The Distribution of Integral Points on the Wonderful Compactification by Height

by

Dylon Chow
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Defense Committee:
Ramin Takloo-Bighash, Chair and Advisor
Izzet Coskun
James Freitag
Nathan Jones
Djordje Milicevic, Bryn Mawr College

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SUMMARY

0.1 Rational Points and Integral Points by Height

Let $X \subset \mathbb{P}^n$ be a smooth projective variety over a number field F. If $X(F)$ is infinite, one can try to measure its size by a height function. For example, given $P \in \mathbb{P}^n(\mathbb{Q})$, one can write $P = (a_0 : ... : a_n)$ with $a_i \in \mathbb{Z}$ and $\gcd(a_0, ..., a_n) = 1$, and define the height of P as $H(P) = \max|a_i|$. Then, for $X \subseteq \mathbb{P}^n_{\mathbb{Q}}$, one defines

$$N(X, B) = \#\{x \in X(F) : H(x) \leq B\}.$$

For certain varieties X, the Batyrev-Manin conjecture and its refinements predict precise asymptotic formulas for $N(X^\circ, B)$ as $B \to \infty$, where $X^\circ \subset X$ is an appropriate Zariski open subset of X. A very accessible survey on counting integer and rational points on varieties is (1). When the variety is a bi-equivariant compactification, approaches from dynamics and harmonic analysis on adele groups can be used. For example, the papers (2) and (3) prove Manin's conjecture for wonderful compactifications of semisimple groups. In this paper, we study (D, S)-integral points on the wonderful compactification of a semisimple adjoint group by extending the techniques used to prove cases of the Batyrev-Manin conjecture.

Let X be a smooth projective variety over a number field F, and let $D \subset X$ be a divisor. To discuss integral points, we choose models $\mathcal{X}, \mathcal{D}$ of X, D over the ring of integers $\mathcal{O}_F$ of F. Let $\mathcal{U} = \mathcal{X} \backslash \mathcal{D}$. We denote by $\mathcal{U}(\mathcal{O}_F)$ the $\mathcal{O}_F$-valued points in U. For example, if $D = \emptyset$ and one chooses a proper model $\mathcal{X}$ over $\mathcal{O}_F$, then since $\mathcal{O}_F$ is a Dedekind domain, the valuative criterion of properness implies that $\mathcal{X}(\mathcal{O}_F) = X(F)$. So rational points on projective varieties over number fields are in this sense a special case of integral points.

To state our result, let G be a semisimple adjoint group of rank at least 2 and let X be the wonderful compactification of G. Assume that S and D are such that the set of (D, S)-integral points is Zariski dense. Let

$$b = \text{rk Pic}(X \backslash D) + \textstyle\sum_{v \in S} d_v$$

Here d_v is the dimension of a certain simplicial complex defined in (4).

Theorem 1 *With the notation above, the number of (S, D)-integral points of bounded height on X with respect to $-(K_X + D)$ is asymptotic to*

$$cB \log(B)^{b-1}(1 + o(1)), \qquad B \to \infty,$$

for some positive constant c.

0.2 Outline of paper and method of proof.

The proof relies on a strategy developed in the earlier papers on rational points and integral points for split groups. We introduce the height zeta function $Z(s) = \sum_{x \in G(F) \cap \mathcal{U}(\mathcal{O}_{F,S})} H(x)^{-s}$ for a complex parameter s. By a Tauberian theorem, it is sufficient to establish certain analytic properties of the function $Z(s)$. In particular, we prove its convergence for $\text{Re}(s) > a(\lambda)$ and then establish its meromorphic continuation in some half-plane $\text{Re}(s) > a(\lambda) - \delta$. This is achieved by rewriting the expression in terms of the spectral decomposition of $L^2(G(F) \backslash G(\mathbb{A}))$.

SUMMARY (Continued)

One of the terms in the spectral expansion is a sum, over a finite set of automorphic characters, of products of local v-adic integrals and adelic integrals. For the semisimple case, unlike the case of commutative groups, automorphic forms like Eisenstein series have to be considered. As in the case of rational points, uniform estimates need to be established.

The rest of the paper is organized as follows. In section 2, we review some background and set up notation. In section 3 we write out a spectral expansion for the height zeta function and establish the required analytic properties.

CHAPTER 1

BACKGROUND AND NOTATION

Let F be a number field. Let $\mathrm{Val}(F)$ be the set of normalized absolute values of F. For $v \in \mathrm{Val}(F)$, we let F_v be the v-adic completion of F. Its absolute value is denoted by $|.|_v$. For finite places v of F, denote by $\mathcal{O}_v$ the ring of v-adic integers and by $\mathfrak{m}_v$ its unique maximal ideal. The residue field $\mathcal{O}_v/\mathfrak{m}_v$ is denoted by k_v, and we write q_v for its cardinality. We fix a uniformizing element ϖ; one has $|\varpi|_v = q_v^{-1}$. We fix an algebraic closure $\overline{F}$ of F. and denote the absolute Galois group of F by Γ_F. We denote by $\mathbb{A}$ the adele ring of F.

1.1 Algebraic groups.

We will need to use various results from the structure theory of reductive groups over fields of characteristic zero. A good reference for the theory of algebraic groups is (5). Although we will mostly be concerned with semisimple groups with trivial center, the facts recalled here apply to any connected reductive group G over F. There exists a maximal torus T in G, and T remains maximal over every field extension of F. There exists a finite Galois extension E of F such that T splits over E - we fix this extension and let Γ denote $\mathrm{Gal}(E/F)$. For each place v of F, T_{F_v} is a maximal torus in G_{F_v}. There is a unique maximal split subtorus of T_{F_v} in G_{F_v}; denote it by S_v. Let $\mathfrak{X}^*(S_v)$ denote the set of characters of S_v, and let $\Phi(G_{F_v}, S_v)$ denote the subset of $\mathfrak{X}^*(S_v)$ consisting of the roots of S_v in G_{F_v}. A choice of minimal parabolic in G_{F_v} determines a set $\mathfrak{X}^+(S_v)$ of positive characters, a set $\Phi(G_{F_v}, S_v)$ of positive roots in $\mathfrak{X}^*(S_v)$, and a set $\Delta(G_{F_v}, S_v)$ of simple roots. Choose a Borel subgroup in G_E and denote by $\Delta(G_E, T_E)$ the corresponding set of simple roots. We define constants κ_α for $\alpha \in \Delta(G_E, T_E)$ by

$$\textstyle\sum_{\alpha>0, \alpha\in\Phi(G_E,T_E)} \alpha = \sum_{\alpha\in\Delta(G_E,T_E)} \kappa_\alpha \alpha.$$

There is a perfect pairing $\langle , \rangle : \mathfrak{X}^*(S_v) \times \mathfrak{X}_*(S_v) \to \mathbb{Z}$. For a simple root $\theta \in \Delta(G_{F_v}, S_v)$, define a cocharacter $\overset{\vee}{\theta}$ by $\langle \alpha, \overset{\vee}{\theta} \rangle = -\delta_{\alpha\theta}$, where $\delta_{\alpha\theta}$ is 1 if and only if $\alpha = \theta$.

1.2 Cartan decomposition

Our computations require the Cartan decomposition. If v is archimedean we set $F_v^0 = \{x \in \mathbb{R} : x \geq 0\}$ and $\hat{F}_v = \{x \in \mathbb{R} : x \geq 1\}$. If v is non-archimedean, we fix a uniformizer ϖ_v of F_v, and set $\hat{F}_v = \{\varpi^{-n} : n \in \mathbb{N}\}, F_v^0 = \{\varpi^n : n \in \mathbb{Z}\}$, and $S_v(F_v)^+ = \{a \in S_v(F_v) : \alpha(a) \in \hat{F}_v$ for each $\alpha \in \Phi^+(S_v)\}$. Then for each place v, there is a maximal compact subgroup K_v of $G(F_v)$ and a finite set $\Omega_v \subset G(F_v)$ such that $G(F_v) = K_v S_v(F_v)^+ \Omega_v K_v$. That is, for each $g \in G(F_v)$, there exist unique elements $a \in S_v(F_v)^+$ and $d \in \Omega_v$ such that $g \in K_v a d K_v$. If F_v is archimedean, or if G is unramified (i.e., quasisplit, and splits over an unramified extension), then $G(F_v) = K_v S_v(F_v)^+ K_v$.

