U = I(h_1,h_2)_U\pi_2\chi_{D_{E_2}}\iota_2$ in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(U)}$.*

Definition 4.2.8. Let X be a differentiable manifold, and let E_1, E_2 be differentiable vector bundles on X. Let $\alpha : E_1 \to E_2$ be a smooth bundle map. Choose metrics g, h_1, h_2 on X, E_1, E_2 respectively. We say α *preserves the metrics*, if the dual map of bundles $\beta : E_2^* \to E_1^*$ on the dual vector bundles (defined by $\beta(e_2^*)(e_1) = e_2^*(\alpha(e_1))$.) makes the diagram below commute.

$$\begin{array}{ccc} E_1 & \xrightarrow{\alpha} & E_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ E_1^* & \xleftarrow[\beta]{} & E_2^*. \end{array}$$

Lemma 4.2.9. *Let $(X, g; E_1, h_1; E_2, h_2)$ be a metric pair (definition 4.2.1), and let $\alpha : E_1 \to E_2$ be a smooth isomorphism of vector bundles that preserves the metrics (definition 4.2.8). Let $D \in Diff_1(E_1, E_1)$ be an essentially self-adjoint differential operator of order one. Then $\chi(D) = \alpha^{-1}\chi(\alpha D\alpha^{-1})\alpha$.*

Proof. Since α preserves the metric on each fiber, then the induced map $\alpha : L^2(X, E_1) \to L^2(X, E_2)$ is a unitary map, i.e. $\alpha^{-1} = \alpha^*$. Since D is symmetric, then $\alpha D\alpha^{-1} = \alpha D\alpha^*$ is also symmetric. Also if $x \in \text{Domain}(\alpha D^*\alpha^{-1}) \subset L^2(X, E_2)$, then there exists a constant M so that for each $y \in \text{Domain}(\alpha D\alpha^{-1}) \subset L^2(X, E_2)$, we have $|\langle x, \alpha D^*\alpha^{-1}y\rangle| \leq M\|y\|$ (HR00, 1.8.2.). But if $x' = \alpha^{-1}x, y' = \alpha^{-1}y \in L^2(X, E_1)$, then this is equivalent to saying that $|\langle x', D^*y'\rangle| \leq M\|y\|$, i.e. $x' \in \text{Domain}(D^*)$. Since D is essentially self-adjoint, then $x' \in \text{Domain}(D)$, hence $x \in \text{Domain}(\alpha D\alpha^{-1})$, i.e. $\alpha D\alpha^{-1}$ is also essentially self-adjoint. Hence $\chi(\alpha D\alpha^{-1}) \in \mathfrak{B}(L^2(X, E_2))$ is well-defined.

Assume that the Fourier transform of the bounded Borel function ϕ is compactly supported, then for small values of $s > 0$, we have $e^{\sqrt{-1}s\alpha D\alpha^{-1}} = \alpha e^{\sqrt{-1}sD}\alpha^{-1}$. Hence by (HR00, 10.3.5.) it is easy to argue that $\phi(\alpha D\alpha^{-1}) = \alpha\phi(D)\alpha^{-1}$. Now if ϕ is a normalizing function, then $\phi(D) - \chi(D), \phi(\alpha D\alpha^{-1}) - \chi(\alpha D\alpha^{-1})$ are locally compact.

□

Lemma 4.2.10. *Let $(X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2})$ be a Hermitian pair (definition 4.2.1), and let $\alpha : E_1 \to E_2$ be a smooth isomorphism of vector bundles that preserves the metrics. Then $\alpha D_{E_1} \alpha^{-1} - D_{E_2}$ is locally bounded (definition 4.2.2).*

Proof. Recall from definition A.0.6 that the metrics induce conjugate linear smooth bundle isomorphisms $h_i : E_i \to E_i^*$ to the dual bundle, for $i = 1, 2$, and $h_i^* : E_i^* \to E_i$ is the inverse. Let $\beta : E_2^* \to E_1^*$ be the map of bundles dual to α. Since α is a smooth isomorphism of vector bundles, then $\alpha\bar{\partial}_{E_1} - \bar{\partial}_{E_2}\alpha$ is locally bounded. Therefore by Equation A.3, to prove the lemma it suffices to show that the term below is locally compact, where $f \otimes e_2$ is a smooth section of $\wedge^{0,*}T^*X \otimes E_2$, and $e_1 = \alpha^{-1}e_2, e_2^* = h_2(e_2)$

$$
\begin{aligned}
&(\alpha \bar{\star}_{E_1^*} \bar{\partial} \bar{\star}_{E_1} \alpha^{-1} - \bar{\star}_{E_2^*} \bar{\partial} \bar{\star}_{E_2})(f \otimes e_2) \\
=&\alpha \bar{\star}_{E_1^*} \bar{\partial}(\bar{\star} f \otimes h_1(e_1)) - \bar{\star}_{E_2^*} \bar{\partial}(\bar{\star} f \otimes h_2(e_2)) \\
=&\alpha \bar{\star}_{E_1^*} \left(\bar{\partial}(\bar{\star} f) \otimes h_1(e_1) + \bar{\star} f \otimes \bar{\partial} h_1(e_1)\right) - \bar{\star}_{E_2^*} \left(\bar{\partial}(\bar{\star} f) \otimes h_2(e_2) + \bar{\star} f \otimes \bar{\partial} h_2(e_2)\right) \\
=&\left((\bar{\star}\bar{\partial}\bar{\star}) f \otimes e_2 + \bar{\star}\bar{\star} f \otimes \alpha h_1^* \bar{\partial} h_1(e_1)\right) - \left((\bar{\star}\bar{\partial}\bar{\star}) f \otimes e_2 + \bar{\star}\bar{\star} f \otimes h_2^* \bar{\partial} h_2(e_2)\right) \\
=&\bar{\star}\bar{\star} f \otimes (\alpha h_1^* \bar{\partial} h_1(e_1) - h_2^* \bar{\partial} h_2(e_2)) \\
=&\bar{\star}\bar{\star} f \otimes (h_2^* \beta^{-1} \bar{\partial} \beta(e_2^*) - h_2^* \bar{\partial}(e_2^*))
\end{aligned}
$$

But β is also a smooth isomorphism of vector bundles hence $\bar{\partial}\beta - \beta\bar{\partial}$ is locally bounded. □

Corollary 4.2.11. *Let $0 \to E_1 \xrightarrow{\varphi_1} E_2 \xrightarrow{\varphi_2} E_3 \to 0$ be a short exact sequence of holomorphic vector bundles on the complex manifold X. Choose a Hermitian metric g on X, and h_1 on E_1.*

Then we get an induced Hermitian metric on the subbundle $\varphi_1(E_1)$ of E_2. Extend this metric to Hermitian metric h_2 on all of E_2. Then there exists a smooth map of bundles $\sigma_2 : E_3 \to E_2$ which is an isomorphism from E_3 to the orthogonal complement of $\varphi_1(E_1)$ in E_2. Let h_3 be the Hermitian metric induced by this isomorhpism.

Hence $(X, g; E_1 \oplus E_3, h_1 \oplus h_3, D_{E_1} \oplus D_{E_3}; E_2, h_2, D_{E_2})$ is a Hermitian pair (definition 4.2.1), and we have a smooth isomorphism $(\varphi_1, \sigma_2) : E_1 \oplus E_3 \to E_2$. By definition of the metrics, it is easy to check that this isomorphism preserves metrics. Therefore as a corollary of 4.2.10,

$$D_{E_1} \oplus D_{E_3} - (\varphi_1, \sigma_2)^{-1} D_{E_2} (\varphi_1, \sigma_2) \tag{4.3}$$

is locally bounded.

Corollary 4.2.12. *Let $(X, g; E_1, h_1, D_{E_1}; E_2, h_2, D_{E_2})$ be a Hermitian pair (definition 4.2.1), and let $\alpha : E_1 \to E_2$ be a smooth isomorphism of vector bundles on X. Let U be a relatively compact open subset of X, let $\pi_i : L^2(X, \wedge^{0,*} T^*X \otimes E_i) \to L^2(U, \wedge^{0,*} T^*X \otimes E_i|_U)$ be the projection and let ι_i be its adjoint. Then*

$$\alpha_U \pi_1 \chi(D_{E_1}) \iota_1 = \pi_2 \chi(D_{E_2}) \iota_2 \alpha_U$$

in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(U)}$, where by abuse of notation, we are denoting the map induced by α_U from $L^2(U, \wedge^{0,*} T^*X \otimes E_1|_U) \to L^2(U, \wedge^{0,*} T^*X \otimes E_2|_U)$ by α_U as well.

Proof. Let h_2' be the Hermitian metric on E_2 defined through the diagram in definition 4.2.8, so that the bundle isomorphism $\alpha : E_1 \to E_2$ preserves the metrics. Let $D'_{E_2} = \bar{\partial}_{E_2} + \bar{\partial}^*_{E_2}$ be the Dolbeault operator with respect to h_2', let $\pi_2' : L^2(X, \wedge^{0,*}T^*X \otimes E_2; h_2') \to L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2')$ be the projection, and let ι_2' be its adjoint. Denote the map induced by α from $L^2(U, \wedge^{0,*}T^*X \otimes E_1|_U)$ to $L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2')$ by α'_U, and let $I(h_2, h_2')$ denote the locally bounded map induced by the identity of E_2 (Example 4.2.4). Therefore $\alpha_U = I(h_2, h_2')_U \alpha'_U$ and:

$$
\begin{aligned}
\alpha_U \pi_1 \chi(D_{E_1}) \iota_1 &= I(h_2, h_2')_U \alpha'_U \pi_1 \chi(D_{E_1}) \iota_1 \\
&= I(h_2, h_2')_U \pi_2' \alpha' \chi(D_{E_1}) \iota_1 \\
&= I(h_2, h_2')_U \pi_2' \chi(\alpha D_{E_1} \alpha^{-1}) \alpha' \iota_1 && \text{By Lemma 4.2.9} \\
&= I(h_2, h_2')_U \pi_2' \chi(D'_{E_2}) \alpha' \iota_1 && \text{By Lemma 4.2.10 and (HR00, 10.9.5.)} \\
&= I(h_2, h_2')_U \pi_2' \chi(D'_{E_2}) \iota_2' \alpha'_U \\
&= \pi_2 \chi(D_{E_2}) \iota_2 I(h_2, h_2')_U \alpha'_U && \text{By Corollary 4.2.7} \\
&= \pi_2 \chi(D_{E_2}) \iota_2 \alpha_U .
\end{aligned}
$$

□

Remark 4.2.13. One may wonder if we can change the metric g on X in the corollary above as well. Consider the case where $E_1 = E_2$ is the trivial bundle of rank one, α is the identity map, and $h_1 = h_2$. When g_1, g_2 are any two Hermitian metrics on X, and U is a

relatively compact open subset of X, then the bounded operator induced by the identity $I(g_2, g_1)_U : L^2(U, E_1|_U; g_1, h_1) \to L^2(U, E_2|_U; g_2, h_2)$ is the identity on the underlying vector spaces (although these Hilbert spaces have different inner products). However, if g_1, g_2 are different metrics, then the symbols of $\bar{\partial}+\bar{\partial}^*_{g_1}$ and $\bar{\partial}+\bar{\partial}^*_{g_2}$ are not equal to each other. Hence it is not clear if $I(g_2, g_1)$ induces a map between the chain complexes $\hat{\tau}^D_{X,g_1}(E_1, h_1)$ and $\hat{\tau}^D_{X,g_2}(E_2, h_2)$.[1]

Proof of Proposition 4.1.8. We have already defined $\hat{\tau}^D_{X,g}$ on the objects of the category, and showed that $\hat{\tau}^D_{X,g}(E, h)$ is an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$. We need to show functoriality and exactness. Before going further, let us fix some notation.

Let $\varphi : (E_1, h_1) \to (E_2, h_2)$ be a morphism in $\mathcal{P}_{\mathrm{m,\,b}}(X)$. Choose a good cover (definition 4.1.5) $\{U_j\}_j$ so that for $i = 1, 2$ and for each j, there exists an open subset V_j of X that contains closure of U_j, and that $E_i|_{V_j}$ and $\wedge^{0,*}T^*X|_{V_j}$ is isomorphic to the trivial bundle on V_j. In other words, there exists holomorphic isomorphisms of bundles $\alpha_j : \wedge^{0,*}T^*X \otimes E_1|_{V_j} \to V_j \times \mathbb{C}^k$ and $\beta_j : \wedge^{0,*}T^*X \otimes E_2|_{V_j} \to V_j \times \mathbb{C}^m$ where k, m are ranks of the corresponding bundles. Then $\psi_j = \beta_j \varphi|_{V_j} \alpha_j^{-1} : V_j \to M_{m,k}(\mathbb{C})$ is a holomorphic matrix valued function. let $D_{i,j} = \gamma_j D_{E_i} \gamma_j$ for $i = 1, 2$. Let $\{\lambda_j\}_j$ be a partition of unity subbordinate to the cover $\{U_j\}_j$, and let γ_j be smooth cutoff functions which are equal to one on U_j. Also, let $\pi_{i,j} : L^2(X, \wedge^{0,*}T^*X \otimes E_i) \to L^2(U_j, \wedge^{0,*}T^*X \otimes E_i|_{U_j})$ be the projection and let $\iota_{i,j}$ be its adjoint. For $n \in \mathbb{Z}^{>0}$, let D^n denote the Dolbeault operator corresponding to the trivial rank n bundle, let $\pi^n_j : L^2(X, X \times \mathbb{C}^n) \to$

---

[1]Recall from (HR00, 10.9.5.) that if two first order symmetric elliptic differential operators have the same symbol, then the bounded operators obtained through applying functional calculus to them, will be equivalent to each other modulo locally compact operators.

