On the Well-Posedness and Long Time Behaviour of the Hall-Magnetohydrodynamics System

by

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THESIS

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DEDICATED TO ALL THOSE WHO STRUGGLE FOR THE TRUTH

إلى أميرة

To the lovely Δ_q s,

Each wave of the sea has a different light, just as the beauty of who we love. -- Virginia Woolf

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"Gracias a la vida que me ha dado tanto; Me dio dos luceros que cuando los abro
perfecto distingo lo negro del blanco, y en el alto cielo su fondo estrellado
y en las multitudes el hombre que yo amo..."

HL

CONTRIBUTIONS OF AUTHORS

Chapter 2.2 and Chapter 3.3 are extracted from manuscripts (Dai and Liu, 2019b) and (Liu, 2019), respectively. Both manuscripts have been uploaded onto arXiv.org and are in the process of submission to scholarly journals. Chapter 3.2 contains excerpts from a published article (Dai and Liu, 2019a).

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SUMMARY

We study the incompressible Hall-MHD system, an important model in plasma physics akin to the Navier-Stokes equations, using harmonic analysis tools. Chapter 1 consists of an introduction of the Hall-MHD system and its derivation from a two-fluid Euler-Maxwell system, along with a review of the mathematical preliminaries.

Chapter 2 concerns the well-posedness of the Hall-MHD system. For completeness, a proof of the global-in-time existence of the Leray-Hopf type weak solutions is included. In addition, we include a proof of the regularity criterion in (Dai, 2016), which is of particular interest as it highlights the dissipation wavenumbers formulated via Littlewood-Paley theory. We then exploit the regularizing effect of diffusion and use a classical fixed point theorem to prove local-in-time existence of solutions to the generalized Hall-MHD system in certain Besov spaces as well as global-in-time existence of solutions to the hyper-dissipative electron MHD equations for small initial data in critical Besov spaces.

Long time behaviour of solutions to the Hall-MHD system is studied in Chapter 3. We reproduce the proof of algebraic decay of weak solutions to the fully dissipative Hall-MHD system in (Chae and Schonbek, 2013); we then present our study of strong solutions to the Hall-MHD systems with mere one diffusion featuring the Fourier splitting technique. Under certain moderate assumptions, we show that the magnetic energy decays to 0 and the kinetic energy converges to a certain constant in the resistive inviscid case, while the opposite happens in the viscous non-resistive case. Inspired by (Cheskidov et al., 2018), we study the long time behaviour of solutions to the Hall-MHD system from the viewpoint of the determining Fourier modes. Via Littlewood-Paley theory, we formulate the determining wavenumbers, which bounds the low frequencies essential to the long time behaviour of the solutions. The fact that the determining wavenumbers can be estimated in a certain average sense suggests that the Hall-MHD system has finite degrees of freedom in a certain sense.

CHAPTER 1

INTRODUCTION AND PRELIMINARIES

This chapter is divided into three parts. In Section 1.1, we give an overview of the Hall-magnetohydrodynamics (Hall-MHD) system including its basic properties. We then derive the Hall-MHD system from a two-fluid Euler-Maxwell system in Section 1.2. Section 1.3 consists of introductions to the mathematical tools featured in this thesis.

1.1 An introduction to the Hall-magnetohydrodynamics (Hall-MHD) system

The incompressible Hall-HMD system, describing the evolution of a system consisting of a magnetic field b and charged particles, i.e., electrons and ions, whose collective motion under b is approximated as an electrically conducting fluid with velocity field u, can be written as

$$u_t + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p = \nu \Delta u, \tag{1.1}$$

$$b_t + (u \cdot \nabla)b - (b \cdot \nabla)u + d_i \nabla \times ((\nabla \times b) \times b) = \mu \Delta b, \tag{1.2}$$

$$\nabla \cdot u = 0, \ \nabla \cdot b = 0. \tag{1.3}$$

Here the coefficients ν, μ and d_i stand for the fluid viscosity, magnetic resistivity and ion inertial length, respectively. We are interested in the Cauchy problem on $\Omega = \mathbb{T}^3$ or $\mathbb{R}^3$, i.e., given divergence-free initial data $(u_0, b_0) : (\Omega)^2 \mapsto \left(\mathbb{R}^3\right)^2$, we would like to solve for the unknown functions

$$u(t,x) : [0,T) \times \Omega \mapsto \mathbb{R}^3,\; b(t,x) : [0,T) \times \Omega \mapsto \mathbb{R}^3 \text{ and } p(t,x) : [0,T) \times \Omega \mapsto \mathbb{R}.$$

We notice that it is sufficient to solve for u and b, as the scalar pressure p can be recovered from (u, b) by solving the Poisson equation

$$-\Delta p = \sum_{i,j=1}^{3} \left(\partial_i u^j \partial_j u^i - \partial_i b^j \partial_j b^i\right) \text{ on } \Omega.$$

If $b \equiv 0$, the Hall-MHD system reduces to the Navier-Stokes equations in hydrodynamics, whereas the case $u \equiv 0$ corresponds to the following electron-MHD (EMHD) equations

$$b_t + d_i \nabla \times ((\nabla \times b) \times b) = \mu \Delta b, \tag{1.4}$$

$$\nabla \cdot b = 0, \tag{1.5}$$

which highlight the Hall term $d_i(\nabla \times ((\nabla \times b) \times b))$, the essential nonlinearity of the Hall-MHD system. The presence of the Hall term, an intrinsically three dimensional term which is both quasilinear and of the highest order in System 1.1 - 1.3, distinguishes the Hall-MHD system from the conventional MHD system in highly nontrivial ways. In many situations, it is easier to first study the EMHD equations and then extend the results to the Hall-MHD system by incorporating the fluid parts of the equations.

The Hall-MHD system is a vital model with applications in a wide range of topics in plasma physics and astrophysics, e.g., solar flares, star formation, neutron stars, tokamak and geo-dynamo. In particular, it is indispensable to the interpretation of the magnetic reconnection phenomenon, a fundamental process in plasma physics involving topological reorganizations of the magnetic field lines accompanied by energy transfers from the magnetic field to the plasma in the forms of kinetic energy, thermal energy or particle acceleration. For the ideal MHD system ($\nu = \mu = d_i = 0$), the possibility of magnetic reconnection seems ruled out by Alfvén's theorem, which asserts that the topology of the magnetic field lines is preserved as the magnetic field lines are frozen into the MHD fluid. This can be seen by applying Kelvin's circulation theorem to any material surface $\mathcal{S}$ moving with the MHD fluid, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathcal{S}} B \cdot \mathrm{d}S = 0,$$

that is, the magnetic flux through any material surface advected by the fluid is conserved. It is thus necessary to take the Hall effect into account to explain the violation of Alfvén's theorem, especially in the collisionless setting where $\nu = \mu = 0$.

Equation 1.4 - 1.5 can be seen as the small-scale limit of the Hall-MHD system. At spatial scale $\ell \ll d_i$, the ions and electrons become decoupled as the ions are too heavy to move, simply forming a neutralizing background, rendering the system determined entirely by the electrons. In this case, the magnetic field lines are frozen into the electron fluid only. There are several applications of the EMHD equations in the study of celestial objects, e.g., accretion flows around black holes and strongly magnetized neutron stars known as magnetars. We refer to (Galtier, 2016; Lyutikov, 2013) for more physical backgrounds.

In this thesis, we include several variants of the Hall-MHD system and the EMHD equations. Replacing the Laplacians Δ by generalized dissipation terms $(-\Delta)^\alpha$ and $(-\Delta)^\beta$ leads to the following generalization of the Hall-MHD system –

$$u_t + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p = -\nu(-\Delta)^\alpha u, \tag{1.6}$$

$$b_t + (u \cdot \nabla)b - (b \cdot \nabla)u + d_i \nabla \times ((\nabla \times b) \times b) = -\mu(-\Delta)^\beta b, \tag{1.7}$$

$$\nabla \cdot u = 0, \ \nabla \cdot b = 0. \tag{1.8}$$

Similarly, we have a generalized version of the EMHD equations

$$b_t + d_i \nabla \times ((\nabla \times b) \times b) = -\mu(-\Delta)^\beta b, \tag{1.9}$$

$$\nabla \cdot b = 0. \tag{1.10}$$

Besides the fully dissipative case ($\nu, \mu > 0$) of the Hall-MHD system, we also consider the viscous, non-resistive case ($\nu > 0, \mu = 0$) and the inviscid, resistive case ($\nu = 0, \mu > 0$).

Included in this thesis are a few of our results concerning the solvability of the Hall-MHD (or EMHD) system and the long time behaviour of the solutions. More specifically, we have obtained results on well-posedness for a class of generalized Hall-MHD and EMHD systems, on temporal decay of solutions to the generalized Hall-MHD systems with mere one dissipation term. In addition, we study the long time behaviour of solutions to the Hall-MHD system from the viewpoint of determining Fourier modes.

As the Hall term introduces a new scale into the standard MHD system, the Hall-MHD system (or System 1.6 - 1.8), unlike the standard MHD system, lacks a genuine scale invariance. However, one can still try to extrapolate from the scaling property of the fluid-free system, as the EMHD equations or System 1.9 - 1.10 is invariant under the scaling transformation $b(t,x) \mapsto b_\lambda(t,x) := b(\lambda^2 t, \lambda x)$ or $b(t,x) \mapsto b_\lambda(t,x) := \lambda^{2\beta-2} b(\lambda^{2\beta} t, \lambda x)$, respectively. This is one of the key heuristics in our studies.

Besides the scaling symmetry above, the EMHD equations enjoy a number of symmetries, which the Hall-MHD system also enjoys. Notably, the Hall-MHD and EMHD systems are invariant under

1. the translation $(u,b) \mapsto (u,b)(t-t_0, x-x_0, y-y_0, z-z_0), \forall (t_0,x_0,y_0,z_0) \in \mathbb{R} \times \Omega$.
2. the rotation $(u,b) \mapsto (\mathcal{O}^\intercal u, \mathcal{O}^\intercal b)(\mathcal{O}(x,y,z)^\intercal)$ for any rotation matrix $\mathcal{O}$.
3. the reflection about any hyperplane, e.g., the reflection about $\{y=0\}$:

$$(u,b) \mapsto \begin{pmatrix} u^1(x,-y,z) & -b^1(x,-y,z) \\ -u^2(x,-y,z) & b^2(x,-y,z) \\ u^3(x,-y,z) & -b^3(x,-y,z) \end{pmatrix}.$$

4. the time reversal $(u,b) \mapsto (-u,-b)(-t,x,y,z)$.
5. Galilean transformation $(u,b) \mapsto (u-\bar{u}, b)(t, (x,y,z)+t\bar{u})$.

The fundamental conserved quantities in the inviscid, non-resistive setting of the Hall-MHD system are energy and magnetic helicity. Assuming sufficient regularity and spatial decay of a solution (u, b), multiplying Equation 1.1 and Equation 1.2 by u and b respectively and integrating by parts lead to the following energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|u|^2 + |b|^2\right)\mathrm{d}x = -\nu \int_\Omega |\nabla u|^2 \mathrm{d}x - \mu \int_\Omega |\nabla b|^2 \mathrm{d}x,$$

as the flux from the Hall term vanishes due to the identity

$$\int_\Omega b \cdot \nabla \times ((\nabla \times b) \times b)\,\mathrm{d}x = \int_\Omega (\nabla \times b) \cdot ((\nabla \times b) \times b)\,\mathrm{d}x = 0.$$

For the EMHD equations, the energy identity is just

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |b|^2\,\mathrm{d}x = -\mu \int_\Omega |\nabla b|^2\,\mathrm{d}x.$$

Intuitively, we can already see that the energy identities imply temporal decay for solutions in the energy space. We omit the details pertaining to the conservation of magnetic helicity as we do not intend to cover related results in this thesis.

1.2 Derivation of the Hall-MHD system from a two-fluid Euler-Maxwell system

In (Acheritogaray et al., 2011), the Hall-MHD system was derived from the two-fluid isothermal Euler-Maxwell system for ions and electrons consisting of

1. (Conservation of species)

$$\partial_t n_i + \nabla \cdot (n_i u_i) = 0 \text{ and } \partial_t n_e + \nabla \cdot (n_e u_e) = 0,$$

   where n_i and u_i are the density and velocity of the ions, while n_e and u_e those of the electrons, respectively;

2. (Conservation of momenta)

$$m_i\left(\partial_t(n_i u_i) + \nabla(n_i u_i \otimes u_i)\right) + \nabla(n_i\theta) = en_i(E + u_i \times B) - e^2\eta n_i n_e(u_i - u_e);$$

$$m_e\left(\partial_t(n_e u_e) + \nabla(n_e u_e \otimes u_e)\right) + \nabla(n_e\theta) = -en_e(E + u_e \times B) - e^2\eta n_i n_e(u_e - u_i),$$

   with m_i and m_e being the ion and electron masses, respectively, e the elementary charge, η the resistivity due to ion-electron collisions, θ the common ion and electron temperature, E the electric field and B the magnetic field;

3. (Gauss' laws)

$$\epsilon_0 \nabla \cdot E = \rho \text{ and } \nabla \cdot B = 0,$$

   with the constants ϵ_0 being the vacuum permittivity and $\rho = e(n_i - n_e)$ the charge density;

4. (Faraday's law of induction)

$$\partial_t B + \nabla \times E = 0;$$

5. (Ampère's circuital law)

$$c^{-2}\partial_t E - \nabla \times B = -\mu_0 j,$$

where the current density $j = e(n_i u_i - n_e u_e)$, the constant μ_0 satisfying $\epsilon_0\mu_0 c^2 = 1$ is the vacuum permeability and c is the speed of light.

In order to convert the above two-fluid system into dimensionless form, we introduce the units n_0, u_0, E_0, B_0, x_0, t_0, ρ_0 and j_0 for particle density, particle velocity, electric field, magnetic field, spatial length, time, charges and current, respectively. which are satisfy the following relations

$$x_0 = u_0 t_0, \quad u_0 = \sqrt{\frac{\theta}{m_i}}, \quad E_0 = u_0 B_0, \quad \rho_0 = e n_0.$$

The dimensionless two-fluid Euler-Maxwell system is written as

$$\partial_t n_i + \nabla \cdot (n_i u_i) = 0, \quad \partial_t n_e + \nabla \cdot (n_e u_e) = 0,$$

$$(\partial_t(n_i u_i) + \nabla(n_i u_i \otimes u_i)) + \nabla(n_i\theta) = \alpha^2 n_i(E + u_i \times B) - \beta n_i n_e(u_i - u_e),$$

$$\varepsilon^2 \left(\partial_t(n_e u_e) + \nabla(n_e u_e \otimes u_e)\right) + \nabla(n_e\theta) = \alpha^2 n_e(E + u_e \times B) - \beta n_i n_e(u_e - u_i),$$

$$\alpha^2\lambda^2 \nabla \cdot E = \rho, \quad \nabla \cdot B = 0,$$

$$\partial_t B + \nabla \times E = 0, \quad c^{-2}\partial_t E - \nabla \times B = -\frac{\gamma^2\eta}{\alpha^2\lambda^2} j,$$

$$\rho = n_i - n_e, \quad j = \eta^{-1}(n_i u_i - n_e u_e),$$

where the parameters α, β, γ, ε, λ and η are defined as follows

$$\varepsilon^2 = \frac{m_e}{m_i}, \quad \alpha^2 = \frac{eE_0x_0}{\theta}, \quad \beta = \frac{e^2\eta n_0 u_0 x_0}{\theta}, \quad \gamma = \frac{u_0}{c}, \quad \lambda^2 = \frac{\epsilon_0\theta}{e^2 n_0 x_0^2}, \quad \eta = \frac{j_0}{en_0u_0}.$$

We assume that the electron to ion mass ratio $\varepsilon^2 \to 0$, the scaled Debye length $\lambda^2 \to 0$ and the ratio of fluid velocity to the speed of light $\gamma \to 0$ while satisfying $\frac{\gamma^2\eta}{\alpha^2\lambda^2} = 1$, which lead to

1. (Generalized Ohm's law) $\nabla(n_e\theta) = \alpha^2 n_e(E + u_e \times B) - \beta n_i n_e(u_e - u_i)$,
2. (Quasi-neutrality) $\rho = 0$, i.e., $n_e = n_i = n$,
3. (The standard magnetostatic Ampère's law) $\nabla \times B = j$.

Denoting the ion velocity u_i by u for simplicity, we write the resulting system in the following manner –

$$\begin{aligned}
\partial_t n + \nabla\cdot(nu) &= 0,\\
\partial_t(nu) + \nabla(nu\otimes u) + \nabla(2n\theta) &= \alpha^2\eta j\times B, \qquad (1.11)\\
\nabla\times B &= j,\\
\partial_t B + \nabla\times E &= 0,\\
\nabla\cdot B &= 0,\\
j &= \frac{n}{\eta}(u-u_e), \qquad (1.12)\\
E + u\times B &= -\frac{\theta}{\alpha^2}\nabla(\ln n) + \eta\frac{j\times B}{n} + \frac{\beta\eta}{\alpha^2}j. \qquad (1.13)
\end{aligned}$$

Letting the Lorentz force in Equation 1.11 be of order 1 by setting $\alpha^2\eta = 1$, we further rewrite Equation 1.11, Equation 1.12 and Equation 1.13 as

$$\partial_t(nu) + \nabla(nu \otimes u) + \nabla(2n\theta) = j \times B, \tag{1.14}$$

$$\frac{1}{\alpha^2} j = n(u - u_e), \tag{1.15}$$

$$E + u \times B = \frac{1}{\alpha^2}\left(-\theta\nabla(\ln n) + \frac{j \times B}{n}\right) + \frac{\beta}{\alpha^4} j, \tag{1.16}$$

in which only two parameters $\frac{1}{\alpha^2}$ and $\frac{\beta}{\alpha^4}$ are present. If $\frac{1}{\alpha^2} \to 1$, then the velocities of ions and electrons are different and the Hall term shall appear in the resulting system, where as whether $\frac{\beta}{\alpha^4} \to 1$ or 0 determines if $\mu = 1$ or 0 in Equation 1.2. Assuming that the fluid is incompressible, we obtain the Hall-MHD system as Equation 1.1 - Equation 1.3.

1.3 Mathematical preliminaries

Littlewood-Paley theory, originally due to J.E. Littlewood and R. Paley in the 1930s, has been applied to the analysis of partial differential equations and borne numerous results in the last three decades. Together with the paradifferential calculus, introduced by J.-M. Bony in 1982, they constitute a powerful set of tools in the study of nonlinear PDEs. We refer to the virtuoso survey articles (Bahouri, 2017; Cannone, 2004) for a more detailed overview.

1.3.1 Littlewood-Paley theory

We shall give a comprehensive review of Littlewood-Paley theory, a fundamental tool in our study of the well-posedness and long time behaviour of the Hall-MHD system, by including both the homogeneous and the inhomogeneous versions of Littlewood-Paley decomposition on $\mathbb{R}^n$ as well as the version on $\mathbb{T}^n$.

