The Integral Hodge Conjecture and Universality of the Abel-Jacobi Maps

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THESIS

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CONTRIBUTION OF AUTHORS

Theorem 1.1.1 and Corollary 1.2.5, which are stated in Chapter 1 and elaborated in Chapter 2 and Chapter 3, are the main results of the paper [35] co-authored with John Christian Ottem. It is hard to further specify contribution of each author because the results were obtained through our frequent exchanges of ideas. I would like to thank John Christian Ottem for allowing me to include the content of our joint work in this thesis.

TABLE OF CONTENTS

SUMMARY

The rational Hodge conjecture states that rational Hodge classes are algebraic. This long-standing heavily studied conjecture has remained widely open since it was proposed in the nineteen fifties. In contrast, the integral Hodge conjecture is known to fail in general. To better understand the rational Hodge conjecture, it is important to ask how the integral Hodge conjecture can fail.

In this thesis, we prove that there exists a pencil of Enriques surfaces defined over $\mathbb{Q}$ with non-algebraic integral Hodge classes of non-torsion type. This gives the first example of a threefold with trivial Chow group of zero-cycles on which the integral Hodge conjecture fails. As an application, we construct a fourfold which gives the negative answer to a classical question posed by Murre on the universality of the Abel-Jacobi maps in codimension three.

This thesis is based on the papers [35] and [45], the first of which is joint with John Christian Ottem.

CHAPTER 1

INTRODUCTION

A smooth complex projective variety X is both algebraic and complex analytic: in addition to being an algebraic variety, it is a Kähler manifold. This fact allows us to define two natural filtrations on the Betti cohomology group $H^i(X, A)$ with coefficient $A = \mathbb{Q}$ or $\mathbb{Z}$: one is the coniveau filtration

$$N^r H^i(X, A) = \mathrm{Ker}\left(H^i(X, A) \to \varinjlim_{Z \subset X} H^i(X - Z, A) \right),$$

where $Z \subset X$ runs through all codimension $\geq r$ closed algebraic subsets of X; the other is the Hodge filtration

$$F^r H^i(X, A) = H^i(X, A) \cap (H^{i,0}(X) \oplus \cdots \oplus H^{r,i-r}(X)),$$

where

$$H^i(X, \mathbb{C}) = \bigoplus_{i=j+k} H^{j,k}(X)$$

is the Hodge decomposition. In the special case of $r = p$ and $i = 2p$, these respectively amount to the classes of algebraic subvarieties

$$H^{2p}_{\mathrm{alg}}(X, A)$$

and the Hodge classes

$$Hdg^{2p}(X,A) = H^{2p}(X,A) \cap H^{p,p}(X).$$

The rational Hodge conjecture states that rational Hodge classes are algebraic, or equivalently, we have

$$H^{2p}_{\mathrm{alg}}(X,\mathbb{Q}) = Hdg^{2p}(X,\mathbb{Q}).$$

While a remarkable piece of evidence was given by Cattanni-Deligne-Kaplan [12], who proved that Hodge loci are algebraic, this long-standing heavily studied conjecture has remained widely open since it was proposed in the nineteen fifties. Another version of the rational Hodge conjecture formulated by Grothendieck [23] in the nineteen sixties, which states that the coniveau filtration $N^rH^i(X,\mathbb{Q})$ is the largest sub Hodge structure of the Hodge filtration $F^rH^i(X,\mathbb{Q})$, again seems far from being resolved.

In contrast, certain integral analogues of the rational Hodge conjecture are known to fail in general. The main purpose of this thesis is to study how such analogues can fail and to provide new counterexamples.

1.1 A pencil of Enriques surfaces with non-algebraic integral classes

For a smooth complex projective variety X, we denote by $CH^p(X)$ the Chow group of codimension p cycles and by $H^{2p}(X,\mathbb{Z})$ the Betti cohomology group of degree $2p$. The image $H^{2p}_{\mathrm{alg}}(X,\mathbb{Z}) \subseteq H^{2p}(X,\mathbb{Z})$ of the cycle class map $\mathrm{cl}^p\colon CH^p(X) \to H^{2p}(X,\mathbb{Z})$ is contained in the group $Hdg^{2p}(X,\mathbb{Z}) \subseteq H^{2p}(X,\mathbb{Z})$ of integral Hodge classes. The integral Hodge conjecture is the statement that these two subgroups of $H^{2p}(X,\mathbb{Z})$ coincide. While this statement holds for

$p = 0, 1$ and $\dim X$, it is known that it can fail in general. The first counterexample was given by Atiyah-Hirzebruch [1], who constructed a projective manifold admitting a non-algebraic degree four torsion class. Later, a different type of counterexample was constructed by Kollár [2, p. 134, Lemma], who proved that for certain high degree hypersurfaces $X \subset \mathbb{P}^4$, the generator of $H^4(X, \mathbb{Z}) = \mathbb{Z}$ is not algebraic. This means that the natural inclusion

$$H^4_{\text{alg}}(X, \mathbb{Z})/\operatorname{tors} \subset Hdg^4(X, \mathbb{Z})/\operatorname{tors}$$

can be strict. Since then, many other examples of non-algebraic integral Hodge classes have been found, both of torsion type [43; 5] and of non-torsion type [14; 46; 17].

In Chapter 2, we study Enriques surface fibrations over curves and show that they can admit non-algebraic integral Hodge classes of non-torsion type.

Theorem 1.1.1 (with J. C. Ottem)**.** *There exists a pencil of Enriques surfaces defined over $\mathbb{Q}$ such that the cohomology groups $H^i(X, \mathbb{Z})$ are torsion-free for all i and the inclusion*

$$H^4_{\text{alg}}(X, \mathbb{Z}) \subsetneq Hdg^4(X, \mathbb{Z})$$

is strict.

One can compare Theorem 1.1.1 with the result of Benoist–Ottem [5], which showed that the integral Hodge conjecture can fail on products $S \times C$ for an Enriques surface S and curve C of genus at least one. In those examples, the non-algebraic classes in question are 2-torsion, but the integral Hodge classes are algebraic modulo torsion classes by the Künneth formula.

Theorem 1.1.1 also relates to certain questions concerning rational points of algebraic varieties. In a letter to Grothendieck, Serre asked whether a projective variety over the function field of a curve always has a rational point if it is $\mathcal{O}$-acyclic, that is, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. This question was answered negatively by Grabber–Harris–Mazur–Starr [19], who constructed an Enriques surface without rational points over the function field of a complex curve. Later, more explicit constructions of such Enriques surfaces were given by Lafon [26] and Starr [44].

According to [44], Esnault expected that the Enriques surfaces of [19] and [26] would satisfy a stronger property that every closed point has even degree over the base field. If that were the case, it would give a pencil of Enriques surfaces with non-algebraic integral Hodge classes of non-torsion type (this follows from [14, Theorem 7.6]). In fact, this observation was the starting point of the joint work with J. C. Ottem.

Another feature of our example is that it has a trivial Chow group of zero-cycles. Indeed, Bloch–Kas–Lieberman [9] proved that $CH_0(S) = \mathbb{Z}$ for any Enriques surface S, and from this one deduces that the same holds for any pencil of Enriques surfaces (see Lemma 2.1.4). To our knowledge, this is the first example of a threefold with trivial Chow group of zero-cycles on which the integral Hodge conjecture fails (see [14, Subsection 5.7] for a threefold constructed by Colliot-Thélène and Voisin which conjecturally satisfies this condition). We emphasize that it is not a priori obvious that such a threefold should exist. For instance, typical examples with trivial Chow groups of zero-cycles are given by rationally connected varieties while the integral Hodge conjecture holds on rationally connected threefolds by a result of Voisin [48].

1.2 The Abel-Jacobi map is not always universal

Let V be a smooth complex projective variety. We denote by $A^p(V) \subset CH^p(V)$ the subgroup of cycles algebraically equivalent to zero. We recall that a homomorphism $\phi\colon A^p(V) \to A$ to an abelian variety A is called *regular* if for any smooth connected projective variety S with a base point s_0 and for any codimension p cycle Γ on $S \times V$, the composition

$$S \to A^p(V) \to A, s \mapsto \phi(\Gamma_*(s - s_0))$$

is a morphism of algebraic varieties (this definition goes back to the work of Samuel [40]). An important example of such homomorphisms is the following. We consider the Abel-Jacobi map

$$AJ^p\colon CH^p(V)_{\mathrm{hom}} \to J^p(V),$$

where $CH^p(V)_{\mathrm{hom}} \subset CH^p(V)$ is the subgroup of cycle classes homologous to zero, and

$$J^p(V) = H^{2p-1}(V, \mathbb{C})/(H^{2p-1}(V, \mathbb{Z}(p)) + F^p H^{2p-1}(V, \mathbb{C}))$$

is the p-th Griffiths intermediate Jacobian (see [49, Section 12] for the definition and properties of the Abel-Jacobi maps). The image $J^p_a(V) \subset J^p(V)$ of the restriction of the Abel-Jacobi map AJ^p to $A^p(V)$ is an abelian variety, and the induced map

$$\psi^p\colon A^p(V) \to J^p_a(V),$$

which we also call Abel-Jacobi, is regular [22][29]. A classical question of Murre [33, Section 7][21, p. 132] asks whether the Abel-Jacobi map $\psi^p\colon A^p(V) \to J_a^p(V)$ is *universal* among all regular homomorphisms $\phi\colon A^p(V) \to A$, that is, whether every such ϕ factors through ψ^p (see [52] for another universality question from a different perspective). It is true for $p = 1$ by the theory of the Picard variety, for $p = \dim V$ by the theory of the Albanese variety, and for $p = 2$ as proved by Murre [32][34] using the Merkurjev-Suslin theorem [31].

Meanwhile, the following theorem was proved by Walker [54] as an application of the theory of the Lawson homology and the morphic cohomology: the Abel-Jacobi map ψ^p factors as

$$\begin{tikzcd} & J(N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))) \arrow[d, "\pi^p"] \\ A^p(V) \arrow[ur, "\widetilde{\psi}^p"] \arrow[r, "\psi^p"'] & J_a^p(V) \end{tikzcd} ,$$

where $J(N^{p-1}H^{2p-1}(V,\mathbb{Z}(p)))$ is the intermediate Jacobian for the mixed Hodge structure given by the coniveau filtration $N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))$[1], π^p is a natural isogeny, and $\widetilde{\psi}^p$ is a surjective regular homomorphism (we will call the homomorphism $\widetilde{\psi}^p$ the Walker map). Consequently, if the Abel-Jacobi map ψ^p is universal, then the kernel

$$\mathrm{Ker}(\pi^p) = \mathrm{Coker}\left(H^{2p-1}(V,\mathbb{Z}(p))_{\mathrm{tors}} \to (H^{2p-1}(V,\mathbb{Z}(p))/N^{p-1}H^{2p-1}(V,\mathbb{Z}(p)))_{\mathrm{tors}}\right)$$

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[1] We denote by $\mathbb{Z}(m)$ the Hodge structure of Tate $(2\pi i)^m \cdot \mathbb{Z}$, which is a pure Hodge structure of weight $-2m$. We keep track of Tate twists in Chapter 3 and Appendix.

is trivial, or equivalently, the sublattice

$$N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))/\operatorname{tors} \subset H^{2p-1}(V,\mathbb{Z}(p))/\operatorname{tors}$$

is primitive.

In Chapter 3, we use the formalism of decomposition of the diagonal [11] to prove an analogue of the Roitman theorem [37, Theorem 3.1] for the Walker maps.

Theorem 1.2.1. *Let V be a smooth projective variety such that $CH_0(V)$ is supported on a three-dimensional closed subset. Let $p \in \{3, \dim V - 1\}$. Then the restriction*

$$\widetilde{\psi}^p|_{\operatorname{tors}} \colon A^p(V)_{\operatorname{tors}} \to J(N^{p-1}H^{2p-1}(V,\mathbb{Z}(p)))_{\operatorname{tors}}$$

is an isomorphism. Moreover the Walker map $\widetilde{\psi}^p$ is universal.