1.3 The $*$-action

In this subsection, let k be an arbitrary field of characteristic 0. Let G be a connected reductive group over k. Let S be a maximal split torus in G and T a maximal torus containing S. Then G_{k^s} is split, and so T_{k^s} is contained in a Borel subgroup B of G_{k^s}. Let $(X, \Phi, \Phi^\vee, \Delta)$ be the based root datum of (G_{k^s}, B, T_{k^s}). As the action of Γ =Gal(k^s/k) preserves T_{k^s}, Γ acts naturally on $X = X^*(T)$ and $X^\vee = X_*(T)$. These actions preserve Φ and $\Phi^\vee$. We recall the $*$-action. If σ is an element of Γ, then there is a unique element w_σ in the Weyl group $W(G_{k^s}, T_{k^s})$ such that $w_\sigma(\sigma(\Delta)) = \Delta$. The $*$-action of Γ on Δ is defined by $\sigma * \alpha = w_\sigma(\sigma\alpha)$, for $\alpha \in \Delta$. We refer to Γ acting by the $*$-action as Γ^*. If G is split, then the $*$-action is trivial. If G is quasi-split, then the $*$-action is the restriction to Δ of the natural action of Γ on $X^*(T)$.

If k' is a field extension of k over which T splits, then we identify $X^*(T_{k^s})$ with $X^*(T_{k'})$, etc. Restriction from $T_{k'}$ to $S_{k'}$ defines a surjective homomorphism res: $X^*(T_{k'}) \to X^*(S_{k'})$. Let $\Phi(G_{k'}, T_{k'})$ be the associated root system and $\Phi(G_k, S) := \mathrm{res}(\Phi) - \{0\}$ be the restricted root system (in general not reduced). The set of simple roots $\Delta(G_{k'}, T_{k'}) \subseteq \Phi(G_{k'}, T_{k'})$ determines a set of simple restricted roots $\Delta_k = \mathrm{res}(\Delta)\backslash\{0\}$. Here 0 denotes the trivial character on S. The restriction map $\Delta \to \Delta_k \cup \{0\}$ is surjective. Let Δ_0 denote the set of elements of Δ that restrict to the trivial character on S. The fibers of the restriction map $\Delta - \Delta_0 \to \Delta_k$ are precisely the orbits of the $*$-action on $\Delta - \Delta_0$. Furthermore, G is quasi-split over k if and only if Δ_0 is trivial, in which case the number of Γ^*-orbits is precisely the cardinality of Δ_k. For all this, see (5), Prop. 25.28. We will apply these considerations when $k = F_v$ and $k' = E_w$, where F and E are number fields, and v and w are places of F and E, respectively, such that $w|v$.

1.4 <u>The Wonderful Compactification</u>

Let G be a semisimple adjoint group and let X be its wonderful compactification. For the construction, a useful reference is chapter 6 of (6).

Here are some of the properties of the wonderful compactification that we need, working initially over an algebraically closed field. Let B be a Borel subgroup of G and let T be a maximal split torus of G contained in B. Let r denote the rank of G, and denote the simple roots by $\alpha_1, ..., \alpha_r$. The compactification X is smooth, and the boundary $X \backslash G$ is the union of r nonsingular prime divisors with normal crossings. Via the isomorphism defined in (6) (Prop. 6.1.11), the Picard group $\mathrm{Pic}(X)$ of X is identified with the Picard group of the flag variety G/B, which is isomorphic to the weight lattice of G, and so $\mathrm{Pic}(X)$ is a free abelian group of rank r, generated by the fundamental weights. This isomorphism identifies the boundary divisors with the simple roots. We shall write $D_{\alpha_1}, ..., D_{\alpha_r}$ for the corresponding boundary divisors. Since the simple roots span the root lattice, we see that the $\mathbb{Z}$-span of the boundary divisors is a sublattice of $\mathrm{Pic}(X)$ with index the order of the center of the simply connected cover of G.

Now we consider the case over number fields. Let G be semisimple adjoint over a number field F, and let X be the corresponding wonderful compactification. The Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ acts on $\mathrm{Pic}(X_{\overline{F}})$. The bijection between the set of simple roots and the boundary divisors if Γ-equivariant (see, for instance, the proof of Theorem 8.1 in (7)), and so $\mathrm{Pic}(X)$ is freely generated by the line bundles corresponding to the orbits of the simple roots under the $*$-action. The F-irreducible boundary components of X are the divisors of the form $D_J = \sum_{\alpha \in J} D_\alpha$ for Γ^*-stable subsets $J \subset \Delta(G_{\overline{F}}, T_{\overline{F}})$. We denote the set of boundary divisors of X by $\mathcal{A}$. The closed cone $\mathrm{Eff}(X) \subset \mathrm{Pic}(X)_{\mathbb{R}}$ of effective divisors on X is generated by the boundary components of X, i.e., $\mathrm{Eff}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{R}_{\geq 0} D_\alpha$.

A dominant weight λ is called regular if $\lambda = \sum_{\alpha \in \Delta} m_\alpha \omega_\alpha$ with all $m_\alpha > 0$ where $\{\omega_\alpha : \alpha \in \Delta\}$ is the set of fundamental weights. The globally generated line bundles correspond to the dominant weights, and the ample line bundles correspond to the regular dominant weights. An anticanonical divisor for X is given by $-K_X = \sum_\alpha (\kappa_\alpha + 1) D_\alpha$. Here κ_α is defined by setting the sum of positive roots to be $\sum_{\alpha\in\Delta} \kappa_\alpha \alpha$.

1.5 Height functions

For the theory of height functions, see sections B.6. and B.8. of (8) and the paper (4). Let X be the wonderful compactification of G. We fix an integral models $\mathcal{X}$ and $\mathcal{D}$ over $\mathcal{O}_S$, for each boundary divisor D. This defines, for all places $v \notin S$, local height functions $H_{L,v}$ for all line bundles L. For places $v \in S$, we fix a v-adic metric on each line bundle L of X. As explained in (4), we have then defined an adelic metrization on each line bundle L. Define $H : \mathrm{Pic}(X)_{\mathbb{C}} \times G(\mathbb{A}) \to \mathbb{C}$ by $H(\mathbf{s}, (g_v)_v) = \prod_v H_v(\mathbf{s}, g_v)$.

1.5.1 Reducing to the simple case.

The adjoint group G decomposes into simple factors, $G = G_1 \times ... \times G_m$. Each G_i is also adjoint, and the wonderful compactification of G is $X = X_1 \times ... \times X_m$, where X_i denotes the wonderful compactification of G_i. By the Bruhat decomposition, G_i is geometrically rational (see, for instance, (9), Prop. 5.1.3), and hence so is X_i. Therefore ((9), Prop. 5.1.2) $\mathrm{Pic}((X_1)_{\overline{F}} \times ... \times (X_m)_{\overline{F}}) = \bigoplus_{i=1}^m \mathrm{Pic}((X_i)_{\overline{F}})$. The height functions on the product are expressed as products of height functions ((8), exercise F. 15). Therefore, we will assume that G is simple.

1.5.2 Maximal compact subgroups.

For each nonarchimedean place v of F, there exists a compact open subgroup $K_v \subset G(F_v)$ such that for all L, $H_{L,v}$ is bi-K_v-invariant. Moreover, one may take $K_v = G(\mathcal{O}_v)$ for all but finitely many v ((2), Prop. 6.3.). Let $K = \prod_v K_v$, and let $K_0 = \prod_{v<\infty} K_v$.

1.5.3 Measures.

For each $v \in \mathrm{Val}(F)$, let dg_v denote the Haar measure on $G(F_v)$ such that K_v has volume 1 whenever $v < \infty$. Then the collection $\{dg_v : v \in \mathrm{Val}(F)\}$ defines a Haar measure, say μ, on $G(\mathbb{A})$. Since G is semisimple, then $G(F)$ is a lattice in $G(\mathbb{A})$. Thus, by replacing $dg_v, v \in F_\infty$, with a suitable multiple of it, we may assume that $\mu(G(F)\backslash G(\mathbb{A})) = 1$.

1.5.4 Domains of convergence.

Let $\mathcal{T}_G$ be the subset of $\mathbb{C}^r$ consisting of vectors $(s_1, ..., s_r)$ such that $s_i = s_j$ whenever α_i and α_j are in the same Galois orbit. For $\epsilon \in \mathbb{R}$, let $\mathcal{T}_\epsilon$ denote the set of $\mathbf{s} = (s_\alpha)_\alpha \in \mathcal{T}_G$ such that $\mathrm{Re}(s_\alpha) > \kappa_\alpha + 1 + \epsilon$, for all α. For each subset R of $\mathbb{C}$, we set $\mathcal{T}(R)$ to be the collection of $\mathbf{s} = (s_\alpha)_\alpha$ with $s_\alpha \in R$ for all α. We set $\mathcal{T}_\epsilon^D = \{\mathbf{s} : \mathrm{Re}(s_\alpha) > \kappa_\alpha + 1 + \epsilon, \text{ for all } \alpha \notin \mathcal{A}_D\}$.

