Rankin-Selberg L-functions for the Unitary Similitude Group of Order Two

by

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B.A., Reed College, 2013

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2019

Chicago, Illinois

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ACKNOWLEDGMENT

First and foremost, I would like to thank my advisor Ramin Takloo-Bighash for his patient mentorship throughout my graduate studies. I am especially grateful for his crucial assistance and advice during all the little problems and roadblocks that appeared during the completion of this thesis.

I would also like to thank my brother, Dejan, and my mother, Zoja, for their unyielding encouragement, understanding and support.

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SUMMARY

Following work by Jacquet on the group GL_2, we construct automorphic L-functions for pairs of representations (π_1, π_2) of the quasi-split unitary similitude group of order two, $\mathrm{GU}_{1,1}$, given by a global zeta integral involving an Eisenstein series.

We show that this integral admits an Euler product expansion, and compute the values of the local zeta integrals when all of the data is unramified at both the split and inert non-Archimedean primes. We take a closer look at the theory at the inert non-Archimedean places, where we classify the irreducible admissible representations, prove the existence of a local functional equation and define the local Langlands factors.

Finally, we show that the global L-functions can be continued as meromorphic functions in the whole complex plane and, assuming the existence of local functional equations at the Archimedean and split non-Archimedean places, satisfy a functional equation of the form conjectured by Langlands.

CHAPTER 1

INTRODUCTION

Robert Langlands revolutionized the study of modular forms in the 1960s by introducing his new approach based on the theory of automorphic L-functions. These functions encode the properties of irreducible cuspidal automorphic representations of an algebraic group, and are expected to satisfy a variety of deep number-theoretical conjectures.

More precisely, suppose that we are given a global field F, and let $\mathbf{A}$ be the ring of adeles of F. Let G be a reductive algebraic group defined over F, π an irreducible cuspidal automorphic representation of $G(\mathbf{A})$ and r a finite-dimensional complex representation of the Langlands dual group ${}^LG$. It follows from a theorem by Flath that $\pi \simeq \bigotimes_v \pi_v$ where the tensor product is taken over the places of F, and π_v is an irreducible admissible representation of $G(F_v)$. Moreover, with the exception of a finite set of places S, including the places at infinity, we know that π_v must be unramified. Thus to each π_v with $v \notin S$ we can, by Satake's isomorphism, canonically attach a semisimple conjugacy class t_v in ${}^LG$. With this data we can define an Euler factor at v by

$$L_v(s, \pi_v, r) = \frac{1}{\det(I - r(t_v)q_v^{-s})},$$

where q_v is the cardinality of the residue field of F_v.

Langlands's most basic conjecture is that we can extend this construction to a function

$$L(s, \pi, r) = \prod_{v} L_v(s, \pi_v, r),$$

where the product is now over all places of F, and such that the resulting function is meromorphic in the complex plane with only a finite number of poles, and such that it satisfies a functional equation of the form

$$L(s, \pi, r) = \epsilon(s, \pi, r) L(1 - s, \pi, \tilde{r}), \tag{1.1}$$

where $\tilde{r}$ is the contragredient representation of r and ϵ is a monomial function of s.

In its full generality, this conjecture is still open, although it has been verified for many particular choices of G and r. Most of the results have been obtained by using one of two main methods: the method of Langlands-Shahidi, which develops the theory of local coefficients and connects them to the global theory via Eisenstein series; and the Rankin-Selberg method, in which L-functions are constructed from an adelic integral admitting an Euler product expansion, where the local integrals give the expected corresponding local Euler factors.

In 1972, Jacquet used the Rankin-Selberg method to construct L-functions for a *pair* (π_1, π_2) of irreducible cuspidal automorphic representations of GL_2. Working with pairs of representations is consistent with the theory outlined above, since we can think of (π_1, π_2) as the representation $\pi_1 \otimes \pi_2$ of $GL_2 \times GL_2$. Following work of Rankin and Selberg from the 30s and

40s on the convolution of two classical modular forms, Jacquet considered a zeta integral of the form

$$\int_{Z(\mathbf{A})\,\mathrm{GL}_2(F)\backslash\,\mathrm{GL}_2(\mathbf{A})} \varphi_1(g)\varphi_2(g)E(s,g)\,dg,$$

where φ_1 and φ_2 are cusp forms of π_1 and π_2 respectively and $E(s,g)$ is a carefully constructed Eisenstein series. He then showed how this integral can be used to construct L-functions $L(s,\pi,r)$ that are absolutely convergent for $\mathrm{Re}(s) > s_0$, can be analytically continued as meromorphic functions of s in the whole complex plane and such that they satisfy Equation 1.1. Since we are working with pairs of representations, Satake's isomorphism attaches to the pair of unramified representations $(\pi_{1,\nu},\pi_{2,\nu})$ a pair of semisimple conjugacy classes $(t_{1,\nu},t_{2,\nu})$. The L-group of interest is then $\mathrm{GL}_2\times\mathrm{GL}_2$, and the finite dimensional complex representation to which Jacquet's L-functions correspond is $r:\mathrm{GL}_2\times\mathrm{GL}_2\to\mathrm{GL}_4$, where $(g_1,g_2)\mapsto g_1\otimes g_2$.

Following Jacquet, our main goal in this paper is to construct analogous automorphic L-functions for pair of representations (π_1,π_2) of the quasi-split unitary similitude group of order two, $\mathrm{GU}_{1,1}$, using the Rankin-Selberg method, and show that these L-functions satisfy the fundamental conjectures from above. The group $\mathrm{GU}_{1,1}$ provides an interesting example since it is closely related to GL_2 (it is a K-rational form of $\mathrm{GL}_2\times\mathrm{GL}_1$), but unlike GL_2, it is not split, but merely quasi-split.

Following the exposition of the "L-function machine" in (Gelbart and Shahidi, 1988), we note that obtaining our result requires us to go through the following steps:

1. Define a global zeta integral and show that it has an Euler product expansion and functional equation.
2. Analyze the meromorphic behavior of the global zeta integral and its functional equation.
3. Prove the existence of a functional equation and meromorphic continuation for the local zeta integrals.
4. Equate the local zeta integrals with the Langlands factors from above at the unramified places.
5. Establish at all places properties of the local zeta integrals like the existence of a common denominator and define Langlands factors $L_v(s, \pi_v, r)$.

In the remainder of Chapter 1 we discuss the unitary similitude group of order two and establish some general results about its Eisenstein series. In Chapter 2 we will cover the global theory and show that our zeta integral admits an Euler product expansion, satisfies a functional equation and admits a meromorphic continuation to the entire complex plane. Finally, in Chapter 3 we will deal with the local non-Archimedean theory by classifying the irreducible admissible representations of $GU_{1,1}$ over a non-Archimedean local field, matching the local zeta integrals at the unramified places to the Euler factors described before and proving the existence of a local functional equation at non-Archimedean inert primes. We will not deal with the existence of the local functional equations and the choice of the local Euler factors at the other primes, noting only that at the split non-Archimedean primes the problem reduces to the more familiar setting of the split group $GL_2 \times GL_1$, whereas at the Archimedean primes the theory is expected to get computationally thornier.

1.1 The Unitary Similitude Group of Order Two

Let F be a field and E a quadratic extension of F. For any F-algebra R, we define the R-rational points of the unitary similitude group of order 2 to be

$$\mathrm{GU}_{1,1}(\mathsf{R}) = \left\{ g \in \mathrm{GL}_2(\mathsf{R} \otimes_{\mathsf{F}} \mathsf{E}) \;\middle|\; {}^{\mathsf{T}}g^{\sigma} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g = \rho(g) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \text{ for } \rho(g) \in \mathsf{R}^{\times} \right\},$$

where σ is the generator of $\mathrm{Gal}(\mathsf{E}/\mathsf{F})$. The map $\rho : \mathrm{GU}_{1,1}(\mathsf{R}) \to \mathsf{R}^{\times}$ is a multiplicative character, which we will call the similitude character of $\mathrm{GU}_{1,1}(\mathsf{R})$.

The center of $\mathrm{GU}_{1,1}(\mathsf{F})$, which we will denote by $\mathsf{Z}(\mathsf{F})$, is given by

$$\mathsf{Z}(\mathsf{F}) = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \;\middle|\; a \in \mathsf{E}^{\times} \right\}.$$

We fix our choice of a maximal torus in $\mathrm{GU}_{1,1}(\mathsf{F})$,

$$\mathsf{T}(\mathsf{F}) = \left\{ \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix} \;\middle|\; a \in \mathsf{E}^{\times}, \lambda \in \mathsf{F}^{\times} \right\},$$

and a Borel subgroup containing it

$$\mathsf{B}(\mathsf{F}) = \left\{ \begin{pmatrix} a & x \\ & \lambda a^{-\sigma} \end{pmatrix} \;\middle|\; x \in \mathsf{F}, a \in \mathsf{E}^{\times}, \lambda \in \mathsf{F}^{\times} \right\}.$$

The unipotent radical of $\mathrm{B}(\mathsf{F})$ is

$$\mathsf{N}(\mathsf{F}) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \middle| \, x \in \mathsf{F} \right\}.$$

Note that while $\mathrm{GU}_{1,1}$ is not split, it is quasi-split.

We will make use throughout of the matrices

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}.$$

The Weyl group of $\mathrm{GU}_{1,1}$ is of order two and is generated by the matrix w.

Let $\chi : \mathsf{E}^\times \to \mathbf{C}^\times$ and $\eta : \mathsf{F}^\times \to \mathbf{C}^\times$ be characters[1]. We define a character $\chi \otimes \eta$ of the torus T by

$$(\chi \otimes \eta) \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix} = \chi(a)\eta(\lambda).$$

There is an action of the Weyl group on the group of characters of $\mathrm{GU}_{1,1}$ given by

$$(\chi \otimes \eta)^w(t) = (\chi \otimes \eta)(wtw^{-1}) = (\chi^{-\sigma} \otimes \chi\eta)(t).$$

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[1] Throughout we will call a character any group homomorphism to the multiplicative group of a field (i.e. what are sometimes called quasi-characters). We make no assumption about whether characters are unitary unless explicitly specified.

If R is an E-algebra, we can identify $R \otimes_F E$ with $R \oplus R$ via the map $r \otimes a \mapsto (ra, r\sigma(a))$. Note that then $\mathrm{GL}_2(R \otimes_F E) \simeq \mathrm{GL}_2(R \oplus R) \simeq \mathrm{GL}_2(R) \times \mathrm{GL}_2(R)$, with $(g_1, g_2)^\sigma = (g_2, g_1)$, from which it follows that $g = (g_1, g_2) \in \mathrm{GU}_{1,1}(R)$ if and only if

$$ {}^{T}g_2 \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g_1 = \rho(g) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \text{and} \quad {}^{T}g_1 \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} g_2 = \rho(g) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, $$

or,

$$ g_2 = \rho(g) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}^{-1} {}^{T}g_1^{-1} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \rho(g) \det{}^{-1}(g_1) g_1. $$

Thus, for R an E-algebra, $\mathrm{GU}_{1,1}(R) \simeq \mathrm{GL}_2(R) \times \mathrm{GL}_1(R)$. Hence the identity component of the L-group, ${}^{L}G^\circ$, which only depends on the structure of the group over a separable closure, is $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_1(\mathbf{C})$. The full L-group, ${}^{L}G$, is then $(\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_1(\mathbf{C})) \rtimes \langle \sigma \rangle$ where the action of σ on ${}^{L}G^\circ$ is given by $(g, \lambda) \mapsto (w^{-1}({}^{T}g^{-1})w, \lambda \det(g))$.

Suppose r is an N-dimensional complex representation of ${}^{L}G$. Then, by the action of σ on $\mathrm{GL}_2 \times \mathrm{GL}_1$ from above, it follows that

$$ r\left((g, \lambda), \sigma\right) = r\Big(((e, 1), \sigma)((w({}^{T}g^{-1})w^{-1}, \lambda \det g), e)\Big). $$

Let $r((e,1),\sigma) = J$, and let ρ and χ be the restriction of r to the GL_2 and GL_1 component of ${}^L G^\circ$ respectively. Then

$$\chi(\lambda)\rho(g)J = J\chi(\lambda \det g)\rho(w({}^T g^{-1})w^{-1}),$$

or,

$$\rho(g) = \chi(\det g)J\rho(w({}^T g^{-1})w^{-1})J^{-1} = (\chi\omega_\rho^{-1})(\det g)J\rho(g)J^{-1},$$

where ω_ρ is the central character of ρ. Taking determinants on both sides, it follows that $(\chi\omega_\rho^{-1})^N \equiv 1$, or, since there are no finite order algebraic characters of $\mathbf{C}^\times$, that $\chi = \omega_\rho$.

Thus we have

$$\rho(g) = J\rho(g)J^{-1}$$

and so it follows that to determine a representation of ${}^L G$, it suffices to specify a representation $\rho : \mathrm{GL}_2(\mathbf{C}) \to \mathrm{GL}_N(\mathbf{C})$, and pick a $J \in \mathrm{GL}_N(\mathbf{C})$ such that $J^2 = e$ and J is in the centralizer of the image of $\mathrm{GL}_2(\mathbf{C})$ under ρ. In particular, the representation of ${}^L G$ that our automorphic L-function will correspond to, is the one where we take ρ to be the contragredient of the standard representation, $\rho(g) = {}^T g^{-1}$, and $J = e$.

We will denote by dg a Haar measure on G. While the group G is unimodular, B is not, and the value of its modular character is given by

$$\delta_B \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix} = \left| \frac{N(a)}{\lambda} \right|,$$

since

$$\begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \lambda^{-1} a a^{\sigma} x \\ & 1 \end{pmatrix}.$$

1.2 Whittaker Models

Let π be an automorphic cuspidal representation of $\mathrm{GU}_{1,1}(\mathbf{A})$ and ψ a nontrivial character of $\mathbf{A}/F$. By a Whittaker model $\mathcal{W}(\pi, \psi)$ we mean a nontrivial space of smooth K-finite functions W on $\mathrm{GU}_{1,1}(\mathbf{A})$ satisfying

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x) W(g).$$

If W is an element of some Whittaker model, we say W is a Whittaker function.