Remark 1.2.2. There is an abelian fourfold V such that $A^3(V)$ has infinite l-torsion elements for all prime numbers l [47] (see also [41; 38]). Therefore the assumption on $CH_0(V)$ is essential.

Then we apply Theorem 1.2.1 to prove the following theorem on the integral Hodge conjecture and the primitivity of the lattice of the coniveau filtration.

Theorem 1.2.3. *Let W be a smooth projective variety such that $CH_0(W)$ is supported on a surface and the inclusion*

$$H^4_{\mathrm{alg}}(W,\mathbb{Z}(2))/\operatorname{tors} \subsetneq Hdg^4(W,\mathbb{Z})/\operatorname{tors}$$

is strict. Then there exists an elliptic curve E such that the sublattice

$$N^2H^5(W \times E, \mathbb{Z}(3))/\operatorname{tors} \subset H^5(W \times E, \mathbb{Z}(3))/\operatorname{tors}$$

is not primitive.

Remark 1.2.4. A "homology counterpart" of Theorem 1.2.3 also holds. See Theorem 3.2.3.

Finally, we apply Theorem 1.2.3 to the pencil of Enriques surfaces of Theorem 1.1.1 to prove that the Abel-Jacobi map is not universal in general. This settles Murre's question.

Corollary 1.2.5 (with J. C. Ottem)**.** *Let X be the pencil of Enriques surfaces of Theorem 1.1.1. Then there exists an elliptic curve E such that the Abel-Jacobi map*

$$\psi^3\colon A^3(X \times E) \to J^3_a(X \times E)$$

is not universal: it factors as

$$\begin{tikzcd} & J(N^2H^5(X \times E, \mathbb{Z}(3))) \arrow[d, "\pi^3"] , \\ A^3(X \times E) \arrow[ur, "\widetilde{\psi}^3"] \arrow[r, "\psi^3"'] & J^3_a(X \times E) \end{tikzcd}$$

where the Walker map $\widetilde{\psi}^3$ is surjective regular, and the natural isogeny π^3 has non-zero kernel, or equivalently, the sublattice

$$N^2H^5(X \times E, \mathbb{Z}(3)) \subset H^5(X \times E, \mathbb{Z}(3))$$

is not primitive.

Remark 1.2.6. The Walker map $\widetilde{\psi}^3$ in the statement is universal by Theorem 1.2.1.

Remark 1.2.7. In fact, we have $N^2H^5(X \times E, \mathbb{Q}(3)) = H^5(X \times E, \mathbb{Q}(3))$ as a consequence of decomposition of the diagonal [11]. In other words, $J^3_a(X \times E) = J^3(X \times E)$ (see [33, Lemma 4.3]).

CHAPTER 2

A PENCIL OF ENRIQUES SURFACES WITH NON-ALGEBRAIC INTEGRAL HODGE CLASSES

We prove Theorem 1.1.1.

This chapter is organized as follows. In Section 2.1, we study the geometry of the pencils of Enriques surfaces appearing in Theorem 1.1.1. These are defined as the rank one degeneracy loci of maps of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$. In particular, we compute their integral cohomology groups and Chow groups of zero-cycles. In Section 2.2, we prove the main theorem, using a specialization argument.

We work over the complex numbers throughout.

This chapter is based on the paper [35] (Ottem, J. C., Suzuki, F. : *A pencil of Enriques surfaces with non-algebraic integral Hodge classes*, Math. Ann. (2020)).

2.1 Geometry of pencils of Enriques surfaces

In this thesis, a *pencil of Enriques surfaces* will mean a smooth complex threefold X with a fibration $X \to \mathbb{P}^1$ over $\mathbb{P}^1$ whose general fibers are Enriques surfaces. In the course of the proof of Theorem 1.1.1, we will give a few explicit constructions of such threefolds. We start with the construction of the Enriques surfaces themselves.

We will fix the following notation[1]:

- $\mathbb{P}_A = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(2,0)\oplus\mathcal{O}(0,2))$, $E_1 = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(2,0))$, $E_2 = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(0,2))$
- $\mathbb{P}_B = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(1,0)\oplus\mathcal{O}(0,1))$, $F_1 = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(1,0))$, $F_2 = \mathbb{P}_{\mathbb{P}^2\times\mathbb{P}^2}(\mathcal{O}(0,1))$
- $\mathbb{P}_C = \mathbb{P}(H^0(\mathbb{P}_B,\mathcal{O}(1)))$, $P_1 = \mathbb{P}(H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(1,0)))$, $P_2 = \mathbb{P}(H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(0,1)))$.

These spaces are related as follows. We can regard P_1 and P_2 as disjoint planes in the five-dimensional projective space $\mathbb{P}_C$ via the idetification

$$H^0(\mathbb{P}_B,\mathcal{O}(1)) = H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(1,0))\oplus H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(0,1)).$$

Then the projective bundle $\mathbb{P}_B$ is identified with the blow-up of $\mathbb{P}_C$ along the union of P_1 and P_2 with the exceptional divisors F_1 and F_2. Moreover, there is a natural involution ι on $\mathbb{P}_C$ induced by the involution on $H^0(\mathbb{P}_B,\mathcal{O}(1))$ with the (± 1)-eigenspaces $H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(1,0))$ and $H^0(\mathbb{P}^2\times\mathbb{P}^2,\mathcal{O}(0,1))$, respectively. The involution ι lifts to an involution on $\mathbb{P}_B$, and we have $\mathbb{P}_A = \mathbb{P}_B/\iota$. Thus there is a double cover $\mathbb{P}_B\to\mathbb{P}_A$ over $\mathbb{P}^2\times\mathbb{P}^2$, which is ramified along F_i, and the divisors F_i are mapped isomorphically onto E_i for $i=1,2$.

The projective models of the Enriques surfaces are defined as follows. On $\mathbb{P}^2\times\mathbb{P}^2$, we consider a map of vector bundles

$$u\colon \mathcal{O}^{\oplus 3}\to\mathcal{O}(2,0)\oplus\mathcal{O}(0,2).$$

[1]We use Grothendieck's notation for projective bundles: for a vector bundle $\mathcal{E}$, $\mathbb{P}(\mathcal{E})$ paramterizes one-dimensional quotients of $\mathcal{E}$.

Let S be the rank one degeneracy locus of u.

Lemma 2.1.1. *If u is general, then S is an Enriques surface.*

Proof. Since the vector bundle $\mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$ is globally generated, S is smooth of dimension two by the Bertini theorem for degeneracy loci.

To show that S is an Enriques surface, we will describe its K3 cover T. The map u defines a global section s of $\mathcal{O}(1)^{\oplus 3}$ on the projective bundle $\mathbb{P}_A$. When u is generic, the zero set $Z(s) \subset \mathbb{P}_A$ maps isomorphically onto S via the bundle projection $\mathbb{P}_A \to \mathbb{P}^2 \times \mathbb{P}^2$.

On the other hand, the map u also defines a global section of $\mathcal{O}(2)^{\oplus 3}$ on $\mathbb{P}_B$ invariant under the action of ι. Indeed, as $(q_*\mathcal{O}_{\mathbb{P}_B}(2))^\iota = (q_*q^*\mathcal{O}_{\mathbb{P}_A}(1))^\iota = \mathcal{O}_{\mathbb{P}_A}(1)$, where $q\colon \mathbb{P}_B \to \mathbb{P}_A = \mathbb{P}_B/\iota$ is a natural projection, we have a natural identification

$$H^0(\mathbb{P}_B, \mathcal{O}(2))^\iota = H^0(\mathbb{P}_A, \mathcal{O}(1)) = H^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)).$$

Let $T \subset \mathbb{P}_B$ denote the zero set of this section. When u is general, we have $S \cap E_i = T \cap F_i = \emptyset$, so T maps isomorphically to a smooth intersection of three quadrics in $\mathbb{P}_C$ via the blow-down map $\mathbb{P}_B \to \mathbb{P}_C$. In particular, T is a K3 surface. Again since $T \cap F_i = \emptyset$, the composition $\mathbb{P}_B \to \mathbb{P}_A \to \mathbb{P}^2 \times \mathbb{P}^2$ restricts to an étale double cover $T \to S$. Hence S is an Enriques surface. $\square$

Remark 2.1.2. The proof of Lemma 2.1.1 shows that the construction of Enriques surfaces introduced above coincides with a classical one from [4, Example VIII.18].

We will now use a variant of the above construction to construct pencils of Enriques surfaces. On $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$, we consider a map of vector bundles

$$v\colon \mathcal{O}^{\oplus 3} \to \mathcal{O}(1,2,0) \oplus \mathcal{O}(1,0,2).$$

Let X be the rank one degeneracy locus of v.

Lemma 2.1.3. *If v is general, then X is a pencil of Enriques surfaces by the first projection $X \to \mathbb{P}^1$. Moreover, we have $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. Since the vector bundle $\mathcal{O}(1,2,0) \oplus \mathcal{O}(1,0,2)$ is globally generated, X is smooth and $\dim X = 3$ by the Bertini theorem for degeneracy loci. Moreover, X is connected since it is defined by three equations of tridegree $(2,2,2)$. The resolution of the ideal sheaf $\mathcal{I}_X$ of X in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$ has the form

$$0 \to \mathcal{O}(-3,-4,-2) \oplus \mathcal{O}(-3,-2,-4) \to \mathcal{O}(-2,-2,-2)^{\oplus 3} \to \mathcal{I}_X \to 0.$$

From this it follows that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. $\square$

We assume that v is general in what follows.

Lemma 2.1.4. *The degree homomorphism $\deg\colon CH_0(X) \to \mathbb{Z}$ is an isomorphism.*

Proof. Let $C \subset X$ be a smooth curve which is a complete intersection of very ample divisors. Then $CH_0(X)$ is supported on C. This follows from the fact that any class in $CH_0(X)$ is represented by a zero-cycle supported on a union of smooth fibers of the first projection $X \to \mathbb{P}^1$

by the moving lemma, and that the Chow group of zero-cycles on any given Enriques surface is trivial due to Bloch–Kas–Lieberman [9].

We consider a natural homomorphism $\phi\colon \operatorname{Ker}(\deg) \to \operatorname{Alb}(X)$ induced by the Albanese map. Since $CH_0(X)$ is supported on a curve, the decomposition of the diagonal [11] implies that $\operatorname{Ker}(\phi)$ is torsion. Moreover $\operatorname{Ker}(\phi)$ is torsion-free by the Roitman theorem [37]. Hence we have $\operatorname{Ker}(\phi) = 0$ and ϕ is an isomorphism. In our situation, $\operatorname{Alb}(X) = 0$ since $H^1(X, \mathcal{O}_X) = 0$ by Lemma 2.1.3. Therefore $\operatorname{Ker}(\deg) = 0$. The proof is complete. $\square$

To study the geometric properties of the threefold X in more detail, it will be convenient to involve its double cover. Recalling the construction above, we get a diagram

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}_B \arrow[r] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}_A \arrow[d] \\ \mathbb{P}^1 \times \mathbb{P}_C & \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \end{tikzcd},$$

where $\mathbb{P}^1 \times \mathbb{P}_B \to \mathbb{P}^1 \times \mathbb{P}_A$ is the quotient by the involution ι (which acts as before on $\mathbb{P}_B$ and as the identity on the first factor) and $\mathbb{P}^1 \times \mathbb{P}_B \to \mathbb{P}^1 \times \mathbb{P}_C$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}_C$ along the union of $\mathbb{P}^1 \times P_1$ and $\mathbb{P}^1 \times P_2$. Restricting to X, we get the following diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X' \arrow[d, "\simeq"] \\ Y_{\min} & X \end{tikzcd}.$$

The varieties appearing in this diagram can be described as follows. The map v induces a global section of $\mathcal{O}(1,1)^{\oplus 3}$ on $\mathbb{P}^1 \times \mathbb{P}_A$ as well as global sections of $\mathcal{O}(1,2)^{\oplus 3}$ on $\mathbb{P}^1 \times \mathbb{P}_B$ and $\mathbb{P}^1 \times \mathbb{P}_C$

which are invariant under the action of ι; the varieties $X', Y, Y_{\min}$ are the zero sets of these sections. By generality, $X', Y, Y_{\min}$ are smooth threefolds; X' is mapped isomorphically onto X, so we can identify X' with X; Y is a double cover of $X' = X$; and $Y_{\min}$ is a minimal model of Y. Note that Y and $Y_{\min}$ are K3 surface fibrations via the first projection.