To start, we choose a radial cut-off function $\chi \in C_0^\infty(\mathbb{R}^n; [0,1])$ in the frequency space, satisfying

$$\chi(\xi) = \begin{cases} 1, \text{ for } |\xi| \le \frac{3}{4}, \\ 0, \text{ for } |\xi| \ge 1. \end{cases}$$

Let $\lambda_q = 2^q$ with $q \in \mathbb{Z}$. We define

$$\psi(\xi) := \chi\left(\frac{\xi}{2}\right) - \chi(\xi) \text{ and } \psi_q(\xi) := \psi\left(\lambda_q^{-1}\xi\right).$$

Notice that functions in $\{\psi_q(\xi)\}_{q=\mathbb{Z}}$ have annular supports that are almost disjoint, i.e., $\mathrm{supp}\psi_i \cap \mathrm{supp}\psi_j = \emptyset$ for indices i, j satisfying $|i-j| \ge 2$. Moreover, $\sum_q \psi_q(\xi) = 1$ on $\mathbb{R}^3/\{0\}$.

The homogeneous version of the dyadic partition of unity $\{\dot{\varphi}_q(\xi)\}_{q=\mathbb{Z}}$ is then defined as

$$\dot{\varphi}_q(\xi) := \psi_q(\xi),$$

while the nonhomogeneous version $\{\varphi_q(\xi)\}_{q=-1}^{\infty}$ is given by

$$\varphi_q(\xi) = \begin{cases} \psi_q(\xi), \text{ for } q \ge 0, \\ \chi(\xi), \text{ for } q = -1. \end{cases}$$

Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Denoting $h := \mathcal{F}^{-1}\psi$ and $\tilde{h} := \mathcal{F}^{-1}\chi$, we introduce the inhomogeneous dyadic blocks $\mathit{\Delta}_q$ and the inhomogeneous low frequency cut-off operators S_q as

$$\begin{aligned}
\mathit{\Delta}_q u &:= \mathcal{F}^{-1}(\varphi_q \hat{u}) = \lambda^{nq} \int_{\mathbb{R}^n} h(\lambda^q y) u(x-y) \mathrm{d}y, \text{ for } q \in \mathbb{N}, \\
\mathit{\Delta}_{-1} u &:= \mathcal{F}^{-1}(\chi \hat{u}) = \lambda^{-n} \int_{\mathbb{R}^n} \tilde{h}(y) u(x-y) \mathrm{d}y, \\
S_q u &:= \sum_{q' \leq q} \Delta_{q'} u,
\end{aligned}$$

while the homogeneous dyadic dyadic blocks and low frequency cut-off operators are defined as

$$\begin{aligned}
\dot{\mathit{\Delta}}_q u &:= \mathcal{F}^{-1}(\dot{\varphi}_q \hat{u}) = \lambda^{nq} \int_{\mathbb{R}^n} h(\lambda^q y) u(x-y) \mathrm{d}y, \\
\dot{S}_q u &:= \mathcal{F}^{-1}\left(\chi\left(\lambda^{-q}\,\cdot\,\right)\hat{u}\right) = \lambda^{nq} \int_{\mathbb{R}^n} \widetilde{h}(\lambda^q y) u(x-y) \mathrm{d}y.
\end{aligned}$$

For $\dot{\mathit{\Delta}}_q$ and $\dot{S}_q$, we restrict u to $\mathscr{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \lim_{q \to -\infty} \dot{S}_q u = 0 \right\}$, the space for which the homogeneous Littlewood-Paley theory makes sense.

The operators introduced above map L^p to L^p with norms independent of p and q. Formally, we have the decompositions

$$\sum_{q=-1}^{\infty} \mathit{\Delta}_q = \mathrm{Id} \quad \text{and} \quad \sum_{q \in \mathbb{Z}} \dot{\mathit{\Delta}}_q = \mathrm{Id};$$

in the inhomogeneous case, the identity makes sense in $\mathscr{S}'(\mathbb{R}^n)$.

With minor changes, the above formalism can be adapted to the periodic domain $\mathbb{T}^n$. Given $u \in \mathscr{S}'(\mathbb{T}^n)$, we can decompose it into Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}_k e^{ik\cdot x}, \text{ with } \hat{u}_k = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} u(x) e^{-ik\cdot x} \mathrm{d}x.$$

We define the periodic dyadic blocks Δ_q^{per} as

$$\Delta_q^{\text{per}} u = \sum_{k \in \mathbb{Z}^n} \varphi_q(k) \hat{u}_k e^{ik\cdot x} = \frac{1}{|\mathbb{T}^n|} \int_{\mathbb{T}^n} h_q^{per}(y) u(x-y) \mathrm{d}y,$$

where $h_q^{per}(x) = \sum_{k \in \mathbb{Z}^n} \varphi_q(k) e^{ik\cdot x}$. In turn, the low frequency cut-off operator on $\mathbb{T}^n$ is defined as $S_q^{\text{per}} = \sum_{q' \le q} \Delta_{q'}^{\text{per}}$.

When there is no confusion about which variant of Littlewood-Paley theory we use, we shall just write the Littlewood-Paley projections of u as u_q or $\Delta_q u$. In addition, we introduce the following notations –

$$u_{\le Q} := \sum_{q=-1}^{Q} u_q, \quad u_{(P,Q]} := \sum_{q=P+1}^{Q} u_q, \quad \tilde{u}_q := \sum_{|p-q|\le 1} u_p.$$

We notice the following quasi-orthogonal relations for the Littlewood-Paley decomposition

$$\Delta_p \Delta_q = 0, \text{ if } |p-q| \ge 2.$$

For a function whose support in the frequency space is an annulus or a ball, we have the following observations on the action of derivatives.

Lemma 1.3.1 (Berstein's inequalities). *(Bahouri et al., 2011) Let n be the space dimension, $s \in \mathbb{R}^+$, $q \in \mathbb{Z}$ and $r, p \in [1, \infty]$ satisfying $1 \leq p \leq r \leq \infty$.*

If supp $\hat{u} \in \mathcal{C}_\lambda = \{\xi \in \mathbb{R}^n : |\xi| \sim \lambda\}$, then $\|D^s u\|_r := \sup_{|\alpha|=s} \|\partial^\alpha u\|_r \sim \lambda^s \|u\|_r$.

If supp $\hat{u} \in B_\lambda = \{\xi \in \mathbb{R}^n : |\xi| \leq \lambda\}$ then $\|u\|_r \leq \lambda^{n\left(\frac{1}{p}-\frac{1}{r}\right)} \|u\|_p$.

In view of the above lemma, we realize that Littlewood-Paley decomposition provides alternative definitions of classical spaces in terms of conditions on the dyadic blocks of functions. For $s \in \mathbb{R}$, the nonhomogeneous Sobolev spaces $H^s(\mathbb{R}^n)$ can thus be characterized via Littlewood-Paley projections –

$$\|u\|_{H^s} = \left(\sum_{q \geq -1} \lambda_q^{2s} \|\varDelta_q u\|_2^2 \right)^{\frac{1}{2}},$$

while the norms of L^2-based homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$, which are Banach spaces if and only if $s < \frac{n}{2}$, can also be give by

$$\|u\|_{\dot{H}^s} = \left(\sum_{q \in \mathbb{Z}} \lambda_q^{2s} \|\dot{\varDelta}_q u\|_2^2 \right)^{\frac{1}{2}}.$$

Littlewood-Paley theory also provides us with a characterization of homogeneous and nonhomogeneous Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. We have $B^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \|u\|_{B^s_{p,q}(\mathbb{R}^n)} < \infty \right\}$ and $\dot{B}^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathscr{S}'_h(\mathbb{R}^n) : \|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} < \infty \right\}$ with

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\displaystyle\sum_{j \geq -1} \lambda_j^{qs} \|\varDelta_j u\|_p^q \right)^{\frac{1}{q}}, & \text{for } q \in [1, \infty), \\ \displaystyle\sup_{j \geq -1} \lambda_j^s \|\varDelta_j u\|_p, & \text{for } q = \infty, \end{cases}$$

and

$$\|u\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} \lambda_j^{qs} \|\dot{\Delta}_j u\|_p^q \right)^{\frac{1}{q}}, \text{ for } q \in [1,\infty), \\ \sup_{j\in\mathbb{Z}} \lambda_j^s \|\dot{\Delta}_j u\|_p, \text{ for } q = \infty. \end{cases}$$

Recalling the cut-off function $\psi(\xi)$, we consider the action of the heat flow over a function with annular support in the Fourier space. Let $u \in \mathscr{S}'$ be such that $\mathrm{supp}\hat{u} \subset \mathrm{supp}\psi$. We then have the following calculations –

$$\begin{aligned} e^{t\Delta}u =& \mathscr{F}^{-1}\left(e^{-t|\xi|^2}\psi(\xi)\hat{u}(\xi)\right) \\ =& g(t,\cdot) * u, \text{ with } g(t,x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)e^{-t|\xi|^2}\mathrm{d}\xi. \end{aligned}$$

Denoting by Δ_ξ the ξ-Laplacian and using the fact that

$$\begin{aligned} (1+|x|^2)^n g(t,x) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left((\mathrm{Id}-\Delta_\xi)^n e^{ix\cdot\xi}\right)\psi(\xi)e^{-t|\xi|^2}\mathrm{d}\xi \\ =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(\mathrm{Id}-\Delta_\xi)^n \left(\psi(\xi)e^{-t|\xi|^2}\right)\mathrm{d}\xi, \end{aligned}$$

and the fact that ψ is supported in an annular region, we can show that there exist constants $C, c > 0$ such that $\|g(t,\cdot)\|_{L^1} \leq Ce^{-ct}$, from which we can deduce

$$\|e^{t\Delta}u\|_{L^p} \leq Ce^{-c\lambda^2 t}\|u\|_{L^p} \tag{1.17}$$

by rescaling.

Given $s>0$ and $u \in \mathscr{S}_h'(\mathbb{R}^n)$, we have, by (1.17)

$$\left\| t^{\frac{s}{2}} \dot{\Delta}_j e^{t\Delta} u \right\|_p \leq t^{\frac{s}{2}} \lambda_j^s C e^{-c\lambda_j^2 t} \lambda_j^{-s} \|\dot{\Delta}_j u\|_p.$$

By the definition of the homogeneous Besov spaces and the integrability of the Gaussian function, it holds that

$$\begin{aligned} \left\| t^{\frac{s}{2}} e^{t\Delta} u \right\|_{L^p} \leq & \sum_{j\in\mathbb{Z}} \left\| t^{\frac{s}{2}} \dot{\Delta}_j e^{t\Delta} u \right\|_{L^p} \\ \leq & \|u\|_{\dot{B}^{-s}_{p,\infty}} \sum_{j\in\mathbb{Z}} t^{\frac{s}{2}} \lambda_j^s C e^{-c\lambda_j^2 t} \lesssim \|u\|_{\dot{B}^{-s}_{p,\infty}}. \end{aligned} \tag{1.18}$$

Invoking the definition of the Gamma function, we may write

$$\dot{\Delta}_j u = \frac{1}{\Gamma\left(\frac{s}{2}+1\right)} \int_0^\infty t^{\frac{s}{2}} (-\Delta)^{\frac{s}{2}+1} \dot{\Delta}_j e^{t\Delta} u \, \mathrm{d}t.$$

Using the identity $e^{t\Delta} = e^{\frac{t}{2}\Delta} e^{\frac{t}{2}\Delta}$, we have

$$\|\dot{\Delta}_j u\|_p \leq C \int_0^\infty t^{\frac{s}{2}} \lambda_j^{s+2} e^{-ct\lambda_j^2} \|e^{t\Delta} \dot{\Delta}_j u\|_p \, \mathrm{d}t \leq C \lambda_j^s \sup_{t>0} t^{\frac{s}{2}} \|e^{t\Delta} u\|_{L^p},$$

which, along with (1.18), implies the norm equivalence $\|\cdot\|_{\dot{B}^{-s}_{p,\infty}} \sim \sup_{t>0} t^{\frac{s}{2}} \|e^{t\Delta}\cdot\|_{L^p}$. In fact, as proven in (Bahouri et al., 2011), for $s > 0$ the homogeneous Besov spaces $\dot{B}^{-s}_{p,q}$ can be characterized by the heat flow as

$$\dot{B}^{-s}_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathscr{S}_h'(\mathbb{R}^n) : \left\| t^{\frac{s}{2}} \|e^{t\Delta} u\|_{L^p} \right\|_{L^q\left(\mathbb{R}^+, \frac{\mathrm{d}t}{t}\right)} \right\} < \infty.$$

In this thesis, the following generalization of the heat flow characterization of the L^∞, ℓ^∞-based Besov spaces is particularly used

Lemma 1.3.2. *(Cheskidov and Dai, 2020) Let $f \in \dot{B}^s_{\infty,\infty}$ for some $s < 0$. The following norm equivalence holds.*

$$\|f\|_{\dot{B}^s_{\infty,\infty}} = \sup_{t>0} t^{-\frac{s}{2\alpha}} \|e^{-t(-\Delta)^\alpha} f\|_{L^\infty}, \ \textit{where } \alpha > 0.$$

More generally, the following lemma, proven in (Kozono et al., 2003; Miao et al., 2008), shall be used.

Lemma 1.3.3. *i) For $\alpha > 0$, the following inequalities hold.*

$$\|e^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le C\|f\|_{L^\infty},$$

$$\|\nabla e^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le Ct^{-\frac{1}{2\alpha}} \|f\|_{L^\infty},$$

$$\|\nabla \mathbb{P} e^{-t(-\Delta)^\alpha} f\|_{L^\infty} \le Ct^{-\frac{1}{2\alpha}} \|f\|_{L^\infty}.$$

ii) For $\alpha > 0$ and $s_0 \leq s_1$, the following inequalities hold.

$$\|e^{-t(-\Delta)^\alpha} f\|_{\dot{B}^{s_1}_{\infty,\infty}} \leq C t^{-\frac{1}{2\alpha}(s_1 - s_0)} \|f\|_{\dot{B}^{s_0}_{\infty,\infty}},$$

$$\|\nabla^k e^{-t(-\Delta)^\alpha} f\|_{\dot{B}^{s_1}_{\infty,\infty}} \leq C t^{-\frac{1}{2\alpha}(s_1 - s_0 + k)} \|f\|_{\dot{B}^{s_0}_{\infty,\infty}}.$$

1.3.2 Paradifferential calculus and commutator estimates

Using Littlewood-Paley decomposition, we can formally write the product of two tempered distributions $u, v \in \mathscr{S}'$ as

$$uv = \sum_{p,q} u_p v_q.$$

The paradifferential calculus provides us with a decomposition of the above sum into three parts

$$\begin{aligned} uv =& \sum_q u_{\leq q-2} v_q + \sum_q u_q v_{\leq q-2} + \sum_q \tilde{u}_q v_q \\ =&: T_u v + T_v u + R(u,v), \end{aligned}$$

with $T_u v$ and $T_v u$ denoting the parts in which the dyadic blocks of u are of significantly lower and higher frequencies than the dyadic blocks of v, respectively, while the remainder $R(u,v)$ denotes the part in which the dyadic blocks of u and v are of comparable frequencies.

Recalling that $\varphi(\xi) = 0$ if $|\xi| \leq \frac{3}{4}$ or $|\xi| \geq 2$, we further observe that

$$(u_q v_{\leq q-2})_p = 0 \text{ if } p \geq q+2 \text{ or } p \leq q-3, \quad (u_q v_{q+1})_p = 0 \text{ for } p \geq q+3.$$

For a generic convection term $u \cdot \nabla v$, the above observation along with Bony's paraproduct decomposition yields

$$\Delta_q(u \cdot \nabla v) = \sum_{|p-q|\leq 2} \Delta_q(u_{\leq p-2} \cdot \nabla v_p) + \sum_{|p-q|\leq 2} \Delta_q(u_p \cdot \nabla v_{\leq p-2}) + \sum_{p\geq q-2} \Delta_q(\tilde{u}_p \cdot \nabla v_p). \quad (1.19)$$

Similarly, for the term $u \times (\nabla \times v)$, the following decomposition holds –

$$\begin{aligned}\Delta_q\left(u \times (\nabla \times v)\right) = &\sum_{|p-q|\leq 2} \Delta_q\left(u_{\leq p-2} \times (\nabla \times v_p)\right) + \sum_{|p-q|\leq 2} \Delta_q\left(u_p \times (\nabla \times v)_{\leq p-2}\right) \\ &+ \sum_{p\geq q-2} \Delta_q\left(\tilde{u}_p \times (\nabla \times v_p)\right).\end{aligned} \quad (1.20)$$

To facilitate the estimation of the nonlinear terms, we introduce several commutators. For the inertial/convection terms, we define

$$[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p = \Delta_q(u_{\leq p-2} \cdot \nabla v_p) - u_{\leq p-2} \cdot \nabla \Delta_q v_p, \quad (1.21)$$

which enjoys the estimate in the following lemma –

Lemma 1.3.4. *Let $\nabla \cdot u_{\leq p-2} = 0$. For r_1, r_2 and $r_3 \in [1, \infty]$ satisfying $\frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3}$, it holds that*

$$\|[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p\|_{r_1} \lesssim \|v_p\|_{r_2} \sum_{p'\leq p-2} \lambda_{p'} \|u_{p'}\|_{r_3}.$$

Proof: By the definition of Δ_q, integration by parts and the fact that $\nabla \cdot u_{\leq p-2} = 0$,

$$
\begin{aligned}
[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p =& \lambda_q^3 \int_{\mathbb{R}^3} h\left(\lambda_q(x-y)\right)\left(u_{\leq p-2}(y) - u_{\leq p-2}(x)\right) \nabla v_p(y)\mathrm{d}y \\
=& -\lambda_q^3 \int_{\mathbb{R}^3} \nabla h\left(\lambda_q(x-y)\right)\left(u_{\leq p-2}(y) - u_{\leq p-2}(x)\right) v_p(y)\mathrm{d}y,
\end{aligned}
$$

Using a change of variables and the first order Taylor's formula, we have

$$
|[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p(x)| \leq \lambda_q^3 \int_0^1 \int_{\mathbb{R}^3} |z| \left|\nabla h\left(\lambda_q(z)\right)\right| \left|\nabla u_{\leq p-2}(x-\tau z)\right| |v_p(x-z)| \mathrm{d}z\mathrm{d}\tau.
$$

We use the fact that the norm of the integral is less than the integral of the norm, along with Hölder's inequality to obtain the following bound on the L^{r_1}-norm of the left hand side –

$$
\left\|[\Delta_q, u_{\leq p-2} \cdot \nabla]v_p\right\|_{r_1} \leq \lambda_q^3 \int_0^1 \int_{\mathbb{R}^3} |z| \left|\nabla h\left(\lambda_q(z)\right)\right| \left\|\nabla u_{\leq p-2}(\cdot - \tau z)\right\|_{r_2} \|v_p(\cdot - z)|\|_{r_3} \mathrm{d}z\mathrm{d}\tau.
$$

The desired result then follows from the above estimate and the translation invariance of the Lebesgue measure.

