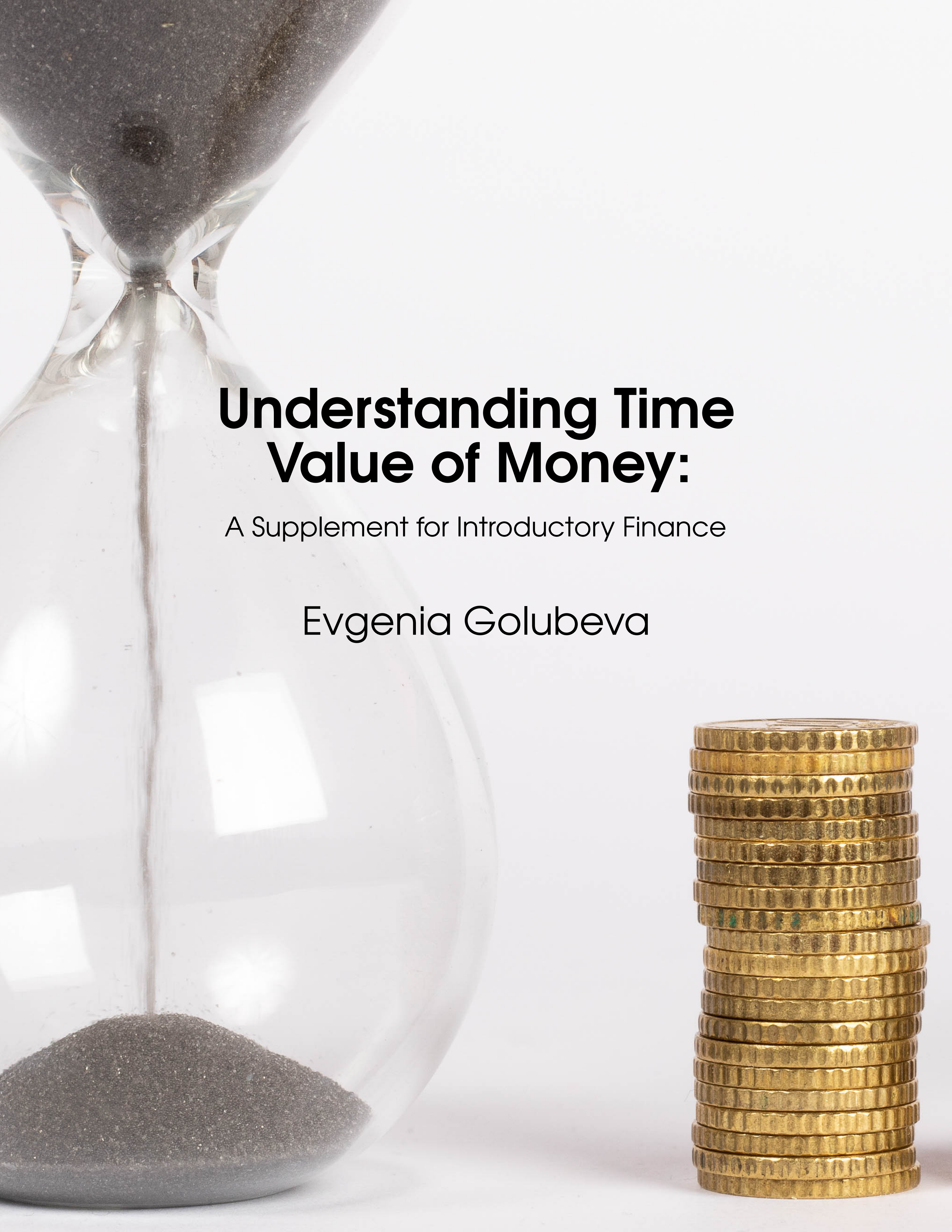

Understanding Time Value of Money:

A Supplement for Introductory Finance

Evgenia Golubeva

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PROLOGUE

I lost count of how many times I have asked my numerous academic finance colleagues over the years, “What do you need your students to know coming into your course?” I do remember the answers that they gave me. These answers are easy to remember because they all boil down to one: “Time Value of Money.” Why do students have such poor retention of that critical material, what is going on, and how shall we think about solving the problem?

Introduction to finance is a very ambitious undertaking. In mere fifteen weeks, students should learn essentially all about finance, from the basics of compounding and discounting to risk and return, to market efficiency, to financial ratios, to budgeting, financing, and even dividend policy. As the instructor focuses on building knowledge of these topics, under a severe time constraint, tradeoffs must be made. Unfortunately, these tradeoffs are not in favor of Time Value of Money, and students are left with very raw knowledge of it, knowledge that is not rooted in deep understanding of the underlying concepts and that is, therefore, poorly retained. Comparing the course to a building, it normally has a straight rectangular shape like a typical skyscraper, whereas it should be a pyramid with Time Value of Money at the base. The more advanced topics will be easier to deliver on this solid foundation, saving time at the end.

Furthermore, in a traditional textbook, every principle is introduced in a specific context: time value of money uses examples from personal finance, interest rates are tied with bonds, risk and return are discussed in the context of equity and using the language of statistics, etc. The danger of this structure is that the student compartmentalizes each principle into its specific context and cannot recognize how those principles are universal. They memorize that risk and return are about stocks, interest rates are for bonds, market efficiency is for the CAPM, and so on. As a specific example (a very low-hanging fruit indeed), students traditionally have a very hard time seeing how time value of money applies to stock valuation or capital budgeting. However, once you ask them a question about a mortgage, they quickly recognize to look up “annuity” on their formula sheet. This deficit in generalization is certainly undesirable.

This book has a modest role. It is intended as a supplementary text in the first introductory finance course taught to undergraduate students. The book attempts to build knowledge of Time Value of Money based on the most fundamental themes and ideas underlying financial decision-making. The purpose is to provide a solid conceptual foundation that would serve finance majors well in their subsequent finance coursework and that would stay with all students – whether finance majors or not – long after they forget the quantitative details and formulae.

The book is recommended for any instructor who would like to reinforce their delivery of Time Value of Money. I recommend taking time going through it. The book has six chapters. Chapter 1 introduces the basic principles of Future Value, Present Value, compounding, discounting, and EAR in the context of lump sums. Chapter 2 expands the library of financial applications. To do so it begins with the discussion of risk and return. This is done because most financial applications of the time value of money techniques incorporate risk, from bond pricing to corporate investment decisions. The chapter proceeds to introduce perpetuity and annuity, with examples varying from a loan to a dividend-paying enterprise. Chapter 3 then builds on this foundation by introducing the concept of growth. A section is included that discusses the conceptual difference between growth and return. The chapter introduces students to growing perpetuity and growing annuity, with examples borrowed from stock pricing, corporate finance, and value investing. Chapter 4 introduces financial calculators. Chapter 5 is a "capstone" chapter where students master combining various elements together. They price a bond. They price a stock using the discounted cash flow method and learn the concept of Terminal Cash Flow. They find NPV of a corporate project. Finally, Chapter 6 is called "Concept Recognition." It gives students five problems with solutions and then five more problems without solutions and framed in a different way. Students are asked to map each question from the first group to a question from the second group to practice recognition of the same concept, and then solve by analogy. Each chapter is complete with a set of end-of-chapter self-check questions and problems and the answer key.

The book presents the material in a different order from what most of us who have studied finance are used to. The idea is to present the foundational principles and rationales of financial decision-making first as a narrative (without any math whatsoever) and then to show how these rationales underlie various financial decisions, from personal finance to investment in stocks and bonds to corporate finance, using hands-on problems. The student is forced to see the principle in general and then to trace various financial scenarios to that same principle over and over, until – the hope is – they learn to recognize the principle regardless of the context and avoid the undesirable compartmentalization mentioned above. This way, time will be saved later when covering bond pricing, stock pricing, capital budgeting and NPV, etc. – because students will already be familiar with these concepts and will be comfortable applying Time Value of Money tools to these contexts. Therefore, time saved on later topics will largely compensate for the longer time it would take the instructor to move through Time Value of Money using this book. The pyramid will have approximately the same total volume as the skyscraper.

The book is as conceptual as it is quantitative. The amount of math is cut to the bare minimum for several reasons. The reality is that most of the students taking introductory finance are not finance majors, do not have math as their strongest suit, and need mostly the intuition anyway; while those majoring in finance will have their fair share of mathematical rigor as they proceed down their curricular path toward more advanced courses. The book provides sufficient rigor to send finance majors a solid message that they will need math skills to study finance in depth; and wherever appropriate, it hints at higher mathematical skills that would be required from someone wishing to tackle a real-world finance problem. But its objective is to maximize intuition subject to the rigor constraint. Ultimately, intuition is what stays while formulae are quickly forgotten – and that is true for the best of us.

It is my hope that many instructors will find this book intriguing enough to experiment with using it for their course; and that ultimately, they will find it helpful in achieving the goal of material retention.

BUILDING THE MOTIVATION

“The Moving Finger writes; and, having writ,
Moves on: nor all thy Piety nor Wit
Shall lure it back to cancel half a Line,
Nor all thy Tears wash out a Word of it.”
— Omar Khayyám

Time is precious and cannot be undone. We hear so much advice about living in the moment and making the best of here and now. But what do you do if you have to make tradeoffs between “now” and “later?” When is your time better spent: when you enjoy life today or when you build the foundation for tomorrow? Can you overinvest in either?

You can choose to have a dollar today or you can save it for the rainy day. Most people will do some of each. At the extremes are the miserly that will spend nothing at all today even if they possess a fortune; and the lavish that borrow beyond their means to live in luxury not thinking about how they will repay the debts. What’s the optimal tradeoff? Is it always the same? If not, what does it depend on? Think about your own life: when do you make the tradeoff in favor of the present, sacrificing some of your future consumption in order to achieve today’s goals; and when is it in favor of the future, denying yourself some pleasures today in order to attain a future target?

Whatever your answers, you would likely agree that you wouldn’t sacrifice much of the present benefit unless there was some incentive or reward for patience! The higher that reward, the higher the fraction of today’s dollar that gets saved for the future. This reward is what time value of money is all about.

As it turns out, no financial decision results in immediate payoffs: ***time is a necessary factor in all decisions***. The same is true about most of the decisions we make in life, whether they involve money or not. We act now in exchange for having something happen at some future time. When we decide to move to another city, to make a change to our lifestyle, to start a new job – whatever the decision may be – we expect that some good will eventually come out of our decision, but surely we do not expect all the benefit to come immediately! It is normal to expect that the results of our decision will be realized over time, sometimes a very long time. Will those future benefits be worth today’s sacrifice? In finance, we quantify the answer to this question.

Here is a simple example, which is perhaps too dry given the philosophical prelude above. You consider installing solar panels on your roof. To generate enough power to provide 100% of your household’s power needs, you are looking at a solar panel that would cost you about $15,000 after solar tax credits. The expected lifespan of the panel is 20 years. The panels should save you about $1000 in electric bills per year, net of additional maintenance and insurance that the panels would require. Should you invest?

Your first reaction may be that you aren't really thinking about this decision as an "investment" but rather you would install solar panels out of desire to be less dependent on the electric grid, to avoid potential power outages, and to consume energy in an environmentally friendly way. Those are important considerations that will certainly have an impact on your ultimate decision. In this example, however, I would like to focus specifically on the investment aspect of it to illustrate the tradeoff between today's cost (investment) and future benefit (savings). This aspect is also important in your overall decision, after all.

Some people might start answering this question as follows. If I save $1000 per year, then over 20 years of the panel's expected service life I will save a total of $20,000. Since $20,000 > $15,000, the investment is worthwhile.

However, some of you might feel uneasy about the solution above. What bothers you? Chances are you realize that the savings accrue over a very long time while the initial investment is paid all at once today. Having to wait a long time for the payoff is not desirable. Instead, you might approach the solution as follows. In order to pay the initial investment back, given the recovery rate of $1000 per year, I will need to wait for $15,000 / $1000 = 15 years! That's too long a time, forget about it.

Yet others may approach the solution in a third way. They would realize that they can invest $15,000 in some alternative investment. That alternative investment should be considered as the opportunity cost for solar panels because it is a potential investment that would be foregone should you invest in solar panels instead. Such an investment should, therefore, be used as the proper framework for evaluating the decision. To make it a proper comparable, the alternative investment must carry approximately the same risk as the investment in solar panels. Investment in solar panels is risky because electricity prices go up and down and your future savings are not plotted with 100% certainty. No plan would fix your electricity rate for 20 years ahead! Should prices fall, so will your savings. So suppose you calculate the risk of the investment in solar panels (how exactly you do that is still beyond the scope of the discussion) and realize that it is roughly the same as the risk of investing in a bond issued by some renewable energy company. Suppose that investing in such a bond should provide a rate of return of approximately 5% per year. So, if the investment in solar panels provides you a return on investment at least that much or higher, it appears worthwhile; otherwise you would be better off investing in the bond of the renewable energy company. So how should you calculate the rate of return to see whether it is high enough or not? That's the "reward" I was talking about earlier.

The purpose of this book is to give you the basic set of tools that you need to evaluate financial decisions and to make an educated tradeoff between today and tomorrow. I should warn you that reading the book itself will take quite a bit of patience from you! I advise to read it slowly and thoroughly, taking breaks as needed. Don’t skip the narrative; don’t rush to write down the formulae and the bullet points: read as you would read a story. That is the way to get the most value out of this text. You will be rewarded for your patience as a reader!

CHAPTER 1: DOLLAR TODAY AND DOLLAR TOMORROW

"For the people not deeply in love with the art of making money out of nothing the legal way (financiers) this might be somewhat boring." – Erle Stanley Gardner, "The Case of the Troubled Trustee"

1.1 Return

You have heard the term "return" before. But have you stopped to give it a thought? What is it really?

Return is magic, no less.

Imagine some object. You put it in the empty room, walk out, then after some time walk back in and see that the object has gotten bigger. What happened? What forces were at work quietly while you were out, to make it expand?

Financial return is a similar occurrence where you put an amount of money in a "room" – except in this case the room will either be a bank, or the stock market, etc. – and after a period of time it becomes a different amount. Your hope is that this new amount will be bigger than the original amount was. Alas, sometimes it gets into something much, much smaller.

What's magic about it? The fact is that you have seemingly done nothing at all to make that amount of money change. You put it in, walk out, walk in – at voila! Job is done.

What job?

I used the quote above as an epigraph to this chapter because financiers are perceived by popular opinion to be the people "making money out of nothing the legal way." They just sit and click on buttons and money goes around. Sure enough, ***someone somewhere*** must be doing ***something*** to make that happen.

As business majors (which most likely is what you are given that you are reading this book), you may think you know better than to share popular perception. You may make an argument that as soon as your money leaves your pocket, it enters the financial market, where it eventually finds its way to some productive assets! Banks lend your money to businesses, which do "real work" to deliver hard-earned profits. Stocks and bonds are a direct way to fund businesses, bypassing banks: they represent claims to business assets sold to investors directly by entrepreneurs to fund their enterprise. So, one way or another, your money ends up working for someone.

That's a good answer, but my next question then is this. Every financial institution (a bank, a fund, a trust) "makes money" even though they aren't "real businesses" made of brick and mortar. Sometime long ago, I heard this definition of a bank: ***"A bank is a big boring grey building, next to which it is impossible to park."*** Today actually, banks aren't even buildings. They exist in virtual reality and money flows in invisible ways, literally billions being moved from place to place by a click of a button. What work do banks, funds, trusts, and other financial institutions do?

You may say, they are the intermediary and they play a very important role in the financial market by moving capital from suppliers (people who deposit money into their account) to users (enterprises and other entities). They ought to earn money for this service.

Very well, let it be so for now. But then my question is what work do *you* do? Ultimately, it is *your* money that turns into some other amount. You ought to make some contribution to the financial market, also, to reap this reward! What is your contribution?

Yes, you make an important contribution, and it is twofold. First, you part with your money for a period of time. Believe me: parting with your money is a painful undertaking for many people! You could have used your money for immediate consumption. Instead, you agree to part with it. The return you are making partly ***rewards you for the patience***.

Second, you take risk. At least if you keep your money under the mattress, the worst that can happen is that it will be eroded by inflation. But if inflation rates are normally low (as they have been in the U.S. for the last few decades), that's not too bad. Compare this risk to the risk of investing in stocks and bonds! A lot of people all around the world prefer not to part with their money and to keep it safe under the mattress instead because they cannot take risks! Financial markets in some countries barely exist. Therefore, the return you are making ***rewards you for the risk***.

Patience and risk-taking are the work you do. Someone needs to do this work for money to be eventually channeled to productive use. Doesn't seem like a lot of work? Well, would you lend me $1,000? If it costs you nothing to do it, I'd better hurry to knock on your door! But something tells me you won't give me the money. You don't know me, even though you'll love me after reading this book. There is too much risk involved. Besides, you may need that $1000 for your own immediate needs. You see, parting with money actually is a job, because there is a cost to you associated with doing so.

In this book, ***we will think of the rate of return as the percentage change from the amount initially invested to the amount eventually collected***.

Payoff: the ending amount

Percentage return

Initial Investment: the beginning amount

> Suppose you give me \$10 today and I will give you back \$11 tomorrow. Your *initial investment* is \$10, your *payoff* is \$11, your *profit* is \$11 - \$10 = \$1, and your percentage *return* is \$1 / \$10 = 10%. You can also express return in decimals: 10% = 0.1

At this point, I normally ask students the following question: ***If you invested \$100 and next year you collected \$120, what percentage rate of return have you made?*** They are always quick to answer, 20 percent. That's the correct answer, of course. But then I ask: if you invested \$593.67 and collected \$691.20, what return have you made now? Some can't tell.

What I'm asking about is the **holding period return (HPR)** over a period t:

$$HPR = \frac{(P_t - P_0)}{P_0} \qquad (1.1)$$

The formula above says that you paid the price P_0 at the start; and then after time t you collected P_t. The percentage change from P_0 to P_t is the holding period return. So, if your beginning amount is \$593.67 and the ending amount is \$691.20, then using formula (1.1), your return is

$$HPR = \frac{(691.20 - 593.27)}{593.27} = 16.5\%$$

Using algebra to transform the formula above, we see that

$$P_t = P_0 * (1 + HPR) \qquad (1.2)$$

And

$$P_0 = \frac{P_t}{(1 + HPR)} \qquad (1.3)$$

Example

I. You paid \$44 today for a share of stock of company ABC. In one year, stock price is \$55. What holding period return have you made on your investment?

If you invested 44 and received 55 back, the return X should be such that

$$44 + X * 44 = 55$$

$$X = \frac{55 - 44}{44} = 0.25 = 25\%$$

You see I have used formula (1.1).

II. You paid \$78 for a share of stock and in one year it fell to \$61. What is your holding period return? Using formula (1.1) one more time, it is $\frac{61-78}{78} = -0.2178 = -21.78\%$. Oops.

III. You paid \$300 and your HPR was 4%, how much money do you receive at the end of your holding period?

$$300 + 4\% * 300 = 300(1 + 0.04) = 312$$

Above, you can see I ultimately used formula (1.2).

IV. You make an investment and keep it for one year. Next year, the value of your investment is \$113.5. If you know that you have earned 13.5% return on your investment, how much must you have paid today?

The amount you paid today must be such that the final amount \$113.5 had better be enough to give you that money back, and 13.5% of that on top.

$$\$113.5 = X + 13.5\% * X$$

$$\$113.5 = X * (1 + 13.5\%)$$

$$\$113.5 = X * (1 + 0.135)$$

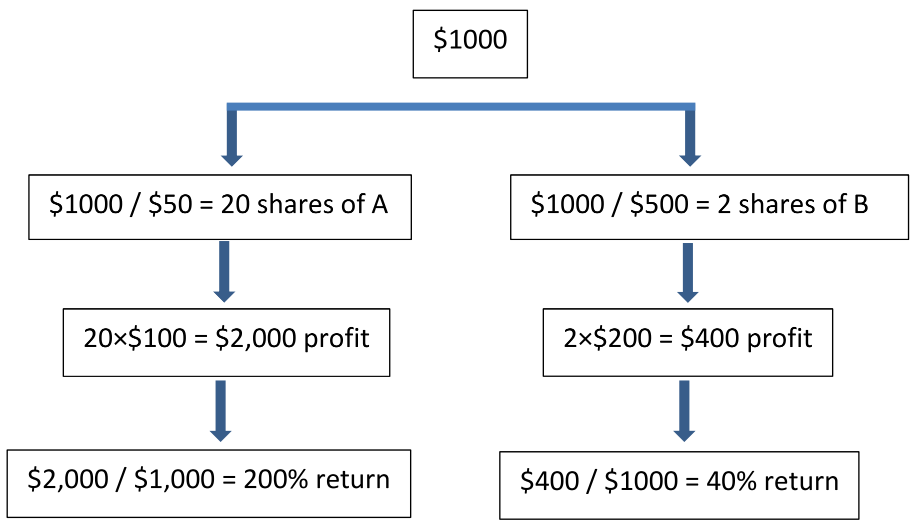
$$X = \frac{113.5}{(1 + 0.135)} = 100$$

You see I used formula (1.3) to solve for X.

This last question is an actual exam problem that I gave my students in more advanced courses than this one; and some of them struggled. One thing I want you to be able to do after reading this whole book is this: I want you to be able to answer a similar question in your sleep with your eyes closed. If you can't, all my efforts have been for nothing. If you learn absolutely nothing else, please promise me you will learn this much.

Why are we so preoccupied with percentage return and not dollar profit? Isn't profit what rewards us for the patience and for the risk?

Consider two companies: A and B. Buying one share of stock of company A would earn a profit of $100 and buying one share of stock of company B would earn a profit of $200. Which is a better investment? Your initial response may be, "Why, of course the one that earns $200!" However, what if the stock of company A cost you only $50 per share today, and the stock of company B cost $500 per share today? Suppose you have $1,000 to invest and can choose to invest it in either A or B. Which would you choose? With $1,000 today you can buy either $1,000 / $50 = 20 shares of stock of company A or $1,000 / $500 = 2 shares of stock of company B.



Each share of A would earn you a profit of $100, for a total profit of $2,000 (see the diagram above). On the other hand, each share of B would earn you $200, for a total profit of $400. This considerably changes the game! Investing in A earns you the rate of return of 200%, whereas investing *the same amount of money* in B earns you only 40%! Which looks better now?

It is the rate of return that ultimately measures the success of an investment because profit depends on the amount invested, while return puts investments on an equal footing. Therefore, you can compare investments of very different sizes based on the rate of return. In the rest of this book, we will be talking about percentage return, and not dollar profit.

1.2 Return vs. Interest

You are familiar with the concept of interest. If I asked you to give me some examples, you might think of a student loan where you borrow a certain amount today and then the lender wants that borrowed amount (called “principal”) plus some interest back. So, we say that the lender charges you interest on the loan.

It is perhaps convenient for you to think about interest in the context of borrowing, and about return in the context of investing. In fact, that’s the normal way to think about these concepts. Given that borrowing and investing are very different financial undertakings, you might think that interest and return have little to do with each other. However, in the numerical examples and problems that follow in this chapter, you will encounter the term “interest” all the time, so it is important to connect it to the concept of return.

When you borrow you pay interest. That much is true. But let’s look at things from the perspective of the lender! What does that interest income represent to the bank? Essentially, that interest income is nothing but the rate of return that the bank is earning on the deal! Suppose I lend you $100 today, and you pay me back 5% interest plus the principal amount next month. Barring default (that is, your failure to pay the money back), I will receive $105 next month in return for lending you $100 today. How is this example conceptually different from the examples of holding period return that we considered in the previous section? Your payment of 5% interest means that I as a lender made 5% return!

Therefore, I encourage you not to get confused too much over the choice of terms. Sometimes I will use the term “return”, sometimes “interest.” Depending on whose perspective we take, one is the mirror image of the other. Later, in more advanced texts, you will learn about interest in depth. You will learn where it comes from and why banks charge higher interest in some instances than others. But the purpose of this book is much humbler: to give you the tools of Time Value of Money; and for that purpose, it is okay to use the terms “return” and “interest” interchangeably.

When there is one starting and one ending amount, calculating holding period return is very easy. However, what about a situation where you pay some amount today and in return you receive many different payments occurring at different times? For example, you take out a loan today but need to pay it back in monthly installments over five years. Or you buy a share of stock and receive dividends every quarter for the entirety of your holding period horizon. How will you calculate the rate of return then?

Alternatively, what if you pay multiple times, and then collect a single amount in the future? For example, you contribute monthly into your retirement plan for 25 years, and then you end up with one sum of money in the account in the future. How do you calculate the rate of return in that case?

Finally, what if you hold the investment for multiple years and cash in at the end, and you want to know what rate of return you have made on average per year?

Soon enough, we are going to consider multiple examples of these situations and you will learn how to determine the rate of return in different scenarios. However, to begin with, let us spend more time on scenarios in which there is one beginning and one ending amount. That “one amount" is called a "lump sum." You will soon discover (if you haven't already) that being patient is essential when reading this book. Let's not rush it.

1.3 Lump Sums: Annual Compounding

Let us begin with some problem-solving.

Problem 1.1 Put \$1 in the Bank for one year. Interest rate is 5% per annum. Assume no risk at all, that is, there is every chance you will in fact receive your money back with the promised amount of interest. How much money will you have at year end?

First, we calculate 5% of the deposited amount:$\$1 * 5\% = \$0.05$. Therefore, at year end, you will receive $\$1 + \$0.05 = \$1.05$. You can, therefore, solve as

$$\$1 + \$1 * 5\% = \$1(1 + 5\%) = \$1(1.05) = \$1.05$$

Problem 1.2 What if you deposit \$2,964.76 for one year?

Similar to the example above, the solution is

$$\$2{,}964.76 + \$2{,}964.76 * 5\% = \$2{,}964.76 * (1.05) = \$3{,}112.99$$

Problem 1.3 What if you deposit \$1 for two years and reinvest the interest received after the first year?

Now we need to be more careful. Let's go step by step. After the first year, you will have \$1.05 as we already know. Given that you will reinvest this amount for the second year, you will be earning the second year's interest of 5% on the amount available at the beginning of that second year, i.e. on the \$1.05! Therefore, at the end of the second year, you will have

$$\$1.05 + \$1.05 * 5\% = \$1.05 * (1.05) = \$1.1025$$

Now, let's remember from Problem 1.1 that the amount available at the beginning of the second year \$1.05 was calculated as \$1 * (1.05) = \$1.05. Therefore, the amount available at the end of the second year can be calculated as

$$\{\$1 * (1.05)\} * (1.05) = \$1 * (1.05)^2 = \$1.1025$$

If you didn't earn interest on interest, i.e. if you withdrew the first \$0.05 at the end of the first year and started the second year with the same \$1 and received another \$0.05 at the end of the second year, then you would have \$1.10 after two years. However, in the reinvestment scenario, you end up with more! The difference may not seem significant but scale it up from one dollar to a million dollars, and then it may seem larger.

What you have just seen in this example is the power of **compounding**. We say the rate compounds when you earn ***interest* on *interest***.

Problem 1.4 What if you deposit $750.32 for three years and reinvest your money every year at the rate of 5% per annum?

At the end of the first year, you will have $\$750.32 * (1.05)$

At the end of the second year, you will have $\{\$750.32 * (1.05)\} * (1.05) = \$750.32 * (1.05)^2$

At the end of the third year, you will have

$\{\$750.32 * (1.05)^2\} * (1.05) = \$750.32 * (1.05)^3 = 868.59$

To verify, consider the following table, which shows how your interest compounds over time. See how your interest earned is larger each year. This is because you leave it invested and it earns interest.

Year	Beginning balance	Interest Calculation	Interest earned	Ending Balance Calculation	Ending Balance
1	750.32	750.32 (0.05) =	37.52	750.32 + 37.52 =	787.84
2	787.84	787.84 (0.05) =	39.39	787.84 + 39.39 =	827.23
3	827.23	827.23 (0.05) =	41.36	827.23 + 41.36 =	868.59

So far, we put a <u>single</u> certain amount in and withdraw a <u>single</u> certain amount later. The single amount is called "**lump sum**." For example, a lump sum of $1 turns into a lump sum of $1.05.

In general, we call the initial amount the **present value (PV)**; and the ending value the **future value (FV)**. Therefore, given ***periodic rate r*** and the ***number of periods T***, we can generalize:

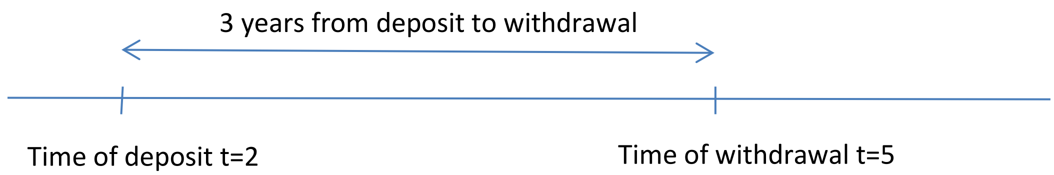
$$FV = PV * (1 + r)^T \qquad (1.4)$$

So what about the idea that a dollar today is not equal to a dollar tomorrow? I hope you just saw for yourself that in our examples, the present amount was never equal to the future amount, so long as there was a non-zero interest rate. For example, in Problem 1.4, we can say that $750.32 today is *the same thing* as $868.59 in three years!!! As an investor, you would be indifferent between collecting $750.32 today and collecting $868.59 in three years. Wow. This insight is very important for everything that follows in this book.