1.5.5 Integral points.

Let S be a finite set of places of F containing the archimedean places. Fix a model $\mathcal{U}$ over the ring $\mathfrak{o}_{F,S}$ of S-integers in F. The S-integral points of U are the elements of $\mathcal{U}(\mathfrak{o}_{F,S})$, in other words, those rational points of $U(F)$ which extend to a section of the structure morphism from $\mathcal{U}$ to $\mathrm{Spec}(\mathfrak{o}_{F,S})$. For $v \notin S$, let $\mathfrak{u}_v = \mathcal{U}(\mathfrak{o}_{F,v})$, and let δ_v be the characteristic function of the subset $\mathfrak{u}_v \subset X(F_v)$. A point $x \in X(F)$ is an S-integral point of U if and only if $x \in \mathfrak{u}_v$ for every place v of F such that $v \notin S$. This condition is equivalent to the condition that $\prod_{v \notin S} \delta_v(x) = 1$. For $v \in S$ we set $\delta_v \equiv 1$. For $g \in G(\mathbb{A})$, we define $\delta_{D,S}(g) = \prod_{v \notin S} \delta_v(g_v)$. For $v \notin S$, $\delta_v(g_v)$ is 1 if and only if the local height with respect to D is 1.

1.6 Eisenstein series

As a function of $G(\mathbb{A})$, the height zeta function $Z(\mathbf{s},.)$ is a function in $L^2(G(F)\backslash G(\mathbb{A}))$. Writing down its spectral expansion requires a great deal of notation. A useful reference is (10)

1. Let G be a reductive group over F. If v is finite, define K_v to be $G(\mathcal{O}_v)$ if this latter group is a special maximal compact subgroup of $G(F_v)$. This takes care of almost all v. For the remaining finite v, we let K_v be any fixed special maximal compact subgroup of $G(F_v)$. We also fix a minimal parabolic subgroup P_0, defined over F, and a Levi component M_0 of P_0. Let A_0 be the maximal split torus in the center of M_0. For each archimedean place v of F, we choose a maximal compact subgroup $K_v \subset G(F_v)$ such that $G(F_v) = K_v A_0(F_v) K_v$. We set $K = \prod K_v$. It is a maximal compact subgroup of $G(\mathbb{A})$. We also set $K_0 = \prod_{v<\infty} K_v$ and $K_\infty = \prod_{v|\infty} K_v$.

2. Fix a parabolic subgroup P defined over F that contains P_0. Let $N = N_P$ be the unipotent radical of P. Let M_P be the unique Levi component of P that contains M_0. Then the split component, A_P, of the center of M_P is contained in A_0. Let $X^*(M_P)_F$ be the group of characters of M_P defined over F, and define $\mathfrak{a}_{M_P}$ =Hom$(X^*(M_P)_F, \mathbb{R})$. For $\lambda \in X^*(M_P)_F$, we have a map

$$H_{M_P} : M_P(\mathbb{A}) \to \mathfrak{a}_{M_P}, \qquad \exp\langle H_M(m), \chi\rangle = \prod_v |\chi(m_v)|_v.$$

Let $M_P(\mathbb{A})^1$ be the kernel of the homomorphism H_{M_P}. Using the Iwasawa decomposition $G(\mathbb{A}) = N_P(\mathbb{A}) M_P(\mathbb{A}) K$, we define a morphism

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_{M_P}, \qquad nmk \mapsto H_{M_P}(m) \qquad ((n,m,k) \in N(\mathbb{A}) \times M_P(\mathbb{A}) \times K)$$

3. Let W be the restricted Weyl group of (G, A_0). Then W acts on the dual space of $\mathfrak{a}_0$. For a pair of standard parabolic subgroups P, P_1, let $W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ be the set of distinct linear isomorphisms from $\mathfrak{a}_P$ onto $\mathfrak{a}_{P_1}$ obtained by restricting elements in W to $\mathfrak{a}_P$. Two parabolic subgroups P and P_1 are said to be associated if $W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$ is not empty. This defines an equivalence relation on the set of parabolic subgroups in G. If P and P_1 are associate, then $M = M_1$. Moreover, w preserves $M = M_1$, where $w \in W(\mathfrak{a}_P, \mathfrak{a}_{P_1})$. In view of this fact, it is therefore natural to expect a relationship between representations of G induced from P and those induced from P'. Suppose that P is a parabolic subgroup. There is a finite number of disjoint open subsets of $\mathfrak{a}_P$, called the chambers of $\mathfrak{a}_P$. We shall write $n(A_P)$ for the number of chambers.

4. Let M be the Levi factor of some standard parabolic P of G. Let $L^2_{\text{cusp}}(M(F)\backslash M(\mathbb{A}_F)^1)$ be the space of functions ϕ in $L^2(M(F)\backslash M(\mathbb{A}_F)^1)$ such that for every parabolic $Q \subseteq P$ we have $\int_{N_Q(F)\cap M(F)\backslash N_Q(\mathbb{A})\cap M(\mathbb{A})} \phi(nm)\ dn = 0$ for almost all m. There is a $G(\mathbb{A})$-invariant orthogonal decomposition $L^2_{\text{cusp}}(M(F)\backslash M(\mathbb{A})^1) = \bigoplus_\sigma L^2_{\text{cusp},\sigma}(M(F)\backslash M(\mathbb{A})^1)$, where σ ranges over irreducible unitary representations of $M(\mathbb{A})^1$, and $L^2_{\text{cusp},\sigma}(M(F)\backslash M(\mathbb{A}))$ is $M(\mathbb{A})$-isomorphic to a finite number of copies of σ. An irreducible unitary representation of $M(\mathbb{A})^1$ is said to be cuspidal if $L^2_{\text{cusp},\sigma}(M(F)\backslash M(\mathbb{A})) \neq 0$.

5. We define an equivalence relation on the set of pairs (M, ρ) with M a Levi factor of some standard parabolic subgroup of G and ρ is an irreducible unitary representation of $M(\mathbb{A})^1$ occurring in $L^2_{\text{cusp}}(M(F)\backslash M(\mathbb{A})^1)$. A *cuspidal automorphic datum* is defined to be an equivalence class of pairs (P, σ), where $P \subset G$ is a standard parabolic subgroup of G, and

σ is an irreducible representation of $M_P(\mathbb{A})^1$ such that the space $L^2_{\mathrm{cusp},\sigma}(M_P(F)\backslash M_P(\mathbb{A})^1)$ is nonzero.

The restricted Weyl group W of (G, A_0) acts naturally on $\mathfrak{a}_{P_0}$ and $\mathfrak{a}^*_{P_0}$. For any $s \in W$, fix a representative w_s in the intersection of $G(F)$ with the normalizer of A_0. The equivalence relation is defined as follows: (P', σ') is equivalent to (P, σ) if there is an element $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ such that the representation $s^{-1}\sigma' : m \mapsto \sigma'(w_s m w_s^{-1})$, for $m \in M_P(\mathbb{A})^1$, of $M_P(\mathbb{A})^1$ is unitarily equivalent to σ. We write $\mathfrak{X}$ of the set of cuspidal automorphic data $\chi = \{(P, \sigma)\}$. For any $\chi \in \mathfrak{X}$ let $\mathcal{P}_\chi$ denote the class of associated parabolic subgroups consisting of those parabolic subgroups P with a Levi subgroup M and a representation ρ such that $(M, \rho) \in \chi$.

If M is the Levi factor of some parabolic subgroup and $\chi \in \mathfrak{X}$, set $L^2_{\mathrm{cusp}}(M(F)\backslash M(\mathbb{A})^1)_\chi = \bigoplus_{(\rho:(M,\rho)\in\chi)} V_\rho$. This is a closed subspace of $L^2_{\mathrm{cusp}}(M(F)\backslash M(\mathbb{A})^1)$. It is zero if $P \notin \mathcal{P}_\chi$ for every parabolic subgroup P that has M as a Levi factor. Then we have

$$L^2_{\mathrm{cusp}}(M(F)\backslash M(\mathbb{A})^1) = \bigoplus_{\chi\in\mathfrak{X}} L^2_{\mathrm{cusp}}(M(F)\backslash M(\mathbb{A})^1)_\chi.$$

6. Any class $\chi = \{(\mathcal{P}, \sigma)\}$ in $\mathfrak{X}$ determines an associated class of standard parabolic subgroups. For any P, let $\Pi(M_P)$ denote the set of equivalence classes of irreducible unitary representations of $M_P(\mathbb{A})$. If $\zeta \in \mathfrak{a}^*_{\mathbb{C}}$ and $\pi \in \Pi(M_P)$, let π_ζ be the product of π with the quasi-character $x \mapsto e^{\zeta(H_P(x))}$, for $x \in G(\mathbb{A})$.

