Model Theory of Differential Fields and Ranks of Underdetermined Systems of Differential Equations

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THESIS

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To my parents.

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Summary

In this thesis we compute the Lascar rank for generic differential equations. First we examine the case of generic linear differential equations. In this case, we show that there is a definable bijection between the solution set of a generic underdetermined system of k linear differential equations in $n \geq 2$ variables and $\mathbb{A}^{n-k}$. We explore how this result can be applied to non-generic linear differential equations.

Next we consider the case of a generic non-linear differential equations. We show that the differential tangent space above a generic point is given by a generic linear differential equation. We compute the Lascar rank by utilizing the relationship between differential tangent spaces and the underlying variety combined with our result for generic linear varieties applied to the tangent space above a generic point.

1

Introduction

A differential ring is given by a pair (R, δ) where R is a ring and $\delta : R \to R$ is an additive ring homomorphism such that for $x, y \in R$,

$$\delta(xy) = \delta(x)y + x\delta(y).$$

The ring of differential polynomials, $R\{X\}$, is a differential ring constructed by extending the differential to the polynomial ring $R[X_0, X_1, \ldots]$ via

$$\delta(X_n) = X_{n+1}.$$

Differential algebraic geometry focuses on the study of solution sets of differential polynomial equations; similar to algebraic geometry and the study of the solution sets of polynomial equations. In the 1930s Ritt began studying differential polynomials from an analytic perspective in [18]. Later, Ritt wrote a foundational text for the subject of differential algebra [19] with some analytic assumptions about the objects appearing. Kolchin expanded upon the work of Ritt by taking a completely algebraic approach to the differential algebra in [6] and [7]. Kolchin's approach to the subject gave rise to the differential algebraic analogs of most notions from classical algebraic geometry (e.g., Zariski topology, Hilbert polynomial, Tangent Bundle). For instance, in classical algebraic geometry algebraically closed fields are used as universal domains for the study of solution sets to polynomial equations. In differential algebra the universal domains are differentially closed fields.

Model theory is the study of definable sets (in first-order structures). The framework of model theory is very general and often used to understand the structure of definable sets. For instance, the definable subsets of a real closed field are finite unions of intervals and points. The tools developed in model theory are very general, however they have several nontrivial applications. One example of this is Hrushovski's proof of the Mordell-Lang conjecture for function fields [4]. Differential algebra is one of the best examples of a theory where the general tools of model theory can be applied to produce interesting results.

The connections between model theory and differential algebra began in the late 1950s with Robinson's first order axioms for the theory of differentially closed fields (DCF) in [20]. Robinson's work shows that DCF has quantifier elimination, hence the definable sets are given by the constructible sets in the Kolchin topology. Blum gave a more concise set of axioms for DCF in [1] and showed that DCF is ω-stable. Lascar rank (RU) and Morley rank (RM) are notions of dimension for definable sets and types coming from stability theory. These notions have useful applications in several areas, including DCF. In our context, these ranks are closely related to differential transcendence, in particular this relationship can be seen in the following inequality

$$\omega \cdot td_\delta(X) \leq RU(X) \leq RM(X) \leq \omega \cdot (td_\delta(X) + 1)$$

Morley rank and Lascar rank have been studied intensely in DCF. For instance, if $RM(X)$ is a limit ordinal then $RM(X) = RU(X)$ [16]. In general these two ranks are not equal within DCF [5]. These notions of rank in DCF are used in Hrushovski's proof of the Mordell-Lang conjecture for function fields [4].

Computing the rank of specific differential polynomials can lead to interesting results. In [3] Freitag and Scanlon compute the Lascar rank for the j-function (over $\mathbb{C}$) by applying several analytic results about the j-function; ultimately they use this towards their result about intersections of elliptic curves with certain isogeny classes. Another example is in [12] where Nagloo and Pillay compute the transcendence degree of extensions of $\mathbb{C}(t)$ by solutions to certain classes of generic Painlevé equations. Their rank computations for Painlevé equations utilize a series of computations applicable specifically to Painlevé equations from the Japanese school of differential algebra.

The objective of this thesis is to compute Lascar rank for generic differential equations.

In chapter 2 we start with a review the preliminary materials needed for this thesis. There are sections for fundamentals of model theory and differential algebra; as well as a section on model theory of differential fields which connects the ideas from the first two sections. Most importantly the last section contains characterizations of forking independence in DCF.

Chapter 3 focuses on generic *linear* differential equations. In this setting we think of the coefficients of the equations as independent differential transcendentals over some differential field K. The main result of this section is the following

Main Theorem 1 (Theorem 31)**.** *Let $n > k \geq 1$. The solution set to a system of k generic linear differential equations in n variables is in definable bijection with $\mathbb{A}^{n-k}$.*

The definable bijection from the above theorem arises as a composition of maps to reduce the order equations within the system; ultimately reducing to a system of linear equations of order 0. In particular this shows that the Lascar rank of the system is $\omega(n-k)$. Moreover, by the Lascar inequality, $\omega(n-k)$ is a lower bound for the Lascar rank of the zero set of any system of k differential polynomials in n variables. We observe that the maps used in the proof of theorem are rational maps defined using the coefficients of the system. Thus it is possible to use the technique more generally (in the cases where the coefficients are not generic) provided that all of the maps are definable; the main issue is that (in the case that the coefficients are not generic) some of the necessary maps could have vanishing denominators.

We explore what these conditions are in the case of a single equation. Lastly, we consider the example of equations with constant coefficients and show that these conditions are met when the coefficients are algebraically independent over $\mathbb{Q}$.

Next, in chapter 4 we review the notions of differential prolongations, arcs, and tangent spaces. The exposition of this chapter follows [10] with slight adaptations to our setting. Differential tangent spaces and prolongations have several applications in differential algebra. For instance, they are used to describe a geometric axiomatization of DCF [13]. They are also used in the proof of Zilber dichotomy for DCF [15]. There are several connections between differential varieties and their tangent spaces. For instance, a differential variety and the tangent space above a generic point have the same Kolchin polynomial.

In chapter 5 we use differential tangent spaces to transition from non-linear differential varieties to linear differential varieties. In particular we show that the tangent space above a generic point of a generic non-linear differential equation is given by a linear differential equations with generic coefficients. Therefore we can use our result for linear differential equations (Theorem 31) to determine the Δ-rank of this variety (the differential tangent space above a generic point). Using this we compute the Lascar rank of the underlying variety.

2

Background

This chapter contains an overview of the background material needed for the thesis. Each section focuses on introducing definitions as well as stating some fundamental results. The areas covered are Model Theory, Differential Algebra, and Model Theory of Differentially Closed Fields.

2.1 Model Theory

This section provides an introduction to some fundamental notions and ideas from Model Theory. Many of the definitions in this section are given with regards to generality of model theory; in section 2.3 we will see many of the notions from this section in the context that is most relevant to this thesis. For a more in detailed exposition of these topics see the following textbooks [8] and [23].

Model theory examines mathematical structures through first-order logic.

Definition 1. *A first-order* ***language****, $\mathcal{L}$, consists of the following:*

1. *A set $\mathcal{F}$ of function symbols and a positive integer n_f for each $f \in \mathcal{F}$.*

2. *A set $\mathcal{R}$ of relation symbols and a positive integer n_R for each $R \in \mathcal{R}$.*

3. *A set $\mathcal{C}$ of constant symbols.*

The n_f and n_R denote the arity of the corresponding functions and relations.

Many languages arise naturally in mathematics. Here are a few examples:

- The language of graphs is $\mathcal{L} = \{E\}$, where E is a binary relation symbol.

- The language of abelian groups is $\mathcal{L} = \{0, +, -\}$, where 0 is a constant, and $+, -$ are binary function symbols.

- The language of rings is $\mathcal{L}_R = \{0, 1, +, -, \cdot\}$, where 0 and 1 are constant symbols, and $+, -, \cdot$ are binary function symbols.

Note that the sets $\mathcal{F}, \mathcal{R}, \mathcal{C}$ can be empty. In fact the empty language in which all 3 sets are empty is also a first-order language.

The symbols of first-order language $\mathcal{L}$ have no limitations imposed their meaning. In order to give meaning to the symbols of a first-order language they must be interpreted within a $\mathcal{L}$-structure.

Definition 2. *A* ***$\mathcal{L}$-structure****, $\mathcal{M}$, consists of the following:*

1. *A nonempty set M which is called the* ***universe*** *or* ***underlying set*** *of $\mathcal{M}$.*
2. *A function $f^{\mathcal{M}} : M^{n_f} \to M$ for every $f \in \mathcal{F}$.*
3. *A set $R^{\mathcal{M}} \subset M^{n_R}$ for every $R \in \mathcal{R}$.*
4. *An element $c^{\mathcal{M}} \in M$ for every $c \in \mathcal{C}$.*

Many naturally occurring first-order languages are motivated by their corresponding first-order structures, where the interpretations are intuitive. For example given an algebraic ring R, we get a $\mathcal{L}_R$-structure by taking the set R as the underlying set, and interpreting the constants $0, 1$ to be the 0 and 1 of the ring, as well as interpreting the functions $+, -, \cdot$ to be the usual addition, subtraction, and multiplication for the ring.

Definition 3. *Let $\mathcal{L}$ be a language. The collection of* ***$\mathcal{L}$-terms*** *is constructed using the function and constant symbols of $\mathcal{L}$ as well as variables $x_0, x_1, \ldots$ according to the following rules:*

1. *Every constant and variable is a $\mathcal{L}$-term.*
2. *Given an n-ary function f and $\mathcal{L}$-terms $t_1, \ldots, t_n$, then $f(t_1, \ldots, t_n)$ is a $\mathcal{L}$-term.*

Terms are the basic elements that can be examine within first-order logic. Given a $\mathcal{L}$-structure $\mathcal{M}$, and a $\mathcal{L}$-term $t(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ contains all of the variables which appear in the term t. Then we can view t as a function,

$t^{\mathcal{M}} : M^n \to M$. Where $t^{\mathcal{M}}(\overline{a})$ is given by evaluating at $\overline{a}$ in $\mathcal{M}$ (i.e., evaluating the function composition by using the interpretation of the function symbols in M after substituting a_i for x_i and substituting $c^{\mathcal{M}}$ for every $c \in \mathcal{C}$ which appears in t.)