$L^2(U_j, U_j \times \mathbb{C}^n)$ be the projection, and let ι_j^n be its adjoint. Then by Corollary 4.2.12, for relatively compact subset U_j of V_j, we get that

$$\alpha_{j,U_j}\pi_{1,j}\chi_{D_{E_1}}\iota_{1,j} = \pi_j^k\chi_{D_j^k}\iota_j^k\alpha_{j,U_j}$$

$$\beta_{j,U_j}\pi_{2,j}\chi_{D_{E_2}}\iota_{2,j} = \pi_j^m\chi_{D_j^m}\iota_j^m\beta_{j,U_j}$$

in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(U_j)}$. Now, let $f \in C_0(X)$ be compactly supported. Then there are only finitely many of the V_j's that intersect support of f, i.e. the sums below are all finite.

$$\begin{aligned}
&(\varphi\chi_{D_{E_1}} - \chi_{D_{E_2}}\varphi)\rho(f)\\
=&(\sum_j \lambda_j^{1/2}\varphi\chi(D_{1,j})\lambda_j^{1/2} && - \sum_j \lambda_j^{1/2}\chi(D_{2,j})\varphi\lambda_j^{1/2})\rho(f)\\
=&(\sum_j \lambda_j^{1/2}\pi_{1,j}\varphi\iota_{1,j}\pi_{1,j}\chi(D_{1,j})\iota_{1,j}\lambda_j^{1/2} && - \sum_j \lambda_j^{1/2}\pi_{2,j}\chi(D_{2,j})\iota_{2,j}\varphi\pi_{2,j}\lambda_j^{1/2})\rho(f)\\
=&(\sum_j \lambda_j^{1/2}\beta_{j,U_j}^{-1}\psi_j\alpha_{j,U_j}\pi_{1,j}\chi(D_{1,j})\iota_{1,j}\lambda_j^{1/2} && - \sum_j \lambda_j^{1/2}\pi_{2,j}\chi(D_{2,j})\iota_{2,j}\beta_{j,U_j}^{-1}\psi_j\alpha_{j,U_j}\pi_{2,j}\lambda_j^{1/2})\rho(f)\\
=&(\sum_j \lambda_j^{1/2}\beta_{j,U_j}^{-1}\psi_j\pi_j^k\chi_{D_j^k}\iota_j^k\alpha_{j,U_j}\lambda_j^{1/2} && - \sum_j \lambda_j^{1/2}\iota_{2,j}\beta_{j,U_j}^{-1}\pi_j^m\chi_{D_j^m}\iota_j^m\psi_j\alpha_{j,U_j}\pi_{2,j}\lambda_j^{1/2})\rho(f)\\
=&(\sum_j \iota_{2,j}\beta_{j,U_j}^{-1}\pi_j^m\chi_{D_j^m}\psi_j\iota_j^k\alpha_{j,U_j}\lambda_j && - \sum_j \iota_{2,j}\beta_{j,U_j}^{-1}\pi_j^m\chi_{D_j^m}\iota_j^m\psi_j\alpha_{j,U_j}\pi_{2,j}\lambda_j)\rho(f).
\end{aligned}$$

Where the last equality holds because, in the first sum $\chi_{D_j^k}$ is pseudo-local, and hence up to compact operators, commutes with multiplication by the matrix valued continuous function $\lambda_j^{1/2}\psi_j$ that vanishes at infinity, therefore $(\lambda_j^{1/2}\psi_j)\chi_{D_j^k} - \chi_{D_j^m}(\lambda_j^{1/2}\psi_j)$ is compact for each j; and in the second sum $\lambda_j^{1/2}\chi_{D_j^m} - \chi_{D_j^m}\lambda_j^{1/2}$ is also compact for each j, and both sums are finite.

We conclude the first part of the proof by noting that

$$\psi_j \iota_j^k \alpha_{j,U_j} \lambda_j = \iota_j^m \psi_j \alpha_j \pi_{2,j} \lambda_j,$$

hence each term in the sum above is zero, and therefore φ induces a map from $\hat{\tau}^D_{X,g}(E_1, h_1)$ to $\hat{\tau}^D_{X,g}(E_2, h_2)$ in the category $Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}$.

It is straightforward to check that $\hat{\tau}^D_{X,g}(\varphi_1 \circ \varphi_2) = \hat{\tau}^D_{X,g}(\varphi_1) \circ \hat{\tau}^D_{X,g}(\varphi_2)$. This shows that $\hat{\tau}^D_{X,g}$ is a functor.

Remark 4.2.14. Note that the condition on $\varphi : L^2(X, \wedge^{0,*} T^*X \otimes E_1) \to L^2(X, \wedge^{0,*} T^*X \otimes E_2)$ being bounded is not used in the proof of why $\hat{\tau}^D_{X,g}$ is functorial. Also holomorphicity of φ was not needed in the argument above, we only needed continuity to show that multiplication by $\lambda_j \psi_j$ commutes with $\chi(D)$ modulo compact operators.

Now, to prove that $\hat{\tau}^D_{X,g}$ is an exact functor, let

$$0 \longrightarrow (E_1, h_1) \underset{\sigma_1}{\overset{\varphi_1}{\rightleftarrows}} (E_2, h_2) \underset{\sigma_2}{\overset{\varphi_2}{\rightleftarrows}} (E_3, h_3) \longrightarrow 0$$

be an exact sequence in $\mathcal{P}_{\mathrm{m,b}}(X)$. Then by definition of exactness in this category, there exists smooth sections $\sigma_2 : E_3 \to E_2, \sigma_1 : E_2 \to E_1$ so that $\sigma_1 \varphi_1 = Id_{E_1}, \varphi_2 \sigma_2 = Id_{E_3}$, and similar to φ_i, σ_i also induce a bounded map of Hilbert spaces $L^2(X, \wedge^{0,*} T^*X \otimes E_{i+1}) \to L^2(X, \wedge^{0,*} T^*X \otimes E_i)$, for $i = 1, 2$.

Let h_2' be the Hermitian metric on E_2 induced by h_1, h_3, i.e.

$$h_2' = \sigma_1^* h_1 \sigma_1 + \varphi_2^* h_3 \varphi_2 : E_2 \to E_2^*$$

where in here, $\sigma_1^* : E_1^* \to E_2^*$ and $\varphi_2^* : E_3^* \to E_2^*$ are the dual maps to σ_1, φ_2 respectively. Then the subbundles $\varphi_1(E_1), \sigma_2(E_3)$ of E_2 are orthogonal with respect to h_2', and the induced metrics on these subbundles match with the metrics h_1, h_3 respectively, i.e. the isomorphism between $E_1 \oplus E_3$ and E_2 preserves the metric (definition 4.2.8). Hence by Corollary 4.2.11 $(\sigma_1, \varphi_2)D'_{E_2}(\varphi_1, \sigma_2) - D_{E_1} \oplus D_{E_3}$ is locally bounded, where D'_{E_2} is the Dolbeault operator on E_2 with respect to the metric h_2'. Therefore by Lemma 4.2.9, we get that

$$\chi_{D_{E_1}} \oplus \chi_{D_{E_3}} = \chi_{D_{E_1} \oplus D_{E_3}} = (\varphi_1, \sigma_2)\chi_{(\sigma_1, \varphi_2)D'_{E_2}(\varphi_1, \sigma_2)}(\sigma_1, \varphi_2).$$

By Corollary 4.2.7, for any relatively compact open subset U of X, we have $I(h_2', h_2)_U \pi_2 \chi_{D_{E_2}} \iota_2 = \pi_2' \chi_{D'_{E_2}} \iota_2' I(h_2', h_2)_U$, where π_2, π_2' are the projections $L^2(X, \wedge^{0,*}T^*X \otimes E_2) \to L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U)$ with respect to the metrics h_2, h_2' and ι_2, ι_2' are their adjoints, respectively. Also $I(h_2', h_2)_U : L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2) \to L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2')$ is the map induced by Id_{E_2} (Example 4.2.4). This factors through

$$\begin{array}{c} L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2) \\ \downarrow (\sigma_1, \varphi_2) \\ L^2(U, \wedge^{0,*}T^*X \otimes E_1|_U) \oplus L^2(U, \wedge^{0,*}T^*X \otimes E_3|_U) \\ \downarrow (\varphi_1, \sigma_2) \\ L^2(U, \wedge^{0,*}T^*X \otimes E_2|_U; h_2') \end{array}$$

Because (ϕ_1, σ_1) has norm one, and (σ_1, φ_2) has a bounded norm (independent of U), then norm of $I(h_2', h_2)_U$ is also independent of U, therefore we can glue all the data to obtain $I(h_2', h_2)\chi_{D_{E_2}} = \chi_{D'_{E_2}} I(h_2', h_2)$. This proves that $\hat{\tau}^D_{X,g}$ is exact.

□

4.3 Restriction to Open Subsets

Lemma 4.3.1. *Let X be a complex manifold, and let U be an open subset. Then the diagram below commutes up to homotopy.*

$$\begin{array}{ccc} K(\mathcal{P}_{m,b}(X,g)) & \xrightarrow{\hat{\tau}^D_{X,g}} & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \\ \downarrow res^X_U & & \downarrow res^x_U \\ K(\mathcal{P}_{m,b}(U,g)) & \xrightarrow{\hat{\tau}^D_{U,g}} & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(U)}) \end{array}$$

Proof. Let (E,h) be an object of $\mathcal{P}_{\mathrm{m,b}}(X,g)$. It suffices to show that in the diagram below (which is *not* commutative on the nose), $res^X_U \hat{\tau}^D_{X,g}(E,h)$ is naturally isomorphic to $\hat{\tau}^D_{U,g} res^X_U(E,h)$.

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{m,b}}(X,g) & \xrightarrow{\hat{\tau}^D_{X,g}} & Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(X)} \\ \downarrow res^X_U & & \downarrow res^X_U \\ \mathcal{P}_{\mathrm{m,b}}(U,g) & \xrightarrow{\hat{\tau}^D_{U,g}} & Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(U)} \end{array}$$

Denote the restriction map $L^2(X, \wedge^{0,*}T^*X \otimes E) \to L^2(U, \wedge^{0,*}T^*X \otimes E|_U)$ given by multiplying with the characteristic function of U by π.

Let u be compactly supported section of $L^2(U, \wedge^{0,*}T^*X \otimes E|_U)$. Then by (HR00, 10.3.1.) there exists $\epsilon > 0$ so that for $|s| < \epsilon$, $e^{\sqrt{-1}sD^U_E}u = e^{\sqrt{-1}sD_E}\pi^*u$ are supported on U. Let ϕ be a normalizing function so that its Fourier transform is supported in the interval $[-\epsilon, \epsilon]$. Then by (HR00, 10.3.5.) we get that $\phi(D^U_E)u = \pi\phi(D_E)\pi^*u$. Since $\phi - \chi \in C_0(\mathbb{R})$, then by Lemma

4.1.1 $\chi(D_E^U) = \pi\chi(D_E)\pi^*$ in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C_0(U)}$. Therefore π is a chain map from $res_U^X \hat{\tau}_{X,g}^D(E,h)$ to $\hat{\tau}_{U,g}^D res_U^X(E,h)$.

Since $\pi\pi^* = Id$ and $\pi^*\pi - Id$ is multiplication by the characteristic function of $X \setminus U$ which is locally compact in $(\mathfrak{D}/\mathfrak{C})_{C_0(U)}$, then π induces an isomorphism.

Therefore there is a natural transformation from $res_U^X \hat{\tau}_{X,g}^D$ to $\hat{\tau}_{U,g}^D res_U^X$, meaning these two functors induce homotopic maps of K-theory spectra. □

Proposition 4.3.2. *Let X be a complex manifold. Then for each relatively compact open subset V of X there exists an exact functor τ_V^D that makes the square below commute up to homotopy. Furthermore, these functors are compatible with further restriction to open subsets, i.e. for an open subset W of V, the triangle on the bottom of the diagram commutes up to homotopy as well.*

$$\begin{array}{ccc} K(\mathcal{P}_{m,b}(X,g)) & \xrightarrow{\hat{\tau}_{X,g}^D} & K(Ac^b(\mathfrak{D}/\mathfrak{C})_{C_0(X)}) \\ \downarrow & & \downarrow{\scriptstyle res_U^X} \\ K(\mathcal{P}(X)) & \xrightarrow{\exists \tau_U^D} & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) \\ & \searrow^{\exists \tau_W^D} & \downarrow{\scriptstyle res_V^U} \\ & & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(W)}) \end{array} \tag{4.4}$$

Proof. For each object E of $\mathcal{P}(X)$, *choose* a Hermitian metric $h(E)$[1]. Then define $\tau_{V,h}^D(E) = res_V^X \hat{\tau}_{X,g}^D(E, h(E))$. Also, for a morphism of bundles $\varphi : E_1 \to E_2$, define $\tau_{V,h}^D(\varphi)$ through the

---

[1]Note that we are assuming the axiom of choice. Also, we are only working over a small skeletal subcategory of $\mathcal{P}(X)$.

composition below, where the first map is given by projection, and the last one is given by extension by zero.

$$
\begin{array}{c}
L^2(X, \wedge^{0,*} T^*X \otimes E_1) \\
\downarrow \\
L^2(V, \wedge^{0,*} T^*X \otimes E_1|_V) \\
\downarrow \hat{\tau}^D_{V,g}(\varphi|_V) \\
L^2(V, \wedge^{0,*} T^*X \otimes E_2|_V) \\
\downarrow \\
L^2(X, \wedge^{0,*} T^*X \otimes E_2)
\end{array}
$$

Note that $\hat{\tau}^D_{X,g}(\varphi)$ is *not* necessarily defined, as φ could induce an unbounded map of Hilbert spaces, however by restricting to the relatively compact open subset V, the composition above is indeed a well-defined map.