An easy computation shows that each of the intersections $Y_{\min} \cap (\mathbb{P}^1 \times P_i)$ consists of twelve points $y_{i,1}, \cdots, y_{i,12}$. Then the map $Y \to Y_{\min}$ is the blow-up of $Y_{\min}$ along $y_{i,j}$ whose exceptional divisors $F_{i,j}$ are the components of $Y \cap (\mathbb{P}^1 \times F_i)$. Moreover the double cover $Y \to X$ is ramified along $F_{i,j}$ which are mapped isomorphically onto $E_{i,j}$, the components of $X \cap (\mathbb{P}^1 \times E_i)$.

Lemma 2.1.5. *The threefold X has Kodaira dimension one.*

Proof. Let S be the class of a fiber of the first projection $X \to \mathbb{P}^1$. It is straightforward to compute that

$$2K_X = 2S + \sum_{i=1}^{2} \sum_{j=1}^{12} E_{i,j}.$$

As the normal bundles $N_{E_{i,j}/X} = \mathcal{O}_{\mathbb{P}^2}(-2)$ are negative, we obtain that $\kappa(X) = 1$. □

Lemma 2.1.6. *The Hodge numbers of X are given by $h^{0,0}(X) = h^{3,3}(X) = 1$, $h^{1,1}(X) = h^{2,2}(X) = 26$, $h^{1,2}(X) = h^{2,1}(X) = 45$, and $h^{p,q}(X) = 0$ otherwise.*

Proof. We first compute the Picard number $\rho(X)$. Using the Lefschetz hyperplane section theorem, $Y_{\min}$ has Picard number two, so $\rho(Y) = \rho(Y_{\min}) + 24 = 26$. Moreover, the action of ι on the Picard group of Y is trivial, so also $\rho(X) = 26$.

We next compute the Betti numbers $b_i(X)$. It is straightforward to compute the topological Euler characteristic $\chi_{\text{top}}(X) = c_3(T_X) = -36$. Obviously $b_0(X) = b_6(X) = 1$. Moreover, $b_1(X) = b_5(X) = 0$ and $b_2(X) = b_4(X) = \rho(X) = 26$ using Lemma 2.1.3. Therefore $b_3(X) = 90$.

Now the computation of the Hodge numbers are immediate using Lemma 2.1.3 again. □

We next study the topology of X. We fix the following notation:

- $X_{\min} = Y_{\min}/\iota$;
- $Y^\circ = Y_{\min} - \{y_{i,j}\}_{i,j}$;
- $X^\circ = Y^\circ/\iota$;
- $V_{i,j} \subset Y$, a small ball around $y_{i,j}$;
- $U_{i,j} = V_{i,j}/\iota$.

We have $Y_{\min} = Y^\circ \cup \left(\bigcup_{i,j} V_{i,j}\right)$ and $X_{\min} = X^\circ \cup \left(\bigcup_{i,j} U_{i,j}\right)$.

Lemma 2.1.7. *The threefold X is simply connected, and the cohomology groups $H^i(X,\mathbb{Z})$ are torsion-free for all i.*

Proof. By the universal coefficient theorem, it is enough to prove that $\pi_1(X) = 0$ and $H^3(X,\mathbb{Z})$ is torsion-free.

We first prove that $\pi_1(X) = 0$. We have a natural pushout diagram

$$\begin{tikzcd}
\pi_1(U_{i,j} \cap X^\circ) \arrow[r] \arrow[d] & \pi_1(X^\circ) \arrow[d] \\
\pi_1(U_{i,j}) \arrow[r] & \pi_1(X_{\min})
\end{tikzcd} .$$

By Lefschetz, Y and hence Y° is simply connected. So since the quotient map $\pi\colon Y^\circ \to X^\circ$ is étale, we have $\pi_1(X^\circ) = \mathbb{Z}/2$. The neighborhood $U_{i,j} \subset X$ is homotopic to the affine cone over a Veronese surface, so we have $\pi_1(U_{i,j}) = 0$. Finally, since the map $V_{i,j} \cap Y^\circ \to U_{i,j} \cap X^\circ$ is homotopic to the universal covering map $(\mathbb{C}^3 - 0) \to (\mathbb{C}^3 - 0)/\pm$, we have $\pi_1(U_{i,j} \cap X^\circ) = \mathbb{Z}/2$. In fact, this cover is induced by the restriction of π to $V_{i,j} \cap Y^\circ$, so the map $\pi_1(U_{i,j} \cap X^\circ) \to \pi_1(X^\circ)$ is non-zero, hence an isomorphism. From the pushout diagram above, we then get $\pi_1(X_{\min}) = 0$. Resolving a finite cyclic quotient singularity does not change the fundamental group ([25, Theorem 7.8]), so we also get $\pi_1(X) = 0$.

We next prove that $H^3(X, \mathbb{Z})$ is torsion-free. The long exact sequence for cohomology groups with supports gives

$$\bigoplus_{i,j} H^3_{E_{i,j}}(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \to H^3(X^\circ, \mathbb{Z}).$$

Since $H^3_{E_{i,j}}(X, \mathbb{Z}) = H_3(E_{i,j}, \mathbb{Z}) = 0$, the group $H^3(X, \mathbb{Z})$ injects into $H^3(X^\circ, \mathbb{Z})$. In particular, we are reduced to showing that $H^3(X^\circ, \mathbb{Z})$ is torsion-free.

Since X° is the quotient of Y° by the group $\langle \iota \rangle \simeq \mathbb{Z}/2$, we can apply the Cartan–Leray spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/2, H^q(Y^\circ, \mathbb{Z})) \Rightarrow H^{p+q}(X^\circ, \mathbb{Z})$$

to compute the cohomology groups of X°. We need to compute $H^q(Y^\circ, \mathbb{Z})$ for $0 \leq q \leq 3$ and the action of ι on these groups. Since Y° is obtained from $Y_{\min}$ by removing finitely many points, we have an identification $H^q(Y^\circ, \mathbb{Z}) = H^q(Y_{\min}, \mathbb{Z})$. Clearly $H^0(Y_{\min}, \mathbb{Z}) = \mathbb{Z}$. By the Lefschetz hyperplane theorem, $H^1(Y_{\min}, \mathbb{Z}) = 0$, and the groups $H^2(Y_{\min}, \mathbb{Z})$ and $H^3(Y_{\min}, \mathbb{Z})$

are torsion-free. Moreover, the action of ι on $H^q(Y_{\min}, \mathbb{Z})$ is trivial for $0 \leq q \leq 2$. Since the group cohomology $H^p(\mathbb{Z}/2, \mathbb{Z}) = 0$ for p odd, it follows that $E_2^{p,3-p} = 0$ for $p \neq 0$. Therefore there is an injection

$$H^3(X^\circ, \mathbb{Z}) \hookrightarrow E_2^{0,3} = H^0(\mathbb{Z}/2, H^3(Y^\circ, \mathbb{Z})) = H^3(Y^\circ, \mathbb{Z})^\iota,$$

where the right hand side is torsion-free. This completes the proof. □

2.2 Failure of the integral Hodge conjecture for pencils of Enriques surfaces

We are now ready to prove our main result in this chapter:

Theorem 2.2.1. *There exists a map of vector bundles on* $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$

$$\mathcal{O}^{\oplus 3} \to \mathcal{O}(1, 2, 0) \oplus \mathcal{O}(1, 0, 2)$$

defined over $\mathbb{Q}$ *such that the rank one degeneracy locus* X *is a pencil of Enriques surfaces such that the cohomology groups* $H^i(X, \mathbb{Z})$ *are torsion-free for all* i *and there is a strict inclusion*

$$H^4_{\text{alg}}(X, \mathbb{Z}) \subsetneq Hdg^4(X, \mathbb{Z}).$$

Proof. We set $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[S, T] \times \operatorname{Proj} \mathbb{C}[X_0, X_1, X_2] \times \operatorname{Proj} \mathbb{C}[Y_0, Y_1, Y_2]$. Fix a sufficiently large prime number p. We consider a map of vector bundles as above given by the matrix

$$M = \begin{pmatrix} P_1 & Q_1 & R_1 \\ SP_2 + pP_3 & SQ_2 + pQ_3 & SR_2 + pR_3 \end{pmatrix},$$

where P_1, Q_1, R_1 (resp. P_2, Q_2, R_2; P_3, Q_3, R_3) are general tri-homogeneous polynomials of tri-degree $(1, 2, 0)$ (resp. $(0, 0, 2)$; $(1, 0, 2)$) over $\mathbb{Q}$. The degeneracy locus X is a pencil of Enriques surfaces defined by the 2×2-minors of M. The torsion-freeness of the cohomology groups follows from Lemma 2.1.7, so it remains to prove that the integral Hodge conjecture does not hold on X.

The closed subscheme defined by $P_1 = Q_1 = R_1 = 0$ is a disjoint union of twelve components $E_{1,1}, \ldots, E_{1,12}$ isomorphic to $\mathbb{P}^2$. We note that this union is defined over $\mathbb{Q}$, even though each $E_{i,j}$ may not be. First we prove that for a given algebraic one-cycle α on X, we have

$$\deg(\alpha/\mathbb{P}^1) \equiv \alpha \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2. \tag{2.1}$$

We use a specialization argument. We spread out $X_{\overline{\mathbb{Q}}}$ over a valuation ring R with the maximal ideal containing p. The ideal of the flat closure of $X_{\overline{\mathbb{Q}}}$ in $(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2)_R$ is generated by the 2×2-minors of M and

$$F = \det \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{pmatrix}.$$

The specialization over $\overline{\mathbb{F}}_p$ consists of two components: one is a pencil of Enriques surfaces $\widetilde{X}_0$ defined by the 2×2-minors of the matrix

$$N = \begin{pmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{pmatrix};$$

the other is defined by $S = F = 0$. It is straightforward to check that $\widetilde{X}_0$ is smooth.

The closed subscheme defined by $P_1 = Q_1 = R_1 = 0$ is again a disjoint union of twelve components $E_{1,1}, \ldots, E_{1,12}$ isomorphic to $\mathbb{P}^2$ and disjoint from the fiber over $S = 0$ by the generality of P_1, Q_1, R_1. We prove that for a given one-cycle α_0 on the specialization over $\overline{\mathbb{F}}_p$, we have

$$\deg(\alpha_0/\mathbb{P}^1) \equiv \alpha_0 \cdot \left(\sum_{j=1}^{12} E_{1,j}\right) \mod 2. \tag{2.2}$$

We may assume that α_0 is supported on $\widetilde{X}_0$. Let D_1 be the Cartier divisor on $\widetilde{X}_0$ defined by $P_1 = 0$. Since D_1 is of type $(1, 2, 0)$, we have

$$\deg(\alpha_0/\mathbb{P}^1) \equiv \alpha_0 \cdot D_1 \mod 2.$$

On the other hand, we have

$$D_1 = D_2 + \sum_{j=1}^{12} E_{1,j},$$

where D_2 is the Cartier divisor on $\widetilde{X}_0$ defined by $P_2 = 0$. Indeed, expanding the 2×2-minors of N, it is easily seen that the identity holds on each of the open subsets $P_2, Q_2, R_2 \neq 0$; these open subsets form an open cover of $\widetilde{X}_0$ by the generality of P_2, Q_2, R_2. Since D_2 is of type $(0, 0, 2)$, we have

$$\alpha_0 \cdot D_1 \equiv \alpha_0 \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) \mod 2.$$

The congruence (2.2) follows, so does the congruence (2.1) by the specialization homomorphism [18, Section 20.3].