Definition 4. *Let $\mathcal{L}$ be a language. The collection of **$\mathcal{L}$-formulas** is built up using the following inductive construction:*

1. *$t_1 = t_2$, where t_1 and t_2 are $\mathcal{L}$-terms*

2. *$R(t_1, \ldots, t_n)$, where R is a n-ary relation symbol and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms*

3. *$\neg\psi$, where ψ is an $\mathcal{L}$-formula*

4. *$(\psi_1 \wedge \psi_2)$, where ψ_1 and ψ_2 are $\mathcal{L}$-formulas*

5. *$\exists x\, \psi$, where x is a variable and ψ is an $\mathcal{L}$-formula*

Definition 5. *Let $\varphi(\overline{x})$ be a formula and $\mathcal{M}$ a $\mathcal{L}$-structure. Let $\overline{a} \in \mathcal{M}$, then we define $\mathcal{M} \models \varphi(\overline{a})$ by induction as follows:*

1. *If φ is $t_1 = t_2$, then $\mathcal{M} \models \varphi(\overline{a})$ if $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$.*

2. *If φ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \models \varphi(\overline{a})$ if $(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_{n_R}^{\mathcal{M}}(\overline{a}) \in R^{\mathcal{M}}$.*

3. *If φ is $\neg\psi$, then $M \models \varphi(\overline{a})$ is $\mathcal{M} \not\models \psi(\overline{a})$.*

4. *If φ is $(\psi \wedge \theta)$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a})$ and $\mathcal{M} \models \theta(\overline{a})$.*

5. *If φ is $(\psi \vee \theta)$, then $\mathcal{M} \models \varphi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a})$ or $\mathcal{M} \models \theta(\overline{a})$.*

6. *If* φ *is* $\exists x_j \psi(\overline{x}, x_j)$*, then* $\mathcal{M} \models \varphi(\overline{a})$ *if there is* $b \in M$ *such that* $\mathcal{M} \models \psi(\overline{a}, b)$*.*

7. *If* φ *is* $\forall x_j \psi(\overline{x}, x_j)$*, then* $\mathcal{M} \models \varphi(\overline{a})$ *if* $\mathcal{M} \models \psi(\overline{a}, b)$ *for all* $b \in M$*.*

If $\mathcal{M} \models \varphi(\overline{a})$ *we say that* $\mathcal{M}$ ***satisfies*** $\varphi(\overline{a})$*.*

Definition 6. *Let* $\mathcal{M}$ *be a* $\mathcal{L}$*-structure. We say that* $X \subset M^n$ *is* ***definable*** *if there is a formula* $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ *and* $\overline{b} \in M^m$ *such that*

$$X = \{\overline{a} \in M : \mathcal{M} \models \varphi(\overline{a}, \overline{b})\}.$$

We say that a set X *is* ***definable over*** A *if there is exists a formula* $\phi(\overline{x}, \overline{y})$ *and* $\overline{b} \in A$ *such that* $\phi(\overline{x}, \overline{b})$ *defines* X*.*

For $A \subset M$ *we have the following notions of closures for* A*:*

1. *Let* $b \in M$*. We say that* b *is definable over* A *if* $\{b\}$ *is definable over* A*. The* ***definable closure of*** A*, denoted* $\mathrm{dcl}(A)$*, is* $\{b \in M : \{b\}$ *is definable over* $A\}$*.*

2. *Let* $b \in M$*. We say that* b *is* ***algebraic over*** A *if there is a finite set* X *such that* $b \in X$ *and* X *is definable over* A*. The* ***algebraic closure of*** A*, denoted* $\mathrm{acl}(A)$*, is* $\{b \in M : b$ *is algebraic over* $A\}$*.*

Example 7. *Let* K *be an algebraically closed field of characteristic 0 (in the language of fields). Then for* $A \subset K$*,* $\mathrm{dcl}(A)$ *is the field generated by* A *and* $\mathrm{acl}(A)$ *is the algebraic closure of the field generated by* A*.*

Understanding the definable sets of a structure is a key aspect of model theory. The definable sets of a structure can be wildly complicated e.g., the natural numbers $\mathbb{N}$ in $\mathcal{L} = \{+, \cdot, 0, 1\}$ can define a universal Turing machine via a first order formula (and a Gödel encoding), hence the definable sets of this structure are not computable. On the other hand the definable sets of a structure can be tame e.g., for $\mathbb{C}$ in $\mathcal{L} = \{+, \cdot, 0, 1\}$ the definable sets are the Zariski-constructible sets (this follows from Chevellay's theorem).

Definition 8. *Let $\mathcal{L}$ be a language. A $\mathcal{L}$-formula with no free variables (i.e., every variable x_i appearing is inside of a $\forall x_i$ or $\exists x_i$) is called a **$\mathcal{L}$-sentence**. A set of $\mathcal{L}$-sentences T is called a **$\mathcal{L}$-theory**. A $\mathcal{L}$-theory is **satisfiable** if there is an $\mathcal{L}$-structure $\mathcal{M}$, such that $\mathcal{M} \models T$.*

For a $\mathcal{L}$-sentence ϕ we use the notation $T \models \phi$ if and only if $\mathcal{M} \models \phi$ for every $\mathcal{M} \models T$.

*A $\mathcal{L}$-theory T is a **complete theory** if for every $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$.*

Many concepts in model theory occur on of the level of theories (i.e., properties shared by all models of a theory). A few examples of such properties are **quantifier elimination, elimination of imaginaries, ω-stable, simple, o-minimal, and NIP**.

Definition 9. *Let $\mathcal{M}$ be a $\mathcal{L}$-structure and $A \subset M$. Let $\mathcal{L}_A$ denote the language obtained by adding constant symbols to $\mathcal{L}$ for every $a \in A$. Let p be a set of $\mathcal{L}_A$-formulas in the variables $x_1, \ldots, x_n$. We call p a **type over** A if $p \cup Th_A(\mathcal{M})$ is satisfiable. We call p a **complete type** if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ in the variables $x_1, \ldots, x_n$. We let $S_n^{\mathcal{M}}(A)$ denote the set of all complete types over A in n variables.*

*Given a type $p \in S_n^{\mathcal{M}}(A)$, we say that p is **realized in** $\mathcal{M}$ if there is some $\overline{a} \in M$ such that $\mathcal{M} \models \phi(\overline{a})$ for all $\phi(\overline{x}) \in p$.*

For $A \subset M$ and $\overline{a} \in M$, we consider the type of $\overline{a}$ over A given by

$$tp(\overline{a}/A) = \{\phi(\overline{x}) \in \mathcal{L}_A | \mathcal{M} \models \phi(\overline{a})\}$$

Note that this is a complete type since for every $\mathcal{L}_A$-formula either $\mathcal{M} \models \phi(\overline{a})$ or $\mathcal{M} \models \neg\phi(\overline{a})$.

We can view $S_n^{\mathcal{M}}(A)$ as a topological space with a basis generated by sets of the form

$$[\phi] := \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$$

This is called the **Stone Topology**. It follows from the compactness theorem that this topology is compact.

Definition 10. *Let λ be an infinite cardinal and let T be a $\mathcal{L}$-theory. We say that T is λ-**stable** if whenever $\mathcal{M} \models T$ and $A \subset M$ with $|A| \leq \lambda$, then $|S_n^{\mathcal{M}}(A)| \leq \lambda$.*

*We say that T is **stable** if it is stable some λ.*

Definition 11. *Let κ be an infinite cardinal and let $\mathcal{M} \models T$. We say that $\mathcal{M}$ is κ-**saturated** if for every $A \subset M$, if $|A| < \kappa$ and $p \in S_n^{\mathcal{M}}(A)$ then p is realized in $\mathcal{M}$.*

κ-saturated models exist for arbitrarily large κ.* If $\mathcal{M} \models T$ is κ-saturated, then for all $\mathcal{N} \models T$ with $|N| \leq \kappa$, there is an elementary embedding of $\mathcal{N}$ into $\mathcal{M}$.

*Note that often set theoretic assumptions are needed to show that κ-saturated models exist for particular κ (i.e., κ-saturated models exist for strongly inaccessible cardinals).

It is common to let $\mathbb{U}$ be a κ-saturated model for some sufficiently large κ and consider the models we are interested in as being embedding into this model.

Next we introduce the model theoretic notion of forking. Our presentation of forking for types is based on [14, Chapters 2, 3]. There is a more general definition of forking for formulas which can be found in the previously mentioned sources ([8], [23]).

For the rest of this section we make the following technical assumption that T is a complete and stable theory in a countable language, and T has infinite models.

Definition 12. *1. Let $p(\overline{x}) \in S_n(A)$. Then the **class of** p is*

$$cl(p) := \{\phi(\overline{x}, \overline{y}) : \text{ there is some } \overline{a} \in A \text{ such that } \phi(\overline{x}, \overline{a}) \in p\}$$

*Note that $cl(p)$ is the set $\mathcal{L}$-formulas which are **represented** in p.*

*2. The **fundamental order for n-types** is denoted*

$$O_n(T) := (\{cl(p) : p \in S_n(M), \mathcal{M} \models T\}, \subseteq)$$

3. For $p(\overline{x}), q(\overline{x})$ n-types over models of T, we write $p \leq q$ if $cl(p) \subseteq cl(q)$, and $p \sim q$ if $cl(p) = cl(q)$.

4. Let $p \in S_n(A)$. Then we define

$$C_p = \{cl(q) : q \supset p, q \in S_n(M), \text{ where } A \subset M \text{ and } \mathcal{M} \models T\}$$

Lemma 13. *For any $p \in S(A)$, C_p has a minimal element. We denote this class by $\beta(p)$ i.e., $\beta(p)$ is the least class of a type over a model which extends p.*

Definition 14. *Let $A \subset B, p \in S_n(A), q \in S_n(B)$ and $p \subset q$. Then we say that q* ***does not fork over*** *A, or equivalently q* ***is a nonforking extension of*** *p, if $\beta(p) = \beta(q)$.*

We say that A ***is independent from*** *B* ***over*** *C, denoted*

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$$

if for every finite tuple $\overline{a}$ from A, $tp(\overline{a}/B \cup C)$ is a nonforking extension of $tp(\overline{a}/C)$.

Remark 15. *Nonforking is a notion of independence that generalizes the notion of algebraic independence for algebraically closed fields. The following are some basic properties of forking independence.*

1. *(Symmetry) $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$*
2. *(Invariance) If $\sigma \in \mathrm{Aut}(\mathbb{U})$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, then $\sigma(A) \mathop{\smile\!\!\!\!\!\!\mid}_{\sigma(C)} \sigma(B)$*
3. *(Transitivity) Let $C \subset B \subset D$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}_C D$ if and only if $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_B D$.*
4. *(Existence) For all a, B, and C there exists b such that $tp(a/C) = tp(b/C)$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_C B$*

Definition 16. *Let T be an ω-stable theory. Let $A \subset \mathbb{U}$ and $p \in S_n(A)$. The* U***-rank of*** p *is defined inductively (over the collection of ordinals) by*

$$RU(p) = \sup\{RU(q) + 1 : \exists B, A \subset B \subset M, q \in S_n(B), p \subset q \text{ and } q \text{ forks over } A\}$$

We will often refer to this as the ***Lascar rank of*** p*. We often write $RU(a/A)$ for $RU(tp(a/A))$.*

Remark 17. *Here are a few properties of Lascar rank:*

1. *$RU(a/A)$ is an ordinal.*

2. *$RU(a/A) = 0$ if and only if $a \in acl(A)$.*

3. *(**Lascar inequality**)*

   $$RU(a/A, b) + RU(b/A) \leq RU(a, b/A) \leq RU(a/A, b) \oplus RU(b/A)$$

   the $\oplus$ denotes the Cantor sum of ordinals.

4. *Let X be a definable set. $RU(X) = \sup\{RU(a/B) : a \in X\}$ where B is any small set of parameters over which X is defined.*

5. *Let X, Y be definable sets. If $f : X \to Y$ is a definable bijection then $RU(X) = RU(Y)$.*

2.2 Differential Algebra

This section reviews some fundamental definitions from differential algebra. For a more detailed exposition of Differential Algebra see [6], [7] and [19]. The presentation here is limited to the case of differential rings with a single derivation, as this is the case we will need for this thesis (opposed to the more general setting of rings with multiple commuting derivations). Throughout this section all rings are assumed to be commutative.