Problem 1.5 What if you deposit $750.32 <u>two years from now</u>, for three years, and reinvest your money every year at the rate of 5% per annum? How much money will you have after three years, that is, <u>five years from now</u>?

Be careful here. You have three different numbers for time: two years from now (time of deposit), three years (length of deposit), and five years from now (time of withdrawal). Which is the correct T to plug in the formula for future value?

You guessed right! $T = 3$. It is the number of periods <u>between</u> the time when present value is given and the time when future value is given. We can make the initial deposit at any time, that doesn't matter at all. What matters is how many periods our interest compounds.



Since $T = 3$, just like in Problem 1.4, and the starting amount and interest are the same as well, nothing changes about the answer! It is still $868.59.

Now, let's try to go backwards.

Problem 1.6 How much do you need to deposit in the Bank today to receive $1000 in one year if interest rate is 5% per annum?

Denote the unknown present amount as X. Then,

$$X * (1.05) = \$1,000$$

$$X = \frac{1,000}{1.05} = \$952.38$$

Problem 1.7 What if you need $1000 in two years?

$$X * (1.05)^2 = \$1,000$$

$$X = \frac{1,000}{1.05^2} = \$907.03$$

Problem 1.8 What if you need $5000 in three years?

$$X * (1.05)^3 = \$5,000$$

$$X = \frac{5,000}{1.05^3} = \$4,319.19$$

As I hope you see by now, the general formula, given the same definitions and notations as before, is the following:

$$PV = \frac{FV}{(1+r)^T} \qquad \textit{(1.5)}$$

When we divide Future Value by the factor $(1+r)^T$, we are doing what is called **discounting**. Therefore, Present Value is discounted Future Value. To get Present Value of any future cash flow we need to discount it. In this context, and in all calculations of present value, *r* is referred to as **discount rate**.

Again, T is the number of periods <u>between</u> the time when present value is given and the time when future value is given. The beginning and the end can be arbitrary; what matters are their positions relative to each other.

In this last example, we can say that $5,000 received in three years is the same thing as $4,319.19 received right now and that you would be indifferent between these two amounts. Even though they look different, the fact that they apply to different points in time makes them equivalent. One is just the present value of the other!

By the way, what happens when $T = 1$?

The formula (1.4) for future value becomes $FV = PV * (1+r)$, and

$r = \frac{FV}{PV} - 1$, or $r = \frac{(FV-PV)}{PV}$

Please go to section 1.1, check out formula for Holding Period Return (1.1), and tell me what difference you notice between that formula and the formula for discount rate r when T=1. For all purposes, there is no difference whatsoever! **When there is only one period T=1, then holding period return and discount rate are one and the same thing!**

Do you remember my initial example investing X, earning a return 13.5% and ending up with a value of $113.5? We solved $X = 100$. Guess what: X is just the present value of $113.5, discounted back one year at the rate 13.5%. Check: $\frac{\$113.5}{(1+0.135)} = \$100$. End of proof. Remember you promised me to learn this much.

To sum up this little section, a dollar today is not equal to a dollar tomorrow. This is the fundamental principle of time value of money. Essentially, time value of money gives us a “time machine” in which we can travel along time and evaluate any amount of money at any time in terms of the equivalent amount at some other time! The self-check problems at the end of this chapter will give you plenty of wild rides in the time machine!

1.4 Lump Sums: Non-Annual Compounding

In all the examples above, “period” was equal to a year and “periodic rate” was given per year. We call this scenario annual compounding. However, “period” may be equal to a quarter, a month, a week, or even a single day; and that, correspondingly, “periodic rate” may be per quarter, per month, and so on. In that case we will talk about quarterly (monthly, weekly, daily) compounding instead of annual. This more frequent compounding is often more representative of reality than annual compounding is, as in many financial products the rate compounds more frequently.

Problem 1.9 Suppose I deposit \$1 in a Bank for one year, but the Bank pays me interest monthly. Suppose I do not withdraw the interest but reinvest it every month. How much money will I have at year-end?

Periodic rate per month is $\frac{5\%}{12} = 0.4167\%$. With interest compounded monthly at the periodic rate of 0.4167% per month over 12 months, the ending lump sum will be

$$\$1 * \left(1 + \frac{0.05}{12}\right)^{12} = \$1.051162$$

Problem 1.10 What if I deposit \$634.59 for four years with interest reinvested monthly? (Assume same interest 5% per year.)

This is equivalent to depositing the money with the $\frac{5\%}{12}$ interest monthly, for $4 * 12 = 48$ months.

$$\$634.59 * \left(1 + \frac{0.05}{12}\right)^{4*12} = \$774.77$$

Problem 1.11 What if I deposit \$237.51 for three years with interest paid quarterly?

This is equivalent to depositing the money with the $\frac{5\%}{4}$ interest quarterly, for $3 * 4 = 12$ quarters

$$\$237.51 * \left(1 + \frac{0.05}{4}\right)^{3*4} = \$275.69$$

Problem 1.12 How much money do you need to deposit in a Bank today to receive \$1000 in one year with the rate of 4.5% compounded monthly?

$$X * \left(1 + \frac{0.045}{12}\right)^{12} = \$1{,}000$$

$$X = \frac{1{,}000}{\left(1+\frac{0.045}{12}\right)^{12}} = \$956.08$$

Problem 1.13 What if you need $6000 in three years and the rate is 3.1% per annum compounded monthly?

Continuing with monthly compounding,

$$X * \left(1+\frac{0.031}{12}\right)^{3*12} = 6{,}000$$

$$X = \frac{6{,}000}{\left(1+\frac{0.031}{12}\right)^{3*12}} = \$5{,}467.82$$

With more frequent than annual compounding, the formula for FV and PV of a lump sum (remember we're still talking about lump sum!) become:

$$FV = PV * \left(1+\frac{SR}{m}\right)^{n*m} \qquad (1.6)$$

$$PV = \frac{FV}{\left(1+\frac{SR}{m}\right)^{n*m}} \qquad (1.7)$$

Above, SR is the **Stated Rate**, which is always given per year (e.g., 3.1% in the most recent example above). Sometimes you may see the name **Annual Percentage Rate** for it, also. Additionally, ***m*** is the ***compounding frequency*** (number of compounding periods per year). For example, $m = 12$ with monthly compounding, $m = 4$ with quarterly compounding, etc. Finally, n stands for the number of years.

Okay, now you can go back and forth solving for future value and present value of a lump sum, whether we compound annually or more frequently.

1.5 Effective Annual Rate

You and a friend have investment accounts. Yours compounds monthly at 5% stated rate, and your friend's compounds quarterly at 6% stated rate. Which is a better investment: yours or your friend's?

At first, you may just look at the stated rate and decide that your friend is earning more. However, the two investments are not directly comparable because of the difference in compounding frequencies. We need some common yardstick before we can tell which is better. This common yardstick is called Effective Annual Rate, or *EAR*.

Problem 1.14 Put $1 in a Bank with Stated Rate = 5% for one year with monthly compounding. Forgetting about the compounding details, what interest have I *effectively* earned on my $1 during the year?

Let's begin with figuring out the ending value. We already solved for that in Problem 1.9:

$$\$1 * \left(1 + \frac{0.05}{12}\right)^{12} = \$1.051162$$

But we're asked to forget about the compounding details! The rate you effectively earned on your investment over the year is simply your annual holding period return, which was defined earlier at the start of this chapter.

Well, effectively, if you put in $1 at the start of the year, and then withdraw $1.051162 at the end, you have made a holding period return over the year of:

$$\frac{(\$1.051162 - \$1)}{\$1} = 5.1162\%$$

That holding period return is called **Effective Annual Rate (EAR).**

Clearly, in this example, EAR > Stated Rate. This is due to the power of compounding! The general rule is, **if compounding is more frequent than annual, EAR > SR. If compounding is annual, then EAR = SR**.

General formula for EAR is

$$EAR = \left(1 + \frac{SR}{m}\right)^{m} - 1 \qquad \textit{(1.8)}$$

Above, m is the compounding frequency, just like before. Notice that EAR depends on two inputs only: SR and m, and doesn't depend on anything else, including the amount invested and the number of years in the account.

Problem 1.15 Suppose you put $1 in a Bank for 2 years and 3 months. SR=5%. Compounding is quarterly. Find EAR.

$SR = 5\%\ , m = 4$

$$EAR = (1 + \frac{0.05}{4})^4 - 1 = 5.0945\%$$

Problem 1.16 How about keeping everything as in the previous problem but changing the investment horizon to 9 months and the initial deposit $590.68?

Solution: Same as above. Recall that EAR does NOT depend on either the length of the investment horizon or the dollar amount of the deposit!

Why is it important to consider EAR? Investments with different compounding intervals provide different effective returns. To compare investments with different compounding intervals, you must look at their effective returns. Let's go back to the example that started this section. Which investment is better: yours, earning 5% stated rate with monthly compounding, or your friend's, earning 6% with quarterly compounding?

For your investment,

$$EAR = (1 + \frac{0.05}{12})^{12} - 1 = 5.1162\%$$

For your friend,

$$EAR = (1 + \frac{0.06}{4})^{4} - 1 = 6.1364\%$$

Now that you compared apples to apples, you can indeed state that your friend is earning a higher rate.

Example You may have received a credit card solicitation in the mail or seen one. They would offer you to open a credit card account under the terms that the balance on the account (up to your credit limit) will carry a certain interest rate, and that, unless you pay off your balance in full at the end of the month, interest will be charged. Consumer credit is risky and credit cards normally carry a high rate. Suppose you receive an offer with a 25% stated rate (APR). If you need to be making monthly payments on any outstanding balances, what's the EAR that your account carries?

$$EAR = (1 + \frac{0.25}{12})^{12} - 1 = 28.07\%$$

Remember, your Effective Annual Rate is higher than the stated rate! Keep that in mind when signing up for that credit card.

1.6 Continuous Compounding

Until now, time was discrete, i.e. we were able to measure it in units (one unit of time could be a year, or it could be a day, but the point is those units were defined). Now let's imagine time as continuous flow. In reality time flows through our fingers like beach sand, and just because we choose to measure it in hours, minutes, and seconds, doesn't change that material property of time as a flow. The difference between continuous and discrete compounding is that under discrete compounding interest is added up on the principal at specific intervals, while under continuous compounding interest is constantly added up.

Continuous compounding, quite obviously, is the fastest of all and results in the greatest appreciation of the present amount because interest compounds faster than we can blink!

To approach continuous compounding let us first take the familiar formulae for future value and present value:

$$FV = PV * \left(1 + \frac{SR}{m}\right)^{n*m}$$

$$PV = \frac{FV}{\left(1 + \frac{SR}{m}\right)^{n*m}}$$

We know that in these formulae, m is the compounding frequency (number of compounding periods per year). What if we made m higher and higher, so that m → ∞? In the limit, the formula becomes the following:

$$FV = PV * e^{r*T} \qquad (1.9)$$

$$PV = FV * e^{-r*T} \qquad (1.10)$$

In this formula, *r* is the rate per year and *T* is the number of years.

Continuous compounding is very handy.

Problem 1.17 Suppose I invest $10,000 and it compounds continuously for 93 days, with annual rate of return 10%. With continuous compounding, what will my money become after 93 days?

To use the continuous compounding formula, all you need to do is convert days into years. The number of days, 93, is equal to $\frac{93}{365} = 0.25479$ years. We are ready to use the formula:

$$FV = \$10{,}000 * e^{(0.1*0.254795)} = \$10{,}258.07$$

Similarly, the formula for present value applies when the future amount is known, and we need to solve for the required amount today. You will see a few practice problems at the end of this chapter.

1.7 Solving for the rate of return

I promised you at the beginning of this book that you will learn how to solve for the rate of return, and I always try my best to keep my word! So, let's learn how to solve for the rate of return in case of lump sums.

Example Suppose I invest $100,000 for thirty years with annual compounding, and my target amount is $750,000 at the end. What should the rate be per year, to turn a mere $100,000 today into $750,000 after thirty years of compounding?

We know that in general, with annual compounding,

$$PV = \frac{FV}{(1+r)^T}$$

In this case, you probably figured that $PV = 100{,}000$ and $FV = 750{,}000$, and that $T = 3$. We just need to solve for r!

$$r = (\frac{FV}{PV})^{\frac{1}{T}} - 1 \qquad (1.11)$$

Solving,

$$r = (\frac{750{,}000}{100{,}000})^{\frac{1}{30}} - 1 = 0.06947 = 6.947\%\, per\ year$$

If your investment earns on average this return per year, you will achieve your investment objective.

What if compounding is more frequent, say monthly? That's better because the annual rate doesn't need to be quite so high for you to meet your target: the power of compounding will help!

If compounding happens m times per year, we know that $PV = \frac{FV}{\left(1+\frac{SR}{m}\right)^{n*m}}$

Solving for *r* (which is the same as stated rate) per year will yield the following solution:

$$r = \{(\frac{FV}{PV})^{\frac{1}{n*m}} - 1\} * m \qquad (1.12)$$

Let's apply the formula to your investment problem for $m = 12$ (monthly compounding).

$$r = \{(\frac{\$750{,}000}{\$100{,}000})^{\frac{1}{(30*12)}} - 1\} * 12 = 0.06735 = 6.735\%\, per\ year$$

Notice that the result confirms our intuition: the annual rate need not be as high as before because frequent compounding will make up for the lower annual rate to still get you to meet your investment target.

Lastly, how about continuous compounding? In that case, the formula for annual rate becomes:

$$r = \frac{1}{T} * \ln(\frac{FV}{PV}) \qquad (1.13)$$

With your investment compounded continuously, an even lower annual rate would be enough compared to monthly compounding, not to mention annual. Let's verify:

$$r = \frac{1}{30} * \ln\left(\frac{\$750{,}000}{\$100{,}000}\right) = 0.06716 = 6.716\%$$

That's the lowest it can get, in order to meet your investment objective. That will apply if compounding is the most powerful of all: continuous. For all other types of compounding, the annual rate would need to be higher, as you saw from the example above.

1.8 Solving for T

What if you know your target amount and the rate of return – but you are trying to figure out how long it will take for you to make it to the target?

Example You put in $100,000 and the rate is 5% per year. How many years (with annual compounding) will it take your investment to double?

With annual compounding, we can take the formula for PV (or FV, whichever you like) and solve for T:

$$T = Log_{(1+r)} * (\frac{FV}{PV}) \qquad (1.14)$$

Solving, we realize that money "doubling" means $\frac{FV}{PV} = 2.$

$$T = Log_{1.05} * (2) = 14.21$$

If compounding is more frequent than annual, then we can still solve using the same formula as above, but the rate should be the periodic rate, and the solution will be the number of compounding intervals and will need to be converted into years.

For example, compounding is monthly and annual rate is 5%. Then periodic rate is
$\frac{5\%}{12} = 0.4167\%$

$$T = Log_{1.004167} * (2) = 166.7$$

The answer above is in months. That's 13.89 years: shorter time than if compounding were annual. Again, as you let the power of compounding work for you, the time needed to double your investment becomes shorter.

Finally, what if compounding was continuous? Then the formula below will do the trick, and the good news is that T will already come out in years, so no additional conversion will be required.

$$T = \frac{1}{r} * \ln(\frac{FV}{PV}) \qquad (1.15)$$

Plugging in the numbers from our example,

$$T = \frac{1}{0.05} * \ln(2) = 13.86\ years$$

Look: as compounding gets progressively more and more frequent, all the way to continuous, the time necessary to double your investment under a given annual rate becomes progressively shorter. That's another way to see the power of compounding!

1.9 Next Chapter: Sneak Peek

Let us take account of what we have covered so far. We have covered a lot of ground. You learned the concepts of present value and future value, compounding, discounting, the effect of compounding frequency, and effective annual rate. That is a lot indeed, and you deserve an accolade for the hard work you have put into studying this chapter and working with me through the sample problems. Before taking the well-deserved coffee break, however, allow me to have your attention for a bit more to give you a preview of what's coming next.

Until now, we have just talked about simple examples such as bank deposits, and we assumed away any risk to keep things simple. By now, you probably associate "time value of money" with "riskless interest rate on a bank deposit." The purpose of the next chapter is to demonstrate that the concepts of compounding and discounting, present value and future value, are much broader and apply in the context of every financial decision. To expand our library of financial decisions, however, we will have to work through examples that involve risk because financial decisions do involve some risk after all. What would change in all the above examples if we turned risk on and allowed for these deposits, investments, etc. to be risky, which they are in real life? Also, what if we talked not about a bank deposit, but say about a risky venture such as investment in solar panels?

Okay, now is a good time to take a break from this book. (And something tells me this won't be your first.)

Chapter 1 Self-Check Problems

1. The EAR of a monthly-compounded investment with an APR of 22% is closest to which of the following?

a. 22%
b. 24%
c. 26%
d. 28%

Answer: b

2. You put $200,000 into an investment account earning an average annual rate of return of 8% and leave the money there for 10 years. What is the expected value of your account at the end of your investment horizon?

Answer: 431,785

3. You put $100,000 in a retirement account, make no additional contributions after that, and end up with $482,770 after twenty-five years.

a. What is the average annual return you have earned on your account, assuming annual compounding?

Answer: 6.5%

b. What if compounding was monthly instead?

Answer: 6.31%

c. What if compounding was continuous instead?

Answer: 6.29%

4. What's the present value of $5,000 received in three years, if $APR = 7\%$ and compounding is quarterly?

Answer: $4,060.29

5. You are considering the purchase of a really nice home *in three years*. You have $100,000 in your bank savings account today but would rather wait to buy the house. You plan to leave the money there for three years under 4% APR with *monthly* compounding. How much money will you have *in your bank savings account* in three years?

Answer: $112,727.19

6. You plan to buy a house in five years. The house of your dreams costs $200,000 today but you expect real estate prices to appreciate at a rate of 3% annually. What is the expected price of the house in five years?
Answer: $231,854.81

7. Suppose you will inherit $1,000,000 in 25 years and will put that amount in an investment account earning 6% average annual return, for 10 years. How much will you have in your investment account at the end of that 10-year time period?

Answer: $1,790,848

8. If a lump sum of $500.00 is available $t = 8$, find the Future Value of that amount as of $t = 16$ under the interest rate of 3% per year with monthly compounding.

Answer: $635.43

9. Suppose you put $1 in a Bank for 2 years and 3 months. APR=5%. Compounding is quarterly.

a. Find EAR.

Answer: 5.0945%

b. How about 9 months deposit time?

Answer: Same as before

c. How about a $590.68 deposit?

Answer: Same as before

10. If the stated rate is 10% per annum, how long will it take for your money to grow from $5,000 to $15,000?

a. Assuming annual compounding?

Answer: 11.53 years

b. What if compounding was monthly?

Answer: 11.03 years

c. What if compounding was continuous?

Answer: 10.99 years

11. Today is the end of 2019. You are an entrepreneur looking to fund your enterprise. You hope to secure the funds by the end of 2020. You plan to run the business for 10 years and sell it at the end of 2030. If the expected sale price is $1,000,000 and you will give it all back to investors, how much money will they provide in 2020 assuming they want a 20% annual return?

Answer: $161,505.60

12. Find the amount needed to deposit in the bank today if stated rate is 3%, compounding is monthly, and you want to have $10,000 in your account after three years.

Answer: $4,110.94

13. You inherited $50,000 five years ago and spent it all. Now, five years later, you regret your choice. You say, "I wish I had invested at least half of that money back then! Money market investments provide a safe return of 2% per year!" So, if you had invested half of the money five years ago at 2% annually, how much would you have saved by now?

Answer: $27,602.02

14. You buy one stock of company XYZ for $250. If you sell the stock for $300 one year later, what is your holding period return?

Answer: 20%

15.If you buy a share of stock of company XYZ today and sell it for $56 next year, how much must you have paid today knowing that you made a 7% return on your investment?

Answer: $52.34

16. In your own words, explain the relationship between "return" and "interest."

17. When you earn interest on a riskless bank deposit, what does this return reward you for?

18. Find future value of $100 deposited for 80 days into an account that earns 5% per annum, with continuous compounding.

Answer: $101.10

19. You buy a Treasury bill, which promises to pay you $10,000 in 78 days. If the annual yield on this bill is 3.5%, and compounding is continuous, what is the price of the bill today?
Answer: $9,925.48

20. Increasing compounding frequency does which of the following?

a. Shortens the time horizon needed to achieve your investment goal, all else equal
b. Reduces the stated rate necessary to achieve your investment goal, all else equal
c. Increases effective annual rate EAR
d. All of the above

21. You and a friend are making monthly contributions into two different investment funds. You contribute $200 per month and your friend contributes $300 per month. You plan to invest for the next two years, and your friend plans to invest for the next year and a half. Both funds compound monthly interest at 8% APR. Based on this information, your EAR is ________ your friend's.

a. higher than
b. the same as
c. lower than

CHAPTER 2: EXPANDING THE LIBRARY

"Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things." – Isaac Newton

2.1 Risk and Return

Real-life financial decisions, with rare exceptions, involve risk. Remember solar panels? The concepts learned in the previous chapter apply in a much broader universe than just riskless transactions. Therefore, in order to expand our library of financial applications of those concepts, we first must discuss financial risk and its impact on how we think about a financial problem.

What is risk? Being an author of a book, you may expect from me *the one and only correct* definition. But I would like for you to think of your own definition instead. Risk is intuitively known by all of us. Risk is whatever your intuition likely screams at you when it is trying to tell you to be careful. At this point, I invite you to stop and think – and write down – what *you think* risk is.

If I were to make a guess, it would be that most of you defined risk in connection with something bad or undesirable. Some of you may identify risk with thrill or excitement, but still, risk is when something bad can happen. The precise definition of risk may vary but that's the theme.

In finance, risk usually refers to the possibility of a financial loss. A financial undertaking is, therefore, risky if there is a chance that it would lose you money. In very few instances you can think of a (nearly) loss-proof proposition. (And of those few, 99% are not particularly thrilling; they are boring things, like putting a dollar in an FDIC insured large bank and earning a meager return.)

With risk turned on, two things would change. First, instead of saying "future amount of money" we must now say "***expected*** future amount of money." With risk, we can never be certain, right? So, we must clarify that we are talking about risky projects, products, and so on.

Second, the ***rate of return must go up*** if risk is present; as we know intuitively, higher risk means higher reward. We already know from Chapter 1 that the proper measure of investment reward is the rate of return. The rate now must compensate us not just for the patience but also for the risk! "High risk, high return" is a phrase you have undoubtedly heard before.

However, the "higher risk – higher return" principle is trickier than many might think. Here is how you maybe think about that. I invest $1 into some project. If it's risk-free, I may want a couple cents back in return ($1.02 payoff). However, if it is risky, I want higher return and I'll want 20 cents back ($1.20 payoff). In other words, you are thinking about return in terms of what you are receiving. The purpose of this section is to challenge this conventional thinking and to convince you that in finance, we think about return more in terms of what's paid, rather than what's received.

What??? Doesn't the very word, "return" imply "what is coming back to me?"

Example Imagine you are an entrepreneur with a brilliant idea. To make the idea work, all you need is a machine that will produce your new amazing product. The machine is able to produce 100,000 units, and your marketing research suggests that the profit-maximizing price will be $15 per unit. Variable cost of production is $5 per unit. In addition, you will incur fixed costs of $250,000 for overhead and other similar expenses. You think that your idea won't be profitable for more than one year though: after one year, the novelty of your product will wear off, and no one will want it anymore. You need to raise money to buy the machine.

If all goes as expected, you anticipate generating 100,000 ($15 - $5) - $250,000 = $750,000 profit next year. However, things can potentially go wrong. The machine may break down (albeit with a very low probability) and you would incur additional expenses, not to mention lost production. The estimated price may prove too high and you may need to lower it. Overhead can run over the estimate. And so on: the point is that $750,000 is *expected* profit, but not guaranteed profit. Yours is a **risky** enterprise.

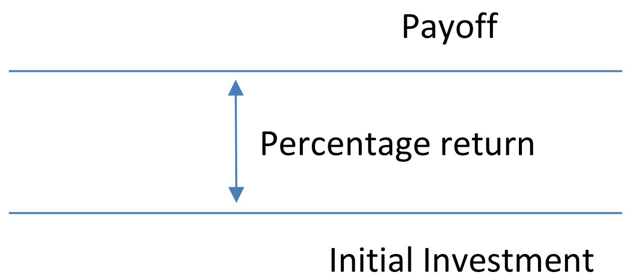
To raise money for the machine, you approach wealthy investors. You figure that they receive similar propositions from many entrepreneurs and invest in various enterprises across different industries, so they are not overly concerned about the risk of your enterprise because they wouldn't put all of their money in just one basket: they are well diversified. If one enterprise fails, another one will succeed, and they care about their overall portfolio of investments rather than just one enterprise.

Your gut feeling is correct: investors are diversified and do not care about the risk of your enterprise the same way you would. However, no matter how diversified they are, the fact remains that there is *some* risk to them. After reading your business plan carefully and conducting due diligence with the help of qualified experts, and after seeing how adding your enterprise would impact the risk of their overall portfolio of investments, they conclude that there is enough risk carried by your enterprise that the fair rate of return of 10% will compensate them sufficiently for this risk.

Suppose you are told that the machine is for sale at $700,000. Will you be able to raise that amount from the investors? Let's think carefully. If investors give you $700,000 but they want 10% return, that means they will require a payoff of $\$700{,}000 + 10\% * \$700{,}000 = \$770{,}000$ next year. Do you expect to be able to provide them with that payoff? No, you do not. Your expected number is $750,000, remember? The machine has its limitations, and you face various costs, and consumers are moody, and the like. You cannot, with a straight face, tell investors that they can expect $770,000 back! How can they get the 10% return then?

Well, as the saying goes, if the mountain will not come to Muhammad, then Muhammad must go to the mountain. There is another way that investors can get themselves an expected rate of return of 10%. If the expected payoff is stuck at $750,000 and there is nothing that they can do about it, then ***there is*** an amount that they ***can*** do something about. That amount is how much they are willing to give you in the first place. That amount is fully under their control!

Remember that "return" depends on not only what's coming, but also what's being invested in the first place! Both of those amounts matter. To leave yourself room for return, you either raise the payoff; or lower today's investment! In our example, if the former is not feasible, the latter is quite within investors' reach!



Refer to the diagram above. It is generic but it shows that return will be higher if either (1) payoff is higher, or (2) initial investment is lower, or (3) both. If option (1) is not under the control of investors, then option (2) is fully under their control. That's the option they will use.

In the real world, is it true that investors really have no control over the profits and cash flows coming from the enterprise? We know shareholders of a company can vote to influence managerial decisions. Yes, to some degree they can exercise monitoring and control. However, we must first note that monitoring and control are costly to investors. In addition, the impact of uncontrollable factors that can potentially affect future payoff is too high relative to the impact that shareholder voting can have on the outcome. Bottom line is that there is a limit to how much investors can use option (1). In reality after exhausting their means to control the future outcome, ultimately it is the second option – control how much is being paid today – that investors use to give themselves the expected return they deem fair in compensation for the risk.

If their expected payoff next year is \$750,000 and they want a return of 10%, how much money will your investors provide to you today? The amount X must be such that the payoff of \$750,000 will pay that initial investment back plus 10% on top of that.

$$\$750{,}000 = X + 10\% X$$

Solving,

$$X = \$750{,}000 / (1.1) = \$681{,}818$$

We see that X is found just like the present value of a lump sum! Recall the formula for present value of a lump sum,

$$PV = FV / (1 + r)T$$

In this case, $FV = \$750{,}000$; $r = 0.1 = 10\%$; and $T = 1$ period.

This similarity should come as no surprise because that's exactly what it is.

Investors will be willing to provide you the amount equal to the present value of the future expected payoff, where the discount rate is equal to the rate of return they are requiring in compensation for the risk.

In general, once I know the rate of return that I want for the risk, I will use that rate of return as my discount rate to find present value: the amount I'm comfortable paying now.

One remaining question is what if the machine is more expensive and the funds secured from investors are not enough to purchase it? The answer is that supply meets demand. Imagine that you are not the only one buying this machine. There are other potential users: Joe, Jane, Jack, Jill, and whoever else, trying to raise funds to buy a machine like that for their projects. If they go to raise funds and investors won't give them a penny over $681,818 *given the risk* of the project for which the machine is being used, then that would have to become the price for that machine! So, the price will converge to the present value of the future payoff that the machine is expected to generate, with discount rate adjusted to reflect the risk of the enterprise.

This is a general rule in finance, and you will see it again many times later:

Price of an asset (a project, a stock, a bond, anything) is equal to the present value of the future expected cash flows, with the discount rate set to reflect the risk of that asset.

Another question you may have is how does the entrepreneur make sure that the return is delivered to the investors? Well, by working hard enough that they receive their expected cash flow! If you said you expect to deliver $750,000 cash flow for the $681,818 investment, you had better do just that!