The Hodge structure of $H^4(X, \mathbb{Z})$ is trivial since we have $H^2(X, \mathcal{O}_X) = 0$ by Lemma 2.1.3. The proof of the theorem is reduced to proving that there exists a class $\beta \in H^4(X, \mathbb{Z}) = H_2(X, \mathbb{Z})$ such that

$$\deg(\beta/\mathbb{P}^1) = \pm 1,\ \beta \cdot \left(\sum_{j=1}^{12} E_{1,j} \right) = 0;$$

such β is not algebraic according to the congruence (1). Since $E_{1,1}, \ldots, E_{1,12}$ are the images of $F_{1,1}, \ldots, F_{1,12}$ under the double cover $Y \to X$, it is enough to prove that there exists $\gamma \in H^4(Y, \mathbb{Z}) = H_2(Y, \mathbb{Z})$ such that

$$\deg(\gamma/\mathbb{P}^1) = \pm 1,\ \gamma \cdot \left(\sum_{j=1}^{12} F_{1,j} \right) = 0;$$

the class β will be the push-forward of γ. By the Lefschetz hyperplane section theorem, the push-forward $H_2(Y_{\min}, \mathbb{Z}) \to H_2(\mathbb{P}^1, \mathbb{Z})$ is surjective. Let $\gamma_{\min} \in H^4(Y_{\min}, \mathbb{Z}) = H_2(Y_{\min}, \mathbb{Z})$ be an element mapped to a generator of $H_2(\mathbb{P}^1, \mathbb{Z})$. Then the pullback $\gamma \in H^4(Y, \mathbb{Z})$ of $\gamma_{\min}$ satisfies the desired property. The proof is complete. $\square$

Remark 2.2.2. The specialization used in the proof of Theorem 2.2.1 deserves a few more comments. The specialization consists of two components: $\widetilde{X}_0$ defined by the 2×2-minors of N, and R defined by $S = F = 0$. The component $\widetilde{X}_0$ is smooth, and it is a pencil of Enriques surfaces by the first projection $\widetilde{X}_0 \to \mathbb{P}^1$. On the other hand, R has isolated singularities, and a smooth model $\overline{R}$ of R is another pencil of Enriques surfaces with a small contraction $\overline{R} \to R$ contracting $\mathbb{P}^1$s over the singular points of R. In addition, $\widetilde{X}_0$ and R intersect in a fiber over $S = 0$, and the intersection is an Enriques surface Z in $\mathbb{P}^2 \times \mathbb{P}^2$.

Remarkably, both of the components $\widetilde{X}_0$ and R are rationally connected: the projections

$$\widetilde{X}_0 \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_2} \mathbb{P}^2, \;\; R \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{pr_3} \mathbb{P}^2$$

are conic bundles, therefore this follows from [20, Corollary 1.3]. In particular, the integral Hodge conjecture holds on $\widetilde{X}_0$ and $\overline{R}$ by a result of Voisin [48]. As a consequence, $H_2(\widetilde{X}_0, \mathbb{Z})$ and $H_2(R, \mathbb{Z})$ are generated by algebraic cycles.

It turns out, however, that this is not the case for the union $\widetilde{X}_0 \cup R$. A key point here is the subtle difference between the Mayer-Vietoris sequence for homology groups and Chow groups. For the homology groups, we have an exact sequence

$$H_2(\widetilde{X}_0, \mathbb{Z}) \oplus H_2(R, \mathbb{Z}) \to H_2(\widetilde{X}_0 \cup R, \mathbb{Z}) \to H_1(Z, \mathbb{Z}) = \mathbb{Z}/2 \to 0.$$

For the Chow groups, on the other hand, we obviously have a surjection

$$CH_1(\widetilde{X}_0) \oplus CH_1(R) \twoheadrightarrow CH_1(\widetilde{X}_0 \cup R)$$

(see also [18, Example 1.8.1]). It follows that $H_2(\widetilde{X}_0 \cup R, \mathbb{Z})$ is not generated by algebraic cycles.

A small modification of the above arguments yields a generalization of Theorem 2.2.1 to higher dimensions:

Theorem 2.2.3. *For a given positive integer n, there exists a map of vector bundles on $\mathbb{P}^1 \times \mathbb{P}^{2n} \times \mathbb{P}^{2n}$*

$$\mathcal{O}^{\oplus(2n+1)} \to \mathcal{O}(1,2,0) \oplus \mathcal{O}(1,0,2)$$

defined over $\mathbb{Q}$ such that the rank one degeneracy locus X is a smooth $(2n+1)$-fold with a fibration over $\mathbb{P}^1$ whose general fibers are $2n$-folds M with $H^i(M, \mathcal{O}_M) = 0$ for all $i > 0$ and universal Calabi-Yau double covers $N \to M$ such that

(i) $H^i(X, \mathcal{O}_X) = 0$ *for all* $i > 0$*;*

(ii) $\kappa(X) = 1$*;*

(iii) X *is simply connected, and the cohomology group* $H^3(X, \mathbb{Z})$ *is torsion-free;*

(iv) *the inclusion* $H_{2,\mathrm{alg}}(X, \mathbb{Z}) \subsetneq Hdg_2(X, \mathbb{Z})$ *is strict.*

CHAPTER 3

THE ABEL-JACOBI MAP IS NOT ALWAYS UNIVERSAL

We prove Theorem 1.2.1, 1.2.3 and Corollary 1.2.5.

This chapter is organized as follows. In Section 3.1, we study regular homomorphisms on the torsion subgroup $A^p(V)_{\mathrm{tors}}$. Then we prove Theorem 1.2.1 and its corollary. In Section 3.2, we prove a proposition on non-algebraic integral Hodge classes of non-torsion type and non-zero torsion algebraic cycles in the Abel-Jacobi kernel. Then we prove Theorem 1.2.3 and its "homology counterpart". Corollary 1.2.5 follows immediately from Theorem 1.2.1 applied to the pencil of Enriques surfaces of Theorem 2.2.1. We end the section by explaining how to produce counterexamples to Murre's question in higher dimensions and for other values of p.

We work over the complex numbers throughout.

This chapter is based on the papers [35] (Ottem, J. C., Suzuki, F. : *A pencil of Enriques surfaces with non-algebraic integral Hodge classes*, Math. Ann. (2020)) and [45] (Suzuki, F.: *A remark on a 3-fold constructed by Colliot-Thélène and Voisin*, Math. Res. Lett. **27** (2020), no1, 301–317).

3.1 Regular homomorphisms on the torsion subgroup $A^p(V)_{\mathrm{tors}}$

Lemma 3.1.1. *Let V be a smooth projective variety and $\phi\colon A^p(V) \to A$ be a surjective regular homomorphism. Assume that the restriction $\phi|_{\mathrm{tors}}\colon A^p(V)_{\mathrm{tors}} \to A_{\mathrm{tors}}$ is an isomorphism. Then ϕ is universal.*

Proof. First we prove the existence of a universal regular homomorphism $\phi_0\colon A^p(V) \to A_0$. By Saito's criterion [39, Theorem 2.2] (see also [34, Proposition 2.1]), it is enough to prove $\dim B \leq \dim A$ for any surjective regular homomorphism $\psi\colon A^p(V) \to B$. Such a homomorphism ψ restricts to a surjection $\psi|_{\text{tors}}\colon A^p(V)_{\text{tors}} \to B_{\text{tors}}$. Indeed, by [39, Proposition 1.2] (see also [34, Lemma 1.6.2] and [27, Chapter III, Proposition 1]), there exists an abelian variety C and $\Gamma \in CH^p(C \times V)$ such that the map

$$C \to A^p(V), s \mapsto \Gamma_*(s - s_0)$$

is a homomorphism of groups and the composition

$$C \to A^p(V) \to B, s \mapsto \psi(\Gamma_*(s - s_0))$$

is an isogeny; it follows that the restriction $C_{\text{tors}} \to B_{\text{tors}}$ is a surjection, so is $\psi|_{\text{tors}}$. By assumption, we have $A^p(V)_{\text{tors}} \cong A_{\text{tors}}$. Then we have

$$\dim B = \frac{1}{2}\operatorname{co-rank} B_{\text{tors}} \leq \frac{1}{2}\operatorname{co-rank} A_{\text{tors}} = \dim A.$$

The existence follows.

The map ϕ_0 should be surjective since the image of a regular homomorphism is an abelian variety [34, Lemma 1.6.2]. Thus ϕ_0 restricts to a surjection $\phi_0|_{\text{tors}}\colon A^p(V)_{\text{tors}} \to (A_0)_{\text{tors}}$ by a similar argument as above. The induced map $A_0 \to A$ is surjective and restricts to an isomorphism $(A_0)_{\mathrm{tors}} \cong A_{\mathrm{tors}}$, therefore it is an isomorphism. The proof is done. $\square$

We review the Bloch-Ogus theory on the coniveau spectral sequence [10]. For a smooth projective variety V, we define $\mathcal{H}^q(\mathbb{Z}(r))$ to be the Zariski sheaf on V associated to the presheaf $U \mapsto H^q(U, \mathbb{Z}(r))$. Then the E_2 term of the coniveau spectral sequence is given by

$$E_2^{p,q} = H^p(V, \mathcal{H}^q(\mathbb{Z}(r))) \Rightarrow N^\bullet H^{p+q}(V, \mathbb{Z}(r)),$$

and we have $E_2^{p,q} = 0$ if $p > q$ [10, Corollary 6.2, 6.3]. We also have

$$E_2^{p,q} = 0 \text{ if } (p,q) \notin [0, \dim V] \times [0, \dim V].$$

Indeed, this follows from the fact that a smooth affine variety of dimension d has the homotopy type of a CW complex of real dimension d.

Let $f^p \colon H^{p-1}(V, \mathcal{H}^p(\mathbb{Z}(p))) \to H^{2p-1}(V, \mathbb{Z}(p))$ be the edge homomorphism.

Lemma 3.1.2. *There is a short exact sequence*[1]*:*

$$\begin{aligned} 0 &\to H^{p-1}(V,\mathcal{K}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l \\ &\to \operatorname{Ker}\left(f^p\otimes\mathbb{Q}_l/\mathbb{Z}_l\colon H^{p-1}(V,\mathcal{H}^p(\mathbb{Z}(p)))\otimes\mathbb{Q}_l/\mathbb{Z}_l\to N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l\right) \\ &\to \operatorname{Ker}\left(\widetilde{\psi}^p|_{l\text{-}\mathrm{tors}}\colon A^p(V)_{l\text{-}\mathrm{tors}}\to J(N^{p-1}H^{2p-1}(V,\mathbb{Z}(p)))_{l\text{-}\mathrm{tors}}\right)\to 0 \end{aligned}$$

for any smooth projective variety V and any prime number l, where $\mathcal{K}_p$ is the Zariski sheaf on X associated to the Quillen K-theory.