Definition 18. *Let R be a ring. A* ***derivation*** *is an additive map $\delta : R \to R$ that satisfies the Leibniz rule,*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

A ***differential ring*** *is a ring with a derivation. When R is a field we call it a* ***differential field****.*

One example of a differential ring is to let R be any ring and equip it with the trivial derivation $\delta : R \to 0$. Another example is C^∞, the ring of infinitely differential real value functions on $(0, 1)$, with the standard derivative.

The kernel of the derivation is called the ring of **constants**, denoted C_R (often denoted by C when R is implicit). That is $C_R = \{a \in R : \delta(a) = 0\}$.

For a differential ring R we construct the ring of **differential polynomials** $R\{x\}$ by $R\{x\} = R[x_0, x_1, \ldots]$ with the structure that $\delta(x_n) = x_{n+1}$. In this ring we identify x_n as the nth derivative of x.

Throughout this thesis we use the following notation for derivatives,

$$\delta(x) = x', \delta^2(x) = x'', \text{ and } \delta^n(x) = x^{(n)} \text{ for } n \in \mathbb{N}.$$

For $f \in R\{x\} \setminus R$, the **order of** f is the largest n such that $x^{(n)}$ appears in f.

Definition 19. *An ideal I in $R\{x\}$ is a* ***differential ideal*** *if $\delta(f) \in I$ for all $f \in I$. Given $A \subset R\{x\}$, we will use $[A]$ to denote the differential ideal generated by A, and we use $\{A\}$ to denote the radical differential ideal generated by A (i.e., $\{A\} = \sqrt{[A]}$).*

Let $L \supset K$ be differential fields and $a \in L$. We let $\mathcal{I}(a/K)$ denote the differential ideal of differential polynomials in $K\{x\}$ that vanish at a. We say that a is ***differentially transcendental over*** *K if $\mathcal{I}(a/K) = \{0\}$. Otherwise a is* ***differentially algebraic over*** *K.*

Theorem 20 (Ritt-Raudenbush Basis Theorem)**.** *Let $R \supset \mathbb{Q}$ be a differential ring such that every radical differential ideal is finitely generated. Then every radical differential ideal in $R\{x\}$ is finitely generated.*

Definition 21. *Let K be a differential field. We say that $X \subset K^n$ is* ***Kolchin closed*** *if there are $f_1, \ldots, f_m \in K\{x\}$ such that*

$$X = \{a \in K^n : f_1(a) = \cdots = f_m(a) = 0\}$$

We refer to the topology generated by the Kolchin closed sets as the ***Kolchin topology****. The previous theorem tells us that an intersection of Kolchin closed* *sets is given by a finite number of differential polynomials.*

The Kolchin topology is the differential analog of the Zariski topology for algebraic geometry.

For $X \subset \mathbb{A}^n$ we will use $\overline{X}$ to denote the Zariski closure of X. We will use $\overline{X}^{Kol}$ for the closure of X in the Kolchin topology.

Definition 22. *An* ***affine differential variety*** *V defined over K is a Kolchin closed subset of $\mathbb{A}^n$ defined over K (i.e., the zero set of a collection of differential polynomials over K).*

For $I \subset K\{x\}$ and $V \subset \mathbb{A}^n$, we have the following notions,

1. $\mathcal{V}(I) := \{a \in \mathbb{A}^n : f(a) = 0$ *for all* $f \in I\}$

2. $\mathcal{I}(V/K) := \{f \in K\{x\} : f(a) = 0$ *for all* $a \in V\}$.

Theorem 23 (Differential Nullstellensatz)**.** *Let K be a differential field. Let $\Sigma \subset K\{x\}$ be a set of differential polynomials. Then $\mathcal{I}(\mathcal{V}(\Sigma)/K) = \{\Sigma\}$.*

An affine differential variety V is **irreducible over** K if it is not equal to the union of two proper closed differential subvarieties defined over K. Every affine differential variety over K has a unique decomposition into irreducible differential subvarieties.

Definition 24. *Let V be an irreducible differential variety over K. We call a point $a \in V$* ***generic over*** *K if a is not contained in any proper differential subvariety of V over K.*

Next we introduce the notion of the Kolchin polynomial from [6].

Theorem 25. *Let a be a finite tuple from an extension of K. Then there exists a numerical polynomial $\omega_{a/K}(t)$ with the following properties.*

1. *For sufficiently large $t \in \mathbb{N}$, $\omega_{a/K}(t)$ is equal to the transcendence degree of $K((\delta^j(a))_{0\leq j\leq t})$.*

2. *The degree of $\omega_{a/K}$ is ≤ 1.*

3. *We can write $\omega_{a/K}(t)$ in the following form*

$$\omega_{a/K}(t) = d_1(t+1) + d_2$$

   where $d_i \in \mathbb{Z}$, in this case d_1 is the differential transcendence degree of $K\langle a\rangle$ over K.

4. *If b is a tuple from $K\langle a\rangle$, then there is $t_0 \in \mathbb{N}$ such that for sufficiently large $t \in N$, $\omega_{b/K}(t) \leq \omega_{a/K}(t+t_0)$.*

We call $\omega_{a/K}$ the **Kolchin polynomial of a over K**. The degree of $\omega_{a/K}(t)$ is called the **differential type of a over K**, denoted Δ-type(a/K). Similarly, the leading coefficient of $\omega_{a/K}(t)$ is called the **typical differential dimension of a over K**, denoted Δ-dim(a/K). In general the Kolchin polynomial is not a differential birational invariant, however the Δ-type and Δ-dim are both differential birational invariants.

A result in [22] shows that Kolchin polynomials are well ordered under eventual domination (i.e., $f \leq g$ if and only if $f(t) \leq g(t)$ for all sufficiently large $t \in \mathbb{N}$). Thus we can extend the notion of Kolchin polynomials to V a differential variety as follows

$$\omega_V := \sup\{\omega_{a/F} : a \in V\} \text{ where } F \text{ is any field over which } V \text{ is defined.}$$

2.3 Model Theory of Differential Fields

In this section we review model theory of differential fields which combines notions from sections 2.1 and 2.2. For a more in depth exposition of Model Theory of Differential Fields see [9, Chapter 2].

We focus on working within the theory of differentially closed fields (DCF). We will use the language $\mathcal{L} = \{+, \cdot, -, \delta, 0, 1\}$. The theory of DCF has the following axioms

1. axioms for algebraically closed fields of characteristic zero.
2. $\forall x, y \quad \delta(x + y) = \delta(x) + \delta(y)$
3. $\forall x, y \quad \delta(xy) = x\delta(y) + y\delta(x)$
4. For any non-constant differential polynomials $f(x)$ and $g(x)$ where the order of g is less than the order of f, there is a y such that $f(y) = 0 \wedge g(y) \neq 0$.

For the rest of this thesis we will assume that all rings are characteristic 0. We also let $\mathbb{U} \models DCF$ be a κ-saturated model for a sufficiently large κ.

Theorem 26. *The theory DCF has the following model theoretic properties:*

1. *Quantifier Elimination,*

2. *ω-stable,*

3. *Elimination of Imaginaries*

One consequence of Theorem 26 is that the definable sets are the constructible sets in the Kolchin topology.

Quantifier elimination also gives a bijective correspondence between complete types over K, differential prime ideals, and irreducible varieties over K as follows

$$p \mapsto I_p := \{f(x) \in K\{x\} : \text{"}f(x) = 0\text{"} \in p\} \mapsto V_p := \mathcal{V}(I_p)$$

The correspondence between varieties and types is given by $V \mapsto tp(a/K)$ where a is a generic point of V. Thus for $a \in \mathbb{U}$, $\mathcal{V}(I_{tp(a/K)})$ is a differential variety over K where a is a generic point.

The forking relation has several characterizations in DCF; the following characterizations are useful for our considerations:

1. If $K \subset F$ and $p \in S_n(F)$, then p does not fork over K if and only if V_p is an irreducible component of $V_{p|_K}$ over F.

2. Let $K \subset F_1, F_2$ be differential fields, then $F_1 \downarrow_K F_2$ if F_1 and F_2 are algebraically disjoint over K i.e., if $\overline{a} \in F_1$ is algebraically independent over K then it is algebraically independent over F_2. For $A, B, C \subset \mathbb{U}$ we write $A \downarrow_C B$ if $\mathbb{Q}\langle AC\rangle \downarrow_{\mathbb{Q}\langle C\rangle} \mathbb{Q}\langle BC\rangle$.

It follows from the first characterization that if $q \supset p$ is a forking extension, then $\omega_q(t) < \omega_p(t)$.

These characterizations are used in [17] to prove the following facts:

1. $RU(a/B) = \omega \iff a$ is differentially transcendental over $\mathbb{Q}\langle B\rangle$

2. $RU(\mathbb{A}^n) = \omega n$

3

Linear Differential Equations

3.1 Introduction

In this chapter we present our results for defining a bijection between differential varieties given by linear differential equations and affine spaces. We are then able to make conclusion about the Lascar rank of these varieties since it is preserved by definable bijections. The main result is motivated by generalizing an algorithm used to define a bijection between a generic linear differential equation of order 2 in 2 variables and $\mathbb{A}^1$. We first present that example then generalize the method to a fully generic linear equation of arbitrary order in any fixed number of variables.

The results are then further generalized to systems of generic linear differential equations.

Upon understanding the algorithm in the context of generic linear equations we analyze the process to see how it can be applied to define similar bijections for certain non-generic linear equations.

3.2 Generic Linear Equations

The main result of this section is Theorem 31 which establishes a definable bijection between the solution set a system of generic linear differential equations and $\mathbb{A}^\ell$, where ℓ depends on the number of equations and variables in the system. We begin by considering some warm-up examples to motivate the techniques applied throughout this section.

Definition 27. *A generic linear differential equation of order m in n variables (over K) is given by an equation of the form:*

$$\sum_{i=0}^{n-1} \sum_{j=0}^{m} a_{i,j} x_i^{(j)} + c = 0 \tag{3.1}$$

where the coefficients $a_{i,j}, c$ are independent differential transcendentals (over K).

Example 28. *For a warm-up example we consider a generic linear differential equation of order 1 in two variables.*

$$a_1 x' + a_0 x + b_1 y' + b_0 y + c = 0 \tag{3.2}$$

Now we apply the definable map $(x, y) \mapsto (z, y)$ *where* $z = x + \frac{b_1}{a_1}y$. *Thus,*

$$x = z - \frac{b_1}{a_1}y \text{ and } x' = z' - \left(\frac{b_1}{a_1}\right)' y - \frac{b_1}{a_1}y'.$$

Substituting these expressions for x, x' *into equation* (3.2) *we get the following equation in* z, y

$$a_1 z' + a_0 z + \left(b_0 - a_1 \left(\frac{b_1}{a_1}\right)' - \frac{a_0 b_1}{a_1}\right) y + c = 0 \tag{3.3}$$

In particular,

$$y = -\frac{a_1 z' + a_0 z + c}{b_0 - a_1 \left(\frac{b_1}{a_1}\right)' - \frac{a_0 b_1}{a_1}}$$

Thus y *is definable from* z. *Therefore the solutions of equation* (3.3) *are parameterized by* z, *and there are no restrictions on the choice of* z, *so the solution set is in definable bijection with* $\mathbb{A}^1$. *In particular, this shows that the solution set of* (3.2) *has Lascar rank* ω.