An important takeaway from this section is that the risk-return tradeoff mechanism works through impacting the price today. Higher risk implies higher required return, which in turn implies lower price paid in exchange for the expected future payoff. We don't secure ourselves a higher return by pushing the entrepreneur to work harder than is reasonable for our money and delivering higher future payoff than can realistically be expected after we have taken every reasonable effort and precaution to control that future outcome. Rather, we make return higher by paying less for the claim to the expected future cash flow.

In this sense, there is nothing conceptually different between "required rate of return" and "discount rate." Higher risk implies higher discount rate, and then of course the price becomes lower (as it must when the discount is larger). In the discussion of real-life risky financial decisions that follows, I encourage you to always think about the discount rate as the rate of return that someone is requiring to be earned on the transaction.

One more thing remains to be clarified. You may have heard accounting terms such as return on equity (ROE), return on assets (ROA), or return on investment (ROI). If you have not yet, you undoubtedly will soon enough if you continue to study business disciplines. Are those conceptually different from the return we are talking about? Yes, they are. The difference is that we normally talk about those returns in accounting sense, i.e. when we want to measure the *actual* realized return over a *past* period. For example, suppose your investors gave you $681,818 for the expected payoff of $750,000. Their *expected* return, as they evaluate *the future*, is 10% as we already know. However, after one year, your actual payoff may end up different from the initial expectation, right? That means you may deliver an amount higher or lower than expected, depending on how things work out. Suppose things went better than expected and your actual cash flow ends up being $800,000! Then you can calculate return on investment $ROI = (800{,}000 - 681{,}818)/681{,}818 = 17.33\%$ which is of course very different from the expected 10%. If you have no debt and your enterprise is financed entirely by equity, then your return on equity ROE and your return on assets ROA will also be 17.33%. But that's not the return that investors required! They only required 10%. Whatever you may potentially deliver on top of that (or any potential shortfall) is something that only becomes known with time. The return that properly compensates your investors for the risk is still only 10%. Therefore, they will use 10% to determine just how much they are willing to give you today for the expected cash flow.

In all problems that are presented in this book, I want you to think about the rate of return ***r*** as being the same as discount rate. In other words, whenever you spot "***r***" in the formula, I want you to remember that it is the 10%, not the 17.33%. It is determined based on risk and future expectations, not based on the actual cash flows.

Finally, you may be curious about how we came up with the number 10% at first place. The accounting measures of return such as ROE are very easy to calculate. We know the initial investment and the resulting payoff, and here we go – we can calculate return. But when we came up with the 10% required return that investors wanted for the risk, we didn't know the present investment, did we? In fact, we found the present investment $681,818 *after* being given the required return! (If you forgot, please go back to the example and see for yourself that the number 10% came out of thin air.) Where does the number come from in real life? Surely it doesn't come out of thin air! (Or so we hope!)

If you are curious about this question, congratulations! You are hitting on one of the most important – and troubling – questions in all of finance. We know intuitively that the rate 10% must compensate investors for the risk in our little example. But that's about all we know. How do we know that the proper rate is 10% and not 11%? How do we measure risk? How do we know what rate we should ask for, per unit of risk? At this point, let me just say that these questions are beyond the scope of this book and belong in far more advanced texts. If you continue to study finance, you will undoubtedly see models that measure risk and then relate it to required return. **Look out for the chapter in your main course textbook, titled "Risk and Return." That chapter begins to explain where the 10% comes from**. So, let me leave it at that.

At this point you may be thinking, "Will this stuff be on the exam?" I cannot tell you that, it depends on your professor. However, I can tell you that if you want to know anything at all about finance, you need to know the causal relationship between risk, return, and price, a sketch of which is presented above. Exam or not, it will ultimately work in your favor in that job interview and in your finance career!

2.2 Multiple Cash Flows

So far, you have learned how to find Present Value and Future value of a lump sum, either with annual or with more frequent compounding of interest. However, financial decisions and scenarios normally involve much more complex cash flow structures than just a one-time lump sum payment. For example, investing in a bond will yield you regular interest payments (called "coupons") paid by the bond issuer, so that one investment generates multiple sums of money instead of just one. Similarly, investing in a stock may result in receiving regular dividends from the company, which again means multiple sums of money instead of just one. When a firm invests in a new project, it normally expects the payoff to occur over some time instead of all at once. Therefore, we next need to learn how to find Present Value and Future Value when it applies to series of payments, or series of cash flows.

Before we learn how to, however, let's think conceptually what exactly we're trying to learn here. ***What is, conceptually, the Present Value of a series of payments?*** With lump sums, the concept is more intuitive: lump sum today is the present value, and the future amount, to which it is equivalent, is the future value. How should we think of present and future value when there are multiple sums of money paid (received) over your time horizon?

Here is a simple example. ***Suppose you put $100 in the bank*** at 5% per year today and added another $100 next year. How much will you have in the account at the end of a two-year period? Note, as we already know, that future value will not just be the sum of those amounts, or $200, because each of those contributions will earn interest! <u>We expect the future amount to be higher</u>. That question is asking for the future value of a series of two cash flows: $100 now and another $100 in one year, where stated rate is 5%.

Another example of future value of a series of cash flows would be ***a retirement plan***. Suppose you open an individual retirement account with some wealth management firm. In order to save for retirement, you will be making regular contributions of $100 out of your paycheck every month for the next 30 years (which is 360 months). How much money will you have saved in your account after 30 years of contributions if your account earns, on average, 8% per year? This question is asking for the future value of a series of 360 cash flows of $100 each with a stated rate of 8%.

What about present value? Suppose you plan your future retirement. You expect to live for 30 years post-retirement, and you would like to withdraw $100,000 per year, every year, for 30 years, from your retirement saving account. Those withdrawals are called "distributions." ***How much wealth will you need to retire with*** in order to meet this goal, if your money earns 5% per year? One may be quick to multiply and offer $100,000 * 30 = $3,000,000 as the answer. But that would be a wrong answer given what we know about time value of money! Since your money earns interest every year, you don't have to retire with so much money to begin with: <u>you can start with less</u> and let compounded interest work for you to grow your leftover amount in the account after every distribution to provide just enough extra income to meet your goal every year, for 30 years, so that at the end your account is depleted. That initial amount that is just enough for you to begin your retirement with would be the present value of a series of 30 annual retirement distributions of $100,000 each, in which stated rate is 5%.

At this point, a very common confusion that many students share is this: why is that amount a present value, rather than a future value, given that my retirement isn't until a future point in time? After all, I'm only still a student now. The answer is the following: the amount is a present value because we're considering a retiree's problem, not a college student's problem. For a retiree, retirement is already the present reality. Alas: everyone's future is someone's present! Whether this fact is for better or worse is beyond the scope of this book.

Or how about ***buying a car*** and financing it fully with a loan? Suppose you plan to pay the loan off over five years, the interest rate stated on the loan is 6% per year, and you figure you can afford a $400 payment each month. How expensive car can you afford today? While this is a more complicated example (in particular, interest is stated per year, but payments will be made per month), what you are essentially looking for is the today's dollar equivalent of a series of future loan payments! The car must have the same value today as all of those future loan payments combined, or else either you will refuse to buy it (if the car is worth less) or the bank will refuse to finance it (if the car is worth more). So, what is the "value of the future loan payments combined"? Is it just the sum of all the future payments, or $400 per month * 5 years * 12 months per year = $24,000? Absolutely not! Remember time value of money and compounded (in this case, monthly) interest! The $400 payment includes interest that you will be charged by the bank! In order for the $400 to accommodate *both* the initially borrowed principal *and* the interest, you can't borrow as much as $24,000: <u>it has to be less</u> to leave room for interest in every monthly payment like you leave room for cream in your Starbucks coffee. Finding that precise amount is finding the present value of a series of 5*12 = 60 monthly payments, $400 each with a stated rate of 6%.

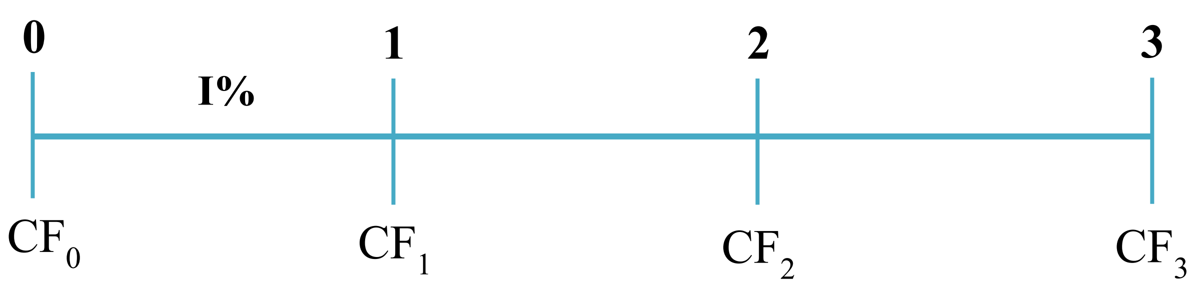
Here is one last example. As an ***entrepreneur***, you dream of starting your own business. You need to raise a certain amount of capital to launch it, which you plan to do three years from now. Once launched, you plan to keep the company for 15 years because you think your entrepreneurial idea will generate you profits for that period but not afterwards. Between the launch and the liquidation, you (as do many beginner entrepreneurs) expect the enterprise to generate positive cash flow every year. Your investors are wary of risk, however, as they have some experience from investing in new ventures in the past, and they know a chance of losing money is substantial. Hence, they agree to provide the initial capital on the condition that it should generate, on average, 20% return per year. You go to the drawing board and figure out, given the size and scope of the planned enterprise, that you would be able to generate about $100,000 per year (on average) after all expenses, for the investors. How much capital can you expect to get from your investors at the time for launching your business if all you can pay them is $100,000 per year but that amount must incorporate a 20% rate of return? Most certainly, they won't give you $100,000 * 15 = $1,500,000! Otherwise, the annual payments will leave no room for return! <u>They will provide less</u>. The exact amount will be the present value of a series of 15 annual cash flows of $100,000 each with a stated rate of 20%.

What is the common thread in all of these examples? It is that in every one of these examples we are looking for a lump sum amount at some point in time (either at the very beginning of the project or at the very end, depending on whether we're looking for present or for future value) that is equivalent to a series of cash flows that occur over a period of time; in other words, such amount that you would be indifferent between *receiving that amount at that time* and *receiving the series of cash flows instead*. Very importantly, note that the rate earned (or paid) on the project (whether it is interest rate on the loan, or the rate of return to investors, or the rate at which your retirement wealth compounds) is always a consideration!

Okay, now that I hope I have conceptually depicted the idea of present and future value of a series of cash flows, let's figure out how to solve for it. To do so, we need the help of timelines. That's right, we will draw illustrations first.

Timelines are very useful tool to visualize the passing of time and the events that take place at different points in time. I strongly encourage – and I cannot stress this point enough – the use of timelines in your (at least initial) problem solving. Cash flow series can be quite complicated and drawing timelines helps to see both the big picture and the detail.

A generic timeline would like something like this:



The numbers on the line mark periods. That is, 0 means "beginning of time horizon", 1 represents "the end of the first period", 2 is "the end of the second period" and so on. The numbers on the scale grow as the time horizon gets longer, and in principle may go all the way to infinity, since we consider time to be endless. Note again that, except 0, all other numbers represent the end of the corresponding period, which might be a year, a quarter, a month, etc.

The abbreviation ***"CF" with a numeric index stands for "cash flow"*** occurring at the corresponding point in time. **From now on, we are going to assume that all cash flows always occur at the end of the corresponding period**. In the diagram above, the first of the future cash flows is paid (received) at the end of period 1, the second at the end of period 2, and so forth.

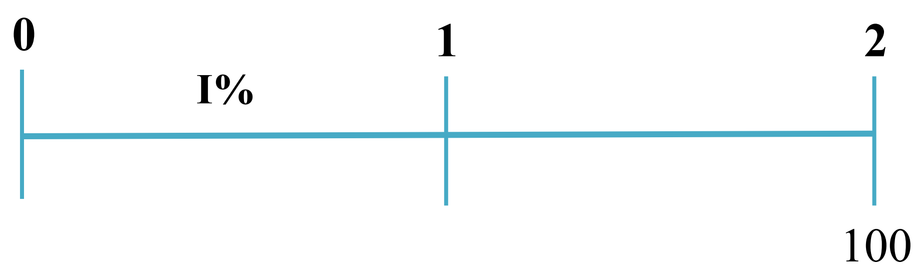
Why do we assume the cash flows to occur *at the end* of the period rather than at some point (or several points) during that period? Quite frankly, we do so partly to simplify our financial analysis because our calculations would be a mess if we accounted for every actual point in time when cash flows are paid or received. However, simplification is not the only reason: many financial securities are structured to make regular payments to investors at the end of a certain period.

For example, bonds pay interest (coupons) to bondholders regularly on specific dates that are spaced in time by regular intervals (e.g., every six months). Companies that pay dividends typically do it on a quarterly basis, again with equal spacing in time. Firms make earnings announcements and release financial statements regularly on a quarterly basis (even though their profits and losses accrue randomly throughout the quarter) so we can think of their cash flows as being periodic. Retirement plans often pay retirees equal installments, or distributions, on a periodic basis (those are called “annuities”). Loans are typically structured so that equal periodic installments are made to pay the loan off. For this reason, it is not only convenient but appropriate to consider cash flows occurring on a regular basis and being spaced evenly through time. There can be, and are, exceptions from this general rule, of course. But we will follow this rule in our examples.

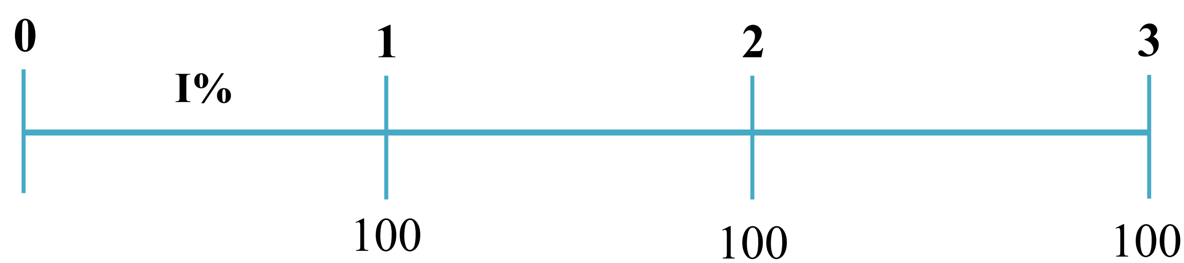
Finally, I% means “periodic rate.”

Let’s consider a few examples of timelines.

A $100 lump sum due in 2 years would look like the diagram below. The number 0 on the line represents the beginning of the first period. The numbers 1 and 2 on the line mark the end of period 1 and 2, respectively. The number 100 at the end of period 2 means we pay a lump sum of 100 at that time. Finally, I% means periodic rate.



A series of three cash flows, $100 each, would look like the diagram below. The number 0 on the line marks the beginning of the first period. The numbers 1, 2, and 3 on the line mark the end of period 1, 2, 3, respectively. The number 100 that appears at each mark means that we pay a lump sum of 100 at the end of period 1; then again at the end of period 2, then again at the end of period 3. Finally, I% means periodic rate.



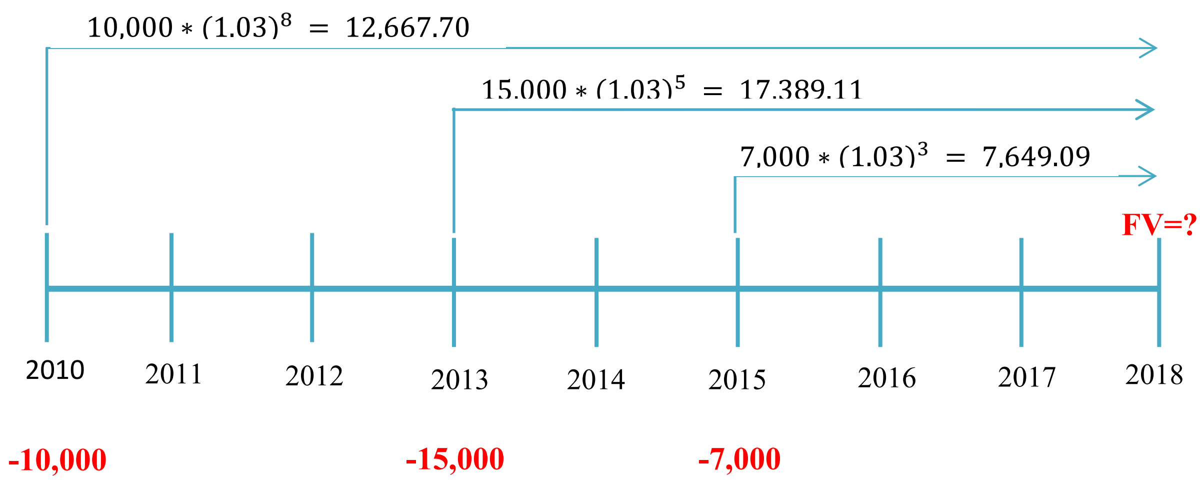
Some uneven cash flow stream, for example an investment of $50 today followed by three cash inflows $100, $75, and $50, would look like the following diagram. The number 0 on the line marks the beginning of the first period. The numbers 1, 2, and 3 on the line mark the end of periods 1, 2, and 3, respectively. Notice the negative amount (-50) at the beginning (at mark 0) as it represents an *outflow* while the three positive amounts (10 at mark 1, 75 at mark 2, and 50 at mark 3) represent *inflows*. In this case, the outflow is the amount of expenses you bear to start an investment and the inflows are the operating cash flows that your investment will generate over time. Finally, I% is the periodic rate.

In what follows, we will work with five main types of cash flow series: uneven cash flows, perpetuity, growing perpetuity, ordinary annuity, and growing ordinary annuity.

2.3 Uneven Cash Flows

Example The year was 2010. Joe decided to start saving towards the purchase of a house that he planned to accomplish in 2018. He put $10,000 into his savings account in 2010. Then, in 2013 he was able to put another $15,000. Finally, in 2015 he managed to save another $7,000. If the savings account is earning 3% interest per year, how much money did Joe end up with by 2018? Hint: it *isn't* $32,000.

We are talking about an uneven cash flow stream:



Given that 2018 is the future point in time relative to these cash flows, the question is really asking to find Future Value of the uneven cash flow stream. Let us realize that we are just dealing with three separate lump sums: the first deposited 8 years prior to the target date ($2018 - 2010 = 8$), the second deposited 5 years prior to the target date ($2018 - 2013 = 5$), and the third deposited 3 years before the target date ($2018 - 2015 = 3$). We already know how to find future value of a lump sum that sits in an account with compounded interest for several years. There is nothing different here, except we have three such lump sums, each sitting in the account for a different length of time. We can find future values of those individually, and then just add them up!

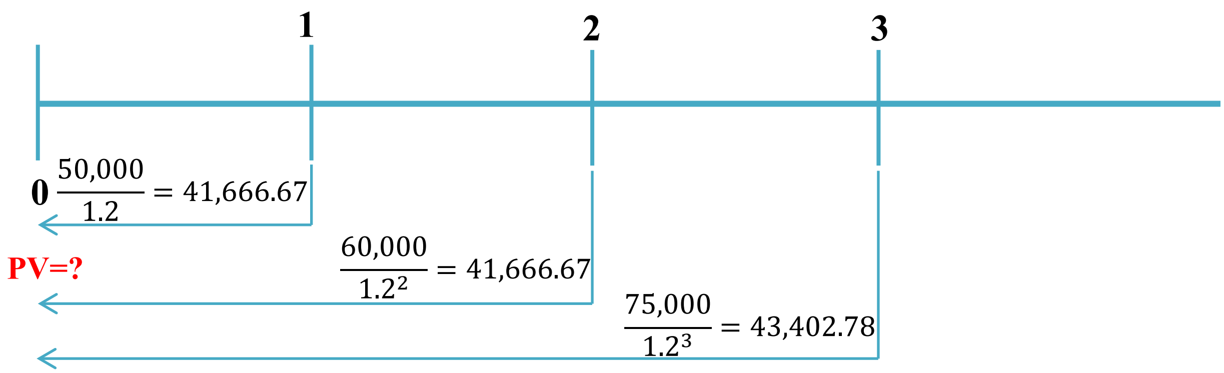
Solving,

FV = \$10,000 $(1.03)^8$ + \$15,000 $(1.03)^5$ + \$7,000 $(1.03)^3$ = \$37,705.90

Future value of uneven cash flow stream is equal to the sum of future values of individual lump sums in the stream.

Something tells me that you already anticipate that my next example will be on present value. If that's right, consider yourself receiving bonus points (from me).

Example I invest in an enterprise and require a high 20% rate of return annually because I believe that the risk is high. The enterprise is expected to generate a cash flow of \$50,000 in year 1, \$60,000 in year 2, and \$75,000 in year three.



What investment is needed from me today?

Note that the target date, “today” precedes the cash flow stream. Hence, we are talking about finding present value. Why is it true by the way that the investment required today is the same concept as the present value of the cash flows? Well, think about it like this. The investment made today is the amount of money out of your pocket that you will be exchanging, today, for the claim to these future cash flows. In other words, you should be indifferent between that amount of money and the amount that the future cash flows are worth in today’s terms. It follows that the investment required today is nothing but the present value of the future cash flows that you’re claiming by making that investment.

Just like the future value of a cash flow stream is equal to the sum of future values of the individual cash flows, guess what? The same is true about present value.

Present Value of uneven cash flow stream is equal to the sum of present values of individual lump sum amounts.

Solving,

$$PV = \frac{\$50,000}{1.2} + \frac{\$60,000}{1.2^2} + \frac{\$75,000}{1.2^3} = \$126,736.10$$

This is really all there is to be said about uneven cash flow streams. Remember that **you cannot ever add up the cash flows themselves if they occur at different points in time. You can add up present values or future values, however.**

Let us practice some more.

Problem 2.1 You put 100,000 in an account to save for the next ten years. Compounding is annual and interest rate is 6% per year. Three years into the investment, you withdraw $30,000 from the account for a home improvement project and leave the rest of the money there until the end. How much money will you have in your account at the end of the ten-year mark?

Let us figure out how much you will have accumulated in the account by the time of the first withdrawal, $t = 3$. That would be

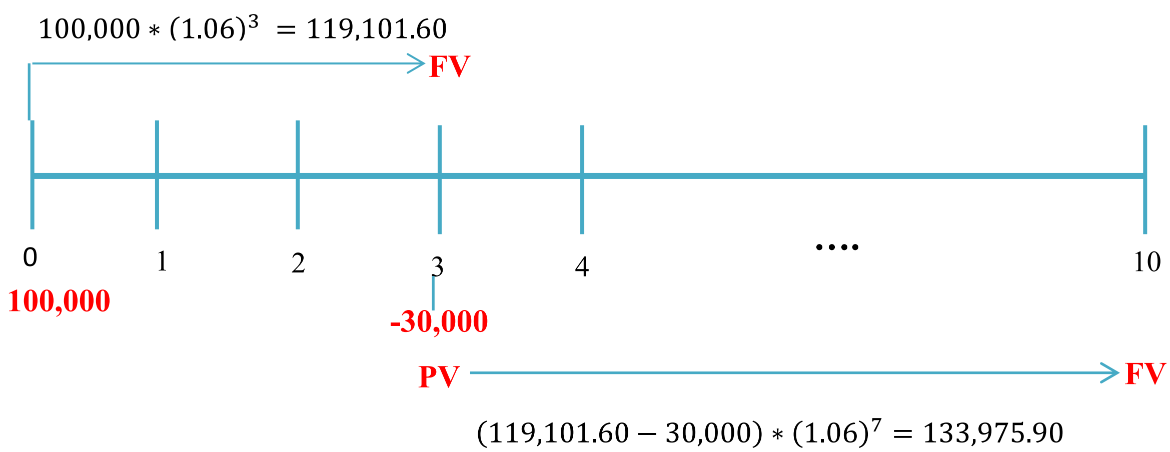
$$100{,}000 * (1.06)^3 = 119{,}101.60$$

At that time, you withdraw 30,000 to be left with

$$119{,}101.60 - 30{,}000 = 89{,}101.60$$

Then you leave the balance in the account for another seven years. Finally, at the $t = 10$ mark, you have

$$89{,}101.60 * (1.06)^7 = 133{,}975.90$$



Problem 2.2 You put \$100,000 in an account to save for the next ten years. Compounding is annual and interest rate is 7% per year. Four years into the investment, you will need to make a withdrawal for a home improvement project. How much can you afford to withdraw at that time if your goal is to have \$150,000 in your account at the end of year 10?

In order to end up with 150,000 at the 10-year mark, how much do you need to start with at $= 4$? Note that I am interested in $t = 4$ because that's when you are making your withdrawal, and I need to know how much you must have left in the account upon that withdrawal in order to still make your ultimate investment objective.

$$\frac{150{,}000}{1.07^6} = 99{,}951.33$$

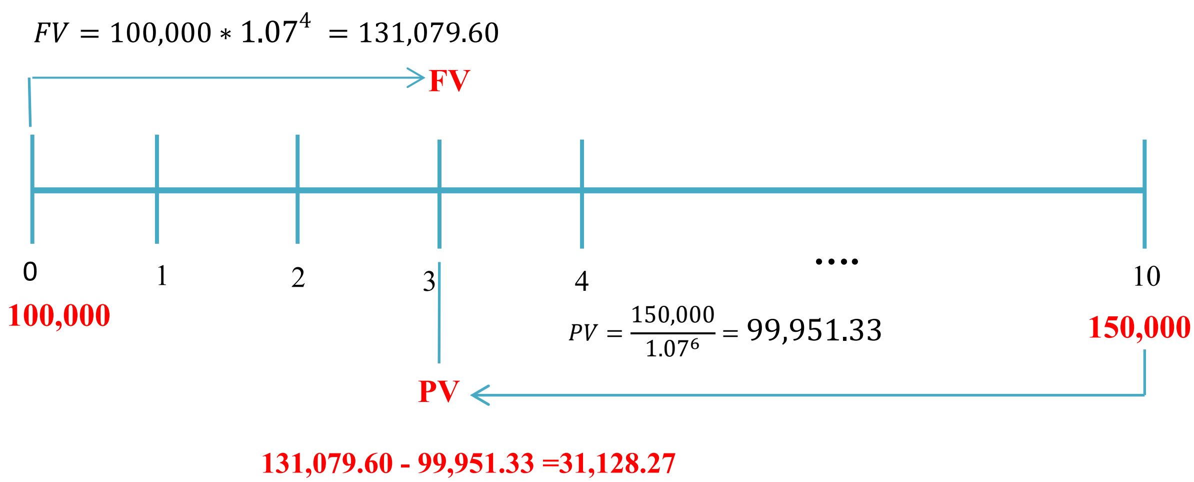
That is how much you must have *after* the withdrawal at $t = 4$ (six years prior to the ultimate 10-year mark).

But how much will you have in the account before the withdrawal at $t = 4$?

$$100{,}000 * 1.07^4 = 131{,}079.60$$

So by $t = 4$ you will have saved 131,079.60, and you can withdraw only enough to still be left with 99,951.33. Therefore, the amount of withdrawal is the difference:

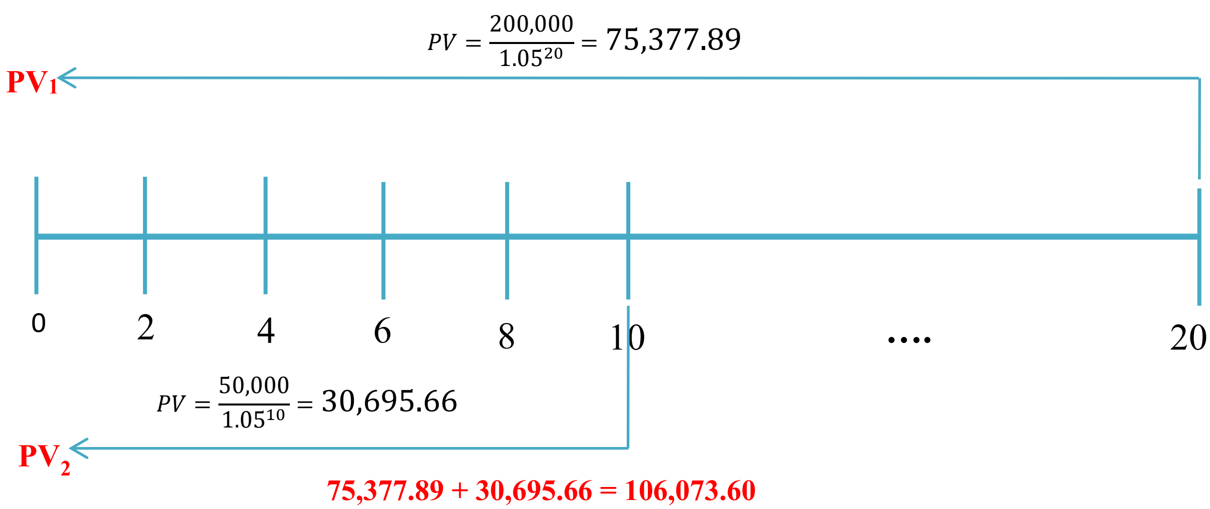
$$131{,}079.60 - 99{,}951.33 = 31{,}128.27$$



Problem 2.3 Your goal is to have \$200,000 in your investment account in 20 years, so you start an investment account today and deposit a certain amount in there. You know that at the 10-year mark however, you will need to withdraw \$50,000 to buy a house. Compounding is annual, and stated annual rate is 5%. How much do you need to deposit today, in order to afford both the planned withdrawal and still to end up with \$200,000 at the end of 20 years?