Proof. We use the Bloch map $\lambda_l^p\colon CH^p(V)_{l\text{-}\mathrm{tors}}\to H^{2p-1}(V,\mathbb{Q}_l/\mathbb{Z}_l(p))$ [7] (see also [13]). By the construction of the Bloch map and [30, Theorem 5.1], we have a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H^{p-1}(V,\mathcal{K}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r]\arrow[d] & H^{p-1}(V,\mathcal{H}^p(\mathbb{Z}(p)))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r]\arrow[d, "f^p\otimes\mathbb{Q}_l/\mathbb{Z}_l"] & A^p(V)_{l\text{-}\mathrm{tors}} \arrow[r]\arrow[d, "-\lambda_l^p"] & 0\,. \\ & 0 \arrow[r] & H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r] & H^{2p-1}(V,\mathbb{Q}_l/\mathbb{Z}_l(p)) & \end{tikzcd}$$

We prove that it induces another commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^{p-1}(V,\mathcal{K}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r]\arrow[d] & H^{p-1}(V,\mathcal{H}^p(\mathbb{Z}(p)))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r]\arrow[d, "f^p\otimes\mathbb{Q}_l/\mathbb{Z}_l"] & A^p(V)_{l\text{-}\mathrm{tors}} \arrow[r]\arrow[d, "-\widetilde{\lambda}_l^p"] & 0\,. \\ & 0 \arrow[r] & N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r, equal] & N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[r] & 0 \end{tikzcd}$$

---

[1] *For an abelian group G and a prime number l, we denote by $G_{l\text{-}\mathrm{tors}}$ the subgroup of l-primary torsion elements of G.*

It is enough to prove the image of $H^{p-1}(V,\mathcal{K}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l$ in $N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l$ is zero. This follows by observing that $H^{p-1}(V,\mathcal{K}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l$ is divisible and

$$\begin{aligned}&\mathrm{Ker}\left(N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l\to H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l\right)\\ &=\mathrm{Coker}\left(H^{2p-1}(V,\mathbb{Z}(p))_{l\text{-}\mathrm{tors}}\to(H^{2p-1}(V,\mathbb{Z}(p))/N^{p-1}H^{2p-1}(V,\mathbb{Z}(p)))_{l\text{-}\mathrm{tors}}\right)\end{aligned}$$

is finite. We prove that $\widetilde{\lambda}^p$ coincides with the restriction $\widetilde{\psi}^p|_{l\text{-}\mathrm{tors}}$. In commutative triangles

$$\begin{tikzcd} & N^{p-1}H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l\ , \arrow[d] \\ A^p(V)_{l\text{-}\mathrm{tors}} \arrow[ur, "\widetilde{\lambda}_l^p\ (\text{resp. }\widetilde{\psi}^p|_{l\text{-}\mathrm{tors}})"] \arrow[r, "\lambda_l^p\ (\text{resp. }\psi^p|_{l\text{-}\mathrm{tors}})"'] & H^{2p-1}(V,\mathbb{Z}(p))\otimes\mathbb{Q}_l/\mathbb{Z}_l \end{tikzcd}$$

$\widetilde{\lambda}_l^p$ (resp. $\widetilde{\psi}^p|_{l\text{-}\mathrm{tors}}$) is the unique lift of λ_l^p (resp. $\psi^p|_{l\text{-}\mathrm{tors}}$) since $A^p(V)_{l\text{-}\mathrm{tors}}$ is l-divisible [10, Lemma 7.10]. Therefore it is enough to prove that λ_l^p coincides with $\psi^p|_{l\text{-}\mathrm{tors}}$. This follows from [7, Proposition 3.7]. The proof is done by the snake lemma. □

Proof of Theorem 1.2.1. The second statement follows from the first one by Lemma 3.1.1.

We prove that $\widetilde{\psi}^3|_{\mathrm{tors}}$ is an isomorphism. By Lemma 3.1.2, it is enough to prove that

$$\mathrm{Ker}(f^3)=\mathrm{Im}\left(H^0(V,\mathcal{H}^4(\mathbb{Z}(3)))\to H^2(V,\mathcal{H}^3(\mathbb{Z}(3)))\right)$$

is torsion. The group $H^0(V,\mathcal{H}^4(\mathbb{Z}(3)))$ is torsion by [14, Proposition 3.3 (i)] (it is actually zero as a consequence of the Bloch-Kato conjecture, see [14, Theorem 3.1]), so the result follows.

Let $d = \dim V$. We prove that $\widetilde{\psi}^{d-1}|_{\mathrm{tors}}$ is an isomorphism. By Lemma 3.1.2, it is enough to prove that

$$\mathrm{Ker}(f^{d-1}) = \mathrm{Im}\left(H^{d-4}(V, \mathcal{H}^d(\mathbb{Z}(d-1))) \to H^{d-2}(V, \mathcal{H}^{d-1}(\mathbb{Z}(d-1)))\right)$$

is torsion. The group $H^{d-4}(V, \mathcal{H}^d(\mathbb{Z}(d-1)))$ is torsion by [14, Proposition 3.3 (ii)], so the result follows. □

Corollary 3.1.3. *Under the assumptions of Theorem 1.2.1, the following are equivalent:*

(i) the Abel-Jacobi map ψ^p is universal;

(ii) the sublattice $N^{p-1}H^{2p-1}(V, \mathbb{Z}(p))/\,\mathrm{tors} \subset H^{2p-1}(V, \mathbb{Z}(p))/\,\mathrm{tors}$ is primitive;

(iii) the restriction $\psi^p|_{\mathrm{tors}} \colon A^p(V)_{\mathrm{tors}} \to J^p_a(V)_{\mathrm{tors}}$ is an isomorphism.

Proof. It is enough to prove that (ii) and (iii) are equivalent. By Theorem 1.2.1, we have an isomorphism

$$\mathrm{Ker}\,(\pi^p) \cong \mathrm{Ker}(\psi^p|_{\mathrm{tors}} \colon A^p(V)_{\mathrm{tors}} \to J^p_a(V)_{\mathrm{tors}})\,.$$

The result follows. □

3.2 Non-algebraic integral Hodge classes of non-torsion type and non-zero torsion cycles in the Abel-Jacobi kernel

Inspired by the work of Soulé and Voisin [43], we prove:

Proposition 3.2.1. *Let W be a smooth projective variety such that the sublattice*

$$H^{2p}_{\mathrm{alg}}(W, \mathbb{Z}(p))/\operatorname{tors} \subset Hdg^{2p}(W, \mathbb{Z})/\operatorname{tors}$$

is not primitive. Then there exists a smooth elliptic curve E such that the restriction

$$\psi^{p+1}|_{\mathrm{tors}} \colon A^{p+1}(W \times E)_{\mathrm{tors}} \to J_a^{p+1}(W \times E)_{\mathrm{tors}}$$

is not an isomorphism.

Remark 3.2.2. The assumption of Proposition 3.2.1 for $p = 2$ is satisfied by Kollár's example [2, p.134, Lemma] (see also [43, Section 2]). It is a very general hypersurface in $\mathbb{P}^4$ of degree l^3 for a prime number $l \geq 5$. When it contains a certain smooth degree l curve, the same conclusion follows from [43, Theorem 4]. The details are given in [43, Section 4].

Proof of Proposition 3.2.1. We define

$$\overline{Z}^{2p}(W) = \operatorname{Coker}\Big(H^{2p}(W, \mathbb{Z}(p))_{\mathrm{tors}} \to Hdg^{2p}(W, \mathbb{Z})/H^{2p}_{\mathrm{alg}}(W, \mathbb{Z}(p))\Big).$$

Then we have the following exact sequence:

$$0 \to \overline{Z}^{2p}(W)_{\mathrm{tors}} \to H^{2p}_{\mathrm{alg}}(W, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z} \to Hdg^{2p}(W, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}.$$

We have $\overline{Z}^{2p}(W)_{\mathrm{tors}} \neq 0$ by the assumption. Let $\alpha \in \overline{Z}^{2p}(W)_{\mathrm{tors}}$ be a non-trivial element; we use the same notation for its image in $H^{2p}_{\mathrm{alg}}(W, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z}$. Let $\widetilde{\alpha} \in CH^p(W) \otimes \mathbb{Q}/\mathbb{Z}$ be an element which maps to α via the surjection

$$cl^p \otimes \mathbb{Q}/\mathbb{Z}\colon CH^p(W) \otimes \mathbb{Q}/\mathbb{Z} \to H^{2p}_{\mathrm{alg}}(W, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z}.$$

Let $k \subset \mathbb{C}$ be an algebraically closed field such that $\operatorname{tr.deg}_{\mathbb{Q}} k < \infty$ and both W and $\widetilde{\alpha}$ are defined over k. Let E be a smooth elliptic curve such that $j(E) \notin k$. We fix one component $\mathbb{Q}/\mathbb{Z}$ of $CH^1(E)_{\mathrm{tors}} = (\mathbb{Q}/\mathbb{Z})^2$, and we identify $\widetilde{\alpha}$ with an element in $CH^p(W) \otimes CH^1(E)_{\mathrm{tors}}$. By the Schoen theorem [41, Theorem 0.2], the image β of $\widetilde{\alpha}$ by the exterior product map

$$CH^p(W) \otimes CH^1(E)_{\mathrm{tors}} \to CH^{p+1}(W \times E)$$

is non-zero. Then $\beta \in A^{p+1}(W \times E)_{\mathrm{tors}}$. We prove

$$\beta \in \operatorname{Ker}\left(\psi^{p+1}\colon A^{p+1}(W \times E) \to J^{p+1}_a(W \times E)\right).$$

It is enough to prove that β is in the kernel of the cycle class map of the Deligne cohomology:

$$cl^{p+1}_{\mathcal{D}}\colon CH^{p+1}(W \times E) \to H^{2p+2}_{\mathcal{D}}(W \times E, \mathbb{Z}(p+1)).$$

The composition of $cl_{\mathcal{D}}^{p+1}$ with the exterior product map factors through

$$cl_{\mathcal{D}}^{p} \otimes cl_{\mathcal{D}}^{1} \colon CH^p(W) \otimes CH^1(E)_{\mathrm{tors}} \to H_{\mathcal{D}}^{2p}(W, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^{2}(E, \mathbb{Z}(1))_{\mathrm{tors}}.$$

Now it is enough to prove that $\widetilde{\alpha}$ is in the kernel of this map. Since we have an extension

$$0 \to J^p(W) \to H_{\mathcal{D}}^{2p}(W, \mathbb{Z}(p)) \to Hdg^{2p}(W, \mathbb{Z}) \to 0$$

and the complex torus $J^p(W)$ is divisible, we have an isomorphism

$$H_{\mathcal{D}}^{2p}(W, \mathbb{Z}(p)) \otimes H_{\mathcal{D}}^{2}(E, \mathbb{Z}(1))_{\mathrm{tors}} \cong Hdg^{2p}(W, \mathbb{Z}) \otimes H_{\mathcal{D}}^{2}(E, \mathbb{Z}(1))_{\mathrm{tors}}.$$

The proof is done by the choice of $\widetilde{\alpha}$. □

Proof of Theorem 1.2.3. For any smooth projective curve E, the group $CH_0(W \times E)$ is supported on a 3-dimensional closed subset. The proof is done by applying Corollary 3.1.3 for $p = 3$ to $V = W \times E$ and Proposition 3.2.1 for $p = 2$. □

The same arguments yield a "homology counterpart" of Theorem 1.2.3:

Theorem 3.2.3. *Let W be a smooth projective variety such that $CH_0(W)$ is supported on a surface and the inclusion*

$$H_{2,\mathrm{alg}}(W, \mathbb{Z}(1))/\operatorname{tors} \subsetneq Hdg_2(W, \mathbb{Z})/\operatorname{tors}$$

is strict. Then there exists a smooth elliptic curve E such that the sublattice

$$N_2H_3(W \times E, \mathbb{Z}(1))/\operatorname{tors} \subset H_3(W \times E, \mathbb{Z}(1))/\operatorname{tors}$$

is not primitive.

Proof of Corollary 1.2.5. Let X be the pencil of Enriques surfaces of Theorem 2.2.1. We have $CH_0(X) = \mathbb{Z}$ by Lemma 2.1.4. Moreover, the cohomology group $H^4(X, \mathbb{Z})$ is torsion-free and the inclusion $H^4_{\mathrm{alg}}(X, \mathbb{Z}) \subsetneq Hdg^4(X, \mathbb{Z})$ is strict by Theorem 2.2.1. Now the assertion follows by applying Theorem 1.2.3 to $W = X$. The proof is complete. □

Finally, we explain how to produce counterexamples to Murre's question in higher dimensions and for other values of p. We take X and E as in Corollary 1.2.5, and let $d \geq 4$. Then, on the d-fold $X \times E \times \mathbb{P}^{d-4}$, for all $3 \leq p \leq d-1$, the sublattice

$$N^{p-1}H^{2p-1}(X \times E \times \mathbb{P}^{d-4}, \mathbb{Z}(p)) \subset H^{2p-1}(X \times E \times \mathbb{P}^{d-4}, \mathbb{Z}(p))$$

is not primitive (this follows from the formula [3, Theorem 3.1] for the Bloch-Ogus spectral sequence [10] under taking the product with a projective space). In particular, for all $3 \leq p \leq d-1$, the Abel-Jacobi map

$$\psi^p \colon A^p(X \times E \times \mathbb{P}^{d-4}) \to J^p_a(X \times E \times \mathbb{P}^{d-4})$$

is not universal.