Proposition 1.1 (Existence of Global Whittaker Models). Let π be an automorphic cuspidal representation of $\mathrm{GU}_{1,1}(\mathbf{A})$ with central character ω. For any $\varphi \in \pi$, let

$$W_{\varphi}(g) = \int_{\mathbf{A}/F} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx.$$

Then the space $\{W_{\varphi} \mid \varphi \in \pi\}$ is a Whittaker model for π, and φ admits the Fourier expansion

$$\varphi(g) = \sum_{a \in F^{\times}} W_{\varphi}\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right).$$

Proof. The function

$$F(x) = \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$$

is continuous and satisfies $F(x+a) = F(x)$ for any $a \in F$. Hence we can think of it as a function on the compact group $\mathbf{A}/F$, which admits a Fourier expansion of the form

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{a \in F} C_a \psi(ax), \tag{1.2}$$

where

$$C_a = \int_{\mathbf{A}/F} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-ax)\, dx.$$

Since π is cuspidal, it follows that $C_0 = 0$. For $a \neq 0$,

$$\begin{aligned} C_a &= \int_{\mathbf{A}/F} \varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-ax)\, dx \\ &= \int_{\mathbf{A}/F} \varphi\left(\begin{pmatrix} 1 & ax \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) \psi(-ax)\, dx \\ &= \int_{\mathbf{A}/F} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx. \qquad (x \mapsto a^{-1}x) \end{aligned}$$

The result follows by substituting this expression for C_a in Equation 1.2 and setting $x = 0$. $\square$

Suppose ψ_v is a nontrivial unitary additive character of a local field F_v. We can define a local Whittaker model $\mathcal{W}_v(\pi_v, \psi_v)$ as a space of smooth K-finite functions W_v on $\mathrm{GU}_{1,1}(F_v)$ satisfying

$$W_v\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi_v(x) W_v(g).$$

Note that for any $\varphi \in \pi_v$,

$$W_{\varphi,v}(g) = \lim_{\substack{N' \subseteq N(F_v) \\ N' \text{ open, compact}}} \int_{N'} \varphi\left(wn'g\right) \psi_v^{-1}(n')\, dn'$$

is clearly in $W_v(\pi_v, \psi_v)$, and so to show the existence of local Whittaker models it suffices to show that at least one of these is not identically zero. However, we will later show that if $g \in \mathrm{GL}_2(F_v)$, $W_{\varphi,v}(g)$ is given by the value of a certain Whittaker function of $\mathrm{GL}_2(F_v)$, and so the existence of local Whittaker models for $\mathrm{GU}_{1,1}(F_v)$ follows from the existence of local Whittaker models for $\mathrm{GL}_2(F_v)$.

Both of the following propositions follow directly from (Shalika, 1974).

Proposition 1.2 (Uniqueness of Local Whittaker Models). There exists at most one local Whittaker model for π_v.

Proposition 1.3 (Uniqueness of Global Whittaker Models). There exists at most one global Whittaker model for π.

In particular, if $\varphi \in \pi$ is a pure tensor, $W_\varphi(g) = \prod_v W_{\varphi,v}(g_v)$.

1.3 Eisenstein Series

In order to be able to define Eisenstein series, we first need a way to vary characters continuously as a function of a complex parameter. We begin with χ, a unitary character of $\mathbf{A}_E^\times/E^\times$ and η, a unitary character of $\mathbf{A}^\times/F^\times$.

Consider the induced representation $\mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})}(\chi \otimes \eta)$ consisting of all smooth, K-finite functions f on $G(\mathbf{A})$ such that

$$f\left(\begin{pmatrix} a & x \\ & \lambda a^{-\sigma} \end{pmatrix} g\right) = \left(\frac{|N(a)|}{|\lambda|}\right)^{1/2} \chi(a)\eta(\lambda)f(g).$$

Given any such f, there is a natural assignment $s \mapsto f_s$ where $f_s \in \mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})}(\chi|N(\cdot)|^{s-1/2} \otimes \eta|\cdot|^{1/2-s})$, given by

$$f_s\left(\begin{pmatrix} a & x \\ \lambda a^{-\sigma} & \end{pmatrix} k\right) = \left(\frac{|N(a)|}{|\lambda|}\right)^{s} \chi(a)\eta(\lambda)f(k).$$

Note that by Iwasawa decomposition, for any $g \in G(\mathbf{A})$ we have that $g = bk$, where b is an element of the Borel subgroup $B(\mathbf{A})$ and $k \in K$. Moreover, f is completely determined by its value on K and therefore this correspondence is actually a bijection.

Let $f \in \mathrm{Ind}_{B(\mathbf{A})}^{G(\mathbf{A})}(\chi \otimes \eta)$. We define the Eisenstein series

$$E(s, f, g) = \sum_{\gamma \in B(F)\backslash G(F)} f(s, \gamma g),$$

where $f(s,g) = f_s(g)$. This is well-defined because $f(b_0 g) = f(g)$ for $b_0 \in B(F)$, since χ and η are trivial on $E^\times$ and $F^\times$ respectively. This summation is not guaranteed to converge, but it is a well-known fact from the general theory of Eisenstein series (Langlands, 1976) that there exists an $s_0 \in \mathbf{R}$ such that the above summation converges whenever $\mathrm{Re}(s) > s_0$.

Let $a \in F$. We will consider the integral

$$C_a(g) = \int_{\mathbf{A}/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-ax)\, dx,$$

and note that by an argument identical to the one in the proof of Proposition 1.1,

$$E\,(g) = \sum_{a \in F} c_a(g)\psi(ax).$$

The group $G(F)$ admits a Bruhat decomposition $G(F) = B(F) \sqcup B(F)wN(F)$, and therefore we may split up the summation defining the Eisenstein series into

$$E(s,f,g) = f(s,g) + \sum_{n \in N(F)} f(s,wng),$$

where the first term corresponds to $B(F)\backslash B(F)$ and the second to a summation over a set of representatives for $B(F)\backslash B(F)wN(F) \simeq wN(F)$.

Unless $a = 0$, the contribution of the first term is zero, and combining the definition of C_a and the previous display, we get

$$C_a(g) = \int_{\mathbf{A}} f\left(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-ax)\, dx.$$

If $a = 0$, the first term cannot be ignored and we instead get

$$C_0(g) = f(g) + \int_{\mathbf{A}} f\left(w \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx,$$

where we will later show that the integral of the second term is actually an intertwining operator.

Our next goal is to establish some properties of Eisenstein series, assuming that the sections f are carefully normalized. In particular, we assume that f is a pure tensor $f = \prod_v f_v$ such that for almost all v, $f_v = L_v(2s, \chi_v) f_v^\circ$, where f_v° is the unique $K(F_v)$-fixed vector that is identically equal to 1 over $K(F_v)$, and such that at the remaining places v, one of the following is true:

1. f_v restricted to $K(F_v)$ is independent of s, or
2. f_v is of the form $M_v^*(1-s)(h_v)$, where $h_v \in \mathrm{Ind}_{B(F_v)}^{G(F_v)}(\chi_v^{-\sigma}|N(\cdot)|^{-s-1/2} \otimes \chi_v \eta_v |\cdot|^{s+1/2})$, $h_v|_{K(F_v)}$ is independent of s, and $M_v^*(1-s)$ is the intertwining operator, suitably normalized as to have $M_v^*(s)M_v^*(1-s)$ be the identity.

Proposition 1.4. Let $f = \prod_v f_v$ as above. Then $M(s)f$, the image of f under the standard intertwining operator, is meromorphic with only finitely many poles in $\mathbf{C}$, namely those of $L(2s-1, \chi)$, the Hecke L-function of the restriction of χ to $\mathbf{A}^\times / F^\times$.

Proof. Let S be the finite set of places v where $f_v \neq L_v(2s, \chi_v) f_v^\circ$. Consider the global intertwining operator

$$M(s)f(g) = \int_{N(\mathbf{A})} f(wng)\, dn$$

and note that since f is a pure tensor, the expression above can be written as a product of local intertwining operators

$$M(s)f(g) = \prod_v \int_{N(F_v)} f_v(wng_v)\, dn = \prod_v M(s)f_v(g).$$

By the analysis of the local intertwining operators in Chapter Three, it follows that

$$\begin{aligned} M(s)f &= \left(\prod_{v\in S} M(s)f_v\right)\left(\prod_{v\notin S} M(s)f_v\right) \\ &= \left(\prod_{v\in S} M(s)f_v\right)\left(\prod_{v\notin S} L_v(2s-1,\chi_v)\tilde{f}_v^\circ\right) \\ &= L(2s-1,\chi)\left(\prod_{v\in S}\frac{M(s)f_v}{L_v(2s-1,\chi_v)}\right)\left(\prod_{v\notin S}\tilde{f}_v^\circ\right), \end{aligned}$$

and therefore it suffices to show that

$$\frac{M(s)f_v}{L_v(2s-1,\chi_v)},$$

where $v \in S$, is entire.

Since

$$M(s)f_v = \int_{F_v} f_v\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx$$
$$= \int_{|x|\leq 1} f_v\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx + \int_{|x|>1} f_v\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx,$$

where the first integral converges absolutely for all s, it follows that the analytic behavior of $M(s)f_v$ is fully determined by the second integral. Moreover, recall that $g = bk$, where we can factor out the action of b out of the integral, and therefore the meromorphicity of $M(s)f_v$ can be evaluated at $g = e$.

Since f_v is right invariant by $\left(\begin{smallmatrix}1 & \\ -x^{-1} & 1\end{smallmatrix}\right)$ for $|x| > c$,

$$w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -x^{-1} & 1\end{pmatrix} = \begin{pmatrix}-x^{-1} & 1\\ & (-x^{-1})^{-1}\end{pmatrix},$$

and so the second integral evaluated at $g = e$ equals

$$f_v(e)\int_{|x|>1} \chi_v^{-1}(x)|x|^{-2s}\,dx.$$

Thus, if $f_v(e)$ is independent of s, it suffices to consider

$$\frac{\int_{|x|>1} \chi_v^{-1}(x)|x|^{-2s}\,dx}{L_v(2s-1,\chi_v)},$$

but the integral in the numerator has the same meromorphic behavior as $L_v(2s-1,\chi_v)$, and so the quotient is entire.

Finally, suppose that $f_v = M^*(1-s)h_v$. Then

$$\begin{aligned}\frac{M(s)f_v}{L_v(2s-1,\chi_v)} &= \frac{M^*(s)f_v}{\gamma(2s-1,\chi_v)L_v(2s-1,\chi_v)} \\ &= \frac{M^*(s)M^*(1-s)h_v}{\epsilon(2s-1,\chi_v)L_v(2-2s,\chi_v^{-1})} \\ &= \frac{h_v}{\epsilon(2s-1,\chi_v)L_v(2-2s,\chi_v^{-1})},\end{aligned}$$

where h_v is entire since its restriction to $K(F_v)$ is independent of s and $1/\epsilon(2s-1,\chi_v)L_v(2-2s,\chi_v^{-1})$ has the desired meromorphic properties. $\square$

It is well-known in the general theory of Eisenstein series that the poles of an Eisenstein series occur at the poles of its constant coefficients. In particular, it follows that an Eisenstein series normalized as above has only finitely many poles—those of $L(2s-1,\chi)$.

Proposition 1.5 (Functional Equation of Eisenstein Series). The Eisenstein series $E(s,f,g)$, normalized as above, satisfies the functional equation

$$E(s,f,g) = E(1-s, M(s)f, g).$$

Proof. Recall that the constant term of $E(s, f, g)$ equals

$$f(g) + \int_{\mathbf{A}} f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx = f(g) + L(2s-1,\chi)\left(\prod_{v\in S}\frac{M(s)f_v}{L_v(2s-1,\chi_v)}\right)\left(\prod_{v\notin S}\tilde{f}_v^\circ\right),$$

whereas that of $E(1-s, M(s)f, g)$ is

$$M(s)f(g) + \int_{\mathbf{A}} M(s)f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx.$$

Focusing on the second term, notice that

$$\begin{aligned}
\int_{\mathbf{A}} M(f)\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx &= M(f)(g) + L(\chi, 2s-1)\left(\prod_{v\in S}\frac{M(M(f_v))}{L_v(\chi_v, 2s-1)}\right)\left(\prod_{v\notin S}\frac{L(\chi_v^{-1},-2s)}{L(\chi_v^{-1},2-2s)}f_v^\circ\right)\\
&= \left(\prod_{v\in S}\gamma_v(\chi_v, 2s-1)M(M(f_v))\right)\left(\prod_{v\notin S}L(\chi_v^{-1},-2s)f_v^\circ\right)\\
&= \left(\prod_{v\in S}\gamma_v(\chi_v^{-1},-2s)\gamma_v(\chi_v,2s-1)M(M(f_v))\right)\left(\prod_{v\notin S}L(\chi_v,2s)f_v^\circ\right)\\
&= \left(\prod_{v\in S}(M^*\circ M^*)(f_v)\right)\left(\prod_{v\notin S}L(\chi_v,2s)f_v^\circ\right)\\
&= f,
\end{aligned}$$

and therefore, that $E(s, f, g)$ and $E(1-s, M(s)f, g)$ have the same constant term.

This implies that $E(s, f, g) - E(1-s, M(s)f, g)$ is a cusp form, but by a result from the general theory of Eisenstein series, the spaces of cusp forms and Eisenstein series are orthogonal to each other. Therefore, $E(s, f, g) - E(1 - s, M(s)f, g) = 0$, which was to be proven. $\square$

CHAPTER 2

GLOBAL THEORY

2.1 Notation

Let E/F be a quadratic extension of number fields, with $\mathrm{Gal}(\mathsf{E}/\mathsf{F}) = \langle \sigma \rangle$. We let $\mathbf{A}$ be the ring of adeles of F, $\mathbf{A}_\mathsf{E}$ the ring of adeles of E, and fix a nontrivial additive character ψ of $\mathbf{A}$ that is trivial on F.