You need to deposit enough today to achieve two cash flows: \$50,000 at $t = 10$ and another \$200,000 at $t = 20$. Therefore, the amount deposited today must be equal to the sum of the present values of those two cash flows.

$$\frac{\$50{,}000}{1.05^{10}} + \frac{\$200{,}000}{1.05^{20}} = \$75{,}377.89 + \$30{,}695.66 = \$106{,}073.60$$



Another way to see this is to realize that you need to have the following amount at $t = 10$ upon the withdrawal, in order to still accumulate \$200,000 after another ten years:

$$\frac{\$200,000}{1.05^{10}} = \$122,782.70$$

Before the withdrawal, therefore, you will have at $t = 10$:

$$122,782.70 + 50,000 = 172,782.70$$

To accumulate that from the beginning, you should deposit the following amount initially:

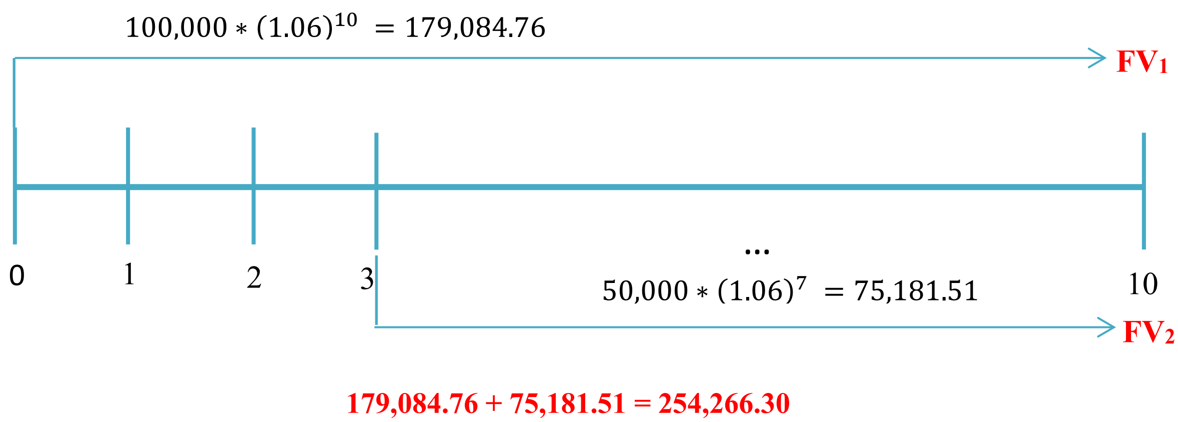
$$\frac{\$172,782.70}{1.05^{10}} = \$106,073.60$$

The same answer!

Problem 2.4 A wealthy alumnus of the XYZ business school donates \$100,000 to the school to start a student managed investment fund, and students invest this money into selected stocks to earn 6% annually, with annual compounding. Three years into the fund's life, the same person donates another \$50,000 to the fund, which gets invested at the same average rate of return. How much money is expected to be in the fund after 10 years from its inception?

This time, we are looking for future value as of $t = 10$, of the two cash flows: 100,000 at $t = 0$, and another 50,000 at $t = 3$. The future value is just the sum of the future values of the individual amounts, as follows:

$$100{,}000 * (1.06)^{10} + 50{,}000 * (1.06)^{7} = 179{,}084.76 + 75{,}181.51 = 254{,}266.30$$



Problem 2.5 Refer to Problem 2.4 above. Over the same ten-year period, the overall stock market happened to grow from 1,800 to 2,600. Did the students beat the market?

This is a problem for Chapter 1, really, because all we need to know is the average annual return earned on the stock market; that we need to then compare to the 6% return that the student investment fund has earned over the same period.

With annual compounding, as per formula 1.11,

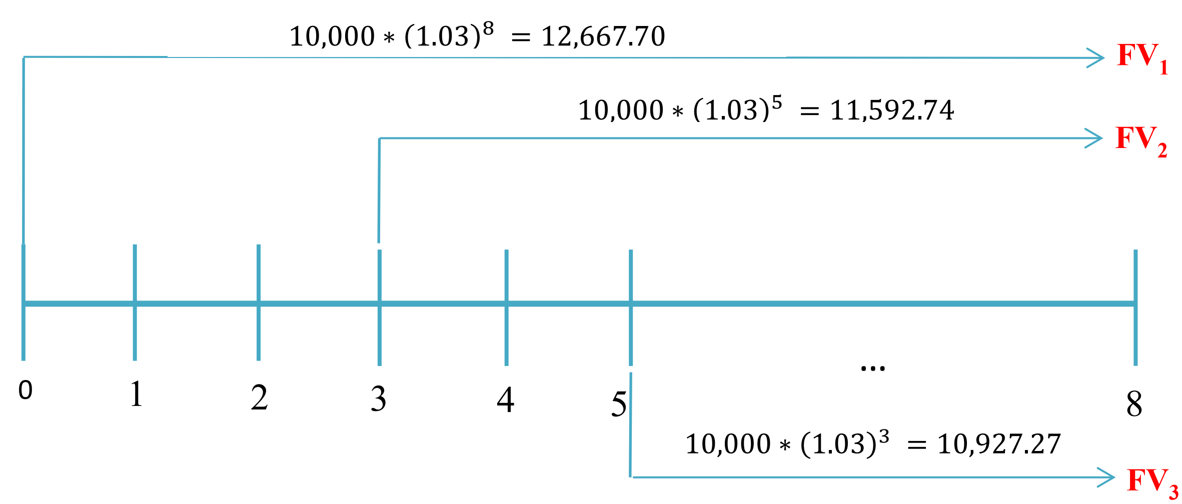
$$r = \left(\frac{2{,}600}{1800}\right)^{\frac{1}{10}} - 1 = 3.75\%$$

Yes, the student investment fund has beaten the market, at least on the raw return measure.

Problem 2.6 You make three equal deposits of $10,000 into your bank account: one today, one more at the end of year 3, and the last one at the end of year 5. Compounding is annual, and stated annual rate is 3%. You close your account at the end of year 8 and withdraw everything. How much do you withdraw?

We are looking for the future value of three cash flows made at three different points in time: the first cash flow is made 8 years prior to the target date, the second 5 years prior to the target date, and the third 3 years prior to the target date. The total future value is

$$100{,}000 * (1.03)^{8} + 10{,}000 * (1.03)^{5} + 10{,}000 * (1.03)^{3} = 35{,}187.71$$



12,667.70 +11,592.74 +10,927.2 = 35,187.71

To conclude this little section, uneven cash flow streams are very popular in financial analysis. If you think about sales forecast of some company for the next 5 years, or 20 quarters, are cash flows forecasted to be the same or grow monotonically at a constant rate? Most likely not: some quarters' sales will be higher than others, e.g. due to the seasonal nature of the business or some other reasons. Maybe you know that the firm is making a major investment and it will not start to pay off until year 3, so you will see very different sales after that time than before. You need to be comfortable find present and future values of uneven cash flow streams.

2.4 Perpetuity

Perpetuity is a cash flow series where the exact same amount is paid (received) at regular intervals and never ceases. A *British Consol* would be an example of a financial security that provides indefinite payments if they still existed today. British Consols (originally short for Consolidated Annuities) were first issued in 1751 and existed through 2015 when the U.K. Government finally redeemed them in full. They were meant to be perpetual government bonds redeemable at the option of the government.

Another decent example would be a corporation that pays constant regular dividends and is expected to do so forever (we think of a corporation as a going concern since its ownership is easily transferable from one set of investors to another through trading of its shares of stock). So, we should learn how to find present values of perpetuities.

How do we solve for present value of perpetuity? Seriously, how do we find present value of an *infinite* number of cash flows? Some math is in order!

Example Imagine that you put X dollars in the bank at the periodic interest rate 5% forever (meaning, your heirs will continue to benefit from the account after you pass away), and that today's lump amount X is such that you should be able to withdraw \$1,000 per year from your account every year forever. What??? How is that even possible, you might ask? Wouldn't I have to put in an infinite amount of money today?

Similar to this, shouldn't a share of corporate stock that pays constant dividends in perpetuity be worth an infinite amount today, since the income from it is expected to extend into eternity?

But we are forgetting something here. As usual, the forgotten element is interest rate! Think about time value of money! How much do you need to put in the bank today to receive \$1000 next year if the interest rate is 5% per year? Clearly, you need to put in $\frac{\$1000}{1.05} = \$952.38$. But what if this \$1000 amount is to be received 10 years from today? With annually compounded interest, \$1000 in ten years is the same as putting $\frac{1000}{1.05^{10}} = \$613.91$ in today, which is much less but still a sizable amount. Well, how about \$1000 in 100 years? That's $\frac{1000}{1.05^{100}} = \$7.60$ today, which is not so much. Do you see where I'm going with this? Yes, these cash flows will continue forever and ever but the distant cash flows are going to be worth progressively less and less in today's value terms. These lucrative infinite future payments will eventually get to be worth so little today that we can neglect these amounts as infinitely small numbers.

The present value of any series can always be calculated as present value of a bunch of lump sums discounted back to the beginning. Let C denote the regularly paid cash flow and let *r* stand for periodic rate. This way, present value of perpetuity is:

$$PV\ (perpetuity) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^{1000}} + \cdots = \frac{C}{r}$$

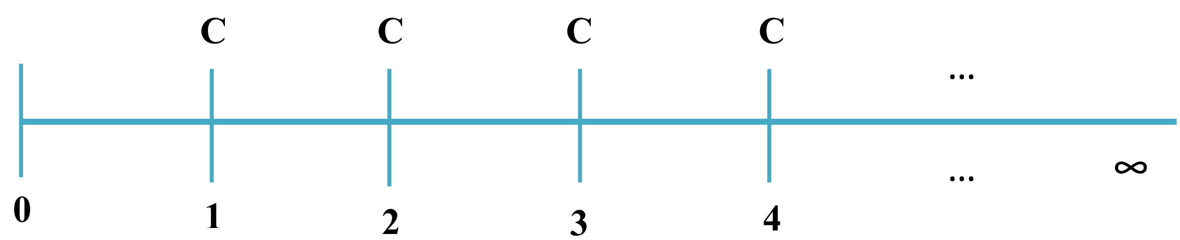
The last equality follows from the mathematical expression for the sum of the terms of an infinite geometric series.

Let's go back to your bank example. Today, the equivalent of the infinite number of \$1000 annual payments is nothing but the sum of an infinite geometric series.

$$PV = \frac{1000}{1.05} + \frac{1000}{1.05^2} + \frac{1000}{1.05^3} + \ldots = \frac{1000}{0.05} = 20{,}000$$

Believe it or not, if interest rate is 5% per year, to be earned forever, all you need today is put a mere \$20,000 in the account so that you and your heirs can enjoy a distribution of \$1000 per year every year until some meteorite destroys the planet. Nice, right? That's the power of compounding for you. That's the power of time value of money!

On a timeline, perpetuity looks like the following diagram. The number 0 marks the beginning of the first period. The numbers 1, 2, 3, and 4 mark the end of periods 1, 2, 3, and 4, respectively. The letter C appearing at each mark (1, 2, 3, 4) indicates that there is a cash flow equal to \$C at the end of each period. The sign ∞ at the end of the line means these cash flows continue into infinity.

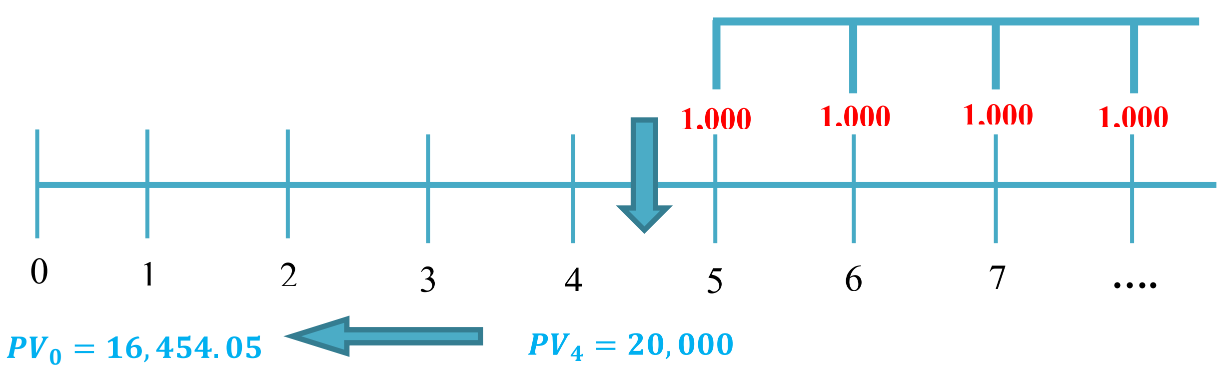


Hence, general formula for present value of perpetuity is

$$PV_0 = \frac{C_1}{r} \qquad (2.1)$$

Note that present value is evaluated at the beginning of the series, at time 0, and the first future cash flow C_1 in the formula is paid at time 1. That is, **there is exactly one period gap between when the cash flow is first paid and the time at which present value is determined**.

What if there is a longer gap? Let us suppose that, like in the previous example, you deposit a certain amount in the bank at 5% rate per year, to be able to make eventual regular \$1000 withdrawals perpetually. However, unlike in that example, suppose the rules do not allow withdrawals for the first five years. That is, your first withdrawal will be at time 5! How do you evaluate the necessary deposit?



Well, there is no reason to be discouraged. The amount will be

$$PV = \frac{1000}{1.05^5} + \frac{1000}{1.05^6} + \frac{1000}{1.05^7} + \ldots$$

$$= \frac{\frac{1000}{1.05} + \frac{1000}{1.05^2} + \frac{1000}{1.05^3} + \ldots}{1.05^4}$$

$$= \frac{\frac{1000}{0.05}}{1.05^4} = \$16{,}454.05$$

Another way to solve for the answer is to apply formula for perpetuity (2.1). Given the first withdrawal is made at $t = 5$, formula (2.1) will give you the value at $t = 4$.

$$PV_4 = \frac{C_5}{r}$$

Solving, we obtain $\frac{1000}{0.05} = 20{,}000$

The amount of \$20,000 is the lump sum, at $t = 4$, which you would be willing to exchange for the entire perpetuity starting at $t = 5$. If we wanted to find the value of that perpetuity at $t = 4$, we would have been done. However, we want the present value today, at $t = 0$. Therefore, we need to discount the lump sum of \$20,000 additionally by four periods to value it today.

$$\frac{20{,}000}{1.05^4} = \$16{,}454.05$$

In this case, we are talking about a **delayed perpetuity**, where the first payment is delayed and occurs at time $t > 1$. Notice in the example above the first withdrawal is not until $t = 5$, but we divide by 1.05^4 – not by 1.05^5! In general, if the first payment in perpetuity occurs at time t, then

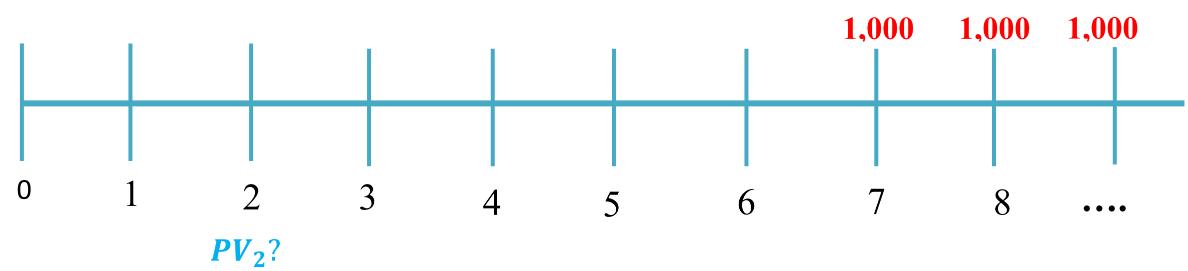
$$PV_0 = \frac{\left(\frac{C_t}{r}\right)}{(1+r)^{t-1}} \qquad \textit{(2.2)}$$

Of course, when $t = 1$, the formula is reduced to $PV_0 = \frac{C_1}{r}$.

Delayed perpetuities are relevant. Suppose you buy a share of stock and expect to receive dividends, but the company is not expected to start paying dividends until year 10! How will you determine the present value equivalent of all those dividends today, at the time when you are buying? After all, it is today that you are exchanging your out-of-pocket dollars for the claim to these future dividends; so, you must know what they are worth in today's terms. The formula for delayed perpetuity will be handy in that case.

Let me give you one more example, at the risk of driving you crazy.

Example Suppose that, like in the examples above, you deposit money in the bank in exchange for an infinite series of $1000 annual withdrawals. Except that you do not have the money right now; you will only make the deposit at $t = 2$. After that, you would have to wait for five years before your first withdrawal, that is until $t = 7$.



How much will you have to deposit at $t = 2$?

Well, if you compare the time line for this example to the time line for the one before, what difference do you notice other than the fact that what used to be 0 is now 2; and what used to be 5 is now 7? There is no other difference! We basically take the project and slide it along the time axis. However, we are not changing the project's structure at all! The main fact remains that the first cash flow will take place five years after the initial deposit and will then continue in perpetuity. That is all that matters! Time index can change. "The beginning" can be time 0 or time 2 or time 100 or whatever for that matter. I can move the project back in time or forward in time: so long as I do not change the size and timing of the project's cash flows *relative to each other*, I am changing nothing at all!

Imagine a toy car that you are rolling forward. Just because the position of the car has changed, nothing has changed about the car itself. It still has the same dimensions and the same characteristics.

So, in our current example, just because you moved the project by two years, the initial deposit will be the same as before, $16,454.05 to be equivalent to all the cash flows that ensue.

So far, we have been considering series of cash flows that last forever. It is obvious, however, that most financial undertakings will not provide an endless stream of cash flows. At some point everything comes to an end. The easiest example is taking out a loan. Sure enough, you don't expect to be paying it off forever! At some point it will be paid off and you will be free and clear of debt. So, it is time to discuss *finite* streams of regularly paid cash flows, which are called annuities.

2.5 Annuity

I wanted to open this section by saying that annuities are all around us, but I figured that sounds kind of creepy. In truth, however, we do deal with annuities more often than we realize. In general, **annuities** are financial products structured to exchange a lump amount at some point for a finite number of regular constant payments.

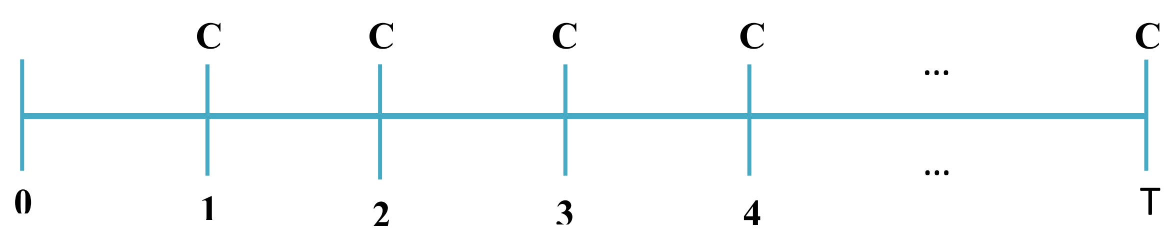
What are some examples? Think about any loan such as a student loan, a mortgage loan, a car loan, etc. As a lender, you give someone a lump amount of money today in exchange for regular installments as they pay the loan off over time.

What about retirement benefits? A retiree has a certain amount of money at the time of retirement, which gets distributed to them in equal distributions over a period of many years, so that they have regular income to live on.

What about a bond? A corporation, or a government, or a municipality can issue bonds, essentially borrowing money from the public. The bonds sell for a certain amount of money; and then the issuer makes fixed payments (called coupons) to its bondholders over the term called the maturity of the bond.

All of these are examples of annuities. The main difference between annuity and perpetuity is that annuity has a certain number of payments in it and then it's over; whereas perpetuity is an infinite series of cash flows.

An annuity has a timeline that looks like the following diagram. The number 0 marks the beginning of the first period. The numbers 1, 2, 3, and 4 mark the end of periods 1, 2, 3, and 4, respectively. The letter T at the last mark indicates the end of the terminal period. The letter C appearing at each mark (1, 2, 3, 4, …, T) indicates that there is a cash flow equal to $C at the end of each period until time T.



Note that cash flows, at usual, are assumed to occur *at the end* of the period. How shall we find the present value and the future value of an annuity?

The formulae are rather simple.

For an annuity that pays a regular cash flow C for T periods, where periodic rate is *r*, we can calculate present value as the sum of the present values of the individual cash flows as lump sums, as follows:

$$PV(\text{annuity}) \ = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C}{(1+r)^T}$$

After massaging this formula with the help of some math, we see that it is nicely simplified to

PV annuity formula: $$PV = \frac{C}{r} * [1 - \frac{1}{(1+r)^T}] \qquad (2.3)$$

Like in the case of perpetuity, **present value is found one period prior to the first cash flow**. That is, if the first cash flow in the series occurs at time 100, for example, the formula for present value will give you the lump sum equivalent of that series as of time 99.

By the same token, we can calculate future value of annuity to be

FV annuity formula $$FV = \frac{C}{r} * [(1+r)^T - 1] \qquad (2.4)$$

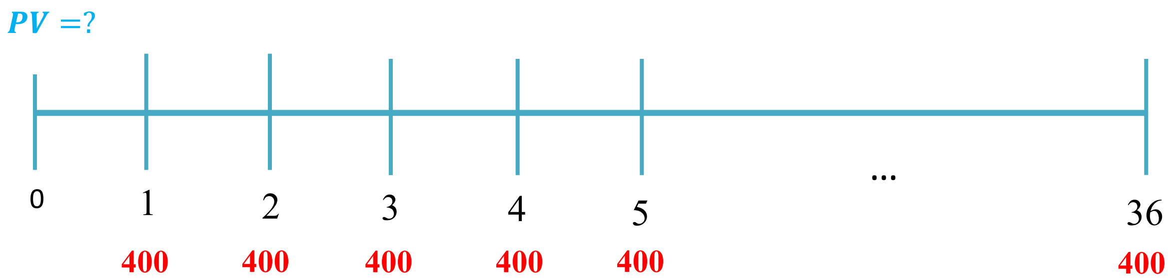
The formula gives you the lump sum equivalent of the cash flow series **at the same time at which the last payment of the series occurs and immediately upon receipt of that last payment.** For example, if the payments in the series occur from year 10 to year 50, then the future value formula will give you the lump sum equivalent value as of year 50, immediately upon the final payment.

Let us do a few examples of annuities, and then you will see what they all have in common.

Problem 2.7 If you can afford a $400 monthly car payment, how expensive is the car you can afford today if stated interest rates are 7% on 36-month loans and you will finance 100% of the car price?

To solve, let us analyze the context. You will be buying a car today and financing it fully with a loan. A loan is the amount of money received today from the lender to be paid in equal installments over a future period. A "36-month loan" means that the loan is to be paid off in 36 monthly installments (3 years). Stated rate of 7% means that the lender is charging you 7% annually. That is the cost of the loan to you and the rate of return that the lender is earning. Essentially, you are being asked, what is the amount of the loan today if the monthly payment is $400?

The setup in this problem is a classic annuity. There are 36 regular $400 cash flows (loan payments) spread over the future period. Each cash flow is the same as all others, and the rate is constant throughout.



At this point, you need to locate the "lump sum" relative to "series of payments." The lump sum is the bulk amount of the loan taken today. The series of payments will follow over the next 36 months. Given that the lump sum precedes the series of payments, this is a question about present value of annuity.

Are we ready to apply the formula above, throwing in C = 400, r = 7%, T = 36, and solving for PV? Not quite! The problem is that we're given the stated rate, but the formula calls for periodic rate! Remember monthly compounding from Chapter 1?

$$Periodic\ rate\ = \frac{7\%}{12} = 0.5833\%$$

Now we are ready.

$$PV\ = \frac{400}{0.005833}\left[1 - \frac{1}{1.005833^{36}}\right] = \$12{,}954.59$$

(If you do this problem with a calculator, your answer may be slightly different from mine due to rounding off. The same note applies to the next few problems.)

Problem 2.8 You take out a mortgage loan for 15 years with stated rate 4%. Your monthly payments are $998.58 per month. What's the amount of the loan you take out today?

Again, you are dealing with annuity. How do we know that? Monthly payments are fixed and made regularly every month for 15 years, which is $15 * 12 = 180$ months. Whenever we have a structure like that (equal cash flows regularly spaced through time and paid for a finite number of period), that's annuity.

This problem is just like the previous one, except you are buying not a car but a house. We already know monthly payment is $998.58 and $T = 180$. We only need to solve for periodic rate.

Periodic rate = $\frac{4\%}{12} = 0.33\%$

Since the lump sum (the loan amount taken) precedes the series of payments, we are dealing with present value of annuity.

$$PV = \frac{\$998.58}{0.0033}\left[1 - \frac{1}{1.0033^{180}}\right] = \$135{,}366.72$$

Problem 2.9 You finance the purchase of a car worth $30,000, at 7% stated rate, to be paid in monthly installments over 5 years. Find the monthly payment you will be making.

Again, you are dealing with an annuity. By now you must have figured out that a loan is an annuity. There are regular monthly cash flows, this time paid over 5 years, which is 5*12 = 60 months. Periodic rate is $\frac{7\%}{12} = 0.5833\%$.

However, what's the amount $30,000? Is it the present value, future value, or payment?

To answer the question, you must see whether this amount is paid / received only once, or whether it occurs periodically. Well, the car price is paid only once at the time of purchase, and the loan is taken once. Therefore, the loan amount of $30,000 cannot be "payment". It is then either future value or present value. To tell which, you see whether this lump amount precedes the series of payments or follows it. The loan is taken out first, and the payments follow. Hence, $30,000 is the present value of annuity. The payment C is unknown, and we must solve for it.

$$\$30{,}000 = \frac{C}{0.005833} * \left(1 - \frac{1}{1.005833^{60}}\right)$$

Solving,

$$C = \$594.03$$

Problem 2.10 You have just landed a job and are going to start saving for a house. You want to have \$100,000 saved in 10 years. How much should you contribute into your saving account monthly if stated rate is 4%?

Given equal regular monthly contributions into the savings account for 10 years (120 months), we are dealing with annuity again. We know the drill: find periodic rate first. That would be $\frac{4\%}{12} = 0.33\%$

What is \$100,000? (We already know better than thinking it's the monthly payment). It is the target amount. Clearly, this amount will occur in your account following the 10 years of diligent monthly contributions! Given that the lump sum follows the series of payments, we are now in the territory of future value. Therefore, we need the future value formula.

$$\$100{,}000 = \frac{C}{0.0033} * (1.0033^{120} - 1)$$

Solving,

$$C = \$680.55$$

Problem 2.11 You are saving for retirement by making monthly contributions of \$100 into your retirement account. Stated rate is 7%. If you plan to retire in 40 years, how much money do you expect to have available at retirement?

If I'm doing a good job explaining these problems, you should recognize that we are dealing with future value of annuity again. This time you are saving for 40 years (480 months), cash flow is \$100 per month, and periodic rate is $\frac{7\%}{12} = 0.5833\%$ as before. This time, the amount of monthly contribution is known, so $C = 100$, and what we are looking for is the future value.

The timeline here is the same as in the previous problem, except we have a total of 480 periods, the cash flow is known, and the future value is unknown. Solving,

$$FV = \frac{100}{0.005833} * (1.005833^{480} - 1) = \$262{,}451.86$$

Problem 2.12 You will buy a house in 10 years. At that time, you will take out a 30-year mortgage loan for $250,000 financed at 5% stated rate. What is the monthly payment you will be making on the loan?

This time I need you to pay very close attention. Take a break if you need it at this point.

First, we know we have annuity here. This is the easy part. What else do we know? Payments are monthly, so periodic rate is $\frac{5\%}{12} = 0.4167\%$. The number of payments in the annuity is $30 * 12 = 360$ months. Why? Because it is a 30-year loan, meaning you will pay if off over 360 months. You will take the loan first, and pay it off later, meaning ***the lump sum of $250,000 is present value.***

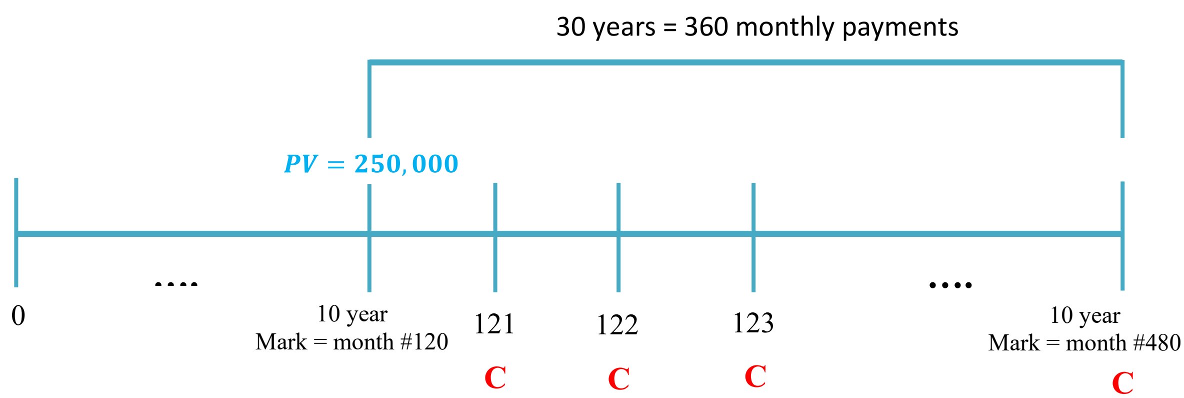
Hmm, what about the 10 years you wait to take out the loan? How will this affect the monthly payment you're looking for? And why doesn't it make the amount $250,000 the future value instead?