APPENDIX

SOME FUNDAMENTAL RESULTS

This chapter is organized as follows. In Section A.1, we give a direct proof of a theorem of Walker on the factorization of the Abel-Jacobi maps. In Section A.2, we discuss stable birational invariants related to our problems. In Section A.3, we prove the Roitman theorem for the Walker maps by using the formalism of decomposition of the diagonal.

We work over the complex numbers throughout.

Section A.2 and A.3 are based on the paper [45] (Suzuki, F.: *A remark on a* 3*-fold constructed by Colliot-Thélène and Voisin*, Math. Res. Lett. **27** (2020), no1, 301–317).

A.1 Factorization of the Abel-Jacobi maps

We give a direct proof of the following theorem of Walker, which was originally proved as an application of the theory of the Lawson homology and the morphic cohomology.

Theorem A.1.1 ([54])**.** *For a smooth projective variety X, the Abel-Jacobi map ψ^p factors as*

$$\begin{tikzcd} & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))) \arrow[d, "\pi^p"] \\ A^p(X) \arrow[ur, "\widetilde{\psi}^p"] \arrow[r, "\psi^p"'] & J_a^p(X) \end{tikzcd} ,$$

APPENDIX (Continued)

where $J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))$ *is the intermediate Jacobian for the mixed Hodge structure given by the coniveau filtration* $N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))$*,* π^p *is a natural isogeny, and* $\widetilde{\psi}^p$ *is a surjective regular homomorphism.*

Remark A.1.2. The Walker map $\widetilde{\psi}^p$ is the uniques lift of the Abel-Jacobi map ψ^p. This follows from the fact that $A^p(X)$ is divisible [10, Lemma 7.10] and $\mathrm{Ker}(\pi^p)$ is finite.

Before beginning the proof, we review the construction of the Abel-Jacobi maps using mixed Hodge structures [24] (the reader can consult [15; 16] for basic knowledge about mixed Hodge structures).

For a mixed Hodge structure $(H, W_\bullet, F^\bullet)$, we define its intermediate Jacobian $J(H)$ as the extension group

$$J(H) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H)$$

in the abelian category MHS of mixed Hodge structures. If H is pure of weight -1, then $J(H)$ is isomorphic to a complex torus

$$H_{\mathbb{C}}/(H_{\mathbb{Z}} + F^0 H_{\mathbb{C}}).$$

For a smooth projective variety X, the cohomology group $H^{2p-1}(X,\mathbb{Z}(p))$ has a pure Hodge structure of weight -1, therefore we have $J^p(X) = J(H^{2p-1}(X,\mathbb{Z}(p)))$. On the other hand, for a codimension p closed subset $Y \subset X$, the long exact sequence for cohomology groups with supports gives a short exact sequence[1]

$$0 \to H^{2p-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(X - Y, \mathbb{Z}(p)) \to Z^p_Y(X)_{\mathrm{hom}} \to 0.$$

This is a short exact sequence of mixed Hodge structures, where $Z^p_Y(X)_{\mathrm{hom}}$ has the trivial Hodge structure. Then the boundary map in the long exact sequence for $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}(0), -)$ determines a map

$$Z^p_Y(X)_{\mathrm{hom}} \to J^p(X).$$

Now we take the direct limit about all codimension p closed subsets of X to obtain a map

$$Z^p(X)_{\mathrm{hom}} \to J^p(X).$$

This coincides with the Abel-Jacobi map AJ^p defined by using currents.

---

[1]For a variety X, we denote by $Z^p(X)$ the group of codimension p cycles on X and by $Z^p(X)_{\mathrm{rat}}$ (resp. $Z^p(X)_{\mathrm{alg}}$, $Z^p(X)_{\mathrm{hom}}$) the subgroup of cycles rationally equivalent to zero (resp. algebraically equivalent to zero, homologous to zero) on X. For a codimension p closed subset $Y \subset X$, we denote by $Z^p_Y(X)$ the subgroup of cycles supported on Y; the groups $Z^p_Y(X)_{\mathrm{rat}}, Z^p_Y(X)_{\mathrm{alg}}$, and $Z^p_Y(X)_{\mathrm{hom}}$ are accordingly defined.

Proof of Theorem A.1.1. First we construct the Walker map. Let X be a smooth projective variety X. For a codimension p closed subset $Y \subset X$, the long exact sequence for cohomology groups with supports gives

$$\begin{tikzcd}
0 \arrow[r] & H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] \arrow[d, "f"] & H^{2p-1}(X-Y,\mathbb{Z}(p)) \arrow[r] \arrow[d] & Z^p_Y(X)_{\mathrm{hom}} \arrow[r] \arrow[d] & 0 , \\
0 \arrow[r] & \varinjlim_{Z\in\mathcal{Z}^{p-1}} H^{2p-1}(X-Z,\mathbb{Z}(p)) \arrow[r, equal] & \varinjlim_{Z\in\mathcal{Z}^{p-1}} H^{2p-1}(X-Y-Z,\mathbb{Z}(p)) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{Z}^{p-1}$ is the set of codimension $p-1$ closed subsets of X. By the snake lemma, we have an exact sequence

$$0 \to N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X-Y,\mathbb{Z}(p)) \to Z^p_Y(X)_{\mathrm{hom}} \xrightarrow{\delta_Y} \mathrm{Coker}(f).$$

We prove that $\mathrm{Ker}(\delta_Y) = Z^p_Y(X)_{\mathrm{alg}}$. We have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & \varinjlim_{(Y,Z)\in\mathcal{Z}^p/\mathcal{Z}^{p-1}} H^{2p-1}_{Z-Y}(X-Y,\mathbb{Z}(p)) \arrow[r, equal] \arrow[d] & \varinjlim_{(Y,Z)\in\mathcal{Z}^p/\mathcal{Z}^{p-1}} H^{2p-1}_{Z-Y}(X-Y,\mathbb{Z}(p)) , \arrow[d, "\partial"] \\
H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] \arrow[d, equal] & \varinjlim_{Y\in\mathcal{Z}^p} H^{2p-1}(X-Y,\mathbb{Z}(p)) \arrow[r] \arrow[d] & \varinjlim_{Y\in\mathcal{Z}^p} H^{2p}_Y(X,\mathbb{Z}(p)) = Z^p(X) \arrow[d] \\
H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r, "f"] & \varinjlim_{Z\in\mathcal{Z}^{p-1}} H^{2p-1}(X-Z,\mathbb{Z}(p)) \arrow[r] & \varinjlim_{Z\in\mathcal{Z}^{p-1}} H^{2p}_Z(X,\mathbb{Z}(p))
\end{tikzcd}$$

where $\mathcal{Z}^p$ is the set of codimension p closed subsets of X and $\mathcal{Z}^p/\mathcal{Z}^{p-1}$ is the set of pairs $(Y,Z) \in \mathcal{Z}^p \times \mathcal{Z}^{p-1}$ such that $Y \subset Z$. Then the result follows from the diagram and the fact that the image of the map ∂ is the subgroup $Z^p(X)_{\text{alg}} \subset Z^p(X)$ [10, Theorem 7.3]. As a consequence, we have a short exact sequence

$$0 \to N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X - Y, \mathbb{Z}(p)) \to Z^p_Y(X)_{\text{alg}} \to 0.$$

This is a short exact sequence of mixed Hodge structures, where $N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))$ has a pure Hodge structure of weight -1 and $Z^p_Y(X)_{\text{alg}}$ has the trivial Hodge structure. Then the boundary map in the long exact sequence for $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}(0), -)$ determines a map

$$\widetilde{\psi}^p_Y \colon Z^p_Y(X)_{\text{alg}} \to J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)),$$

where $J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))$ is a complex torus. Now we take the direct limit to obtain a map

$$\widetilde{\psi}^p \colon Z^p(X)_{\text{alg}} \to J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))),$$

which we call the Walker map.

Next we establish several basic properties of the Walker map $\widetilde{\psi}^p$.

Lemma A.1.3. *We have a commutative diagram*

$$\begin{tikzcd}
Z^p(X)_{\text{alg}} \arrow[r, "\widetilde{\psi}^p"] \arrow[d, hook] & J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))) \arrow[d, "\pi^p"] \\
Z^p(X)_{\text{hom}} \arrow[r, "AJ^p"] & J^p(X)
\end{tikzcd},$$

where π^p is induced by the inclusion $N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \subseteq H^{2p-1}(X,\mathbb{Z}(p))$.

Proof. We have a commutative diagram of short exact sequences of mixed Hodge structures

$$\begin{tikzcd}
0 \arrow[r] & N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] \arrow[d] & N^{p-1}H^{2p-1}(X-Y,\mathbb{Z}(p)) \arrow[r] \arrow[d] & Z^p_Y(X)_{\mathrm{alg}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] & H^{2p-1}(X-Y,\mathbb{Z}(p)) \arrow[r] & Z^p_Y(X)_{\mathrm{hom}} \arrow[r] & 0
\end{tikzcd}$$

for any codimension p closed subset $Y \subset X$. The assertion follows by applying $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}(0),-)$ and taking the direct limit. $\square$

Lemma A.1.4. *Let C be a smooth projective curve and Γ be a codimension p cycle on $C \times X$ each of whose components dominates C. Then we have a commutative diagram:*

$$\begin{tikzcd}
Z^1(C)_{\mathrm{hom}} \arrow[r, "AJ^1"] \arrow[d, "\Gamma_*"] & J^1(C) \arrow[d, "\Gamma_*"] \\
Z^p(X)_{\mathrm{alg}} \arrow[r, "\widetilde{\psi}^p"] & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))
\end{tikzcd} .$$

Proof. We freely use the fact that the Betti cohomology and the Borel-Moore homology form a Poincaré duality theory with supports (see [3; 10] for the axioms). Let $\pi_C\colon C\times X \to C$ (resp. $\pi_X\colon C \times X \to X$) be the projection to C (resp. X). For a codimension one closed subset $Y \subset C$, setting $Y' = \pi_C^{-1}(Y)$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(C,\mathbb{Z}(1)) \arrow[r] \arrow[d, "(\pi_C)^*"] & H^1(C-Y,\mathbb{Z}(1)) \arrow[r] \arrow[d, "(\pi_C)^*"] & Z^1_Y(C)_{\mathrm{hom}} \arrow[r] \arrow[d, "(\pi_C)^*"] & 0 \\
0 \arrow[r] & H^1(C\times X,\mathbb{Z}(1)) \arrow[r] & H^1(C\times X-Y',\mathbb{Z}(1)) \arrow[r] & Z^1_{Y'}(C\times X)_{\mathrm{hom}} \arrow[r] & 0
\end{tikzcd} . \tag{A.1}$$

APPENDIX (Continued)

Similarly, setting $G = \mathrm{Supp}(\Gamma)$ and $Y'' = Y' \cap G$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(C \times X, \mathbb{Z}(1)) \arrow[r] \arrow[d, "\cup\Gamma"] & H^1(C \times X - Y', \mathbb{Z}(1)) \arrow[r] \arrow[d, "(\cup\Gamma)'"] & Z^1_{Y'}(C \times X)_{\mathrm{hom}} \arrow[r] \arrow[d, "\cup\Gamma"] & 0\ , \\
0 \arrow[r] & H^{2p+1}(C \times X, \mathbb{Z}(p+1)) \arrow[r] & H^{2p+1}(C \times X - Y'', \mathbb{Z}(p+1)) \arrow[r] & Z^{p+1}_{Y''}(C \times X)_{\mathrm{hom}} \arrow[r] & 0
\end{tikzcd}$$

where, letting $i\colon G - Y'' \to X \times C - Y'$ be a closed immersion and denoting by H^{BM}_* the Borel-Moore homology, the middle vertical map $(\cup\Gamma)'$ is the composition

$$\begin{aligned}
H^1(C \times X - Y', \mathbb{Z}(1)) &\xrightarrow{i^*} H^1(G - Y'', \mathbb{Z}(1)) \xrightarrow{\cap(\Gamma|_{G-Y''})} H^{BM}_{2\dim G-1}(G - Y'', \mathbb{Z}(\dim G - 1)) \\
&\xrightarrow{i_*} H^{BM}_{2\dim G-1}(C \times X - Y'', \mathbb{Z}(\dim G - 1)) = H^{2p+1}(C \times X - Y'', \mathbb{Z}(p)).
\end{aligned}$$