2.2 The Zeta Integral

Let π_1 and π_2 be cuspidal automorphic representations of G with central characters ω_1 and ω_2 respectively, let φ_1 and φ_2 be cusp forms of π_1 and π_2, and let f be a section of $\mathrm{Ind}_{\mathsf{B}(\mathbf{A})}^{\mathsf{G}(\mathbf{A})}\left(\omega_1^{-1}\omega_2^{-1}|\mathsf{N}(\cdot)|^{s-1/2} \otimes |\cdot|^{1/2-s}\right)$ with the normalization given in the section on Eisenstein series. We can then define the *global Rankin-Selberg zeta integral*,

$$Z(s, \varphi_1, \varphi_2, f) = \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \varphi_1(g)\varphi_2(\varepsilon g)E(s, f, g)\,dg.$$

PROPOSITION 2.1 (Euler Product). Let W_1 and W_2 be the Whittaker functions of φ_1 and φ_2 respectively. Assuming φ_1, φ_2 and f are pure tensors, the global Rankin-Selberg zeta integral $Z(s, \varphi_1, \varphi_2, f)$ admits an Euler product expansion

$$Z(s, \varphi_1, \varphi_2, f) = \prod_v \left(\int_{N(F_v)Z(F_v)\backslash G(F_v)} W_{1,v}(g)W_{2,v}(\varepsilon g)f_v(s, g)\,dg \right),$$

where the product is taken over all places v of F.

Proof. For Re(s) sufficiently large, we can substitute the definition of $E(s,f,g)$ into the definition of the global zeta integral, to see that

$$\begin{aligned} Z(s,\varphi_1,\varphi_2,f) &= \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \varphi_1(g)\varphi_2(\varepsilon g)E(s,f,g)\,dg \\ &= \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \varphi_1(g)\varphi_2(\varepsilon g) \sum_{\gamma\in B(F)\backslash G(F)} f(s,\gamma g)\,dg \\ &= \int_{G(F)Z(\mathbf{A})\backslash G(\mathbf{A})} \varphi_1(g)\varphi_2(\varepsilon g) \sum_{\gamma\in Z(\mathbf{A})B(F)\backslash Z(\mathbf{A})G(F)} f(s,\gamma g)\,dg \\ &= \int_{Z(\mathbf{A})B(F)\backslash G(\mathbf{A})} \varphi_1(g)\varphi_2(\varepsilon g)f(s,g)\,dg \\ &= \int_{T(F)N(\mathbf{A})Z(\mathbf{A})\backslash G(\mathbf{A})} \int_{Z(\mathbf{A})B(F)\backslash T(F)N(\mathbf{A})Z(\mathbf{A})} \varphi_1(ng)\varphi_2(\varepsilon ng)f(s,g)\,dn\,dg \\ &= \int_{T(F)N(\mathbf{A})Z(\mathbf{A})\backslash G(\mathbf{A})} f(s,g) \int_{N(F)\backslash N(\mathbf{A})} \varphi_1(ng)\varphi_2(\varepsilon ng)\,dn\,dg, \end{aligned}$$

since $Z(\mathbf{A})B(F)\backslash T(F)N(\mathbf{A})Z(\mathbf{A}) \simeq Z(\mathbf{A})T(F)N(F)\backslash T(F)N(\mathbf{A})Z(\mathbf{A}) \simeq N(F)\backslash N(\mathbf{A})$.

Note that by Proposition 1.1, φ_1 and φ_2 admit a Fourier expansion in terms of Whittaker functions,

$$\begin{aligned} \varphi_1(ng) &= \sum_{t\in Z(F)\backslash T(F)} W_1(tng) = \sum_{t\in Z(F)\backslash T(F)} W_1(tg)\psi(tnt^{-1}) \\ \varphi_2(\varepsilon ng) &= \sum_{\tau\in Z(F)\backslash T(F)} W_2(\varepsilon\tau ng) = \sum_{\tau\in Z(F)\backslash T(F)} W_2(\varepsilon\tau g)\psi(\varepsilon\tau n\tau^{-1}\varepsilon^{-1}) \end{aligned}$$

and, therefore, since

$$\psi(tnt^{-1}\varepsilon\tau n\tau^{-1}\varepsilon^{-1}) = \psi((\alpha-\beta)x),$$

where

$$t = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} \beta & \\ & 1 \end{pmatrix}, \text{ and } n = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix},$$

it follows that

$$\begin{aligned}
&\int_{N(F)\backslash N(\mathbf{A})} \varphi_1(ng)\varphi_2(\varepsilon ng)\,dn\,dg \\
&\quad = \sum_{t\in Z(F)\backslash T(F)} \sum_{\tau\in Z(F)\backslash T(F)} W_1(tg)W_2(\varepsilon\tau g)\int_{N(F)\backslash N(\mathbf{A})} \psi(tnt^{-1}\varepsilon\tau n\tau^{-1}\varepsilon^{-1})\,dn \\
&\quad = \sum_{t\in Z(F)\backslash T(F)} W_1(tg)W_2(\varepsilon tg).
\end{aligned}$$

Hence

$$\begin{aligned}
Z(s,\varphi_1,\varphi_2,f) &= \int_{T(F)N(\mathbf{A})Z(\mathbf{A})\backslash G(\mathbf{A})} f(s,g) \sum_{t\in Z(F)\backslash T(F)} W_1(tg)W_2(\varepsilon tg)\,dg \\
&= \int_{N(\mathbf{A})Z(\mathbf{A})\backslash G(\mathbf{A})} W_1(g)W_2(\varepsilon g)f(s,g)\,dg,
\end{aligned}$$

since $Z(\mathbf{A})\backslash T(F)Z(\mathbf{A}) \simeq Z(F)\backslash T(F)$ by the second group isomorphism theorem.

Note that for φ_1 and φ_2 pure tensors, W_1 and W_2 admit an Euler product by Proposition 1.3. Therefore, if $f = \otimes_\nu f_\nu$,

$$Z(s, \varphi_1, \varphi_2, f) = \prod_\nu \left(\int_{N(F_\nu)Z(F_\nu)\backslash G(F_\nu)} W_{1,\nu}(g) W_{2,\nu}(\varepsilon g) f_\nu(s, g)\, dg \right).$$

The results follows for general s by analytic continuation. □

We will denote by $Z_\nu(s, W_{1,\nu}, W_{2,\nu}, f_\nu)$ the expression

$$\int_{N(F_\nu)Z(F_\nu)\backslash G(F_\nu)} W_{1,\nu}(g) W_{2,\nu}(\varepsilon g) f_\nu(s, g)\, dg,$$

which gives the previous result the standard form

$$Z(s, \varphi_1, \varphi_2, f_\nu) = \prod_\nu Z_\nu(s, W_{1,\nu}, W_{2,\nu}, f_\nu).$$

We will require a slightly more explicit form of the local zeta integrals for the unramified computations. To that end, we have the following lemma.

LEMMA 2.2. Let $W_{1,\nu}$, $W_{2,\nu}$ and f_ν be as above. Then

$$Z_\nu(s, W_{1,\nu}, W_{2,\nu}, f_\nu) = \int_{K(F_\nu)} \int_{T_1} W_{1,\nu}(tk) W_{2,\nu}(\varepsilon tk) f_\nu(s, tk) \delta_B^{-1}(t)\, dt\, dk$$

where $T_1 \simeq T(F_\nu)/Z(F_\nu)$.

Proof.

$$\begin{aligned}
Z_v(s, W_{1,v}, W_{2,v}, f_v) &= \int_{N(F_v)Z(F_v)\backslash G(F_v)} W_{1,v}(g)W_{2,v}(\varepsilon g)f_v(s,g)\,dg \\
&= \int_{N(F_v)Z(F_v)\backslash B(F_v)} \int_{K(F_v)} W_{1,v}(bk)W_{2,v}(\varepsilon bk)f_v(s,bk)\,d_r b\,dk \\
&= \int_{T_1(F_v)} \int_{K(F_v)} W_{1,v}(tk)W_{2,v}(\varepsilon tk)f_v(s,tk)\delta_B^{-1}(t)\,dt\,dk
\end{aligned}$$

□

2.3 The Global Functional Equation

THEOREM 2.3. The integrals $Z(s, \varphi_1, \varphi_2, f)$ and $Z(1-s, \varphi_1, \varphi_2, M(s)f)$ are absolutely convergent for large enough (resp. small enough) Re(s). They can be analytically continued as meromorphic functions of s in the whole complex plane, and as such they satisfy the functional equation

$$Z(s, \varphi_1, \varphi_2, f) = Z(1-s, \varphi_1, \varphi_2, M(s)f).$$

Proof. This is immediate from the functional equation of the Eisenstein series, the analysis of the possible poles of the Eisenstein series and the fact that cusp forms are rapidly decreasing. □

Set

$$L(s, \pi_1 \times \pi_2) = \prod_v L_v(s, \pi_{1,v} \times \pi_{2,v}),$$

where we will describe how to select the local factors L_v in a later chapter. Then there exists an $s_0 \in \mathbf{R}$ such that for all s with $\mathrm{Re}(s) > s_0$, all of the local factors are holomorphic and their product is absolutely convergent. Hence $L(s,\pi)$ is holomoprhic on the right half-space $\mathrm{Re}(s) > s_0$. The same is true for the L-function

$$L(s,\widetilde{\pi}_1 \times \widetilde{\pi}_2) = \prod_{v} L_v(s,\widetilde{\pi}_{1,v} \times \widetilde{\pi}_{2,v}).$$

The following theorem is conditional on the existence of a local functional equation at all places v.

THEOREM 2.4 (Main Theorem). Under the above assumptions, the Euler products $L(s,\pi_1 \times \pi_2)$ and $L(1-s,\widetilde{\pi}_1 \times \widetilde{\pi}_2)$ are absolutely convergent for $\mathrm{Re}(s)$ large enough. They can be analytically continued as meromorphic functions of s in the whole complex plane. As such they satisfy a functional equation

$$L(s,\pi_1 \times \pi_2) = \epsilon(s,\pi_1 \times \pi_2) L(1-s,\widetilde{\pi}_1 \times \widetilde{\pi}_2).$$

Proof. We may assume without loss of generality that for each v, we may pick $W_{1,v}$, $W_{2,v}$ and f_v such that

$$L(s,\pi_1 \times \pi_2) = \prod_{v} Z_v(s, W_{1,v}, W_{2,v}, f_v).$$

Then

$$\begin{aligned}
L(s,\pi_1\times\pi_2) &= Z(s,W_1,W_2,f)\\
&= Z(1-s,W_1,W_2,M(s)f)\\
&= \left(\prod_{v\in S} Z_v(1-s,W_{1,v},W_{2,v},M^*(s)f)\right)\left(\prod_{v\notin S} L_v(1-s,\widetilde{\pi}_1\times\widetilde{\pi}_2)\right)\\
&= \left(\prod_{v\in S}\frac{Z_v(1-s,W_{1,v},W_{2,v},M^*(s)f)}{L_v(1-s,\widetilde{\pi}_1\times\widetilde{\pi}_2)}\right) L(1-s,\widetilde{\pi}_1\times\widetilde{\pi}_2)\\
&= \left(\prod_{v\in S}\epsilon_v(s,\pi_1\times\pi_2)\right) L(1-s,\widetilde{\pi}_1\times\widetilde{\pi}_2).
\end{aligned}$$

$\square$

CHAPTER 3

NON-ARCHIMEDEAN LOCAL THEORY

3.1 Notation

For the rest of this chapter and unless specified otherwise, let E/F be a quadratic extension of local nonarchimedean fields with norm $N : E \to F$, let $\mathcal{O}_F$ be the ring of integers of F, ϖ the maximal ideal of $\mathcal{O}_F$ and ord the ϖ-adic valuation on F. We let q be the cardinality of the residue field $\mathcal{O}_F/\varpi$ and define an absolute value on F by $|x| = q^{-\operatorname{ord}(x)}$.

We will also make use of a maximal ideal ϖ_E of $\mathcal{O}_E$ with an associated ord_E, a ϖ_E-adic valuation on E, and an absolute value on E given by $|x|_E = q_E^{-\operatorname{ord}_E(x)}$, where q_E is the cardinality of $\mathcal{O}_E/\varpi_E$.

We will denote by G the group of F-rational points $\mathrm{GU}_{1,1}(F)$, B the Borel subgroup of G, and K the group $\mathrm{GU}_{1,1}(\mathcal{O}_F)$, a maximal compact subgroup. We will sometimes use $\mathbf{H}_{\chi,\eta}(s)$ as shorthand for the induced representation $\mathrm{Ind}_B^G(\chi|N(\cdot)|^{s-1/2} \otimes \eta|\cdot|^{1/2-s})$.

3.2 Intertwining Operators

The existence of a nonzero local G-intertwining operator $M : \mathrm{Ind}_B^G(\chi_1 \otimes \eta_1) \to \mathrm{Ind}_B^G(\chi_2 \otimes \eta_2)$ is equivalent to the existence of a nonzero linear functional $\Lambda : \mathrm{Ind}_B^G(\chi_1 \otimes \eta_1) \to \mathbf{C}$ such that

$$\Lambda\left(\begin{pmatrix} a & x \\ & \lambda a^{-\sigma} \end{pmatrix} \cdot \phi\right) = \delta_B^{1/2}\begin{pmatrix} a & x \\ & \lambda a^{-\sigma} \end{pmatrix} \chi_2(a)\eta_2(\lambda)\Lambda(\phi),$$

since given such a Λ we can define $M(\varphi)(g) = \Lambda(g \cdot \phi)$.

Our candidate for such a functional is $\Lambda(\varphi) = \int_N f(wn)\,dn$. When defined, it is clear that this integral satisfies the condition in the last display. Our next goal is to get an analytic continuation of this expression.

In order to get an analytic continuation, we consider $\varphi \in \mathrm{Ind}_B^G(\chi|N(\cdot)|^s \otimes \eta|\cdot|^{-s})$. Note that

$$\begin{aligned}
\Lambda(\varphi) &= \int_N \varphi(wn)\,dn \\
&= \int_F \varphi\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\,dx \\
&= \int_{|x|\leq 1} \varphi\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\,dx + \int_{|x|>1} \varphi\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\,dx,
\end{aligned}$$

where the first integral is over a compact domain, and thus it is only the second integral that can fail to converge. However,

$$\begin{aligned}
\int_{|x|>1} \varphi\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)\,dx &= \int_{|x|>1} \varphi\left(\begin{pmatrix} -x^{-1} & 1 \\ & (-x) \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}\right)\,dx \\
&= \int_{|x|>1} \chi(-1)\chi^{-1}(x)|x|^{-2s-1}\varphi\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}\,dx \\
&= \int_{|x|\leq 1} \chi(-1)\chi(x)|x|^{2s}\varphi\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\,d^\times x \\
&= \chi(-1)Z\left(2s, 1_{\varpi\mathcal{O}_F^\times}(x)\varphi\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \chi\right),
\end{aligned}$$

where Z is a local zeta integral $Z(s,\Phi,\tau)$ of the type studied by Tate in (Tate, 1967), defined by

$$Z(s,\Phi,\tau)=\int_{F^\times}\Phi(x)\tau(x)|x|^s\,d^\times x$$

for $\Phi\in\mathcal{S}(F)$, a character $\tau: F^\times\to\mathbf{C}$ and $s\in\mathbf{C}$. Thus the existence of an analytic continuation for Λ follows from (Tate, 1967).