Well, should it?

Think about the toy car example again. Just because you roll the toy car forward, it changes nothing about the car itself. Hence, just because the loan will be taken out in 10 years, it changes nothing about the structure or the amount of the loan. The payment on the loan depends only on the loan amount, the rate, and the term.

You may object. Wait a minute, you may say. How do I know, in real life, that the rate will be 5% in as long as 10 years' time? I can observe mortgage rates today, but come on, this example is unrealistic. The time I wait to take out the loan will affect the rate, because rates are random and fluctuate over time.

You know what? If the above thought crossed your mind, I would give you bonus points if I were your professor. My example is unrealistic, I admit it. But as an author of this book, I reserve the right to make unrealistic examples sometimes: I need to trade realism of examples for transparency of concepts. These concepts are still very new.



Anyway, enough talking, here is the solution.

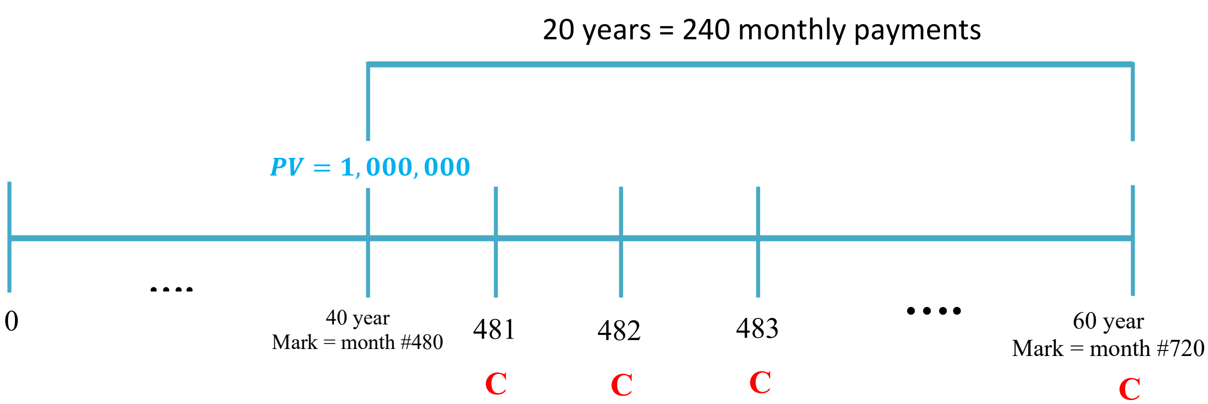
$$\$250{,}000 = \frac{C}{0.004167} * \left(1 - \frac{1}{1.004167^{360}}\right)$$

Solving,

$$C = \$1{,}342.12$$

Problem 2.13 You will retire in 40 years. At that time, you will have $1,000,000 saved in your retirement account. Annual rate your retirement account is 7%. If you plan to live for 20 years after retirement, what's the amount of monthly distribution you will be receiving?

This is again the present value of annuity problem. We are talking about monthly distributions (withdrawals) from the retirement account post-retirement given the lump sum available at the start of retirement is $1,000,000. Just like in the previous problem, the fact that retirement is delayed by 40 years has no bearing on the problem. The lump sum is available prior to the series of cash flows, so it is again a present value problem. We are solving for the cash flow (monthly payment). Periodic rate is $\frac{7\%}{12} = 0.5833\%$, and the number of payments is $20 * 12 = 240$ months.

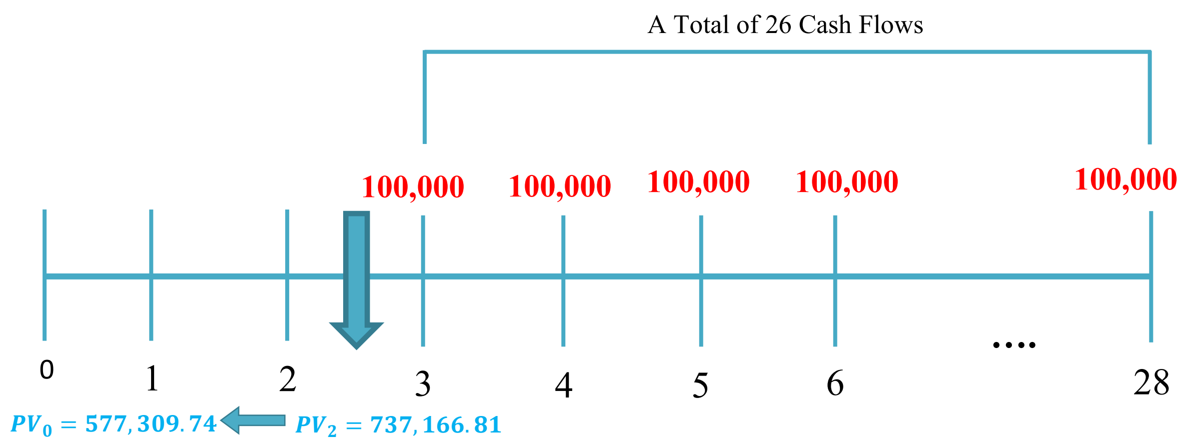


$$\$1{,}000{,}000 = \frac{C}{0.005833} * \left(1 - \frac{1}{1.005833^{240}}\right)$$

Solving,

$$C = \$7{,}752.75$$

Problem 2.14 (Challenge) This is the last problem, I promise, for this chapter. You are an entrepreneur, financing an enterprise. You manage to find wealthy investors who are considering your project. If they invest today, they will not see any cash flows for the next three years. At the end of the third year, and for 25 years thereafter, you expect to pay them regular dividends of $100,000 per year. Given the rate of return required to be 13% per year, how much will the investors give you today?



Finally, this is something different than loans and retirement savings! We are dealing with annuity given there will be regular $100,000 cash flows. How many of them are there? The first cash flow will occur at the end of year 3 ($t = 3$), plus there will be 25 more cash flows, ending at $t = 28$. Hence, there are a total of $1 + 25 = 26$ cash flows. Please refer to the timeline and pay close attention. So, we know now that $T = 26$ and $C = 100{,}000$. We also know that $r = 13\%$, because we already know that the rate of return required by investors is the discount rate that we must use to find the amount they are paying (remember section 2.1). So, we need to find present value of the annuity.

If you are like me, the first thing you will do is grab the formula for present value of annuity and find the result:

$$PV = \frac{100{,}000}{0.13} * \left(1 - \frac{1}{1.13^{26}}\right) = \$737{,}166.81$$

Is this the final answer? No, it isn't. Be patient, go back to the beginning of this section and find where it says, "Present value is found one period prior to the first cash flow. That is, if the first cash flow in the series occurs at time 100, for example, the formula for present value will give you the lump sum equivalent of that series as of time 99."

Aha! This means, given the first cash flow occurs at $t = 3$, the formula just gave us the value at $t = 2$!

But the investors are financing your enterprise today, not in two years. So, we need to discount the amount $737,166.81 additionally for two more years, in order to get to the final answer! We can now treat is as a lump sum, and discount as such.

$$\frac{\$737{,}166.81}{1.13^2} = \$577{,}309.74$$

That's the amount you will receive today from the investors. And you were hoping for $737,166.81, right? Well, life isn't all that fun.

After seeing all of these examples, I hope you see that annuities come in different shapes and sizes, they have different names and numbers, but the same concept always applies, they are finite series of regularly occurring fixed cash flows.

2.6 Loan Amortization

We have done quite a few examples on loans in this chapter. Let us figure out how the interest is folded into each installment payment.

You may think that each installment payment contains a little bit of the principal and a little bit of interest mixed in. Or perhaps you were thinking that all principal gets paid out first, and that the payments after that contain pure interest (or vice versa). How does it really work?

Normally, a loan gets **amortized** over time in the following way. Yes, each installment contains a little bit of principal and a little bit of interest mixed . However, the proportion in which they are mixed is not constant from one payment to the next. It so happens that the initial payments contain more interest and less principal; whereas the later payments contain mostly principal and very little interest.

Below is a comprehensive example of loan amortization.

Example Suppose you buy a car and finance it with a $15,000 loan. Stated rate is 6% and the term is 3 years. Payments will be made monthly. Let us calculate the monthly payment from realizing that the loan is simply an annuity, $T = 3 * 12 = 36$, and $r = \frac{6\%}{12} = 0.5\%$

$$\$15{,}000 = \frac{X}{0.005} * \left(1 - \frac{1}{1.005^{36}}\right)$$

Solving,

$$X = 456.33$$

You will make 36 payments of that amount over the life of the loan until it is paid in full. We know each payment has room for interest in it, just like you leave room for cream in your coffee. How much is interest and how much is principal?

Please refer to Table 2.1 below. In the first month, you begin with the principal amount of \$15,000 (before your first payment). Interest on that principal is $\$15{,}000 * 0.005 = \$75$. Monthly payment, however, is \$456.33, meaning that $\$456.33 - \$75 = \$381.33$ will be the amount of the principal paid that month. The remaining principal balance at the end of the first month will be $\$15{,}000 - \$381.33 = \$14{,}618.67$.

In the next month, interest due is $\$14{,}681.67 * 0.005 = \$73.09$; which is a little less than the first month's interest was. However, the monthly payment is still the same as before! Hence, a little bit more of the principal is paid now, compared to the first month.

Table 2.1 Loan Amortization

Time	Beginning Principal Balance	Interest	Payment	Principal Paid	Ending Principal Balance
0					15,000.00
1	15,000.00	75.00	456.33	381.33	14,618.67
2	14,618.67	73.09	456.33	383.24	14,235.44
3	14,235.44	71.18	456.33	385.15	13,850.28
4	13,850.28	69.25	456.33	387.08	13,463.21
5	13,463.21	67.32	456.33	389.01	13,074.19
6	13,074.19	65.37	456.33	390.96	12,683.23
7	12,683.23	63.42	456.33	392.91	12,290.32
8	12,290.32	61.45	456.33	394.88	11,895.44
9	11,895.44	59.48	456.33	396.85	11,498.59
10	11,498.59	57.49	456.33	398.84	11,099.76
11	11,099.76	55.50	456.33	400.83	10,698.93
12	10,698.93	53.49	456.33	402.83	10,296.09
13	10,296.09	51.48	456.33	404.85	9,891.24
14	9,891.24	49.46	456.33	406.87	9,484.37
15	9,484.37	47.42	456.33	408.91	9,075.46
16	9,075.46	45.38	456.33	410.95	8,664.51
17	8,664.51	43.32	456.33	413.01	8,251.50
18	8,251.50	41.26	456.33	415.07	7,836.43
19	7,836.43	39.18	456.33	417.15	7,419.29
20	7,419.29	37.10	456.33	419.23	7,000.05
21	7,000.05	35.00	456.33	421.33	6,578.72
22	6,578.72	32.89	456.33	423.44	6,155.29
23	6,155.29	30.78	456.33	425.55	5,729.74
24	5,729.74	28.65	456.33	427.68	5,302.06
25	5,302.06	26.51	456.33	429.82	4,872.24
26	4,872.24	24.36	456.33	431.97	4,440.27
27	4,440.27	22.20	456.33	434.13	4,006.14
28	4,006.14	20.03	456.33	436.30	3,569.84
29	3,569.84	17.85	456.33	438.48	3,131.36
30	3,131.36	15.66	456.33	440.67	2,690.69
31	2,690.69	13.45	456.33	442.88	2,247.82
32	2,247.82	11.24	456.33	445.09	1,802.73
33	1,802.73	9.01	456.33	447.32	1,355.41
34	1,355.41	6.78	456.33	449.55	905.86
35	905.86	4.53	456.33	451.80	454.06
36	454.06	2.27	456.33	454.06	(0.00)

And so on, notice that exactly by the end of the 36^{th} month the loan is paid in full. This process is called "loan amortization". Notice that the interest becomes smaller and smaller with each installment, being replaced by principal payment, which becomes larger and larger with each installment.

2.7 Next Chapter: Sneak Peek

To summarize your learning in this chapter, you have learned that the concepts of future value and present value, compounding and discounting, apply to a broad set of financial undertakings. We even introduced risk and showed how it affects the discount rate at which we find present (or future) value of a project.

Before taking the next long break, here is what's coming next. So far, we talked mostly about cash flows that either have some erratic pattern (uneven cash flows) or are constant through time (perpetuity and annuity). How will our analysis be enriched by introducing growth into the cash flow pattern? Growth is such an important concept! The economy grows. Businesses grow. And so on: now you are ready to be introduced to this very important element of financial analysis.

Time for a break!

Chapter 2 Self-Check Problems

1. You are raising money for a new project, which is expected to pay $500,000 as a lump sum in five years. There will be no other cash flows. You approach investors for funding, and they inform you that they require an average annual return of 15% because your enterprise is rather risky to them. How much money will the investors provide today in order to be able to earn the required rate of return from your project?

Answer: $248,588.40

2. In order to secure an average return that properly compensates investors for the risk, investors will

a. Control and monitor the firm to make sure they can get the cash flows that can reasonably be expected given the nature of the enterprise
b. Provide the amount of initial investment that secures the desired average return given the expected future cash flow
c. Both a and b

3. In your own words, explain how investors achieve higher return for higher risk.

4. In which context did this chapter use the comparison with leaving room for cream in your Starbucks coffee?

5. The project has the following cash flows: an inflow of $5 million at the end of year 4 ($t = 4$), and outflow of $10 million at the end of year 11 ($t = 11$), and an inflow of $35 million at the end of year 15 ($t = 15$). Compounding is annual and the rate is 6% per annum.

a. Draw the project's cash flows on a timeline.
b. Find the lump sum equivalent to the project's cash flows, as of $t = 0$

Answer: 13.3 million

c. Find the lump sum equivalent to the project's cash flows, as of $t = 20$

Answer: 42.6 million

d. Find the lump sum equivalent to the project's cash flows as of $t = 8$

Answer: 21.2 million

6. You deposit money in an investment account today ($t = 0$). You plan to withdraw no money from your account for the first three years. At the end of the third year ($t = 3$), you want to make your first withdrawal of $5,000, and then you want to continue to make equal withdrawals of $5,000 every year indefinitely. If the rate you earn on this investment is 8% per year, and you believe that this rate will continue forever, how much money do you need to deposit today to provide enough capital for all your planned future withdrawals?

Answer: 53,583.68

7. Consider the previous problem. Everything is the same as before except you will deposit the money at $t = 2$. (The first withdrawal is still at $t = 3$.) What is the amount of money that needs to be deposited at $t = 2$?

Answer: 62,500

8. Consider the previous problem. Everything is the same (you deposit money at $t = 2$), but the first withdrawal will be at $t = 5$. Find the needed deposit amount.

9. You are an entrepreneur, financing an enterprise. You manage to find wealthy investors who are considering your project. If they invest today, they will not see any cash flows for the next four years. At the end of the fourth year, and for 15 years thereafter, you expect to pay them regular dividends of $200,000 per year. Given the rate of return required to be 11% per year, how much will the investors give you today?

Answer: $1,079,116

10. The project pays regular cash flows of $100 every year for 10 years, with semi-annual compounding of 4% per annum stated rate. The first cash flow will be paid at the end of year 6 ($t = 6$).

a. Find EAR

Answer: 4.04%

b. Using EAR as the discount rate, find present value of the project's cash flows as of $t = 5$

Answer: 809.48

c. Using EAR as the discount rate, find present value of the project's cash flows as of t = 0

Answer: 664.05

11. You take out a three-year loan, which you will amortize fully using annual installments. (That is, payments are annual.) Interest rate on the loan is 5%, and the amount borrowed is $50,000.

a. Find the annual payment amount.
b. Fill out the following table

Year	Beg. Principal Balance	Interest	Payment	Principal Paid	End. Principal Balance
1					
2					
3					

12. You deposit three equal amounts of money into your investment account: one today $(t = 0)$, another three years from now $(t = 3)$, and the last one seven years from now (t=7). You leave the money in the account until $t = 12$. Stated rate is 5% and compounding is monthly. If your target amount in the account is $100,000, how much money do you need to deposit each time?

Answer: 21,413.03

13. You invest in a project that pays the following cash flows: $50,000 in year 4, $70,000 in year 6, and $100,000 in year 10. What is the breakeven lump amount today such that you would be indifferent between getting that amount today and getting the project instead? Stated rate is 6% and compounding is continuous.

Answer: 143,049.90

14. You deposit $100,000 in an account earning 7% annually, with annual compounding. After five years (at $t = 5$), you withdraw an amount needed to finance college. How much can you afford to withdraw if you need to have saved $150,000 by $t = 10$?

Answer: 33,307.25

15. The formula for a simple annuity is: $PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$.

You take a $100,000 mortgage loan at 6% fixed annual interest, with monthly payments, for 30 years. Which of the following is correct to plug into the formula above (if you're calculating it by hand)?

a. $C = \$100{,}000;\quad T = 30;\quad r = 0.06;$
b. $PV = \$100{,}000;\ T = 30;\quad r = 0.06;$
c. $C = \$100{,}000;\quad T = 360;\ r = 0.005;$
d. $PV = \$100{,}000;\ T = 360;\ r = 0.005.$

16. A common stock pays annual dividends of $2.20 per share in perpetuity. The discount rate is 10%. Which of the following is the value of the stock today?

a. $10
b. $11
c. $20
d. $22

17. You are considering the purchase of a really nice home *in three years*. You have $100,000 in your bank savings account today but would rather wait to buy the house. You plan to leave the money there for three years under 4% APR with *monthly* compounding. You understand that this money alone will not be enough to buy the house of your dreams, so when it's time to buy the house you will have to take out a mortgage loan. Based on your credit score, you can obtain a loan at 7% APR for 30 years. You calculate that you will be able to afford *monthly* mortgage payments of $1,500.00.

a. How much money will you have *in your bank savings account* in three years? (This has nothing to do with the house or the loan.)

b. What is the amount of mortgage *loan* that you will be able to afford given a 30-year term, $1,500.00 *monthly* payments, and 7% APR?

c. Out of your very first $1,500 mortgage payment, how much will go towards interest, and how much toward the principal balance on your loan? (Remember that payments are *monthly*.)

d. So, in three years you will have a certain amount in your bank savings account (part a) plus the loan you take from a bank (part b). What is the price of the house that you can afford then?

18. You consider investing in a project with the following expected future cash flows. For the first three years, there will be no cash flows. Then starting at the end of year 4, and through the end of year 20, you will receive $1000 every year. Finally, at the end of year 20, you will sell the investment for $100,000. You consider 10% per year a fair rate of return.

a. Calculate the present value of the $1000-per-year annuity *as of today*. (Hint: notice that the $1000 annuity is delayed by *three years*! Instead of receiving the first cash flow in year 1, you will receive it in year 4.)

b. Calculate the present value of the lump $100,000 received in year 20.

c. What is the total value of the investment project today?

d. The value of the project today would be higher, all else equal, if the rate of return were lower. (True / False)

19. Calculate the monthly payment on a $250,000 mortgage loan taken out for 15 years at 4.5% stated rate.

CHAPTER 3: GROWTH

"My dear, here, we must run as fast as we can, just to stay in place. And if you wish to go anywhere you must run twice as fast as that." – Lewis Carroll, "Alice Through the Looking Glass"

3.1 Growing perpetuity

Welcome back. As the title of this chapter suggests, we are going to discuss growth and its impact on our financial decisions. Before getting into a long philosophical discussion about growth in general, however (by now you must have figured I love those), let us consider an example.

Example Suppose there is a project where cash flows in the infinite series do not stay constant from one period to the next but rather increase at some constant growth rate *g*. For example, if $g = 3\%$, and the cash flow today is $C_0 \ = \ \$100$, then the next one will be

$$C_1 \ = \ 100 * (1.03) \ = \ \$103$$

The one after that will be

$$C_2 \ = \ \$103 \ * (1.03) = \{\$100 * 1.03\} * 1.03 = \ \$100 \ * (1.03)^2 \ = \ \$106.09$$

And so on. The cash flow at an arbitrary point in time T will then be

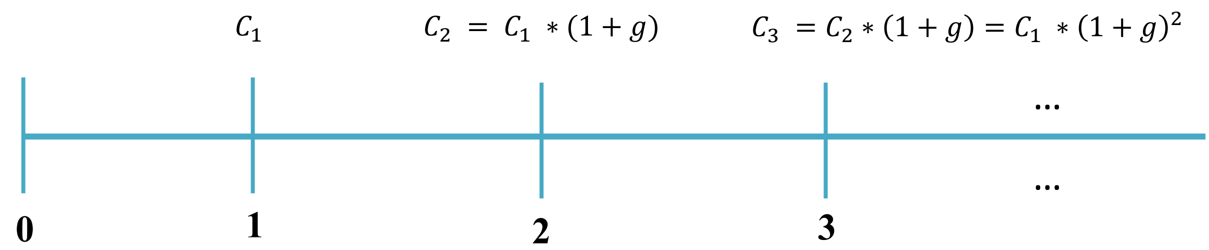
$$C_T \ = \ C_0 \ * (1 \ + \ g)^T \qquad (3.1)$$

Or,

$$C_T \ = \ C_1 \ * (1 \ + \ g)^{T-1} \qquad (3.2)$$

And cash flows won't stop there! Rather, they will continue to come perpetually, each one being *g*% higher than the previous one. We call such a cash flow series a *growing perpetuity*.

On a timeline, growing perpetuity looks as follows. The number 0 marks the beginning of the first period. The numbers 1, 2, 3 mark the end of periods 1, 2, 3, respectively. The letter C_1 at mark 1 indicates that there is a cash flow equal to $\$C_1$ at the end of the first period. The expression $C_2 \ = \ C_1 \ * (1 + g)$ at mark 2 indicates that the cash flow at the end of the second period is higher than C_1 by the factor equal to growth rate g. The expression $C_3 \ = C_2 * (1 + g) = C_1 \ * (1 + g)^2$ at mark 3 indicates that the cash flow at the end of period 3 is higher than the previous cash flow by the factor equal to the growth rate g; and so on.



What might be a good example of this cash flow structure? Imagine a company that pays dividends every year, but as the company's earnings grow from one year to the next so do the dividends. Given that the company is assumed to be a going concern, rather than have a finite lifetime, a series of such constantly growing dividends would be considered to fit the framework above.

To find the present value of this growing infinite series, apply what we already know about discounting lump sums and realize that we can represent the present value of the whole series as the sum of present values of the individual payments, as follows:

$$PV = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \ldots + \frac{C_1(1+g)^{999}}{(1+r)^{1000}} + \cdots = \frac{C_1}{r-g}$$

The last equality follows from properties of infinite geometric series.

Finally, we can write this expression in a more compact form. In general, present value of growing perpetuity is:

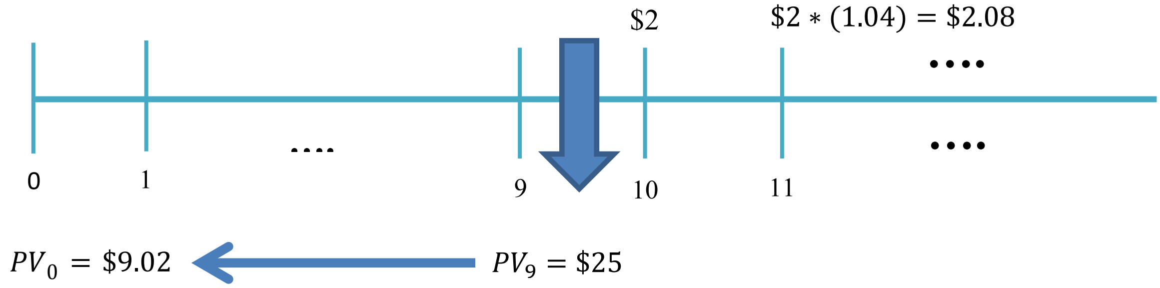
$$PV_0 = \frac{C_1}{r-g} \qquad (3.3)$$

Note, just like in the case of regular perpetuity, that present value is evaluated at time 0, and the cash flow C_1 in the formula above is paid one period later, at time 1. That means **there is one period gap between when the first cash flow occurs and when the present value is determined**.

Just like in the case of regular perpetuity, for a **delayed growing perpetuity** in which the first payment in the series occurs at time t,

$$PV_0 = \frac{\frac{C_t}{r-g}}{(1+r)^{t-1}} \qquad (3.4)$$

Example Suppose you buy a share of stock of company X. The first dividend is expected at time 10 and it will be \$2 per share. Thereafter, dividends are expected to grow at 4% per year indefinitely. You want to discount all those dividends at the rate of 12% because that is the rate of return on an alternative investment of similar risk; so, you require that your investment in stock X earn the same rate of return over time. What is the present value of these dividends, i.e. the equivalent lump amount of money today?



In this example, $C_t = C_{10} = 2, t = 10, g = 4\%$, and $r = 12\%$

$$PV_0 = \frac{\frac{2}{0.12 - 0.04}}{(1.12)^9} = 9.02$$

In other words, you would be indifferent between receiving all those future dividends starting with $2 in year 10 and growing at 4% forever; and receiving $9.02 today!

Another way to solve for the answer is to apply formula for growing perpetuity (3.3). Given the first dividend is paid at $t = 10$, formula (3.3) will give you the value of the stock at $t = 9$.

$$PV_9 = \frac{C_{10}}{r-g}$$

Solving, we obtain $\frac{2}{0.12 - 0.04} = 25$

The amount of $25 is the lump sum, at $t = 9$, which an investor would be willing to exchange for the entire growing perpetuity starting at $t = 10$. If we wanted to find the price of the stock at $t = 9$, we would have been done. However, we want the present value today, at $t = 0$. Therefore, we need to discount the lump sum of $25 additionally by nine periods to value it today.

$$\frac{25}{(1 + 0.12)^9} = 9.02$$

Does this sound unfair? I mean, it seems like you will be receiving a lot of money in the future, and what, all they are worth today is some $9? The answer is yes, and this is because your discount rate of 12% is so high that those future dividend payments soon turn miniscule in present value terms! You don't believe me? See for yourself:

The first dividend of $2 paid at time 10 has present value of

$$\frac{2}{1.12^{10}} = 0.64$$

The second dividend, $\$2 * (1.04) = \$2.08$, paid in year 11, has present value of

$$\frac{2.08}{1.12^{11}} = 0.60$$

And so on: let's take the 50th consecutive dividend of $2 * (1.04)^{49} = \$13.67$, paid in year 59. Its present value is

$$\frac{13.67}{1.12^{59}} = 0.02$$

As you can see, the power of time value of money has just turned \$13.67 into \$0.02! Distant dividends add exponentially less and less to the total present value! That is why the total adds up to only about \$9. Had the discount rate not been so sky high, the present value would have been higher. If you want to verify that, try this example with the lower discount rate of 10% and see that the answer will be \$14.14, which is higher than before.

3.2 Rate of Growth versus Rate of Return

Alright, it is time for the dreaded philosophical discussion.

If there is one confusing concept that I have seen students struggle with over the years, it ought to be the distinction between "growth rate" and "rate of return." This fact is to no surprise perhaps given that we casually think of our investments "growing" over time in the same sense as we are thinking about them "earning return" over time. Markets grew at an astonishing rate last year; your investment portfolio is expected to grow at a certain rate; etc., etc. – we really are talking here about return, aren't we?

It is time to resolve this confusion.

In this chapter, in the rest of this book, and in finance in general, the term **"growth rate" generally refers to the rate at which some value appreciates (or declines) over time**. Here are a few examples:

A stock pays dividends that increase in amount from one quarter to the next. We say dividends "grow." For example, they grow by 0.5% per quarter. That would be the growth rate.

An enterprise generates profits that increase over time: low profits at first, followed by higher profits later. We say profits "grow." For example, they grow by 10% per year. That would be the growth rate.

Where does growth come from? Generally, two things must be present for a business to grow. First, the business must invest. Growth does not happen without investment. If you just sit on your money and do not put it to some productive use, you will never see your money grow. Second, the investment must be profitable. These two factors may be obvious, but you will eventually (not in this book though) see this simple concept treated in much more depth. Bottom line is – growth comes from the nature of the enterprise and from willingness to invest in that enterprise.

Now, what about the rate of return? **The rate of return is always – *always*, *<u>always</u>* – defined in connection with the *<u>price</u>* of something**. You simply do not talk about rate of return outside of that context. Notice that when we talked about the growth in dividends or growth in profits of an enterprise, we never needed to know the price paid by investors for a claim to those dividends, or the investment made by a venture capitalist in that enterprise. Dividends grow. Profits grow. Okay, whatever. But let us now consider the following example.