From the Lascar inequality gives the follow bounds for the Lascar rank of this variety

$$\omega \leq RU(x, y/K) \leq \omega + 1.$$

Thus we see that in this case the lower bound is realized. The results throughout this section will continue to realize the lower bounds from the Lascar inequality.

Example 29. *Now we consider the case of a generic linear equation of order 2* *in 2 variables*

$$a_2x_0'' + a_1x_0' + a_0x_0 + b_2x_1'' + b_1x_1' + b_0x_1 + c = 0 \tag{3.4}$$

Let V denote the linear differential variety over K given by (3.4).

Now we apply the following definable map which reduces the order of equation (3.4) *in the variable x_1.*

$$(x_0, x_1) \mapsto (y_0, x_1), \ \textit{where}\ y_0 = x_0 + \frac{b_2}{a_2}x_1 \tag{3.5}$$

The map (3.5) *is a definable bijection. Moreover we represent equation* (3.4) *in terms of the variables y_0, x_1 by making the following substitutions*

$$x_0 = y_0 - \frac{b_2}{a_2}x_1 \tag{3.6}$$

$$x_0' = y_0' - \frac{b_2}{a_2}x_1' - \left(\frac{b_2}{a_2}\right)' x_1 \tag{3.7}$$

$$x_0'' = y_0'' - \frac{b_2}{a_2}x_1'' - 2\left(\frac{b_2}{a_2}\right)' x_1' - \left(\frac{b_2}{a_2}\right)'' x_1 \tag{3.8}$$

The resulting equation is

$$a_2y_0''+a_1y_0'+a_0y_0+\left(b_1 - 2a_2\left(\frac{b_2}{a_2}\right)' - a_1\frac{b_2}{a_2}\right)x_1'+\left(b_0 - a_2\left(\frac{b_2}{a_2}\right)'' - a_1\left(\frac{b_2}{a_2}\right)' - a_0\frac{b_2}{a_2}\right)x_1+c = 0 \tag{3.9}$$

To keep the notation more compact we substitute $b_{1,j}$ for the coefficients of $x_1^{(j)}$

in equation (3.9)*. That is*

$$b_{1,1} = b_1 - 2a_2\left(\frac{b_2}{a_2}\right)' - a_1\left(\frac{b_2}{a_2}\right)$$
$$b_{1,0} = b_0 - a_2\left(\frac{b_2}{a_2}\right)'' - a_1\left(\frac{b_2}{a_2}\right)' - a_0\left(\frac{b_2}{a_2}\right)$$

This substitution allows us to express equation (3.9) *as*

$$a_2 y_0'' + a_1 y_0' + a_0 y_0 + b_{1,1} x_1' + b_{1,0} x_1 + c = 0 \tag{3.10}$$

Now we apply another following definable map to reduce the order of equation (3.10) *in the variable* y_0*.*

$$(y_0, x_1) \mapsto (y_0, y_1) \text{ where } y_1 = x_1 + \frac{a_2}{b_{1,1}} y_0' \tag{3.11}$$

Again we note that the map given in (3.11) *is a definable bijection. To express equation* (3.10) *in terms* y_0, y_1 *we make the following substitutions*

$$x_1 = y_1 - \frac{a_2}{b_{1,1}} y_0' \tag{3.12}$$
$$x_1' = y_1' - \left(\frac{a_2}{b_{1,1}}\right)' y_0' - \frac{a_2}{b_{1,1}} y_0'' \tag{3.13}$$

This substitution allows us to express equation (3.10) *as*

$$\left(a_1 - b_{1,1}\left(\frac{a_2}{b_{1,1}}\right)' - b_{1,0}\frac{a_2}{b_{1,1}}\right) y_0' + a_0 y_0 + b_{1,1} y_1' + b_{1,0} y_1 + c = 0 \tag{3.14}$$

Again, for ease of notation we will substitute $a_{1,j}$ *for the coefficient of* $y_0^{(j)}$ *in* (3.14)*. So*

$$a_{1,1} = a_1 - b_{1,1}\left(\frac{a_2}{b_{1,1}}\right)' - b_{1,0}\frac{a_2}{b_{1,1}}$$
$$a_{1,0} = a_0$$

Thus equation (3.14) *becomes the following linear equation of order 1 in the variables* y_0, y_1*.*

$$a_{1,1}y_0' + a_{1,0}y_0 + b_{1,1}y_1' + b_{1,0}y_1 + c = 0 \tag{3.15}$$

Thus by applying the composition of the maps (3.5) *and* (3.11) *we see that the solutions of equation* (3.4) *are in definable bijection with the solutions of* (3.15)*. In Example 28 we showed that the Lascar rank of* (3.15) *is* ω*, hence the Lascar rank of* (3.4) *is* ω*.*

In this case the bounds for the Lascar rank of (3.4) *coming from the Lascar inequality are*

$$\omega \leq RU(x, y/K) \leq \omega + 2$$

Again we note that the lower bound is realized.

Using the maps from example 29 for motivation we generalize the result to the following lemma.

Lemma 30. *Let* $n \geq 2$*. The solution set of a generic linear differential equation in* n *variables is in definable bijection with* $\mathbb{A}^{n-1}$*.*

Proof. We prove this by induction on the order of the generic linear differential equation.

For the case of order 0 the linear differential equation is

$$a_{0,0}x_0 + a_{1,0}x_1 + \cdots + a_{n-1,0}x_{n-1} + c = 0$$

We observe that

$$x_0 = -\frac{1}{a_{0,0}}\left(a_{1,0}x_1 + \cdots + a_{n-1,0}x_{n-1} + c\right)$$

so $x_0 \in dcl(x_1, \ldots, x_{n-1})$. Thus this equation is dependent only on the variables $x_1, \ldots, x_{n-1}$. Since the coefficients are generic the solution set is in definable bijection with $\mathbb{A}^{n-1}$.

For the inductive step we start with a generic linear equation of order m and apply definable maps and change the variables to end up with a generic linear equation of order $m-1$. The maps we want to use are generalizations of the maps applied in Example 29.

First we apply the map

$$(x_0, x_1, \ldots, x_{n-1}) \mapsto (y_0, x_1, \ldots, x_n) \text{ where } y_0 = x_0 + \sum_{i=1}^{n} \frac{a_{i,m}}{a_{0,m}} x_i \tag{3.16}$$

Note that $x_0^{(m)} = y_0^{(m)} - \sum_{i=1}^{n-1} \frac{a_{i,m}}{a_{0,m}} x_i^{(m)}$ + lower order terms in x_i.
The lower order derivatives of x_0 will contribute to the lower order terms of

$x_1, \ldots, x_{n-1}$, thus altering the coefficients of these variables. We apply the map (3.16) to equation (3.1) and substitute y_0 for x_0. We also make a substitution for the coefficients where we let $b_{i,j}$ be the coefficient for $x_i^{(j)}$ for $i \geq 1$. This gives the following equation,

$$\sum_{j=0}^{m} a_{0,j} y_0^{(j)} + \sum_{i=1}^{n-1} \sum_{j=0}^{m-1} b_{i,j} x_i^{(j)} + c = 0. \tag{3.17}$$

Thus the variables $x_1, \ldots, x_{n-1}$ now all appear with order $m-1$ in equation (3.17).

We need to show that the coefficients of equation (3.17) are generic over K. We show that the set coefficients is differentially algebraically independent over K.

Suppose not, then there is some differential polynomial $p \in K\{\overline{x}\}$ such that $p(a_{0,0}, \ldots, a_{0,m}, b_{0,0}, \ldots, b_{n-1,m-1}) = 0$. However the coefficients $b_{i,j}$ are given by rational differential expressions in terms of $a_{i,j}$. Therefore we can make this substitution and then clear the denominators. This gives a differential polynomial relationship over K among the $\{a_{i,j}\}$ contradicting that the $\{a_{i,j}\}$ are generic over K.

Next we apply the map

$$(y_0, x_1, \ldots, x_{n-1}) \mapsto (y_0, y_1, x_2, \ldots x_{n-1}) \text{ where } y_1 = x_1 + \frac{a_{0,m}}{b_{1,m-1}} y_0' \tag{3.18}$$

Note that $x_1^{(m-1)} = y_1^{(m-1)} - \dfrac{a_{0,m}}{b_{1,m-1}} y_0^{(m)}$ + lower order terms in y_0.
Again we see that the lower order derivatives of x_1 will introduce addition lower order terms in y_0. By applying the map (3.18) to equation (3.17) and substituting

$b_{0,j}$ for the coefficients of $y_0^{(j)}$ we get an equation of the form.

$$\sum_{i=0}^{1}\sum_{j=0}^{m-1} b_{i,j}y_i^{(j)} + \sum_{i=2}^{n-1}\sum_{j=0}^{m-1} b_{i,j}x_i^{(j)} + c = 0 \tag{3.19}$$

Therefore the resulting equation has order $m-1$ in all n variables. Again we need to check that the coefficients of equation (3.19) are generic. When we applied the map to produce (3.19) this introduced the coefficients $b_{0,j}$, which are given by differential rational expressions in $a_{0,j}$ and $b_{1,j}$. Thus as above we see that if there was a differential polynomial relationship among the coefficients of (3.19) then we could express the $b_{0,j}$ in terms of $a_{0,j}$ and $b_{1,j}$ then clear the denominators from this rational expression. Contradicting that the coefficients of (3.17) are generic.

The composition of the maps (3.16) and (3.18) give a definable bijection between the solutions of (3.1) and (3.19). By induction there is a definable bijection between (3.19) and $\mathbb{A}^{n-1}$. Therefore taking the composition of these two maps gives desired definable bijection between (3.1) and $\mathbb{A}^{n-1}$. □

Next we extend the result of Lemma 30 to systems of generic linear equations with the following theorem.

Theorem 31. *Let $n > k \geq 1$. The solution set to a system of k generic linear differential equations in n variables is in definable bijection with $\mathbb{A}^{n-k}$.*

Proof. We prove this by induction on the number of equations in the system. The base case of $k = 1$ is done by Lemma 30.

Now we consider a system of k generic linear differential equations in n variables (i.e., the set of all coefficients appearing in the system is a differentially independent set)

$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{m_0} a_{0,i,j}x_i^{(j)} + c_0 = 0 \\
&\quad\vdots \\
&\sum_{i=0}^{n-1}\sum_{j=0}^{m_{k-1}} a_{k-1,i,j}x_i^{(j)} + c_{k-1} = 0
\end{aligned} \tag{3.20}$$

We reduce the first equation of the system (3.20) to an equation of order 0 by applying the maps from the proof of lemma 30.

$$\sum_{i=0}^{n-1} b_{0,i,0}y_i + c_0 = 0 \tag{3.21}$$

Throughout this process we have applied several maps to the variables x_i and we must apply these to the other $k-1$ equations of (3.20). We make substitutions for the coefficients of these equations so that the system is now of the form

$$\begin{aligned}
&\sum_{i=0}^{n-1} b_{0,i,0}y_i + c_0 = 0 \\
&\sum_{i=0}^{n-1}\sum_{j=0}^{m_1} b_{1,i,j}y_i^{(j)} + c_1 = 0 \\
&\quad\vdots \\
&\sum_{i=0}^{n-1}\sum_{j=0}^{m_{k-1}} b_{k-1,i,j}y_i^{(j)} + c_{k-1} = 0
\end{aligned} \tag{3.22}$$

From equation (3.21) we see that

$$y_0 = \frac{1}{b_{0,0,0}}(-c_0 - \sum_{i=1}^{n-1} b_{0,i,0} y_i) \tag{3.23}$$

Hence, y_0 is definable from $y_1, \ldots, y_{n-1}$.