We can explicitly compute the value of the intertwining operator at the unramified vectors. To this end we have the two following results:

Proposition 3.1. Let χ and η be unramified characters of $E^\times$ and $F^\times$ respectively. Suppose φ° is the unique K-fixed vector of $\mathrm{Ind}_B^G(\chi|N(\cdot)|^{s-1/2}\otimes\eta|\cdot|^{t+1/2})$ normalized so that $\varphi^\circ(e)=e$. Then

$$M(\varphi)=\frac{L(2s-1,\chi)}{L(2s,\chi)}\tilde{\varphi}^\circ,$$

where $\tilde{\varphi}^\circ$ is the unique K-fixed vector of $\mathrm{Ind}_B^G(\chi^{-\sigma}|N(\cdot)|^{-s+1/2}\otimes\chi\eta|\cdot|^{2s+t-1/2})$ such that $\tilde{\varphi}^\circ(e)=e$.

Proof. It is clear that M takes K-fixed vectors to K-fixed vectors, so it suffices to evaluate $M(\varphi^\circ)(e)=\Lambda(\varphi^\circ)$.

Note that if $n\in N\cap K$, $wn\in K$ and so $\varphi^\circ(wn)=1$. Otherwise, if $n\notin K$,

$$n=\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\text{ for }x\notin\mathcal{O}_F\text{, and so }\begin{pmatrix}1 & \\ -x^{-1} & 1\end{pmatrix}\in K.$$

Thus

$$\varphi^\circ(wn) = \varphi^\circ\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -x^{-1} & 1\end{pmatrix}\right) = \varphi^\circ\begin{pmatrix}-x^{-1} & 1\\ & (-x^{-1})^{-1}\end{pmatrix} = \chi^{-1}(x)|x|^{-2s} \text{ for } x \notin \mathcal{O}_F.$$

We have

$$\begin{aligned}
\int_F \varphi^\circ\left(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\right)\,dx &= \int_{|x|\le 1}\varphi^\circ\left(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\right)\,dx + \int_{|x|>1}\varphi^\circ\left(w\left(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right)\right)\,dx\\
&= 1 + \int_{|x|>1}\chi^{-1}(x)|x|^{-2s}\,dx\\
&= 1 + \sum_{n=1}^{\infty}\frac{q-1}{q}q^n\chi(\varpi^n)(q^{-n})^{2s}\\
&= 1 + \frac{q-1}{q}\frac{\chi(\varpi)q^{1-2s}}{1-\chi(\varpi)q^{1-2s}}\\
&= \frac{1-\chi(\varpi)q^{1-2s} + (q-1)\chi(\varpi)q^{-2s}}{1-\chi(\varpi)q^{1-2s}}\\
&= \frac{1-q\chi(\varpi)q^{-2s} + (q-1)\chi(\varpi)q^{-2s}}{1-q^{1-2s}}\\
&= \frac{1-\chi(\varpi)q^{-2s}}{1-\chi(\varpi)q^{1-2s}}.
\end{aligned}$$

Finally, we can determine the action of T on $\tilde{\varphi}^\circ$ from the following computation:

$$\begin{aligned}
\int_{N(F)} f(wntg)\,dn &= \int_{N(F)} f(wtt^{-1}ntg)\,dn \\
&= \delta_B(t)\int_{N(F)} f(wtng)\,dn \\
&= \delta_B(t)\int_{N(F)} f(wtw^{-1}wng)\,dn \\
&= \delta_B(t)(\chi^{-\sigma}|N(\cdot)|^{-s}\otimes\chi\eta|\cdot|^{2s+t})(t)\int_{N(F)} f(wng)\,dn \\
&= \delta_B^{1/2}(t)(\chi^{-\sigma}|N(\cdot)|^{-s+1/2}\otimes\chi\eta|\cdot|^{2s+t-1/2})(t)\int_{N(F)} f(wng)\,dn.
\end{aligned}$$

□

PROPOSITION 3.2. Let E/F be a global field extension of degree two. Let v be a place of F that splits over E, and let χ and η be unramified characters of $\mathbf{A}_E^\times/E^\times$ and $\mathbf{A}^\times/F^\times$. Suppose φ° is the unique K-fixed vector of $\mathrm{Ind}_{B(F_v)}^{G(F_v)}(\chi_v|N(\cdot)|^{s-1/2}\otimes\eta_v|\cdot|^{t+1/2})$ normalized so that $\varphi^\circ(e)=1$. Then

$$M(\varphi^\circ) = \frac{L(\chi_{1,v}\chi_{2,v}, 2s-1)}{L(\chi_{1,v}\chi_{2,v}, 2s)}\tilde{\varphi}^\circ,$$

where $\tilde{\varphi}^\circ$ is the unique K-fixed vector of $\mathrm{Ind}_B^G(\chi_v^{-\sigma}|N(\cdot)|^{-s+1/2}\otimes\chi_v\eta_v|\cdot|^{2s+t-1/2})$ such that $\tilde{\varphi}^\circ(e)=e$.

The proof of the split prime version of the proposition proceeds almost identically to the previous proof.

Our next goal is to analyze the composition of intertwining operators. Since $\mathrm{Ind}_B^G(\chi|N(\cdot)|^{s-1/2}\otimes\eta|\cdot|^{t+1/2})$ is irreducible for almost all s and t, the composition $M\circ M$ must be a scalar by Schur's lemma. This calculation is important because it allows us to explicitly express the normalized intertwining operator which we introduced in the section on Eisenstein series in terms of the standard interwining operator.

Let Λ' be the Whittaker functional

$$\Lambda' f = \lim_{N\to\infty} \int_{\varpi^{-N}} f\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi(-x)\,dx.$$

Lemma 3.3. Let $f \in \mathrm{Ind}_B^G(\chi|N(\cdot)|^{s-1/2} \otimes \eta|\cdot|^{t+1/2})$. Then

$$(\Lambda' \circ M)f = \chi(-1)\gamma(2-2s, \chi^{-1})\Lambda' f.$$

Proof. Note that $\Lambda' \circ M$ is a Whittaker functional, so by Proposition 1.2, there exists λ, a meromorphic function of s and t such that

$$(\Lambda' \circ M)f = \lambda(s,t)\Lambda' f.$$

By analytic continuation, we may assume without loss of generality that s is large enough for M to be given by the integral expression from above. Then

$$\begin{aligned}(\Lambda' \circ M)f &= \lim_{N\to\infty} \int_F \int_{\varpi^{-N}} f\left(\begin{pmatrix} y^{-1} & -1 \\ & y \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & x - y^{-1} \end{pmatrix}\right)\psi(-x)\,dx\,dy \\ &= \lim_{N\to\infty} \int_F \int_{\varpi^{-N}} \varphi(x - y^{-1})\chi^{-1}(y)|y|^{-2s}\psi(-x)\,dx\,dy,\end{aligned}$$

where

$$\varphi(x) = f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right).$$

Thus

$$\lim_{N\to\infty}\int_F\int_{\varpi^{-N}}\varphi(x-y^{-1})\chi^{-1}(y)|y|^{-2s}\psi(-x)\,dx\,dy = \lambda(s,t)\lim_{N\to\infty}\int_{\varpi^{-N}}\varphi(x)\psi(-x)\,dx,$$

or, making the change of variables $y \mapsto y^{-1}$,

$$\lim_{N\to\infty}\int_F \chi(y)|y|^{2s-2}\psi(-y)\int_{\varpi^{-N}}\varphi(x-y)\psi(-(x-y))\,dx\,dy = \lambda(s,t)\lim_{N\to\infty}\int_{\varpi^{-N}}\varphi(x)\psi(-x)\,dx.$$

Let $\varphi = \mathrm{ch}_{\varpi^M}$, where M is large enough that ψ is trivial on ϖ^M. Then, for $N \geq -M$,

$$\int_{\varpi^{-N}}\varphi(x-y)\psi(-(x-y))\,dx = \mathrm{vol}(\varpi^M)\,\mathrm{ch}_{\varpi^{-N}}(x).$$

Thus we have

$$\lim_{N\to\infty}\mathrm{vol}(\varpi^M)\int_{\varpi^{-N}}\chi(y)|y|^{2s-2}\psi(-y)dy = \lambda(s,t)\,\mathrm{vol}(\varpi^M).$$

Hence

$$\lambda(s,t) = \int_{\varpi^{-N}}\chi(y)|y|^{2s-2}\psi(-y)dy = \chi(-1)\gamma(2-2s,\chi^{-1})$$

by Exercise 3.1.10 in (Bump, 1997). $\square$

Applying the lemma twice to $\Lambda' \circ M \circ M$ gives us the following result.

PROPOSITION 3.4. The composition $M \circ M$ is given by multiplication by the scalar

$$\gamma(2-2s, \chi^{-1})\gamma(2s-1, \chi).$$

3.3 Classification of Admissible Non-Archimedean Representations

LEMMA 3.5 (Restriction to GL_2). Let E/F be a quadratic extension of local (possibly Archimedean) fields, $G = \mathrm{GU}_{1,1}(F)$ and $B = B_{\mathrm{GU}_{1,1}}(F)$. Let $\pi = \mathrm{Ind}_B^G(\chi \otimes \eta)$. Then $\pi|_{\mathrm{GL}_2(F)}$ is contained in $\pi' = \mathrm{Ind}_{B_{\mathrm{GL}_2(F)}}^{\mathrm{GL}_2(F)}(\chi\eta \otimes \eta)$. If π is unramified, then so is π' and the restriction of a K-spherical vector in π is given by a $K_{\mathrm{GL}_2(F)}$-spherical vector in π'.

Moreover, for any $\varphi \in \pi$, the value of the restriction to GL_2 of the Whittaker function associated to φ is given by the Whittaker function associated to $\varphi' = \varphi|_{\mathrm{GL}_2(F)}$ in π', i.e.

$$W_\varphi|_{\mathrm{GL}_2(F)}(g) = W_{\varphi'}(g).$$

Proof. Let $\varphi \in \pi$, let $\left(\begin{smallmatrix} a & * \\ & b \end{smallmatrix}\right)$ be a matrix in the Borel subgroup of $\mathrm{GL}_2(F)$, and let $g \in \mathrm{GL}_2(F)$. Then

$$\delta_B \begin{pmatrix} a & * \\ & b \end{pmatrix} = \delta_B \begin{pmatrix} a & * \\ & (ba)a^{-\sigma} \end{pmatrix} = \frac{|N(a)|}{|ba|} = \frac{|a|}{|b|} = \delta_{B_{\mathrm{GL}_2(F)}} \begin{pmatrix} a & * \\ & b \end{pmatrix},$$

$$\varphi\left(\begin{pmatrix} a & * \\ & b \end{pmatrix} g\right) = \varphi\left(\begin{pmatrix} a & * \\ & (ba)a^{-\sigma} \end{pmatrix} g\right) = \delta_{B_{\mathrm{GL}_2(F)}}^{1/2} \begin{pmatrix} a & * \\ & b \end{pmatrix} \chi\eta(a)\eta(b)\varphi(g),$$

and therefore $\varphi\big|_{\mathrm{GL}_2(F_v)} \in \pi'$.

The assertion about spherical vectors follows from the fact that $K_{\mathrm{GL}_2(F)} \subseteq K$.

Finally, note that

$$\begin{aligned} W_\varphi\big|_{\mathrm{GL}_2(F_v)}(g) &= \lim_{\substack{N' \subseteq N \\ N' \text{ open, compact}}} \int_{N'} \varphi\left(wn'g\right)\psi^{-1}(n')\,dn' \\ &= \lim_{\substack{N' \subseteq N_{\mathrm{GL}_2} \\ N' \text{ open, compact}}} \int_{N'} \varphi'\left(wn'g\right)\psi^{-1}(n')\,dn' \\ &= W'_\varphi(g), \end{aligned}$$

since $N = N_{\mathrm{GL}_2(F_v)}$. $\square$

Let $\mathrm{I} = \mathrm{Ind}_B^G(|N(\cdot)|^{-1/2} \otimes |\cdot|^{1/2})$. Note that the space of constant functions is a subspace of I, since a function f is in the representation space of I if it is smooth and

$$f\left(\begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix} g\right) = \left(\frac{|N(a)|}{|\lambda|}\right)^{1/2} |N(a)|^{-1/2}|\lambda|^{1/2} f(g) = f(g),$$

which the constant functions satisfy. Moreover, since the action of G on the constant functions is trivial, they clearly form an invariant subspace. However, by (Casselman, 1995), the length of a composition series for I is at most two, so therefore the quotient of I by the constant functions must be an irreducible representation. The restriction of this quotient representation to GL_2 is precisely the Steinberg representation, so we will call it the Steinberg representation of $\mathrm{GU}_{1,1}$ and denote it by St.

Let π be a representation of G and χ a character of $E^\times$. We define the twist of π by χ as

$$(\pi \otimes \chi)(g) = (\chi \circ \det)(g)\pi(g).$$

Theorem 3.6 (Classification of Irreducible Admissible Representations). Let π be an admissible irreducible representation of G. Then one of the following is true:

- π is a supercuspidal representation,
- π is a one-dimensional representation of the form $\mathbf{C}(\chi \circ \det)$ where χ is a character of $E^\times$,
- π is a special representation, i.e. isomorphic to a twisted Steinberg representation $\mathrm{St} \otimes \chi$ where χ is a character of $E^\times$, or
- π is isomorphic to $\mathrm{Ind}_B^G(\chi \otimes \eta)$ where $\chi\Big|_F \neq |\cdot|^{\pm 1}$.

Proof. Consider the Jacquet module of π. If it is trivial, π is supercuspidal. Otherwise, by Frobenius reciprocity, π is a subrepresentation of a parabolically induced representation $\mathrm{Ind}_B^G(\chi' \otimes \eta')$. Thus it suffices to describe the irreducible subspaces of $\mathrm{Ind}_B^G(\chi' \otimes \eta')$.