□

To handle the Hall term, we introduce

$$
[\Delta_q, u \times \nabla \times]v = \Delta_q(u \times (\nabla \times v)) - u \times (\nabla \times v_q),
$$

$$
[\Delta_q, (\nabla \times u) \times]v = \Delta_q((\nabla \times u) \times v) - (\nabla \times u) \times v_q.
$$

More specifically, for the terms $b_{\le p-2} \times (\nabla \times h_p)$ and $(\nabla \times b_{\le p-2}) \times h_p$, the above commutators take the forms of

$$[\Delta_q, b_{\le p-2} \times \nabla \times]h_p = \Delta_q(b_{\le p-2} \times (\nabla \times h_p)) - b_{\le p-2} \times (\nabla \times \Delta_q h_p), \tag{1.22}$$

$$[\Delta_q, (\nabla \times b_{\le p-2}) \times]h_p = \Delta_q((\nabla \times b_{\le p-2}) \times h_p) - (\nabla \times b_{\le p-2}) \times \Delta_q h_p. \tag{1.23}$$

Associated with the above commutators are the estimates in the following lemma, whose proof is omitted here due to its resemblance to that of Lemma 1.3.4.

Lemma 1.3.5. *(Dai, 2016) Let $r \in [1, \infty]$. Let the vector fields b and h vanish at infinity and $\nabla \cdot b_{\le p-2} = 0$. The following estimates hold –*

$$\|[\Delta_q, b_{\le p-2} \times \nabla \times]h_p\|_r \lesssim \|h_p\|_r \sum_{p' \le p-2} \lambda_{p'} \|b_{p'}\|_\infty,$$

$$\|[\Delta_q, (\nabla \times b_{\le p-2}) \times]h_p\|_r \lesssim \|h_p\|_r \sum_{p' \le p-2} \lambda_{p'} \|b_{p'}\|_\infty.$$

CHAPTER 2

WELL-POSEDNESS RESULTS FOR THE HALL-MHD SYSTEM

2.1 Some existing results on the well-posedness of the Hall-MHD system

We briefly review the mathematical results concerning the solvability of the Hall-MHD system. (Acheritogaray et al., 2011) proved global-in-time existence of Leray-Hopf type weak solutions on periodic domains, which was extended to case of the whole space by (Chae et al., 2014), where local-in-time existence of classical solutions in the space $(H^s)^2$ with $s > \frac{5}{2}$ was also proven. Local well-posedness in Sobolev spaces has also been established via the Littlewood-Paley approach by (Dai, 2020). (Chae and Lee, 2014) proved global well-posedness for small initial data in $\left(\dot{H}^{\frac{3}{2}}\right)^2$ as well as in $\dot{B}^{\frac{1}{2}}_{2,1} \times \dot{B}^{\frac{3}{2}}_{2,1}$, in addition to the following blow-up criteria.

Theorem 2.1.1. *Let $s > \frac{5}{2}$ be an integer and $u_0, b_0 \in H^s(\mathbb{T}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then for the first blow-up time $T^* < \infty$ of the classical solution to System 1.1 - 1.3, it holds that*

$$\limsup_{t \nearrow T^*} (\|u(t)\|^2_{H^s} + \|b(t)\|^2_{H^s}) = \infty,$$

i) if and only if

$$\int_0^{T^*} \left(\|u\|^2_{BMO} + \|\nabla b\|^2_{BMO}\right) \mathrm{d}t = \infty;$$

ii) and if and only if

$$\|u\|_{L^q(0,T^*;L^p(\mathbb{T}^3))} + \|\nabla b\|_{L^\gamma(0,T^*;L^\beta(\mathbb{T}^3))} = \infty,$$

where p, q, β *and* γ *satisfy the relation*

$$\frac{3}{p} + \frac{2}{q} \leq 1, \ \frac{3}{\beta} + \frac{2}{\gamma} \leq 1, \ \textit{with } p, \beta \in (3, \infty].$$

Local and global well-posedness results for large or small initial data can also be found in the works by (Danchin and Tan, 2019), (Kwak and Lkhagvasuren, 2018), (Benvenutti and Ferreira, 2016), (Wu et al., 2017) and (Chae et al., 2015). For a variety of regularity criteria, we refer to (Dai, 2016; Fan et al., 2015; He et al., 2016; Wan and Zhou, 2015; Ye, 2017). Partial regularity for the $2\frac{1}{2}$-dimensional Hall-MHD system were studied in (Chae and Wolf, 2016). As seen in (Chae and Wolf, 2015), in sharp contrast to the steady-state solutions to the Navier-Stokes equations and to the MHD system, solutions to the stationary 3D Hall-MHD system are only known to be partially regular, with the singular set being compact and of Hausdorff dimension no more than 1, which is alluded to by the absence of a satisfactory bound on the determining wavenumber in Section 3.3.3.

On the other hand, there are striking ill-posedness results in the irresistive setting due to (Chae and Weng, 2016) as well as (Jeong and Oh, 2019). Recently, (Dai, 2018) proved the non-uniqueness of weak solutions in the Leray-Hopf class via a convex integration scheme.

2.1.1 Global existence of Leray-Hopf type weak solutions to the Hall-MHD system

Recalling the low-frequency truncation operator S_q introduced in Section 1.3, we approximate Equation 1.1 - Equation 1.3 by the following system with initial data $u_0^N = S_N u_0$ and $b_0^N = S_N b_0$ –

$$u_t + S_N\left((S_N u\cdot\nabla)S_N u\right) - S_N\left((S_N b\cdot\nabla)S_N b\right) + S_N\nabla p = \nu S_N\Delta u, \tag{2.1}$$

$$\begin{aligned} b_t + S_N\left((S_N u\cdot\nabla)S_N b\right) - S_N\left((S_N b\cdot\nabla)S_N u\right) & \\ + d_i S_N\left(\nabla\times(\nabla\cdot(S_N b\otimes S_N b))\right) &= \mu S_N\Delta b. \end{aligned} \tag{2.2}$$

We notice that the pressure is given by $p = \sum_{1\le j,k\le 3}(-\Delta)^{-1}\partial_j\partial_k(u^j u^k - b^j b^k)$ and the system consisting of Equation 2.1 - Equation 2.2 is in fact a system of ordinary differential equations, written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ b\end{pmatrix} = \begin{pmatrix} F_N^1(u,b) \\ F_N^2(u,b)\end{pmatrix}. \tag{2.3}$$

By Picard-Lindelöf theorem, we can show that there exists a time $T_N > 0$ such that the above system has a unique maximal solution $(u^N, b^N) \in C^1\left(0, T_N; L^2_{\sigma,N}(\mathbb{R}^3)\right)$, where $L^2_{\sigma,N}(\mathbb{R}^3) :=$

$\left\{f \in L^2(\mathbb{R}^3) : \nabla \cdot f = 0,\ \operatorname{supp} \hat{f} \in B_{\lambda^N}\right\}$. This amounts to showing that F_N^1 and F_N^2 are locally Lipschitz for finite $N \in \mathbb{N}$, which straightforwardly follows from the estimates below –

$$\|S_N \Delta u\|_{L^2} \leq \lambda^{2N} \|u\|_{L^2},\ \|S_N \Delta b\|_{L^2} \leq \lambda^{2N} \|b\|_{L^2},$$

$$\|S_N\left((S_N u \cdot \nabla) S_N u\right)\|_{L^2} \leq \lambda^N \|S_N u\|_\infty \|S_N u\|_2 \leq \lambda^N \lambda^{\frac{3N}{2}} \|u\|_{L^2}^2,$$

$$\|S_N\left((S_N b \cdot \nabla) S_N b\right)\|_{L^2} \leq \lambda^{\frac{5N}{2}} \|b\|_{L^2}^2,\ \|S_N \nabla p\|_{L^2} \leq \lambda^{\frac{5N}{2}} \left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2\right)$$

$$\|S_N\left((S_N u \cdot \nabla) S_N b\right)\|_{L^2} \leq \lambda^{\frac{5N}{2}} \|u\|_{L^2} \|b\|_{L^2},\ \|S_N\left((S_N b \cdot \nabla) S_N u\right)\|_{L^2} \leq \lambda^{\frac{5N}{2}} \|u\|_{L^2} \|b\|_{L^2},$$

$$\|S_N\left(\nabla \times \left(\nabla \cdot (S_N b \otimes S_N b)\right)\right)\|_{L^2} \leq \lambda^{\frac{7N}{2}} \|b\|_{L^2}^2.$$

Moreover, the uniqueness of (u^N, b^N) implies that $(u^N, b^N) = (S_N u^N, S_N b^N)$ since $(S_N u^N, S_N b^N)$ also solves the same system.

We multiply Equation 2.1 and Equation 2.2 by u^N and b^N, respectively. The smoothness of (u^N, b^N) allows integration by parts on the two equations, which we sum up to obtain

$$\begin{aligned}\|u^N(t)\|_{L^2}^2 + \|b^N(t)\|_{L^2}^2 + 2\nu \int_0^t \left\|\nabla u^N(\tau)\right\|_{L^2}^2 \mathrm{d}\tau + 2\mu \int_0^t \left\|\nabla b^N(\tau)\right\|_{L^2}^2 \mathrm{d}\tau =& \|u_0^N\|_{L^2}^2 + \|b_0^N\|_{L^2}^2 \\ \leq& \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.\end{aligned}$$

The above bound then ensures that $T_N = +\infty$. It also implies that the sequence $\left\{(u^N, b^N)\right\}_{N \in \mathbb{N}}$ is uniformly bounded in $L^\infty\left(0, \infty; \left(L^2(\mathbb{R}^3)\right)^2\right) \cap L^2\left(0, \infty; \left(H^1(\mathbb{R}^3)\right)^2\right)$.

To proceed, we need to derive a time compactness result. For the convection terms we have, by duality and Gagliardo-Nirenberg inequality, that

$$
\begin{gathered}
\left\|S_N\left((u^N\cdot\nabla)u^N\right)\right\|_{H^{-1}} \lesssim \left\|u^N\otimes u^N\right\|_{L^2} \lesssim \|u^N\|_{L^4}^2 \lesssim \|u^N\|_{L^2}^{\frac{1}{2}}\left\|\nabla u^N\right\|_{L^2}^{\frac{3}{2}},\\
\left\|S_N\left((u^N\cdot\nabla)b^N\right)\right\|_{H^{-1}} \lesssim \left\|u^N\otimes b^N\right\|_{L^2} \lesssim \|u^N\|_{L^2}^{\frac{1}{4}}\|b^N\|_{L^2}^{\frac{1}{4}}\left\|\nabla u^N\right\|_{L^2}^{\frac{3}{4}}\left\|\nabla b^N\right\|_{L^2}^{\frac{3}{4}},\\
\left\|S_N\left((b^N\cdot\nabla)b^N\right)\right\|_{H^{-1}} \lesssim \|b^N\|_{L^2}^{\frac{1}{2}}\left\|\nabla b^N\right\|_{L^2}^{\frac{3}{2}},\\
\left\|S_N\left((b^N\cdot\nabla)u^N\right)\right\|_{H^{-1}} \lesssim \|u^N\|_{L^2}^{\frac{1}{4}}\|b^N\|_{L^2}^{\frac{1}{4}}\left\|\nabla u^N\right\|_{L^2}^{\frac{3}{4}}\left\|\nabla b^N\right\|_{L^2}^{\frac{3}{4}}.
\end{gathered}
$$

Similarly, it follows that

$$
\|\nabla p^N\|_{H^{-1}} \lesssim \|u^N\|_{L^2}^{\frac{1}{2}}\left\|\nabla u^N\right\|_{L^2}^{\frac{3}{2}} + \|b^N\|_{L^2}^{\frac{1}{2}}\left\|\nabla b^N\right\|_{L^2}^{\frac{3}{2}}.
$$

As for the Hall term, it holds that

$$
\left\|S_N\left(\nabla\times\left(\nabla\cdot(b^N\otimes b^N)\right)\right)\right\|_{H^{-2}} \lesssim \left\|b^N\otimes b^N\right\|_{L^2} \lesssim \|b^N\|_{L^4}^2 \lesssim \|b^N\|_{L^2}^{\frac{1}{2}}\left\|\nabla b^N\right\|_{L^2}^{\frac{3}{2}}.
$$

Therefore, the sequence $\left\{(u_t^N, b_t^N)\right\}_{N\in\mathbb{N}}$ is uniformly bounded in $L^{\frac{4}{3}}_{\text{loc}}\left(0,\infty; H^{-1}\times H^{-2}(\mathbb{R}^3)\right)$. For $T>0$, $\left\{(u^N, b^N)\right\}_{N\in\mathbb{N}}$ is compactly embedded in $L^2\left(0,T;\left(L^2_{\text{loc}}(\mathbb{R}^3)\right)^2\right)$ by virtue of Lions-Aubin lemma. We can thus extract a subsequence, which we relabel as (u^N, b^N), such that

1. $(u^N, b^N) \overset{*}{\rightharpoonup} (u,b)$ in $L^\infty\left(0,T;\left(L^2(\mathbb{R}^3)\right)^2\right)$,
2. $(u^N, b^N) \rightharpoonup (u,b)$ in $L^2\left(0,T;\left(H^1(\mathbb{R}^3)\right)^2\right)$,

3. $(u^N, b^N) \to (u, b)$ in $L^2\left(0, T; \left(L^2_{\text{loc}}(\mathbb{R}^3)\right)^2\right)$,

4. $(u^N_t, b^N_t) \to (u_t, b_t)$ in $L^{\frac{4}{3}}\left(0, T; H^{-1} \times H^{-2}(\mathbb{R}^3)\right)$,

for some $(u, b) \in L^2\left(0, \infty; \left(H^1(\mathbb{R}^3)\right)^2\right)$ with $(u_t, b_t) \in L^{\frac{4}{3}}_{\text{loc}}\left(0, \infty; H^{-1} \times H^{-2}(\mathbb{R}^3)\right)$.

To show that (u, b) is a weak solution to Equation 1.1 - Equation 1.3, we just need to verify the weak convergence of the nonlinear terms. Recalling that the sequence $\left\{u^N\right\}_{N\in\mathbb{N}}$ is bounded in $L^2(0, T; L^4_{\text{loc}}(\mathbb{R}^3))$ for any $T > 0$, we have, by Gagliardo-Nirenberg inequality,

$$\left\|u^N - u\right\|_{L^2(0,T;L^4_{\text{loc}}(\mathbb{R}^3))} \lesssim \left\|u^N - u\right\|^{\frac{1}{4}}_{L^2(0,T;L^2(\mathbb{R}^3))} \left\|\nabla u^N - \nabla u\right\|^{\frac{3}{4}}_{L^2(0,T;L^2(\mathbb{R}^3))},$$

from which we infer that

$$\lim_{k\to\infty} \left\|u^N \otimes u^N - u \otimes u\right\|_{L^1(0,T;L^2_{\text{loc}}(\mathbb{R}^3))} = 0.$$

It follows that

$$\int_0^T \int_{\mathbb{R}^3} u^N \otimes u^N : \nabla\varphi \,\mathrm{d}x\mathrm{d}t \longrightarrow \int_0^T \int_{\mathbb{R}^3} u \otimes u : \nabla\varphi \,\mathrm{d}x\mathrm{d}t, \;\forall \varphi \in \left(\mathscr{D}(\mathbb{R}^3)\right)^3.$$

Similarly, we have $b^N \otimes b^N \overset{*}{\rightharpoonup} b \otimes b$, $u^N \otimes b^N \overset{*}{\rightharpoonup} u \otimes b$, and $b^N \otimes u^N \overset{*}{\rightharpoonup} b \otimes u$.

As $\nabla b^N \rightharpoonup \nabla b$ in $L^2\left(0, T; L^2(\mathbb{R}^3)\right)$ and $b^N \to b$ in $L^2\left(0, T; L^2_{\text{loc}}(\mathbb{R}^3)\right)$, it is clear that

$$\int_0^T \int_{\mathbb{R}^3} \left(\nabla \cdot (b^N \otimes b^N)\right) \cdot (\nabla \times \varphi) \,\mathrm{d}x\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^3} \left(\nabla \cdot (b \otimes b)\right) \cdot (\nabla \times \varphi) \,\mathrm{d}x\mathrm{d}t, \;\forall \varphi \in \left(\mathscr{D}(\mathbb{R}^3)\right)^3.$$

We have thus reproduced a proof to the following result.

Theorem 2.1.2. *(Acheritogaray et al., 2011; Chae et al., 2014) Let* (u_0, b_0) *be a divergence vector field in* $L^2(\mathbb{R}^3)$*. Then there exists a weak solution to the Hall-MHD system,* $(u, b) \in L^\infty\left(0, \infty; \left(L^2(\mathbb{R}^3)\right)^2\right) \cap L^2\left(0, \infty; \left(H^1(\mathbb{R}^3)\right)^2\right)$*, which satisfies the energy inequality*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \,\mathrm{d}\tau + 2\mu \int_0^t \|\nabla b(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.$$

2.1.2 A low-modes regularity criterion for the Hall-MHD system

In what follows, a weak solution (u, b) to the Hall-MHD system is said to be regular on the time interval $[0, T]$ if $(u, b) \in C\left(0, T; \left(H^s(\mathbb{R}^3)\right)^2\right)$ for some $s > \frac{5}{2}$. Concerning the regularity problem, an interesting result is the condition on the integrability of certain low frequency parts of the solutions, proven in (Dai, 2016). Here we replicate the result along with its proof as it demonstrates the robustness of the application of Littlewood-Paley theory in fluid dynamics. Regularity criteria of this type have also been obtained for the Navier-Stokes and MHD equations as well as for the chemotaxis-Navier-Stokes system in (Cheskidov and Dai, 2015; Dai and Liu, 2020) using the same wavenumber splitting approach first formulated by (Cheskidov and Shvydkoy, 2014) based on Kolmogorov's 1941 theory of isotropic turbulence. A review of results in this direction is given by (Dai and Liu, To appear).

To start, we define the time-dependent dissipation wavenumbers in terms of the conditions of smallness of the dyadic blocks of each individual solution in certain spaces critical in the sense of scaling invariance. The prototypical concept of a wavenumber that separates the inertial range from the dissipation range was due to (Kolmogorov, 1941). As we shall see in Section 3.3, the notion of the dissipation wavenumber is intimately connected to that of the determining wavenumber.

Definition 2.1.3. Let (u, b) be a weak solution to the Hall-MHD system. Let $\kappa := \min\{\mu, \nu, d_i^{-1}\mu\}$. We define the dissipation wavenumbers associated with u and b as

$$\Lambda_u(t) = \min\left\{\lambda_q : \lambda_p^{-1}\|u_p\|_\infty \le c_0\kappa,\ \forall p > q\right\},$$

$$\Lambda_b(t) = \min\left\{\lambda_q : \lambda_{p-q}^{\delta}\|b_p\|_\infty \le c_0\kappa,\ \forall p > q\right\},$$

where $c_0 > 0$ is some small constant to be specified later and λ_{p-q}^{δ} with $\delta > s > 0$ represents a certain kernel. We let $Q_u(t)$ and $Q_b(t)$ denote the integers such that $\Lambda_u(t) = \lambda_{Q_u(t)}$ and $\Lambda_b(t) = \lambda_{Q_b(t)}$.

For simplicity, we denote $f(t) := \|u_{\le Q_u(t)}(t)\|_{B^1_{\infty,\infty}} + \Lambda_b(t)\|b_{\le Q_b(t)}(t)\|_{B^1_{\infty,\infty}}$. We proceed to state and prove the regularity criterion.