Since $\hat{\tau}^D_{V,g}$ is a functor, then $\tau^D_{V,h}$ is also a functor, i.e. for composable maps of bundles φ_1, φ_2, we have $\tau^D_{V,h}(\varphi_2 \circ \varphi_1) = \tau^D_{V,h}(\varphi_2) \circ \tau^D_{V,h}(\varphi_1)$. Exactness of $\tau^D_{V,h}$ also follows from that of $\hat{\tau}^D_{V,g}$. Hence we have an induced map of spectra

$$\tau^D_{V,h} : K(\mathcal{P}(X)) \to K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(V)}). \tag{4.5}$$

The square in the diagram from Equation 4.4 commutes (up to homotopy) because for any object (E_1, h_1) of $\mathcal{P}_{\mathrm{m,b}}(X, g)$, by Corollary 4.2.7, the identity map of E induces an isomorphism from $res_V^X \hat{\tau}_{X,g}^D(E_1, h_1)$ to $res_V^X \hat{\tau}_{X,g}^D(E_1, h(E_1)) = \tau_{V,h}^D(E)$. Also for a morphism $\varphi : (E_1, h_2) \to (E_2, h_2)$ in $\mathcal{P}_{\mathrm{m,b}}(X, g)$, the difference $res_V^X \hat{\tau}_{X,g}^D(\varphi) - \tau_{V,h}^D(\varphi)$ is locally compact, because multiplying by characteristic function of $X \setminus V$ is locally compact in $(\mathfrak{D}/\mathfrak{C})_{C_0(V)}$.

The functor defined $\tau_{-,h}^D$ commutes (up to homotopy) with restriction to further open subset $W \subset V$ because multiplying by characteristic function of $V \setminus W$ is locally compact in $(\mathfrak{D}/\mathfrak{C})_{C_0(W)}$. Therefore the triangle in the diagram from Equation 4.4 commutes as well.

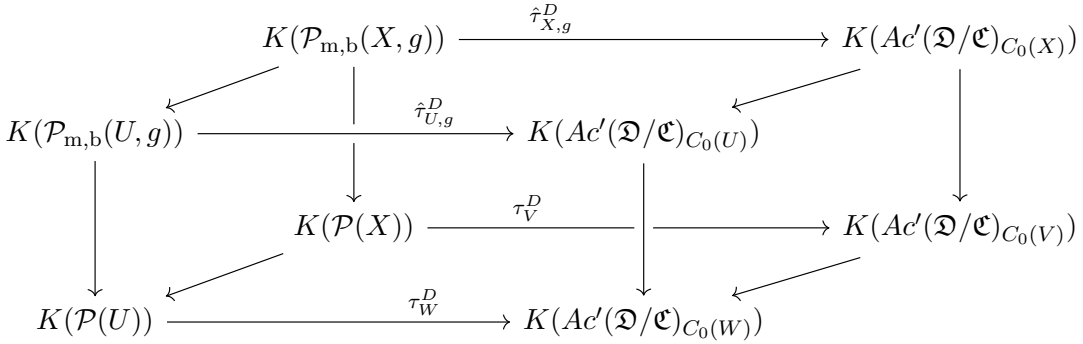
Note that the choices of metrics $h(E)$ on E do not affect the map from Equation 4.5 up to homotopy, because again by Corollary 4.2.7, for any two choices h_1, h_2, the objects $\tau_{V,h_1}^D(E), \tau_{V,h_2}^D(E)$ are naturally isomorphic, hence all the different functors are homotopic.

□

Corollary 4.3.3. *Let X be a complex manifold. Then the functor τ^D defined in Proposition 4.3.2 commutes with restriction to open subsets, i.e. for open subset U of X and relatively compact open subset V of X and open subset W of $U \cap V$ which is relatively compact as an open subset of U, the diagram below commutes up to homotopy.*

$$\begin{array}{ccc} K(\mathcal{P}(X)) & \xrightarrow{\tau_V^D} & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) \\ \downarrow{\scriptstyle res_U^X} & & \downarrow{\scriptstyle res_W^V} \\ K(\mathcal{P}(U)) & \xrightarrow{\tau_W^D} & K(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(W)}) \end{array}$$

Proof. Consider the diagram below, where all the arrows with no labels are the natural ones.



The squares on the left and the one on the right commute because restriction maps (and the forgetful functors $\mathcal{P}_{\mathrm{m,b}} \to \mathcal{P}$) are natural. By Lemma 4.3.1 the square on the top commutes up to homotopy. By Proposition 4.3.2 the squares in the back and on the front commute up to homotopy as well. This proves that the square on the bottom commutes up to homotopy. □

Remark 4.3.4. All the results in this section hold whether we use K^{alg} or K^{top}.

CHAPTER 5

MAIN RESULTS

5.1 K-theory of the Paschke category

In this section, we will compute the K-theory groups of the Paschke category and the Calkin-Paschke category.

Let A be a C^*-algebra. Let $K^{top}_i(A)$ denote the topological K-homology groups of A, which are contravariant functors of the C^*-algebra, and let $K^{top}_i(A)$ denote the topological K-theory groups of A, which are covariant functors. The reason for the unusual naming is that we are primarily interested in the case when A is the C^*-algebra $C_0(X)$ of continuous complex valued functions on the (locally compact and Hausdorf) topological space X which vanish at infinity, and in this case functoriality matches the expectations.

Let $\mathcal{A}$ be a topological exact category, and recall that $K^{top}(\mathcal{A})$ denotes the K-theory spectrum of $\mathcal{A}$ with respect to the fat geometric realization. Since the additivity theorem holds for K-theory of topological categories, this is a *connective spectrum*, i.e. there are no negative K-theory groups.

Here is the main result of this section.

Theorem 5.1.1. *The $(1-i)$'th topological K-homology group $K^{1-i}_{top}(A)$ of a C^*-algebra A, is isomorphic to the i'th topological K-theory groups of the exact C^*-categories $(\mathfrak{D}/\mathfrak{C})_A$ and*

$(\mathfrak{D}/\mathfrak{C})'_A$ *for* $i \geq 1$. *If* A *is unital and nuclear, then* $K^1_{top}(A) = K_0^{top}((\mathfrak{D}/\mathfrak{C})'_A)$ *as well. For a* $*$*-morphism* $f : A \to B$, *this isomorphisms commutes with respect to the pull-back maps* f^*.

In particular if $A = C_0(X)$ *for a locally compact Hausdorf topological space* X, *then the topological K-homology groups* $K^{top}_{i-1}(X)$ *are isomorphic to* $K_i((\mathfrak{D}/\mathfrak{C})'_{C_0(X)})$ *for* $i \geq 0$. *This isomorphism also commutes with the restriction maps to open subsets.*

Proof. Recall that the K_0 group of a Waldhausen category is the free abelian group generated by the weak equivalence classes of objects of the category, modulo the relations induced by the cofibration sequences. The same is true for *topological* Waldhausen categories (cf. (Wei13, 4.8.4.)). In particular since all the non-zero objects of the category $\mathfrak{Q}_A$ are isomorphic to each other by 2.2.5, $K_0(\mathfrak{Q}_A) = 0$.

In the case when A is a unital C^*-algebra, recall that $Ext(A)$ (BDF77) is defined as the semi-group of unitary equivalence classes of unital injective representations of A to the Calkin algebra $(\mathfrak{B}/\mathfrak{K})(H)$ (cf. (HR00, 2.7.1.)). When A is nuclear, then as a corollary of Voiculescu's theorem[1] we know that $Ext(A)$ is in fact a group, and isomorphic to the first K-homology group $K^1_{top}(A)$.

Lemma 5.1.2. *Let* A *be a unital and nuclear* C^**-algebra. Then* $K_0(\mathfrak{Q}'_A) = K^1_{top}(A)$.

---

[1]Note that for a nuclear C^*-algebra A and C^*-algebra B with a C^*-ideal K, a $*$-morphism $A \to B/K$ lifts to a *completely positive* map $A \to B$ (cf. (HR00, 3.3.6.)), and *Stinespring's theorem* (cf. (HR00, 3.1.3.)) shows that each completely positive map to $\mathfrak{B}(H)$, can be written as $V^*\rho V$, where $V : H \to H'$ is an isometry and ρ is a representation to $\mathfrak{B}(H')$. Then the restriction of ρ to the orthogonal complement of image of V induces a representation of A to the Calkin algebra (cf. (HR00, 3.1.6.)).

Proof. Let $\rho'_i : A \to (\mathfrak{B}/\mathfrak{K})(H_i)$ be non-zero objects in $\mathfrak{Q}'_A$ for $i = 1, 2$, which are isomorphic, i.e. there exists isomorphisms $T : \rho'_1 \to \rho'_2$ and $S : \rho'_2 \to \rho'_1$ which are inverses to each other. By definition of a C^*-category, for a positive operator $T^*T \in \mathfrak{Q}'_A(\rho'_1)$, there exists an operator $F \in \mathfrak{Q}'_A(\rho'_1)$ so that $F^*F = T^*T$. Since S, T are invertible, then so are F and $FS : \rho'_2 \to \rho'_1$. We have $(FS)^*(FS) = S^*F^*FS = S^*T^*TS = Id$ and $((FS)(FS)^*)F = (FSS^*F^*)F = FSS^*T^*T = FST = F$ in the category $\mathfrak{Q}'_A$. Hence FS is a unitary isomorphism in this category. Choose representatives for S, F in the category $\mathfrak{D}_A$. Because ρ'_1, ρ'_2 are unital, then it means FS is also a Fredholm operator, and in particular has closed image and finite dimensional kernel and cokernel. Hence there exists closed subspaces $H'_i \subset H_i$ of finite codimension so that $\pi_1 FS\iota_2 : H'_2 \to H'_1$ is a isomorphism of Hilbert spaces, where $\iota_i : H'_i \to H_i$ is the inclusion and $\pi_i : H_i \to H'_i$ is the projection for $i = 1, 2$. Since $\ker(FS) = coker(S^*F^*), \ker(S^*F^*) = coker(FS)$ then $\pi_1 FS\iota_2$ is a unitary map of Hilbert spaces. Let $\nu'_i = \pi_i \rho'_i \iota_i : A \to (\mathfrak{B}/\mathfrak{K})(H'_i)$ for $i = 1, 2$. Then we just showed that ν'_1, ν'_2 are unitarily equivalent. Since the difference between ρ'_i, ν'_i is a finite dimensional Hilbert space, then they represent the same class in $Ext(A)$ [1]. This shows that $Ext(A) \to K_0(\mathfrak{Q}'_A)$ is injective. Surjectivity follows from the definition. □

By Propositions 3.1.14 and 2.2.3, the maps of spectra induced by inclusion of subcategories $K(\mathfrak{Q}_A) \to K((\mathfrak{D}/\mathfrak{C})_A)$ and $K(\mathfrak{Q}'_A) \to K((\mathfrak{D}/\mathfrak{C})'_A)$ are both homotopy equivalences. By remark 2.2.6 and Proposition 3.1.14, the map $K((\mathfrak{D}/\mathfrak{C})_A) \to K((\mathfrak{D}/\mathfrak{C})'_A)$ induces an isomorphism on the i'th K-groups for $i \geq 1$.

---

[1] Let H''_i denote the orthogonal complement of H'_i in H_i, which is finite dimensional. Then the direct sum of the zero representation from A to $(\mathfrak{B}/\mathfrak{K})(H''_i)$ and ν'_i is equal to ρ'_i.

Let A be a C^*-algebra, let ρ be an ample representation of A, and let $\mathfrak{R}$ be a full subcategory of $\mathfrak{Q}_A$ with two objects: the zero representation $A \to 0$ and ρ. Then this is a C^*-category and also a skeleton for the category $\mathfrak{Q}_A$, as all ample representaions are isomorphic to each other. But since every short exact sequence in $\mathfrak{Q}_A$ splits, then by a result of Mitchener (Mit01), $\Omega\|wS.\mathfrak{Q}_A\|$ is homotopy equivalent to $BGL_\infty(Sk(\mathfrak{Q}_A))$, where the latter is defined in (Mit01, 6.1.) and $Sk(\mathfrak{Q}_A)$ denotes the skeleton of the additive category $\mathfrak{Q}_A$. Hence the K-theory space of $\mathfrak{Q}_A$ is homotopy equivalent to $BGL(\mathfrak{R})$, which by definition is homotopy equivalent to $BGL_\infty(\mathfrak{Q}(A))$.

Therefore when $i \geq 1$ we have the following sequence of isomorphisms of abelian groups, where the first isomorphism is given by Paschke duality (Pas81), the second isomorphism is one of the equivalent definitions of topological K-theory groups, the third and the fourth one were explained above, and the last one follows from Propositions 3.1.14 and 2.2.3.

$$\begin{aligned} K_{top}^{1-i}(A) \cong K_i^{top}(\mathfrak{Q}(A)) = \pi_i(BGL(\mathfrak{Q}(A))) \cong \pi_i(BGL_\infty(\mathfrak{R})) \\ \cong K_i^{top}(\mathfrak{Q}_A) \cong K_i^{top}((\mathfrak{D}/\mathfrak{C})_A). \end{aligned} \tag{5.1}$$

This means we have proved the first part of the theorem.