If ζ belongs to $i\mathfrak{a}^*_P$, then π_ζ is unitary, and so we obtain a free action of the group $i\mathfrak{a}^*_P$ on $\Pi(M_P)$. Then $\Pi(M_P)$ becomes a differentiable manifold whose connected components are the orbits of $i\mathfrak{a}^*_P$. We can transfer our Haar measure on $i\mathfrak{a}^*_P$ to each of the orbits in $\Pi(M_P)$. This allows one to define a measure $d\pi$ on $\Pi(M_P)$.

7. Let $\pi \in \Pi(M_P)$. We will define certain representations of $G(\mathbb{A})$ induced from π. First we describe the space of the representation. Let $H_P^0(\pi)$ be the space of smooth functions $\phi : N_P(\mathbb{A})M_P(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ which satisfy the following conditions:

(i) ϕ is right K-finite, i.e., the span of the set of functions $\phi_k : x \mapsto \phi(xk), x \in G(\mathbb{A})$, indexed by $k \in K$, is finite dimensional.

(ii) For every $x \in G(\mathbb{A})$, the function $m \mapsto \phi(mx), m \in M_P(\mathbb{A})$ is a matrix coefficient of π.

(iii) If we define $||\phi||^2$ by the equation $||\phi||^2 = \int_K \int_{M_P(F)\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2 \; dm \; dk$, then $||\phi||^2 < \infty$.

Let $H_P(\pi)$ be the Hilbert space completion of $H_P^0(\pi)$. Now we describe the action of $G(\mathbb{A})$ on this space. More precisely, we will define representations $I_P(\pi_\zeta)$ of $G(\mathbb{A})$, one for each $\zeta \in \mathfrak{a}^*_{P,\mathbb{C}}$, as follows. For each $y \in G(\mathbb{A})$, $I_P(\pi_\zeta)(y)$ maps a function ϕ of $H_P(\pi)$ to the function given by $(I_P(\pi_\zeta)(y)\phi)(x) = \phi(xy)e^{(\zeta+\rho_P)(H_P(xy))}e^{-(\lambda+\rho_P)(H_P(x))}$.

For $\phi \in H_P(\pi)$ and $\zeta \in \mathfrak{a}^*_{\mathbb{C}}$, define $\phi_\zeta(x) = \phi(x)e^{\zeta(H_P(x))}, \;\; x \in G(\mathbb{A})$. Recall that the function $e^{\rho_P(H_P(.))}$ is the square root of the modular function of the group $P(\mathbb{A})$. It is included in the definition so that the representation $I_P(\pi_\zeta)$ is unitary whenever the inducing representation is unitary, which is to say, whenever ζ belongs to the subset $i\mathfrak{a}^*_P$ of $\mathfrak{a}^*_{P,\mathbb{C}}$. The above equation can be rewritten as $(I_P(\pi_\zeta)(y))(\phi)(x) = \phi_\zeta(xy)\delta_P(xy)^{1/2}\delta_P(x)^{-1/2}$. We have put the twist by ζ into the operator $I_P(\pi_\zeta)(y)$ rather than the underlying Hilbert space $H_P(\pi)$, so that $H_P(\pi)$ is independent of ζ. We remark that $H_P(\pi) = \{0\}$ unless there is a subrepresentation of the regular representation of $M(\mathbb{A})^1$ on $L^2(M(F)\backslash M(\mathbb{A})^1)$ which is equivalent to the restriction of π to $M(\mathbb{A})^1$.

8. Note that $I_P(\pi_\zeta)$ is a representation of $G(\mathbb{A})$ on a space of functions on $G(\mathbb{A})$, but the functions in the representation space $H_P(\pi)$ are not left invariant under $G(F)$. The functions in the space are left invariant under $P(F)$, however. We will average the functions to make them invariant under $G(F)$. We are now ready to define Eisenstein series. They provide an intertwining map from $I_P(\pi_\zeta)$ to the regular action of $G(\mathbb{A})$ on functions on $G(\mathbb{A})$.

Suppose that $\pi \in \Pi(M)$. For $\phi \in H_P^0(\pi)$ and $\zeta \in (\mathfrak{a}_P^*)_\mathbb{C}$, we formally define $E(x, \phi, \zeta) = \sum_{\gamma \in P(F)\backslash G(F)} \phi_\zeta(\gamma x)\delta_P(\gamma x)^{1/2}$. This expression is defined by a sum over a noncompact space. In general, such an expression does not converge. For any P, we can form the chamber $(\mathfrak{a}_P^*)^+ = \{\Lambda \in \mathfrak{a}_P^* : \Lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in \Delta_P\}$ in $\mathfrak{a}_P^*$. Here Δ_P is the set of roots of (P, A_P), where A_P is the split component of the center of M_P. As before, suppose that $\pi \in \Pi(M), \phi \in H_P^0(\pi)$, and $\zeta \in \mathfrak{a}_{P,\mathbb{C}}^*$. It is a theorem of Langlands that if ζ lies in the open subset of $\mathfrak{a}_{P,\mathbb{C}}^*$ with $\mathrm{Re}(\zeta) \in \rho_P + (\mathfrak{a}_P^*)^+$, then the sum that defines $E(x, \phi, \zeta)$ converges absolutely to an analytic function of ζ.

The set of points ζ for which $I_P(\zeta)$ is unitary, i.e. such that ζ belongs to the real subspace $i\mathfrak{a}_P^*$ of $\mathfrak{a}_{P,\mathbb{C}}^*$, is never inside the domain of absolute convergence of the Eisenstein series. It is a theorem of Langlands that the series $E(x, \phi, \zeta)$ have analytic continuations to this space. More precisely, if $\phi \in H_P^0(\pi)$, then $E(x, \phi, \zeta)$ can be analytically continued to a meromorphic function of $\zeta \in \mathfrak{a}_{P,\mathbb{C}}^*$. If $\zeta \in i\mathfrak{a}_P^*$, then $E(x, \phi, \zeta)$ is analytic. We denote the value of this analytically continued function at $\zeta = 0$ by $E(x, \phi)$.

9. We need more notation to write down a spectral decomposition of the height zeta function. Suppose that P is fixed and that $\chi \in \mathfrak{X}$. Suppose first of all that there is a group P_1 in $\mathcal{P}$ which is contained in P. Let ψ be a smooth function on $N_{P_1}(\mathbb{A})M_{P_1}(F)\backslash G(\mathbb{A})$ such that

$$\Psi_a(m,k) := \psi(amk), k \in K, m \in M_{P_1}(F)\backslash M_{P_1}(\mathbb{A})^1, a \in A_{P_1}(F)\backslash A_{P_1}(\mathbb{A})$$

vanishes for a outside a compact subset of $A_{P_1}(F)\backslash A_{P_1}(\mathbb{A})$, transforms under K_∞ according to an irreducible representation W, and, as a function of m, belongs to $L^2_{\mathrm{cusp}}(M_{P_1}(F)\backslash M_{P_1}(\mathbb{A})^1)$. The function $\hat{\psi}^M(m) := \sum_{\delta \in P_1(F)\cap M_P(F)\backslash M_P(F)} \psi(\delta m)$, for $m \in M_P(F)\backslash M_P(\mathbb{A})^1$, is square integrable on $M_P(F)\backslash M_P(\mathbb{A})^1$. We define $L^2(M_P(F)\backslash M_P(\mathbb{A})^1)_\chi$ to be the closed span of all functions of the form $\hat{\psi}^M$, where P_1 runs through those groups in $\mathcal{P}$ which are contained in P, and W is allowed to vary over all irreducible representations of K_∞. If there does not exist a group $P_1 \in \mathcal{P}$ which is contained in P, define $L^2(M_P(F)\backslash M_P(\mathbb{A})^1)_\chi$ to be $\{0\}$. Then we have an orthogonal direct sum decomposition $L^2(M_P(F)\backslash M_P(\mathbb{A})^1) = \bigoplus_{\chi\in\mathfrak{X}} L^2(M_P(F)\backslash M_P(\mathbb{A})^1)_\chi$.