Therefore we can use equation (3.23) to eliminate y_0 from the other $k-1$ equations of (3.22). Therefore we have a definable bijection between the solutions of the system (3.20) and the solutions of a system of $k-1$ equations in $n-1$ variables. As in the proof of lemma 30 we see that the coefficients for this system are still generic as any nontrivial differential algebraic relationship among them yields a relation among the original coefficients by expanding their expressions in terms of the original coefficients and clearing denominators.

By induction there is a definable bijection between the solutions of this system of $k-1$ equations in $n-1$ variables and $\mathbb{A}^{n-k}$. Taking the composition of this map and the map constructed above gives the desired definable bijection between the solutions of (3.20) and $\mathbb{A}^{n-k}$. □

Corollary 32. *Let $n > k \geq 1$. Let V be the differential algebraic variety corresponding to a system of k generic linear differential equations in n variables. Then V has no proper subvarieties of rank $\omega \cdot (n-k)$ nor any subvarieties of Δ-dim $n-k$.*

Proof. Suppose towards a contradiction that $W \subset V$ is a proper subvariety of rank $\omega \cdot (n-k)$ (resp. Δ-dim $n-k$). By Theorem 31 there is a definable bijection between V and $\mathbb{A}^{n-k}$. Under this bijection the image of W is a proper subvariety of $\mathbb{A}^{n-k}$ with rank $\omega\cdot(n-k)$ (resp. Δ-dim $n-k$), but no such subvariety exists. $\square$

Corollary 33. *Let $n > k \geq 1$. Let V be the differential algebraic variety corresponding to a system of k generic linear differential equations in n variables. Then V has Lascar rank $\omega \cdot (n-k)$.*

3.3 Non-generic Coefficients

In the proof of Theorem 31 we do not need fully generic coefficients to construct the bijection. Instead what we need is that in each application of (3.16) and (3.18) the coefficient appearing in the denominator has not vanished. In this section we present a recursive system of conditions that the coefficients of a system of linear differential equations must satisfy in order to be able to utilize the process from the proof of Theorem 31. We say that the coefficients for a system of linear differential equations are **sufficiently generic** if they satisfy the conditions necessary to construct a definable bijection to $\mathbb{A}^{n-k}$.

3.3.1 Analysis of a Single Equation

Let's examine what occurs when reducing a linear differential equation of order 3 in two variables.

Example 34. *Throughout this example we apply maps and make substitutions similar to example 29 in order to reduce the order of the equation. At the end we see what conditions the coefficients need to satisfy for all of the maps to be definable and thus produce a bijection with $\mathbb{A}^1$.*

Notation: We use the subscripts of the variables to keep track of how many iterations of the maps we have applied (i.e., the maps we use be such that $x_i \mapsto x_{i+1}$). Also we will be making the substitutions $a_{i,j}$ (resp. $b_{i,j}$) for the coefficient of $x_i^{(j)}$ (resp. y_i^j) where appropriate.

We start with a linear differential equation of order 3 in 2 variables

$$a_{0,3}x_0''' + a_{0,2}x_0'' + a_{0,1}x_0' + a_{0,0}x_0 + b_{0,3}y_0''' + b_{0,2}y_0'' + b_{0,1}y_0' + b_{0,0}y_0 + c = 0 \quad (3.24)$$

Next we apply the following definable map to equation (3.24)

$$(x_0, y_0) \mapsto (x_1, y_0) \text{ where } x_1 = x_0 + \frac{b_{0,3}}{a_{0,3}}y_0 \quad (3.25)$$

Note that in order to define the map (3.25) *we require that* $a_{0,3} \neq 0$.

We can make the following substitutions

$$\begin{aligned}
x_0 &= x_1 - \frac{b_{0,3}}{a_{0,3}}y_0 \\
x_0' &= x_1' - \left(\frac{b_{0,3}}{a_{0,3}}\right)' y_0 - \frac{b_{0,3}}{a_{0,3}}y_0' \\
x_0'' &= x_1'' - \left(\frac{b_{0,3}}{a_{0,3}}\right)'' y_0 - 2\left(\frac{b_{0,3}}{a_{0,3}}\right)' y_0' - \frac{b_{0,3}}{a_{0,3}}y_0'' \\
x_0''' &= x_1''' - \left(\frac{b_{0,3}}{a_{0,3}}\right)''' y_0 - 3\left(\frac{b_{0,3}}{a_{0,3}}\right)'' y_0' - 3\left(\frac{b_{0,3}}{a_{0,3}}\right)' y_0'' - \frac{b_{0,3}}{a_{0,3}}y_0'''
\end{aligned} \quad (3.26)$$

After applying this map and making the substitutions for the resulting coeffi-

cients we write equation (3.24) *in the following form*

$$a_{0,3}x_1''' + a_{0,2}x_1'' + a_{0,1}x_1' + a_{0,0}x_1 + b_{1,2}y_0'' + b_{1,1}y_0' + b_{1,0}y_0 + c = 0 \tag{3.27}$$

Next we apply the following map to equation (3.27)

$$(x_1, y_0) \mapsto (x_1, y_1) \text{ where } y_1 = y_0 + \frac{a_{0,3}}{b_{1,2}}x_1' \tag{3.28}$$

Note that in order to define the map (3.28) *requires that* $b_{1,2} \neq 0$.

Thus we make the following substitutions,

$$\begin{aligned} y_0 &= y_1 - \frac{a_{0,3}}{b_{1,2}}x_1' \\ y_0' &= y_1' - \left(\frac{a_{0,3}}{b_{1,2}}\right)' x_1' - \frac{a_{0,3}}{b_{1,2}}x_1'' \\ y_0'' &= y_1'' - \left(\frac{a_{0,3}}{b_{1,2}}\right)'' x_1' - 2\left(\frac{a_{0,3}}{b_{1,2}}\right)' x_1'' - \frac{a_{0,3}}{b_{1,2}}x_1''' \end{aligned} \tag{3.29}$$

After applying map (3.28) *and making the substitutions for the resulting coefficients we write equation* (3.27) *in the following form*

$$a_{1,2}x_1'' + a_{1,1}x_1' + a_{1,0}x_1 + b_{1,2}y_1'' + b_{1,1}y_1' + b_{1,0}y_1 + c = 0 \tag{3.30}$$

Next we apply the following map to (3.30)

$$(x_1, y_1) \mapsto (x_2, y_1) \text{ where } x_2 = x_1 + \frac{b_{1,2}}{a_{1,2}}y_1 \tag{3.31}$$

Note that in order to define the map (3.31) *requires* $a_{1,2} \neq 0$.

Thus we make the following substitutions

$$\begin{aligned} x_1 &= x_2 - \frac{b_{1,2}}{a_{1,2}} y_1 \\ x_1' &= x_2' - \left(\frac{b_{1,2}}{a_{1,2}}\right)' y_1 - \frac{b_{1,2}}{a_{1,2}} y_1' \\ x_1'' &= x_2'' - \left(\frac{b_{1,2}}{a_{1,2}}\right)'' y_1 - 2\left(\frac{b_{1,2}}{a_{1,2}}\right)' y_1' - \frac{b_{1,2}}{a_{1,2}} y_1'' \end{aligned} \tag{3.32}$$

After applying map (3.31) *and making the substitutions for the resulting coefficients we write equation* (3.30) *in the following form*

$$a_{1,2}x_2'' + a_{1,1}x_2' + a_{1,0}x_2 + b_{2,1}y_1' + b_{2,0}y_1 + c = 0 \tag{3.33}$$

Next we apply the following map to equation (3.33)

$$(x_2, y_1) \mapsto (x_2, y_2) \text{ where } y_2 = y_1 + \frac{a_{1,2}}{b_{2,1}} x_2' \tag{3.34}$$

Note that map (3.34) *requires* $b_{2,1} \neq 0$ *in order to be applicable.*

Thus

$$\begin{aligned} y_1 &= y_2 - \frac{a_{1,2}}{b_{2,1}} x_2' \\ y_1' &= y_2' - \left(\frac{a_{1,2}}{b_{2,1}}\right)' x_2' - \frac{a_{1,2}}{b_{2,1}} x_2'' \end{aligned} \tag{3.35}$$

After applying map (3.34) *and making the substitutions for the resulting coeffi-*

cients we write equation (3.33) as

$$a_{2,1}x_2' + a_{2,0}x_2 + b_{2,1}y_2' + b_{2,0}y_2 + C = 0 \tag{3.36}$$

Finally we apply the following map to (3.36)

$$(x_2, y_2) \mapsto (x_3, y_2) \text{ where } x_3 = x_2 + \frac{b_{2,1}}{a_{2,1}}y_1 \tag{3.37}$$

Note that in order to define the map (3.37) requires $a_{2,1} \neq 0$.

Thus we can make the substitutions

$$\begin{aligned} x_2 &= x_3 - \frac{b_{2,1}}{a_{2,1}}y_2 \\ x_2' &= x_3' - \left(\frac{b_{2,1}}{a_{2,1}}\right)' y_2 - \frac{b_{2,1}}{a_{2,1}}y_2' \end{aligned} \tag{3.38}$$

This simplifies to the following equation

$$a_{2,1}x_3' + a_{2,0}x_3 + b_{3,0}y_2 + c = 0 \tag{3.39}$$

Note that if $b_{3,0} \neq 0$ then in (3.39) y_2 is definable from x_3. As in the end of example 28 we see that the set of solutions is in definable bijection with $\mathbb{A}^1$.

To complete the entire process we require that all of the following coefficients $a_{0,3}, b_{1,2}, a_{1,2}, b_{2,1}, a_{2,1}$ and $b_{3,0}$ do not vanish. Each of these coefficients can be expressed by a rational differential expression in terms of the original coefficients $a_{0,i}$ and $b_{0,i}$ by unpacking the recursive substitutions we made throughout the process.