Recall that $\mathrm{Ind}_{B_{\mathrm{GL}_2}(F)}^{\mathrm{GL}_2(F_v)}(\chi_1 \otimes \chi_2)$ is reducible if and only if $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$, and that therefore, by the preceding lemma, it follows that the restriction of $\mathrm{Ind}_B^G(\chi' \otimes \eta')$ to GL_2 is irreducible if $\chi\Big|_F \neq |\cdot|^{\pm 1}$. Hence $\mathrm{Ind}_B^G(\chi' \otimes \eta')$ is itself irreducible.

Recall that for $g \in G$,

$${}^{T}g^{\sigma} w g = \rho(g) w,$$

or, taking determinants on both sides, $\mathsf{N}(\det(g)) = (\rho(g))^2$. In particular, this implies that $\det(g)/\rho(g)$ is of norm one, and that therefore for any character χ of $E^\times$, we can write $(\chi\circ\det)(g)$ as $\chi_0(\rho(g))\chi_1(\det(g))$ where χ_0 is a character of $F^\times$ and χ_1 is a character of $E^\times/F^\times$.

Fix a character χ and χ_0, χ_1 as above and consider $\upsilon = \mathrm{Ind}_B^G(\chi_1^2|\mathsf{N}(\cdot)|^{-1/2} \otimes \chi_0|\cdot|^{1/2})$. Let

$$t = \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix},$$

and note that

$$\begin{aligned}
(\chi\circ\det)(tg) &= \chi_0(\rho(t))\chi_1(\det(t))(\chi\circ\det)(g) \\
&= \chi_0(\lambda)\chi_1(\lambda a a^{-\sigma})(\chi\circ\det)(g) \\
&= \chi_0(\lambda)\chi_1^2(a)(\chi\circ\det)(g) \\
&= \left(\frac{|\mathsf{N}(a)|}{|\lambda|}\right)^{1/2} \chi_1^2(a)|\mathsf{N}(a)|^{-1/2}\chi_0(\lambda)|\lambda|^{1/2}(\chi\circ\det)(g),
\end{aligned}$$

and so it follows that $\mathbf{C}(\chi\circ\det) \subseteq \upsilon$. Moreover, since $g\cdot(\chi\circ\det) = \chi(\det(g))(\chi\circ\det)$, this is clearly an invariant subspace of υ. This gives us the 1-dimensional representations of G.

Finally, consider the contragredient representation to υ, and note that it contains the dual of $\mathbf{C}(\chi^{-1}\circ\det)$, an irreducible representation isomorphic to $\mathrm{St}\otimes\chi^{-1}$. $\square$

3.4 The Unramified Computation (Inert Prime)

Our goal in this section is to explicitly compute the value of the local zeta integral at the places that remain inert in the quadratic extension of the global field that we defined our group over, and where all of the relevant data is unramified. In particular, for all but finitely many inert places, the conductor of the local character ψ is $\mathcal{O}$, W_1 and W_2 are the unique K-fixed vectors identically equal to 1 over K of $\mathcal{W}(\pi_1, \psi)$ and $\mathcal{W}(\pi_2, \psi)$ respectively, and f is equal to $L(2s, (\omega_1\omega_2)^{-1})f^\circ$, where f° is the unique K-fixed vector identically equal to 1 over K.

To this end, we need an explicit expression for W_1 and W_2. However, we will later show that we will only need to evaluate W_1 and W_2 over a subgroup of $\mathrm{GL}_2(F)$, and therefore we will be able to make do with the restriction lemma from earlier in this chapter and the following result for GL_2.

Theorem 3.7. Let π be an unramified infinite-dimensional representation of $\mathrm{GL}_2(F)$. Then there exist unramified characters μ_1, μ_2 of $F^\times$ such that $\pi = \mathrm{Ind}_{B_{\mathrm{GL}_2(F)}}^{\mathrm{GL}_2(F)}(\mu_1 \otimes \mu_2)$, and π is not a special representation (i.e. $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$).

Moreover, if ϖ^{-d} is the largest ideal on which ψ is trivial, the unique right K-invariant function in the Whittaker model of π that takes the value 1 at e is given by

$$W^\circ \begin{pmatrix} x & \\ & 1 \end{pmatrix} = \begin{cases} |x|^{1/2} \frac{\mu_1(\varpi)^{\mathrm{ord}(x)+1} - \mu_2(\varpi)^{\mathrm{ord}(x)+1}}{\mu_1(\varpi) - \mu_2(\varpi)} & \text{if } x \in \varpi^{-d}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See Theorem 11 in Section 16 of (Godement, 1970). $\square$

Proposition 3.8 (Unramified Computation — Inert Prime). Let π_1 and π_2 be irreducible unramified representations of G with central characters ω_1 and ω_2 respectively, W_1° and W_2°

the unique spherical Whittaker functions of π_1 and π_2 respectively that are identically equal to 1 over K, and $f = L(2s, (\omega_1\omega_2)^{-1})f^\circ$ where we take f° to be identically equal to 1 on K. Then

$$Z_v(s, W_1^\circ, W_2^\circ, f) = \frac{1}{\det\left(1 - r(t_1) \otimes r(t_2) q^{-s}\right)},$$

where t_i is the conjugacy class associated to π_i in ${}^LG$ and

$$\begin{aligned} r : {}^LG &\to \mathrm{GL}_2(\mathbf{C}) \\ \left(\beta, \left(\begin{smallmatrix} \alpha & \\ & 1 \end{smallmatrix}\right)\right) \rtimes \sigma &\mapsto \beta^{-1} \left(\begin{smallmatrix} 1 & \\ & \alpha^{-1} \end{smallmatrix}\right). \end{aligned}$$

Proof. Note that f is K-invariant by construction and that therefore

$$\begin{aligned} Z_v(s, W_1^\circ, W_2^\circ, f) &= \int_K \int_{F^\times} W_1^\circ\left(\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) k\right) W_2^\circ\left(\left(\begin{smallmatrix} -y & \\ & 1 \end{smallmatrix}\right) k\right) f\left(s, \left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) k\right) \delta_B^{-1}\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) \, d^\times y \, dk \\ &= f(s, e) \int_{y \in F^\times} W_1^\circ\left(\begin{smallmatrix} y & \\ & 1 \end{smallmatrix}\right) W_2^\circ\left(\begin{smallmatrix} -y & \\ & 1 \end{smallmatrix}\right) |y|^{s-1} \omega^{-1}(y) \, d^\times y, \end{aligned}$$

where

$$f(s, e) = L\left(2s, (\omega_1\omega_2)^{-1}\right).$$

Suppose $\pi_1 \simeq \mathrm{Ind}_B^G(\alpha \otimes \beta)$ and $\pi_2 \simeq \mathrm{Ind}_B^G(\gamma \otimes \delta)$. For the rest of this proof, if χ is a character, we will write χ to mean $\chi(\varpi)$. Then by Lemma 3.5 we have

$$\begin{aligned}\int_{y\in F^\times} & W_1^\circ \begin{pmatrix} y & \\ & 1\end{pmatrix} W_2^\circ \begin{pmatrix} -y & \\ & 1\end{pmatrix} (\omega_1\omega_2)^{-1}(y)|y|^{s-1}\, d^\times y \\ &= \sum_{n=0}^{\infty} (\omega_1\omega_2)^{-n}|\varpi^n|^s \frac{(\alpha\beta)^{n+1}-\beta^{n+1}}{\alpha\beta-\beta}\frac{(\gamma\delta)^{n+1}-\delta^{n+1}}{\gamma\delta-\delta},\end{aligned}$$

or, setting $(\omega_1\omega_2)^{-1}|\varpi|^s = X$,

$$\begin{aligned}&\sum_{i=0}^{\infty} X^n \frac{(\alpha\beta)^{n+1}-\beta^{n+1}}{\alpha\beta-\beta}\frac{(\gamma\delta)^{n+1}-\delta^{n+1}}{\gamma\delta-\delta} \\ &\quad = \frac{1}{(\alpha\beta-\beta)(\gamma\delta-\delta)}\left(\frac{\alpha\beta\gamma\delta}{1-\alpha\beta\gamma\delta X} - \frac{\alpha\beta\delta}{1-\alpha\beta\delta X} - \frac{\beta\gamma\delta}{1-\beta\gamma\delta X} + \frac{\beta\delta}{1-\beta\delta X}\right) \\ &\quad = \frac{1}{(\alpha\beta-\beta)(\gamma\delta-\delta)}\left(\frac{\alpha\beta(\gamma\delta-\delta)}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)} - \frac{\beta(\gamma\delta-\delta)}{(1-\beta\gamma\delta X)(1-\beta\delta X)}\right) \\ &\quad = \frac{1}{(\alpha\beta-\beta)}\left(\frac{\alpha\beta-\omega_1\delta X-\omega_1\gamma\delta X+\omega_1\omega_2\beta X^2-\left(\beta-\omega_1\gamma\delta X-\omega_1\delta X+\omega_1\omega_2\alpha\beta X^2\right)}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)(1-\beta\gamma\delta X)(1-\beta\delta X)}\right) \\ &\quad = \frac{1-\omega_1\omega_2 X^2}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)(1-\beta\gamma\delta X)(1-\beta\delta X)} \\ &\quad = \frac{L^{-1}\left(2s,(\omega_1\omega_2)^{-1}\right)}{\det\left(1-r(t_1)\otimes r(t_2)q^{-s}\right)}.\end{aligned}$$

Therefore

$$Z_v(W_1^\circ, W_2^\circ, f_\Phi, s) = \frac{f(s,e)}{L\left(2s,(\omega_1\omega_2)^{-1}\right)} \cdot \frac{1}{\det\left(1-r(t_1)\otimes r(t_2)q^{-s}\right)} = \frac{1}{\det\left(1-r(t_1)\otimes r(t_2)q^{-s}\right)}.$$

$\square$

3.5 The Unramified Computation (Split Prime)

Our goal in this section is to evaluate the local zeta integral where all of the data is unramified, but the prime is split.

For this section only, suppose that E/F is a quadratic extension of global fields. Let v be a prime of F that splits in E as $v = w_1 w_2$. Recall that then $G(F_v) = \mathrm{GU}_{1,1}(F_v)$ is a subgroup of $\mathrm{GL}_2(F_v) \times \mathrm{GL}_2(F_v)$ isomorphic to $\mathrm{GL}_2(F_v) \times \mathrm{GL}_1(F_v)$ via $(g, \lambda) \mapsto (g, \lambda \det^{-1}(g) g)$. The image of the torus $T(F_v)$ in $\mathrm{GL}_2(F_v) \times \mathrm{GL}_1(F_v)$ is $\left\{ \left(\begin{pmatrix} a_1 & \\ & \lambda a_2^{-1} \end{pmatrix}, \lambda \right) \;\middle|\; \lambda, a_1, a_2 \in F_v^\times \right\}$, and therefore, as a quotient of subgroups of $\mathrm{GL}_2(F_v) \times \mathrm{GL}_1(F_v)$, $T_1(F_v) = T(F_v)/Z(F_v)$ equals $\left\{ \left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, y \right) \;\middle|\; y \in F_v^\times \right\}$.

Analogous to the restriction lemma (Lemma 3.5) from the previous section, we have a restriction lemma for the split primes that allows us to express the Whittaker functions of $G(F_v)$ in terms of those of $\mathrm{GL}_2(F_v)$.

Lemma 3.9. Let $\varphi \in \mathrm{Ind}_{B(F_v)}^{G(F_v)} (\chi_v \otimes \eta_v)$. Then

$$W_\varphi(g, \lambda) = \eta_v \chi_{v,2}(\lambda) W_{\varphi'}(g)$$

where $\varphi'(g) = \varphi(g, 1) \in \mathrm{Ind}_{B_{\mathrm{GL}_2}(F_v)}^{\mathrm{GL}_2(F_v)} (\chi_{v,1} \otimes \chi_{v,2}^{-1})$.

Proof. Note that

$$(\chi_v \otimes \eta_v)\left(\begin{pmatrix} a & * \\ & b \end{pmatrix}, \lambda\right) = (\chi_v \otimes \eta_v)\left(\begin{pmatrix} a & * \\ & \lambda(\lambda b^{-1})^{-1} \end{pmatrix}, \lambda\right)$$
$$= \eta_v(\lambda)\chi_{v,1}(a)\chi_{v,2}(\lambda b^{-1})$$
$$= \eta_v\chi_{v,2}(\lambda)\chi_{v,1}(a)\chi_{v,2}^{-1}(b),$$

and that therefore the character $\chi_v \otimes \eta_v$ defined with respect to our parametrization of $T(F)$, corresponds to the character $\chi_{v,1} \otimes \chi_{v,2}^{-1} \otimes \eta_v\chi_{v,2}$ with respect to the standard parametrization of the torus in $\mathrm{GL}_2 \times \mathrm{GL}_1$:

$$(\chi_1 \otimes \chi_2 \otimes \eta)\left(\begin{pmatrix} a & \\ & b \end{pmatrix}, \lambda\right) = \chi_1(a)\chi_2(b)\eta(\lambda).$$

We then have

$$W_\varphi(g, \lambda) = \lim_{\substack{N' \subseteq N \\ N' \text{ open, compact}}} \int_{N'} \varphi\left(wn'g, \lambda\right) \psi^{-1}(n')\, dn'$$
$$= \eta_v\chi_{v,2}(\lambda) \lim_{\substack{N' \subseteq N \\ N' \text{ open, compact}}} \int_{N'} \varphi\left(wn'g, 1\right) \psi^{-1}(n')\, dn'$$
$$= \eta_v\chi_{v,2}(\lambda) \lim_{\substack{N' \subseteq N \\ N' \text{ open, compact}}} \int_{N'} \varphi'\left(wn'g\right) \psi^{-1}(n')\, dn'$$
$$= \eta_v\chi_{v,2}(\lambda)W_{\varphi'}(g).$$

□

Note that the center of $\mathrm{GL}_2(F_v) \times \mathrm{GL}_1(F_v)$ is

$$Z(F_v) = \left\{ \left(\begin{pmatrix} z & \\ & z \end{pmatrix}, \lambda \right) \middle| z, \lambda \in F_v^\times \right\}$$

and that therefore for any representation π of $G(F_v)$, the center acts by a *pair* of characters (ω', ω''):

$$\pi \left(\begin{pmatrix} z & \\ & z \end{pmatrix}, \lambda \right) \cdot f(g) = \omega'(z)\omega''(\lambda) f(g).$$

Proposition 3.10 (Unramified Computation — Split Prime). Let π_1 and π_2 be irreducible unramified representations of $G(F_v)$ through which the center acts by the pairs of characters (ω_1', ω_1'') and (ω_2', ω_2'') respectively. Let W_1° and W_2° be the unique spherical Whittaker functions of π_1 and π_2 respectively that are identically equal to 1 over $K(F_v)$. Let $\omega' = \omega_1'\omega_2'$ and $\omega'' = \omega_1''\omega_2''$ and suppose that $f = L((\omega'(\omega'')^2)^{-1}, 2s) f^\circ$, where $f^\circ\big|_{K(F_v)} \equiv 1$.