Theorem 2.1.4. *(Dai, 2016) Let (u, b) be a weak solution to the Hall-MHD system on $[0, T]$. Assume that $(u(t), b(t))$ is regular on $[0, T)$ and*

$$\int_0^T \left(\|u_{\le Q_u(t)}(t)\|_{B^1_{\infty,\infty}} + \Lambda_b(t)\|b_{\le Q_b(t)}(t)\|_{B^1_{\infty,\infty}}\right) \mathrm{d}t < \infty,$$

then (u, b) is a regular solution beyond time T.

Proof: Multiplying Equation 1.1 by $\lambda_q^{2s}\Delta_q^2 u$ and Equation 1.2 by $\lambda_q^{2s}\Delta_q^2 b$, respectively, integrating by parts, summing over $q \geq -1$ and adding the two equations, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\lambda_q^{2s}\left(\|u_q\|_2^2+\|b_q\|_2^2\right) \leq &-\sum_{q\geq -1}\lambda_q^{2s}\left(\nu\|\nabla u_q\|_2^2+\mu\|\nabla b_q\|_2^2\right)\\ &+I_1+I_2+I_3+I_4+I_5,\end{aligned}\tag{2.4}$$

where

$$I_1=\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla u)\cdot u_q\,\mathrm{d}x,\quad I_2=-\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b\cdot\nabla b)\cdot u_q\,\mathrm{d}x,$$
$$I_3=\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u\cdot\nabla b)\cdot b_q\,\mathrm{d}x,\quad I_4=-\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(b\cdot\nabla u)\cdot b_q\,\mathrm{d}x,$$
$$I_5=-\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q((\nabla\times b)\times b)\cdot(\nabla\times b_q)\,\mathrm{d}x.$$

To eventually obtain the regularity criterion, we shall prove a Grönwall-type inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\lambda_q^{2s}\left(\|u_q\|_2^2+\|b_q\|_2^2\right)\lesssim \max\{Q_u,Q_b\}f(t)\sum_{q\geq -1}\lambda_q^{2s}\left(\|u_q\|_2^2+\|b_q\|_2^2\right).\tag{2.5}$$

To this end, we will estimate the terms I_1, I_2,..., I_5, with the goal of showing

$$\sum_{k=1}^{5}|I_k|\lesssim \max\{Q_u,Q_b\}f(t)\sum_{q\geq -1}\lambda_q^{2s}\left(\|u_q\|_2^2+\|u_q\|_2^2\right)+c_0\kappa\sum_{q\geq -1}\lambda_q^{2s+2}\left(\|u_q\|_2^2+\|b_q\|_2^2\right),\tag{2.6}$$

where Q_u, Q_b, $f(t)$, c_0 and κ are as previously defined.

We divide I_1 into three terms using Bony's paraproduct decomposition

$$\begin{aligned} I_1 =& \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_{\leq p-2}\cdot\nabla u_p)\cdot u_q\,\mathrm{d}x \\ &+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla u_{\leq p-2})\cdot u_q\,\mathrm{d}x \\ &+\sum_{q\geq -1}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{u}_p)\cdot u_q\,\mathrm{d}x \\ =&:I_{11}+I_{12}+I_{13}. \end{aligned}$$

We then rewrite I_{11} using the commutator (1.21).

$$\begin{aligned} I_{11} =& \sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\leq p-2}\cdot\nabla]u_p\cdot u_q\,\mathrm{d}x \\ &+\sum_{q\geq -1}\lambda_q^{2s}\int_{\mathbb{R}^3}(u_{\leq q-2}\cdot\nabla u_q)\cdot u_q\,\mathrm{d}x \\ &+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\lambda_q^{2s}\int_{\mathbb{R}^3}((u_{\leq p-2}-u_{\leq q-2})\cdot\nabla\Delta_q u_p)\cdot u_q\,\mathrm{d}x \\ =&:I_{111}+I_{112}+I_{113}. \end{aligned}$$

Integrating I_{112} by parts, we notice that it vanishes due to the fact $\nabla\cdot u_{\leq q-2}=0$.

We split I_{111} into three terms by the wavenumber Q_u.

$$\begin{aligned}
I_{111} = & \sum_{1\le p\le Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\le p-2}\cdot\nabla]u_p\cdot u_q\,\mathrm{d}x \\
& + \sum_{p>Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\le Q_u}\cdot\nabla]u_p\cdot u_q\,\mathrm{d}x \\
& + \sum_{p>Q_u+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{(Q_u,p-2]}\cdot\nabla]u_p\cdot u_q\,\mathrm{d}x \\
=: & I_{111}^{\flat} + I_{111}^{\natural} + I_{111}^{\sharp}.
\end{aligned}$$

By Lemma 1.3.4 and Hölder's inequality, we have

$$\begin{aligned}
|I_{111}^{\flat}| \le & \sum_{1\le p\le Q_u+2}\|\nabla u_{\le p-2}\|_\infty\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{2s}\|u_q\|_2 \\
\lesssim & Q_u f(t)\sum_{-1\le p\le Q_u+2}\lambda_p^s\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^s\|u_q\|_2 \\
\lesssim & Q_u f(t)\sum_{q\ge -1}\lambda_q^{2s}\|u_q\|_2^2.
\end{aligned}$$

Similarly, $I_{111}^{\natural}$ enjoys the following estimate.

$$\begin{aligned}
|I_{111}^{\natural}| \le & \sum_{p>Q_u+2}\|\nabla u_{\le p-2}\|_\infty\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^{2s}\|u_q\|_2 \\
\lesssim & Q_u f(t)\sum_{p>Q_u+2}\lambda_p^s\|u_p\|_2\sum_{|q-p|\le 2}\lambda_q^s\|u_q\|_2 \\
\lesssim & Q_u f(t)\sum_{q\ge -1}\lambda_q^{2s}\|u_q\|_2^2.
\end{aligned}$$

We estimate $I_{111}^{\sharp}$ using Lemma 1.3.4 and Hölder's inequality.

$$\begin{aligned}
|I_{111}^{\sharp}| \leq & \sum_{p>Q_u+2} \|u_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{2s} \|u_q\|_2 \sum_{Q_u<p'\leq p-2} \lambda_{p'} \|u_{p'}\|_\infty \\
\lesssim & c_0 \kappa \sum_{p>Q_u+2} \|u_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{2s} \|u_q\|_2 \sum_{Q_u<p'\leq p-2} \lambda_{p'}^2 \\
\lesssim & c_0 \kappa \sum_{p>Q_u+2} \lambda_p^{s+1} \|u_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{s+1} \|u_q\|_2 \sum_{Q_u<p'\leq p-2} \lambda_{p'-p}^2 \\
\lesssim & c_0 \kappa \sum_{q\geq -1} \lambda_q^{2s+2} \|u_q\|_2^2.
\end{aligned}$$

For I_{113}, we have the following.

$$\begin{aligned}
|I_{113}| \lesssim & \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |u_{\leq p-2} - u_{\leq q-2}| \, |\nabla \Delta_q u_p| \, |u_q| \, \mathrm{d}x \\
& + \sum_{q>Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |u_{\leq p-2} - u_{\leq q-2}| \, |\nabla \Delta_q u_p| \, |u_q| \, \mathrm{d}x \\
=: & I_{113}^{\flat} + I_{113}^{\sharp}.
\end{aligned}$$

For the low frequency part $I_{113}^{\flat}$, we have

$$\begin{aligned}
I_{113}^{\flat} \lesssim & \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s+1} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \|u_q\|_\infty \\
\lesssim & Q_u f(t) \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|u_p\|_2 \\
\lesssim & Q_u f(t) \sum_{-1\leq q\leq Q_u} \lambda_q^{2s} \|u_q\|_2^2.
\end{aligned}$$

The high frequency part $I_{113}^{\sharp}$ can be estimated as follows.

$$
\begin{aligned}
I_{113}^{\sharp} \lesssim & \sum_{q>Q_u}\sum_{|p-q|\leq 2}\lambda_q^{2s+1}\|u_{\leq p-2}-u_{\leq q-2}\|_2\|u_p\|_2\|u_q\|_\infty \\
\lesssim & c_0\kappa\sum_{q>Q_u}\sum_{|p-q|\leq 2}\lambda_q^{2s+2}\|u_{\leq p-2}-u_{\leq q-2}\|_2\|u_p\|_2 \\
\lesssim & c_0\kappa\sum_{q>Q_u}\sum_{|p-q|\leq 2}\lambda_q^{2s+2}\left(\|u_{p-3}\|_2+\|u_{p-2}\|_2+\|u_{p-1}\|_2+\|u_p\|_2\right)\|u_p\|_2 \\
\lesssim & c_0\kappa\sum_{q\geq -1}\lambda_q^{2s+2}\|u_q\|_2^2.
\end{aligned}
$$

We omit the estimate for I_{12} as it is identical to that for I_{111}.

I_{13} is split into two terms

$$
\begin{aligned}
I_{13} = & \sum_{-1\leq q\leq Q_u}\sum_{|p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{u}_p)\cdot u_q\,\mathrm{d}x \\
& +\sum_{q>Q_u}\sum_{p\geq q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{u}_p)\cdot u_q\,\mathrm{d}x \\
=: & I_{13}^{\flat}+I_{13}^{\sharp}.
\end{aligned}
$$

$I_{13}^{\flat}$ can be estimated as follows.

$$
\begin{aligned}
|I_{13}^{\flat}| \leq & \sum_{-1\leq q\leq Q_u}\lambda_q^{2s}\|u_q\|_\infty\sum_{p\geq q-2}\|u_p\|_2\|\nabla\tilde{u}_p\|_2 \\
\lesssim & Q_u f(t)\sum_{-1\leq q\leq Q_u}\sum_{p\geq q-2}\lambda_p^{2s}\|u_p\|_2^2\lambda_{q-p}^{2s-1} \\
\lesssim & Q_u f(t)\sum_{q\geq -1}\lambda_q^{2s}\|u_q\|_2^2.
\end{aligned}
$$

For $I_{13}^{\sharp}$, we have.

$$
\begin{aligned}
|I_{13}^{\sharp}| \leq & \sum_{q>Q_u} \lambda_q^{2s} \|u_q\|_\infty \sum_{p\geq q-2} \|u_p\|_2 \|\nabla \tilde{u}_p\|_2 \\
\lesssim & c_0 \kappa \sum_{q>Q_u} \sum_{p\geq q-2} \lambda_p^{2s+2} \|u_p\|_2^2 \lambda_{q-p}^{2s+1} \\
\lesssim & c_0 \kappa \sum_{q\geq -1} \lambda_q^{2s+2} \|u_q\|_2^2.
\end{aligned}
$$

By Bony's paraproduct decomposition, we have

$$
\begin{aligned}
I_2 = & -\sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{\leq p-2} \cdot \nabla b_p) \cdot u_q \, \mathrm{d}x \\
& - \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla b_{\leq p-2}) \cdot u_q \, \mathrm{d}x \\
& - \sum_{q\geq -1} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla \tilde{b}_p) \cdot u_q \, \mathrm{d}x \\
=: & I_{21} + I_{22} + I_{23}.
\end{aligned}
$$

To estimate I_{21}, we rewrite it using the commutator (1.21)

$$
\begin{aligned}
I_{21} = & -\sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\leq p-2} \cdot \nabla] b_p \cdot u_q \, \mathrm{d}x \\
& - \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \Delta_q \nabla b_p) \cdot u_q \, \mathrm{d}x \\
& - \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b_{\leq p-2} - b_{\leq q-2}) \cdot \Delta_q \nabla b_p) \cdot u_q \, \mathrm{d}x \\
=: & I_{211} + I_{212} + I_{213}.
\end{aligned}
$$

We postpone the estimate for I_{212} as it cancels a part of I_4.

We can further split I_{211} into three terms using the wavenumber Q_b.

$$\begin{aligned}
I_{211} =& -\sum_{1\le p\le Q_b+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{\le p-2}\cdot\nabla]b_p\cdot u_q\,\mathrm{d}x\\
&-\sum_{p>Q_b+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{\le Q_b}\cdot\nabla]b_p\cdot u_q\,\mathrm{d}x\\
&-\sum_{p>Q_b+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, b_{(Q_b,p-2]}\cdot\nabla]b_p\cdot u_q\,\mathrm{d}x\\
=:& I_{211}^{\flat}+I_{211}^{\natural}+I_{211}^{\sharp}.
\end{aligned}$$

By Lemma 1.3.4 and Hölder's inequality, we can estimate $I_{211}^{\flat}$ and $I_{211}^{\natural}$.

$$\begin{aligned}
|I_{211}^{\flat}| \le& \sum_{1\le p\le Q_b+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\|\nabla b_{\le p-2}\|_\infty\|b_p\|_2\|u_q\|_2\\
\lesssim& Q_b f(t)\sum_{1\le p\le Q_b+2}\lambda_p^{s}\|b_p\|_2\sum_{|q-p|\le 2}\lambda_q^{s}\|u_q\|_2\\
\lesssim& Q_b f(t)\sum_{q\ge -1}\lambda_q^{2s}\left(\|b_q\|_2^2+\|u_q\|_2^2\right);
\end{aligned}$$

$$\begin{aligned}
|I_{211}^{\natural}| \le& \sum_{p>Q_b+2}\sum_{|q-p|\le 2}\lambda_q^{2s}\|\nabla b_{\le Q_b}\|_\infty\|b_p\|_2\|u_q\|_2\\
\lesssim& Q_b f(t)\sum_{p>Q_b+2}\lambda_p^{s}\|b_p\|_2\sum_{|q-p|\le 2}\lambda_q^{s}\|u_q\|_2\\
\lesssim& Q_b f(t)\sum_{q\ge -1}\lambda_q^{2s}\left(\|b_q\|_2^2+\|u_q\|_2^2\right).
\end{aligned}$$

$I_{211}^{\sharp}$ is estimated with the help of Hölder's, Young's and Jensen's inequalities.

$$
\begin{aligned}
|I_{211}^{\sharp}| \leq & \sum_{p>Q_b+2} \sum_{|q-p|\leq 2} \lambda_q^{2s} \|\nabla b_{(Q_b,p-2]}\|_\infty \|b_p\|_2 \|u_q\|_2 \\
\lesssim & \sum_{p>Q_b+2} \lambda_p^s \|b_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^s \|u_q\|_2 \sum_{Q_b<p'\leq p-2} \lambda_{p'} \|b_{p'}\|_\infty \\
\lesssim & c_0\kappa \sum_{p>Q_b+2} \lambda_p^s \|b_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^s \|u_q\|_2 \sum_{Q_b<p'\leq p-2} \lambda_{p'} \lambda_{Q_b-p'}^{\delta} \\
\lesssim & c_0\kappa \sum_{p>Q_b+2} \lambda_p^{s+1} \|b_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{s+1} \|u_q\|_2 \sum_{Q_b<p'\leq p-2} \lambda_{p'-p} \lambda_{Q_b-p'}^{\delta} \lambda_p^{-1} \\
\lesssim & c_0\kappa \sum_{q\geq -1} \lambda_q^{2s+2} \left(\|b_q\|_2^2 + \|u_q\|_2^2\right).
\end{aligned}
$$

For I_{213}, we have

$$
\begin{aligned}
|I_{213}| \leq & \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(b_{\leq p-2} - b_{\leq q-2})| \, |\Delta_q \nabla b_p| |u_q| \, \mathrm{d}x \\
& + \sum_{q>Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |(b_{\leq p-2} - b_{\leq q-2})| \, |\Delta_q \nabla b_p| |u_q|, \mathrm{d}x \\
=: & I_{213}^{\flat} + I_{213}^{\sharp}.
\end{aligned}
$$

For the low frequency part $I_{213}^{\flat}$, it holds that

$$
\begin{aligned}
I_{213}^{\flat} \leq & \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s+1} \|b_{\leq p-2} - b_{\leq q-2}\|_2 \| \|b_p\|_2 \|u_q\|_\infty \\
\lesssim & Q_u f(t) \sum_{-1\leq q\leq Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \|b_{\leq p-2} - b_{\leq q-2}\|_2 \| \|b_p\|_2 \\
\lesssim & Q_u f(t) \sum_{q\geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{aligned}
$$

For the high frequency part $I_{213}^{\sharp}$, we have

$$
\begin{aligned}
I_{213}^{\sharp} \leq & \sum_{q>Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s} \|b_{\leq p-2} - b_{\leq q-2}\|_2 \|\nabla b_p\|_2 \|u_q\|_\infty \\
\lesssim & c_0 \kappa \sum_{q>Q_u} \sum_{|p-q|\leq 2} \lambda_q^{2s+1} \|b_{\leq p-2} - b_{\leq q-2}\|_2 \|\nabla b_p\|_2 \\
\lesssim & c_0 \kappa \sum_{q\geq -1} \lambda_q^{2s+2} \|b_q\|_2^2.
\end{aligned}
$$

We omit the estimate for I_{22}, as it is in the same vein as I_{211}.