Doing this gives the following conditions *

$$a_{0,3} \neq 0 \tag{3.40}$$

$$b_{1,2} = b_{0,2} - 3a_{0,3}\left(\frac{b_{0,3}}{a_{0,3}}\right)' - \frac{a_{0,2}\cdot b_{0,3}}{a_{0,3}} \neq 0 \tag{3.41}$$

$$a_{1,2} = \frac{a_{0,3}\cdot\left(b_{0,1} - a_{0,1}\cdot\left(\frac{b_{0,3}}{a_{0,3}}\right) - 2\cdot a_{0,2}\cdot\left(\frac{b_{0,3}}{a_{0,3}}\right)' - 3\cdot a_{0,3}\cdot\left(\frac{b_{0,3}}{a_{0,3}}\right)''\right)}{b_{0,2} - 3a_{0,3}\left(\frac{b_{0,3}}{a_{0,3}}\right)' - \frac{a_{0,2}\cdot b_{0,3}}{a_{0,3}}}$$
$$- 2\left(b_{0,2} - 3a_{0,3}\left(\frac{b_{0,3}}{a_{0,3}}\right)' - \frac{a_{0,2}\cdot b_{0,3}}{a_{0,3}}\right)\left(\frac{a_{0,3}}{b_{0,2} - 3a_{0,3}\left(\frac{b_{0,3}}{a_{0,3}}\right)' - \frac{a_{0,2}\cdot b_{0,3}}{a_{0,3}}}\right)' \neq 0 \tag{3.42}$$

$$b_{2,1} = b_{1,1} - \frac{a_{1,1}\cdot b_{1,2}}{a_{1,2}} - 2a_{1,2}\left(\frac{b_{1,2}}{a_{1,2}}\right)' \neq 0 \tag{3.43}$$

$$a_{2,1} = a_{1,1} - \frac{b_{2,0}\cdot a_{1,2}}{b_{2,1}} - b_{2,1}\cdot\left(\frac{a_{1,2}}{b_{2,1}}\right)' \neq 0 \tag{3.44}$$

$$b_{3,0} = b_{2,0} - \frac{a_{2,0}\cdot b_{2,1}}{a_{2,1}} - a_{2,1}\cdot\left(\frac{b_{2,1}}{a_{2,1}}\right)' \neq 0 \tag{3.45}$$

Next we perform a recursive analysis to determine sufficient conditions for the solution set of linear differential equation of the following form to be in definable

*We did not expand all of the conditions out to be in terms of the original coefficients. We include this example to illustrate the point that it is possible to explicitly determine what the necessary conditions on the coefficients for a given order and degree. If one wants to verify the conditions for examples beyond the case of order two in two variables in this manner then the use of computers is strongly recommended.

bijection with $\mathbb{A}^{n-1}$.

$$\sum_{i=0}^{n}\sum_{j=0}^{m} a_{0,i,j} x_{0,i}^{(j)} + c = 0 \tag{3.46}$$

The strategy is to apply appropriate versions of the maps (3.16) and (3.18) to reduce the order of the resulting equation on each application.

The purpose of map (3.16) is to make the variable of highest order unique; we can denote this coordinate by $x_{k,i'}$. To reduce the order of $x_{k,i'}$ we use the map (3.18) which requires some non-zero $a_{k,i,j}$ where $i \neq i'$. After k iterations of applying these maps the composition of the maps will give a bijection between the solution sets of (3.46) and an equation with the following form:

$$\sum_{i=0}^{n}\sum_{j=0}^{m-k} a_{k,i,j} x_{k,i}^{(j)} + c = 0 \tag{3.47}$$

WLOG (up to applying a permutation of the coordinates) we may assume that $x_{k,0}$ is of the highest order and if some $a_{k,i,m-k} \neq 0$ then $a_{k,1,m-k} \neq 0$.

Then the coefficients $a_{k,i,j}$ in (3.47) can be computed as follows:

For $i > 0$ and $0 \leq j \leq m - k$:

$$a_{k+1,i,j} = a_{k,i,j} - \sum_{\ell=j}^{m-k} a_{k,0,\ell} \binom{\ell}{j} \left(\frac{a_{k,i,m-k}}{a_{k,0,m-k}} \right)^{(\ell-j)}$$

For $i = 0$ and $0 \leq j \leq m - k$:

$$\begin{aligned} a_{k+1,0,0} &= a_{k,0,0} \\ a_{k+1,0,j} &= a_{k,0,j} - \sum_{\ell=j-1}^{m-k-1} a_{k+1,1,\ell} \binom{\ell}{j-1} \left(\frac{a_{k,0,m-k}}{a_{k+1,1,m-k-1}} \right)^{(\ell-j+1)} \end{aligned} \tag{3.48}$$

In order to produce a bijection between the solution sets of (3.46) and a linear equation of order 0 we need to apply m iterations of the maps which reduce the order, hence need $a_{\ell,0,m-\ell} \neq 0$ and $a_{\ell,1,m-\ell} \neq 0$ for $0 \leq \ell \leq m$. Note that such a bijection will give the desired result of being a bijection to $\mathbb{A}^{n-1}$.

3.3.2 Constant Coefficients

Example 35. *In this example we consider the case where the coefficients of* (3.46) *are all constants of the differential field K.*

When working with constants all of the derivatives vanish, hence the conditions *of* (3.48) *simply to the following*

$$\begin{aligned}
&\textit{For } i > 0 \textit{ and } 0 \leq j \leq m-k: \\
&\qquad a_{k+1,i,j} = a_{k,i,j} - a_{k,0,j}\left(\frac{a_{k,i,m-k}}{a_{k,0,m-k}}\right) \\
&\textit{For } i = 0 \textit{ and } 0 \leq j \leq m-k: \\
&\qquad a_{k+1,0,0} = a_{k,0,0} \\
&\qquad a_{k+1,0,j} = a_{k,0,j} - a_{k+1,1,j-1}\left(\frac{a_{k,0,m-k}}{a_{k+1,1,m-k-1}}\right)
\end{aligned} \tag{3.49}$$

From this example we see that the coefficients are given by rational algebraic expressions. Hence if all the coefficients are algebraically independent over $\mathbb{Q}$, then none the coefficients appearing the transformations will vanish. This proves the following Theorem.

Theorem 36. *Let $n \geq 2$ and let V be a differential variety given by a linear equation in n variables with constant coefficients. If the coefficients are algebraically independent over $\mathbb{Q}$, then there is a definable bijection between V and $\mathbb{A}^{n-1}$.*

We present one last example here to show that it is not always the case that the lower bound from the Lascar inequality is always obtained.

Example 37. *Consider the differential equation*

$$x' + y' = 0 \tag{3.50}$$

Note that here if we try to apply the reduction map $(x, y) \mapsto (z, y)$ where $z =$

$x + y$, then we end up with the equation

$$z' = 0 \tag{3.51}$$

In particular we see that we do not have a y which we can use to reduce the order of z. Hence we cannot perform further reductions and construct a map to an affine space. Instead, what we see is that in equation (3.51) *(in terms of z, y) there are no restrictions on y and that z satisfies an equation of order* 1*. Moreover there is no relationship between the variables, hence equation* (3.51) *(in z, y) has Lascar rank $\omega + 1$. Therefore,* (3.50) *has Lascar rank $\omega + 1$, which is the upper bound for the Lascar rank of this equation coming from the Lascar inequality.*

Moreover, this example can be generalized to

$$x^{(m)} + y^{(m)} = 0 \tag{3.52}$$

A similar argument shows that (3.52) *has Lascar rank $\omega + m$, which is the upper bound for the Lascar rank of this equation coming from the Lascar inequality.*

4

Differential Tangent and Arc Spaces

4.1 Introduction

This chapter is an exposition of differential arc and tangent spaces. Differential arc spaces were originally developed in [15].

The presentation of differential arc spaces used for this chapter follows the presentation in [11] and [10]. For differential tangent spaces we use [6] and [7]. The presentation in this chapter is adapted to the setting of differential rings with a single derivation. Analogous definitions for the case of several commuting derivations can be found in the cited texts.

4.2 Arc Spaces

In this section we review the construction of Arc spaces in the algebraic setting. For this section let S and T be schemes and $\pi : T \to S$ be a map of schemes.

Definition 38. *Let Y be a scheme over T, then we construct a set-valued functor on the category of schemes over S, $R_{T/S}(Y)$, given by $S' \mapsto Y(S' \times_S T)$. Here $Y(S' \times_S T)$ denotes the set of $(S' \times_S T)$ valued points of Y over T (i.e., $Hom_T(S' \times_S T, Y)$). If this is a representable functor, then we denote the representing object by $R_{T/S}(Y)$; this is the* ***Weil restriction of** Y **from** T **to** S.*

When T is finite over S and Y satisfies that every finite set of points is contained in an affine open subset, then the Weil restriction is representable. We are working in a setting where these conditions are satisfied, therefore we implicitly assume these conditions hold whenever necessary.

The relevant setting is the following. Let k be a field and $S = Spec(k)$. Let $T = Spec(k^{(m)})$ where

$$k^{(m)} := k[\epsilon]/(\epsilon^{m+1}).$$

Let $Y = X \times_k k^{(m)}$, where X is an affine variety over k. $k^{(m)}$ is a k-algebra via the natural map,

$$a \mapsto a + 0\epsilon + \cdots + 0\epsilon^m.$$

Definition 39. *The m^{th}* ***arc bundle of** X **over** k is $R_{k^{(m)}/k}(X \otimes_k k^{(m)})$, the Weil restriction of $X \otimes_k k^{(m)}$ from $Spec(k^{(m)})$ to $Spec(k)$. We will denote this by $\mathcal{A}_m(X/k)$ or $\mathcal{A}_m X$ when k is implicit.*

Note that $\mathcal{A}_m X$ is not necessarily a reduced or irreducible scheme over k.

Example 40. *Given a k-algebra, R, $\mathcal{A}_m X(R)$ can be identified with $X(R[\epsilon]/(\epsilon^{m+1}))$. In particular, $\mathcal{A}_m X(k)$ is identified with $X(k^{(m)})$. Hence when $X \subset \mathbb{A}^\ell$ is an affine variety we get equations for $\mathcal{A}_m X \subset \mathbb{A}^{\ell(m+1)}$ by the following process:*

$$\text{Let } X = Spec(k[x_1, \ldots, x_\ell]/(\{f_j\}_{j\in J}))$$

then

$$\mathcal{A}_m X = Spec(k[\{x_{i,s}\}_{1\le i\le \ell, 0\le s\le m}]/(\{f_{j,t}\}_{j\in J, 0\le t\le m}))$$

where $f_{j,t} \in k[\{x_{i,s}\}_{1\le i\le \ell, 0\le s\le m}]$ is given by

$$f_j\left(\left(\sum_{t=0}^{m} x_{i,t}\epsilon^t\right)_{1\le i\le \ell}\right) = \sum_{t=0}^{m} f_{j,t}\epsilon^t$$

computed in the ring $k[\{x_{i,s}\}_{1\le i\le \ell, 0\le s\le m}, \epsilon]/(\epsilon^{m+1})$.

Let $f : X \to Y$ be a regular map of varieties over k. Then we have an induced map, $\mathcal{A}_m(f) : \mathcal{A}_m X \to \mathcal{A}_m Y$ given by f evaluated on $X(k[\epsilon]/(\epsilon^{m+1})])$. More specifically, let $X \subset \mathbb{A}^\ell, Y \subset \mathbb{A}^r$ and $f = (f_1, \ldots, f_r)$. Let $b \in \mathcal{A}_m X(k)$ we can view b as a point in $\mathbb{A}^\ell(k[\epsilon]/(\epsilon^{m+1}))$. Then $\mathcal{A}_m(f)(b) = (f_1(b), \ldots, f_r(b))$ where we compute $f_i(b)$ in $k[\epsilon]/(\epsilon^{m+1})$.

For $\ell \ge m$ there is a natural map $\rho_{\ell,m} : \mathcal{A}_\ell \to \mathcal{A}_m$ induced by the quotient map $k^{(\ell)} \to k^{(m)}$. For $a \in X$, the m^{th} **arc space of X at a**, denoted $\mathcal{A}_m X_a$ is the fiber over a of the map $\rho_{m,0}$.