Then

$$Z_v\left(W_1^\circ, W_2^\circ, f, s\right) = \frac{1}{\det\left(I_4 - r(t_1) \otimes r(t_2) q^{-s}\right)},$$

where t_i is the conjugacy class associated to π_i in ${}^L G$ and

$$r : {}^L G \to \mathrm{GL}_2(\mathbf{C})$$

$$\left(\beta, \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}\right) \rtimes \sigma \mapsto \beta^{-1} \begin{pmatrix} \alpha^{-1} & \\ & 1 \end{pmatrix}.$$

Proof. Suppose $\pi_1 \simeq \mathrm{Ind}_B^G((\alpha_1, \alpha_2) \otimes \beta)$ and $\pi_2 \simeq \mathrm{Ind}_B^G((\gamma_1, \gamma_2) \otimes \delta)$. For the rest of this proof, if χ is a character, we will write χ to mean $\chi(\varpi)$. Note that then

$$\begin{aligned}
W_1^\circ\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, y\right) &= (\beta\alpha_2)^{\mathrm{ord}_v(y)}|y|^{1/2}\left(\frac{\alpha_1^{\mathrm{ord}_v(y)+1} - \alpha_2^{-(\mathrm{ord}_v(y)+1)}}{\alpha_1 - \alpha_2^{-1}}\right) \\
&= \frac{(\beta\alpha_2)^{\mathrm{ord}_v(y)+1}}{\beta\alpha_2}|y|^{1/2}\left(\frac{\alpha_1^{\mathrm{ord}_v(y)+1} - \alpha_2^{-(\mathrm{ord}_v(y)+1)}}{\alpha_1 - \alpha_2^{-1}}\right) \\
&= |y|^{1/2}\frac{(\alpha_1\alpha_2\beta)^{\mathrm{ord}_v(y)+1} - \beta^{\mathrm{ord}_v(y)+1}}{\alpha_1\alpha_2\beta - \beta} \\
&= |y|^{1/2}\frac{(\alpha\beta)^{\mathrm{ord}_v(y)+1} - \beta^{\mathrm{ord}_v(y)+1}}{\alpha\beta - \beta},
\end{aligned}$$

and,

$$W_2^\circ\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, y\right) = |y|^{1/2}\frac{(\gamma\delta)^{\mathrm{ord}_v(y)+1} - \delta^{\mathrm{ord}_v(y)+1}}{\gamma\delta - \delta},$$

where $\alpha = \alpha_1\alpha_2$ and $\gamma = \gamma_1\gamma_2$.

Additionally,

$$f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}, y\right) = f\left(\begin{pmatrix} y & \\ & y(y)^{-1} \end{pmatrix}, y\right) = \left((\omega')(\omega'')^2\right)^{-1}(y)|y|^s f(e, 1),$$

where

$$f(1, e) = L\left(s, (\omega'(\omega'')^2)^{-1}\right).$$

Then we have

$$\int_{F_v^\times} W_1^\circ\begin{pmatrix} y & \\ & 1\end{pmatrix} W_2^\circ\begin{pmatrix} y & \\ & 1\end{pmatrix}\left(\omega'(\omega'')^2\right)^{-1}(y)|y|^{s-1}\,d^\times y$$
$$= \sum_{n=0}^{\infty}\left(\omega'(\omega'')^2\right)^{-n}|\varpi^n|^s\frac{(\alpha\beta)^{n+1}-\beta^{n+1}}{\alpha\beta-\beta}\frac{(\gamma\delta)^{n+1}-\delta^{n+1}}{\gamma\delta-\delta},$$

or, setting $\left(\omega'(\omega'')^2\right)^{-1}(\varpi)\,|\varpi|^s = X$,

$$\sum_{i=0}^{\infty} X^n\frac{(\alpha\beta)^{n+1}-(\beta)^{n+1}}{\alpha\beta-\beta}\frac{(\gamma\delta)^{n+1}-(\delta)^{n+1}}{\gamma\delta-\gamma}$$
$$= \frac{1}{(\alpha\beta-\beta)(\gamma\delta-\delta)}\left(\frac{\alpha\beta\gamma\delta}{1-\alpha\beta\gamma\delta X}-\frac{\alpha\beta\delta}{1-\alpha\beta\delta X}-\frac{\beta\gamma\delta}{1-\beta\gamma\delta X}+\frac{\beta\delta}{1-\beta\delta X}\right)$$
$$= \frac{1}{(\alpha\beta-\beta)(\gamma\delta-\delta)}\left(\frac{\alpha\beta(\gamma\delta-\delta)}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)}-\frac{\beta(\gamma\delta-\delta)}{(1-\beta\gamma\delta X)(1-\beta\delta X)}\right)$$
$$= \frac{1}{(\alpha\beta-\beta)}\left(\frac{\alpha\beta-\alpha\beta^2\delta X-\alpha\beta^2\gamma\delta X+\alpha\beta^3\gamma\delta^2X^2-\beta+\alpha\beta^2\delta X+\alpha\beta^2\gamma\delta X-\alpha^2\beta^3\gamma\delta^2X^2}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)(1-\beta\gamma\delta X)(1-\beta\delta X)}\right)$$
$$= \frac{1-\alpha\beta^2\gamma\delta^2X^2}{(1-\alpha\beta\gamma\delta X)(1-\alpha\beta\delta X)(1-\beta\gamma\delta X)(1-\beta\delta X)}$$
$$= \frac{1}{L(2s,(\omega'(\omega'')^2)^{-1})}\cdot\frac{1}{\det(I_4-r(t_1)\otimes r(t_2)q^{-s})}.$$

$\square$

3.6 The Local Functional Equation

The proofs in this section follow closely Jacquet's work for GL_2 in (Jacquet, 1972), requiring only some minor modifications for $GU_{1,1}$.

For any $\Phi \in \mathcal{S}(E^2)$, χ a character of $E^\times$ and η a character of $F^\times$, let

$$f_\Phi^{\chi,\eta}(s,g) = (\chi \circ \det)(g)(\eta \circ \rho)(g)|\rho(g)|^s \int_{E^\times} \Phi(y(0,1)g)\chi(y)|N(y)|^s \, d^\times y$$

and note that for

$$t = \begin{pmatrix} a & \\ & \lambda a^{-\sigma} \end{pmatrix},$$

we have

$$\begin{aligned}
(\chi \circ \det)(tg)(\eta \circ \rho)(tg)|\rho(tg)|^s &= \chi(\lambda a a^{-\sigma})\eta(\lambda)|\lambda|^s(\chi \circ \det)(g)(\eta \circ \rho)(g)|\rho(g)|^s \\
\int_{E^\times} \Phi(y(0,1)tg)\chi(y)|N(y)|^s \, d^\times y &= \int_{E^\times} \Phi(\lambda a^{-\sigma} y(0,1)g)\chi(y)|N(y)|^s \, d^\times y \\
&= \chi(\lambda^{-1} a^\sigma)|\lambda|^{-2s}|N(a)|^s \int_{E^\times} \Phi(y(0,1)g)\chi(y)|N(y)|^s \, d^\times y,
\end{aligned}$$

and therefore

$$f_\Phi^{\chi,\eta}(s,tg) = \chi(a)\eta(\lambda)\left(\frac{|N(a)|}{|\lambda|}\right)^s f_\Phi^{\chi,\eta}(s,g),$$

or, $f_\Phi^{\chi,\eta} \in \mathbf{H}_{\xi,\eta}(s)$.

This gives us a convenient way to generate explicit examples of properly normalized local sections f, and, we will later demonstrate that it will suffice to only consider such $f_\Phi = f_\Phi^{\omega^{-1},1}$ for the rest of this section.

Our first main goal in this section is to prove the following theorem.

Theorem 3.11.

- There is a $s_0 \in \mathbf{R}$ such that whenever $\mathrm{Re}(s) > s_0$, the integrals $Z(s, W_1, W_2, f_\Phi)$ and $Z(1-s, W_1, W_2, M(s)f_\Phi)$ are absolutely convergent.
- $Z(s, W_1, W_2, f_\Phi)$ and $Z(1-s, W_1, W_2, M(s)f_\Phi)$ are rational functions of q^{-s}. More precisely, they can be written as the quotient of an element of $\mathbf{C}[q^{-s}, q^s]$ by an element of $\mathbf{C}[q^{-s}]$, where the denominator is independent of the choice of $W_1 \in \mathcal{W}(\pi_1, \psi)$, $W_2 \in \mathcal{W}(\pi_2, \psi)$ and $\Phi \in \mathcal{S}(E^2)$.
- There exists a rational function of q^{-s}, $\gamma(s)$, such that for all $W_1 \in \mathcal{W}(\pi_1, \psi)$, $W_2 \in \mathcal{W}(\pi_2, \psi)$ and $\Phi \in \mathcal{S}(E^2)$,

$$Z(1-s, W_1, W_2, M^*(s)f_\Phi) = \gamma(s)Z(s, W_1, W_2, f_\Phi).$$

To prove Theorem 3.11, we will require a series of lemmas.

Lemma 3.12. Let π be an irreducible infinite-dimensional representation of $G(F)$, and let $\mathcal{W}(\pi, \psi)$ be the Whittaker model of π. To each $W \in \mathcal{W}(\pi, \psi)$ we associate a function $\varphi_W : F^\times \to \mathbf{C}$ by letting

$$\varphi_W(x) = W\big|_{\mathrm{GL}_2(F)}\begin{pmatrix} x & \\ & 1 \end{pmatrix}.$$

Then if π is a principal series or special representation induced from $\chi \otimes \eta$,

$$\varphi_W(a) = \begin{cases} |a|^{1/2}(\chi\eta)(a)f_1(a) + |a|^{1/2}\eta(a)f_2(a), & \text{if } \chi\big|_F \neq 1, |\cdot|^{\pm 1}, \\ |a|^{1/2}(\chi\eta)(a)f_1(a) + |a|^{1/2}\eta(a)\operatorname{ord}(a)f_2(a), & \text{if } \chi\big|_F = 1, \\ |a|^{1/2}(\chi\eta)(a)f(a), & \text{if } \chi\big|_F = |\cdot|, \\ |a|^{1/2}\eta(a)f(a), & \chi\big|_F = |\cdot|^{-1} \end{cases}$$

for some f, f_1 and f_2 in $\mathcal{S}(F)$.

If π is supercuspidal, $\varphi_W(a) = f(a)$ for some $f \in \mathcal{S}(F)$.

Proof. Apply Lemma 3.5 to the table at the end of Section 10 of (Godement, 1970). □

We will denote by $\mathcal{K}(\pi, \psi)$ the space $\{\varphi_W \mid W \in \mathcal{W}(\pi, \psi)\}$ (i.e. it is the Kirillov model of the restriction of the representation π to GL_2). Let

$$U(F) = \left\{ \begin{pmatrix} a & x \\ & 1 \end{pmatrix} \middle| \, a \in F^\times, x \in F \right\},$$

and note that it is a subgroup of G. $U(F)$ acts on the space $\mathcal{K}(\pi, \psi)$ by

$$\xi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \varphi(b) = \varphi(ab),$$
$$\xi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \varphi(b) = \varphi(b)\psi(bx).$$

LEMMA 3.13. Let f be a function in $\mathcal{S}(F)$ and μ a character of $F^\times$. Let $i \in \{0, 1, 2\}$. Then

$$\int_{F^\times} f(a) \operatorname{ord}^i(a) |a|^s \mu(a)\, d^\times a$$

is absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large. Moreover, it is a rational function of q^{-s}.

Proof. Recall that f is a locally constant function of compact support, and so there exists a $c_1 \in \mathbf{R}$ such that $f(a) = 0$ whenever $\operatorname{ord}(a) < c_1$. If f vanishes at 0, there exists a $c_2 \in \mathbf{R}$ such that $f(a) = 0$ whenever $\operatorname{ord}(a) > c_2$. Thus $f(a) = 0$ unless $c_1 \leq \operatorname{ord}(a) \leq c_2$, and so the integral expands into a finite Laurent series in q^{-s}.

It remains to prove the lemma for f such that $f(0) \neq 0$. In fact, by linearity and the translation invariance of the Haar measure, it suffices to prove it for the characteristic function of the ring of integers of F, $1_{\mathcal{O}_F}$. If μ is ramified, the integral vanishes. Otherwise, let $\mu(\varpi) = q^{-s_0}$ and note that

$$\int_{F^\times} 1_{\mathcal{O}_F}(a) \operatorname{ord}^i(a) |a|^s \mu(a)\, d^\times a = \sum_{n=0}^{\infty} q^{-n(s+s_0)} n^i,$$

where the right-hand side is absolutely convergent for $\mathrm{Re}(s) > -\mathrm{Re}(s_0)$.

Finally, note that this summation simplifies to

$$\begin{cases} (1-X)^{-1} & \text{if } i = 0, \\ X(1-X)^{-2} & \text{if } i = 1, \\ X(1+X)(1-X)^{-3} & \text{if } i = 3, \end{cases}$$

where $X = q^{s_0} q^s$, completing the lemma. $\square$

PROPOSITION 3.14. For φ_1 in $\mathcal{K}(\pi_1, \psi)$ and φ_2 in $\mathcal{K}(\pi_2, \psi)$, let

$$\beta_s(\varphi_1, \varphi_2) = \int_{F^\times} \varphi_1(a)\varphi_2(-a)|a|^{s-1}\omega^{-1}(a)\, d^\times a.$$

Then there exists a $s_0 \in \mathbf{R}$ such that the integral above is convergent for any s with $\mathrm{Re}(s) > s_0$. Moreover, $\beta_s(\varphi_1, \varphi_2)$ is a rational function of q^{-s} and can be written as a quotient of an element of $\mathbf{C}[q^{-s}, q^s]$ by a polynomial in $\mathbf{C}[q^{-s}]$ that does not depend on the choice of φ_1 and φ_2.