I paid \$100 for a financial security (be it a share of stock of some company or anything else) last year. This year, the security can be sold at \$125. The rate at which the value of this security appreciated is, of course, $\frac{125-100}{100} =$ 25%. What is this rate, growth or return?

This rate is the rate of return *because it is related to the price paid*. Yes, you can say, "The price of the security grew by 25% over the past year" and you wouldn't lie by saying so. The price did grow. But there is *more to it* than just "grew!" The fact is that the investor *paid* something, and it turned into some other amount, tells us that the investor has *earned the rate of return* of 25%. We cannot ever say what rate of return is earned unless we know how much was paid! On the other hand, we can talk about growth generically, e.g. growth in sales from year to year, even if we don't know what amount was paid: but then we cannot find the rate of return. Can you see the difference?

Example Here is the ultimate example. A corporation is paying annual dividends, which grow at a constant rate $g = 4\%$ per year. This growth rate and these dividend payments are expected to continue into perpetuity. (I hope you recognize at this point that we are talking about a growing perpetuity.) The upcoming dividend to be paid at the end of the first year, D_1 = \$2 per share. Today, one share of stock of this firm sells for \$20. What is the annual rate of return that investors are earning on this stock?

At first, you may say, 4% is the rate of return because dividends get higher and higher at that rate. But that information never used the fact that the price of the stock is \$20, did it? Dividends grow at 4% whether the price is \$20 or \$30 or whatever. That should immediately cause you to become suspicious over 4% being the answer. It doesn't mean right away that the answer is wrong; but the fact that it isn't tied to the stock price is worrisome.

How do we use the fact that the stock price is \$20? Well, what we know is that \$20 is the lump amount of money that investors are willing to part with, today, in exchange for a claim to all those future dividends. That means that the present value of all those dividends had better also be \$20, otherwise they won't buy (if the present value is lower) or the selling party won't sell (if the present value is higher). Now remember that we are talking about a growing perpetuity. It isn't delayed and there are no complications, so we can apply the familiar simple formula

$$PV_0 = \frac{C_1}{r-g}$$

In that formula, PV_0 must be the same as the price of the stock today, \$20. C_1 is the first upcoming dividend, \$2. Growth rate g is given to be 4%. What is *r*?

Indeed, what is *r*?

You guessed it. The unknown rate *r* is precisely the rate of return that investors are earning on this investment over time! Yes: the rate of return earned on an investment is the same as discount rate. It is the link that connects the future cash flows (dividends) and the current price. Recall Chapter 2, section 2.1, and the discussion about the equivalence of the rate of return and discount rate.

Solving,

$$r = \frac{C_1}{PV_0} + g = \frac{2}{20} + 0.04 = 0.14 = 14\%$$

The investors are earning 14% per year on this investment!

Imagine putting $20 into an account that earns 14% return per year. Starting at the end of the first year, and every year after that, you will be withdrawing ever growing amounts of money from your account. The first withdrawal will be $2. Then the withdrawals will be increasing at 4%. If this is the setup, you will be able to maintain this withdrawal pattern *forever*. This is the same as paying $20 for a stock that earns a return of 14% per year and turns your investment into a stream of dividends starting at $2 and growing at 4% forever.

You don't believe me? Okay, so you put $20 into the account. It earns 14% per year. By the time of your first withdrawal, how much money will you have in the account? $\$20 * (1.14) = \$22.80$. Okay, you make your first withdrawal of $2. You are left with $20.80. Another year goes by. By the end of the second year, you have $20.80(1.14) = $23.71 and withdraw $\$2 * (1.04) = \$2.08$. You are left with $\$23.71 - \$2.08 = \$21.63$. Okay, after one more (third) year you will have $\$21.63 * (1.14) = \$24.66$, and withdraw $\$2.08 * (1.04) = \$2.16$, being left with $\$24.66 - \$2.16 = \$22.50$; and so on, and so forth. Please refer to the Table 3.1 below for the first twenty years of the investment's life.

Table 3.1 Rate of return 14%

Year	Balance before withdrawal	Withdrawal	Balance after withdrawal	Balance Growth Rate
-	20.00	-	20.00	
1	22.80	2.00	20.80	0.04
2	23.71	2.08	21.63	0.04
3	24.66	2.16	22.50	0.04
4	25.65	2.25	23.40	0.04
5	26.67	2.34	24.33	0.04
6	27.74	2.43	25.31	0.04
7	28.85	2.53	26.32	0.04
8	30.00	2.63	27.37	0.04
9	31.20	2.74	28.47	0.04
10	32.45	2.85	29.60	0.04
11	33.75	2.96	30.79	0.04
12	35.10	3.08	32.02	0.04
13	36.50	3.20	33.30	0.04
14	37.96	3.33	34.63	0.04
15	39.48	3.46	36.02	0.04
16	41.06	3.60	37.46	0.04
17	42.70	3.75	38.96	0.04
18	44.41	3.90	40.52	0.04
19	46.19	4.05	42.14	0.04
20	48.04	4.21	43.82	0.04

Note a couple of things. First, your balance after withdrawal keeps growing! So, you will never run out of money for withdrawals. That was the first point I was trying to make: that the investment will last forever. However, also importantly, your balance grows steadily at a constant 4% per year. (Keep in mind that the 14% rate of return includes <u>both</u> the growth in your leftover balance <u>*and*</u> the withdrawals!) This way, you're set for life, provided, of course, that the investment vehicle doesn't cease to exist (e.g., due to bankruptcy)!

You might say that this steady growth in your balance is the result of the fact that your investment return 14% is higher than the growth rate of withdrawals 4%: as long my money grows faster than my withdrawals do, I'm guaranteed to be able to make these withdrawals endlessly. Wrong: see for yourself in Table 3.2 what will happen to your balance if, for example, your rate of return is 13%.

Table 3.2 Rate of return 13%

Year	Balance before withdrawal	Withdrawal	Balance after withdrawal	Balance Growth Rate
-	20.00	-	20.00	
1	22.60	2.00	20.60	0.03
2	23.28	2.08	21.20	0.03
3	23.95	2.16	21.79	0.03
4	24.62	2.25	22.37	0.03
5	25.28	2.34	22.94	0.03
6	25.92	2.43	23.49	0.02
7	26.55	2.53	24.01	0.02
8	27.14	2.63	24.50	0.02
9	27.69	2.74	24.95	0.02
10	28.20	2.85	25.35	0.02
11	28.65	2.96	25.69	0.01
12	29.03	3.08	25.95	0.01
13	29.32	3.20	26.12	0.01
14	29.51	3.33	26.18	0.00
15	29.59	3.46	26.12	(0.00)
16	29.52	3.60	25.92	(0.01)
17	29.29	3.75	25.54	(0.01)
18	28.86	3.90	24.96	(0.02)
19	28.21	4.05	24.16	(0.03)
20	27.30	4.21	23.08	(0.04)

Ouch! At year 14 your balance reaches its maximum at $26.18 and then enters a state of permanent decline! That way, you're guaranteed to run out of money soon enough. You can see for yourself that it isn't enough that the rate of return should exceed the growth rate of withdrawals: it is imperative that it be 14%, no less!

What if it were more than 14%? Let's say 15%? Then you would surely never run out of money! True, but soon enough the growth rate in your balance will be so high that no amount of money in the world will be enough to support it! See for yourself in the Table 3.3 below, where rate of return is 15%. The rate of growth in your balance is ever increasing, year after year, starting with 5%, then 6%, then 7%, then 8% - and so on, soon enough it will become *unsustainable*.

The result is that only 14% can be the rate of return that will provide you sustainable growth of 4% forever! How's that for a cool result. So now I hope I have convinced you that 14% is the answer to the question in my ultimate example. I also hope that now you can tell between the rate of growth and the rate of return. **Growth is all about future cash flow**. **Return is all about discount rate**.

Table 3.3 Rate of return 15%

Year	Balance before withdrawal	Withdrawal	Balance after withdrawal	Balance Growth Rate
-	20.00	-	20.00	
1	23.00	2.00	21.00	0.05
2	24.15	2.08	22.07	0.05
3	25.38	2.16	23.22	0.05
4	26.70	2.25	24.45	0.05
5	28.12	2.34	25.78	0.05
6	29.64	2.43	27.21	0.06
7	31.29	2.53	28.76	0.06
8	33.08	2.63	30.44	0.06
9	35.01	2.74	32.27	0.06
10	37.12	2.85	34.27	0.06
11	39.41	2.96	36.45	0.06
12	41.92	3.08	38.84	0.07
13	44.66	3.20	41.46	0.07
14	47.68	3.33	44.35	0.07
15	51.00	3.46	47.54	0.07
16	54.67	3.60	51.07	0.07
17	58.73	3.75	54.98	0.08
18	63.23	3.90	59.33	0.08
19	68.23	4.05	64.18	0.08
20	73.81	4.21	69.60	0.08

3.3 Growing annuity

A growing annuity is just like a growing perpetuity, expect it ends at some point T. Cash flows grow at the rate *g*, which is constant over time.

What would be a real-life example of a growing annuity? Imagine that a start-up company grows at a high rate of 15% per year for the first 10 years after its inception. What does that mean? For a start-up, the kind of company that invests every dollar of its earnings back into the business to grow faster, that would mean that for every invested dollar they generate 15 cents profit (earnings). If they start with a dollar today, they will have $1.15 to invest next year, then $1.15^2 = 1.3225$ the year after, and so on. However, this high rate of growth cannot persist indefinitely! Sooner or later, the company (if it survives, which is what we assume) will gradually mature, competition will catch up to its innovative products, and its growth is likely to taper off over time until the company turns into a “cash cow” that pays all of its earnings out as dividends to its shareholders, retaining nothing at all to shove back into the business for lack of profitable growth opportunities, which have by then been exhausted.

So, if we are trying to figure out what is the value of the company, we will need to discount those future earnings to the present. *The first 10 years of constantly growing earnings would represent a growing annuity*. Beyond that, growth rate will change, etc. and we won't worry for now about how to discount those more distant earnings. You will see a comprehensive example at the end of this book.

The formula for finding present value of a growing annuity is a bit cumbersome. We derive it like any other present value formula by first realizing that the growing annuity can be represented as a stream of lump sums, discounting those lump sums separately, and then adding up the present values:

$$PV_0 = \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \ldots + \frac{C_1(1+g)^{T-1}}{(1+r)^T}$$

This long series of additive terms reduces to the following:

$$PV_0 = \frac{C1}{r-g} * \left[1 - \left(\frac{1+g}{1+r}\right)^T\right] \qquad (3.5)$$

The usual rule applies: the formula gives you the value of growing annuity **one period prior to the first cash flow**. If you want the value more than one period prior to the first cash flow, you will need to discount the result of the formula additionally by the needed number of periods. You will practice with examples at the end of the chapter.

Example (Fairytale) Once upon a time, a company was born, and a lot of folks were invited to its birthday except the bad witch… You know how it goes. The bad witch shows up uninvited, of course, just when the celebration is in full swing, and casts the following spell:

"This company will grow as a beauty. It will generate a lucrative cash flow of $10,000 in year 1, and then its cash flows will grow at a constant rate of 15% per year for 10 years. After that, immediately after paying its final cash flow in year 10, the company will make a fatal strategic mistake and will cease to exist. There will not be the 11th year in its life!" (Demonic laughter)

Most guests are taken aback by this horrible spell. "I will not invest in this company!" they exclaim in utter disappointment. However, among those invited, there are a few shrewd investors. They realize that most people are now going to avoid the company – a company that in fact is worth something! It will be generating cash flow for 10 full years! If most people will avoid it, it can potentially come as a very good investment because it will be cheap for lack of demand.

The shrewd investors figure out that the company's cash flow will be somewhat risky, even though the bad witch seemed to forecast them with certainty. What if the witch was off a few thousand in either direction? So, given that cash flow is expected but not certain, some compensation for the risk is in order. The investors decide that a fair rate of return would be 25% per year.

How much is the company worth? The cash flows it will generate are nothing but a growing annuity, right? $T = 10$, $g = 15\%$, $C_1 = \$10{,}000$, $r = 25\%$. Throwing all of this into the formula for growing annuity, we obtain,

$$PV_0 = \frac{10{,}000}{0.25 - 0.15} * \left[1 - \left(\frac{1 + 0.15}{1 + 0.25}\right)^{10}\right] = \$56{,}561.15$$

Recall that a lot of other people are avoiding the company, discouraged by the evil spell. The low demand results in the market price of the company of only $50,000. The shrewd investors are quick to snatch it! They can now buy for $50,000 something that is worth much more!

Then 10 years go by, and a good witch comes to undo the evil spell. The company doesn't cease to exist after all, and its stock price skyrockets because everyone's expectations have been surpassed. The shrewd investors that bought in initially cash in and proceed to buy private islands and retire to live happily ever after. *The end*.

While this was an example of a growing annuity, it also told a stylized story of what's called "value investing." Value investors look for firms that have been beaten down by the markets for some bad news, figure out what those firms are really worth (that's hard, much harder than my example), and make money (some of the time). If you want to get rich like Warren Buffet, you need to learn how to value companies and find good deals.

Oh, and by the way, the other name for the bad witch is "equity analyst."

3.4 Negative Growth

Can the growth rate g be negative? Yes, absolutely. Sometimes, cash flows decline over time instead of growing. Don't worry: if the growth (or in this case, decline) rate is constant, and if it less than 100%, we can have cash flows declining indefinitely and never quite reaching zero.

Let's say growth rate is (-4%). That means, each subsequent cash flow will be 4% smaller than the previous one, or in other words,

$C_2 = C_1 (1 - 0.04) = C_1 (0.96)$

$C_3 = C_2 (1 - 0.04) = C_1 (0.96)^2$

$C_4 = C_3 (1 - 0.04) = C_1 (0.96)^3$

…

$C_t = C_{t-1} (1 - 0.04) = C_1 (0.96)^{t-1}$

…

For example, if $C_1 = 100$ and $t = 25$, then $C_{25} = 100 (0.96)^{24} = 37.54$

After 25 years, cash flow will decline from the initial 100 to a mere 37.54, but it will still be positive! What about 250 years? Just plug in,

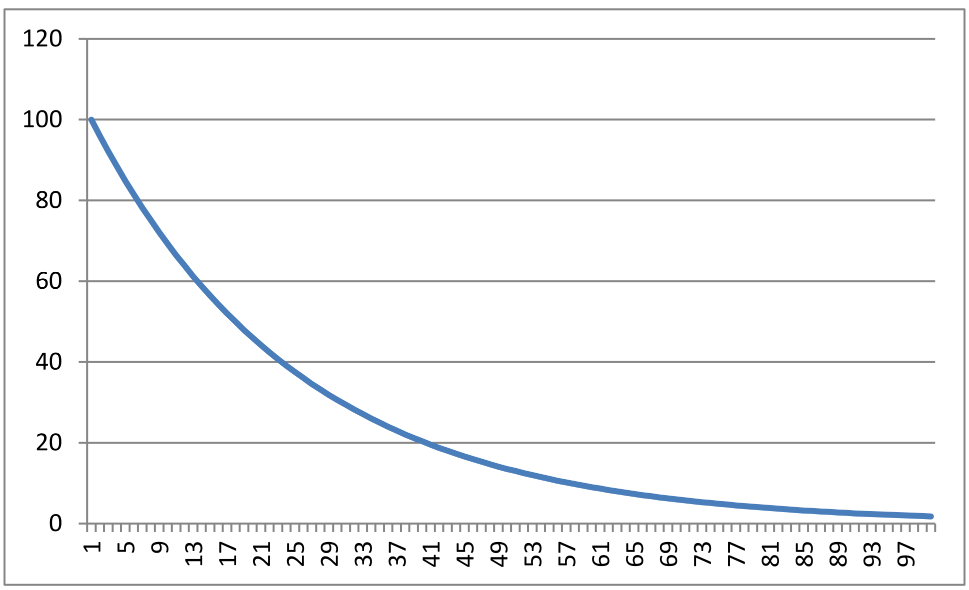
$$C_{250} = 100\,(0.96)^{249} = 0.003851$$

You can say that after 250 years of declining at a 4% annual rate, cash flow is zero for all practical purposes, but it's still positive and the firm is still worth *something*.

The graph below (Figure 3.1) shows the cash flows starting with $C1 = 100$ and for the next 100 years. You see they steadily decline from year to year, but the decline is not linear, and the cash flows never quite reach zero.

For example, we would like to find the value of this company at *t=50*. Will it be worth something at that point? Sure, it will because there will be cash flows (even though they will be small and declining) after that point in time.

Figure 3.1 Cash flows declining at 4% per year



Let's consider the firm as a going concern with zero debt. Let's further assume that its shareholders require a rate of return of 10%. We can calculate the value of the firm at *t=50* using the formula (3.3) for growing perpetuity:

$$PV_{50} = \frac{C_{51}}{r - g}$$

$$C_{51} = 100\,(0.96)^{50} = 12.99$$

Next, remember that g = -4%! That means,

$$PV_{50} = \frac{12.99}{0.1 - (-0.4)} = \frac{12.99}{0.14} = 92.78$$

As you can see from the example above, as miniscule as the cash flows become eventually, the firm will still be worth $92.78 as far out as 50 years from now.

What is the firm worth today? Well, applying the same formula for growing perpetuity (3.3),

$$PV_0 = \frac{C_1}{r - g}$$

$$PV_0 = \frac{100}{0.1 - (-0.4)} = \frac{100}{0.14} = 714.29$$

You might say who would ever want to buy this firm for $714.29 today knowing that its value is going to *decline* over time? Indeed, if you hold this company for 50 years and sell it at $t = 50$, then your stock value will decline from 714.29 to 92.78, and our per-year change in stock price would be found from formula (1.11) – see Chapter 1 if you forgot.

$$change\ in\ price\ per\ year = (\frac{FV}{PV})^{\frac{1}{T}} - 1$$

In our example, $FV = 92.78$, $PV = 714.29$, $T = 50$. Plugging the values into the formula above we get stock price change per year as (-4%)! Who in their right mind would want this stock? Why is it worth so much today??

However, you are forgetting about the cash flows! By buying the firm today you are getting the claim to ***all*** those cash flows starting from the first one. You will experience a decline in stock price – that much is true – but you will receive the cash flows in the interim!

What will be your annual rate of return from $t = 0$ to $t = 50$? Why, you might say formula (1.11) gives you that rate of return, and it is therefore (-4%). But wait! Formula (1.11) gives you the "rate of return" only in the case when there is one lump sum at the beginning, one lump sum at the end, and nothing in between! Go back to Chapter 1 to verify. In our case, however, there are plenty of cash flows in between! So, in this case, while the formula gives you the percentage change in stock price per year, it does not yet give you the full picture.

The rate of return shouldn't just be the change in the stock price from year to year. **That return must also include the cash flow**. See Table 3.4 below for the calculation of your annual rate of return. As you can see, your rate of return is 10% each year, just as you require as a shareholder, even though your stock price declines 4% per year. See again Section 3.2 in this chapter. Growth rate is (-4%), but rate of return is 10%! There is more to return than just growth.

Table 3.4 Calculation of the annual rate of return

Time	Cash Flow	Firm Value Calculation	Firm Value	Return Calculation	Return
0		=100.00/.14 =	714.29		
1	100.00	= 96.00/.14 =	685.71	= (100 + 685.71 - 714.29)/714.29 =	0.1
2	96.00	= 92.16/.14 =	658.29	= (96 + 658.29 - 685.71) / 685.71 =	0.1
3	92.16		631.95		0.1
4	88.47		606.68		0.1
5	84.93		582.41		0.1
6	81.54		559.11		0.1
7	78.28		536.75		0.1
8	75.14		515.28		0.1
9	72.14		494.67		0.1
10	69.25		474.88		0.1
…	…	…	…	…	…
50	13.53	= 12.99/.14 =	92.78		0.1
51	12.99		89.06	=(12.99 +89.06 - 92.78)/92.78 =	0.1
…	…	…	…	…	…

3.5 Next Chapter: Sneak Peek

Up until now, we always solved either for the present value, or for the future value, or for the payment. What if you need to solve for the number of payments T or for the discount rate r? You know how to solve for the rate in the case of lump sums (if you forgot, please review sections 1.7 and 1.8), but what about annuities or growing annuities or delayed growing perpetuities, etc.?

The solution would be too difficult to find analytically (from a cumbersome formula). At this point, it is time to introduce the power of financial calculators. This is, after all, the moment you may have been waiting for. Finally, finally you will learn how to calculate the rate of return on your investment in solar panels!

But even though I know you can't wait to turn the page, now is a good time for a break!

Chapter 3 Self-Check Problems

1. A share of stock is paying dividends that grow at an annual rate of 2% and are expected to continue in perpetuity. The dividend expected at the end of this year ($t = 1$) is $1.50 per share. If the shareholders want a return of 11% per year on this stock, what is the price of the stock today?

Answer: $16.67

2. Company XYZ is not paying dividends today. Analysts expect that it will pay its first dividend five years from now ($t = 5$). The dividend amount is expected to be $2 per share. From that point, dividends are expected to grow at an annual rate of 4%. If the shareholders want a return of 13% on this stock, what is the price of the stock today ($t = 0$)?

Answer: $13.63

3. Refer to the previous problem. You intend to hold the stock for 15 years and then you plan to sell it. What is the expected price of this stock at t=15?

Answer: $34.21

4. Today, you put $X of your money into an investment account earning 10% return per year. In one year ($t = 1$) you would withdraw $500 and afterwards your withdrawals would increase by 2% each year. (That is, your withdrawal at $t = 2$ will be $\$500 * 1.02 = \$510$; and so on.) You would like for these withdrawals to continue indefinitely with the same growth rate. Find X.

Answer: $6,250

5. Today, you put $X of your money into an investment account earning 10% return per year. In five years ($t = 5$) you would withdraw $500 and afterwards your withdrawals would increase by 2% each year. (That is, your withdrawal at $t = 6$ will be $500 * 1.02 = $510; and so on.) You would like for these withdrawals to continue indefinitely. Find X.

Answer: $4,268.83

6. An investment is expected to generate its first cash flow of $10,000 after one year ($t = 1$). After that, cash flows are expected to grow at a steady rate of 5% per year every year for 25 years. If the rate of return that fairly compensates investors for the risk is 15%, find the amount that they are willing to invest today.

Answer: $89,713.05

7. In your own words, explain the difference between the rate of return and the rate of growth.

CHAPTER 4: FINANCIAL CALCULATOR AND SOLAR PANELS REVISITED

"Technology is a useful servant but a dangerous master." – Christian Lous Lange

4.1 A handy tool, indeed!

Financial calculators are very handy tools. I feel comfortable introducing them now that you know the concepts for which they are applied.

In many calculators, there are buttons for P/Y (periods per year), I/Y (rate per year), PMT (periodic payment), PV (present value), FV (future value), and N or NPER (number of periods). I am basically going to show you which buttons to push to solve any of the problems we have covered by hand so far, and more!

I am not familiar with the exact model of your financial calculator, and you are responsible for figuring out the manual. My examples in this chapter are very generic.

In addition, I will show you the corresponding functions in Microsoft Excel.

A "generic setup" will be depicted somewhat like the following.

Example Suppose you want to solve for Future Value of a $100 lump sum deposited for 5 years at the stated rate 10%.

For this problem, you will set P/Y = 1 since compounding is annual. In addition, you will set the following inputs and solve for future value: N=5, I/Y = 10, PV = -100, PMT = 0, and solve for FV.

INPUT	5	10	-100	0	
KEY	N	I/Y	PV	PMT	FV
OUTPUT					161.05

Notice a couple things. First, in order to obtain a positive FV, you need to enter PV as a negative amount. Second, you need to enter the stated rate in percent. You will enter "0" for payment because this is not an annuity and there is no periodic payment.

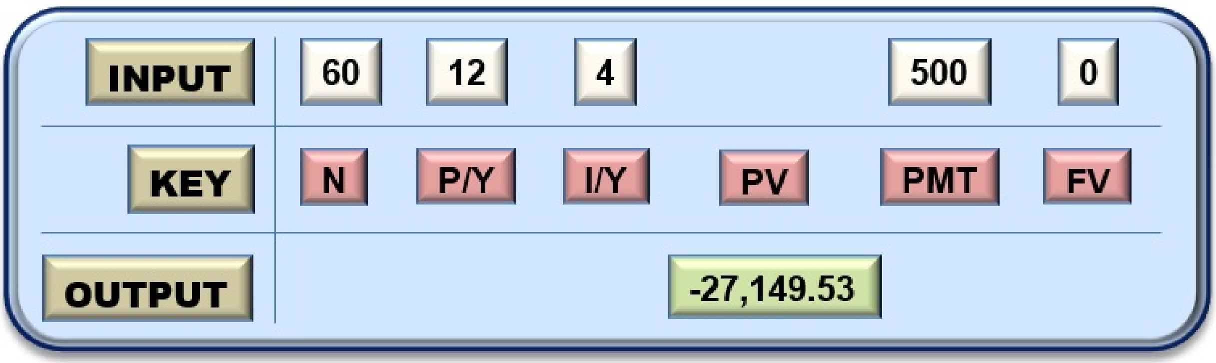
Notice that in Excel, the interest rate is entered as decimal. In Excel, the corresponding function will be

=FV(0.1,5,0,-100) = 161.05

Let us now practice a few problems.

Problem 4.1 Find present value of annuity, with monthly payment $500, stated rate 4%, and a 5-year term.

For this problem, you will set P/Y = 12 since compounding is monthly. In addition, you will set the following inputs and solve for future value: N=60, I/Y = 4, PMT = 500, FV = 0, and solve for PV.



Solve for $PV = -27{,}149.53$

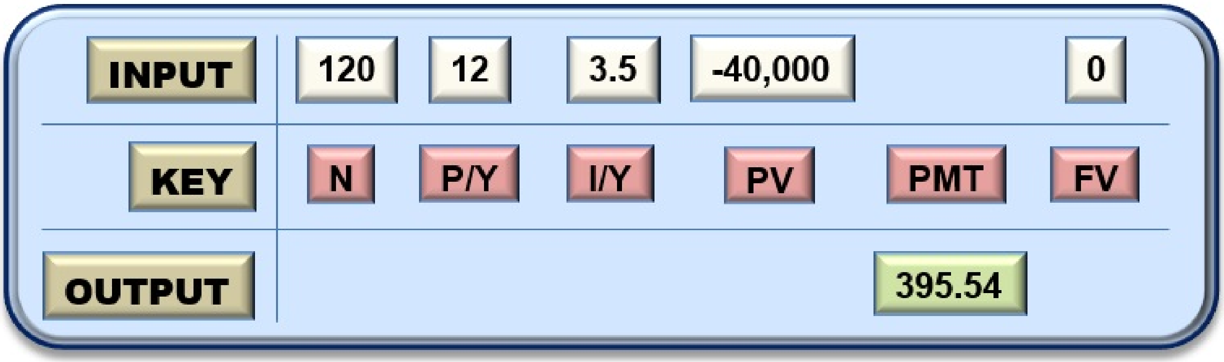
Note that if you enter PMT as a positive value, PV comes out negative. This makes sense, for example, if you make an investment now and receive payments later. On the other hand, if you receive a lump amount now (for example, borrow) and make monthly payments over a 5-year period, then you may want to enter PMT as a negative amount, in which case PV will come out positive.

Note also: in **TI-83 Plus** and **Excel** there is no *P/Y,* so the interest rate needs to be divided by 12. In addition, some calculators, such as **HP 10B II**, do not require even to multiply by 12 the number of years to get the number of periods; *N* would be simply imputed from the number years, and everything is automatic afterwards.

In Excel, =PV(**0.04/12**, 60, 500,0) = -27,149.53

Problem 4.2 Find monthly payment on a $40,000 loan taken for 10 years with a 3.5% stated rate.

For this problem, you will set P/Y = 12 since compounding is monthly. In addition, you will set the following inputs and solve for payment: N=120, I/Y = 3.5, PV = -40,000, FV = 0, and solve for PMT.



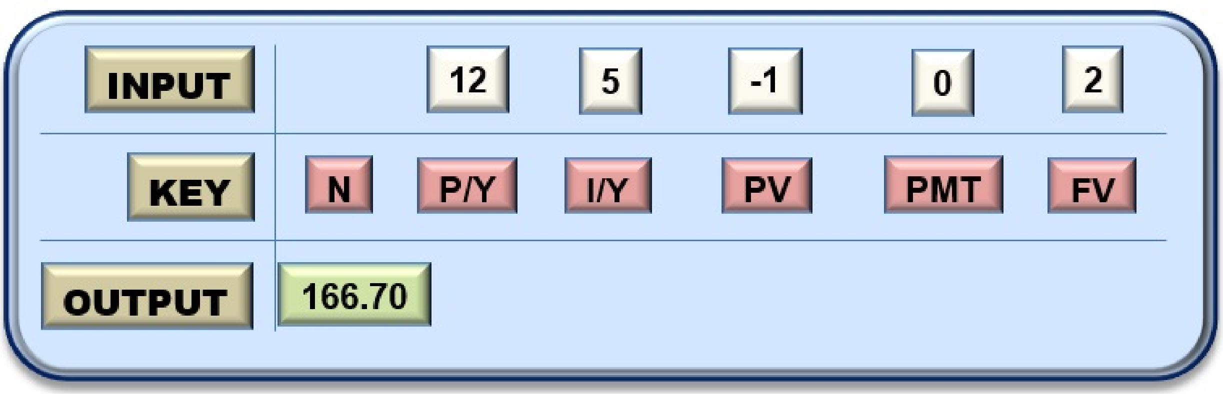
Solve for $PMT \;= 395.54$

In Excel, =PMT(0.053/12,120,-40000) = 395.54

If you would rather enter the present value as a positive amount (it is a loan, so the amount received today), then the answer for PMT will come out negative.