Splitting I_{23} by the wavenumber Q_u, we have

$$
\begin{aligned}
I_{23} = & \sum_{-1\leq q\leq Q_u} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla \tilde{b}_p) \cdot u_q \, \mathrm{d}x \\
& + \sum_{q>Q_u} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla \tilde{b}_p) \cdot u_q \, \mathrm{d}x \\
=: & I_{23}^{\flat} + I_{23}^{\sharp}.
\end{aligned}
$$

The estimate for $I_{23}^{\flat}$ is as follows.

$$
\begin{aligned}
|I_{23}^{\flat}| \leq & \sum_{-1\leq q\leq Q_u} \sum_{p\geq q-2} \lambda_q^{2s} \|b_p\|_2 \|\nabla \tilde{b}_p\|_2 \|u_q\|_\infty \\
\lesssim & Q_u f(t) \sum_{-1\leq q\leq Q_u} \sum_{p\geq q-2} \lambda_q^{2s-1} \|b_p\|_2 \|\nabla \tilde{b}_p\|_2 \\
\lesssim & Q_u f(t) \sum_{q\geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{aligned}
$$

$I_{23}^\sharp$ is estimated as follows.

$$
\begin{aligned}
|I_{23}^\sharp| \le & \sum_{q>Q_u}\sum_{p\ge q-2}\lambda_q^{2s}\|b_p\|_2\|\nabla\tilde b_p\|_2\|u_q\|_\infty\\
\lesssim & c_0\kappa\sum_{q>Q_u}\sum_{p\ge q-2}\lambda_q^{2s+1}\|b_p\|_2\|\nabla\tilde b_p\|_2\\
\lesssim & c_0\kappa\sum_{q>Q_u}\sum_{p\ge q-2}\lambda_p^{2s+2}\|b_p\|_2^2\lambda_{q-p}^{2s+1}\\
\lesssim & c_0\kappa\sum_{q\ge -1}\lambda_q^{2s+2}\|b_q\|_2^2.
\end{aligned}
$$

We have the following decompositions, similar to the case of I_1.

$$
\begin{aligned}
I_3 = & \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_{\le p-2}\cdot\nabla b_p)\cdot b_q\,\mathrm{d}x\\
& + \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla b_{\le p-2})\cdot b_q\,\mathrm{d}x\\
& + \sum_{q\ge -1}\sum_{p\ge q-2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde b_p)\cdot b_q\,\mathrm{d}x\\
=: & I_{31}+I_{32}+I_{33},
\end{aligned}
$$

$$
\begin{aligned}
I_{31} = & \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}[\Delta_q, u_{\le p-2}\cdot\nabla]b_p\cdot b_q\,\mathrm{d}x\\
& + \sum_{q\ge -1}\lambda_q^{2s}\int_{\mathbb{R}^3}(u_{\le q-2}\cdot\nabla b_q)\cdot b_q\,\mathrm{d}x\\
& + \sum_{q\ge -1}\sum_{|p-q|\le 2}\lambda_q^{2s}\int_{\mathbb{R}^3}((u_{\le p-2}-u_{\le q-2})\cdot\nabla\Delta_q b_p)\cdot b_q\,\mathrm{d}x\\
=: & I_{311}+I_{312}+I_{313}.
\end{aligned}
$$

Using the wavenumber Q_u, we can estimate I_{311} in the same way as I_{111}, whereas we know that I_{312} vanishes by the divergence-free condition. For I_{313}, we can explicitly calculate $(u_{\leq p-2} - u_{\leq q-2})$ for $|p-q| \leq 2$ and obtain

$$\begin{aligned}
I_{313} \leq &\sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |u_{p-3} + u_{p-2} + u_{p-1} + u_p| \, |\nabla \Delta_q b_p||b_q| \,\mathrm{d}x \\
\lesssim &\sum_{1\leq p\leq Q_u+2} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |u_{p-2}||\nabla \Delta_q b_p||b_q| \,\mathrm{d}x \\
&+ \sum_{p>Q_u+2} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} |u_{p-2}||\nabla \Delta_q b_p||b_q| \,\mathrm{d}x \\
=: &I_{313}^{\flat} + I_{313}^{\sharp}.
\end{aligned}$$

We estimate $I_{313}^{\flat}$ as

$$\begin{aligned}
I_{313}^{\flat} \leq &\sum_{-1\leq p\leq Q_u+2} \sum_{|q-p|\leq 2} \lambda_q^{2s+1} \|u_{p-2}\|_\infty \|b_p\|_2 \|b_q\|_2 \\
\lesssim &Q_u f(t) \sum_{-1\leq p\leq Q_u+2} \sum_{|q-p|\leq 2} \lambda_q^{2s} \|b_p\|_2 \|b_q\|_2 \\
\lesssim &Q_u f(t) \sum_{q\geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{aligned}$$

For $I^\sharp_{313}$, we have

$$
\begin{aligned}
I^\sharp_{313} \leq & \sum_{p>Q_u+2}\sum_{|q-p|\leq 2}\lambda_q^{2s+1}\|u_{p-2}\|_\infty\|b_p\|_2\|b_q\|_2\\
\lesssim & c_0\kappa\sum_{p>Q_u+2}\lambda_p\sum_{|q-p|\leq 2}\lambda_q^{2s+1}\|b_p\|_2\|b_q\|_2\\
\lesssim & c_0\kappa\sum_{q\geq -1}\lambda_q^{2s+2}\|b_q\|_2^2.
\end{aligned}
$$

We can estimate I_{32} in a similar manner as I_{211} and I_{22}, so we do not include the details here for the sake of conciseness.

We can split I_{33} using the wavenumber Q_u.

$$
\begin{aligned}
I_{33} = & \sum_{-1\leq p\leq Q_u}\sum_{q\leq p+2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{b}_p)\cdot b_q\,\mathrm{d}x\\
& + \sum_{p>Q_u}\sum_{q\leq p+2}\lambda_q^{2s}\int_{\mathbb{R}^3}\Delta_q(u_p\cdot\nabla\tilde{b}_p)\cdot b_q\,\mathrm{d}x\\
=: & I_{331}+I_{332}.
\end{aligned}
$$

By Hölder's, Young's and Jensen's inequalities, we have

$$
\begin{aligned}
|I_{331}| \leq & \sum_{-1\leq p\leq Q_u}\sum_{q\leq p+2}\lambda_q^{2s}\|u_p\|_\infty\|\nabla\tilde{b}_p\|_2\|b_q\|_2\\
\lesssim & Q_u f(t)\sum_{-1\leq p\leq Q_u}\sum_{q\leq p+2}\lambda_q^{2s-1}\lambda_p\|\tilde{b}_p\|_2\|b_q\|_2\\
\lesssim & Q_u f(t)\sum_{-1\leq p\leq Q_u}\lambda_p^s\|\tilde{b}_p\|_2\sum_{q\leq p+2}\lambda_q^s\|_2\|b_q\|_2\lambda_{q-p}^{s-1}\\
\lesssim & Q_u f(t)\sum_{q\geq -1}\lambda_q^{2s}\|b_q\|_2^2.
\end{aligned}
$$

To estimate I_{332}, we use Hölder's, Young's and Jensen's inequalities.

$$
\begin{aligned}
|I_{332}| \leq & \sum_{p>Q_u} \sum_{q\leq p+2} \lambda_q^{2s} \|u_p\|_\infty \|\nabla \tilde{b}_p\|_2 \|b_q\|_2 \\
\lesssim & c_0\kappa \sum_{p>Q_u} \sum_{q\leq p+2} \lambda_q^{2s+1} \lambda_p \|\tilde{b}_p\|_2 \|b_q\|_2 \\
\lesssim & c_0\kappa \sum_{p>Q_u} \lambda_p^{s+1} \|\tilde{b}_p\|_2 \sum_{q\leq p+2} \lambda_q^{s+1} \|_2 \|b_q\|_2 \lambda_{q-p}^{s} \\
\lesssim & c_0\kappa \sum_{q\geq -1} \lambda_q^{2s+2} \|b_q\|_2^2.
\end{aligned}
$$

By Bony's paraproduct decomposition, we have

$$
\begin{aligned}
I_4 = & -\sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_{\leq p-2} \cdot \nabla u_p) \cdot b_q \, \mathrm{d}x \\
& -\sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla u_{\leq p-2}) \cdot b_q \, \mathrm{d}x \\
& -\sum_{q\geq -1} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q(b_p \cdot \nabla \tilde{u}_p) \cdot b_q \, \mathrm{d}x \\
=: & I_{41} + I_{42} + I_{43}
\end{aligned}
$$

We rewrite I_{41} as

$$\begin{aligned}
I_{41} =& - \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\leq p-2} \cdot \nabla] u_p \cdot b_q \,\mathrm{d}x \\
&- \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \Delta_q \nabla u_p) \cdot b_q \,\mathrm{d}x \\
&- \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} ((b_{\leq p-2} - b_{\leq q-2}) \cdot \Delta_q \nabla u_p) \cdot b_q \,\mathrm{d}x \\
=:& I_{411} + I_{412} + I_{413}.
\end{aligned}$$

We omit the estimates for I_{411}, I_{42} and I_{43} as they are analogous to those for I_{211}, I_{311} and I_{23}, respectively. We can also see that I_{413} can be estimated the same way as I_{213} upto an integration by parts.

As previously noted, I_{212} and I_{412} cancel each other in the following manner.

$$\begin{aligned}
I_{212} + I_{412} =& - \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \Delta_q \nabla u_p) \cdot (b_q + u_q) \,\mathrm{d}x \\
&- \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \Delta_q \nabla b_p) \cdot (u_q + b_q) \,\mathrm{d}x \\
=& - \sum_{q\geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} (b_{\leq q-2} \cdot \nabla (u_q + b_q)) \cdot (u_q + b_q) \,\mathrm{d}x = 0.
\end{aligned}$$

To estimate I_5, which results from the Hall term, we apply Bony's paraproduct decomposition.

$$
\begin{aligned}
I_5 =&d_i \sum_{q\geq -1}\sum_{|p-q|\leq 2} \lambda_q^{2s}\int_{\mathbb{R}^3} \Delta_q\left(b_{\leq p-2}\times(\nabla\times b_p)\right)\cdot(\nabla\times b_q)\,\mathrm{d}x\\
&+ d_i \sum_{q\geq -1}\sum_{|p-q|\leq 2} \lambda_q^{2s}\int_{\mathbb{R}^3} \Delta_q(b_p\times(\nabla\times b_{\leq p-2}))\cdot(\nabla\times b_q)\,\mathrm{d}x\\
&+ d_i \sum_{q\geq -1}\sum_{p\geq q-2} \lambda_q^{2s}\int_{\mathbb{R}^3} \Delta_q(b_p\times(\nabla\times \tilde{b}_p))\cdot(\nabla\times b_q)\,\mathrm{d}x\\
=:&I_{51}+I_{52}+I_{53}.
\end{aligned}
$$

Using the commutator (1.22), we can rewrite I_{51} as

$$
\begin{aligned}
I_{51} =&d_i \sum_{q\geq -1}\sum_{|p-q|\leq 2} \lambda_q^{2s}\int_{\mathbb{R}^3} [\Delta_q, b_{\leq p-2}\times\nabla\times]b_p))\cdot(\nabla\times b_q)\,\mathrm{d}x\\
&+ d_i \sum_{q\geq -1} \lambda_q^{2s}\int_{\mathbb{R}^3} (b_{\leq q-2}\times(\nabla\times b_q))\cdot(\nabla\times b_q)\,\mathrm{d}x\\
&+ d_i \sum_{q\geq -1}\sum_{|p-q|\leq 2} \lambda_q^{2s}\int_{\mathbb{R}^3} ((b_{\leq q-2}-b_{\leq p-2})\times(\nabla\times \Delta_q b_p))\cdot(\nabla\times b_q)\,\mathrm{d}x\\
=:&I_{511}+I_{512}+I_{513}.
\end{aligned}
$$

By the basic algebraic property of the cross product, $I_{512}\equiv 0$.

We further partition I_{511} by the wavenumber Q_b.

$$
\begin{aligned}
I_{51} =& d_i \sum_{1\le p\le Q_b+2} \sum_{|q-p|\le 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\le p-2} \times \nabla \times] b_p)) \cdot (\nabla \times b_q) \, \mathrm{d}x \\
&+ d_i \sum_{p>Q_b+2} \sum_{|q-p|\le 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{\le Q_b} \times \nabla \times] b_p)) \cdot (\nabla \times b_q) \, \mathrm{d}x \\
&+ d_i \sum_{p>Q_b+2} \sum_{|q-p|\le 2} \lambda_q^{2s} \int_{\mathbb{R}^3} [\Delta_q, b_{(Q_b,p-2]} \times \nabla \times] b_p)) \cdot (\nabla \times b_q) \, \mathrm{d}x \\
=:& I_{511}^{\flat} + I_{511}^{\natural} + I_{511}^{\sharp}.
\end{aligned}
$$

By Lemma 1.3.5, we have

$$
\begin{aligned}
|I_{511}^{\flat}| \le& d_i \sum_{1\le p\le Q_b+2} \|\nabla b_{\le p-2}\|_\infty \|b_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{2s+1} \|b_q\|_2 \\
\lesssim& \Lambda_b \|\nabla b_{\le Q_b}\|_\infty \sum_{1\le p\le Q_b+2} \lambda_p^{s} \|b_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{s} \|b_q\|_2 \\
\lesssim& Q_b f(t) \sum_{q\ge -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{aligned}
$$

To estimate $I_{511}^{\natural}$, we recall Lemma 1.3.5,

$$
\begin{aligned}
|I_{511}^{\natural}| \le& d_i \sum_{p>Q_b+2} \lambda_q^{2s+1} \|\nabla b_{\le Q_b}\|_2 \|b_p\|_\infty \sum_{|q-p|\le 2} \|b_q\|_2 \\
\lesssim& d_i \sum_{q>Q_b} \lambda_q^{2s+1} \|b_q\|_2 \|b_q\|_\infty \sum_{-1\le p'\le Q_b} \|\nabla b_{p'}\|_2 \\
\lesssim& c_0 \kappa \sum_{q>Q_b} \lambda_q^{2s+1} \|b_q\|_2 \lambda_{Q_b-q}^{\delta} \sum_{-1\le p'\le Q_b} \|\nabla b_{p'}\|_2 \\
\lesssim& c_0 \kappa \sum_{q>Q_b} \lambda_q^{s+1} \|b_q\|_2 \sum_{-1\le p'\le Q_b} \lambda_{p'}^{1+s} \|b_{p'}\|_2 \lambda_q^{s-\delta} \Lambda_b^{\delta} \lambda_{p'}^{-s}.
\end{aligned}
$$

By Young's inequality and Jensen's inequality,

$$
\begin{aligned}
|I_{511}^{\natural}| \lesssim & c_0\kappa \sum_{q>Q_b} \lambda_q^{2s+2}\|b_q\|_2^2 + c_0\kappa \sum_{q>Q_b}\left(\sum_{-1\le p'\le Q_b} \lambda_{p'}^{1+s}\|b_{p'}\|_2 \lambda_q^{s-\delta}\Lambda_b^{\delta}\lambda_{p'}^{-s}\right)^2 \\
\lesssim & c_0\kappa \sum_{q\ge -1} \lambda_q^{2s+2}\|b_q\|_2^2.
\end{aligned}
$$

We estimate $I_{511}^{\sharp}$ as follows.

$$
\begin{aligned}
|I_{511}^{\sharp}| \le & d_i \sum_{p>Q_b+2} \|\nabla b_{(Q_b,p-2]}\|_\infty \|b_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{2s}\|\nabla b_q\|_2 \\
\lesssim & d_i \sum_{p>Q_b+2} \|b_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{2s+1}\|b_q\|_2 \sum_{Q_b<p'\le p-2} \|\nabla b_{p'}\|_\infty \\
\lesssim & c_0\kappa \sum_{p>Q_b+2} \lambda_p^{s+1}\|b_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{s+1}\|b_q\|_2 \sum_{Q_b<p'\le p-2} \lambda_{p'-p}\lambda_{Q_b-p'}^{\delta} \\
\lesssim & c_0\kappa \sum_{q\ge -1} \lambda_q^{2s+2}\|b_q\|_2^2.
\end{aligned}
$$

As for I_{52}, we observe that it enjoys the same estimate as I_{511}. Therefore, we omit the detailed estimation.

We divide I_{53} into two parts using the wavenumber Q_b.

$$
\begin{aligned}
I_{53} \lesssim & \sum_{-1\le p\le Q_b+1} \sum_{q\le p+2} \lambda_q^{2s} \int_{\mathbb{R}^3} \left| \Delta_q(b_p \times (\nabla \times \tilde{b}_p)) \cdot (\nabla \times b_q)\right| \,\mathrm{d}x \\
& + \sum_{p>Q_b-1} \sum_{q\le p+2} \lambda_q^{2s} \int_{\mathbb{R}^3} \left| \Delta_q(b_p \times (\nabla \times \tilde{b}_p)) \cdot (\nabla \times b_q)\right| \,\mathrm{d}x \\
=: & I_{531} + I_{532}.
\end{aligned}
$$

The estimate for I_{531} follows from the definition of $f(t)$ and Hölder's inequality.

$$
\begin{aligned}
I_{531} \leq & \sum_{-1 \leq p \leq Q_b} \sum_{q \leq p+2} \lambda_q^{2s} \|\tilde{b}_p\|_2 \|\nabla b_p\|_\infty \|\nabla b_q\|_2 \\
\lesssim & Q_b f(t) \sum_{-1 \leq p \leq Q_b} \sum_{q \leq p+2} \lambda_q^{2s} \|b_p\|_2 \|b_q\|_2 \lambda_{q-Q_b} \\
\lesssim & Q_b f(t) \sum_{-1 \leq p \leq Q_b} \sum_{q \leq p+2} \lambda_p^{s} \|b_p\|_2 \lambda_q^{s} \|b_q\|_2 \lambda_{q-p}^{s} \lambda_{q-Q_b} \\
\lesssim & Q_b f(t) \sum_{q \geq -1} \lambda_q^{2s} \|b_q\|_2^2.
\end{aligned}
$$

The estimate for I_{532} is as follows.

$$
\begin{aligned}
I_{532} \leq & \sum_{p > Q_b} \sum_{q \leq p+2} \lambda_q^{2s} \|\tilde{b}_p\|_2 \|\nabla b_p\|_\infty \|\nabla b_q\|_2 \\
\lesssim & c_0 \kappa \sum_{p > Q_b} \sum_{q \leq p+2} \Lambda_b \lambda_p \|b_p\|_2 \lambda_q^{2s} \|b_q\|_2 \\
\lesssim & c_0 \kappa \sum_{p > Q_b} \sum_{q \leq p+2} \lambda_p^{s+1} \|b_p\|_2 \lambda_q^{s+1} \|b_q\|_2 \lambda_{q-p}^{s-1} \lambda_{Q_b - p} \\
\lesssim & c_0 \kappa \sum_{q \geq -1} \lambda_q^{2s+2} \|b_q\|_2^2.
\end{aligned}
$$

Summarizing all the estimates above, we obtain the desired inequality (2.6). We can choose a sufficiently small c_0, so that Inequality 2.5 holds.

By Definition 2.1.3 and Lemma 1.3.1, we have

$$
\Lambda_u \leq (c_0 \kappa)^{-1} \|u_{Q_u}\|_\infty \lesssim \Lambda_u^{\frac{3}{2} - s} \lambda_{Q_u}^{s} \|u_{Q_u}\|_2,
$$

which indicates that

$$\Lambda_u^{s-\frac{1}{2}} \lesssim \|u\|_{H^s}.$$

Similarly, we can deduce that

$$\Lambda_b^{s-\frac{3}{2}} \lesssim \|b\|_{H^s}.$$

By the fact that $s > \frac{3}{2}$ as well as the definitions of Q_u and Q_b, it holds that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right) \le C(\mu, \nu, d_i, s) f(t) \left(1 + \log\left(\|u\|_{H^s} + \|b\|_{H^s}\right)\right)\left(\|u\|_{H^s}^2 + \|b\|_{H^s}^2\right).$$

Hence, we can conclude that if $f(t) \in L^1([0,T])$, then (u, b) is bounded in $H^s \times H^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$ beyond time T.

□

We also note that via a similar analysis as above, it can be shown that the Hall-MHD system is locally well-posed in $H^{s_1} \times H^{s_2+1}(\mathbb{R}^3)$ with $s_1 > s_2 > \frac{1}{2}$. We refer readers to (Dai, 2020) for the result and its proof.

2.2 Well-posedness results for a class of generalized Hall-MHD system in Besov spaces

Exploiting the regularizing effect of the dissipation terms $-(-\Delta)^\alpha$ and $-(-\Delta)^\beta$, we can overcome the seemingly singular Hall term and establish local well-posedness for a class of generalized Hall-MHD system 1.6 - 1.8 in the Besov space $\dot{B}_{\infty,\infty}^{-(2\alpha-\gamma_1)} \times \dot{B}_{\infty,\infty}^{-(2\beta-\gamma_2)}(\mathbb{R}^3)$ for suitable choices of indices α, β, γ_1 and γ_2.