Proof. By Lemma 3.12, we can write

$$\begin{aligned}
\varphi_1(x) &= |x|^{1/2}\left(\mu_1(x)f_1(x) + s\,\mathrm{ord}^i(x)\mu_2(x)f_2(x)\right),\\
\varphi_2(-x) &= |x|^{1/2}\left(\nu_1(x)g_1(x) + t\,\mathrm{ord}^j(x)\nu_2(x)g_2(x)\right),
\end{aligned}$$

where $s, t, i, j \in \{0, 1\}$, $\mu_1, \mu_2, \nu_1, \nu_2$ are characters of $F^\times$ and $f_1, f_2, g_1, g_2 \in \mathcal{S}(F^\times)$.

Therefore

$$\begin{aligned}
\beta_s(\varphi_1, \varphi_2) &= \int_{F^\times} \varphi_1(a)\varphi_2(-a)|a|^{s-1}\omega^{-1}(a)\, d^\times a\\
&= \int_{F^\times}(f_1g_1)(a)|a|^{s-1/2}(\mu_1\nu_1\omega^{-1})(a)\, d^\times a + s\int_{F^\times}(f_2g_1)(a)|a|^{s-1/2}\,\mathrm{ord}^i(a)(\mu_2\nu_1\omega^{-1})(a)\, d^\times a\\
&\quad + t\int_{F^\times}(f_1g_2)(a)|a|^{s-1/2}\,\mathrm{ord}^j(a)(\mu_1\nu_2\omega^{-1})(x)\, d^\times a\\
&\quad + st\int_{F^\times}(f_2g_2)(a)|a|^{s-1/2}\,\mathrm{ord}^{i+j}(a)(\mu_2\nu_2\omega^{-1})(a)\, d^\times a
\end{aligned}$$

and so the result follows by Lemma 3.13. □

The bilinear form β_s, by analytic continuation, is defined for almost all s and satisfies the identity

$$\beta_s\left[\xi\begin{pmatrix} a & x \\ & 1\end{pmatrix}\varphi_1, \xi\begin{pmatrix} a & x \\ & 1\end{pmatrix}\varphi_2\right] = |a|^{1-s}\omega(a)\beta_s(\varphi_1,\varphi_2) \tag{3.1}$$

for all $a \in F^\times$, $x \in F$ and $\varphi_1, \varphi_2 \in \mathcal{K}(\pi_i, \psi)$.

Our goal is to show that any billinear form γ on $\mathcal{K}(\pi_1,\psi) \times \mathcal{K}(\pi_2,\psi)$ that satisfies Equation 3.1 is proportional to β_s. We separate our proof into three seperate lemmas depending on whether both, one or neither of π_1 and π_2 are supercuspidal.

Lemma 3.15. Let γ be a bilinear form on $\mathcal{S}(F^\times) \times \mathcal{S}(F^\times)$ that satisfies Equation 3.1. Then γ is proportional to β_s.

Proof. Since the action ξ of $U(F)$ on $\mathcal{S}(F^\times)$ is irreducible (Lemma 2.9.1 of (Jacquet and Langlands, 1970)), it suffices to prove that $\gamma(\varphi,\varphi_0) = c\beta_s(\varphi,\varphi_0)$ for at least one nonzero φ_0.

Let $\varphi_0(u) = \omega^{-1}(u)\mathbf{1}_{\mathcal{O}_F^\times}(u)$, and note that then

$$\beta_s(\varphi,\varphi_0) = \int_{\mathcal{O}_F^\times} \varphi(u)\,du.$$

For any $u \in \mathcal{O}_{F^\times}$ we have

$$\gamma \left[\xi \begin{pmatrix} u & 0 \\ & 1 \end{pmatrix} \varphi, \varphi_0 \right] = \gamma(\varphi, \varphi_0),$$

which implies that

$$\gamma(\varphi, \varphi_0) = \sum_{n=-\infty}^{\infty} a_n \int_{\mathcal{O}_F^\times} \varphi(u\varpi^n)\, d^\times u, \tag{3.2}$$

where, for a given φ, all but finitely many a_n are zero. To prove the lemma it now suffices to show that $a_n = 0$ for all $n > 0$.

Without loss of generality, we can take ψ to be of order zero. Let

$$\ell = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

and note that then

$$\gamma(\xi(\ell)\varphi, \varphi_0) = \gamma(\varphi, \varphi_0).$$

Hence, by letting φ be the characteristic function of $\varpi^{-m}\mathcal{O}_F^\times$ and applying Equation 3.2 to both sides, we have

$$a_{-m} \int_{\mathcal{O}_F^\times} \psi(u\varpi^{-m})\, du = a_{-m}.$$

If $m > 0$, the left-hand side vanishes and $a_{-m} = 0$. If $m = -1$, we have $a_{-1}(1-q)^{-1} = a_{-1}$ which implies that $a_{-1} = 0$.

Since

$$\int_{\varpi^{-1}\mathcal{O}_F} \xi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \varphi_0(a)\, dx = \int_{\varpi^{-1}\mathcal{O}_F} \psi(ax)\varphi_0(a)\, dx = 0,$$

it follows that

$$\gamma \left[\int_{\varpi^{-1}\mathcal{O}_F} \xi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \varphi(a)\, dx, \varphi_0 \right] = 0.$$

Letting φ be the characteristic function of $\varpi^m \mathcal{O}_F^\times$ with $m > 1$, it follows that $a_m = 0$, completing the lemma. □

LEMMA 3.16. Let γ be a bilinear form on $\mathcal{K}(\pi_1, \psi) \times \mathcal{S}(F^\times)$ that satisfies Equation 3.1. Then γ is proportional to β_s.

Proof. Similarly to the previous lemma, by the irreducibility of $\mathcal{S}(F^\times)$ under the action ξ of $U(F)$, it suffices to prove that there exists a nonzero $\varphi_2 \in \mathcal{S}(F^\times)$ such that

$$\gamma(\varphi_1, \varphi_2) = c\beta(\varphi_1, \varphi_2, s)$$

for all $\varphi_1 \in \mathcal{K}(\pi_1, \psi)$.

Let

$$\ell = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

and note that then

$$
\begin{aligned}
\gamma(\varphi_1, \varphi_2 - \xi(\ell^{-1})\varphi_2) &= \gamma(\varphi_1, \varphi_2) - \gamma(\varphi_1, \xi(\ell^{-1})\varphi_2) \\
&= \gamma(\varphi_1, \varphi_2) - \gamma(\xi(\ell)\varphi_1, \varphi_2) \\
&= \gamma(\varphi_1 - \xi(\ell)\varphi_1, \varphi_2).
\end{aligned}
$$

Since $\varphi_1 - \xi(\ell)\varphi_1 \in \mathcal{S}(F^\times)$, it follows by the previous lemma that

$$
\gamma(\varphi_1, \varphi_2 - \xi(\ell^{-1})\varphi_2) = c\beta(\varphi_1, \varphi_2 - \xi(\ell^{-1})\varphi_2, s),
$$

which, since we can choose φ_2 and ℓ such that $\varphi_2 - \xi(\ell^{-1})\varphi_2 \neq 0$, completes the proof of the lemma. $\square$

LEMMA 3.17. Let γ be a bilinear form on $\mathcal{K}(\pi_1, \psi) \times \mathcal{K}(\pi_2, \psi)$ that satisfies Equation 3.1. Then γ is proportional to β_s.

Proof. If either of π_1 or π_2 is supercuspidal, the result follows by the previous lemma. Otherwise, by linearity and Lemma 3.12, it suffices to prove the result for the following pairs:

1. $\varphi_1(a) = \chi_1(a)f_1(a),\ \varphi_2(a) = \chi_2(a)f_2(a);$
2. $\varphi_1(a) = \chi_1(a)\,\mathrm{ord}(a)f_1(a),\ \varphi_2(a) = \chi_2(a)f_2(a);$
3. $\varphi_1(a) = \chi_1(a)\,\mathrm{ord}(a)f_1(a),\ \varphi_2(a) = \chi_2(a)\,\mathrm{ord}(a)f_2(a);$

where χ_1 and χ_2 are characters and f_1 and $f_2 \in \mathcal{S}(F)$.

Case (I):

Define φ_i' such that

$$\xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_i = \varphi_i' + \chi_i(a)\varphi_i$$

for $i = 1, 2$. Then φ_i' vanishes for x small enough and so $\varphi_i' \in \mathcal{S}(F^\times)$. We have

$$\begin{aligned}
|a|^{1-s}\omega(a)\gamma(\varphi_1, \varphi_2) &= \gamma\left(\xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_1, \xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_2\right) \\
&= \gamma\left(\varphi_1' + \chi_1(a)\varphi_1, \varphi_2' + \chi_2(a)\varphi_2\right) \\
&= (\chi_1\chi_2)(a)\gamma(\varphi_1, \varphi_2) + \chi_1(a)\gamma(\varphi_1, \varphi_2') + \chi_2(a)\gamma(\varphi_1', \varphi_2) + \gamma(\varphi_1', \varphi_2'),
\end{aligned}$$

or,

$$(|a|^{1-s}\omega(a) - (\chi_1\chi_2)(a))\gamma(\varphi_1, \varphi_2) = \chi_1(a)\gamma(\varphi_1, \varphi_2') + \chi_2(a)\gamma(\varphi_1', \varphi_2) + \gamma(\varphi_1', \varphi_2').$$

By a similar argument, we get

$$(|a|^{1-s}\omega(a) - (\chi_1\chi_2)(a))\beta(\varphi_1, \varphi_2, s) = \chi_1(a)\beta(\varphi_1, \varphi_2', s) + \chi_2(a)\beta(\varphi_1', \varphi_2, s) + \beta(\varphi_1', \varphi_2', s),$$

and the result follows since we know that β and γ are proportional whenever at least one of the vectors is in $\mathcal{S}(F^\times)$.

Case (II):

Let $\mu(a) = \chi_1(a)f_1(a)$ and define φ_1' and φ_2' by

$$\xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_1 = \varphi_1' + \chi_1(a)\varphi_1 + \chi_1(a)\operatorname{ord}(a) + \chi_1(a)\operatorname{ord}(a)\mu(a),$$

$$\xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_2 = \varphi_2' + \chi_2(a)\varphi_2.$$

Then φ_1' and φ_2' both belong in $\mathcal{S}(F^{\times})$. From the the fact that γ is bilinear and satisfies Equation 3.1, it follows that

$$\begin{aligned}(|a|^{1-s}\omega(a) - (\chi_1\chi_2)(a))\gamma(\varphi_1, \varphi_2) =& \gamma(\varphi_1', \varphi_2'), + \chi_1(a)\gamma(\varphi_1, \varphi_2') + \chi_2(a)\gamma(\varphi_1', \varphi_2) \\ &+ \chi_1(a)\operatorname{ord}(a)\gamma(\mu, \varphi_2') + (\chi_1\chi_2)(a)\operatorname{ord}(a)\gamma(\mu, \varphi_2).\end{aligned}$$

We can obtain a similar expression for β, and note that all of the $\gamma(\cdot, \cdot)$ terms on the right are proportional to the corresponding $\beta_s(\cdot, \cdot)$ terms by the previous case. Hence the result holds for Case (II).

Case (III):

Let $\mu_i(a) = \chi_i(a)f_i(a)$ and define φ_i' by

$$\xi\begin{pmatrix} a & \\ & 1 \end{pmatrix}\varphi_1 = \varphi_i' + \chi_i(a)\varphi_i + \chi_i(a)\operatorname{ord}(a) + \chi_i(a)\operatorname{ord}(a)\mu_i(a)$$

for $i = 1, 2$.

Then φ_1' and φ_2' both belong in $\mathcal{S}(F^\times)$. From the the fact that γ_s is bilinear and satisfies Equation 3.1, it follows that

$$\begin{aligned}(|a|^{1-s}\omega(a) - (\chi_1\chi_2)(a))\gamma(\varphi_1, \varphi_2) = &\gamma(\varphi_1', \varphi_2'), +\chi_1(a)\gamma(\varphi_1, \varphi_2') + \chi_2(a)\gamma(\varphi_1', \varphi_2)\\ &+ \chi_1(a)\operatorname{ord}(a)\gamma(\mu_1, \varphi_2') + \chi_2(a)\operatorname{ord}(a)\gamma(\varphi_1', \mu_2)\\ &+ (\chi_1\chi_2)(a)\operatorname{ord}(a)\gamma(\mu_1, \varphi_2) + (\chi_1\chi_2)(a)\operatorname{ord}(a)\gamma(\varphi_1, \mu_2)\\ &+ (\chi_1\chi_2)(a)\operatorname{ord}^2(a)\gamma(\mu_1, \mu_2),\end{aligned}$$

with a similar expansion for β. However, once again, all of the terms on the right hand side are known to be proportional to β by the previous case, completing the proof. □

We now return to the proof of Theorem 3.11.

LEMMA 3.18. There is an $s_0 \in \mathbf{R}$ such that for all s with $\operatorname{Re}(s) > s_0$ and $\Phi \in \mathcal{S}(E^2)$, $f_\Phi^{\omega^{-1},1}(g) = 0$ for all $g \in G$ implies that $f_{I(\Phi)}^{\omega^\sigma,\omega^{-1}}(g) = 0$ for all g, where I is an endomorphism of $\mathcal{S}(E^2)$ described in the proof.

Proof. In fact, we will prove that $M(s)f_\Phi^{\omega^{-1},1} = \gamma_E^{-1}(s-1/2,\omega)f_{I(\Phi)}^{\omega^\sigma,\omega^{-1}}$.

Replacing Φ with $g\Phi$, we see that it suffices to prove this for $g = e$.