Next we give a summary of some basic properties for algebraic arc spaces; proofs of these lemmas can be found in [10].

Lemma 41. *Let X be an algebraic variety over a field k and $a \in X(k)$ a smooth point, then for any pair of natural numbers $\ell > m \geq 0$ the restriction of the map $\rho_{\ell,m} : \mathcal{A}_\ell X \to \mathcal{A}_m X$ to $\mathcal{A}_\ell X_a(k)$ is surjective onto $\mathcal{A}_m X_a(k)$.*

Lemma 42. *Let $f : X \to Y$ be a regular map of algebraic varieties over the field k. Let m be a natural number and $a_m \in \mathcal{A}_m X(k)$ such that $a := \rho_m(a_m)$ is a smooth point of X and $f(a)$ is a smooth point of Y. Let $\tilde{X}$ be the fiber of $\rho_{m+1,m} : \mathcal{A}_{m+1} X \to \mathcal{A}_m X$ over a_m, and $\tilde{Y}$ the fiber of $\rho_{m+1,m} : \mathcal{A}_{m+1} Y \to \mathcal{A}_m Y$ over $\mathcal{A}_m(f)(a_m)$. Then there are biregular maps $\psi_X : \tilde{X} \to T_a X$ and $\psi_Y : \tilde{Y} \to T_{f(a)} Y$ so that the following diagram is commutative*

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\mathcal{A}_{m+1}(f)"] \arrow[d, "\psi_X"'] & \tilde{Y} \arrow[d, "\psi_Y"] \\
T_a X \arrow[r, "df_a"] & T_{f(a)} Y
\end{tikzcd}$$

Lemma 43. *Let $f : X \to Y$ be a dominant map of algebraic varieties over the field k. Suppose $a \in X(k)$ is a smooth point and $f(a) \in Y(k)$ is smooth on Y, and df_a has rank equal to the dimension of Y. Then for every m, the map $\mathcal{A}_m(f) : \mathcal{A}_m X_a(k) \to \mathcal{A}_m Y_{f(a)}(k)$ is surjective.*

Lemma 44. *Let K be an algebraically closed field of characteristic zero and $X, Y \subset Z$ are irreducible algebraic varieties over k. If $a \in X(k) \cap Y(k)$, then $X = Y$ if and only if $\mathcal{A}_m X_a(k) = \mathcal{A}_m Y_a(k)$ for all $m > 0$.*

4.3 Prolongation Sequences

In this section we review the construction of prolongation sequences; continuing towards the development of differential arc spaces as presented in [10].

Let R be a differential ring and let d be the derivation on R. We will use the following notation

$$R_m := R[\eta]/(\eta)^{m+1}$$

This ring is a R-algebra via the exponential map $E : R \to R_m$ given by

$$a \mapsto \sum_{0 \leq \alpha \leq m} \frac{1}{\alpha!} \delta^{\alpha}(a) \eta^{\alpha}$$

Definition 45. *Let X be an algebraic variety over a differential field K, the m^{th}* ***prolongation*** *$\tau_m X$* ***of*** *X is the Weil restriction of $X \times_E k_m$ from Spec(K_m) to Spec(K) (i.e., $\tau_m X = R_{K_m/K}(X \times_E K_m)$).*

Note that when δ is the trivial derivation (i.e., $\delta = 0$), τ_m and $\mathcal{A}_m$ are the same. For $\ell \geq m$, the quotient maps $K_\ell \to K_m$ induce maps $\pi_{\ell,m} : \tau_\ell \to \tau_m$. We often denote the map $\pi_{m,0}$ by π_m. We also have a map, $\nabla_m : X \to \tau_m X$ given by

$$x \mapsto \sum_{0 \leq \alpha \leq m} \frac{1}{\alpha!} \delta^{\alpha}(x) \eta^{\alpha}$$

In some situations we consider $\tau_m X$ under its canonical embedding into the iterated prolongation $\tau^m X := \overbrace{\tau \circ \cdots \circ \tau}^{m \text{ times}} X$. The iterated prolongation has the section map $\nabla^m : X \to \tau^m X := \overbrace{\nabla \circ \cdots \circ \nabla}^{m \text{ times}}$. Then the map from $K[\eta]/(\eta)^{m+1}$ to

$K[\xi_1, \dots, \xi_m]/((\xi_j)^2_{1\le j\le m})$ given by $\eta \mapsto \sum_j \xi_j$ gives an embedding $\tau_m X \hookrightarrow \tau^m X$. This embedding gives an extension of the map $\pi_{\ell,m} : \tau_\ell X \to \tau_m X$ (for $\ell \ge m$) to $\pi_{\ell,m} : \tau^\ell X \to \tau^m X$ given by $\overbrace{\pi_1 \circ \cdots \circ \pi_1}^{\ell - m \text{ times}}$.

We have the following lemma that taking prolongations and taking arcs commute for algebraic varieties. The proof of this lemma is in [10].

Lemma 46. *$\tau_m \mathcal{A}_r(X) = \mathcal{A}_r \tau_m(X)$.*

For a differential variety X over K and $\ell \in \mathbb{N}$, we let $\overline{X}$ be the Zariski closure of X and let $\tau_\ell X$ be the Zariski closure of $\nabla_\ell(X)$ in $\tau_\ell \overline{X}(K)$.

Note that X is determined by the its prolongation sequence

$$\langle \pi_{\ell,m} : \tau_\ell X \to \tau_m X | \ell \ge m \rangle$$

Observe that

$$X = \{a \in \overline{X}(k) : \nabla_\ell(a) \in \tau_\ell X(K), \forall \ell \ge 0\}$$

Conversely, suppose Y is an algebraic variety and $\langle X_\ell \subset \tau_\ell Y | \ell \ge 0 \rangle$ is a sequence of algebraic subvarieties such that:

1. $\pi_{\ell+1}$ restricts to a dominant map from $X_{\ell+1}$ to X_ℓ and

2. $X_{\ell+1}$ is a closed subvariety of τX_ℓ, under the embeddings $\tau_\ell Y$ in $\tau^\ell Y$ and $\tau_{\ell+1} Y$ in $\tau^{\ell+1} Y$,

then there exists a unique differential subvariety X of Y such that $\tau_\ell X = X_\ell$. This establishes an equivalence of categories between differential subvarieties and prolongation sequences.

Definition 47. *Let K be a differentially closed field. Let X be an irreducible differential over K and let $\overline{X}$ be the Zariski closure of X. Then the m^{th}* ***differential arc bundle of*** *X, $\mathcal{A}_m^{\Delta}X$ is given by the following prolongation sequence*

$$\langle \mathcal{A}_m(\pi_{s,t}) : \mathcal{A}_m\tau_s X \to \mathcal{A}_m\tau_t X | s \geq t\rangle$$

For $a \in X$ the m^{th} ***differential arc space of*** *X* ***at*** *a is the fiber above a of $\rho_m : \mathcal{A}_m\overline{X} \to \overline{X}$ restricted to $\mathcal{A}_m^{\Delta}X$.*

The following definition and lemmas establish a connection between differential arc bundles and the differential tangent space developed by Kolchin in [7] (which we discuss in the next section).

Definition 48. *Let $a \in X$, then a is a* ***smooth point*** *if $\nabla_s(a)$ is a smooth point on X_s for every s and $d(\pi_{s,t})_{\nabla_s(a)}$ has full rank for every $s \geq t$.*

If $a \in X$ is a smooth point, then $\overline{\nabla_s(\mathcal{A}_m^{\Delta}X_a)} = \mathcal{A}_m(X_s)_{\nabla_s(a)}$.

Lemma 49 ([10], Lemma 2.11)**.** *Let $X \subset \mathbb{A}^{\ell}$ be a differential variety. If $a \in X$ is a smooth point, then $\mathcal{A}_1^{\Delta}X_a$ is canonically isomorphic to $T_a^{\Delta}X$.*

Lemma 50 ([10], Corollary 2.12)**.** *Let $X \subset \mathbb{A}^{\ell}$ be a differential variety and $a \in X$ a smooth point (as in Definition 48). Then $\omega_{\mathcal{A}_m^{\Delta}X_a}(t) = m\omega_X(t)$.*

In particular, for $a \in X$ smooth we see that $\mathcal{A}_1^{\Delta}X_a$ has the same Kolchin polynomial as X.

The (first) prolongation of a variety $\tau_1 V$ is often useful in applications as generators for the differential variety can be explicitly computed. If $V \subset \mathbb{A}^n$ is a differential variety over K, then $\tau_1 V \subset \mathbb{A}^{2n}$ is generated by

$$f(x) = 0 \text{ and } \sum_{\substack{i \le n \\ j \le ord(f)}} \frac{\partial f}{\partial x_i^{(j)}}(x) u_i^{(j)} + f^\delta(x) = 0 \text{ for } f \in \mathcal{I}(V/K)$$

where $f^\delta(x)$ is the differential polynomial given by applying δ to all of the coefficients of f.

4.4 Differential Tangent Spaces

In this section we review the construction of differential tangent spaces. This is based on the work by Kolchin in [7].

Definition 51. *Let $V \subset \mathbb{A}^n$ be a differential variety over K. The* ***differential tangent bundle of*** *V, is the differential variety given by*

$$f(x) = 0 \text{ and } \sum_{\substack{i \le n \\ j \le ord(f)}} \frac{\partial f}{\partial x_i^{(j)}}(x) \zeta_i^{(j)} = 0, \text{ for } f \in \mathcal{I}(V/K)$$

We denote this by $T^\Delta V \subset \mathbb{A}^{2n}$. We also have the map for projection onto the first n coordinates $\pi : T^\Delta V \to V$.

For a point $a \in V$, the fiber of this projection over a gives the ***differential tangent space over*** *a which we denote by $T_a^\Delta V$.*

Lemma 52. *Let V be a differential variety. If $W \subset V$ is differential subvariety, then $T_a^\Delta W \subset T_a^\Delta V$ for all $a \in W$.*

Proof. Since $W \subset V$, then $\mathcal{I}(W) \supset \mathcal{I}(V)$. Therefore $T^\Delta W \subset T^\Delta V$; in particular this holds on the fiber above a for $a \in W$. □

Definition 53. *A point $a \in V$ is **smooth** if the differential tangent space over a has the same Kolchin polynomial as V, that is*

$$\omega_{V/K}(t) = \omega_{T_a^{\Delta} V/K\langle a \rangle}(t)$$

Note that this definition of a smooth point is different than the definition given in the previous section (Definition 48). However both definitions hold on dense open sets. In particular, if $a \in V$ is generic, then $\omega_V(t) = \omega_{T_a^{\Delta} V}(t) = \omega_{\mathcal{A}_1^{\Delta} V_a}(t)$. In fact, the result at a generic point $a \in V, \omega_V(t) = \omega_{T_a^{\Delta} V}(t)$ was originally shown by Kolchin in [7].

4.5 Applications of Differential Tangent and Arc Spaces

In this section we present a brief overview of some applications of the concepts from this chapter. We want to give a sense of the different types of results that have been shown by using these ideas.

Kolchin defined differential tangent spaces in [6] and [7] where he creates a foundation for differential algebra by establishing differential analogs of algebraic concepts. In particular this was part of his development of Lie Theory for differential algebra.