Given $\chi \in \mathfrak{X}$, let $H_P(\pi)_\chi$ be the closed subspace of $H_P(\pi)$ consisting of those ϕ such that for all x the function $m \mapsto \phi(mx), m \in M_P(F)\backslash M_P(\mathbb{A})^1$ belongs to $L^2(M_P(F)\backslash M_P(\mathbb{A})^1)_\chi$. Then $H_P(\pi) = \bigoplus_{\chi\in\mathfrak{X}} H_P(\pi)_\chi$. Suppose that W is an equivalence class of irreducible representations of K_∞. Let $H_P(\pi)_{\chi,K_0}$ be the subspace of functions in $H_P(\pi)_\chi$ which are invariant under $K_0 \cap K$, and let $H_P(\pi)_{\chi,K_0,W}$ be the space of functions in $H_P(\pi)_{\chi,K_0}$ which transform under K_∞ according to W. Each of the spaces $H_P(\pi)_{\chi,K_0,W}$ is finite-dimensional. We shall need orthonormal bases of the spaces $H_P(\pi)_\chi$. We fix such a basis, $B_P(\pi)_\chi$, for each π and χ, in such a way that for every $\zeta \in i\mathfrak{a}^*$, $B_P(\pi_\zeta)_\chi = \{\phi_\zeta : \phi \in B_P(\pi)_\chi\}$, and that every $\phi \in B_P(\pi)_\chi$ belongs to one of the spaces $H_P(\pi)_{\chi,K_0,W}$.

CHAPTER 2

HEIGHTS ON THE WONDERFUL COMPACTIFICATION

2.1 Spectral expansion.

To study the distribution properties of (D,S)-integral points we will establish analytic properties of the height zeta function $Z_{S,D} : \mathcal{T}_G \times G(\mathbb{A}) \to \mathbb{C}$, which is defined by

$$Z_{D,S}(\mathbf{s}, g) = \sum_{\gamma \in G(F)} \delta_{D,S}(g) H(\mathbf{s}, \gamma g)^{-1}.$$

Proposition 1 *Given $g \in G(\mathbb{A})$, the series defining $Z_{D,S}(.,g)$ converges absolutely to a holomorphic function for $\boldsymbol{s} \in \mathcal{T}_{\gg 0}$. For all $\boldsymbol{s}$ in the region of convergence, $Z_{S,D}(\boldsymbol{s},.) \in C^\infty(G(F)\backslash G(\mathbb{A}))$, and for all integers $n \geq 1$ and all $\partial \in U(\mathfrak{g}), \partial^n Z_{S,D}(\boldsymbol{s}.) \in L^2$. Moreover, in this domain, we have an equality*

$$Z_{S,D}(\boldsymbol{s}, g) = \sum_{\chi \in \mathfrak{X}} \sum_P \frac{1}{n(A_P)} \int_{\Pi(M_P)} \int_{G(\mathbb{A})} \delta_{S,D}(g') H(\boldsymbol{s}, g')^{-1} (\sum_{\phi \in B_P(\pi)_\chi} E(g, \phi) \overline{E(g', \phi)}) dg' d\pi. \tag{2.1}$$

Identical to the proof of (2), Proposition 8.2; it suffices to observe that $Z_{S,D}$ is a subsum of the series defining the height zeta function for rational points considered in (2).

Let $\mathcal{X}$ be the set of unramified automorphic characters of G, i.e., continuous $G(F)$-invariant homomorphisms $G(\mathbb{A}) \to S^1$ which are invariant under K_0 on both sides. Only these characters can contribute to the rightmost pole. By (3) Lemma 4.7(2), the number of automorphic characters that are invariant under this compact open subgroup is finite.

When we set $g = e$, the identity in $G(\mathbb{A})$, we obtain

$$Z(\mathbf{s}, e) = \sum_{\chi \in \mathcal{X}} \prod_v \int_{G(F_v)} \delta_v(g_v) H_v(\mathbf{s}, g_v)^{-1} \chi_v(g_v)\ dg_v + S^\flat(\mathbf{s}), \tag{2.2}$$

where $S^\flat(\mathbf{s})$ denotes the subsum corresponding to infinite dimensional representations (restricted to $g = e$). The innermost sum in the definition of $S^\flat(\mathbf{s})$ is uniformly convergent for g in compact sets (see the first half of the proof of Lemma 4.4 of (2)). Therefore, we may interchange the innermost summation with the integral over $G(\mathbb{A})$ and find that $S^\flat(\mathbf{s})$ equals

$$\sum_{\chi \in \mathfrak{X}}^{\flat} \sum_P \frac{1}{n(A_P)} \int_{\Pi(M_P)} \big(\sum_{\phi \in B_P(\pi)_\chi} E(e, \phi) \int_{G(\mathbb{A})} \delta_{S,D}(g) H(\mathbf{s}, g)^{-1} \overline{E(g, \phi)}\ dg \big)\ d\pi_P.$$

2.2 Complexified Height Function.

When G is split, the local height functions can be described in terms of roots. Since E is a splitting field of T, the group G_E is split. Given a boundary divisor D_α of the wonderful compactification of G_E, the local heights are given at all places w by $H_{D_\alpha, w}(g) = |\alpha(t)|_w$, where $g = k_1 t k_2$ for k_1 and k_2 in $G(\mathcal{O}_{E_w})$ ((2) Proposition 6.3).

In general, $H_{D,v}(P) = (\prod_{w|v} H_{D,w}(P)^{[E_w:F_v]})^{\frac{1}{[E:F]}}$ (see (8) Remark B.8.3). For unramified places the equation becomes $H_{D,v}(P) = H_{D,w}(P)$. For almost all places v, $g_v \in G(F_v)$ can be written as $k_v t_v k_v'$ with $t_v \in S_v(F_v)^+$ and $k_v, k_v' \in K_v$, and $H_{D_\alpha,v}(g_v) = H_{D_\alpha,v}(t_v) = |\alpha(t_v)|_v$. Thus, for such places (which we denote by S_F) δ_v is given by $\delta_v(g_v) = \prod_{\alpha \in \mathcal{A}_D} |\alpha(t_v)|_v$.

Considering arbitrary line bundles, we see that the local complexified height function on $G(F_v)$ has the following explicit form: $H_v(\mathbf{s}, g_v) = \prod_{\alpha \in \Delta(G_{E_w}, T_{E_w})} |\alpha(t_v)|_v^{s_\alpha}$. Right now the height function is written in terms of roots of T_{E_w}. We will express the height function in terms of roots of the maximal split F_v-torus S_v. Some of the roots of T_{E_w}, when restricted to S_v, will have the same behavior. Given $\theta_v \in \Delta(G_{F_v}, S_v)$, let $l_v(\theta_v)$ denote the number of $\beta \in \Delta(G_{E_w}, T_{E_w})$ with $r_u(\beta) = \theta_v$. For each $\theta_v \in \Delta(G_{F_v}, S_v)$, there is a simple root β in $\Delta(G_E, T_E)$ such that $r_v(\iota_u^*(\beta)) = \theta_v$. We define local parameters by requiring that s_{θ_v} depend only on the Galois orbit of β in $\Delta(G_{E_w}, T_{E_w})$. Then, for $\mathbf{s} \in \mathcal{T}_G$ and $g_v \in G(F_v)$, we have

$$H_v(\mathbf{s}, g_v) = \prod_{\theta_v \in \Delta(G_{F_v}, S_v)} |\theta_v(t_v)|_v^{l_v(\theta_v) s_{\theta_v}}. \tag{2.3}$$

For $v \notin S_F$, $\delta_v(g_v) = 1$ if and only if, when g_v is written as $k_1 t_v k_2$, we have $\theta_v(t_v) = 1$ for all $\theta_v \in \Delta(G_{F_v}, S_v)$ such that θ_v is the restriction of a root of T_{E_w} in a Galois orbit corresponding to D.

2.3 Integrals from one dimensional representations.

2.3.1 An L-function

In this section we review some of the notation defined in section 2.8 of (2). Fix a simple root $\alpha \in \Delta(G_E, T_E)$ and let Γ_α be the stabilizer of α in Γ. Let E_α be the field of definition of α, *i.e.*, the fixed field of Γ_α. We then have a morphism $\overset{\vee}{\alpha} : \mathbb{G}_m \to T$ defined over E_α, and consequently a continuous homomorphism $\overset{\vee}{\alpha}_{\mathbb{A}} : \mathbb{G}_m(\mathbb{A}_{E_\alpha}) \to T(\mathbb{A}_{E_\alpha})$. Let $N : T(\mathbb{A}_{E_\alpha}) \to T(\mathbb{A}_F)$ be the norm map, as defined in (2), section 1.6. Let ϕ_α be the composite

$$\phi_\alpha = N \circ \overset{\vee}{\alpha}_{\mathbb{A}} : \mathbb{G}_m(\mathbb{A}_{E_\alpha}) \to T(\mathbb{A}_{E_\alpha}) \to T(\mathbb{A}_F).$$

If χ is a character of $T(\mathbb{A}_F)/T(F)$, then $\xi_\alpha(\chi) = \chi \circ \phi_\alpha$ is a Hecke character. For $w \in \mathrm{Val}(E_\alpha)$, define $\xi_\alpha(\chi)_w$ by $\xi_\alpha(\chi) = \prod_{w \in \mathrm{Val}(E_\alpha)} \xi_\alpha(\chi)_w$.