Theorem 2.2.1 (Local well-posedness)**.** *For* $(u_0, b_0) \in \dot{B}^{-(2\alpha-\gamma_1)}_{\infty,\infty} \times \dot{B}^{-(2\beta-\gamma_2)}_{\infty,\infty}(\mathbb{R}^3)$, *there exists a unique local-in-time solution* (u, b) *to system 1.6 - 1.8 such that*

$$(u, b) \in L^\infty\big(0, T; \dot{B}^{-(2\alpha-\gamma_1)}_{\infty,\infty} \times \dot{B}^{-(2\beta-\gamma_2)}_{\infty,\infty}(\mathbb{R}^3)\big)$$

with $T = T\big(\nu, \mu, d_i, \|u_0\|_{\dot{B}^{-(2\alpha-\gamma_1)}_{\infty,\infty}}, \|b_0\|_{\dot{B}^{-(2\beta-\gamma_2)}_{\infty,\infty}}\big)$, *provided that the parameters* α, β, γ_1 *and* γ_2 *satisfy the following constraints*

$$\begin{cases} \gamma_1 \geq \max\{1, \frac{\alpha}{\beta}\}, \\ \gamma_2 \geq \max\{2, \frac{(\gamma_1+1)\beta}{2\alpha}\}, \\ \frac{\gamma_1}{2} < \alpha < \gamma_1, \\ \frac{\gamma_1}{2} < \beta < \gamma_2. \end{cases} \tag{2.7}$$

We proceed to prepare for the proof of Theorem 2.2.1. Viewing the convection terms and the Hall term as perturbations to the generalized heat equations, we introduce the notion of the mild solutions.

Definition 2.2.2 (Mild solutions)**.** A mild solution to System 1.6 - 1.8 is the fix point of the map

$$S(u, b) := \begin{pmatrix} S_1(u, b) \\ S_2(u, b) \end{pmatrix}, \tag{2.8}$$

where $S_1(u,b)$ and $S_2(u,b)$ are given by the following Duhamel's formulae -

$$\begin{aligned} S_1(u,b) :=& e^{-\nu t(-\Delta)^{\alpha}} u_0(x) - \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha}} \mathbb{P}\nabla \cdot (u \otimes u)(s) \mathrm{d}s \\ &+ \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha}} \mathbb{P}\nabla \cdot (b \otimes b)(s) \mathrm{d}s, \end{aligned} \tag{2.9}$$

$$\begin{aligned} S_2(u,b) :=& e^{-\mu t(-\Delta)^{\beta}} b_0(x) - \int_0^t e^{-\mu(t-s)(-\Delta)^{\alpha_2}} \mathbb{P}\nabla \cdot (u \otimes b)(s) \mathrm{d}s \\ &+ \int_0^t e^{-\mu(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla \cdot (b \otimes u)(s) \mathrm{d}s \\ &- d_i \int_0^t e^{-\mu(t-s)(-\Delta)^{\beta}} \nabla \times (\nabla \cdot (b \otimes b))(s) \mathrm{d}s. \end{aligned} \tag{2.10}$$

In (2.10), we have applied the vector identity $\nabla \times (\nabla \cdot (b \otimes b)) = \nabla \times ((\nabla \times b) \times b)$ to the Hall term.

To further simplify notations, we view the integrals in expressions (2.9) and (2.10) as bilinear forms.

Definition 2.2.3 (Bilinear forms)**.** Let $f, g \in \mathcal{S}'$. The bilinear forms $\mathcal{B}_\alpha(\cdot,\cdot)$, $\mathcal{B}_\beta(\cdot,\cdot)$ and $\mathfrak{B}_\beta(\cdot,\cdot)$ are defined as follows.

$$\begin{aligned} \mathcal{B}_\alpha(f,g) =& \int_0^t e^{-\nu(t-s)(-\Delta)^{\alpha}} \mathbb{P}\nabla \cdot (f \otimes g)(s) \mathrm{d}s; \\ \mathcal{B}_\beta(f,g) =& \int_0^t e^{-\mu(t-s)(-\Delta)^{\beta}} \mathbb{P}\nabla \cdot (f \otimes g)(s) \mathrm{d}s; \\ \mathfrak{B}_\beta(f,g) =& d_i \int_0^t e^{-\mu(t-s)(-\Delta)^{\beta}} \nabla \times (\nabla \cdot (b \otimes b))(s) \mathrm{d}s. \end{aligned}$$

In view of the above, we can write the formulae (2.8), (2.9) and (2.10) as

$$\begin{aligned} S_1(u,b) =&\tilde{u}_0(x) - \mathcal{B}_\alpha(u,u) + \mathcal{B}_\alpha(b,b), \\ S_2(u,b) =&\tilde{b}_0(x) - \mathcal{B}_\beta(u,b) + \mathcal{B}_\beta(b,u) - \mathfrak{B}_\beta(b,b). \end{aligned} \tag{2.11}$$

Given the mild solution formulation (2.8), a traditional approach is to find a fixed point by iterating the map $(u,b) \mapsto S(u,b)$. In order to do so, it is essential to find a space $\mathcal{E}$ such that the bilinear forms $\mathcal{B}_\alpha(\cdot,\cdot)$ and $\mathfrak{B}_\alpha(\cdot,\cdot)$ are bounded from $\mathcal{E}\times\mathcal{E}$ to $\mathcal{E}$. We shall use the following lemma, proven in (Lemarié-Rieusset, 2002) as a simple consequence of Banach fixed point theorem.

Lemma 2.2.4. *Let $\mathcal{E}$ be a Banach space. Given a bilinear form $\mathcal{B}:\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ such that $\|\mathcal{B}(u,v)\|_\mathcal{E} \leq C_0\|u\|_\mathcal{E}\|v\|_\mathcal{E}, \forall u,v\in\mathcal{E}$, for some constant $C_0>0$, we have the following assertions for the equation*

$$u = y + \mathcal{B}(u,u). \tag{2.12}$$

i). Suppose that $y \in B_\varepsilon(0) := \{f\in\mathcal{E} : \|f\|_\mathcal{E} < \varepsilon\}$ for some $\varepsilon \in \left(0, \frac{1}{4C_0}\right)$, then the equation (2.12) has a solution $u\in B_{2\varepsilon}(0) := \{f\in\mathcal{E} : \|f\|_\mathcal{E} < 2\varepsilon\}$, which is, in fact, the unique solution in the ball $\overline{B_{2\varepsilon}(0)}$.

ii). On top of i), suppose that $\bar{y}\in B_\varepsilon(0), \bar{u}\in B_{2\varepsilon}(0)$ and $\bar{u} = \bar{y} + \mathbb{B}(\bar{u},\bar{u})$, then the following continuous dependence is true.

$$\|u-\bar{u}\|_\mathcal{E} \leq \frac{1}{1-4\varepsilon C_0}\|y-\bar{y}\|_\mathcal{E}. \tag{2.13}$$

It can be seen from inequality (2.13) that to ensure local well-posedness, it suffices that $C_0 = CT^a$ for some $a > 0$, while global well-posedness would require C_0 to be bounded above by a time-independent constant.

We work within a framework based on the concepts of the "admissible path space" and "adapted value space", as formulated in (Lemarié-Rieusset, 2002). The idea is to first identify an "admissible path space" $\mathcal{E}_T$ in which we may apply Lemma 2.2.4, then characterize the "adapted value space" E_T associated with $\mathcal{E}_T$. In our case, we consider the space

$$E_T = \{f : f \in \mathcal{S}',\ e^{-t(-\Delta)^\sigma} f \in \mathcal{E}_T,\ 0 < t < T\},\ \sigma = \alpha \text{ or } \beta.$$

We define the Banach spaces X_T and Y_T and the admissible path space $\mathcal{E}_T := X_T \times Y_T$.

$$X_T = \left\{ f : \mathbb{R}^+ \to L^\infty(\mathbb{R}^3) : \nabla \cdot f = 0 \text{ and } \sup_{0<t<T} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \|f(t)\|_{L^\infty(\mathbb{R}^3)} < \infty \right\} \tag{2.14}$$

$$Y_T = \left\{ f : \mathbb{R}^+ \to L^\infty(\mathbb{R}^3) : \nabla \cdot f = 0 \text{ and } \sup_{0<t<T} t^{\frac{2\beta-\gamma_2}{2\beta}} \|f(t)\|_{L^\infty(\mathbb{R}^3)} < \infty \right\} \tag{2.15}$$

By formulae (2.9) and (2.10) along with the characterization of homogeneous Besov spaces in terms of the heat flow Lemma 1.3.3, we have the following inequalities -

$$\begin{aligned} \|u\|_{X_T} \leq & \sup_{t>0} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \|\tilde{u}_0\|_\infty + \|\mathcal{B}_\alpha(u,u)\|_{X_T} + \|\mathcal{B}_\alpha(b,b)\|_{X_T} \\ \leq & C_\nu \|u_0\|_{\dot{B}^{-(2\alpha-\gamma_1)}_{\infty,\infty}} + \|\mathcal{B}_\alpha(u,u)\|_{X_T} + \|\mathcal{B}_\alpha(b,b)\|_{X_T}, \end{aligned}$$

$$\|b\|_Y \leq \sup_{t>0} t^{\frac{2\beta-\gamma_2}{2\beta}} \|\tilde{b}_0\|_\infty + \|\mathcal{B}_\beta(u,b)\|_{Y_T} + \|\mathcal{B}_\beta(b,u)\|_{Y_T} + \|\mathfrak{B}_\beta(b,b)\|_{Y_T}$$

$$\leq C_\mu \|b_0\|_{\dot{B}^{-(2\beta-\gamma_2)}_{\infty,\infty}} + \|\mathcal{B}_\beta(u,b)\|_{Y_T} + \|\mathcal{B}_\beta(b,u)\|_{Y_T} + \|\mathfrak{B}_\beta(b,b)\|_{Y_T}.$$

Clearly, $\dot{B}^{-(2\alpha-\gamma_1)}_{\infty,\infty} \times \dot{B}^{-(2\beta-\gamma_2)}_{\infty,\infty}(\mathbb{R}^3)$ is an adapted value space corresponding to the admissible path space $\mathcal{E}_T$ given by Definitions 2.14 and 2.15.

As an intermediate step to Theorem 2.2.1, we prove the following proposition.

Proposition 2.2.5. *Suppose that the parameters α, β, γ_1 and γ_2 satisfy the set of conditions 2.7. If $(u,b) \in \mathcal{E}_T$ for some $0 < T < \infty$, then $\|S(u,b) - (\tilde{u}_0, \tilde{b}_0)\| \in \mathcal{E}_T$. In particular,*

$$\|S(u,b) - (\tilde{u}_0, \tilde{b}_0)\|_{\mathcal{E}_T} \leq CT^a \|(u,b)\|^2_{\mathcal{E}_T} \tag{2.16}$$

for some $a > 0$ and $C = C(\nu, \mu, \eta) > 0$.

Proof: First, we remark that the constraints on the parameters indeed yield a non-empty set, since the combination $\alpha = 1 - \delta, \beta = 2 - 2\delta, \gamma_1 = 1$ and $\gamma_2 = 2$ with $\frac{1}{4} < \delta < \frac{1}{2}$ clearly satisfies (2.7).

To prove (2.16), it suffices to show that the bilinear forms are bounded from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$, with bounds dependent on ν, μ, d_i and T. To this end, we invoke the property of the Beta function. More specifically, for $\alpha > 1$ and $\theta \in [0.\alpha]$, we have

$$\int_0^t (t-\tau)^{-\frac{1}{\alpha}} \tau^{-\frac{\theta}{\alpha}} \mathrm{d}\tau = t^{1-\frac{1}{\alpha}-\frac{\theta}{\alpha}} B\Big(1 - \frac{\theta}{\alpha}, 1 - \frac{1}{\alpha}\Big) \leq Ct^{1-\frac{1}{\alpha}-\frac{\theta}{\alpha}}. \tag{2.17}$$

Let $\gamma_1 \geq 1$ and $\frac{\gamma_1}{2} < \alpha < \gamma_1$. Via integration by parts, Hölder's inequality, identity (2.17) and Definition 2.14, we have the following inequalities.

$$\begin{aligned}\|\mathcal{B}_\alpha(u,u)\|_{X_T} \leq & C_\nu \sup_{0<t<T} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{2\alpha}} \|u(s)\|_\infty \|u(s)\|_\infty \mathrm{d}s \\ \leq & C_\nu \|u\|_{X_T}^2 \sup_{0<t<T} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{2\alpha}} s^{-2+\frac{\gamma_1}{\alpha}} \mathrm{d}s \\ \leq & C_\nu T^{\frac{\gamma_1-1}{2\alpha}} \|u\|_{X_T}^2.\end{aligned}$$

Similarly, the following estimates are true provided that $\gamma_1 \geq 1$, $\frac{\gamma_1}{2} < \alpha < \gamma_1$, $\frac{\gamma_2}{2} < \beta < \gamma_2$ and $\gamma_2 \geq \frac{(\gamma_1+1)\beta}{2\alpha}$.

$$\begin{aligned}\|\mathcal{B}_\alpha(b,b)\|_{X_T} \leq & C_\nu \sup_{0<t<T} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{2\alpha_1}} \|b(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s \\ \leq & C_\nu \|b\|_{X_T}^2 \sup_{0<t<T} t^{\frac{2\alpha-\gamma_1}{2\alpha}} \int_0^t (t-s)^{-\frac{1}{2\alpha}} s^{-2+\frac{\gamma_2}{\beta}} \mathrm{d}s \\ \leq & C_\nu T^{\frac{\gamma_2}{\beta}-\frac{\gamma_1+1}{2\alpha}} \|b\|_{X_T}^2.\end{aligned}$$

To bound the term $\|\mathcal{B}_\beta(b,u)\|_{Y_T}$, we further require that $\beta > \frac{1}{2}$ and $\gamma_1 \geq \frac{\alpha}{\beta}$.

$$\begin{aligned}\|\mathcal{B}_\beta(b,u)\|_{Y_T} \leq & C_\mu \sup_{0<t<T} t^{\frac{2\beta-\gamma_2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} \|u(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s \\ \leq & C_\mu \|u\|_{X_T} \|b\|_{Y_T} \sup_{0<t<T} t^{\frac{2\beta-\gamma_2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{2\beta}} s^{-2+\frac{\gamma_1}{2\alpha}+\frac{\gamma_2}{2\beta}} \mathrm{d}s \\ \leq & C_\mu T^{\frac{\gamma_1}{2\alpha}-\frac{1}{2\beta}} \|u\|_{X_T} \|b\|_{Y_T}.\end{aligned}$$

We note that the term $\|\mathcal{B}_\beta(u,b)\|_Y$ can be estimated in an identical manner.

Finally, we integrate by parts twice to estimate the Hall term. We end up with the condition $\beta > 1$ along with all the constraints from previous estimates.

$$\begin{aligned}\|\mathfrak{B}_\beta(b,b)\|_{Y_T} \leq & C_{\mu,d_i} \sup_{0<t<T} t^{\frac{2\beta-\gamma_2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{\beta}} \|b(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s \\ \leq & C_{\mu,d_i} \|b\|_{Y_T}^2 \sup_{0<t<T} t^{\frac{2\beta-\gamma_2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{\beta}} s^{-2+\frac{\gamma_2}{\beta}} \mathrm{d}s \\ \leq & C_{\mu,d_i} T^{\frac{\gamma_2-2}{2\beta}} \|b\|_{Y_T}^2.\end{aligned}$$

□

Proof of Theorem 2.2.1: By inequality (2.16), Lemma 1.3.3 and Lemma 2.2.4, there exists a solution $(u,b) \in \mathcal{E}_T$ provided that the initial data (u_0, b_0) and the time T satisfy

$$4CT^a\Big(C_\nu \|u_0\|_{\dot{B}_{\infty,\infty}^{-(2\alpha-\gamma_1)}} + C_{\mu,d_i}\|b_0\|_{\dot{B}_{\infty,\infty}^{-(2\beta-\gamma_2)}}\Big) < 1.$$

It remains to be shown that $(u,b) \in L^\infty\big(0,T; \dot{B}_{\infty,\infty}^{-(2\alpha-\gamma_1)} \times \dot{B}_{\infty,\infty}^{-(2\beta-\gamma_2)}(\mathbb{R}^3)\big)$. By (2.9) and Lemma 1.3.3, it holds that

$$\begin{aligned}\|S_1 u(t)\|_{\dot{B}_{\infty,\infty}^{-(2\alpha-\gamma_1)}} = & \sup_{0<\tau<T} \tau^{\frac{2\alpha-\gamma_1}{2\alpha}} \|e^{-\nu\tau(-\Delta)^\alpha} S_1 u(t)\|_{L^\infty} \\ \lesssim & \sup_{0<\tau<T} \tau^{\frac{2\alpha-\gamma}{2\alpha}} \|e^{-\nu(\tau+t)(-\Delta)^\alpha} u_0\|_{L^\infty} \\ & + \sup_{0<\tau<T} \tau^{\frac{2\alpha-\gamma_1}{2\alpha}} \|u\|_{X_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\alpha}} s^{-2+\frac{\gamma_1}{\alpha}} \mathrm{d}s \\ & + \sup_{0<\tau<T} \tau^{\frac{2\alpha-\gamma_1}{2\alpha}} \|b\|_{Y_T}^2 \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\alpha}} s^{-2+\frac{\gamma_2}{\beta}} \mathrm{d}s.\end{aligned}$$

Estimating with the help of (2.17), we have

$$
\begin{aligned}
\|S_1u(t)\|_{\dot B^{-(2\alpha-\gamma_1)}_{\infty,\infty}} \lesssim \sup_{0<\tau<T} \tau^{\frac{2\alpha-\gamma_1}{2\alpha}} \Big(&\|e^{-\nu\tau(-\Delta)^\alpha} u_0\|_{L^\infty} + (\tau+t)^{-1+\frac{2\gamma_1-1}{2\alpha}} \|u\|^2_{X_T} \\
&+ (\tau+t)^{-1-\frac{1}{2\alpha}+\frac{2\gamma_2}{2\beta}} \|b\|^2_{Y_T} \Big) \\
\lesssim &\|u_0\|_{\dot B^{-(2\alpha-\gamma_1)}_{\infty,\infty}} + T^a \|(u,b)\|^2_{\mathcal{E}_T}.
\end{aligned}
$$

In a similar fashion, the following inequalities follow from (2.10) and Lemma 1.3.3.

$$
\begin{aligned}
\|S_2b(t)\|_{\dot B^{-(2\beta-\gamma_2)}_{\infty,\infty}} &= \sup_{0<\tau<T} \tau^{\frac{2\beta-\gamma_2}{2\beta}} \|e^{-\mu\tau(-\Delta)^\beta} S_2b(t)\|_{L^\infty} \\
&\lesssim \sup_{0<\tau<T} \tau^{\frac{2\beta-\gamma_2}{2\beta}} \Big(\|e^{-\mu(\tau+t)(-\Delta)^\beta} b_0\|_{L^\infty} \\
&\quad + 2\|u\|_{X_T}\|b\|_{Y_T} \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{2\beta}} s^{-2+\frac{\gamma_1}{2\alpha}+\frac{\gamma_2}{2\beta}} \mathrm{d}s \\
&\quad + \|b\|^2_{Y_T} \int_0^{\tau+t} (\tau+t-s)^{-\frac{1}{\beta}} s^{-2+\frac{\gamma_2}{\beta}} \mathrm{d}s \Big).
\end{aligned}
$$

The integrals can be evaluated thanks to (2.17), which yields the bound on S_2b.

$$
\begin{aligned}
\|S_2b(t)\|_{\dot B^{-(2\beta-\gamma_2)}_{\infty,\infty}} \lesssim \sup_{0<\tau<T} \tau^{\frac{2\beta-\gamma_2}{2\beta}} \Big(&\|e^{-\nu\tau(-\Delta)^\beta} b_0\|_{L^\infty} \\
&+ (\tau+t)^{-1+\frac{\gamma_1}{2\alpha}+\frac{\gamma_2-1}{2\beta}} \|u\|_{X_T}\|b\|_{Y_T} + (\tau+t)^{-1+\frac{\gamma_2-1}{\beta}} \|b\|^2_{Y_T} \Big) \\
\lesssim &\|b_0\|_{\dot B^{-(2\beta-\gamma_2)}_{\infty,\infty}} + T^a \|(u,b)\|^2_{\mathcal{E}_T}.
\end{aligned}
$$

The inequalities above imply that

$$
(u,b) \in L^\infty\big(0,T; \dot B^{-(2\alpha-\gamma_1)}_{\infty,\infty} \times \dot B^{-(2\beta-\gamma)}_{\infty,\infty}(\mathbb{R}^3)\big).
$$

□

However, the well-posedness of the standard Hall-MHD system, i.e., the case $\alpha_1 = \alpha_2 = 1$, is unattainable as the above method breaks down in this case.