In [13] prolongations spaces are used to give geometric axioms for differentially closed fields of characteristic 0. The authors show that a differential field K is differentially closed if

1. K is algebraically closed.

2. Let $V \subset K^n$ and $W \subset \tau_1(V)$ be irreducible varieties defined over K such that W projections generically on V. Let U be a nonempty Zariski-open subset of W defined over K. Then there is a point of the form $(a, \nabla_1(a)) \in U$ where $a \in V$ is generic over K.

These axioms were later generalized in [21] to produce a geometric axiomatization of differentially closed fields with multiple commuting derivations.

In [15] the authors use differential jet spaces to prove Zilber dichotomy for DCF_0; the result that every type of Lascar rank 1 is either modular or nonorthogonal to C_K. In [10] differential arc spaces are used to prove a version of Zilber dichotomy for differentially closed fields with multiple commuting derivations; the results that every non-locally modular regular type is nonorthogonal to a regular type which is the generic type of a definable subgroup of the additive group.

Another application appears in [2] where prolongation spaces are used to obtain an effective version of uniform bounding for differential fields with multiple commuting derivations. They produce an upper bound for the degree of the Zariski closure of the solution set to a system of differential polynomials. The bound given depends only on the order, degree and number of variables appearing in the differential polynomials. Their proof utilizes prolongations by bounding the degree of irreducible components of $\tau_\ell(V)$. They also produce an effective differential Nullstellensatz, i.e., they give an algorithm for determining if $f \in \{f_1, \ldots, f_r\}$ with a bounded search space.

In chapter 5 we use differential tangent spaces to apply our results for linear differential equations to non-linear differential equations. We do this by using the result that at generic points V and $T_a^{\Delta}V$ have the same Kolchin polynomial. In chapter 3 we computed the Kolchin polynomial for linear differential varieties with sufficiently generic coefficients. Thus we can bound the Lascar rank of the non-linear variety if the coefficients for the tangent space are sufficiently generic.

5

Non-Linear Differential Equations

5.1 Introduction

In this chapter we combine the results of chapter 3 along with the ideas of chapter 4 to prove results for non-linear differential equations.

5.2 Generic Non-linear Equations

The main result of this section is Theorem 57. We begin with a motivating example.

Definition 54. *A generic non-linear differential equation of order m and degree*

d in n variables (over K) is given an equation of the form

$$\sum_{\substack{\sum e_{i,j} \leq d \\ 0 \leq e_{i,j}}} a_{\overline{e_{i,j}}} \prod_{\substack{1 \leq i \leq n \\ 0 \leq j \leq m}} \left(x_i^{(j)}\right)^{e_{i,j}} = 0$$

where the coefficients $a_{\overline{e_{i,j}}}$ are independent differential transcendentals (over K).

In general a (non-linear) differential polynomial in n variables of order m and degree d will have

$$\sum_{k=0}^{d} \binom{n \cdot (m+1) + k - 1}{k}$$

many terms.

Example 55. *Let $p(x, y)$ be a generic differential polynomial of order 1, degree 2 in two variables over K.*

$$\begin{aligned} p(x,y) = {} & a_0(x')^2 + a_1x'x + a_2x'y' + a_3x'y + a_4x' + a_5x^2 + a_6xy' + a_7xy + a_8x \\ & + a_9(y')^2 + a_{10}y'y + a_{11}y' + a_{12}y^2 + a_{13}y + a_{14} \end{aligned} \tag{5.1}$$

Let (α_0, α_1) be a generic point of the differential variety $V = \mathcal{V}(p(x, y))$ over K.

Thus $T_\alpha^\Delta V$ is given by the following linear equation

$$\begin{aligned} P(w,z) &= (2a_0\alpha_0' + a_1\alpha_0 + a_2\alpha_1' + a_3\alpha_1 + a_4)w' \\ &\quad + (a_1\alpha_0' + 2a_5\alpha_0 + a_6\alpha_1' + a_7\alpha_1 + a_8)w \\ &\quad + (a_2\alpha_0' + a_6\alpha_0 + 2a_9\alpha_1' + a_{10}\alpha_1 + a_{11})z' \\ &\quad + (a_3\alpha_0' + a_7\alpha_0 + a_{10}\alpha_1' + 2a_{12}\alpha_1 + a_{13})z \end{aligned} \tag{5.2}$$

Let us make a substitution on the coefficients to rewrite equation (5.2) *as*

$$\beta_1 w' + \beta_2 w + \beta_3 z' + \beta_4 z.$$

In order to use Theorem 31 to determine the Lascar rank of $T_\alpha^\Delta V$ we need to show that the coefficients $\beta_1, \ldots, \beta_4$ are generic over $k\langle\alpha\rangle$. Thus we need to show that the differential transcendence degree of $K\langle\alpha, \beta_1, \beta_2, \beta_3, \beta_4\rangle / k\langle\alpha\rangle$ is 4.

Notice that $\beta_1 = 2a_0\alpha_0' + a_1\alpha_0 + a_2\alpha_1' + a_3\alpha_1 + a_4$, in particular the coefficient a_4 is only used in the definition of β_1 (i.e., a_4 does not appear in the definitions of β_2, β_3, or β_4). Similarly a_8, a_{11}, and a_{13} appear uniquely in the definitions of β_2, β_3, and β_4 respectively. Also note that a_{14} does not appear in any of the β_i.

Let $\hat{a} := (a_0, a_1, \ldots)$ excluding $a_4, a_8, a_{11}, a_{13}, a_{14}$. Note that $(a_4, a_8, a_{11}, a_{13})$ is definable from $(\beta_1, \beta_2, \beta_3, \beta_4)$ over $K\langle\alpha, \hat{a}\rangle$. The tuple $(a_4, a_8, a_{11}, a_{13})$ are independent differential transcendentals over $K\langle\alpha, \hat{a}\rangle$, since α is a generic solution of $p(x, y)$ which contains a_{14} (and a_{14} is not included in $\hat{a}$). Therefore the differential transcendence degree of $(a_4, a_8, a_{11}, a_{13})$ over $K\langle\alpha, \hat{a}\rangle$ is 4. Hence the *differential transcendence degree of $(\beta_1, \beta_2, \beta_3, \beta_4)$ over $K\langle\alpha, \hat{a}\rangle$ is 4. In particular $(\beta_1, \beta_2, \beta_3, \beta_4)$ are generic coefficients over $K\langle\alpha\rangle$.*

Therefore by Corollary 32 we see that $T^{\Delta}_{\alpha}V$ has no subvarieties of rank ω. We want to use this to see that $RU(\alpha/K) = \omega$.

Suppose $F \supset K$ and $q = tp(\alpha/F)$ is a forking extension of $tp(\alpha/K)$. Let W be the differential variety corresponding to q. Then α is a generic point of W and $T^{\Delta}_{\alpha}W \subset T^{\Delta}_{\alpha}V$. Moreover $\omega_{T^{\Delta}_{\alpha}W} = \omega_W < \omega_V = \omega_{T^{\Delta}_{\alpha}V}$, since α is a generic point of V and W and $W \subset V$ is proper. Therefore $T^{\Delta}_{\alpha}W \subset T^{\Delta}_{\alpha}V$ is proper.

Since $T^{\Delta}_{\alpha}V$ contains no proper subvarieties with Δ-dim 1 we see that $T^{\Delta}_{\alpha}W$ must be finite rank. Hence W has finite rank, thus $RU(\alpha/K) \leq \omega$ as desired.

Now we want to generalize the previous example to arbitrary order and degree. We start with the following lemma to show the coefficients for the tangent space over a generic point are generic.

Lemma 56. *Let V be a differential variety given by a generic differential equation of order m in n variables. Let $\alpha \in V$ be a generic point. Then the coefficients of the equation defining $T^{\Delta}_{\alpha}V$ are generic over $K\langle\alpha\rangle$.*

Proof. Let $f(x)$ be the generic differential equation for V and let $\overline{a}$ denote the coefficients of f. Let $a_{i,j}$ be the coefficient $x_i^{(j)}$ in f. Note that we are looking at the coefficients of the linear terms of f.

We make a substitution to write the equation defining $T^{\Delta}_{\alpha}V$ as

$$\sum_{i=1}^{n}\sum_{j=0}^{m}\beta_{i,j}z_i^{(j)}$$

That is $\beta_{i,j} = \frac{\partial f}{\partial x_i^{(j)}}(\alpha)$. In particular we see that $a_{i,j}$ only appears in the definition of $\beta_{i,j}$, since it is the coefficient of $x_i^{(j)}$ in f.

Let $\hat{a}$ be $\overline{a}$ excluding the $a_{i,j}$ and c. We observe that the $a_{i,j}$ are definable from the $\beta_{i,j}$ over $K\langle \hat{a}, \alpha\rangle$. Also the $a_{i,j}$ are independent differential transcendentals over $K\langle \alpha, \hat{a}\rangle$ since α is a generic solution $f(x)$ which is defined using c and c is not definable over $K\langle \hat{a}, \alpha\rangle$. Therefore the differential transcendence degree of the $\beta_{i,j}$ over $K\langle \alpha, \hat{a}\rangle$ is $n \cdot (m+1)$. In particular the $\beta_{i,j}$ are generic over $K\langle \alpha\rangle$. $\square$

Theorem 57. *Let $n \geq 2$ and let V be a differential variety given by a generic differential equation of order m in n variables. Let $\alpha \in V$ be a generic point. Then $RU(\alpha/K) \leq \omega \cdot (n-1)$.*

Proof. Suppose $F \supset K$ and $q = tp(\alpha/F)$ is a forking extension of $tp(\alpha/K)$. Let W be the differential variety corresponding to q. The α is a generic point of W and $T_\alpha^\Delta W \subset T_\alpha^\Delta V$. Also $\omega_{T_\alpha^\Delta W} = \omega_W < \omega_V = \omega_{T_\alpha^\Delta V}$, since α is a generic point of V and $W \subset V$ is proper. Hence $T_\alpha^\Delta W \subset T_\alpha^\Delta V$ is proper.

By Lemma 56 the linear differential equation for $T_\alpha^\Delta V$ satisfies the conditions of Theorem 31 over $K\langle \alpha\rangle$. Thus by Corollary 32, $T_\alpha^\Delta V$ contains no proper subvarieties of Δ-dim $n-1$. Therefore W must have Lascar rank $< \omega \cdot (n-1)$, since $\omega_W = \omega_{T_\alpha^\Delta W}$. $\square$

5.3 Further Directions and Comments

As in chapter 3 the results of this chapter be applied to compute the Lascar rank for non-generic differential equations. In this case the conditions we need to satisfy are that for some generic point α, the equations defining T_α^Δ are sufficiently generic (over $K\langle\alpha\rangle$). One obstacle is that the coefficients for T_α^Δ depend on α. This poses a challenge to giving a more precise characterization of when these methods can be applied to non-generic equations. Given the large number of coefficients of non-linear differential equations it would be interesting to see if there multiple formulations of sufficient conditions to yield sufficiently generic coefficients for the tangent space above a generic point. In particular, our proof of Lemma 56 ultimately relies on the coefficients of the linear terms being generic with respect to the rest of the coefficients. Are there sufficient conditions that can be described for other subsets of coefficients to produce a similar result? What can be done in the case of a non-linear equation that doesn't have any linear terms?

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