Let $\alpha \in \Delta(G_E, T_E)$, and let $\mathfrak{O} = \Gamma \cdot \alpha$ be the orbit of α. The Hecke L-function $L(s, \xi_\beta(\chi))$ depends only on the Galois orbit $\mathfrak{O}$, and not on the particular β. For this reason, we denote the L-function $L(s, \xi_\beta(\chi))$ by $L(s, \xi_{\mathfrak{O}}(\chi))$. Suppose $(E_\alpha)_w/F_v$ is unramified and that $\xi_\alpha(\chi)_w$ is unramified. Let $L_w(s, \xi_\alpha(\chi)_w) = (1 - \xi_\alpha(\chi)_w(\varpi_w) q_w^{-s})^{-1}$, where ϖ_w is a prime element of $(E_\alpha)_w$. Then with the above notations, $L_w(s, \xi_\alpha(\chi)_w) = (1 - \chi_v(\theta_v^\vee(\varpi_v)) q_v^{-l_v(\theta_v)s})^{-1}$.

2.3.2 Infinite product

Theorem 2 *Let $\chi = \otimes_v' \chi_v$ be a one-dimensional unramified automorphic representation of $G(\mathbb{A})$. There exists a function $f_{S,\chi}$, which depends only on $(s_\alpha)_{\alpha \notin \mathcal{A}_D}$, is holomorphic in $\mathcal{T}^D_{-1/2}$, uniformly bounded in $\mathcal{T}^D_{-1/2+\epsilon}$, for any $\epsilon > 0$, and such that*

$$\int_{G(\mathbb{A}_{S_D \cup S_F})} \delta_{D,S}(g) H_S(\boldsymbol{s}, g)^{-1} \chi(g)\, dg = \prod_{\mathfrak{O} \notin \mathcal{A}_D} L^{S_{\mathfrak{O}}}(s_{\mathfrak{O}} - \kappa_{\mathfrak{O}}, \xi_{\mathfrak{O}}(\chi)) \cdot f_{S,\chi}(\boldsymbol{s}).$$

Here, given an orbit $\mathfrak{O}$, there corresponds a subfield of E. We let $S_{\mathfrak{O}}$ be the finite set of places such that if $w \notin S_{\mathfrak{O}}$, then w does not lie above any place $v \in S_F$. Fix a place v of F such that $v \notin S_D \cup S_F$. For each $\theta_v \in \Delta(G_{F_v}, S_v)$, there is a simple root β in $\Delta(G_E, T_E)$ such that $r_v(\iota_u^*(\beta)) = \theta_v$. We require that s_{θ_v} depend only on the Galois orbit of β in $\Delta(G_E, T_E)$.

Starting with an element $\mathbf{s} = (s_\alpha)_\alpha \in \mathcal{T}_G$ and $v \notin S_F$, we obtain a tuple $\mathbf{s}^v = (s^v_\theta)$ indexed by $\Delta(G_{F_v}, S_v)$ by setting $s^v_{r_v(\iota^*(\alpha))} = s_\alpha$; this is well-defined. For $\mathbf{s}, \mathbf{t} \in \mathcal{T}_G$ and $v \notin S_F$, we set $\langle \mathbf{s}, \mathbf{t} \rangle_v = \sum_{\theta_v \in \Delta(G_{F_v}, S_v)} s^v_\theta t^v_\theta$. For any vector $\mathbf{a} = (a_\alpha)_\alpha \in \mathcal{T}(\mathbb{N})$, we set $t_v(\mathbf{a}) = \prod_{\theta \in \Delta(G_v, S_v)} \theta^\vee(\varpi_v)^{a^v_\theta}$. By the Cartan decomposition,

$$\int_{G(\mathbb{A}_{S_D \cup S_F})} \delta_{D,S}(g) H_S(\mathbf{s}, g)^{-1} \chi(g)\ dg = \sum_{w \in S_v(F_v)^+} \delta_v(w) H_v(\mathbf{s}, w)^{-1} \chi_v(w) \mathrm{vol}(K_v w K_v).$$

The above sum is equal to

$$\sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} \delta_v(t_v(\mathbf{a})) q_v^{-\langle \mathbf{s}, \mathbf{a} \rangle_v} \chi_v(t_v(\mathbf{a})) \mathrm{vol}(K_v t_v(\mathbf{a}) K_v).$$

This can be rewritten as

$$\sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} \delta_v(t_v(\mathbf{a})) q_v^{-\langle \mathbf{s} - 2\rho, \mathbf{a} \rangle_v} \chi_v(t_v(\mathbf{a})) + b_v(\mathbf{s}),$$

where

$$b_v(\mathbf{s}) = \sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} \delta_v(t_v(\mathbf{a})) q_v^{-\langle \mathbf{s}, \mathbf{a} \rangle_v} \chi_v(t_v(\mathbf{a})) (\mathrm{vol}(K_v t_v(\mathbf{a}) K_v) - \delta_{B_v}(t_v(\mathbf{a}))).$$

First, we must bound the infinite product

$$\prod_{v \notin S_D \cup S_F} \sum_{\mathbf{a} \in \mathcal{T}(\mathbb{N})} \delta_v(t_v(\mathbf{a})) q_v^{-\langle \mathbf{s} - 2\rho, \mathbf{a} \rangle_v} \chi_v(t_v(\mathbf{a})).$$

This expression is equal to

$$\prod_{v\notin S_D\cup S_F}\prod_{\theta\in\mathcal{A}_v-\mathcal{A}_{D_v}}\sum_{a_\theta^v=0}^{\infty}\chi_v(\theta^\vee(\varpi_v)^{a_\theta})q_v^{-(s_\theta^v-\kappa_\theta^v)a_\theta^v l(\theta)}$$

$$=\prod_{v\notin S_D\cup S_F}\prod_{\theta\in\mathcal{A}_v-\mathcal{A}_{D_v}}(1-\chi_v(\theta^\vee(\varpi_v))q_v^{-(s_\theta-\kappa_\theta)l(\theta)})^{-1}.$$

By (2) Proposition 2.9, this infinite product is equal to

$$\prod_{\mathfrak{D}\in\mathcal{A}-\mathcal{A}_D}L^{S_\mathfrak{D}}(s_\mathfrak{D}-\kappa_\mathfrak{D},\xi_\mathfrak{D}(\chi)).$$

Now we turn to $\sum_{v\notin S_D\cup S_F}b_v(\mathbf{s})$. Let $\sigma=(\mathrm{Re}(s_\alpha))_\alpha$. Observe that in the definition of b_v we may assume that $\mathbf{a}\neq\underline{0}$. Since for each $v\notin S_F\cup S_D$,

$$\{\mathbf{a}|\mathbf{a}\neq\underline{0}\}=\bigcup_{\theta\in\Delta(G_{F_v},S_v)}\{\mathbf{a}:a_\theta^v\neq 0\},$$

we have

$$\sum_{v\notin S_D\cup S_F}|b_v(\mathbf{s})|\leq\sum_{v\notin S_D\cup S_F}\sum_\theta\sum_{a_\alpha\neq 0}\delta_v(t_v(\mathbf{a}))q_v^{-\langle\sigma,\mathbf{a}\rangle_v}|(\mathrm{vol}(K_vt_v(\mathbf{a})K_v)-\delta_{B_v}(t_v(\mathbf{a})))|$$

$$\ll\sum_{v\notin S_D\cup S_F}q_v^{-1}\sum_{\alpha\notin\mathcal{A}_D}\sum_{a_\alpha\neq 0}q_v^{-\langle\sigma,\mathbf{a}\rangle_v}\delta_{B_v}(t_v(\mathbf{a}))$$

$$=\sum_{v\notin S}q_v^{-1}\sum_{\theta\notin\mathcal{A}_D}(\sum_{a_\theta=1}^{\infty}q_v^{-(\sigma_\alpha-\kappa_\theta)a_\theta l(\theta)})\prod_{\beta\neq\theta,\beta\notin\mathcal{A}_D}(\sum_{a_\beta=0}^{\infty}q_v^{-(\sigma_\beta-\kappa_\beta)a_\beta l(\beta)})$$

$$=\sum_{v\notin S_D\cup S_F}q_v^{-1}\sum_{\theta\notin\mathcal{A}_D}\frac{q_v^{-(\sigma_\theta-\kappa_\theta)a_\theta l(\theta)}}{\prod_{\beta\notin\mathcal{A}_D}(1-q_v^{-(\sigma_\beta-\kappa_\beta)l(\beta)})}$$

$$\ll\sum_{\theta\notin\mathcal{A}_D}\sum_{v\notin S_D\cup S_F}q_v^{-3/2}<\infty.$$

We need to show the existence of a $C>0$ such that $|1+a_v|\geq C>0$ for all v. For this,

$$|1+a_v| \geq \prod_{\theta \in \mathcal{A}_D} \frac{1}{1+q_v^{-\sigma_\beta+\kappa_\beta}} \geq \prod_{\theta \notin \mathcal{A}_D} \frac{1}{2} \geq \frac{1}{2^r},$$

with $r = |\Delta(G_E, T_E) \backslash \mathcal{A}_D|$. For $\mathbf{s} \in \mathcal{T}^D_{-1/2+\epsilon}$ the estimates are uniform, i.e. the quotient

$$\frac{\prod_{v \notin S} I_v(\chi)}{\prod_{v \notin S_D \cup S_F}(1+a_v)}$$

is holomorphic in $\mathcal{T}^D_{-1/2+\epsilon}$. This completes the proof.