Let A, B be unital C^*-algebras with ample representations $\rho_A : A \to \mathfrak{B}(H_A)$ and $\rho_B : B \to \mathfrak{B}(H_B)$. Let $\alpha : A \to B$ be a unital map of C^*-algebras, then by Voiculescu's theorem there exists an isometry $V : H_B \to H_A$ so that $V^*\rho_A(a)V - \rho_B(\alpha(a))$ is compact for all $a \in A$. Note that $VV^* \in \mathfrak{B}(H_A)$ is a projection which commutes with the representation ρ_A (HR00, 3.1.6.)

modulo compact operators. Also note that $V : H_B \to VV^*H_A$ is an isomorphism of Hilbert spaces. Now the function $Ad_V(T) = VTV^*$ gives a map

$$Ad_V(-) : (\mathfrak{D}/\mathfrak{C})_B(\rho_B) \cong \mathfrak{Q}(B) \to \mathfrak{Q}(A) \cong (\mathfrak{D}/\mathfrak{C})_A(\rho_A),$$

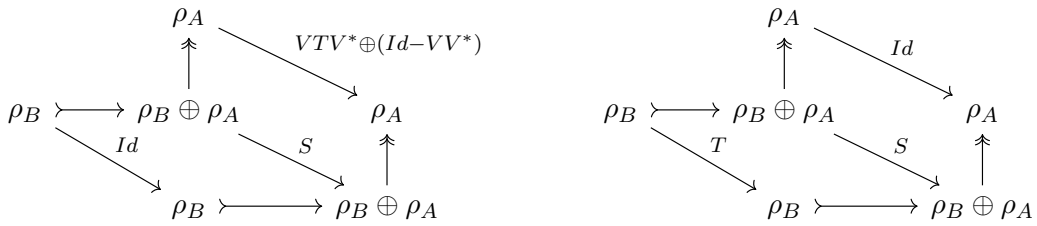
and hence induces a map on the K-homology groups which only depends on α, i.e. does not depend on the choices of ample representations ρ_A, ρ_B and the isometry V.

Let $T \in (\mathfrak{D}/\mathfrak{C})_B(\rho_B) \cong \mathfrak{Q}(B)$ be a unitary element. The pull-back map of K-homology groups sends T to the unitary

$$V(T - Id_{H_B})V^* + Id_{H_A} = VTV^* \oplus (Id - VV^*) \in (\mathfrak{D}/\mathfrak{C})_A(\rho_A) \cong \mathfrak{Q}(A).$$

On the other hand, pulling back in the Paschke category is given by precomposing with the representation. Hence $T \in (\mathfrak{D}/\mathfrak{C})_B(\rho_B)$ is sent to $T \in (\mathfrak{D}/\mathfrak{C})_A(\rho_B \circ \alpha)$. These two procedures give two different maps from the unitaries (or invertible elements) in $(\mathfrak{D}/\mathfrak{C})_B(\rho_B)$ to the topological space $\Omega^2\|S_\cdot(\mathfrak{D}/\mathfrak{C})_A\|$.

Let $S = Id_{H_B} \oplus (VTV^* \oplus (Id_{H_A} - VV^*)) \in \mathfrak{B}(H_B \oplus H_A)$. Consider the following two "prisms" in $wS_\cdot(\mathfrak{D}/\mathfrak{C})_A$ where the cofibration and the quotient maps on the left diagram are the trivial ones, but the cofibrations on the right diagram are given by $(0, V) : H_B \to H_B \oplus H_A$ and the quotient maps are given by $V + (Id_{H_A} - VV^*) : H_B \oplus H_A \to H_A$. By (HR00, 3.1.6.) these maps are pseudo-local. It is also easy to check that both diagrams below commute.



Since the fat geometric realization of a point is the infinite dimensional ball which is contractible, then in the fat geometric realization of $wS_\cdot(\mathfrak{D}/\mathfrak{C})_A$, the side of the prism $\Delta^1_{top}\times\Delta^2_{top}$ corresponding to the identity map is contractible. Hence we get a homotopy from S to $VTV^*\oplus(Id-VV^*)$ induced by the left diagram and also from S to T induced by the right diagram, by "sliding" one side of the prism towards the other along the contractible side (corresponding to identity map).[1] Therefore the two different push-forward maps are homotopic to each other and they induce the same map on the level of K-homology groups.[2].

In the context of Paschke categories restriction maps are defined similar to pull-back maps, i.e. by precomposing with the representation. To be more precise, let X be a locally compact

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[1]To be more precise, by the additivity theorem we know $t, s+q : \mathcal{E}(\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})_A$ are homotopic, where $\mathcal{E}$ is temporarily denoting the category of cofibration sequences, and s, t, q refer to the first (source), second (target), and the third (quotient) object in the cofibration sequence. By applying this to the prism on the right we get that $T + Id$ is homotopic to S, and by applying it on the prism on the left we get that $(VTV^* \oplus (Id - VV^*)) + Id$ is homotopic to S, but since Id is contractible in the fat geometric realization, then we get that T is homotopic to $VTV^* \oplus (Id - VV^*)$.

[2]One may wonder why we did not simply say that $VTV^* \oplus (Id - VV^*)$ is direct sum of VTV^* and $Id - VV^*$, and argue similar to above that the identity on the Hilbert space $(Id - VV^*)H_A$ corresponds to a contractible side of a prism, and then "slide" $VTV^* \oplus (Id_{H_A} - VV^*)$ directly onto VTV^* which is isomorphic to T in the category $(\mathfrak{D}/\mathfrak{C})_A$, to obtain a homotopy. The reason is that the restriction of ρ_A to $(Id - VV^*)H_A$ is only a representation up to compact operators, and the Hilbert space $(1 - VV^*)H_A$ does not come with a representation, which means we can not simply consider $(1 - VV^*)H_A$ as an object in $(\mathfrak{D}/\mathfrak{C})_A$. Note that however, we can consider the Hilbert space VV^*H_A together with the representation $V(\rho_B\alpha)V^*$.

and Hausdorf topological space, and let U be an open subset. The inclusion $j : U \hookrightarrow X$ induces an inclusion $j_* : C_0(U) \hookrightarrow C_0(X)$ of C^*-algebras, given by extending functions by zero. Then the restriction map sends the object $\rho : C_0(X) \to \mathfrak{B}(H)$ to $j^*(\rho) \coloneqq \rho \circ j_* : C_0(U) \to \mathfrak{B}(H)$.

We follow (RS13) to recall the process of defining the (wrong-way) restriction maps on the classical topological K-homology. Let $X, U, j, \rho : C_0(X) \to \mathfrak{B}(H)$ be as before. If we extend ρ to the Borel functions on X, then $\rho(\mathbb{1}_U)$ is a self-adjoint projection, where $\mathbb{1}_U$ is the characteristic function of the open subset U of X. Let H_U be the image of this projection, and define the representation $\rho_U : C_0(U) \to \mathfrak{B}(H_U)$ by $\rho_U(f) = \pi_U \rho(j_*(f))\iota_U$ where $\iota_U : H_U \to H$ is the inclusion, $\pi_U : H \to H_U$ is the projection, and $j_* : C_0(U) \to C_0(X)$ is extension by zero. The linear map $\mathfrak{B}(H) \to \mathfrak{B}(H_U)$ defined by $T \mapsto \pi_U T \iota_U$ maps $\mathfrak{D}_{C_0(X)}(\rho)$ to $\mathfrak{D}_{C_0(U)}(\rho_U)$, and $\mathfrak{C}_{C_0(X)}(\rho)$ to $\mathfrak{C}_{C_0(U)}(\rho_U)$. Hence there is an induced map $\mathfrak{Q}_\rho(C_0(X)) \to \mathfrak{Q}_{\rho_U}(C_0(U))$, which induces the restriction map from the K-homology groups of X to the K-homology groups of U.

But the representations $j^*\rho, \rho_U$ are naturally isomorphic; in fact the maps π_U, ι_U induce the isomorphisms. To show this, first note that π_U, ι_U commute with these representations since for $f \in C_0(U)$, we have $\rho_U(f)\pi_U = \rho_U(f)\rho(\mathbb{1}_U) = \rho(j_* f) = \pi_U j^* \rho(f)$. Also $\rho_U(\pi_U \iota_U - Id_{H_U})$ and $j^*\rho(\iota_U \pi_U - Id_H)$ are both zero. Therefore these two restriction functors are homotopic to each other as a maps to the $\Omega \| wS.(\mathfrak{D}/\mathfrak{C})_{C_0(U)} \|$, which means the two induced restriction maps on K-homology groups are equal to each other. □

5.2 Descent and The Riemann-Roch Transformation

Definition 5.2.1. Let $\mathcal{V}_\mathbb{C}$ denote the topological category where the objects are finite dimensional complex vector spaces, and morphisms are invertible linear maps. Let A be a nuclear

C^*-algebra. Then there is a biexact functor $\mathcal{V}_{\mathbb{C}} \times (\mathfrak{D}/\mathfrak{C})'_A \to (\mathfrak{D}/\mathfrak{C})'_A$ induced by taking the tensor product of the corresponding Hilbert space with the finite dimensional vector space. This induces a map of spectra

$$ku \wedge K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \to K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \tag{5.2}$$

where ku is the K-theory spectrum $K^{top}(\mathcal{V}_{\mathbb{C}})$, also known as the *connective complex K-theory spectrum.*

Let KU denote the *(non-connective) complex K-theory spectrum.*

Since we have only defined the functor $\tau^D_{X,g}$ on relatively compact open subsets, we will need a descent argument to glue them together. So far we have defined a connective K-homology spectrum for C^*-algebras. We need a non-connective K-homology spectrum to make the descent work. It is well known that the process below will give us the non-connective spectrum we need. We will only provide a sketch proof for this lemma.

Lemma 5.2.2. *By employing the above definition, we can consider the smash product of spectra* $K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \wedge_{ku} KU$. *This has the same homotopy groups as the* non-connective *topological K-homology of A.*

Sketch Proof: Recall that for a ring spectrum $X.$ and $b \in X_n$, multiplication by b induces a map $X. \to \Sigma^{-n}X.$. Then we define $X.[b^{-1}]$ to be the homotopy colimit of the telescope

$$X. \xrightarrow{b} \Sigma^{-n}X. \xrightarrow{\Sigma^{-n}b} \Sigma^{-2n}X. \xrightarrow{\Sigma^{-2n}b} \Sigma^{-3n}X. \to \dots.$$

Also, (stable) homotopy groups commute with this mapping telescope (cf. (EKMM97, 5.1.14.)). Let $\beta \in ku_2$ denote the *bott element*. Then it is well-known that KU is naturally homotopy equivalent to $ku[\beta^{-1}]$ (cf. (Sna81)).

Since $K^{top}((\mathfrak{D}/\mathfrak{C})'_A)$ is a ku-module, then there is a natural weak equivalence (cf. (EKMM97, 5.1.15.))

$$K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \wedge_{ku} ku[\beta^{-1}] \to K^{top}((\mathfrak{D}/\mathfrak{C})'_A)[\beta^{-1}].$$

Then it is easy to see that the homotopy groups of the latter is 2-periodic, as one could disregard the first n temrs in the mapping telescope, and that positive homotopy groups of $K^{top}((\mathfrak{D}/\mathfrak{C})'_A)$ are 2-periodic. The fact that positive homotopy groups of $K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \wedge_{ku} KU$ are isomorphic to the positive homotopy groups of $K^{top}((\mathfrak{D}/\mathfrak{C})'_A)$ follows from the (strongly converging) Atiyah-Hirzebruch spectral sequence (cf. (EKMM97, 4.3.7.)) and the fact that positive homotopy groups of ku and KU agree with each other. This finishes the proof. □

Proposition 5.2.3. *Let A be a C^*-algebra, and let $I \subset A$ be an ideal, so that the projection $\pi : A \to A/I$ has a completely positive section. Then*

$$K^{top}((\mathfrak{D}/\mathfrak{C})'_{A/I}) \wedge_{ku} KU \to K^{top}((\mathfrak{D}/\mathfrak{C})'_A) \wedge_{ku} KU \to K^{top}((\mathfrak{D}/\mathfrak{C})'_I) \wedge_{ku} KU$$

is a homotopy fiber sequence.

Proof. It is easy to observe that the composition of the two maps above is null-homotopic. Hence it suffices to show that the homotopy groups of the sequence above induce a long exact sequence of homotopy groups. This is a direct consequence of the six-term exact sequence of

K-homology groups (HR00, 5.3.10.), Lemma 5.2.2 which says that the homotopy groups agree with the K-homology groups and also Theorem 5.1.1, which says that the pull-back maps of the Paschke category agree with the classical pull-backs. □

We will give definition of descent with respect to *hypercovers* (DHI04, 4.2.) below. The definition essentially states when a presheaf of spectra is in fact a sheaf (up to homotopy). A hypercover over a *site* X is a *simplicial presheaf* $U.$ (Jar87, Sec 1.) with an augmentation $U. \to X$, which satisfies certain conditions. This can be thought of as a generalization of *Čech covers*, which have the form

$$\ldots \qquad \prod_{j_0,j_1,j_2}(U_{j_0} \cap U_{j_1} \cap U_{j_2}) \Longrightarrow \prod_{j_0,j_1}(U_{j_0} \cap U_{j_1}) \rightrightarrows \prod_{j_0}(U_{j_0}) \longrightarrow X$$

In fact, Čech covers are hypercovers of *height* zero. Also, for our purposes, *homotopy limits* refer to limits in the homotopy category of spectra.

Definition 5.2.4. (DHI04, 4.3.) Let X be an object in the site $\mathcal{C}$ (which can be thought of as a topological space). An object-wise fibrant simplicial presheaf $\mathscr{F}$ *satisfies descent* for a hypercover $U. \to X$ if the natural map from $\mathscr{F}(X)$ to the homotopy limit of the diagram

$$\prod_i \mathscr{F}(U_0^j) \rightrightarrows \prod_j \mathscr{F}(U_1^j) \Longrightarrow \prod_j \mathscr{F}(U_2^j) \ldots \tag{5.3}$$

is a weak equivalence. Here the products range over the representable summands of each U_n. If $\mathscr{F}$ is not object-wise fibrant, we say it satisfies descent if some object-wise fibrant replacement for $\mathscr{F}$ does.