Problem 4.3 How long will it take for your money to double, if compounding is monthly and stated rate is 5%?

For this problem, you will set P/Y = 12 since compounding is monthly. In addition, you will set the following inputs and solve for number of periods: I/Y = 5, PV = -1, FV = 2, PMT = 0, and solve for N.



Solve for $NPER\ (N) \;= 166.70$

In Excel, *=NPER (0.05/12,0,-1,2) = 166.70*

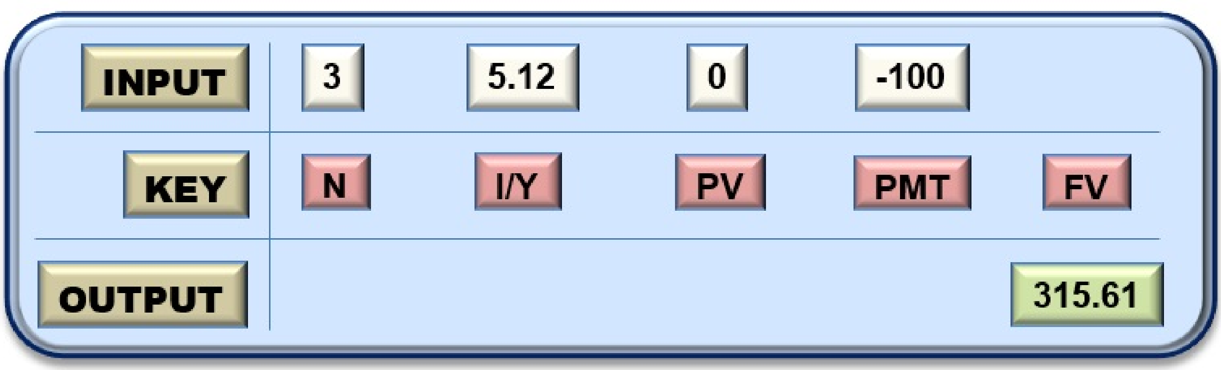
Of course, this is the exact same answer we obtained in Chapter 1, section 1.8! Except we solve much faster! Notice it is given as the number of MONTHS, not years! If you want to convert it into years, you will have to divide by 12 by hand!

Problem 4.4 There are three \$100 payments: the first at the end of year 1, the second at the end of year 2, the third at the end of year 3. However, even though payments are annual, compounding is monthly. Stated rate is 5%. Find future value.

If payments are annual but compounding is monthly, your calculator won't handle it immediately, but you can first solve for the EAR!

$$EAR = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 5.11619\%$$

Now you are ready to plug numbers into your calculator. For this problem, you will set P/Y = 1 since cash flows are annual and the EAR is the annual rate. In addition, you will set the following inputs and solve for future value: N=3, I/Y = 5.12, PV = 0, PMT = -100, and solve for FV.



Solve for $FV = 315.61$

In Excel, $= FV(0.0511619, 3, -100) = 315.61$

Problem 4.5 You deposit \$1000 in an account at $t = 0$. Additionally, starting at $t = 1$ and for 40 years total, you put in \$500 every year. If the rate earned on your investment is 7%, what's the ending amount in your account at the end of 40 years?

This is complicated: we have a lump sum of \$1000 at the start, plus the 40-year-long annuity of \$500 per year. Compounding is annual. How do we treat this problem?

By hand, you would have to find future value of the lump sum first; future value of annuity next, and then add them up.

$$FV\ (lump\ sum)\ =\ 1000\ (1.07)^{40}\ =\ 14{,}974.46$$

$$FV\ (annuity) = \frac{500}{0.07}\ [\ 1.07^{40} - \ 1] =\ 99{,}817.55$$

Total $FV\ =\ 14{,}974.46\ +\ 99{,}817.56\ =\ 114{,}792.01$

That's kind of long, to do it by hand. Let's use the financial calculator! For this problem, you will set P/Y = 1 since compounding is annual. In addition, you will set the following inputs and solve for future value: N=40, I/Y = 7, PV = -1,000, PMT = -500, and solve for FV.

INPUT	40	7	-1000	-500	
KEY	N	I/Y	PV	PMT	FV
OUTPUT					114,792.01

Solve for $FV\ = 114{,}792.01$

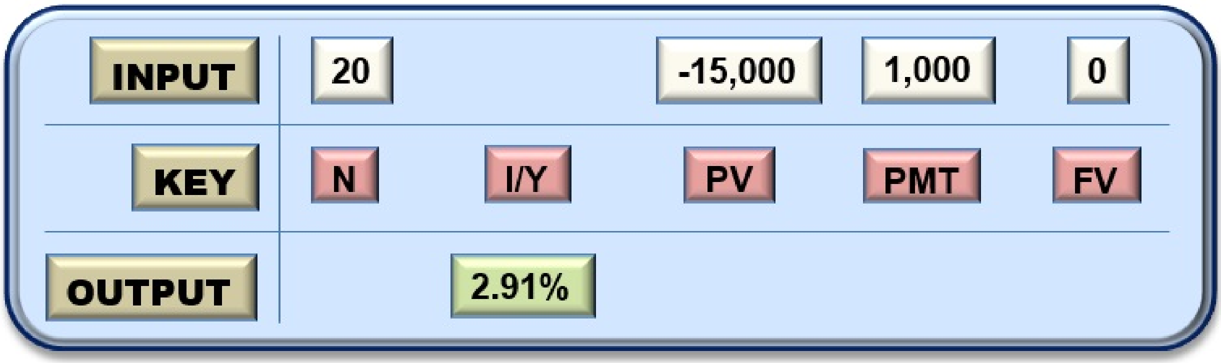
In Excel, $= FV(0.07{,}40, -500, -1000)\ =\ 114{,}792.01$

Aren't calculators a blessing (in disguise)?

Problem 4.6 You invest in solar panels. (Finally!!!) You pay $15,000 today and expect savings of $1000 per year for 20 years. Find the rate of return you are earning on this investment.

If you were solving for the rate of return by hand, you could spend hours plugging in different values to guess the answer. Luckily, you have a calculator!

For this problem, you will set P/Y = 1 since compounding is annual. In addition, you will set the following inputs and solve for rate: N=20, PV = -15,000, PMT = 1,000, FV = 0, and solve for I/Y.



Solve for $I/Y = 2.91\%$

In Excel, =RATE(20,1000,-15000,0) = 0.0291

Congratulations! Now you can calculate the rate of return on your investment and decide whether you like it or not.

This chapter will not have end-of-chapter self-check problems. Instead, go to the previous chapters and try to tackle any one of your choice that you have previously solved by hand, using a financial calculator this time!

4.2 Next Chapter: Sneak Peek

You are basically done with all the concepts I wanted to introduce in this book. It's just a matter of practice now, and the next chapter puts your skills to the ultimate test by throwing a few tough problems at you. Each one of those problems consists of the simple elements that you already learned, and it's just a matter of combining those elements together to solve the whole puzzle. Good luck!

CHAPTER 5: PUTTING IT TOGETHER

"Complex ideas may, perhaps, be well known by definition, which is nothing but an enumeration of those parts of simple ideas, that compose them." – David Hume, An Enquiry Concerning Human Understanding (1748)

Before anything else, I owe you one word: congratulations. The book is very difficult, it presents many challenging concepts heretofore unfamiliar, and you have gone through it thoroughly. This chapter can be considered as the capstone course. We are going to put the different elements learned in this book together to tackle the most complex financial structures. Hang in there!

5.1 Uneven Cash Flows: Calculating Net Present Value

You are evaluating a proposal to buy a new machine. The price is $120,500. In addition, shipment and installation costs are $5,500. Both these investments will take place today, at the time of purchase. The machine will last for ten full years.

The machine will generate the following cash flow.

1. Because you will depreciate the machine, you will save $3,615 each year in taxes. (Depreciation is not a cash flow by itself, but it is a tax-deductible expense for your business.)
2. The machine will generate after-tax savings of labor costs of $44,000 in year 1, and these savings will grow at the rate of 2% per year.
3. The machine will require maintenance, and you anticipate spending $10,000 in year 6 and another $20,000 in year 7.
4. At the end of ten years, you will sell the machine and the inventory, for a total of $25,000 after tax.

Is the machine worth buying if you require a rate of return 10%?

To answer this question, you will have to calculate Present Value of all future cash flows and compare them to the amount you pay today. If the investment needed today is "too high", that is, higher than the present value of all the future cash flows from the machine, then don't buy. Otherwise, buy.

All the cash flows from the project are provided and summarized in Table 5.1.

Table 5.1 Cash Flows

Time	Machine	Shipment and installation	Tax savings	Labor cost savings	Maintenance expenses	Sale of Machine	Total Cash Flow
0	(120,500)	(5,500)					(126,000)
1			3,615	44,000			47,615
2			3,615	44,880			48,495
3			3,615	45,778			49,393
4			3,615	46,693			50,308
5			3,615	47,627			51,242
6			3,615	48,580	(10,000)		42,195
7			3,615	49,551	(20,000)		33,166
8			3,615	50,542			54,157
9			3,615	51,553			55,168
10			3,615	52,584		25,000	81,199

The labor cost savings are calculated as 44,000 in year 1, then $44{,}000\ (1.02)\ =\ 44{,}880$ in year 2, then $44{,}880\ (1.02)\ =\ 45{,}778$ in year 3, and so on until year 10.

The negative amounts are your cash outflows (initial investments and maintenance expenses).

Finally, the Total Cash Flow column provides the sum across all columns, for each year.

It looks like every year the cash flow will be different and you may immediately recognize the "uneven cash flow" pattern. You can then discount each individual cash flow as a lump sum:

$$PV\ (future\ cash\ flows) = \frac{47{,}615}{1.1} + \frac{48{,}495}{1.1^2} + \frac{49{,}393}{1.1^3} + \dots + \frac{81{,}199}{1.1^{10}} = 307{,}456.98$$

Since this machine generates $307,456.98 in present value terms, but only costs $126,000 in present value terms, then of course the machine is certainly worth buying.

How else can you approach this solution? You can recognize that the Tax Savings column shows nothing but an *annuity*; and that the Labor Cost Savings column shows nothing but a *growing annuity*. Maintenance expenses are two *lump sums* in years 6 and 7, and sale of machine is a *lump sum* in year 10. You can find present value of each separately, then add them up, as follows. Please check the formulae for lump sum (Chapter 1), annuity (Chapter 2), and growing annuity (Chapter 3) if you have forgotten them!

$$PV\ (tax\ savings\ only) = \frac{3{,}615}{0.1}\left[1 - \frac{1}{1.1^{10}}\right] = 22{,}212.61$$

$$PV\ (labor\ cost\ savings\ only) = \frac{44{,}000}{(0.1 - 0.02)}\left[1 - \left(\frac{1.02}{1.1}\right)^{10}\right] = 291{,}513.68$$

$$PV\ (maintenance\ only) = \frac{-10{,}000}{1.1^{6}} - \frac{20{,}000}{1.1^{7}} = -15{,}907.90$$

$$PV\ (sale\ of\ machine\ only) = \frac{25{,}000}{1.1^{10}} = 9{,}638.58$$

$$Total\ PV = 22{,}212.61 + 291{,}513.68 - 15{,}907.90 + 9{,}638.58 = \$307{,}456.98$$

The value is the same as before. I personally don't have a preference over how exactly you choose to find present value of the project's cash flows, so long as you arrive at the same result. That said, it is imperative that you are perfectly comfortable with doing it either way.

By the way, the difference between the total present value of all the future cash flows, on the one hand, and the current investment, on the other hand, has a name. It is called ***Net Present Value (NPV)***.

$$NPV = 307{,}456.98 - 126{,}000 = 181{,}456.98$$

That's how much value the machine will add to your business if you buy it!

NPV can be found using financial calculator or Microsoft Excel. For this particular example, when cash flows are different every period, on a **TI-83 plus** scientific calculator, you will have to go to the APPS button and select NPV() function. In our example we will have to write the following function:

NPV (Rate, Initial Investment, {Cash Flows})

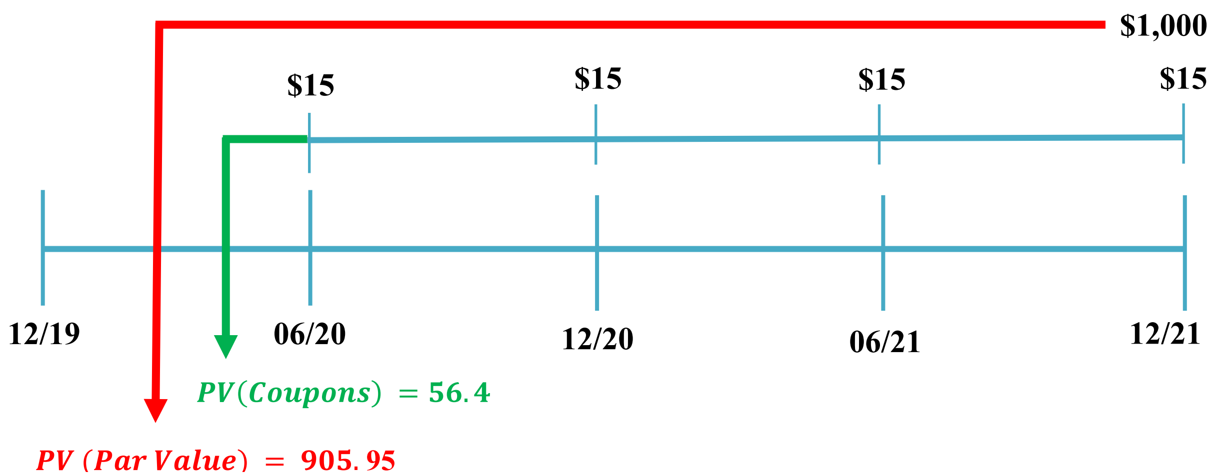
For "Input" you will type, *NPV(10,-126600,{47615, 48495, 49393, 50308, 51242, 42195, 33166, 54157, 55168, 81199})*

Then "Output" will be the result of solving for present value.

INPUT	**NPV(10,-126600,{47615, 48495, 49393, 50308, 51242, 42195, 33166, 54157, 55168, 81199})**
OUTPUT	**181,456.98**

Learn the NPV function on your own calculator! It is very handy in many financial calculations.

While I'm only presenting a little example here, NPV is a concept you will see again and again in finance when you analyze corporate capital budgeting decisions! This tiny example is just a preview of what's coming in your studies of finance.



$$Total\ PV = 56.40 + 905.95 = 962.38$$

5.2 Annuity and Lump Sum: Valuing a Bond

Today is December 2019. You are a bondholder of Amazon. The bond "matures" in December of 2021, which means it will be paying you regular interest payments (coupons) twice a year, each June and December, with the first coupon coming up in June 2020 and the last coupon to be paid in December 2021. The bond has a "par value" of $1,000, which means in December 2021 you will also receive a check for $1,000, in addition to the final coupon payment. The coupons are $15 each. The discount rate is 5% per year. What's the value of the bond today?

By now you know that the value today is the present value of the future cash flows. Looking at the timeline, we notice that there are two types of cash flows in this case. The first type is an annuity, which has four regular $15 cash flows (coupon payments). The second is the lump sum of $1,000 to be paid at the end.

You already know that if your cash flow structure consists of some elements, you can find present value as the sum of present values of those elements. Hence, we need to find (1) present value of the annuity, (2) present value of the lump sum, and (3) add them together.

Doing this calculation by hand, present value of annuity is found using the number of payments $T = 4$, periodic rate $r = \frac{5\%}{2} = 2.5\%$, and payment $C\ =\ 15$.

$$PV\ (coupons\ only) = \frac{15}{0.025}\left[1 - \frac{1}{1.025^4}\right] = \$56.43$$

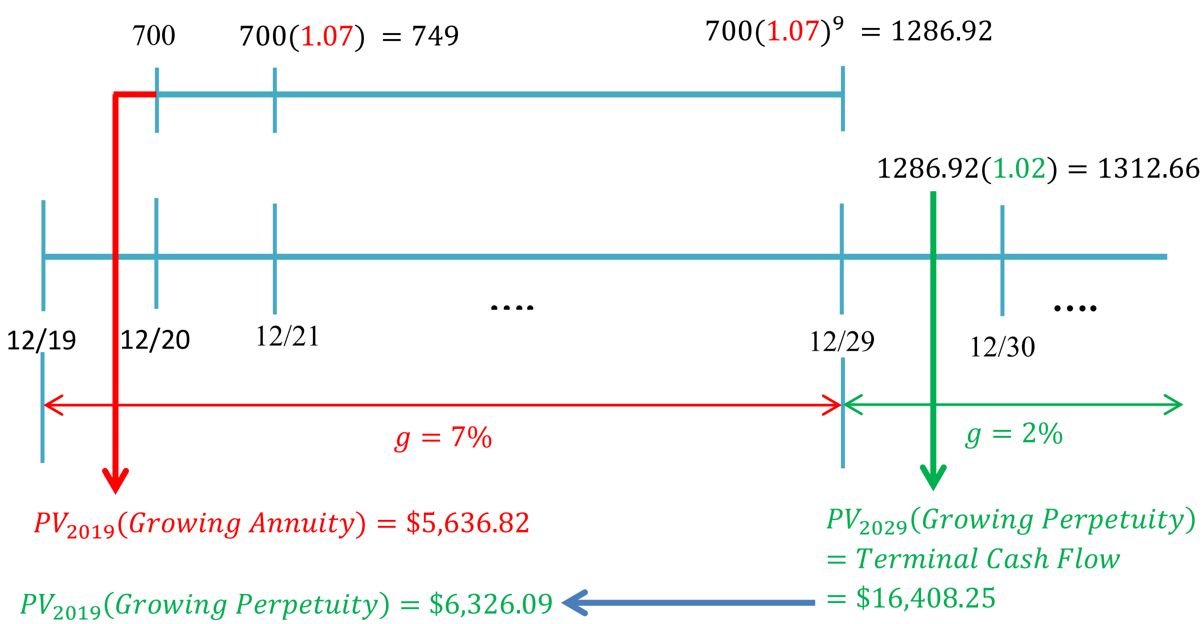
Next, present value of the lump sum is

$$PV\ (par\ value\ only) = \frac{1000}{1.025^4} = \$905.95$$

Finally, the total value of the bond is $\$56.43\ +\ \$905.95\ =\ \$962.38$

5.3 Growing Annuity and Delayed Growing Perpetuity: Valuing a Company

Today is December 31, 2019. You are valuing a company. As soon as you're done you will meet friends to celebrate New Year's Eve. But for now, you are diligently focused on the task. You forecast that the company will generate cash flow of $700 million in 2020, and you estimate that the cash flows will grow at about 7% rate per year until 2029. After that, cash flows will still grow, but not quite so fast. You make a conservative forecast that growth rate will slow down to only 2% per year. You forecast that the company can maintain that low stable growth rate perpetually. If you require a rate of return of 10%, find the price of the stock today.



$$\boldsymbol{Total\ PV_{2019} = 5{,}636.82 + 6{,}326.09 = 11{,}926.91}$$

What are we dealing with here? We observe two distinct elements in the cash flow structure: a growing annuity from 2020 to 2029, followed by a growing perpetuity from 2030 on. As before, we can find present value of the whole thing by adding up present values of the parts. There are ten payments in the growing annuity.

Present value of growing annuity only =

$$PV_{2019}\ (growing\ annuity) = \frac{700}{0.10 - 0.07}\left[1 - \left(\frac{1.07}{1.10}\right)^{10}\right] = \ \$5{,}636.82$$

Note that the first cash flow \$700 occurs in 2020, and we are finding present value in 2019, so there is exactly one period gap, which means the formula gives us the answer we're looking for.

Next, we value the second part, the growing perpetuity. The first payment of that series occurs in 2030. What is the amount of that payment? We know that post-2029 growth rate will be 2% per year. Therefore,

$$C_{2030} \ = \ C_{2029}\,(1.02)$$

Oh, but we aren't given the cash flow in 2029! Should it bother us? Not really, since we can figure it out given that C_{2020} = 700 and that growth rate until 2029 is 7% per year. Hence,

$$C_{2029} \ = \ C_{2020}\,(1.07)^9 = 700(1.07)^9 = \$1{,}286.92$$

Now we're ready:

$$C_{2030} \ = 1{,}286.92(1.02) = \$1{,}312.66$$

The formula for present value of a growing perpetuity is

$$PV_0 \ = \frac{C_1}{r - g}$$

In this case, C_1 is the first cash flow of the growing perpetuity, $C_{2030} = \$1{,}312.66$; $g \ = \ 2\%$, and *r* is still 10%.

Plugging in,

$$PV_{2029}\ (growing\ perpetuity) = \ \frac{1{,}312.66}{(0.10 - 0.02)} = \ \$16{,}408.25$$

Are we done? Not yet!! (Sorry.) We just found the present value of the growing perpetuity as of 2029! This is because the first cash flow occurs in 2030, and the formula only takes us one period back!

Let's stop and think about the amount \$16,408.25. What's that amount? It is the amount of money such that you would be indifferent between receiving that amount as a lump sum in 2029 and receiving the claim to the growing perpetuity instead. In other words, you can choose to receive the lump sum in 2029, or you can choose to receive the growing perpetuity that begins in 2030 and continues forever. Same difference, as they say.

This lump sum amount in 2029, which is equivalent to the entire growing perpetuity starting 2030, has a name. It is called ***Terminal Cash Flow (TCF)***.

But we need the value of this growing perpetuity today, in 2019! Therefore, we need to take \$16,408.25 and discount it additionally ten more years before we finally land in 2019.

$$PV_{2019}\,(growing\ perpetuity) = \frac{16{,}408.25}{1.10^{10}} = \$6{,}326.09$$

Now, we are ready to add up the present value of growing annuity \$5,636.82, and the present value of *delayed* growing perpetuity \$6,326.09.

The total amount is

$PV_{2019}\,(total) = 5{,}636.82 + 6{,}326.09 = \$11{,}962.91$

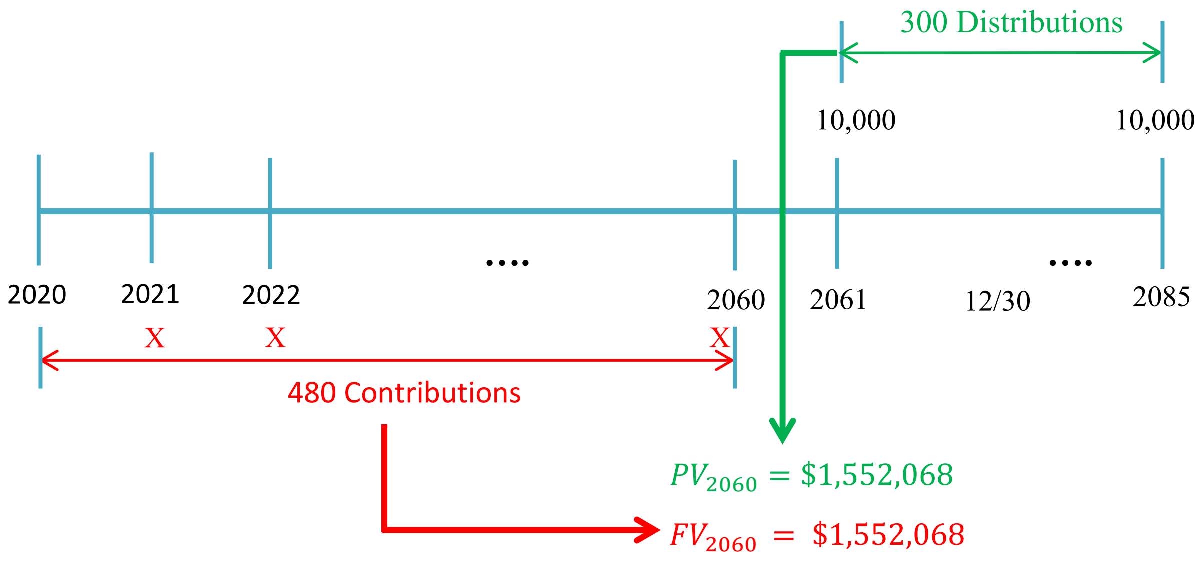
The value of the company today is \$11,962.91 million, or roughly \$12 billion.

This example is one of the most complex examples you will encounter in the whole introductory finance course, so I commend you for plowing through it. It is very important that you see not only how, but also why we perform every step of the solution the way we do.

5.4 Two annuities: Retirement Planning

Today is year 2020. I am saving for retirement, putting X into my retirement account every month. I would like to retire in 40 years, and by the time I retire I would like to have enough money saved to be able to withdraw \$10,000 per month for 25 years. If my retirement money is invested at the average annual rate of 6%, find X.

Wow. This time we are dealing with two annuities: one consisting of retirement contributions (monthly for 40 years or $40 * 12 = 480$ contributions) and the other consisting of retirement distributions (monthly for 25 years or $25 * 12 = 300$ distributions). Stated rate is 6%, so periodic rate is $\frac{6\%}{12} = 0.5\%$



What's the amount \$10,000? That is the cash flow of the second annuity. Then the unknown amount X is the cash flow of the first annuity.

To solve for X, you need to know the target amount to be available at retirement, which is the future value of the first annuity. But the key to seeing the solution is realizing that the future value of the first annuity is the present value of the second one!

Using the second annuity, solve for the lump sum available at retirement:

$$PV_{2060} = \frac{10{,}000}{0.005}\left[1 - \frac{1}{1.005^{300}}\right] = 1{,}552{,}068.64$$

Now we know that the future value of the first annuity is 1,552,068.64 – and we are ready to solve for X!

$$FV_{2060} = 1{,}552{,}068.64 = \frac{X}{0.005}[1.005^{480} - 1]$$

Solving,

$$X = 779.35$$

That's the amount you need to contribute into your retirement account, monthly for 40 years straight, in order to save the amount that would allow you to live on \$10,000 per month every month for 25 years post-retirement!

5.5 Next Chapter: Sneak Peak

What, there is the next chapter??? Didn't I just complete the "capstone" one? I'm ready to graduate from this book!

The next chapter is for extra practice if you feel you need it. (I do strongly encourage you to work through it, but it will ultimately depend on your professor whether it is assigned.)

Chapter 5 Self-Check Problems

1. You want to start a retirement account, intending to contribute $500 towards retirement monthly. You plan to retire in 40 years. Your adviser suggests that you invest in mutual funds, expecting an average annual return of 6%. Calculate the amount of the monthly distribution post-retirement over a 20-year period.

Answer: $7,133.83

2. All parts of this problem are related and follow from each other.

a. You plan to buy a house in five years. The house of your dreams costs $200,000 today but you expect real estate prices to appreciate at a rate of 3% annually. What is the expected price of the house in five years?

Answer: $231,854.81

b. At the time when you buy the house, you will pay 20% of the house price out of pocket. For the remaining 80%, you will need a mortgage loan for 15 years with monthly mortgage payments. You believe the rate will be 5% APR. Find the amount of the mortgage loan you will need.

Answer: $185,483.85

c. Find the monthly payment you will be making on the mortgage loan.

Answer: $1,466.79

d. Find the EAR of the mortgage loan

Answer: 5.1162%

3. Corporate bonds of ABC, Inc. pay $35 coupons every six months, have 10 years to maturity, and pay additionally the "par value" of $1000 at maturity. The bond investors price the bond to earn a 10-year yield of 8% per year. Find the current market price of the bond.

4. A firm considers a new project. The project will require initial investment (t=0) of 1,000,000 and will have an economic life of ten years. The project will generate the following cash flows:

- Tax savings from depreciation, $25,000, per year for ten years starting t=1
- Additional revenues of $75,000 after tax, per year for ten years starting t=1
- Additional cost savings of $30,000 after tax, per year for ten years, starting t=1
- Operating expense of $50,000 after tax, once in year 5
- Resale value of the equipment of $350,000 after tax, once in year t=10

If investors require an annual rate of return of 8%, will they pay $1,000,000 in exchange for these future cash flows?

5. You buy a share of stock today. The stock is paying dividends, and the first dividends to be paid at the end of the first year (t=1) is D_1 = 1.50. Dividends are expected to grow at the annual rate of 8% for ten years. Thereafter, starting in year 11, growth rate will fall to 2% per year and will ensue indefinitely.

a. Find the present value today (t=0) of the growing annuity represented by the first ten years of dividends.
b. Find D_{10}
c. Find D_{11}
d. Find Terminal Cash Flow, i.e. the present value at t=10 of the growing perpetuity that starts with D_{11}
e. Find the present value today (t=0) of the Terminal Cash Flow
f. Find the total present value of all future dividends today, which should be the price of the stock

CHAPTER 6: CONCEPT RECOGNITION

"Keep calm, it's only another panic attack." - Unknown

Before leaving you, dear reader, let me help you with one more thing. I find that students often struggle with concept recognition. That is, the same concept is not recognized if framed differently. Let me illustrate this problem with the following example.