Since the images of the vertical maps are supported on G, we have another commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(C \times X, \mathbb{Z}(1)) \arrow[r] \arrow[d, "\cup\Gamma"] & H^1(X \times C - Y', \mathbb{Z}(1)) \arrow[r] \arrow[d, "(\cup\Gamma)'"] & Z^1_{Y'}(C \times X)_{\mathrm{hom}} \arrow[r] \arrow[d, "\cup\Gamma"] & 0 \\
0 \arrow[r] & N^pH^{2p+1}(C \times X, \mathbb{Z}(p+1)) \arrow[r] & N^pH^{2p+1}(C \times X - Y'', \mathbb{Z}(p+1)) \arrow[r] & Z^{p+1}_{Y''}(C \times X)_{\mathrm{alg}} \arrow[r] & 0
\end{tikzcd} \tag{A.2}$$

Finally, setting $Y''' = \pi_X(Y'')$ and letting $j\colon X \times C - \pi_X^{-1}(Y''') \to X \times C - Y''$ be an open immersion, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{2p+1}(C \times X, \mathbb{Z}(p+1)) \arrow[r] \arrow[d, "(\pi_X)_*"] & H^{2p+1}(X \times C - Y'', \mathbb{Z}(p+1)) \arrow[r] \arrow[d, "(\pi_X)_* j^*"] & Z^{2p+1}_{Y''}(C \times X)_{\mathrm{hom}} \arrow[r] \arrow[d, "(\pi_X)_*"] & 0\ , \\
0 \arrow[r] & H^{2p-1}(X, \mathbb{Z}(p)) \arrow[r] & H^{2p-1}(X - Y''', \mathbb{Z}(p)) \arrow[r] & Z^p_{Y'''}(X)_{\mathrm{hom}} \arrow[r] & 0
\end{tikzcd}$$

APPENDIX (Continued)

which restricts to

$$\begin{tikzcd} 0 \arrow[r] & N^pH^{2p+1}(C\times X,\mathbb{Z}(p+1)) \arrow[r] \arrow[d, "(\pi_X)_*"] & N^pH^{2p+1}(X\times C-Y'',\mathbb{Z}(p+1)) \arrow[r] \arrow[d, "(\pi_X)_*j^*"] & Z^{2p+1}_{Y''}(C\times X)_{\mathrm{alg}} \arrow[r] \arrow[d, "(\pi_X)_*"] & 0 \\ 0 \arrow[r] & N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] & N^{p-1}H^{2p-1}(X-Y''',\mathbb{Z}(p)) \arrow[r] & Z^p_{Y'''}(X)_{\mathrm{alg}} \arrow[r] & 0 \end{tikzcd} \quad (A.3)$$

By the diagrams (A.1), (A.2), and (A.3), we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(C,\mathbb{Z}(1)) \arrow[r] \arrow[d, "\Gamma_*"] & H^1(C-Y,\mathbb{Z}(1)) \arrow[r] \arrow[d, "(\pi_X)_*j^*(\cup\Gamma)'(\pi_C)^*"] & Z^1_Y(C)_{\mathrm{hom}} \arrow[r] \arrow[d, "\Gamma_*"] & 0\ . \\ 0 \arrow[r] & N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)) \arrow[r] & N^{p-1}H^{2p-1}(X-Y''',\mathbb{Z}(p)) \arrow[r] & Z^p_{Y'''}(X)_{\mathrm{alg}} \arrow[r] & 0 \end{tikzcd}$$

This is a commutative diagram of mixed Hodge structures. The assertion follows by applying $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}(0),-)$ and taking the direct limit. □

Corollary A.1.5. *The Walker map $\widetilde{\psi}^p$ factors through $A^p(X)$. Moreover we have a commutative diagram*

$$\begin{tikzcd} A^p(X) \arrow[r, "\widetilde{\psi}^p"] \arrow[d, hook] & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))\ . \arrow[d, "\pi^p"] \\ CH^p(X)_{\mathrm{hom}} \arrow[r, "AJ^p"] & J^p(X) \end{tikzcd}$$

Proof. By Lemma A.1.4, we have a commutative diagram

$$\begin{tikzcd} \bigoplus_\Gamma Z^1(\mathbb{P}^1)_{\mathrm{hom}} \arrow[r, "(AJ^1)"] \arrow[d, "(\Gamma_*)"] & \bigoplus_\Gamma J^1(\mathbb{P}^1) = 0 \arrow[d, "(\Gamma_*)"] \\ Z^p(X)_{\mathrm{alg}} \arrow[r, "\widetilde{\psi}^p"] & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))) \end{tikzcd} \quad ,$$

where Γ runs through all codimension p cycles on $\mathbb{P}^1 \times X$ with the components dominating $\mathbb{P}^1$. Since the image of the left vertical map is the subgroup $Z^p(X)_{\mathrm{rat}} \subset Z^p(X)$, the first assertion follows. The second assertion is immediate by using Lemma A.1.3. $\square$

The source of the Walker map $\widetilde{\psi}^p$ will be $A^p(X)$ in the following.

Lemma A.1.6. *The Walker map $\widetilde{\psi}^p$ is functorial for correspondences.*

Proof. The result follows from an argument similar to that of Lemma A.1.4 and the moving lemma. $\square$

Corollary A.1.7. *The Walker map $\widetilde{\psi}^p$ is surjective. Moreover $J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))$ is an abelian variety.*

Proof. Let $Z \subset X$ be a closed subset of codimension $p-1$ such that the natural map

$$H_Z^{2p-1}(X, \mathbb{Z}(p)) \to H^{2p-1}(X, \mathbb{Z}(p))$$

induces a surjection

$$H_Z^{2p-1}(X, \mathbb{Z}(p)) \to N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)).$$

By the right exactness of the intermediate Jacobian functor $J(-)$ [6], we have a surjection

$$J(H_Z^{2p-1}(X, \mathbb{Z}(p))) \to J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))). \tag{A.4}$$

Let $\widetilde{Z}$ be a resolution of Z and $\widetilde{Z}_i$ be the components of $\widetilde{Z}$. An easy computation shows that the natural map

$$\bigoplus_i H^1(\widetilde{Z}_i, \mathbb{Z}(1)) \to H_Z^{2p-1}(X, \mathbb{Z}(p))$$

is an injection with the cokernel having the trivial Hodge structure. This induces a surjection

$$\bigoplus_i J^1(\widetilde{Z}_i) \to J(H_Z^{2p-1}(X, \mathbb{Z}(p))). \tag{A.5}$$

Then we combine (A.4) and (A.5) to obtain a surjection

$$\bigoplus_i J^1(\widetilde{Z}_i) \to J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))),$$

which coincides with the map induced by the graphes Γ_i of $\widetilde{Z}_i \to Z \to X$. By Lemma A.1.6, we have a commutative diagram

$$\begin{tikzcd}
\bigoplus_i CH^1(\widetilde{Z}_i)_{\mathrm{hom}} \arrow[r, phantom, "\overset{AJ^1}{\cong}"] \arrow[d, "((\Gamma_i)_*)"] & \bigoplus_i J^1(\widetilde{Z}_i) \arrow[d, two heads, "((\Gamma_i)_*)"] \\
A^p(X) \arrow[r, "\widetilde{\psi}^p"] & J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))
\end{tikzcd}.$$

The results follow. $\square$

Corollary A.1.8. *The Walker map $\widetilde{\psi}^p$ is regular.*

APPENDIX (Continued)

Proof. By Lemma A.1.6, we have a commutative diagram

$$\begin{tikzcd}
CH^{\dim S}(S)_{\mathrm{hom}} \arrow[r, "AJ^{\dim S}"] \arrow[d, "\Gamma_*"] & J^{\dim S}(S) \arrow[d, "\Gamma_*"] \\
A^p(X) \arrow[r, "\widetilde{\psi}^p"] & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))
\end{tikzcd}$$

for any smooth projective variety S and codimension p cycle Γ on $S \times X$. Now the result is immediate using the Albanese map. $\square$

It remains to show that the natural map

$$\pi^p \colon J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))) \to J^p(X)$$

has finite kernel. It is straightforward to compute that

$$\mathrm{Ker}(\pi^p) = \mathrm{Coker}\left(H^{2p-1}(X,\mathbb{Z}(p))_{\mathrm{tors}} \to (H^{2p-1}(X,\mathbb{Z}(p))/N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))_{\mathrm{tors}}\right).$$

The result immediately follows. $\square$

APPENDIX (Continued)

A.2 Stable birational invariants

Let X be a smooth projective variety. By Lemma 3.1.1, the Abel-Jacobi map ψ^p is universal if the restriction $\psi^p|_{\mathrm{tors}} \colon A^p(X)_{\mathrm{tors}} \to J^p_a(X)_{\mathrm{tors}}$ is an isomorphism. We recall the factorization of the Abel-Jacobi map ψ^p due to Walker [54]:

$$\begin{tikzcd} & J(N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))) \arrow[d, "\pi^p"] . \\ A^p(X) \arrow[ur, "\widetilde{\psi}^p"] \arrow[r, "\psi^p"'] & J^p_a(X) \end{tikzcd}$$

The kernel

$$\mathrm{Ker}(\pi^p) = \mathrm{Coker}\left(H^{2p-1}(X,\mathbb{Z}(p))_{\mathrm{tors}} \to (H^{2p-1}(X,\mathbb{Z}(p))/N^{p-1}H^{2p-1}(X,\mathbb{Z}(p)))_{\mathrm{tors}}\right)$$

is trivial if and only if the sublattice

$$N^{p-1}H^{2p-1}(X,\mathbb{Z}(p))/\,\mathrm{tors} \subset H^{2p-1}(X,\mathbb{Z}(p))/\,\mathrm{tors}$$

is primitive.

Lemma A.2.1. *The groups* $\mathrm{Ker}(\psi^3|_{\mathrm{tors}})$, $\mathrm{Ker}(\psi^{d-1}|_{\mathrm{tors}})$, $\mathrm{Ker}(\pi^3)$, *and* $\mathrm{Ker}(\pi^{d-1})$, *where* $d = \dim X$, *are stable birational invariants of smooth projective varieties* X.

Remark A.2.2. A related result is proved by Voisin [51, Lemma 2.2].

Proof of Lemma A.2.1. For each group, it is enough to check

(i) the invariance under the blow-up along a smooth subvariety;

(ii) the invariance under taking the product with $\mathbb{P}^n$.

By the formulas under these operations for the Chow groups and the Deligne cohomology groups (resp. the coniveau spectral sequence and the integral cohomology groups) and by their compatibility with the cycle class maps (resp. the differentials and the edge homomorphisms), (i) and (ii) are reduced to the triviality of the groups $\mathrm{Ker}(\psi^i|_{\mathrm{tors}})$ and $\mathrm{Ker}(\pi^i)$ for $i \leq 2$ and $i = \dim Y$ on a smooth projective variety Y. The triviality of $\mathrm{Ker}(\psi^2|_{\mathrm{tors}})$ (resp. $\mathrm{Ker}(\psi^{\dim Y}|_{\mathrm{tors}})$) follows from the Roitman theorem for codimension 2-cycles due to Murre [34, Theorem 10.3] (resp. the Roitman theorem [37, Theorem 3.1]). The triviality of $\mathrm{Ker}(\pi^2)$ (resp. $\mathrm{Ker}(\pi^{\dim Y})$) follows from the universality of ψ^2 (resp. $\psi^{\dim Y}$). The rest is clear. The proof is done. $\square$

Corollary A.2.3. *Let X be a smooth projective stably rational variety. Let $p \in \{3, \dim X - 1\}$. Then $\mathrm{Ker}(\psi^p|_{\mathrm{tors}}) = \mathrm{Ker}(\pi^p) = 0$. Therefore the Abel-Jacobi map ψ^p is universal and the sublattice*

$$N^{p-1}H^{2p-1}(X, \mathbb{Z}(p))/\,\mathrm{tors} \subset H^{2p-1}(X, \mathbb{Z}(p))/\,\mathrm{tors}$$

is primitive.