The fact that topological K-homology satisfies descent is essentially a result of the *Atiyah-Hirzebruch spectral sequence* (cf. (Bro73)(AH61)). In fact, by (DI04, 4.3.), for a hypercover $U_\cdot \to X$, we have weak equivalences $\mathrm{hocolim} U_\cdot \to |U_\cdot| \to X$, where the second map is induced by taking geometric realization. Since taking smash product in the homotopy category of spectra preserves colimits, then by applying the smash product with an Ω-spectrum $E_\cdot$, the colimit of the diagram from Equation 5.3 smashed with the spectrum $E_\cdot$ is weakly equivalent to $E_\cdot \wedge X$. When the Ω-spectrum $E_\cdot = KU$ is the (non-connective) topological K-theory spectrum, this proves descent. (Also see (AW14, 2.2.) for the case of twisted topological K-theory of CW-complexes.)

Now we are ready to define the Riemann-Roch transformation over the relatively compact open subsets of a complex manifold, which in turn induce the Riemann-Roch transformation over the manifold itself.

Definition 5.2.5. Let X be a complex manifold, and let V be a relatively compact open subset of X. Define the functor $\tau_{X,V}$ in the homotopy category of spectra as the composition below[1]

---

[1]When X is a collection of finitely many points, then $L^2(X)$ is a finite dimensional vector space, which is isomorphic to the zero Hilbert space in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C(X)}$. Hence the map of spectra τ_X^D is the zero map, and instead of using the composition

$$K^{alg}(\mathcal{P}(X)) \xrightarrow{\tau_X^D} K^{alg}(Ac'(\mathfrak{D}/\mathfrak{C})_{C(X)}) \xrightarrow{\tau_{C(X)}^H} K^{alg}(Bi'(\mathfrak{D}/\mathfrak{C})_{C(X)}),$$

we use the map

$$K^{alg}(\mathcal{P}(X)) \xrightarrow{\tau_{C(X)}^H \circ \tau_X^D} K^{alg}(Bi'(\mathfrak{D}/\mathfrak{C})_{C(X)}).$$

$$
\begin{array}{ll}
K^{alg}(\mathcal{P}(X)) & \\
\quad \downarrow \tau^{D}_{V,g} & \text{Defined in Proposition 4.3.2.} \\
K^{alg}(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) & \\
\quad \downarrow \tau^{H}_{C_0(V)} & \text{Defined in Equation 3.3.} \\
K^{alg}(Bi'(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) & \\
\quad \downarrow & \\
K^{alg}(Bi^{b}(\mathfrak{D}/\mathfrak{C})_{C_0(V)}) & \\
\quad \downarrow \tau^{G}_{(\mathfrak{D}/\mathfrak{C})_{C_0(V)}} & \text{Defined in Equation 3.1.} \\
\Omega K^{alg}((\mathfrak{D}/\mathfrak{C})_{C_0(V)}) & \\
\quad \downarrow c & \text{Definition 3.1.5.} \\
\Omega K^{top}((\mathfrak{D}/\mathfrak{C})_{C_0(V)}) & \\
\quad \downarrow & \\
\Omega K^{top}((\mathfrak{D}/\mathfrak{C})'_{C_0(V)}) \wedge_{ku} KU & \\
\quad \downarrow \cong & \text{By Lemma 5.2.2.} \\
K^{top}(V) &
\end{array}
$$

Note that except for the first one, all the maps above are functorially defined. Also, we showed in Proposition 4.3.2 that in the homotopy category of spectra, τ^D is compatible with

restriction to further open subsets. Therefore for a hypercover $V_\cdot$ so that all the open sets in V_n are relatively compact subsets of the open sets in V_{n-1} (and in particular, all are relatively compact in X), there is an induced map $\tau : K^{alg}(\mathcal{P}(X)) \to \text{holim } K^{top}(V_\cdot)$ in the homotopy category of spectra, where the latter is referring to the homotopy colimit of the diagram from Equation 5.3 for the hypercover $V_\cdot \to X$. Since topological K-homology satisfies descent, then there is an induced map in the homotopy category of spectra

$$\tau_X : K^{alg}(\mathcal{P}(X)) \to K^{top}(X). \tag{5.4}$$

By taking a finer cover if necessary, one can see that the map above is independent of the choice of the hypercover $V_\cdot$.

Proposition 5.2.6. *The Riemann-Roch transformation defined above commutes with restriction to open subsets. In other words, for a complex manifold X and an open subset U of X, the diagram below commutes in the homotopy category of spectra.*

$$\begin{array}{ccc} K^{alg}(\mathcal{P}(X)) & \longrightarrow & K^{alg}(\mathcal{P}(U)) \\ \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_U} \\ K^{top}(X) & \longrightarrow & K^{top}(U) \end{array}$$

Proof. Let $V_\cdot \to X$ be a hypercover so that all the open sets in V_n are relatively compact in V_{n-1}. Choose a hypercover $W_\cdot \to U$ with the same condition as $V_\cdot$ so that W_n is finer than $V_n \cap U$, i.e. each open set in W_n (which is a relatively compact open subset of U) is contained in (the intersection of U with) some open set in V_n. Hence for any relatively compact open set W_n^j in the hypercover $W_\cdot$ there exists relatively compact open set V_n^j of the hypercover $V_\cdot$

so that $W_n^j \subset V_n^j \cap U$ is relatively compact. Hence by Corollary 4.3.3 and by naturality of all the other maps in definition of τ, the diagram below commutes in the homotopy category of spectra.

$$\begin{array}{ccccc} & & K^{alg}(\mathcal{P}(X)) & \longrightarrow & K^{alg}(\mathcal{P}(U)) \\ & \overset{\tau_X}{\swarrow} & \downarrow{\scriptstyle \tau^D_{V_n^j}} & & \downarrow{\scriptstyle \tau^D_{W_n^j}} \\ & & K^{alg}(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(V_n^j)}) & \longrightarrow & K^{alg}(Ac'(\mathfrak{D}/\mathfrak{C})_{C_0(W_n^j)}) \\ & & \downarrow & & \downarrow \\ K^{top}(X) & \longrightarrow & K^{top}(V_n^j) & \longrightarrow & K^{top}(W_n^j) \end{array}$$

Hence there is a unique map from $K^{top}(X)$ to the holim $K^{top}(W_\cdot)$ that makes the diagram above commute (in the homotopy category of spectra), where the homotopy limit is taken on the diagram from Equation 5.3 for the hypercover $W_\cdot \to U$. But this homotopy limit is weakly equivalent to $K^{top}(U)$ because topological K-homology satisfies descent. This finishes the proof. □

Proposition 5.2.7. *Let X be a compact complex manifold, and let $f : X \to *$ be the map to a single point. Then the diagram below commutes in the homotopy category of spectra.*

$$\begin{array}{ccc} K^{alg}(\mathcal{P}(X)) & \xrightarrow{Rf_*} & K^{alg}(\mathcal{P}(*)) \\ \downarrow{\scriptstyle \tau_X} & & \downarrow{\scriptstyle \tau_*} \\ K^{top}(X) & \xrightarrow{f_*} & K^{top}(*) \end{array}$$

Proof. First note that every coherent sheaf over a point is also a locally free sheaf, and that Rf_* sends a locally free sheaf (corresponding to the holomorphic vector bundle E) over X to the cohomology of its Dolbeault complex (considered as a complex of vector bundles over a point). Choose Hermitian metrics g, h on X, E, and recall that $\hat{\tau}^D_{X,g}(E,h)$ denotes the chain complex in Equation 4.1. As a result of the Hodge decomposition A.0.5, there is a map of

chain complexes from $\hat{\tau}^D_{X,g}(E,h)$ to the cohomology of the corresponding Dolbeault complex, induced by sending the subspace orthogonal to the kernel of $\chi(D_E)$ in $L^2(X, \wedge^{0,*}T^*X \otimes E)$ to zero, which in turn (because of naturality of the Dolbeault cohomology and $\hat{\tau}^D$ with respect to holomorphic bundle maps), induces a natural transformation from the exact functor

$$t \coloneqq \tau^H_{C(*)} \circ f_* \circ \hat{\tau}^D_{X,g} : \mathcal{P}_{\mathrm{m,b}}(X,g) \to Bi'(\mathfrak{D}/\mathfrak{C})_{C(*)}$$

to the exact functor $q \coloneqq \tau^H_{C(*)} \circ \tau^D_* \circ Rf_*$.

Let $s_g(E,h) \in Ac'(\mathfrak{D}/\mathfrak{C})_{C(*)}$ denote the kernel of the map from $\hat{\tau}^D_{X,g}(E,h)$ to the cohomology of the corresponding Dolbeault complex. Similar to $\hat{\tau}^D_{X,g}$, one can show that $s_g : \mathcal{P}_{\mathrm{m,b}}(X,g) \to Ac'(\mathfrak{D}/\mathfrak{C})_{C(*)}$ is an exact functor. By Proposition 4.1.3, $s_g(E,h)$ is isomorphic to a finite direct sum of the trivial chain complex $0 \to H \xrightarrow{Id} H \to 0$ and its shifts in $Ac'(\mathfrak{D}/\mathfrak{C})_{C(*)}$ (for example, take H to be the image of $\chi(D_E)$ in $L^2(X, \wedge^{0,i}T^*X \otimes E)$ for different values of i), hence the kernel $\tau^H_{C(*)} \circ s_g$ of $t \twoheadrightarrow q$ is null-homotopic. The assertion now follows from the additivity theorem, and the fact that every other map in τ is natural.

□

5.3 Cap Product

In the last section, let us emphasize on the case when the C^*-algebra A is unital, and how one could define a pairing between the K-theory and K-homology of A using the Paschke category.

When A is unital, we will define the *Euler characteristic* of an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. Recall that if

$$\ldots \to \rho_i \xrightarrow{T_i'} \rho_{i+1} \xrightarrow{T_{i+1}'} \rho_{i+2} \to \ldots$$

is a chain complex in $(\mathfrak{D}/\mathfrak{C})_A$, and T_i are representatives for T_i' in $\mathfrak{D}_A$, then the composition $T_{i+1} \circ T_i$ may not be zero; it only has to be locally compact. If the representations are all unital, then $T_{i+1} \circ T_i$ has to be compact. This is still not enough for us to be able to take the quotient of $\ker(T_{i+1})$ by the image of T_i, as the image may not even be a subset of the kernel. However, we can get around this issue. Recall the following definition from (Seg70, Sec 1.) and the main result of (Tar07).

Definition 5.3.1. Let $\ldots \to V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \to \ldots$ be a complex of Hilbert spaces and bounded linear operators. Then it is called a *Fredholm complex* if all the T_i's have closed images and the cohomology is finite dimensional at every step.

Equivalently, we may define a complex of bounded operators between Hilbert spaces as before to be a Fredholm complex if there exists bounded operators $S_i : V_{i+1} \to V_i$ so that $T_{i-1}S_{i-1} + S_iT_i - Id_{V_i}$ is a compact operator for all i.

If the complex is bounded, i.e. $V_i = 0$ for all but finitely many values of i, then define its *Euler characteristic* by

$$\chi(V_\cdot) = \textstyle\sum_i (-1)^i H^i(V_\cdot)$$

We can consider the Euler characteristic as a formal difference of two finite dimensional subspaces of $\oplus_i V_i$.

Proposition 5.3.2. *(Tar07, 3.1, 4.2.) Let* $\ldots \xrightarrow{T'_{i-1}} V_i \xrightarrow{T'_i} V_{i+1} \xrightarrow{T'_{i+1}} \ldots$ *be a bounded above exact sequence in* $(\mathfrak{B}/\mathfrak{K})$*. Then there are morphisms* $T_i \in \mathfrak{B}(V_i, V_{i+1})$ *so that* T_i *is a representative for* T'_i*, and also* $T_{i+1} \circ T_i = 0$*. Hence the new complex has a well-defined cohomology. Also the sequence* $\ldots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} \ldots$ *is a Fredholm complex, and if the complex is both bounded above and below, then the Euler characteristic of the complex is independent of the choices of* T_i*'s.(In the sense that for the finite dimensional subspaces* V^+, V^-, W *of the Hilbert space* H*, we consider the formal differences* $V^+ - V^-$ *and* $V^+ \oplus W - V^- \oplus W$ *to be equivalent.)*

Let A be a unital C^*-algebra, and let

$$\ldots \to \rho_i \xrightarrow{T'_i} \rho_{i+1} \xrightarrow{T'_{i+1}} \rho_{i+2} \to \ldots$$

be a bounded exact sequence (i.e. there are only finitely many non-zero objects) in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. Then by the argument proving Proposition 2.2.3, we know there exists natural choices of unital representations $\hat{\rho}_i$ which are isomorphic to ρ_i. This induces a new exact sequence

$$\ldots \to \hat{\rho}_i \xrightarrow{\hat{T}_i} \hat{\rho}_{i+1} \xrightarrow{\hat{T}_{i+1}} \hat{\rho}_{i+2} \to \ldots$$

where all the representations are unital and hence $\hat{T}_{i+1}\hat{T}_i$ is compact for all i, and this induces an exact sequence in $(\mathfrak{B}/\mathfrak{K})$. Therefore by Proposition 5.3.2, the exact sequence above has a well-defined euler characteristic. Note that this process can not be replicated for the Calkin-Paschke category, as the choice of the projection $\pi \in \mathfrak{B}(H)$ corresponding to $\rho'(1)$ may affect the index. To sum it all up:

Corollary 5.3.3. *Let A be a unital C^*-algebra, and let $(\rho., T.)$ be an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ with finitely many non-zero objects. Then the procedure above defines the Euler characteristic of this complex. The Euler characteristic defined is additive with respect to exact sequences in the category $Ac'(\mathfrak{D}/\mathfrak{C})_A$. (This is* not *true for the category $Ac^b(\mathfrak{D}/\mathfrak{C})_A$.)*

Remark 5.3.4. When A is not unital, the argument in (Tar07) does not work anymore; as it relies on the fact that if $Id_H - \Delta \in \mathfrak{K}(H)$, then Δ has a closed image. This is no longer the case if we replace $\mathfrak{K}(H)$ by locally compact operators.