An interesting byproduct of the proof is a small data global well-posedness result for the hyper-resistive electron-MHD equations, i.e., System 1.9 - 1.10 with $1 < \beta < 2$..

Theorem 2.2.6 (Global existence for small data)**.** *Let $1 < \beta < 2$. There exists some $\varepsilon = \varepsilon(\mu) > 0$ such that if $\|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)} \leq \varepsilon$, then there exists a solution b to the generalized EMHD equations, i.e., System 1.9 - 1.10 with $u \equiv 0$, satisfying*

$$b \in L^\infty\big(0, +\infty; \dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)\big) \text{ and } \sup_{t>0} t^{\frac{\beta-1}{\beta}} \|b\|_{L^\infty(\mathbb{R}^3)} < \infty.$$

We recall that System 1.9 - 1.10 possesses the property of scale invariance. We can see that the space $L^\infty\big(0, \infty; \dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)\big)$ is the largest critical space according to the scaling property. Unfortunately, our pathway to small data global well-posedness fails just when $\beta = 1$, leaving the question of the standard EMHD equations' solvability in its largest critical space $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$ unanswered.

We proceed to prove Theorem 2.2.6 by finding a ball $B \subset Y_T$ where the solution map S_2 is a contraction mapping. We have the following two propositions.

Proposition 2.2.7. *Let $\beta \in (1, 2)$ and $\gamma_2 = 2$. For $0 < T \leq \infty$, the map S_2 satisfies*

$$\|S_2 b - \tilde{b}_0\|_{Y_T} \leq C\|b\|^2_{Y_T}. \tag{2.18}$$

Therefore, there exists some $\varepsilon_1 > 0$*, such that* S_2 *is a self-mapping on the ball*

$$B_{\varepsilon_1}\big(\tilde{b}_0\big) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_1\},$$

provided that $\|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)} < \varepsilon_1$.

Proof: The inequality (2.18) follows from the following estimate.

$$\begin{aligned}
\|\mathfrak{B}_\beta(b,b)\|_{Y_T} \leq & \sup_{t>0} t^{\frac{2\beta-2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{\beta}} \|b(s)\|_\infty \|b(s)\|_\infty \mathrm{d}s \\
\leq & \|b\|_{Y_T}^2 \sup_{t>0} t^{\frac{2\beta-2}{2\beta}} \int_0^t (t-s)^{-\frac{1}{\beta}} s^{-2+\frac{2}{\beta}} \mathrm{d}s \\
\leq & C_{\mu,d_i} \|b\|_{Y_T}^2.
\end{aligned}$$

Since it is assumed that $b \in B_{\varepsilon_1}\big(\tilde{b}_0\big)$ and $\|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)} < \varepsilon_1$, it follows from inequality (2.18) and lemma (1.3.3) that

$$\|S_2 b - \tilde{b}_0\|_{Y_T} \leq C\|b\|_{Y_T}^2 \leq C\big(\|b - \tilde{b}_0\|_{Y_T}^2 + \|\tilde{b}_0\|_{Y_T}^2\big) \leq C\varepsilon_1^2.$$

□

Proposition 2.2.8. *Let* $1 < \beta < 2$ *and* $\gamma_2 = 2$*. For any* $T \in (0, \infty]$*, there exists some* $\varepsilon_2 \in (0, \varepsilon_1)$ *such that if* $\|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)} < \varepsilon_2$*, then the solution map* S_2 *is a contraction mapping on the ball*

$$B_{\varepsilon_2}\big(\tilde{b}_0\big) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_2\}.$$

Proof: Let $b, \bar{b} \in B_{\varepsilon_2}\big(\tilde{b}_0\big)$. Clearly, the following inequalities hold.

$$\begin{aligned}
\|S_2 b - S_2 \bar{b}\|_{Y_T} =& \|\mathfrak{B}_\beta(b,b) - \mathfrak{B}_\beta(\bar{b},\bar{b})\|_{Y_T} \\
\leq& \|\mathfrak{B}_\beta(b,b) - \mathfrak{B}_\beta(b,\bar{b})\|_{Y_T} + \|\mathfrak{B}_\beta(b,\bar{b}) - \mathfrak{B}_\beta(\bar{b},\bar{b})\|_{Y_T} \\
\leq& C_{\mu,d_i} \max\{\|b\|_{Y_T}, \|\bar{b}\|_{Y_T}\} \|b - \bar{b}\|_{Y_T} \\
\leq& C_{\mu,d_i} \varepsilon_2 \|b - \bar{b}\|_{Y_T}.
\end{aligned}$$

We can ensure that S_2 is a contraction mapping by choosing $\varepsilon_2 < 1/2C_{\mu,d_i}$.

□

Proof of Theorem 2.2.6. As a result of Proposition 2.2.8, we know that for some $\varepsilon_2 > 0$, S_2 has a fixed point, which is a mild solution to System 1.9 - 1.10, in

$$B_{\varepsilon_2}\big(\tilde{b}_0\big) =: \{f \in Y_T : \|f - \tilde{b}_0\|_{Y_T} < \varepsilon_2,\ T = +\infty\},$$

provided that $\|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3)} < \varepsilon_2$.

To see that the solution b is in $L^\infty(0,\infty; \dot{B}^{-(2\beta-2)}_{\infty,\infty}(\mathbb{R}^3))$, we just calculate

$$\begin{aligned}
\|S_2 b(t)\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}} \lesssim& \sup_{\tau>0} \tau^{\frac{2\beta-2}{2\beta}} \bigg(\|e^{-\mu(\tau+t)(-\Delta)^\beta} b_0\|_{L^\infty} \\
&+ \|b\|_{Y_T}^2 \int_0^{\tau+t} (\tau + t - s)^{-\frac{1}{\beta}} s^{-2+\frac{2}{\beta}} \mathrm{d}s \bigg) \\
\lesssim& \|b_0\|_{\dot{B}^{-(2\beta-2)}_{\infty,\infty}} + \|b\|_{Y_T}^2.
\end{aligned}$$

□

CHAPTER 3

LONG TIME BEHAVIOUR OF SOLUTIONS TO THE HALL-MHD SYSTEM

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3.1 Temporal decay for the fully dissipative Hall-MHD system

Let (u, b) be a strong solution to System 1.6 - 1.8. Multiplying Equation 1.6 and Equation 1.7 by u and b, respectively, integrating by parts and adding the resulting identities lead to the following differential energy equality –

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_2^2 + \|b(t)\|_2^2\right) = -\left(\nu\|\nabla^\alpha u(t)\|_2^2 + \mu\|\nabla^\beta b(t)\|_2^2\right). \tag{3.1}$$

Integrating in time further leads to the integral energy equality –

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu\int_0^t \|\nabla^\alpha u(s)\|_2^2\,\mathrm{d}s + 2\mu\int_0^t \|\nabla^\beta b(s)\|_2^2\,\mathrm{d}s = \|u(0)\|_2^2 + \|b(0)\|_2^2.$$

Heuristically, the above equalities already seem to imply decay of the total energy. For dissipative systems satisfying certain energy inequalities, it is classical to establish decay results via the Fourier splitting technique, which was first employed to obtain algebraic decay rates for solutions to parabolic conservation laws and the Navier-Stokes equations in (Schonbek, 1985; Schonbek, 1986a; Schonbek, 1986b). A thorough review of researches in this direction can be found in (Brandolese and Schonbek, 2018). Via the Fourier splitting technique, algebraic decay in L^2 for weak solutions to the fully dissipative case of System 1.1 - 1.3 was obtained in (Chae and Schonbek, 2013). We shall reproduce the result here for the sake of completeness –

Theorem 3.1.1. *For $(u_0, b_0) \in \left(L^1 \cap L^2(\mathbb{R}^3)\right)^2$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, there exists a weak solution (u, b) to System 1.1 - 1.3 such that*

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 \lesssim (1+t)^{-\frac{1}{2}}.$$

To apply the Fourier splitting technique, we need the following lemma concerning the bounds on $\hat{u}$ and $\hat{b}$, which is particularly relevant to the decay of the high frequency parts.

Lemma 3.1.2. *Let (u, b) be a mild solution to System 1.6 - 1.8. Assume the initial data (u_0, b_0) belongs to $\left(L^2(\mathbb{R}^3)\right)^2$. If $\mu > 0$ and additionally $b_0 \in L^1(\mathbb{R}^3)$, then we have*

$$|\hat{b}(t, \xi)| \lesssim 1 + \frac{1 + |\xi|}{|\xi|^{2\beta - 1}}.$$

If $\nu > 0$ and additionally $u_0 \in L^1(\mathbb{R}^3)$, then we have

$$|\hat{u}(t, \xi)| \lesssim 1 + |\xi|^{1-2\alpha}.$$

In particular, let $\alpha = \beta = 1$ $\mu, \nu > 0$ *and* $(u_0, b_0) \in \left(L^1 \cap L^2(\mathbb{R}^3)\right)^2$, *then*

$$|\hat{u}(t,\xi)| + |\hat{b}(t,\xi)| \lesssim \left(1 + \frac{1}{|\xi|}\right).$$

Proof: Taking Fourier transform of Equation 1.7 yields

$$\hat{b}_t + |\xi|^{2\beta}\hat{b} = G(t,\xi)$$

where $G(t,\xi) = -\widehat{u \cdot \nabla b} + \widehat{b \cdot \nabla u} - \mathscr{F}\left(\nabla \times ((\nabla \times b) \times b)\right)$. Thus, we have

$$\hat{b}(t) = e^{-|\xi|^{2\beta}t}\hat{b}(0) + \int_0^t e^{-|\xi|^{2\beta}(t-s)}G(s,\xi)\mathrm{d}s.$$

As a consequence of the vector identity $(\nabla \times b) \times b = b \cdot \nabla b - \nabla \frac{|b|^2}{2} = \nabla \cdot (b \otimes b) - \nabla \frac{|b|^2}{2}$, it holds that $\nabla \times ((\nabla \times b) \times b) = \nabla \times (\nabla \cdot (b \otimes b))$, which leads to

$$\begin{aligned} |G(s,\xi)| \lesssim & \sum_{i,j} \left(|\xi||\widehat{u^i b^j}| + |\xi|^2|\widehat{b^i b^j}|\right) \\ \lesssim & (|\xi|\|u_0\|_2\|b_0\|_2 + |\xi|^2\|b_0\|_2^2) \\ \lesssim & |\xi|(1 + |\xi|). \end{aligned}$$

It then follows that

$$
\begin{aligned}
|\hat{b}(t,\xi)| \leq & |\hat{b}(0)| + C|\xi|(1+|\xi|)\int_0^t e^{-|\xi|^{2\beta}(t-s)}\,\mathrm{d}s \\
\leq & C\|b_0\|_1 + C\frac{1+|\xi|}{|\xi|^{2\beta-1}}(1-e^{-|\xi|^{2\beta}t}) \\
\lesssim & 1 + \frac{1+|\xi|}{|\xi|^{2\beta-1}}.
\end{aligned}
$$

The estimate for $\hat{u}$ can be established in a similar way.

□

Given Lemma 3.1.2 along with the energy equality, we proceed to prove Theorem 3.1.1.

Proof: For the Leray-Hopf type weak solutions in the case $\alpha = \beta = 1$, we rather have the energy inequality

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_2^2 + \|b(t)\|_2^2\right) \leq -\left(\nu\|\nabla u(t)\|_2^2 + \mu\|\nabla b(t)\|_2^2\right).
$$

The Fourier transform of the differential energy inequality along with Plancherel's theorem yields

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{u}(t)\|_2^2 + \|\hat{b}(t)\|_2^2\right) \leq -\int_{\mathbb{R}^3}\left(\nu|\xi|^2|\hat{u}(t,\xi)|^2 + \mu|\xi|^2|\hat{b}(t,\xi)|^2\right)\mathrm{d}\xi. \tag{3.2}
$$

Introducing the set

$$
\mathcal{S} = \left\{\xi : |\xi| \leq \left(\frac{3}{2\min\{\mu,\nu\}(1+t)}\right)^{\frac{1}{2}}\right\},
$$

we rewrite (3.2) as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2\right)\leq &-\int_{\mathcal{S}}\left(\nu|\xi|^2|\hat{u}(t,\xi)|^2+\mu|\xi|^2|\hat{b}(t,\xi)|^2\right)\mathrm{d}\xi\\ &-\int_{\mathcal{S}^c}\left(\nu|\xi|^2|\hat{u}(t,\xi)|^2+\mu|\xi|^2|\hat{b}(t,\xi)|^2\right)\mathrm{d}\xi\end{aligned}$$

and then discard the low frequency part to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2\right)\leq &-\frac{3}{(1+t)}\int_{\mathcal{S}^c}\left(|\hat{u}(t,\xi)|^2+|\hat{b}(t,\xi)|^2\right)\mathrm{d}\xi\\ \leq &-\frac{3}{(1+t)}\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2-\int_{\mathcal{S}}\left(|\hat{u}(t,\xi)|^2+|\hat{b}(t,\xi)|^2\right)\mathrm{d}\xi\right).\end{aligned}$$

We infer from the pointwise bound in Lemma 3.1.2 that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2\right)+\frac{3}{(1+t)}\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2\right)\lesssim &-\frac{1}{(1+t)}\int_{\mathcal{S}}\left(1+\frac{1}{|\xi|}\right)^2\mathrm{d}\xi\\ \lesssim &-(1+t)^{-3/2}.\end{aligned}$$

Multiplying both sides of the above inequality by $(1+t)^3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^3\left(\|\hat{u}(t)\|_2^2+\|\hat{b}(t)\|_2^2\right)\right)\lesssim -(1+t)^{3/2}.$$

The desired result follows from integrating in time and dividing both sides of the above inequality by $(1+t)^3$.

□

3.2 Temporal decay for the Hall-MHD system with mere one dissipation

As a continuation of (Chae and Schonbek, 2013), in (Dai and Liu, 2019a), we further studied the long time behaviour of solutions to the generalized Hall-MHD system without either the velocity dissipation or the magnetic dissipation term. In these cases, the energy inequality becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_2^2+\|b(t)\|_2^2\right)\leq \nu\|\nabla^\alpha u\|_2^2 \text{ and } \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_2^2+\|b(t)\|_2^2\right)\leq \mu\|\nabla^\beta b\|_2^2,$$

respectively, which, at a glance, only implies decay of the total energy but seems to reveal little about the behaviours of the individual energies. It is not obvious whether or not the kinetic energy and magnetic energy might oscillate in a way that compensate each other, despite that their sum decays. Using a strategy similar to that in (Agapito and Schonbek, 2007) for the MHD system without magnetic diffusion, we demonstrate that such compensatory oscillations do not occur in either case.

In order to study the energy decay problem for System 1.6 - 1.8 with mere one dissipation term, we introduce the cut-off functions φ and ψ in the Fourier space. In the inviscid resistive case $\nu=0,\mu>0$, we take functions $\varphi(\xi):=e^{-|\xi|^2}$ and $\psi(\xi):=1-\varphi(\xi)$. Obviously, $\varphi\hat{b}$ and $\psi\hat{b}$ represent the low and high frequency parts of b, respectively. It follows from Plancherel's theorem that

$$\|b(t)\|_2=\|\hat{b}(t)\|_2\leq\|\varphi\hat{b}(t)\|_2+\|\psi\hat{b}(t)\|_2.$$

Corresponding to the viscous non-resistive case $\nu > 0, \mu = 0$ are $\varphi(\xi, t) := e^{-|\xi|^{2\beta}t}$ and $\psi(\xi, t) := 1 - \varphi(\xi, t)$ instead. We split $\|u(t)\|_2$ as

$$\|u(t)\|_2 = \|\hat{u}(t)\|_2 \leq \|\varphi(t)\hat{u}(t)\|_2 + \|\psi(t)\hat{u}(t)\|_2 .$$

We will need the generalized energy inequalities in the following lemma to estimate the above low and high frequency parts.

Lemma 3.2.1. *Let $\mu > 0$, $\varphi(\xi) = e^{-|\xi|^2}$, and $\psi(\xi) = 1 - \varphi(\xi)$. Let $E(t)$ be a weight function such that $E(t) \in C^1(\mathbb{R}; \mathbb{R}_+)$ and $E(t) \geq 0$. Then, a weak solution (u, b) to System 1.6 - 1.8 satisfies the generalized energy inequalities –*

$$\begin{aligned}\left\|\mathscr{F}^{-1}\varphi * b(t)\right\|_2^2 \leq & \left\|e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * b(s)\right\|_2^2 \\ & - 2\int_s^t \left\langle u\cdot\nabla b(\tau), e^{2\mu(t-\tau)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle \mathrm{d}\tau \\ & + 2\int_s^t \left\langle b\cdot\nabla u(\tau), e^{2\mu(t-\tau)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle \mathrm{d}\tau \\ & - 2d_i\int_s^t \left\langle \nabla\times(\nabla\cdot(b\otimes b)(\tau)), e^{2\mu(-\Delta)^\beta(t-\tau)}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle \mathrm{d}\tau,\end{aligned} \tag{3.3}$$

$$\begin{aligned}E(t)\|\psi\hat{b}(t)\|_2^2 \leq & E(s)\|\psi\hat{b}(s)\|_2^2 - 2\mu\int_s^t E(\tau)\left\||\xi|^\beta\psi\hat{b}(\tau)\right\|_2^2 \mathrm{d}\tau \\ & + \int_s^t E'(\tau)\|\psi\hat{b}(\tau)\|_2^2 \mathrm{d}\tau - 2\int_s^t E(\tau)\left\langle \widehat{u\cdot\nabla b}(\tau), \psi^2\hat{b}(\tau)\right\rangle \mathrm{d}\tau \\ & + 2\int_s^t E(\tau)\left\langle \widehat{b\cdot\nabla u}(\tau), \psi^2\hat{b}(\tau)\right\rangle \mathrm{d}\tau \\ & - 2d_i\int_s^t E(\tau)\left\langle \mathscr{F}\left(\nabla\times(\nabla\cdot(b\times b))\right)(\tau), \psi^2\hat{b}(\tau)\right\rangle \mathrm{d}\tau.\end{aligned} \tag{3.4}$$

Proof: The estimates will be established formally for classical solutions. Multiplying Equation 1.7 by $e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*b(s)$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned}&\left\langle b_t, e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*b(s)\right\rangle+\mu\left\langle\nabla^\beta b, e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*\nabla^\beta b(s)\right\rangle\\&+\left\langle u\cdot\nabla b, e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*b(s)\right\rangle-\left\langle b\cdot\nabla u, e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*b(s)\right\rangle\\&+d_i\left\langle\nabla\times(\nabla\cdot(b\times b)), e^{2\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*\mathscr{F}^{-1}\varphi*b(s)\right\rangle=0.\end{aligned}$$

Using the fact that $\partial_t\left(e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi\right)=\mu(-\Delta)^\beta\left(e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi\right)$, we rewrite the first two terms in the above equality as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*b(s)\right\|_2^2-\left\langle e^{\mu(t-s)\Delta^\beta}\mathscr{F}^{-1}\varphi*b(s),\partial_t\left(e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi\right)*b\right\rangle\\&+\mu\left\langle e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*b(s),(-\Delta)^\beta\left(e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi\right)*b\right\rangle\\=&\frac{1}{2}\frac{d}{dt}\left\|e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi*b(s)\right\|_2^2.\end{aligned}$$

Integrating over the time interval $[s,t]$ yields the generalized energy inequality for the low frequency part.