For a smooth projective variety X, let $Z^{2p}(X) = Hdg^{2p}(X, \mathbb{Z})/H^{2p}_{\mathrm{alg}}(X, \mathbb{Z}(p))$ be the defect of the integral Hodge conjecture in degree $2p$. We define

$$\overline{Z}^{2p}(X) = \mathrm{Coker}\left(H^{2p}(X, \mathbb{Z}(p))_{\mathrm{tors}} \to Z^{2p}(X)\right).$$

APPENDIX (Continued)

Then $\overline{Z}^{2p}(X)_{\mathrm{tors}} = 0$ if and only if the sublattice

$$H^{2p}_{\mathrm{alg}}(X, \mathbb{Z}(p))/\operatorname{tors} \subset Hdg^{2p}(X, \mathbb{Z})/\operatorname{tors}$$

is primitive.

Lemma A.2.4. *The groups* $\overline{Z}^4(X)$ *and* $\overline{Z}^{2d-2}(X)$*, where* $d = \dim X$*, are stable birational invariants of smooth projective varieties* X*.*

Remark A.2.5. The groups $Z^4(X)$ and $Z^{2d-2}(X)$, where $d = \dim X$, are stable birational invariants of smooth projective varieties X [50][53] and related to the unramified cohomology groups [14].

Proof of Lemma A.2.4. The proof is reduced to the triviality of the groups $\overline{Z}^2(Y)$ and $\overline{Z}^{2\dim Y}(Y)$ on a smooth projective variety Y. The triviality of $\overline{Z}^2(Y)$ follows from the Lefschetz $(1,1)$-theorem. The triviality of $\overline{Z}^{2\dim Y}(Y)$ is clear. The proof is done. □

We recall the following question (see [14, Subsection 5.6]):

Question A.2.6. *Let* X *be a smooth projective rationally connected variety. Is the group* $\overline{Z}^4(X)$ *trivial? Equivalently, is the inclusion* $H^4_{\mathrm{alg}}(X, \mathbb{Z}(2))/\operatorname{tors} \subset Hdg^4(X, \mathbb{Z})/\operatorname{tors}$ *strict?*

The negative answer to this question would provide us with another example to which we can apply Theorem 1.2.3. There is a unirational fourfold X constructed by Schreieder [42] with $Z^4(X) \neq 0$, that is, the integral Hodge conjecture fails in degree four on X. The fourfold X is a smooth model of a conic bundle Y over $\mathbb{P}^3$. It is hard to analyze $\overline{Z}^4(X)$ while the construction of Y is explicit.

A.3 Decomposition of the diagonal and the Roitman theorem for the Walker maps

A smooth projective variety with $CH_0(X)$ supported on a proper closed subset admits a decomposition of the diagonal due to Bloch [8] and Bloch-Srinivas [11]. This result is generalized by Paranjape [36] and Laterveer [28]. We follow Laterveer's formulation here. Let X be a smooth projective variety of dimension d. For non-negative integers r and s, we consider the following condition: $CH_i(X)_{\mathbb{Q}}$ is supported on an $(i+r)$-dimensional closed subset for any $0 \leq i \leq s$. We call this condition $L_{r,s}$. Assume that $L_{r,s}$ holds for X. Then X admits a generalized decomposition of the diagonal [28, Theorem 1.7] (see also [36, Proposition 6.1]): there exist closed subsets $V_0, \cdots, V_s$ and $W_0, \cdots, W_{s+1}$ of X with $\dim V_j \leq j + r$ $(j = 0, \cdots, s)$ and $\dim W_j \leq d - j$ $(j = 0, \cdots, s+1)$ such that we have a decomposition

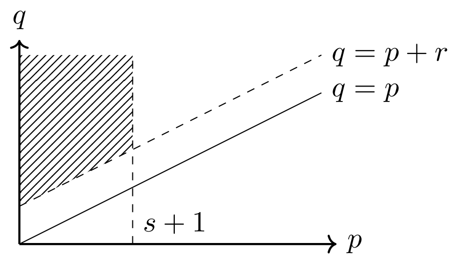
$$\Delta_X = \Delta_0 + \cdots + \Delta_s + \Delta^{s+1}$$

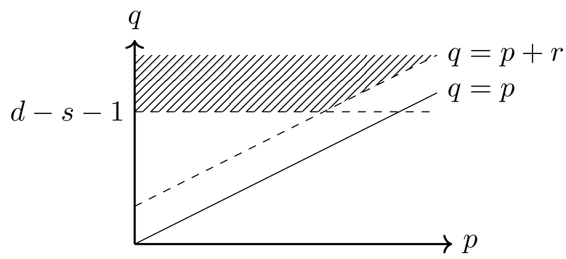
in $CH^d(X \times X)_{\mathbb{Q}}$, where Δ_j is supported on $V_j \times W_j$ $(j = 0, \cdots, s)$ and Δ^{s+1} is supported on $X \times W_{s+1}$.

For a smooth projective variety Y, let $E_2^{p,q}(Y) = H^p(Y, \mathcal{H}^q(\mathbb{Z}))$. For the action of correspondences on the coniveau spectral sequence, we refer the reader to [14, Appendice A].

Lemma A.3.1. *Let X be a smooth projective variety of dimension d such that $L_{r,s}$ holds for X. Then $E_2^{p,q}(X)$ is torsion if $p + r < q$ and $p < s + 1$, or if $p + r < q$ and $q > d - s - 1$.*

APPENDIX (Continued)





Remark A.3.2. The case $s = 0$ is [14, Proposition 3.3 (i)(ii)].

Proof of Lemma A.3.1. We may assume that the inequalities about the dimensions of V_j, W_j are equal. Let N be a positive integer such that

$$N\Delta_X = N\Delta_0 + \cdots + N\Delta_s + N\Delta^{s+1} \in CH^d(X \times X).$$

Let $\widetilde{V_j}(j = 0, \cdots, s)$ and $\widetilde{W_j}(j = 0, \cdots, s+1)$ be resolutions of V_j and W_j, and $\widetilde{\Delta}_j$ be d-cycles on $\widetilde{V_j} \times \widetilde{W_j}$ pushed forward to $c_j\Delta_j$ for some positive integer c_j. We may assume that $c_0 = \cdots = c_{s+1}$. Let $N' = N \cdot c_0$. We prove $N' \cdot E_2^{p,q}(X) = 0$ if $p + r < q$ and $p < s + 1$, or if $p + r < q$ and $q > d - s - 1$.

For $0 \le j \le s$, we prove that

(i) $(N'\Delta_j)_* = 0$ if $(p, q) \notin [j, j + r] \times [j, j + r]$;

(ii) $(N'\Delta_j)^* = 0$ if $(p, q) \notin [d - j - r, d - j] \times [d - j - r, d - j]$.

APPENDIX (Continued)

We have a commutative diagram

$$\begin{tikzcd}
E_2^{p,q}(X) \arrow[r, "(N'\Delta_j)_*"] \arrow[d] & E_2^{p,q}(X) \\
E_2^{p,q}(\widetilde{V}_j) \arrow[d] & E_2^{p-j,q-j}(\widetilde{W}_j) \arrow[u] \\
E_2^{p,q}(\widetilde{V}_j \times \widetilde{W}_j) \arrow[r, "\cup(N\widetilde{\Delta}_j)"] & E_2^{p+r,q+r}(\widetilde{V}_j \times \widetilde{W}_j) \arrow[u]
\end{tikzcd}.$$

To prove (i), it is enough to observe that $E_2^{p,q}(\widetilde{V}_j) = 0$ if $p > j + r$ or $q > j + r$, and $E_2^{p-j,q-j}(\widetilde{W}_j) = 0$ if $p < j$ or $q < j$. Similarly, we have a commutative diagram

$$\begin{tikzcd}
E_2^{p,q}(X) \arrow[r, "(N'\Delta_j)^*"] \arrow[d] & E_2^{p,q}(X) \\
E_2^{p,q}(\widetilde{W}_j) \arrow[d] & E_2^{p+r-d+j,q+r-d+j}(\widetilde{V}_j) \arrow[u] \\
E_2^{p,q}(\widetilde{V}_j \times \widetilde{W}_j) \arrow[r, "\cup(N\widetilde{\Delta}_j)"] & E_2^{p+r,q+r}(\widetilde{V}_j \times \widetilde{W}_j) \arrow[u]
\end{tikzcd}.$$

To prove (ii), it is enough to observe that $E_2^{p,q}(\widetilde{W}_j) = 0$ if $p > d - j$ or $q > d - j$, and $E_2^{p+r-d+j,q+r-d+j}(\widetilde{V}_j) = 0$ if $p < d - r - j$ or $q < d - r - j$.

For Δ^{s+1}, we prove that

(iii) $(N'\Delta^{s+1})_* = 0$ if $(p, q) \notin [s + 1, d] \times [s + 1, d]$;

(iv) $(N'\Delta^{s+1})^* = 0$ if $(p, q) \notin [0, d - s - 1] \times [0, d - s - 1]$.

APPENDIX (Continued)

We have a commutative diagram

$$\begin{tikzcd}
E_2^{p,q}(X) \arrow[rr, "(N'\Delta^{s+1})_*"] \arrow[dd] & & E_2^{p,q}(X) \\
& & E_2^{p-s-1,q-s-1}(\widetilde{W}_{s+1}) \arrow[u] \\
E_2^{p,q}(X \times \widetilde{W}_{s+1}) \arrow[r, "\cup(N\widetilde{\Delta}^{s+1})"] & E_2^{p+d-s-1,q+d-s-1}(X \times \widetilde{W}_{s+1}) \arrow[ur] &
\end{tikzcd}.$$

To prove (iii), it is enough to observe that $E_2^{p-s-1,q-s-1}(\widetilde{W}_{s+1}) = 0$ if $p < s+1$ or $q < s+1$.

Similarly, we have a commutative diagram

$$\begin{tikzcd}
E_2^{p,q}(X) \arrow[rr, "(N'\Delta^{s+1})^*"] \arrow[d] & & E_2^{p,q}(X) \\
E_2^{p,q}(\widetilde{W}_{s+1}) \arrow[d] & & \\
E_2^{p,q}(X \times \widetilde{W}_{s+1}) \arrow[r, "\cup(N\widetilde{\Delta}^{s+1})"] & E_2^{p+d-s-1,q+d-s-1}(X \times \widetilde{W}_{s+1}) \arrow[uur] &
\end{tikzcd}.$$

To prove (iv), it is enough to observe that $E^{p,q}(\widetilde{W}_{s+1}) = 0$ if $p > d-s-1$ or $q > d-s-1$.

The proof is done by (i), (ii), (iii) and (iv). $\square$

Theorem A.3.3. *Let X be a smooth projective variety of dimension d such that $L_{3,s}$ holds for X. Let $p \in [3, s+3] \cup [d-s-1, d-1]$. Then the restriction*

$$\widetilde{\psi}^p|_{\mathrm{tors}} \colon A^p(X)_{\mathrm{tors}} \to J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{\mathrm{tors}}$$

is an isomorphism. Moreover the Walker map $\widetilde{\psi}^p$ *is universal.*

Remark A.3.4. The case $s = 0$ is Theorem 1.2.1.

Proof of Theorem A.3.3. The second statement follows from the first one by Lemma 3.1.1.

We prove that the restriction $\widetilde{\psi}^p|_{\mathrm{tors}}$ is an isomorphism. By Lemma 3.1.2, it is enough to prove that $\mathrm{Ker}(f^p)$ is torsion. By Lemma A.3.1, the groups

$$E_2^{p-3,p+1}(X), \cdots, E_2^{0,2p-2}(X)$$

are torsion, so the result follows. $\square$

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