In a slightly different direction, let us define a natural pairing between projective modules and representations of a C^*-algebra.

Definition 5.3.5. Let R be a ring. Recall that a bi-module M over R is equipped with an action of R from both the left and the right side, though these actions do not have to commute. Denote the exact category of finitely generated projective *bi-modules* on R by $\mathcal{P}(R)$.

Note that for any finitely generated projective R-module, there exists an integer n, and an inclusion $\iota : P \to R^n$ of R-modules.

We are interested in the particular case when $R = A$ is a unital C^*-algebra. Let $\mathcal{P}_{\mathrm{m}}(A)$ denote the category of finitely generated projective A-bi-modules with an inner-product structure. Morphisms in $\mathcal{P}_{\mathrm{m}}(A)$ are the (not necessarily norm-preserving) morphisms between the projective modules.

Let X be a compact Hausdorf space, in particular a compact manifold. Then denote the exact category of topological (complex) vector bundles on X by $\mathcal{P}^{\mathrm{t}}(X)$. Recall the category

$\mathcal{P}_{\mathrm{m}}(X)$ from definition 4.1.7. Note that by the Serre-Swan theorem (Swa62, Thm 2.), $\mathcal{P}^{\mathrm{t}}(X) = \mathcal{P}(C(X))$, hence $\mathcal{P}^{\mathrm{t}}_{\mathrm{m}}(X) = \mathcal{P}_{\mathrm{m}}(C(X))$.

Since we will be only working with finitely generated projective bi-modules on a unital C^*-algebra A, from now on, we refer to them as projective A-modules.

Definition 5.3.6. Let A be a unital C^*-algebra and let $P \in \mathcal{P}_{\mathrm{m}}(A)$. Let ρ be an object in $(\mathfrak{D}/\mathfrak{C})_A$. Then we define the representation $P \otimes \rho : A \to \mathfrak{B}(P \otimes_A H)$, where we are considering H as a left A-module through the representation ρ, and the representation $P \otimes_A \rho$ acts on the Hilbert space $P \otimes_A H$ on the *left*.

We follow (Ati70) to show that $P \otimes_A H$ is in fact a Hilbert space, and hence the definition above makes sense.

Since P is a finitely generated projective module, there exists a norm preserving A-module surjection $\pi : A^n \to P$, with a norm preserving A-module section $\iota : P \to A^n$. Without loss of generality, we can assume that $\iota\pi$ is a self-adjoint projection on $A^{\oplus n}$. Now let $\iota\pi(e_i) = \sum_j e_j a_i^j$ for $1 \leq i \leq n$, where $e_1, \ldots, e_n$ are the standard basis for A^n, and $a_i^j \in A$. Define the linear operator $\hat{\pi}$ on $H^{\oplus n} \cong A^n \otimes_A H$ by

$$\hat{\pi}(e_i \otimes_A h) = \sum_j e_j \otimes_A \rho(a_i^j)h.$$

It is easy to check that this is in fact a self-adjoint projection, and since $\iota\pi$ is a map of left A-modules,[1] then $\hat{\pi}$ also commutes with the action of A on the left:

$$\rho^{\oplus n}(a)\hat{\pi}(e_i \otimes_A h) = \rho^{\oplus n}(a)\sum_j e_j \otimes_A \rho(a_i^j)h = \sum_j a.e_j \otimes_A \rho(a_i^j)h = \hat{\pi}\rho^{\oplus n}(a)(e_i \otimes_A h).$$

Now let $V \subset H^{\oplus n}$ be the image of $\hat{\pi}$, and let $\hat{\iota} : V \to H^{\oplus n}$ be the inclusion, then consider the composition $A \xrightarrow{\rho^{\oplus n}} \mathfrak{B}(H^{\oplus n}) \to \mathfrak{B}(V)$, where the last map sends T to $\hat{\pi}T\hat{\iota}$ (we are abusing the notation and denoting the composition of $\hat{\pi}$ with the orthogonal projection $H^{\oplus n} \to V$ by $\hat{\pi}$ as well.). It is easy to check that the compositions below are inverses to each other.

$$V \xrightarrow{\hat{\iota}} H^{\oplus n} \cong A^n \otimes_A H \xrightarrow{\pi \otimes_A Id} P \otimes_A H$$
$$V \xleftarrow{\hat{\pi}} H^{\oplus n} \cong A^n \otimes_A H \xleftarrow{\iota \otimes_A Id} P \otimes_A H$$

Since all the maps above commute with multiplication by A in $\mathfrak{D}_A$, this induces structure of a Hilbert space on $P \otimes_A H$.

Proposition 5.3.7. *Let A be a unital C^*-algebra. The tensor product introduced in definition 5.3.6, induces a biexact functor*

$$\cap_A : \mathcal{P}_m(A) \times (\mathfrak{D}/\mathfrak{C})_A \to (\mathfrak{D}/\mathfrak{C})_A \tag{5.5}$$

---

[1] This is the only part of the argument where we use the left action of the non-commutative C^*-algebra A on the projective module P.

which we will call the cap product. *This induces a pairing on the level of K-theory spectra*

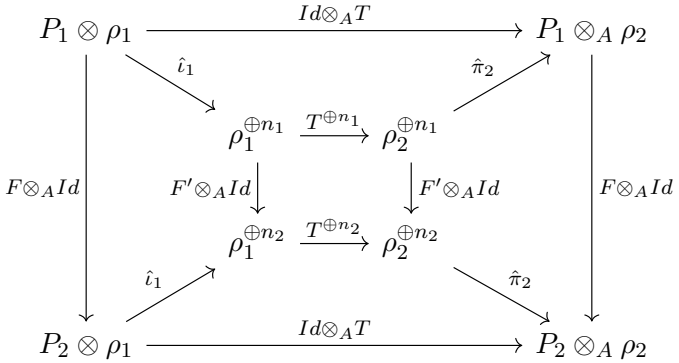
$$\cap_A : K^{alg}(\mathcal{P}_m(A)) \wedge K((\mathfrak{D}/\mathfrak{C})_A) \to K((\mathfrak{D}/\mathfrak{C})_A). \tag{5.6}$$

Proof. First we need to check functoriality. Let $F : P_1 \to P_2$ be a morphism in $\mathcal{P}_{\mathrm{m}}(A)$. Then we can consider the morphism $F \otimes_A Id : P_1 \otimes_A \rho \to P_2 \otimes_A \rho$ in the category $\mathfrak{D}_A$ where $p \otimes_A h \mapsto F(p) \otimes_A h$. Note that for $a \in A$, $F(p) \otimes_A \rho(a)h = F(p).a \otimes_A h = F(p.a) \otimes_A h$. Hence this is well defined and commutes with multiplication by A. It is clear that this process is functorial, i.e. $(F_2 \otimes_A Id) \circ (F_1 \otimes_A Id) = F_2F_1 \otimes_A Id$ in $\mathfrak{D}_A$.

Let $T : \rho_1 \to \rho_2$ be a morphism in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. Then we define $Id \otimes_A T : P \otimes_A \rho_1 \to P \otimes_A \rho_2$ as follows. Let $\pi : A^n \to P$ be a norm preserving surjective map of A-modules, let $\iota : P \to A^n$ be the corresponding inclusion of A-modules, and let $\hat{\pi}_i \in \mathfrak{B}(H_i^{\oplus n})$ be the projection corresponding to $P \otimes_A H_i$ for $i = 1, 2$, and $\hat{\iota}_i$ be the inclusion of its image V_i in the corresponding Hilbert space. Consider $T^{\oplus n} : A^n \otimes_A \rho_1 \cong \rho_1^{\oplus n} \to \rho_2^{\oplus n} \cong A^n \otimes_A \rho_2$, and define $Id \otimes_A T = \hat{\pi}_2 T^{\oplus n} \hat{\iota}_1$. Note that this is a pseudo-local operator, as both $\hat{\pi}_2, \hat{\iota}_1$ commute with the representations and T is pseudo-local. Also, since T commutes with multiplication by $a_i^j \in A$ modulo compact operators, then $\hat{\pi}_2 T^{\oplus n} - T^{\oplus n} \hat{\pi}_1$ is compact. This shows that the definition for $Id \otimes_A T$ is independent of the choice of the projection π up to locally compact operators. This also shows that for morphisms $T_1 : \rho_1 \to \rho_2$ and $T_2 : \rho_2 \to \rho_3$ in the Paschke category, the compositions $(Id \otimes_A T_2) \circ (Id \otimes_A T_1) = \hat{\pi}_3 T_2^{\oplus n} \hat{\iota}_2 \hat{\pi}_2 T_1^{\oplus n} \iota_1 = \hat{\pi}_3 T_2^{\oplus n} T_1^{\oplus n} \hat{\pi}_1 \hat{\iota}_1$ are equal to each other modulo locally compact operators. Hence this process is functorial.

Remark 5.3.8. The map $Id \otimes_A T : P \otimes_A \rho_1 \to P \otimes_A \rho_2$ is *not* well-defined in the category $\mathfrak{D}_A$.

Let ρ_1, ρ_2 be two objects of $(\mathfrak{D}/\mathfrak{C})_A$, let $T : \rho_1 \to \rho_2$ be a morphism, let P_1, P_2, be two objects of $\mathcal{P}_{\mathrm{m}}(A)$, and let $F : P_1 \to P_2$ be a morphism of A-modules. Choose the norm preserving inclusions $\iota_i : P_i \to A^{n_i}$ of A-modules, so that there exist a map of A-modules $F' : A^{n_1} \to A^{n_2}$ that makes the corresponding diagram commute. Then the square on the left and the one on the right commutes in the category $\mathfrak{D}_A$. Consider $e_i \otimes_A h \in A^{n_1} \otimes_A H$, we have $(F' \otimes_A Id)T^{\oplus n_1}(e_i \otimes h) = (b_{1,i}T(h), \ldots, b_{n_2,i}T(h))$, where $b_{j,i}$ is the j'th term in $F'(e_i) \in A^{n_2}$, and $T^{\oplus n_2}(F' \otimes_A Id)(e_i) = (T(b_{1,i}h), \ldots, T(b_{n_2,i}h))$. Since T is pseudo-local, then the square in the center also commutes in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. Hence functoriality in two directions are compatible with each other.



It is easy to check that if T is invertible, then so is $Id \otimes_A T$, and if F is an isomorphism of A-modules, then $F \otimes_A Id$ is a pseudo-local isomorphism of Hilbert spaces. This procedure is exact in both variables, because if $T_2 \circ T_1 = 0$ then $(Id \otimes_A T_2) \circ (Id \otimes_A T_1) = 0$ and similarly for F. Also, an exact sequence $(\rho_\cdot, T_\cdot)$ in $(\mathfrak{D}/\mathfrak{C})_A$ has a contracting homotopy $S_\cdot$, which translates into a contracting homotopy $Id \otimes_A S_\cdot$ for the sequence $(P \otimes_A \rho_\cdot, Id \otimes_A T_\cdot)$. Also a short exact

sequence $(P_\cdot, F_\cdot)$ of projective modules splits, i.e. has a contracting homotopy which again gives a contracting homotopy for the sequence $(P_\cdot \otimes_A \rho, F_\cdot \otimes_A Id)$.

Let $F_1 : P_1 \to P_2$ be an admissible monomorphism in $\mathcal{P}_{\mathrm{m}}(A)$ and consider an exact sequence

$$0 \longrightarrow \rho_1 \underset{S_1}{\overset{T_1}{\rightleftarrows}} \rho_2 \underset{S_2}{\overset{T_2}{\rightleftarrows}} \rho_3 \longrightarrow 0$$

with a choice of contracting homotopy in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. Then there is a map

$$P_1 \otimes_A \rho_3 \oplus P_2 \otimes_A \rho_1 \xrightarrow{(Id\otimes_A S_2)\oplus Id} P_1 \otimes_A \rho_2 \oplus P_2 \otimes_A \rho_1 \to (P_1 \otimes_A \rho_2) \cup_{(P_1\otimes_A \rho_1)} (P_2 \otimes_A \rho_1)$$

which induces an isomorphism between the first object and the last object in the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$. The map $P_1 \otimes_A \rho_3 \oplus P_2 \otimes_A \rho_1 \xrightarrow{(F\otimes_A S_2, Id\otimes_A T_1)^{\mathrm{t}}} P_2 \otimes_A \rho_2$ is an admissible monomorphism, whose cokernel is $P_2 \otimes_A \rho_2 \xrightarrow{F_2\otimes_A T_2} P_3 \otimes \rho_3$, where $F_2 : P_2 \to P_3$ is cokernel of F_1, and the contracting homotopies are the trivial ones induced by contracting homotopies of $\rho_\cdot$ and $F_\cdot$. This proves that $\cap_A$ is biexact, hence by Proposition 3.1.12 induces a map of K-theory spectra. □

Let $f : A \to B$ be a unital map between unital C^*-algebras. Recall there is an exact *push-forward* functor $f_* : \mathcal{P}(A) \to \mathcal{P}(B)$ and $f_* : \mathcal{P}_{\mathrm{m}}(A) \to \mathcal{P}_{\mathrm{m}}(B)$ defined by $f_*(P) = P \otimes_A B$ (for the push-forward to be a map of bi-modules, we need the image of A under f to be in the center of B). There is also a pull-back map $f^* : (\mathfrak{D}/\mathfrak{C})_B \to (\mathfrak{D}/\mathfrak{C})_A$. One could ask about the relation between the pairing defined above and these structures.