2.3.3 Local integrals for places in S_F.

We turn to places $v \in S_F$. If $v \in S_F \notin S_D$, then by Proposition 4.4 in (4) the rightmost pole is at $\max_{\alpha \notin \mathcal{A}_D} \dfrac{\kappa_\alpha}{\lambda_\alpha}$. Such integrals will not contribute to the rightmost pole of the height zeta function. If $v \in S_F \cap S_D$, the analysis in (4) shows that the rightmost pole is at $\max_{\alpha \in \mathcal{A}} \dfrac{\kappa_\alpha}{\lambda_\alpha}$ and that the order of the pole is $1 + \dim \mathcal{C}^{\mathrm{an}}_{F_v,\lambda}(D)$. They also establish analytic continuation to the left of the pole.

2.4 Integrals from infinite dimensional representations.

Lemma 3 *(1) Let H be a connected reductive group over a number field F. Let v be a place of F such that $H(F_v)$ is not compact modulo center. Let π be an automorphic representation of $H(\mathbb{A}_F)$. If π_v is one-dimensional then π is one-dimensional.*

(2) Let F be a non-archimedean local field. Let G be a connected reductive group over F. If the F-simple factors of the derived group of G are F-isotropic, then any irreducible smooth representation V of $G(F)$ is either one-dimensional or infinite-dimensional.

For (1), see Lemma 6.2 in (11). For (2), see (12), Proposition 3.9.

For the integrals

$$\int_{G(\mathbb{A})} \delta_{S,D}(g) H(\mathbf{s}, g)^{-1} E(g, \phi)\ dg,$$

We will follow and briefly sketch the argument presented in section 4.5 of (13). At any finite place v we let $C_c^\infty(G(F_v))$ denote, as usual, the space of functions on $G(F_v)$ that are locally constant and of compact support. For archimedean v we require such functions to be smooth and of compact support. The set $C_c^\infty(G(F_v))$ forms a convolution algebra $\mathcal{H}(G(F_v))$ with respect to the measure dg_v. For each place v we define idempotents ξ_v as in section 4.5 of (13). The global Hecke algebra $\mathcal{H}(G(\mathbb{A}))$ is the space of finite linear combinations of functions $\otimes_v \varphi_v$, where $\varphi_v \in \mathcal{H}(G(F_v))$ and φ_v is ξ_v for almost all v.

Since $\delta \cdot H$ is invariant on the left and right under the compact open subgroup K_v for each non-archimedean place v, there is an associated idempotent $\xi_0 = \otimes'_{v \text{non-arch.}} \xi_v$ in the Hecke algebra $\otimes'_{v\text{non-arch.}} \mathcal{H}(G(k_v))$ such that $\xi_0 * (\delta \cdot H) = (\delta \cdot H) * \xi_0$. By a theorem of Harish-Chandra, $\sum_{W \in \hat{K}_\infty} \xi_W * H_\infty$ converges in the topology of $C^\infty(G_\infty)$ to H_∞. Therefore, we may rewrite the integral above as

$$\int_{G(\mathbb{A})} (\sum_{W \in \hat{K}_\infty} \xi_W \otimes \xi_0) H(\mathbf{s}, g)^{-1} E(g, \phi)\ dg = \sum_{W \in \hat{K}_\infty} \int_{G(\mathbb{A})} (\xi_W \otimes \xi_0) * H(\mathbf{s}, g))^{-1} E(g, \phi)\ dg$$

$$= \sum_{W \in \hat{K}_\infty} \int_{G(\mathbb{A})} H(g, \mathbf{s})^{-1} (\xi_W \otimes \xi_0) * E(g, \phi)\ dg.$$

We define $M_\xi(g, \phi) = (\xi * E)(g, \phi)$. For groups of rank at least two we will use uniform bounds on matrix coefficients obtained by Oh in (14). The same result is expected to hold in general by adapting the proof of Theorem 4.5 in (2). We shall use the following result - see (13), Lemma 4.7. Suppose we are given a strongly orthogonal system $\mathcal{S}_v$ in $G(k_v)$ for each place v. We continue to use the notation in section 4.5 of (13). Let ξ be a non-trivial idempotent in the global Hecke algebra.

Lemma 4 *There is a constant C_ξ, depending only on the idempotent ξ, such that*

$$|M_\xi(g,\phi)| \leq C_\xi \sqrt{\dim H_P(\pi)_{\chi,K_0,W}} \cdot \max_{\phi \in B_P(\pi)_\chi \cap H_P(\pi)_{\chi,K_0,W}} \{|E(e,\phi)|\} \cdot \prod_v \xi_{\mathcal{S}_v}(g_v).$$

Proposition 2 *Let r denote the rank of G. Given $\epsilon > 0$, there is a constant $C_{\xi,\epsilon}$, depending only on ϵ and the idempotent ξ, such that*

$$|M_\xi(g,\phi)| \leq C_{\xi,\epsilon} \sqrt{\dim H_P(\pi)_{\chi,K_0,W}} \cdot \max_{\phi \in B_P(\pi)_\chi \cap H_P(\pi)_{\chi,K_0,W}} \{|E(e,\phi)|\}$$
$$\cdot \prod_v \prod_{\theta_v \in \Delta(G_{F_v},S_v)} |\theta_v(t_v)|_v^{-l_v(\theta)/(2r)+\epsilon}.$$

First, we fix a place v of F. Each root $\theta \in \Delta(G_{F_v}, S_v)$ forms a strongly orthogonal system. By Theorem 5.9(3) of (14), for every $\epsilon > 0$, there is a constant C_ϵ such that for every $a \in S_v^+$, $\xi_{\mathcal{S}}(a) \leq C_\epsilon |\theta(a)|_v^{-1/2+r\epsilon}$. Multiplying these inequalities over all simple roots of G over E and taking the rth root gives the result.

Let φ be an automorphic form of $G(\mathbb{A})$ in the space of an automorphic representation π which is right invariant under the maximal compact subgroup K. We must bound the infinite product $\prod_v I_v(\mathbf{s},\varphi)$, where

$$I_v(\mathbf{s},\varphi) = \int_{G(F_v)} \delta_{S,D}(g_v) H_v(\mathbf{s},g_v)^{-1} \prod_{\theta_v \in \Delta(G_{F_v},S_v)} |\theta_v(t_v)|_v^{-l_v(\theta_v)/(2r)+\epsilon} \, dg_v$$

2.4.1 Local integrals.

Proposition 3 *For all $v \notin S_\infty$, the integral $I_v(\boldsymbol{s}, \varphi)$ is holomorphic for $\boldsymbol{s} \in \mathcal{T}_{-1-1/(2r)}$. Moreover, for all $\epsilon > 0$ there is a constant $C_v(\epsilon)$ such that $|I_v(\boldsymbol{s}, \varphi)| \leq C_v(\epsilon)$ for all $\boldsymbol{s} \in \mathcal{T}_{-1-1/(2r)+\epsilon}$.*

(2) For $v \in S_\infty$ and ∂ in the universal enveloping algebra the integral

$$I_{v,\partial}(\boldsymbol{s}, \varphi) = \int_{G(F_v)} \partial(H_v(\boldsymbol{s}, g_v)^{-1}) \prod_{\theta_v \in \Delta(G_{F_v}, S_v)} |\theta_v(t_v)|_v^{-l_v(\theta_v)/(2r)+\epsilon} \, dg_v$$

is holomorphic for $\boldsymbol{s} \in \mathcal{T}_{-1-1/(2r)}$. Moreover, for all $\epsilon > 0$ there is a constant $C_v(\partial, \epsilon)$ such that $|I_{v,\partial}(\boldsymbol{s}, \varphi_{\pi_v})| \leq C_v(\partial, \epsilon)$ for all $\boldsymbol{s} \in \mathcal{T}_{-1-1/(2r)+\epsilon}$.