Note that $\mathcal{S}(E^2) = \mathcal{S}(E) \otimes \mathcal{S}(E)$, and so it suffices to consider the case when $\Phi(x, y) = \Phi_1(x)\Phi_2(y)$.

$$\begin{aligned}
&\int_F \int_{E^\times} \Phi\left(y(0,1)w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\omega^{-1}(y)|N(y)|^s\, d^\times y\, dx\\
&\quad= \int_F \int_{E^\times} \Phi(-y,-yx)\omega^{-1}(y)|N(y)|^s\, d^\times y\, dx\\
&\quad= \omega^{-1}(-1)\int_F \int_{E^\times} \Phi_1(y)\Phi_2(yx)\omega^{-1}(y)|N(y)|^s\, d^\times y\, dx\\
&\quad= \omega^{-1}(-1)\int_{E^\times} \Phi_1(y)\left(\int_F \Phi_2(yx)\, dx\right)\omega^{-1}(y)|N(y)|^s\, d^\times y
\end{aligned}$$

The Schwartz–Bruhat space $\mathcal{S}(E)$ is spanned by functions of the form $\mathrm{ch}_{u+\varpi_E^r}$ and so we can assume without loss of generality that $\Phi_2(x) = \mathrm{ch}_{u+\varpi_E^r}$. Given $u \in E$, define the r-part of u to be

$$\theta_r\left(\sum_{j=n}^{m} u_j\varpi_E^j\right) = \sum_{j=n}^{r} u_j\varpi_E^j,$$

and note that since ϖ_F is inert in E, we can identify ϖ_F and ϖ_E. Then

$$\begin{aligned}
\int_F \mathrm{ch}_{u+\varpi_E^r}(xy)\, dx &= \mathrm{vol}(y^{-1}(u+\varpi_E^r)\cap F)\\
&= q^{-\mathrm{ord}_E(y)}\,\mathrm{vol}((\varepsilon_y^{-1}u+\varpi_E^r)\cap F) \qquad (y=\varepsilon_y\varpi_E^{\mathrm{ord}_E(y)})\\
&= q^{-\mathrm{ord}_E(y)}\,\mathrm{ch}_F(\theta_r(u\varepsilon_y^{-1}))q^{-r}\\
&= |N(y)|^{-1/2}q^{-r}\,\mathrm{ch}_F(\theta_r(u\varepsilon_y^{-1})),
\end{aligned}$$

and so,

$$\int_{E^\times} \Phi_1(y)\left(\int_F \Phi_2(yx)\,dx\right)\omega^{-1}(y)|N(y)|^s\,d^\times y$$
$$= q^{-r}\int_{E^\times} \Phi_1(y)\,\mathrm{ch}_F(\theta_r(u\varepsilon_y^{-1}))\omega^{-1}(y)|y|_E^{s-1/2}\,d^\times y,$$

which, since $\varphi(y) = \Phi_1(y)\,\mathrm{ch}_F(\theta_r(u\varepsilon_y^{-1}))$ is a Schwartz-Bruhat function of E, is a Tate zeta integral over E. Therefore, by the local functional equation in (Tate, 1967),

$$\int_{E^\times} \varphi(y)\omega^{-1}(y)|y|_E^{s-1/2}\,d^\times y = \varepsilon_E^{-1}(s-1/2,\omega^{-1})\frac{L_E(s-1/2,\omega^{-1})}{L_E(3/2-s,\omega)}\int_{E^\times}\widehat{\varphi}(y)\omega(y)|y|_E^{3/2-s}\,d^\times y.$$

□

Lemma 3.19. There is a $s_0 \in \mathbf{R}$ such that for all $s \in \mathbf{C}$, $\mathrm{Re}(s) > s_0$, there is a unique trilinear form $\gamma_s : \mathcal{W}(\pi_1,\psi)\times\mathcal{W}(\pi_2,\psi)\times\mathbf{H}_{\omega^{-1},1}(s)\to\mathbf{C}$, such that if $f = f_\Phi^{\omega^{-1},1}$,

$$\gamma_s(W_1,W_2,f) = Z(1-s,W_1,W_2,f_{I(\Phi)}^{\omega^\sigma,\omega^{-1}})$$

for all W_1 in $\mathcal{W}(\pi_1,\psi)$ and W_2 in $\mathcal{W}(\pi_2,\psi)$. Moreover, for all $g\in G$,

$$\gamma_s(\pi_1(g)W_1,\pi_2(g)W_2,r(g)f) = \gamma_s(W_1,W_2,f),$$

where $r(g)f$ is the right translate of f under g^{-1}.

Proof. For all s with $\mathrm{Re}(s)$ large enough, any element of $\mathbf{H}_{\omega^{-1},1}(s)$ can be written as $f_{\Phi}^{\omega^{-1},1}$ for some $\Phi \in \mathcal{S}(E^2)$. If s and Φ are such that $f_{\Phi}^{\omega^{-1},1}$ vanishes, so does $f_{I(\Phi)}^{\omega^{\sigma},\omega^{-1}}$ by the previous lemma, and therefore also the integral $Z(1-s, W_1, W_2, f_{I(\Phi)}^{\omega^{\sigma},\omega^{-1}})$.

For $\mathrm{Re}(s)$ sufficiently small, $Z(1-s, W_1, W_2, f_{I(\Phi)}^{\omega^{\sigma},\omega^{-1}})$ is given by a convergent integral against the invariant measure, and therefore

$$Z(1-s, \pi_1(g)W_1, \pi_2(g)W_2, f_{gI(\Phi)}^{\omega^{\sigma},\omega^{-1}}) = \omega^{\sigma}(\det(g))\omega^{-1}(\rho(g))|\rho(g)|^{-s+1} Z(1-s, W_1, W_2, f_{gI(\Phi)}^{\omega^{\sigma},\omega^{-1}}).$$

By analytic continuation this holds for almost all s, from which the second assertion follows. $\square$

Since Whittaker functions are K-finite on the right, it follows that the linear form

$$f \mapsto \gamma_s(W_1, W_2, f)$$

is K-finite and therefore belongs to the contragredient of $\mathbf{H}_{\omega^{-1},1}(s)$, which is $\mathbf{H}_{\omega,1}(1-s)$. Hence there exists a

$$g \mapsto \delta_s(g, W_1, W_2),$$

such that

$$\delta_s\left(\begin{pmatrix} a & x \\ & \lambda a^{-\sigma} \end{pmatrix} g, W_1, W_2\right) = |N(a)|^{1-s}|b|^{s-1}\omega(a)\delta_s(g, W_1, W_2)$$

and

$$\gamma_s(W_1, W_2, f) = \int_{B\backslash G} f(g)\delta_s(g, W_1, W_2)\, dg.$$

By the invariance property of γ_s from the preceding lemma, it follows that δ_s has the invariance property

$$\delta_s(gh^{-1}, \pi_1(h)W_1, \pi_2(h)W_2) = \delta_s(g, W_1, W_2).$$

Thus by letting

$$\varphi_i(a) = W_i \begin{pmatrix} a & \\ & 1 \end{pmatrix},$$

it follows that $\lambda_s(\varphi_1, \varphi_2) = \delta_s(e, W_1, W_2)$ is a bilinear form that satisfies Equation 3.1, and therefore there exists a function $c(s)$, defined for s large enough, such that $\lambda_s = c(s)\beta_s$.

Therefore

$$Z(1-s, W_1, W_2, M(s)f) = c(s) \int_K f(k)\beta_s(\pi_1(k)\varphi_1, \pi_2(k)\varphi_2)\, dk,$$

or,

$$Z(1-s, W_1, W_2, M(s)f) = c(s)Z(s, W_1, W_2, f).$$

To conclude the proof of Theorem 3.11, it remains to show that $\gamma(s)$ is a rational function of q^{-s}. To do that, it suffices to prove the following lemma.

Lemma 3.20. There exists a choice of $W_1 \in \mathcal{W}(\pi_1, \psi)$, $W_2 \in \mathcal{W}(\pi_1, \psi)$ and f such that

$$Z(s, W_1, W_2, f) = 1$$

for all s.

Proof. We begin by choosing W_1 and W_2 such that

$$W_1\begin{pmatrix} a & \\ & 1 \end{pmatrix} = W_2\begin{pmatrix} a & \\ & 1 \end{pmatrix} = 1_{\mathcal{O}_F^\times}(a).$$

There exists an $n \geq 1$ such that W_1 and W_2 are invariant under translations by $\left(\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right)$ for $x \in \varpi^n \mathcal{O}_F$.

Let

$$K' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \,\middle|\, c = 0 \bmod \varpi^n \mathcal{O}_F \right\},$$

and note that for all $k \in K'$,

$$\begin{aligned} W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \\ -d^{-1}c & 1 \end{pmatrix}\right) \\ &= W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a - bcd^{-1} & b \\ & d \end{pmatrix}\right) \\ &= \omega(d)W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} ud^{-1} & bd^{-1} \\ & 1 \end{pmatrix}\right) \qquad (u \in \mathcal{O}_F^\times) \\ &= \omega(d)W\left(\begin{pmatrix} uyd^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & bu^{-1} \\ & 1 \end{pmatrix}\right) \\ &= \omega(d)\psi(byd^{-1})W\begin{pmatrix} uyd^{-1} & \\ & 1 \end{pmatrix}. \end{aligned}$$

Therefore

$$\int W_1\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix} k\right] W_2\left[\begin{pmatrix} -y & \\ & 1 \end{pmatrix} k\right] |y|^{s-1}\, \mathrm{d}^\times y = \int_{y\in\mathcal{O}_F^\times} \omega(d)\, \mathrm{d}^\times y = \omega(d)\, \mathrm{vol}(\mathcal{O}_F^\times).$$

Thus, if for $k \in K$ we let

$$f(k) = \begin{cases} \left(\mathrm{vol}(\mathcal{O}_F^\times)\right)^{-2} (\omega_1\omega_2)^{-1}(d) & \text{if } k \in K', \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$Z(s, W_1, W_2, f) = 1.$$

$\square$

Note that if we substitute the representation π_i with $\widetilde{\pi}_i$, the central character ω_i is replaced by ω_i^{-1}, and there is a correspondence between the Whittaker spaces of the two representations given by

$$W_i(g) \mapsto W_i(g)\omega_i^{-1}(\det g).$$

Then, since

$$\begin{aligned}
Z(s, W_1, W_2, f_\Phi^{\omega^{-1},1}) &= \int_{NZ\backslash G} W_1(g)W_2(g)f_\Phi^{\omega^{-1},1}(g)\,dg \\
&= \int_{NZ\backslash G} W_1(g)W_2(g)\omega_1^{-1}(\det g)\omega_2^{-1}(\det g)\omega(\det g)f_\Phi^{\omega^{-1},1}(g)\,dg \\
&= \int_{NZ\backslash G} \widetilde{W_1}(g)\widetilde{W_2}(g)f_\Phi^{\omega^\sigma,\omega^{-1}}(g)\,dg,
\end{aligned}$$

we can see that the roles of the integrals $Z(s, W_1, W_2, f)$ and $Z(1-s, W_1, W_2, M^*(s)f)$ are interchanged when we replace π_i with $\widetilde{\pi}_i$.

Finally, we summarize our results in a more precise form.

THEOREM 3.21. There exist Euler factors $L(s, \pi_1 \times \pi_2)$ and $L(s, \widetilde{\pi}_1 \times \widetilde{\pi}_2)$ such that:

- $$Z(s, W_1, W_2, f_\Phi) = L(s, \pi_1 \times \pi_2)Z'(s, W_1, W_2, f_\Phi)$$
  $$Z(1-s, W_1, W_2, M^*(s)f_\Phi) = L(s, \widetilde{\pi}_1 \times \widetilde{\pi}_2)Z'(1-s, W_1, W_2, M^*(s)f_\Phi)$$

  where $Z'(s, W_1, W_2, f_\Phi)$ and $Z'(s, W_1, W_2, M^*(s)f_\Phi)$ are polynomials in $\mathbf{C}[q^s, q^{-s}]$.
- We can choose a family $\{(W_{1,i}, W_{2,i}, f_{\Phi_i})\}$ such that

$$\sum_i Z'(s, W_{1,i}, W_{2,i}, f_{\Phi_i}) = 1 \qquad \left(\text{resp. } \sum_i Z'(1-s, W_{1,i}, W_{2,i}, M^*(s)f_{\Phi_i}) = 1\right).$$

- There is a function $\epsilon(s, \pi_1 \times \pi_2, \psi)$ of the form cq^{-sn} such that

$$\frac{Z(1-s, W_1, W_2, M^*(s)f_\Phi)}{L(1-s, \widetilde{\pi}_1 \times \widetilde{\pi}_2)} = \epsilon(s, \pi_1 \times \pi_2, \psi)\frac{Z(s, W_1, W_2, f_\Phi)}{L(s, \pi_1 \times \pi_2)}.$$

Proof. Recall that from the integral representation of f_Φ it follows that

$$Z(s, \pi_1(g)W_1, \pi_2(g)W_2, f_{g\Phi}) = \omega(\det(g))|\rho(g)|^{-s} Z(s, W_1, W_2, f_\Phi).$$

Combining this with the previous theorem, it follows that the subspace of $\mathbf{C}(q^{-s})$ spanned by the $Z(s, W_1, W_2, f_\Phi)$ is actually a fractional ideal of $\mathbf{C}[q^{-s}, q^s]$.

Let P/Q be a generator of this ideal with P and Q relatively prime. We may assume that $Q(0) = 1$, and note that by the previous lemma it follows that $P \equiv 1$. Then

$$L(s, \pi_1 \times \pi_2) = \frac{1}{Q(q^{-s})}$$

is the unique Euler factor satisfying the first two conditions. Likewise, there exists a unique factor $L(s, \widetilde{\pi}_1 \times \widetilde{\pi}_2)$.

Then, by Theorem 3.11, there exists a function of s, which we will denote $\epsilon(s, \pi_1 \times \pi_2, \psi)$, such that

$$Z(1-s, W_1, W_2, M^*(s)f) = \epsilon(s, \pi_1 \times \pi_2, \psi) Z(s, W_1, W_2, f),$$

which, by the first two results, must be a polynomial in $\mathbf{C}[q^s, q^{-s}]$.

Thus by applying the functional equation twice, it follows that

$$\epsilon(s, \pi_1 \times \pi_2, \psi)\epsilon(1 - s, \widetilde{\pi}_1 \times \widetilde{\pi}_2, \psi) = 1,$$

and therefore, that $\epsilon(\pi_1 \times \pi_2, \psi, s)$ must be a monomial. $\square$

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