Proposition 5.3.9. *The pairing defined in Proposition 5.3.7 is natural in the sense that for a unital map $f : A \to B$ of unital C^*-algebras (whose image is in the center of B), the diagram below commutes up to homotopy*

$$\begin{array}{ccc}
 & K(\mathcal{P}_m(A)) \wedge K((\mathfrak{D}/\mathfrak{C})_B) & \\
 {\scriptstyle f_*\times Id}\swarrow & & \searrow{\scriptstyle Id\times f^*} \\
K(\mathcal{P}_m(B)) \wedge K((\mathfrak{D}/\mathfrak{C})_B) & & K(\mathcal{P}_m(A)) \wedge K((\mathfrak{D}/\mathfrak{C})_A) \\
 {\scriptstyle \cap_B}\downarrow & & \downarrow{\scriptstyle \cap_A} \\
K((\mathfrak{D}/\mathfrak{C})_B) & \xrightarrow{f^*} & K((\mathfrak{D}/\mathfrak{C})_A).
\end{array}$$

Proof. Consider the diagram below

$$\begin{array}{ccc}
 & \mathcal{P}_{\mathrm{m}}(A) \times (\mathfrak{D}/\mathfrak{C})_B & \\
 {\scriptstyle f_*\times Id}\swarrow & & \searrow{\scriptstyle Id\times f^*} \\
\mathcal{P}_{\mathrm{m}}(B) \times (\mathfrak{D}/\mathfrak{C})_B & & \mathcal{P}_{\mathrm{m}}(A) \times (\mathfrak{D}/\mathfrak{C})_A \\
 {\scriptstyle \cap_B}\downarrow & & \downarrow{\scriptstyle \cap_A} \\
(\mathfrak{D}/\mathfrak{C})_B & \xrightarrow{f^*} & (\mathfrak{D}/\mathfrak{C})_A.
\end{array}$$

Let $\rho : B \to \mathfrak{B}(H)$ be a representation, and let P be an object in $\mathcal{P}_{\mathrm{m}}(A)$. We can consider H as a left A-module through the representation $f^*\rho : A \to B \to \mathfrak{B}(H)$. It is straightforward to check that the natural map of Hilbert spaces $P \otimes_A H \to (P \otimes_A B) \otimes_B H$ defined by $p \otimes_A h \mapsto (p \otimes_A 1) \otimes_B h$ is well-defined, and has a two-sided inverse given by $(p \otimes_A b) \otimes_B h \mapsto p \otimes_A \rho(f(b))h$. This isomorphism is pseudo-local, hence induces a natural isomorphism between $f^*\left((P \otimes_A B) \otimes_B \rho\right)$ and $P \otimes_A f^*\rho$ in the category $\mathfrak{D}_A$. Hence the diagram above commutes up to natural isomorphisms. □

Remark 5.3.10. One can replace the Paschke category $(\mathfrak{D}/\mathfrak{C})_A$ with the category $Ac^b(\mathfrak{D}/\mathfrak{C})_A$ (or $C(\mathfrak{D}/\mathfrak{C})_A, C^b(\mathfrak{D}/\mathfrak{C})_A, Ac(\mathfrak{D}/\mathfrak{C})_A$) in Propositions 5.3.7 and 5.3.9 and the same result would still hold. However, we can not necessarily replace $Ac'(\mathfrak{D}/\mathfrak{C})_A$, as morphisms in $Ac'(\mathfrak{D}/\mathfrak{C})_A$ come from $\mathfrak{D}_A$, but pairing a morphism with the identity on a projective module is only well-defined up to compact operators.

Fix an object $(\rho_\cdot, T_\cdot)$ of $Ac'(\mathfrak{D}/\mathfrak{C})_A$, since for a morphism $F : P_1 \to P_2$ in $\mathcal{P}_{\mathrm{m}}(A)$, the morphism $F \otimes_A Id : P_1 \otimes_A \rho_\cdot \to P_2 \otimes_A \rho_\cdot$ is well-defined in $\mathfrak{D}_A$, hence we obtain a functor

$$\mathcal{P}_{\mathrm{m}}(A) \to Ac'(\mathfrak{D}/\mathfrak{C})_A \tag{5.7}$$

which maps P to $P \cap_A (\rho_\cdot, T_\cdot) = (P \otimes_A \rho_\cdot, Id \otimes_A T_\cdot)$.

Definition 5.3.11. Let X be a compact complex manifold, let g be a Hermitian metric on X, let $X \times \mathbb{C}$ denote the trivial rank one bundle on X, and let E be a *topological* vector bundle on X. Then denote the functor from Equation 5.7 obtained through pairing with $\tau^D_{X,g}(X \times \mathbb{C}) \in Ac'(\mathfrak{D}/\mathfrak{C})_{C(X)}$ defined in Equation 4.1, by $- \cap \tau^D[X]$.

We have to emphasize that for a non-holomorphic vector bundle, the Dolbeault complex is not well-defined. Let X be a compact complex manifold, let E be a holomorphic vector bundle on X, and let g, h be Hermitian metrics on X, E, respectively. Recall from definition 4.1.2 that we have an exact sequence $\tau^D_{X,g}(E,h)$ in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C(X)}$ corresponding to the Dolbeault complex.

Proposition 5.3.12. *Let X be a compact complex manifold, and let E be a holomorphic vector bundle on X. Choose hertmitian metrics g, h on X, E respectively. Then the chain complexes $\tau^D_{X,g}(E,h)$ and $E \cap \tau^D[X]$ are isomorphic to each other in the category $Ac'(\mathfrak{D}/\mathfrak{C})_{C(X)}$.*

Proof. Let $\pi : X \times \mathbb{C}^m \to E$ be a smooth projection onto the bundle E, and let $\iota : E \to X \times \mathbb{C}^m$ be the inclusion. Denote the Dolbeault operator on the trivial bundle $X \times \mathbb{C}^k$ of rank k by $D^k = \bar{\partial} + \bar{\partial}^*$, and let $D_E = \bar{\partial}_E + \bar{\partial}^*_E$ denote the Dolbeault operator on E. Note that by definition, $Id \otimes \chi(D^1) \in \mathfrak{B}(E \otimes L^2(X, \wedge^{0,*} T^* X))$ is defined as $\pi\chi(D^1)^{\oplus m}\iota = \pi\chi(D^m)\iota$. By remark 4.2.14, $\pi\chi(D^m)\iota - \pi\iota\chi(D_E)$ is locally compact in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C(X)}$. Since $\pi\iota = Id_E$, this proves the assertion. □

Corollary 5.3.13. *Let X be a compact complex manifold, and let E be a topological vector bundle. Then $E \cap \tau^D_{X,g}$ is an exact sequence in the Paschke category $(\mathfrak{D}/\mathfrak{C})_{C(X)}$, and by Corollary 5.3.3 has a well-defined Euler characteristic. By Propositions 5.3.12 and 4.1.3, this concept of Euler characteristic is equal to the classical concept of the Euler characteristic of the Dolbeault complex when E is a holomorphic vector bundle.*

APPENDICES

Appendix A

COMPLEX MANIFOLDS

Let us give the basics and the notation used for complex manifolds in here. A good source for reading more on the topic is (Wel08).

Let X be a complex manifold and let E be a holomorphic vector bundle, then denote the sheaf of holomorphic, real analytic, differentiable, and continuous sections of E by $\mathscr{O}(E), \mathscr{C}^{\omega}(E)$, $\mathscr{C}^{\infty}(E), \mathscr{C}(E)$, respectively. Notice that each of the four sheaves just mentioned, is a subsheaf of the next ones. Also if X is only real analytic, then we can still consider the sheaves $\mathscr{C}^{\omega}(E), \mathscr{C}^{\infty}(E), \mathscr{C}(E)$, and similar statements can be repeated for differentiable or topological manifolds. Let $\mathscr{S}$ be one of the four sheaves above, then for an open subset U of X, denote the space of sections of E on U by $\mathscr{S}(U, E)$. In the case when E is the trivial line bundle $X \times \mathbb{C}$, then we will just denote $\mathscr{S}(U)$ instead of $\mathscr{S}(U, E)$ and also denote the structure sheaf by $\mathscr{S}_X$.

Let T^*X denote the cotangent bundle of the complex manifold X. Then the (almost) complex structure of X induces the decomposition $T^*X \otimes_{\mathbb{R}} \mathbb{C} = T^*(X)^{1,0} \oplus T^*(X)^{0,1}$, which in turn induces the *Dolbeault operator* $\bar{\partial} : \mathscr{C}^{\infty}(\wedge^{p,q} T^*X) \to \mathscr{C}^{\infty}(\wedge^{p,q+1} T^*X)$, that vanishes on the holomorphic sections. Hence for a holomorphic vector bundle E, we get an induced differential operator

$$\bar{\partial} \otimes 1 : \mathscr{C}^{\infty}(\wedge^{p,q} T^*X) \otimes_{\mathscr{O}} \mathscr{O}(E) \to \mathscr{C}^{\infty}(\wedge^{p,q+1} T^*X) \otimes_{\mathscr{O}} \mathscr{O}(E),$$

Appendix A (Continued)

which is also known as the Dolbeault operator. But we have $\mathscr{C}^\infty(\wedge^{p,q}T^*X) \otimes_{\mathscr{O}} \mathscr{O}(E) \cong \mathscr{O}(\wedge^{p,q}T^*X \otimes_{\mathbb{C}} E)$. From now on, we will abbreviate the latter to $\mathscr{A}_X^{p,q}(E)$ (or just $\mathscr{A}^{p,q}(E)$, if X is clear from the context.), and we denote the Dolbeault operator by $\bar{\partial}_E : \mathscr{A}^{p,q}(E) \to \mathscr{A}^{p,q+1}(E)$. We will also call the following as the *Dolbeault complex* with coefficients in E:

$$0 \to \mathscr{A}_X^{0,0}(E) \xrightarrow{\bar{\partial}_E} \mathscr{A}_X^{0,1}(E) \xrightarrow{\bar{\partial}_E} \mathscr{A}_X^{0,2}(E) \xrightarrow{\bar{\partial}_E} \dots \xrightarrow{\bar{\partial}_E} \mathscr{A}_X^{0,n}(E) \to 0 \tag{A.1}$$

where in here, $n = dim_{\mathbb{C}}(X)$.

Definition A.0.1. We follow (Wel08, 4.2.) to recall the definition of *symbol* of a differential operator. First let X be a differentiable manifold, and consider differentiable vector bundle[1] E, F on X. A linear operator $D : \mathscr{C}^\infty(X, E) \to \mathscr{C}^\infty(X, F)$ is a *differential operator of order* k, if no derivations of order $\geq k+1$ appear in its local representation. We denote the vector space of all such operators with $\mathrm{Diff}_k(E, F)$.

Let $T'X$ denote the cotangent bundle T^*X of X with the zero section deleted, and let $\pi : T'X \to X$ denote the projection. For $k \in \mathbb{Z}$ set

$$\mathrm{Smbl}_k(E, F) \coloneqq \{\sigma \in Hom(\pi^*E, \pi^*F) : \sigma(x, \rho v) = \rho^k \sigma(x, v), \text{ where } (x, v) \in T'X, \rho > 0\}.$$

---

[1] *All* the vector bundles and vector spaces we are considering in this section are over the complex numbers. Some of the arguments still hold over the real numbers as well.

Appendix A (Continued)

We now define the *k-symbol of a differential operator* as a linear map $\sigma_k : \mathrm{Diff}_k(E,F) \to \mathrm{Smbl}_k(E,F)$ by

$$\sigma_k(D)(x,v)e = D(\frac{j^k}{k!}(g-g(x))^k f)(x) \in F_x$$

where in here $(x,v) \in T'X, e \in E_x$ are given and $g \in \mathscr{C}^\infty(X), f \in \mathscr{C}^\infty(X,E)$ are chosen so that $f(x) = e, dg_x = v$. We can see that we have a linear mapping $\sigma_k(D)(x,v) : E_x \to F_x$, and that the symbol does not depend on the choices made.

One can also define *pseudo-differential operator* of order k for $k \in \mathbb{Z}$ (which we will denote by PDiff_k), and their symbol, but since definitions are somewhat more technical, and will not be used here, we refer the interested reader to (Wel08, 4.3.).

Symbols of (pseudo-) differential operators have the following important properties:

$$\sigma_{k+m}(D_2D_1) = \sigma_m(D_2)\sigma_k(D_1) \text{ when } D_1 \in \mathrm{PDiff}_k(E_1,E_2), D_2 \in \mathrm{PDiff}_m(E_2,E_3)$$

$$\sigma_k(D^*) = (-1)^k\sigma_k(D)^* \text{ if } D \in \mathrm{PDiff}_k(E,F)$$

where in here $D^* \in \mathrm{PDiff}_k(F,E)$ is the *formal adjoint* of D (Wel08, 4.1.5.), and $\sigma_k(D)^*$ is the adjoint of the linear map $\sigma_k(D)(x,v) : E_x \to F_x$. Note that both D^* and $\sigma(D^*) = \sigma(D)^*$ *depend* on the choice of metric on X and the bundles E, F.

Definition A.0.2. (Wel08, 4.4.) Let E, F be differentiable vector bundles on the differentiable manifold X and let $D \in \mathrm{Diff}_k(E,F)$. Then we say that D is an *elliptic differential operator* if for all $(x,v) \in T'X$, the linear map $\sigma_k(D)(x,v) : E_x \to F_x$ is an isomorphism. In particular

Appendix A (Continued)

both E, F have the same fiber dimension. The same can be defined for pseudo-differential operators.

Let $E_0, E_1, \ldots, E_m$ be a sequence of differentiable vector bundles on X and for some fixed k, let $D_i \in \mathrm{Diff}_k(E_i, E_{i+1})$ for all $i = 0, 1, \ldots, m-1$. We say this is an *elliptic complex* if $D_{i+1} \circ D_i = 0$ for all i, and also if the associated symbol sequence

$$0 \to \pi^* E_0 \xrightarrow{\sigma_k(D_0)} \pi^* E_1 \xrightarrow{\sigma_k(D_1)} \pi^* E_2 \xrightarrow{\sigma_k(D_2)} \ldots \xrightarrow{\sigma_k(D_{m-1})} \pi^* E_m \to 0$$

is exact, where $\pi : T'X \to X$ is the projection.