Frame 1 *You invest in a project. At the end of the year, and for the next 20 years, the project will pay $1000 per year. Then at the end of year 20, you plan to sell the project for $100,000. What is the project worth today if you want to earn 10% return on investment?*

Frame 2 *You put a certain amount of money in the investment account that earns 10% per year. At the end of each year for the next 20 years, you plan to withdraw $1000 from your account. Then after 20 such withdrawals, you want to still have $100,000 available in the account. How much money do you need to invest now to meet this goal?*

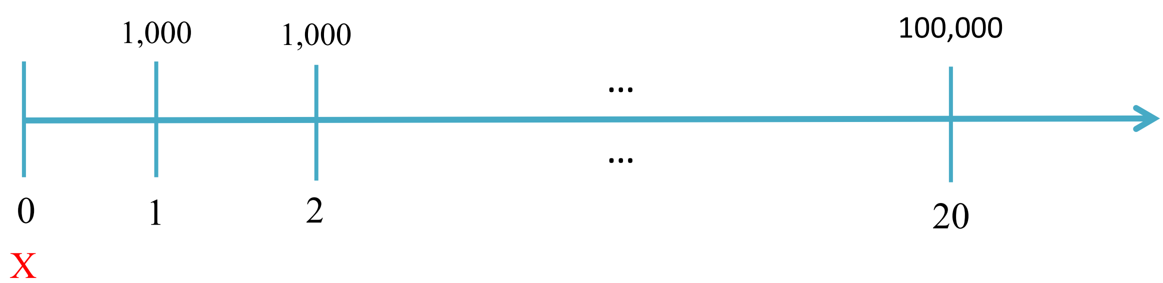
These two frames are identical but asking the same question differently. What I find is that if I solve Frame 1 in class but ask Frame 2 on the exam, students cannot recognize the concept, panic, have a mental block, and fail to solve the problem. This brief tutorial is meant to help you with this issue so you can "see through the frame" down to the concept. That way you will know which formula you need, how to set up the equations, etc. and ultimately solve the question.

1. Never, ever panic. This may sound like a cliché, but I cannot tell you how many times I've seen bright hardworking students literally incapacitated by panic. Rest assured: if your professor asks you a question on the exam, you have solved similar questions in class. You know how to solve it. You've solved it before. It's just the framing that is throwing you off and making the question look like something you've never seen.
2. Get your pen and map the cash flows. Make it a habit. Draw the timeline and mark the years and each cash flow in each year. Illustration:

Frame 1 *You invest in a project. At the end of the year, and for the next 20 years, the project will pay $1000 per year. Then at the end of year 20, you plan to sell the project for $100,000. What is the project worth today if you want to earn 10% return on investment?*

In this problem, you have 20 cash flows $1000 each. In addition, at the end of year 20, you have a single cash flow of $100,000. The question is, "what's the project worth", i.e., how much are you willing to pay today for the claim to all those future cash flows? You will be investing today, so paying some amount. That's a dollar value, it occurs today, you put it on the timeline also. It is unknown, so I'll mark it as X. I'll put X below the line, and the other amounts above the line, to differentiate the outflows from the inflows.

Drawing:



You may ask me, okay I can see how $1000 per year is a cash flow, but what about the final $100,000, is it also just a cash flow or the Future Value of the project? The answer is it doesn't matter for the purpose of drawing the picture. At this point, you're just drawing.

Additionally, you're provided the rate of return of 10% per year, and you're told that time is measured in years (e.g., as opposed to months). The rate of 10% applies to the entire timeline, to each year. This means, whichever compounding or discounting you will perform eventually, that's the rate you use.

Frame 2 *You put a certain amount of money in the investment account that earns 10% per year. At the end of each year for the next 20 years, you plan to withdraw $1000 from your account. Then after 20 such withdrawals, you want to still have $100,000 available in the account. How much money do you need to invest now to meet this goal?*

Get the pen and draw the picture. You will be receiving $1000 each year for 20 years. Then at the end of year 20 you will have another $100,000. You need to put X into the account today in order to make all of that happen. At this point it should sound familiar! It shouldn't take long to recognize that the picture is identical to the one for Frame 1. The rate is the same 10% as before. This is the exact same problem. So, once you've solved Frame 1, you should solve Frame 2 with your eyes closed.

3. Solve.

Once you've translated the problem into the language of cash flows, you are no longer distracted by framing / wording details. In this problem, the initial outflow X is exchanged for all the future inflows. The outflow X occurs first, the other cash flows occur later, hence this is a Present Value setup. Looking at the picture, we recognize that we're dealing with (a) a 20-year annuity of $1000 per year, and (b) a lump sum of $100,000 in year 20. So, finding the present value is accomplished in three steps: (a) present value of the annuity, (b) present value of the lump sum, (c) total present value = (a) + (b).

a. Present value of annuity

$$PV = \frac{1000}{0.1} * \left(1 - \frac{1}{1.1^{20}}\right) = 8{,}513.56$$

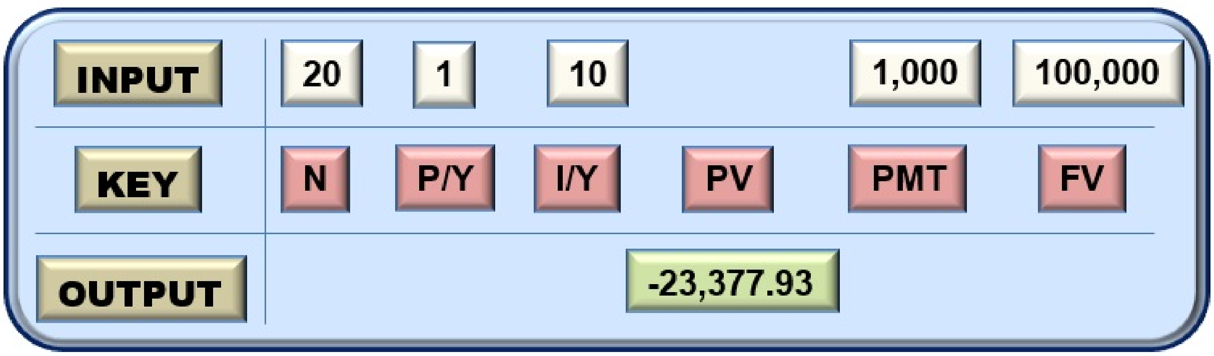
b. Present value of lump sum

$$PV = \frac{100{,}000}{1.1^{20}} = 14{,}864.36$$

c. Total present value

$$PV = 8{,}513.56 + 14{,}864.36 = 23{,}377.93$$

Also, you can use the financial calculator to arrive at the same answer. For this problem, set P/Y = 1 since the compounding is annual. Enter N = 20, I/Y = 10, PMT = 1,000, FV = 100,000, and solve for PV.



Done.

This result may seem weird, what, how come I just need to pay a mere 23K and be able to get 20 withdrawals of \$1000 each (in your mind, $20 * 1000 = 20{,}000$ already) and still have something amounting to \$100,000 left at the end??

Compounding is the answer! The rate of 10% is quite high. See for yourself.

Table 6.1 Result

	Beginning	After earning 10%	Withdrawal	End
1	$23,377.93	$25,715.72	1000	$24,715.72
2	$24,715.72	$27,187.29	1000	$26,187.29
3	$26,187.29	$28,806.02	1000	$27,806.02
4	$27,806.02	$30,586.62	1000	$29,586.62
5	$29,586.62	$32,545.28	1000	$31,545.28
6	$31,545.28	$34,699.81	1000	$33,699.81
7	$33,699.81	$37,069.79	1000	$36,069.79
8	$36,069.79	$39,676.77	1000	$38,676.77
9	$38,676.77	$42,544.45	1000	$41,544.45
10	$41,544.45	$45,698.90	1000	$44,698.90
11	$44,698.90	$49,168.79	1000	$48,168.79
12	$48,168.79	$52,985.66	1000	$51,985.66
13	$51,985.66	$57,184.23	1000	$56,184.23
14	$56,184.23	$61,802.65	1000	$60,802.65
15	$60,802.65	$66,882.92	1000	$65,882.92
16	$65,882.92	$72,471.21	1000	$71,471.21
17	$71,471.21	$78,618.33	1000	$77,618.33
18	$77,618.33	$85,380.17	1000	$84,380.17
19	$84,380.17	$92,818.18	1000	$91,818.18
20	$91,818.18	$101,000.00	1000	$100,000.00

Et voila!

If you want to practice, here are some framing exercises for you. For each problem from Group 1, find the "conceptual match" in Group 2. Well, and solve. To make things harder, I've changed the numbers around so you can't identify the match by the numbers (including dollar amounts, compounding frequency, and the rate). Find the conceptual match, i.e. "the same problem, just different numbers." This is a challenging task. But unless you practice you cannot learn. This is one of those things learned by doing.

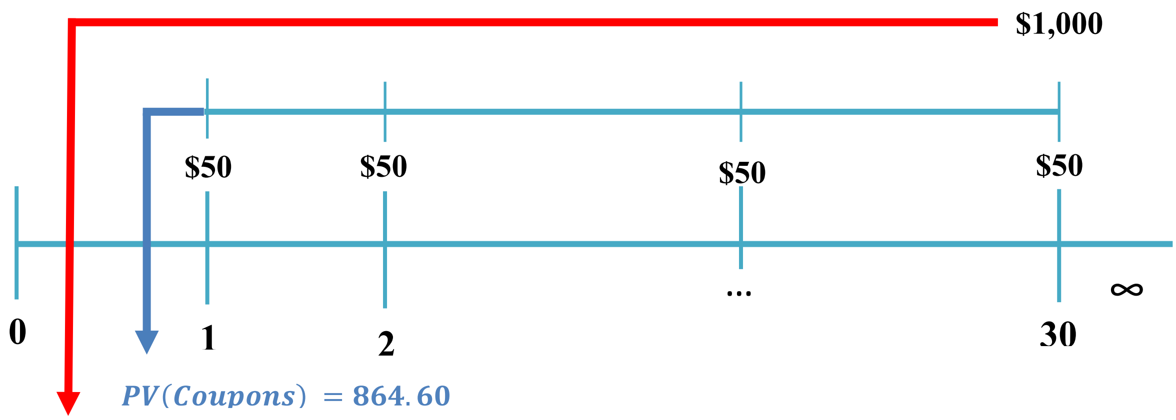
Group 1

Problem 6.1 You invest in a bond. This bond promises to pay $50 (called "coupon") every six months for 15 years. At the end of 15 years (called "maturity") it will additionally pay a $1000 final payment, called "par value." If you expect to earn 8% per year on this bond, what's the price of the bond today? Answer: $1,172.92

Solution

"What is the price of the bond today" means, "what is the amount of cash that you'd be willing to exchange today for the claim to all of those future cash flows?" – or, alternatively, "what is the amount you need to invest today that will generate all these future cash flows?" That's just the present value of those cash flows.

You have 30 cash flows (twice a year for 15 years) of \$50 each (should be recognizable as annuity); and then a single \$1000 cash flow at the end. Since time is measured in six-month steps, we should use semiannual compounding (compounding frequency m=2). Periodic rate then becomes $\frac{8\%}{2} = 4\%$.



$\boldsymbol{PV(Coupons) = 864.60}$

$\boldsymbol{PV\ (Par\ Value) = 308.32}$

$\boldsymbol{Total\ PV = 864.60 + 308.32 = 1{,}172.92}$

$$PV\ (annuity) = \frac{50}{0.04} * \left(1 - \frac{1}{1.04^{30}}\right) = 864.60$$

$$PV\ (lump\ sum) = \frac{1000}{1.04^{30}} = 308.32$$

$$Total = 864.60 + 308.32 = 1{,}172.92$$

Also, you can use the financial calculator to arrive at the same answer. For this problem, set P/Y = 2 since the compounding is semiannual. Enter N = 30, I/Y = 8, PMT = 50, FV = 1,000, and solve for PV. The answer will come out negative, -1,172.92. Disregard the negative sign.

INPUT	30	2	8*		50	-1000
KEY	N	P/Y	I/Y	PV	PMT	FV
OUTPUT				-1,172.92		

*Since we input 2 for P/Y we don't have to make $I/Y = 4\%$, if you put $P/Y = 1$ then you will have to put $I/Y = 4\%$, Try it out!

In Excel, enter = PV(0.04, 30, 50, 1000) = -1,172.92

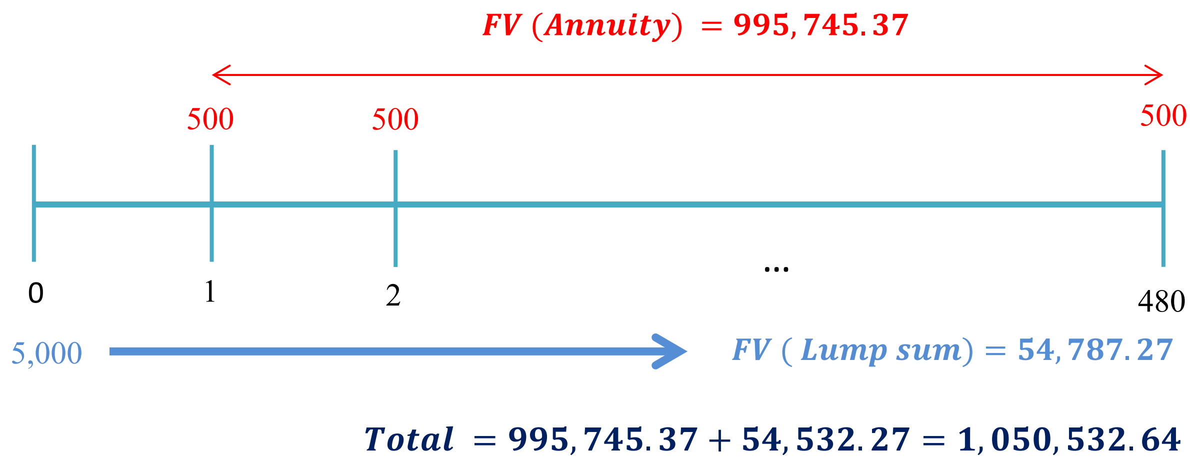
Problem 6.2 You start a business and expect to earn 6% per year. The business requires an initial investment of $5,000 to buy equipment. After that, you will need to make additional investments of $500 per month every month in operating expenses. You plan to sell your business after 40 years. What is the minimum sale price that will compensate for all your expenses? Answer: $1,050,532.64

Solution

"What is the minimum sale price that will compensate for all your expenses?" means, "what's the amount of cash that has the same value as your expenses?" You will be receiving this amount of cash in the future (40 years from now). Hence, the expenses must also be valued as of that point in time. Meaning, we must determine the future value of all those expenses at the 40-year-mark.

This may be confusing, since the expense occurs when it occurs; what's "the future value" of it? The future value is the amount of money that you would have had in the future if you hadn't incurred this expense. In this case, interest rate is 6% and compounding is monthly. So, spending $1 today means I could have had $\$1 * \left(1 + \frac{0.06}{12}\right)^{40*12} = \$10.96$ in 40 years had I not spent that dollar. In future value terms then, $1 expense today is equivalent to $10.96 at the 40-year mark! So, I would require $10.96 at that time to compensate me fairly for the dollar spent today.

Hence, in this problem the solution boils down to finding the Future Value of all my business expenses. We have a lump sum expense of $5000 at the start; and then $40 * 12 = 480$ cash flows of $500 each (annuity). The total value of all expenses at the 40-year mark is the sum of the future values of these components.



$$FV(lump\ sum) = 5000 * \left(1 + \frac{0.06}{12}\right)^{480} = 54{,}787.27$$

$$FV\ (annuity) = \frac{500}{\frac{0.06}{12}} * \left(\left(1 + \frac{0.06}{12}\right)^{480} - 1\right) = 995{,}745.37$$

$$Total = 54{,}787.27 + 995{,}745.37 = 1{,}050{,}532.64$$

Also, you can use the financial calculator to arrive at the same answer. For this problem, set P/Y = 12 since the compounding is monthly. Enter N = 480, I/Y = 6, PV = -5,000, PMT = -500, and solve for FV.

INPUT	480	12	6	-5000	-500	
KEY	N	P/Y	I/Y	PV	PMT	FV
OUTPUT						1,050,532.64

In Excel, enter = FV (0.005, 480, -500, -5000) = 1,050,532.64

Problem 6.3 You put $50,000 in your investment account today, earning 8% per year. You would like to withdraw a certain amount X each year for 10 years, such that you will still have $50,000 available upon the last withdrawal. Find X. Answer: $4,000

Solution

You will be withdrawing *the same* value X every year for 10 years. This should ring "annuity" in your mind. Let's bank that thought and put it aside.

Next, the idea is the following. Suppose you withdraw nothing. If you start with $50,000 and leave it in the account at 8% per year for 10 years, how much would you have ended up with?

$$FV\ (lump\ sum) = 50{,}000 * 1.08^{10} = 107{,}946.25$$

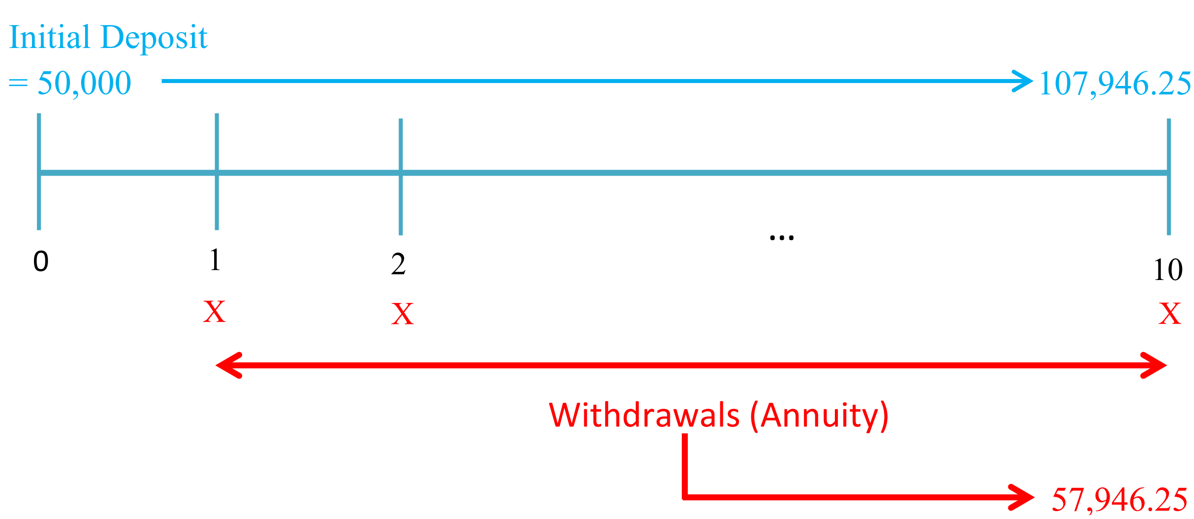
But you're told that you are only going to have $50,000 at the end.

$$107{,}946.25 - 50{,}000 = 57{,}946.25$$

Notice! The amount $57,946.25 applies to the 10-year mark; that's the amount you will be "missing" at THAT TIME because of all the withdrawals you make between now and then.

Question: how much should you withdraw each year, so that the FUTURE VALUE of all these withdrawals would amount to $57,946.25?

Now we can reach into our thought bank and recall that the series of $X withdrawals represents an annuity! Hence, we can use the formula for FV (annuity). Compounding frequency is m=1 (compounding is annual).



$$\boldsymbol{Balance = 107{,}946.50 - 57{,}946.50 = 50{,}000}$$

$$FV = 57{,}946.25 = \frac{X}{0.08} * (1.08^{10} - 1)$$

Solving for X,

$$X = 4000$$

This problem can be solved using a financial calculator or Excel. With a financial calculator, set P/Y = 1 since the compounding is annual. Enter N = 10, I/Y = 8, PV = -50,000, FV = 50,000, and solve for PMT.

INPUT	10	1	8	-50,000		50,000
KEY	N	P/Y	I/Y	PV	PMT	FV
OUTPUT					4,000	

In Excel, enter =PMT(0.08, 10, -50000, 50000) = 4,000

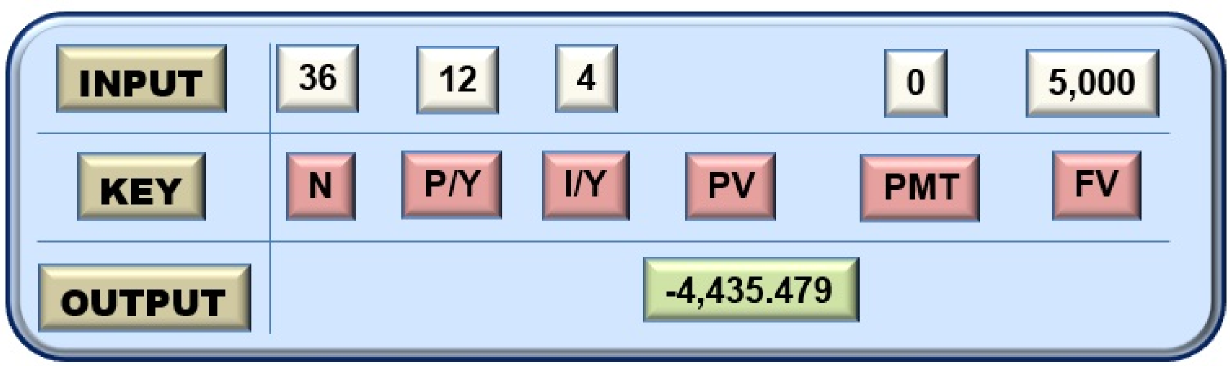
Problem 6.4 You want to have $5,000 in your account in three years. If the rate of return is 4% per year, and compounding is monthly, how much do you need to deposit in your account today? Answer: $4,435.49

Solution

Well, we've done this type of problem many times already. Just thought I'd give you a simple question for a change. You have the target amount given, $5000. You need to find the present value of that amount with monthly compounding.

$$PV\ (lump\ sum) = \frac{5000}{\left(1 + \frac{0.04}{12}\right)^{3*12}} = \$4{,}435.49$$

With a financial calculator, set P/Y = 12 since the compounding is monthly. Enter N = 36, I/Y = 4, PMT = 0, FV = 5,000, and solve for PV.



In Excel, enter = PV (0.04/12, 36, 0, 5000) = -4,435.49

Problem 6.5 Find the future value after 5 years of a lump sum of $100 if $APR = 10\%$ and compounding is monthly. Answer: $164.53

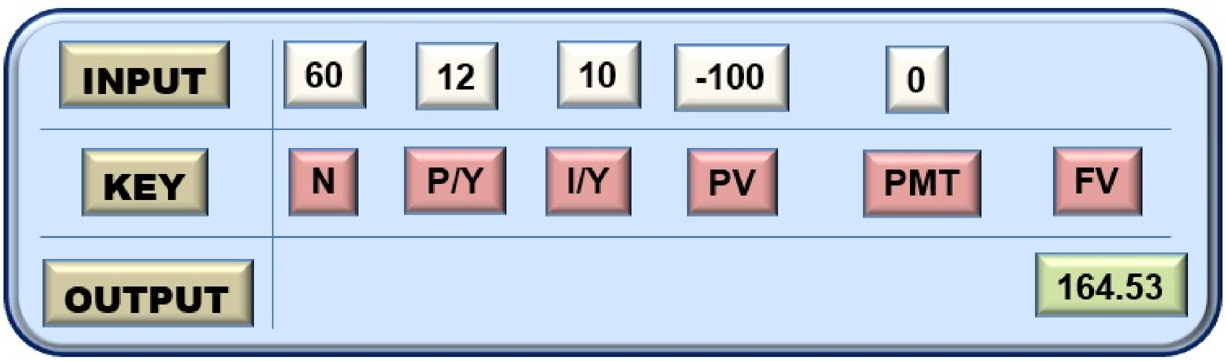
Solution

Do I really need to show you this one? Okay, you can look at the formula for FV of lump sum. $M = 12,\ APR\ =\ 10\%,\ T = 5$

$$FV\ (lump\ sum) =\ 100\ *\ \left(1\ + \frac{0.1}{12}\right)^{5*12}\ =\ 164.53$$

On a financial calculator, set P/Y = 12 since the compounding is monthly. Enter N = 60, I/Y = 10, PV = -100, PMT = 0, and solve for FV.

On a financial calculator:



In Excel, enter = FV (0.1/12, 60, 0, -100) = 164.53

Now that I've provided the solution steps to every question in Group 1, all you have to do is identify the match for each one from Group 2 and solve by analogy. <u>PLEASE DO IT</u>.

Group 2

1. You want to buy a house in ten years, and you would like a $500,000 home. How much money should you have today, if it is invested at 10% per year, compounded annually? Answer: $192,771.64

2. You are retiring with $10,000,000. Indeed, who wouldn't. You would like to withdraw a certain amount each year from this retirement money, to live comfortably but in such a way that you will still have $5,000,000 available to pass on to your heirs 30 year later. If your money is invested at 5% per year, how much can you withdraw per year? Answer: $575,257.18

3. You lend \$5,000 at 5% per year for three years. How much will you receive back (provided you're paid back in full)? Answer: \$5,788.13

4. You are saving for retirement. You begin with \$10,000 initial deposit, and additionally contribute 300 per month for 30 years. If your retirement plan invests the funds at 7% per year, how much money will you have after 30 years in your account? Answer: \$447,156.30

5. You start an investment account earning 9% per year. You initially deposit \$X and withdraw \$100 every six months from the account for 10 years. At the end of 10 years, there is \$1000 in your account. Find X. Answer: \$1,715.44

CONCLUSION: WHAT'S NEXT?

First and foremost, I hope you are still reading. The material in this book is difficult and laborious, and I know you have worked very hard to learn and understand it. Let's summarize what you have learned so far and see what's coming next if your plan is to study finance further.

You have learned that a dollar today is not the same as a dollar tomorrow. We make tradeoffs between different points in time. Exactly how much of today's money we are ready to set aside as an investment today depends on the reward that we expect for the patience and for the risk we take. That reward is called the rate of return.

You have learned the concepts of present value and future value, discounting and compounding. You know that compounding can be annual, monthly, daily, or even continuous. You can calculate present and future value of uneven cash flows, perpetuities, annuities, growing perpetuities, and growing annuities – even delayed ones. You know these streams of cash flows have applications in real life ranging from personal finance (i.e. taking out a loan or saving for retirement) to investments (pricing stocks and bonds) to business finance (raising funds for an enterprise or choosing the best investment). You have worked through multiple examples and self-check problems. You can solve for the loan payment, the time needed to repay the loan, and the rate of return on an investment, whether by hand or using a financial calculator. Indeed, you have mastered many important financial tools that will serve you well whether finance is your planned career or not.

You have learned that return and risk are related, and that the main tool an investor uses to earn the desired expected return is to decide on the price paid today. The higher the risk, all else equal, the lower the price paid today per dollar of expected cash flow tomorrow. Therefore, you know that the rate of return is essentially the same as the discount rate. High risk means high discount rate, which makes present value of the future cash flows lower.

You have learned the difference between return and growth. Return is always considered in connection with the price we pay. Return is ultimately about the discount rate. Growth is a more generic concept that means an increase (or a decrease) over time and applies to cash flow.

At the same time, there is much more left to learn. For example, you know that return depends on risk. How is risk measured exactly? We never talked about measuring risk in this book; we only referred to it generically. Next, once we learn to measure risk, how much return is appropriate to reward me for one unit of risk? One percent? Two? Twenty? If you study finance in more depth, you will learn the answers to these questions backed by the most current research.

We spoke about growth but never learned where it comes from and whether growth is always desirable. Should a business grow just for the sake of growing, or should it grow for the sake of adding value? Well, what is value then, how exactly is it created, and how is growth part of it? Are there better ways to grow? For example, should a company grow through internal innovation or through acquiring other firms? Should a company use its own internal resources to grow or should it turn to external capital markets and raise capital from shareholders and bondholders by issuing equity and debt? Which, equity or debt, is a better source of funds?

Businesses and individuals manage their risk. You know that much because you know investors exercise monitoring and control over the expected cash flows from an enterprise, but to the extent that they can. What exactly are those risk management approaches? Which risks can be controlled and mitigated? Which risks are important for the enterprise given its strategic objectives, and which risks should be avoided?

What does it mean for investors, to monitor and control the management of an enterprise? How is the monitoring accomplished? How effective is it? What are the costs and benefits? How does it depend on the form of business organization, i.e. on whether we have a proprietorship, a partnership, or a publicly owned corporation?

We always treated cash flows as given. But they are not given in real life and are to be estimated. What goes into estimating those future cash flows? What factors affect them, and which of these factors can be estimated with higher certainty?

I could continue with this list as there are far more unknowns than there are knowns. But I hope I've made my point that if you want to study finance in any depth you still have a lot to learn. Let me conclude then, by saying that I hope that after reading this very introductory text you are interested and encouraged to learn more.

Whatever your ultimate choice may be, to continue along the path of learning finance or to pursue other exciting endeavors, I thank you sincerely for being the reader of my little book and I wish you the very best in all your future pursuits.