We will only prove the first part; the second part is similar. Locally, every two local integral structures give rise to essentially equivalent height functions; so, we replace the local integral structure so that the resulting height function is invariant under K_v, a good maximal compact subgroup. Let $\underline{\sigma}$ be the vector consisting of the real parts of the components of $\mathbf{s}$. The local height integral is majorized by

$$\prod_{\theta_v \in \Delta(G_{F_v}, S_v)} \sum_{l=0}^{\infty} \delta_{B_v}(\theta_v^\vee(\varpi_v^l)) H(\underline{\sigma}, \theta_v^\vee(\varpi_v^l))^{-1} q_v^{-(1/(2r)-\epsilon)l}$$

$$\prod_{\theta_v \in \Delta(G_{F_v}, S_v)} \sum_{l=0}^{\infty} q_v^{-(\sigma_{\theta_v} - \kappa_{\theta_v} + 1/(2r) - \epsilon) l l_v(\theta_v)}.$$

2.4.2 Infinite product.

Proposition 4 *The infinite product* $I_{S,D}(\boldsymbol{s},\varphi) = \prod_{v\notin S_F\cup S_D} I_v(\boldsymbol{s},\varphi)$ *is holomorphic for* $\boldsymbol{s} \in \mathcal{T}^D_{-1/(2r)}$*. Moreover, for all* $\epsilon > 0$ *and all compact subsets* $\underline{K} \subset \mathcal{T}^D_{-1/(2r)+\epsilon}$ *there exists a constant* $C(\epsilon,\underline{K})$*, independent of* π*, such that for all* $\boldsymbol{s} \in \underline{K}$*,* $|I_{S,D}(\boldsymbol{s},\varphi)| \leq C(\epsilon,\underline{K})$*.*

For each vector $\mathbf{a} = (a_\alpha)_\alpha \in \mathcal{T}(\mathbb{N})$, we set $t_v(\mathbf{a}) = \prod_{\theta_v\in\Delta(G_{F_v},S_v)} \theta_v^\vee(\varpi_v)^{a_{\theta_v}}$. Let $\epsilon > 0$. $I_{S,D}(\mathbf{s},\varphi)$ is bounded by

$$\prod_{v\notin S_D\cup S_F} \sum_{\mathbf{a}} \delta_v(t_v(\mathbf{a}))\delta_{B_v}(t_v(\mathbf{a})) \prod_{\theta_v\in\Delta(G_{F_v},S_v)} (|\theta_v(t_v(\mathbf{a}))|_v^{-l_v(\theta)/2r+\epsilon-l_v(\theta)s_{\theta_v}}).$$

Therefore, to establish the convergence of the Euler product over places $v \notin S_D \cup S_F$ it suffices to bound

$$\sum_{v\notin S_D\cup S_F} \sum_{(a_\alpha)\in\mathbb{N}^{\mathcal{A}\setminus\mathcal{A}_D}} q_v^{-\sum_{\theta\notin\mathcal{A}_{D_v}} a_{\theta_v}[(s_{\theta_v}-\kappa_{\theta_v}+1/(2r))l_v(\theta_v)-\epsilon]}.$$

Corollary 4.1 $I_{S,D}(\boldsymbol{s},\varphi)$ *has an analytic continuation to a function which is holomorphic on* $\mathcal{T}^D_{-1/2}\cap\mathcal{T}_{-1-1/(2r)}$*. Suppose* φ *is an eigenfunction for* Δ*. Define* $\Lambda(\varphi)$ *by* $\Delta\varphi = \Lambda(\varphi)\varphi$*. Then for each integer* $k > 0$*, all* $\epsilon > 0$*, and every compact subset* $\underline{K} \subset \mathcal{T}^D_{-1/(2r)+\epsilon}\cap\mathcal{T}_{-1-1/(2r)+\epsilon}$*, there exists a constant* $C = C(\epsilon,\underline{K},k)$*, independent of* ϕ*, such that for all* $\boldsymbol{s} \in \underline{K}$*,*

$$|I_{S,D}(\boldsymbol{s},\varphi)| \leq C\Lambda(\varphi)^{-k}|\varphi(e)|. \tag{2.4}$$

The following proposition shows that infinite-dimensional representations will never contribute to the right-most pole of the height zeta function.

Proposition 5 *The function $S^\flat$ admits an analytic continuation to a function which is holomorphic on $\mathcal{T}^D_{-1/2r} \cap \mathcal{T}_{-1-1/2r}$, where r is the rank of G.*

We need to show the convergence of

$\sum^\flat_{\chi\in\mathfrak{X}} \sum_P n(A_P)^{-1} \sum_{W\in\hat{K}_\infty} \int_{\Pi(M_P)} (\sum_{\phi\in B_P(\pi)_\chi \cap H_P(\pi)_{\chi,K_0,W}} \Lambda(\phi)^{-r} |E(e,\phi)| \sqrt{\dim H_P(\pi)_{\chi,K_0,W}}$

$\times \max_{\phi\in B_P(\pi)_\chi\cap H_P(\pi)_{\chi,K_0,W}} \{|E(e,\phi)|\})\ d\pi$

for r large. The proof of this is in the proof of Theorem 4.10 in (13)

2.4.3 The leading pole

We have shown that

$$Z_{S,D}(s\lambda) = \sum_{\chi\in\mathcal{X}(G)} \int_{G(\mathbb{A})} \delta_{S,D}(g) H(s\lambda, g)^{-1} \chi(g)\ dg + f(s)$$

with f holomorphic for $\mathrm{Re}(s) > a(\lambda) - \delta$, for some $\delta > 0$. For $\chi \in \mathcal{X}(G)$ the integral $\int_{G(\mathbb{A})} \delta_{S,D}(g) H_S(s\lambda, g)^{-1}\chi(g)\ dg$ admits a regularization of the shape

$$\prod_{\alpha\in\mathcal{A}(\lambda)-\mathcal{A}_D(\lambda)} L_S(s\lambda_\alpha - \kappa_\alpha, \xi_\alpha(\chi)) \cdot h_\chi(s) \cdot \prod_{v\in S_D\setminus S_D\cap S_F} \prod_{\alpha\in\mathcal{A}_D(\lambda)} L_v(s\lambda_\alpha - \kappa_\alpha, \xi_{\alpha,v}(\chi_v)) \cdot h_{\chi,v}(s),$$

with h_χ and $h_{\chi,v}$ holomorphic for $\mathrm{Re}(s) > a(\lambda) - \delta$, for some $\delta > 0$. It follows that only $\chi \in \mathcal{X}_{S,D,\lambda}(G)$ contribute to the leading term at $s = a(\lambda)$. We can rewrite this contribution as

$$|\mathcal{X}_{S,D,\lambda}(G)| \int_{G(\mathbb{A})^{\mathrm{Ker}_\lambda}} \delta_{S,D}(g) H(s\lambda, g)^{-1}\ dg,$$

where $G(\mathbb{A})^{\mathrm{Ker}_\lambda} = \bigcap_{\chi \in \mathcal{X}_{S,D,\lambda}(G)} \mathrm{Ker}(\chi)$ is the intersection of the kernels of automorphic characters, and where $\chi_{S,D,\lambda}(G)$ denotes the finite set of automorphic characters that contribute to the main pole. Following the proof of Theorem 6.4 in (15), we conclude the following theorem:

Theorem 5 *The number of (S, D)-integral points of bounded height on X with respect to λ is asymptotic to*

$$cB^{a(\lambda)} \log(B)^{b(\lambda)-1}(1 + o(1)), \qquad B \to \infty.$$

In the case of the log-anticanonical line bundle, recall that $D = \bigcup_{\alpha \in \mathcal{A}_D} D_\alpha$. Also, $-K_X = \sum_\alpha (\kappa_\alpha + 1) D_\alpha$. Then $\max_{\alpha \in \mathcal{A}_D} \dfrac{\kappa_\alpha}{\lambda_\alpha} = \max_{\alpha \notin \mathcal{A}_D} \dfrac{\kappa_\alpha + 1}{\lambda_\alpha} = 1$. Thus the right-most pole for $-K_X - D$ is at $s = 1$. In this case, each place $v \in S$ contributes, to the pole at $s = 1$, a pole of order $1 + \dim \mathcal{C}^{an}_{F_v}(D)$. The order of the pole at $s = 1$ is

$$b = \mathrm{rank}(\mathrm{Pic}(X \backslash D)) + \textstyle\sum_{v \in S_D} (1 + \dim \mathcal{C}^{an}_{F_v}(D)).$$

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VITA

NAME: Dylon Chow

EDUCATION: B.A., Mathematics, Swarthmore College, Swarthmore, PA, 2012

Ph.D., Mathematics, UIC, Chicago, IL, finishing July 2019

TEACHING: Summer Enrichment Workshop, Math 165, 170, 180, 181, 210, 220.

PUBLICATIONS: The Distribution of Integral Points on the Wonderful Compactification by Height, preprint.