Remark A.0.3. In the literature, elliptic complexes are usually defined for *compact* differentiable manifolds, since *Sobelov spaces* over non-compact spaces don't behave as well as they do on compact spaces (e.g. Rellich's lemma works for Sobelov spaces over a fixed compact subset of the manifold.), which makes elliptic complexes over non-compact manifolds not as easy to work with. For example, the Hodge decomposition theorem (mentioned later in this section) is no longer true for non-compact complex manifolds.

Example A.0.4. Let E be a holomorphic vector bundle on the complex manifold X. The Dolbeault operator $\bar{\partial}_E : \mathscr{A}_X^{p,q}(E) \to \mathscr{A}_X^{p,q+1}(E)$ is a differential operator of order 1, and for $(x, v) \in T'X$, and $f \otimes e \in \wedge^{p,q} T_x^* X \otimes E$,

$$\sigma_1(\bar{\partial}_E)(x, v) f \otimes e = (i v^{0,1} \wedge f) \otimes e$$

Appendix A (Continued)

where in here $v = v^{1,0} + v^{0,1} \in T_x^*(X)^{1,0} + T_x^*(X)^{0,1}$. It is easy to check that the symbol sequence is exact, and hence the Dolbeault complex in Equation A.1 is an elliptic complex.

Theorem A.0.5 (The Hodge decomposition)**.** *Let X be a* compact *complex manifold, and let E be a holomorphic vector bundle on X. Choose Hermitian metrics on X and on E and let $\bar{\partial}_i^* : \mathscr{A}_X^{0,i+1}(E) \to \mathscr{A}_X^{0,i}(E)$ be the formal adjoint of $\bar{\partial}_i : \mathscr{A}_X^{0,i}(E) \to \mathscr{A}_X^{0,i+1}(E)$ (with respect to the metrics chosen). Let $\Delta_i = \bar{\partial}_{i-1}\bar{\partial}_{i-1}^* + \bar{\partial}_i^*\bar{\partial}_i$ and let $\mathscr{H}^{0,i}(X,E) = \ker \Delta_i \subset \mathscr{A}_X^{0,i}(E)$ denote the* harmonic $(0,i)$*-forms. Then we have the orthogonal decomposition*

$$\mathscr{A}_X^{0,i}(E) \cong \mathscr{H}^{0,i}(X,E) \oplus im(\bar{\partial}_{i-1}) \oplus im(\bar{\partial}_i^*), \tag{A.2}$$

and also there is an isomorphism $\mathscr{H}^{0,i}(X,E) \cong H^i(X,E)$, where the latter, is the cohomology of X with coefficients in E.

Definition A.0.6 (Hodge Star operator)**.** (Wel08, 5.1.) Let V be a (complex) vector space of dimension n. Choose an inner product on V and then choose an orthonormal basis $e_1, \ldots, e_n$ for V. Then define the *Hodge $*$-operator*

$$\star : \wedge^k V \to \wedge^{n-k} V$$

defined by $\star(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \pm(e_{j_1} \wedge \ldots \wedge e_{j_{n-k}})$, where $\{j_1, \ldots, j_{n-k}\}$ is complement of $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$, and we assign the plus sign if $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\}$ is an even permutation of $\{1, \ldots, n\}$, and assign the minus sign if it is an odd permutation.

Appendix A (Continued)

It is easy to extend $\star$ by linearity, and also to observe that $\star$ does not depend on the choice of the orthonormal basis, and depends only on the inner-product structure.

Let X be a complex manifold of dimension n, and choose a Hermitian metric g on X. Then similar to above, we can define the Hodge $\star$-operator

$$\star : \wedge^k T^*X \to \wedge^{n-k} T^*X$$

and it is easy to see that there is an induced $\star$-operator

$$\star : \wedge^{p,q} T^*X \to \wedge^{n-p,n-q} T^*X.$$

Let E be a holomorphic vector bundle on X, and choose a Hermitian metric h on E. We can consider the metric as a linear map $h : E \to E^*$, where E^* is the dual vector bundle to E, and we also have the dual linear map $h^* : E^* \to E$, and these satisfy $h^*h = Id_E, hh^* = Id_{E^*}$. Let $\bar{\star}(f) \coloneqq \star(\bar{f})$ for a section f of $\wedge^{*,*} T^*X$. Define

$$\bar{\star}_E = \bar{\star} \otimes h : \wedge^{p,q} T^*X \otimes E \to \wedge^{n-q,n-p} T^*X \otimes E^*.$$

Then one can show (Wel08, 5.2.4.a.) the following relation between the adjoint $\bar{\partial}^*$ of the Dolbeault operator $\bar{\partial}$ and the Hodge $\star$-operator.

$$\bar{\partial}^* = -\bar{\star}_{E^*} \bar{\partial} \bar{\star}_E : \mathscr{A}_X^{p,q}(E) \to \mathscr{A}_X^{p,q-1}(E). \tag{A.3}$$

Appendix B

FUNCTIONAL CALCULUS

Let us follow (HR00) to give a quick introduction to functional calculus.

Definition B.0.1. Let $T \in \mathfrak{B}(H)$. Then let $C^*(T)$ temporarily denote the Banach subalgebra of $\mathfrak{B}(H)$, generated by T, its adjoint T^*, and the identity operator.

We say that the operator T is *normal* if $TT^* = T^*T$. If T is normal then $C^*(T)$ is a commutative Banach algebra.

Let A be a unital Banach algebra. Then for $a \in A$, we define

$$\text{Spectrum}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda.1 \text{ is not invertible in } A\}.$$

Proposition B.0.2 (Spectral Theorem)**.** *(HR00, 1.1.11.) Let $T \in \mathfrak{B}(H)$ be a bounded normal operator acting on the Hilbert space H, then the map $\alpha \mapsto \alpha(T)$ is a homomorphism from dual of $C^*(T)$ onto Spectrum$_{\mathfrak{B}(H)}(T)$, and the induced* Gelfand transform $C^*(T) \to \mathscr{C}(\textit{Spectrum}_{\mathfrak{B}(H)}(T))$ *is an isometric $*$-isomorphism.*

Definition B.0.3. Let $T \in \mathfrak{B}(H)$ be a bounded normal operator acting on the Hilbert space H, and let $f \in \mathscr{C}(\text{Spectrum}_{\mathfrak{B}(H)}(T))$. Denote the corresponding element in $C^*(T)$ by $f(T)$. The $*$-homomorphism (inverse of the Gelfand transform) $\mathscr{C}(\text{Spectrum}_{\mathfrak{B}(H)}(T)) \to \mathfrak{B}(H)$ defined by $f \mapsto f(T)$ is called *functional calculus* for T.

Appendix B (Continued)

Definition B.0.4. (HR00, 1.8.) Let T be an unbounded operator, defined over a dense subset of the Hilbert space H. Then we say T is *symmetric* if for each $x, y \in H$ which are in domain of T, we have $\langle Tx, y\rangle = \langle x, Ty\rangle$.

We say T is *essentially self-adjoint* if domain of T is a subset of domain of T^*, and for any x in domain of T, $Tx = T^*x$, and also x is in domain of T^* if there is a sequence of points $\{x_i\}_{i=1}^{\infty}$ in domain of T so that x_i's converge to x and $\|T(x_i)\|$ remains bounded.

Note that the first two conditions are equivalent to T being symmetric. In other words, every essentially self-adjoint unbounded operator is symmetric.

Lemma B.0.5. *(HR00, 10.2.6.) Every symmetric differential operator on a compact manifold is essentially self-adjoint. More generally, every compactly supported symmetric differential operator on a (non-compact) manifold is essentially self-adjoint.*

Appendix C

INDEX OF NOTATION

$\mathfrak{A}(A, B)$	The space of morphisms from the object A to the object B of the C^*-category $\mathfrak{A}$
$\mathfrak{A}(A)$	The space of endomorphisms of the object A of the C^*-category $\mathfrak{A}$
$C(\mathcal{A})$	The exact category of chain complexes in the exact category $\mathcal{A}$
$C^b(\mathcal{A})$	The exact category of bounded chain complexes in the exact category $\mathcal{A}$
$Ac(\mathcal{A})$	The exact category of acyclic chain complexes in the exact category $\mathcal{A}$
$Ac^b(\mathcal{A})$	The exact category of bounded acyclic chain complexes in the exact category $\mathcal{A}$
$\mathcal{P}(R)$	The category of finitely generated projective modules over the ring R
$\mathcal{P}_{\mathrm{m}}(A)$	The category of finitely generated projective modules with a choice of norm over the C^*-algebra A

Appendix C (Continued)

$\mathscr{O}_X$	The sheaf of holomorphic functions on the complex manifold X
$\mathscr{C}_X^\infty$	The sheaf of smooth functions on the smooth manifold X
$\mathscr{A}_X^{0,i}$	The sheaf of smooth $(0,i)$-forms on the almost complex manifold X

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CITED LITERATURE

J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

M. F. Atiyah and F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.*, 65:276–281, 1959.

M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I., 1961.

M. F. Atiyah and F. Hirzebruch. The Riemann-Roch theorem for analytic embeddings. *Topology*, 1:151–166, 1962.

M. F. Atiyah. *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

M. F. Atiyah. Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo, 1970.

Benjamin Antieau and Ben Williams. The period-index problem for twisted topological K-theory. *Geom. Topol.*, 18(2):1115–1148, 2014.

L. G. Brown, R. G. Douglas, and P. A. Fillmore. Unitary equivalence modulo the compact operators and extensions of C^*-algebras. pages 58–128. Lecture Notes in Math., Vol. 345, 1973.

L. G. Brown, R. G. Douglas, and P. A. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. of Math. (2)*, 105(2):265–324, 1977.

Paul Baum, William Fulton, and Robert MacPherson. Riemann-Roch for singular varieties. *Inst. Hautes Études Sci. Publ. Math.*, (45):101–145, 1975.

Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973.

Armand Borel and Jean-Pierre Serre. Le théorème de Riemann-Roch. *Bull. Soc. Math. France*, 86:97–136, 1958.

Jeff Cheeger, Mikhail Gromov, and Michael Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17(1):15–53, 1982.

Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. *Math. Proc. Cambridge Philos. Soc.*, 136(1):9–51, 2004.

Daniel Dugger and Daniel C. Isaksen. Topological hypercovers and $\mathbb{A}^1$-realizations. *Math. Z.*, 246(4):667–689, 2004.

A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

Henri Gillet. Riemann-Roch theorems for higher algebraic K-theory. *Adv. in Math.*, 40(3):203–289, 1981.

P. Ghez, R. Lima, and J. E. Roberts. W^*-categories. *Pacific J. Math.*, 120(1):79–109, 1985.

Daniel R. Grayson. Adams operations on higher K-theory. *K-Theory*, 6(2):97–111, 1992.

Daniel R. Grayson. Algebraic K-theory via binary complexes. *J. Amer. Math. Soc.*, 25(4):1149–1167, 2012.

Nigel Higson. Algebraic K-theory of stable C^*-algebras. *Adv. in Math.*, 67(1):140, 1988.

Nigel Higson. On the relative k-homology theory of baum and douglas. *Unpublished preprint*, 1990.

Nigel Higson. C^*-algebra extension theory and duality. *J. Funct. Anal.*, 129(2):349–363, 1995.

Nigel Higson and John Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. Oxford Science Publications.

J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

Tamaz Kandelaki. KK-theory as the K-theory of C^*-categories. *Homology Homotopy Appl.*, 2:127–145, 2000.

Max Karoubi. Algèbres de Clifford et K-théorie. *Ann. Sci. École Norm. Sup. (4)*, 1:161–270, 1968.

Max Karoubi. K-theory, an elementary introduction. In *Cohomology of groups and algebraic K-theory*, volume 12 of *Adv. Lect. Math. (ALM)*, pages 197–215. Int. Press, Somerville, MA, 2010.

G. G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(3):571–636, 719, 1980.

Peter D. Lax. *Linear algebra and its applications*. Pure and Applied Mathematics (Hoboken). Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, second edition, 2007.

Roni N. Levy. The Riemann-Roch theorem for complex spaces. *Acta Math.*, 158(3-4):149–188, 1987.

Roni N. Levy. Riemann-Roch theorem for higher bivariant K-functors. *Ann. Inst. Fourier (Grenoble)*, 58(2):571–601, 2008.

Paul D Mitchener. *Symmetric Waldhausen K-theory spectra of topological categories*. Institut for Matematik og Datalogi, SDU-Odense Universitet, 2001.

Paul D Mitchener. C*-categories. *Proceedings of the London Mathematical Society*, 84(2):375–404, 2002.

Nigel R. O'Brian, Domingo Toledo, and Yue Lin L. Tong. Grothendieck-Riemann-Roch for complex manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 5(2):182–184, 1981.

William L. Paschke. K-theory for commutants in the Calkin algebra. *Pacific J. Math.*, 95(2):427–434, 1981.

Daniel Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

John Roe and Paul Siegel. Sheaf theory and Paschke duality. *J. K-Theory*, 12(2):213–234, 2013.

Khashayar Sartipi. Paschke categories, k-homology, and the riemann-roch transformation. *arXiv preprint arXiv:1810.11951*, 2018.

Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

Graeme Segal. Fredholm complexes. *Quart. J. Math. Oxford Ser. (2)*, 21:385–402, 1970.

Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

Victor Snaith. Localized stable homotopy of some classifying spaces. *Math. Proc. Cambridge Philos. Soc.*, 89(2):325–330, 1981.

Richard G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 105:264–277, 1962.

N. N. Tarkhanov. The Euler characteristic of a Fredholm quasicomplex. *Funktsional. Anal. i Prilozhen.*, 41(4):87–93, 2007.

R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

Dan Voiculescu. A non-commutative Weyl-von Neumann theorem. *Rev. Roumaine Math. Pures Appl.*, 21(1):97–113, 1976.

Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.