We take Fourier transform of Equation 1.7, multiply it by $\psi^2\hat{b}E(t)$ and integrate in space to infer

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}E(t)|\psi\hat{b}|^2\mathrm{d}\xi-\frac{1}{2}\int_{\mathbb{R}^3}E'(t)|\psi\hat{b}|^2\mathrm{d}\xi+\mu E(t)\int_{\mathbb{R}^3}|\xi|^{2\alpha}|\psi\hat{b}|^2\mathrm{d}\xi\\&+E(t)\left\langle\widehat{u\cdot\nabla b},\psi^2\hat{b}\right\rangle-E(t)\left\langle\widehat{b\cdot\nabla u},\psi^2\hat{b}\right\rangle+d_iE(t)\left\langle\mathscr{F}\left(\nabla\times(\nabla\cdot(b\times b))\right),\psi^2\hat{b}\right\rangle=0.\end{aligned}$$

Integrating the last equation over $[s,t]$, we obtain the generalized energy inequality for the high frequency part.

$\square$

Analogous computations shall produce the generalized energy inequalities for the velocity u in the following lemma.

Lemma 3.2.2. *Let* $\nu > 0$, $\varphi(\xi,t) = e^{-|\xi|^{2\beta}t}$ *and* $\psi(\xi,t) = 1-\varphi(\xi,t)$. *Let* $E(t)$ *be a weight function such that* $E(t) \in C^1(\mathbb{R};\mathbb{R}_+)$ *and* $E(t) \geq 0$. *Then, a weak solution* (u,b) *to System 1.6 - 1.8 satisfies the generalized energy inequalities –*

$$\begin{aligned}\left\|\mathscr{F}^{-1}\varphi * u(t)\right\|_2^2 \leq & \left\|e^{\nu(t-s)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * u(s)\right\|_2^2 \\ & - 2\int_s^t \left\langle u\cdot\nabla u(\tau), e^{2\nu(t-\tau)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * u(\tau)\right\rangle \mathrm{d}\tau \\ & + 2\int_s^t \left\langle b\cdot\nabla b(\tau), e^{2\nu(t-\tau)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * u(\tau)\right\rangle \mathrm{d}\tau,\end{aligned} \tag{3.5}$$

$$\begin{aligned}E(t)\|\psi(t)\hat{u}(t)\|_2^2 \leq & E(s)\|\psi(s)\hat{u}(s)\|_2^2 - 2\nu\int_s^t E(\tau)\,\||\xi|^\alpha\psi\hat{u}(\tau)\|_2^2\,\mathrm{d}\tau \\ & + \int_s^t E'(\tau)\|\psi\hat{u}(\tau)\|_2^2\mathrm{d}\tau - 2\nu\int_s^t E(\tau)\langle\psi'(\tau)\hat{u}(\tau),\psi\hat{u}(\tau)\rangle\mathrm{d}\tau \\ & - 2\int_s^t E(\tau)\left\langle\widehat{u\cdot\nabla u}(\tau),\psi^2\hat{u}(\tau)\right\rangle\mathrm{d}\tau + 2\int_s^t E(\tau)\left\langle\widehat{b\cdot\nabla b}(\tau),\psi^2\hat{u}(\tau)\right\rangle\mathrm{d}\tau.\end{aligned} \tag{3.6}$$

3.2.1 The inviscid resistive case

For strong solutions to the inviscid resistive Hall-MHD system, i.e., System 1.6 - 1.8 with $\nu = 0$ and $\mu > 0$, the magnetic energy $\|b(t)\|_2^2$ vanishes eventually despite the lack of velocity diffusion, provided that $u(t)$ is bounded in $W^{1-\beta,\frac{3}{\beta}}(\mathbb{R}^3)$. Our result states as follows.

Theorem 3.2.3. *Let (u, b) be a global strong solution to System 1.6 - 1.8 with $\nu = 0$ and $\mu > 0$. Assume $u_0 \in L^2(\mathbb{R}^3)$, $b_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and one of the following two conditions –*

(i) $u \in L^\infty(0, \infty; W^{1-\beta,\frac{3}{\beta}}(\mathbb{R}^3))$ *with* $\beta \in [1/2, 1]$;

(ii) $b \in L^\infty(0, \infty; W^{1-\beta,\infty}(\mathbb{R}^3))$ *and* $u \in L^\infty(0, \infty; W^{1-\beta,\frac{3}{\beta}}(\mathbb{R}^3))$ *with* $\beta \in (0, 1]$.

Then, we have

$$\lim_{t\to\infty} \|b(t)\|_2^2 = 0, \quad \lim_{t\to\infty} \|u(t)\|_2^2 = C$$

for some absolute constant C.

In view of the generalized energy inequalities, we establish decay for low and high frequency parts separately. We estimate the low frequency part $\|\varphi\hat{b}(t)\|_2$ in the following proposition.

Proposition 3.2.4. *Let $(u_0, b_0) \in \left(L^2(\mathbb{R}^3)\right)^2$ and (u, b) be a strong solution to System 1.6 - 1.8 with $\nu = 0$ and $\mu > 0$. For $\varphi = e^{-|\xi|^2}$, it holds that*

$$\lim_{t\to\infty} \|\varphi\hat{b}(t)\|_2 = 0.$$

Proof: The generalized energy inequality (3.3) implies

$$\begin{aligned}
\left\|\mathscr{F}^{-1}\varphi * b(t)\right\|_2^2 \leq & \left\|e^{\mu(t-s)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * b(t)\right\|_2^2 \\
& + 2\int_s^t \left|\left\langle u\cdot\nabla b(\tau), e^{2\mu(t-\tau)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle\right| \mathrm{d}\tau \\
& + 2\int_s^t \left|\left\langle b\cdot\nabla u(\tau), e^{2\mu(t-\tau)(-\Delta)^\beta}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle\right| \mathrm{d}\tau \\
& + 2d_i\int_s^t \left|\left\langle \nabla\times(\nabla\cdot(b\otimes b)(\tau)), e^{2\mu(-\Delta)^\beta(t-\tau)}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * b(\tau)\right\rangle\right| \mathrm{d}\tau \\
:= & I + II + III + IV.
\end{aligned}$$

One can see immediately that

$$\limsup_{t\to\infty} I = \limsup_{t\to\infty}\left\|e^{-\mu|\xi|^{2\beta}(t-s)}\varphi\hat{b}(s)\right\|_2^2 = 0.$$

By Parseval's identity, the fact that φ^2 is a rapidly decreasing function and Hölder's inequality, we have, for $p\in[2,\infty]$ and $\beta\in(0,1]$, that

$$\begin{aligned}
\left|\left\langle u\cdot\nabla b(\tau), e^{2\mu(-\Delta)^\beta(t-\tau)}\mathscr{F}^{-1}(\varphi^2) * b(\tau)\right\rangle\right| & \leq \left|\left\langle |\xi|\widehat{(u\otimes b)}(\tau), e^{-2\mu|\xi|^{2\beta}(t-\tau)}\varphi^2\hat{b}(\tau)\right\rangle\right| \\
& \leq \left|\left\langle \widehat{(u\otimes b)}(\tau), |\xi|^{1-\beta}e^{-2\mu|\xi|^{2\beta}(t-\tau)}\varphi^2|\xi|^\beta\hat{b}(\tau)\right\rangle\right| \\
& \leq \left\|\widehat{(u\otimes b)}(\tau)\right\|_p \left\||\xi|^{1-\beta}\varphi^2\right\|_{\frac{2p}{p-2}} \left\||\xi|^\beta e^{-2\mu|\xi|^{2\beta}(t-\tau)}\hat{b}(\tau)\right\|_2 .
\end{aligned}$$

Setting $p = \frac{3}{\beta}$ and $p' = \frac{p}{p-1}$, which is compatible with $p \geq 2$ since $\alpha \in (0,1]$, we apply Hausdorff-Young and Sobolev inequalities. It follows from the boundedness of $\|u(t)\|_2$ that

$$\begin{aligned}\left|\left\langle u\cdot\nabla b(\tau), e^{2\mu(-\Delta)^{\beta}(t-\tau)}\mathscr{F}^{-1}(\varphi^2)*b(\tau)\right\rangle\right| \lesssim& \|(u\otimes b)(\tau)\|_{p'}\left\||\xi|^{\beta}\hat{b}(\tau)\right\|_2\\ \lesssim& \|u(\tau)\|_2\|b(\tau)\|_{\frac{2p}{p-2}}\|\nabla^{\beta}b(\tau)\|_2\\ \lesssim& \|u(\tau)\|_2\|\nabla^{\beta}b(\tau)\|_2^2 \lesssim \|\nabla^{\beta}b(\tau)\|_2^2.\end{aligned}$$

Via a similar strategy as above, we have

$$\left|\left\langle b\cdot\nabla u(\tau), e^{2\mu(-\Delta)^{\beta}(t-\tau)}\mathscr{F}^{-1}(\varphi^2)*b(\tau)\right\rangle\right| \lesssim \|\nabla^{\beta}b(\tau)\|_2^2.$$

As for the Hall term IV, we notice that $\left\||\xi|^{2-\beta}\varphi^2\right\|_p$ is finite for any $p > 1$ and $\beta \in (0,1]$. Hence, for $\beta = \frac{3}{2} - \frac{3}{p}$ and $p > 2$, which is again compatible with $\beta \in (0,1]$.

$$\begin{aligned}&\left|\left\langle \nabla\times(\nabla\cdot(b\otimes b))(\tau), e^{2\mu(-\Delta)^{\beta}(t-\tau)}\mathscr{F}^{-1}(\varphi^2)*b(\tau)\right\rangle\right|\\ =&\left|\left\langle \xi\times(\xi\cdot\widehat{(b\otimes b)})(\tau), e^{2\mu|\xi|^{2\beta}(t-\tau)}\varphi^2\hat{b}(\tau)\right\rangle\right|\\ =&\left|\left\langle \varphi^2|\xi|^{-\beta}\xi\times(\xi\cdot\widehat{(b\otimes b)})(\tau), e^{2\mu|\xi|^{2\beta}(t-\tau)}\xi^{\beta}\hat{b}(\tau)\right\rangle\right|\\ \leq&\left\||\xi|^{2-\beta}\varphi^2\right\|_p\left\|\widehat{(b\otimes b)}(\tau)\right\|_{\frac{2p}{p-2}}\left\||\xi|^{\beta}\hat{b}(\tau)\right\|_2\\ \lesssim&\|b(\tau)\|_2\|b(\tau)\|_p\|\nabla^{\beta}b(\tau)\|_2 \lesssim \|\nabla^{\beta}b(\tau)\|_2^2\end{aligned}$$

Combining the last three inequalities yields

$$II + III + IV \lesssim \int_s^t \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau.$$

Thanks to the fact that $\nabla^\beta b \in L^\infty(0,\infty; L^2(\mathbb{R}^3))$, it follows that

$$\lim_{t\to\infty}(II + III + IV) \leq \lim_{s\to\infty}\lim_{t\to\infty}\int_s^t \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau = 0.$$

Therefore, we conclude

$$\lim_{t\to\infty}\|\varphi\hat{b}(t)\|_2 = \lim_{t\to\infty}\left\|\mathscr{F}^{-1}\varphi * b(t)\right\|_2 = 0.$$

□

The decay of the high frequency part $\|\psi\hat{b}(t)\|_2$ is as follows.

Proposition 3.2.5. *Let $u_0 \in L^2(\mathbb{R}^3)$, $b_0 \in L^1 \cap L^2(\mathbb{R}^3)$ and (u,b) be a strong solution to System 1.6 - 1.8 with $nu = 0$ and $\mu > 0$. Let $\psi(\xi) = 1 - e^{-|\xi|^2}$. Under one of the following two conditions*

(i) $u \in L^\infty(0,\infty; W^{1-\beta,\frac{3}{\beta}}(\mathbb{R}^3))$ *with* $\beta \in [1/2, 1]$;

(ii) $b \in L^\infty(0,\infty; W^{1-\beta,\infty}(\mathbb{R}^3))$ *and* $u \in L^\infty(0,\infty; W^{1-\beta,\frac{3}{\beta}}(\mathbb{R}^3))$ *with* $\beta \in (0,1]$,

it holds that

$$\lim_{t\to 0}\|\psi\hat{b}(t)\|_2 = 0.$$

Proof: We start with estimating the last three integrals on the right hand side of the generalized energy inequality (3.4), recalled here,

$$\begin{aligned}
E(t)\|\psi\hat{b}(t)\|_2^2 \leq & E(s)\|\psi\hat{b}(s)\|_2^2 - 2\mu\int_s^t E(\tau)\left\||\xi|^\beta\psi\hat{b}(\tau)\right\|_2^2\mathrm{d}\tau + \int_s^t E'(\tau)\|\psi\hat{b}(\tau)\|_2^2\mathrm{d}\tau \\
& + 2\int_s^t E(\tau)\left\langle\widehat{b\cdot\nabla u}(\tau),\psi^2\hat{b}(\tau)\right\rangle\mathrm{d}\tau - 2\int_s^t E(\tau)\left\langle\widehat{u\cdot\nabla b}(\tau),\psi^2\hat{b}(\tau)\right\rangle\mathrm{d}\tau \\
& - 2d_i\int_s^t E(\tau)\left\langle\mathscr{F}\left(\nabla\times(\nabla\cdot(b\times b))\right)(\tau),\psi^2\hat{b}(\tau)\right\rangle\mathrm{d}\tau \\
:= & J_0 + J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

In order to estimate J_3 where no cancellation presents, we need the additional assumptions on u and b. First, we have by using Hölder's inequality and Plancherel's theorem

$$\begin{aligned}
& \int_s^t E(\tau)\left|\left\langle\widehat{b\cdot\nabla u}(\tau),\psi^2\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau \\
= & \int_s^t E(\tau)\left|\left\langle|\xi|^{-\beta}\xi\cdot\widehat{b\otimes u}(\tau),\psi^2|\xi|^\beta\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau \\
\leq & \int_s^t E(\tau)\left\||\xi|^{1-\beta}\widehat{b\otimes u}(\tau)\right\|_2\left\|\psi^2|\xi|^\beta\hat{b}(\tau)\right\|_2\mathrm{d}\tau \\
\lesssim & \int_s^t E(\tau)\left(\|u\nabla^{1-\beta}b(\tau)\|_2 + \|b\nabla^{1-\beta}u(\tau)\|_2\right)\|\nabla^\beta b(\tau)\|_2\mathrm{d}\tau.
\end{aligned}$$

If $u \in W^{1-\beta,\frac{3}{\beta}}$, it follows from Hölder's inequality and Sobolev embedding that

$$\|b\nabla^{1-\beta}u(\tau)\|_2 \leq \|b\|_{\frac{6}{3-2\beta}}\|\nabla^{1-\beta}u\|_{\frac{3}{\beta}} \lesssim \|\nabla^\beta b\|_2\|\nabla^{1-\beta}u\|_{\frac{3}{\beta}}.$$

Under condition (i), it follows from Hölder's inequality and the Sobolev inequality that

$$\|u\nabla^{1-\beta}b(\tau)\|_2 \le \|u\|_{\frac{3}{2\beta-1}}\|\nabla^{1-\beta}b\|_{\frac{6}{5-4\beta}} \lesssim \|\nabla^{1-\beta}u\|_{\frac{3}{\beta}}\|\nabla^{\beta}b\|_2.$$

Meanwhile, under condition (ii) $\|u\nabla^{1-\beta}b(\tau)\|_2$ can be estimated as

$$\|u\nabla^{1-\beta}b(\tau)\|_2 \le \|u(\tau)\|_2\|\nabla^{1-\beta}b(\tau)\|_\infty.$$

Therefore, assuming condition (i), we have

$$\begin{aligned}&\int_s^t E(\tau)\left\langle \widehat{b\cdot\nabla u}(\tau), \psi^2\hat{b}(\tau)\right\rangle \mathrm{d}\tau\\ \lesssim &\int_s^t E(\tau)\|\nabla^{1-\beta}u\|_{\frac{3}{\beta}}\|\nabla^{\beta}b(\tau)\|_2^2\mathrm{d}\tau \lesssim \int_s^t E(\tau)\|\nabla^{\beta}b(\tau)\|_2^2\mathrm{d}\tau;\end{aligned} \tag{3.7}$$

whereas assuming condition (ii), we have

$$\begin{aligned}&\int_s^t E(\tau)\left\langle \widehat{b\cdot\nabla u}(\tau), \psi^2\hat{b}(\tau)\right\rangle \mathrm{d}\tau\\ \lesssim &\int_s^t E(\tau)\left(\|\nabla^{\beta}b\|_2\|\nabla^{1-\beta}u\|_{\frac{3}{\beta}} + \|u(\tau)\|_2\|\nabla^{1-\beta}b(\tau)\|_\infty\right)\|\nabla^{\beta}b(\tau)\|_2\mathrm{d}\tau\\ \lesssim &\Big(\int_s^t E^2(\tau)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_s^t \|\nabla^{\beta}b(\tau)\|_2^2\mathrm{d}\tau\Big)^{\frac{1}{2}} + \int_s^t E(\tau)\|\nabla^{\beta}b(\tau)\|_2^2\mathrm{d}\tau.\end{aligned} \tag{3.8}$$

To deal with J_4, observing the cancellation $\langle u\cdot\nabla b, b\rangle = 0$, we have

$$\begin{aligned}\int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla b}(\tau), \psi^2\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau &= \int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla b}(\tau), (\psi^2-1)\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau\\ &= \int_s^t E(\tau)\left|\langle \widehat{u\otimes b}(\tau), (\psi^2-1)\widehat{\nabla b}(\tau)\rangle\right|\mathrm{d}\tau\\ &\le \int_s^t E(\tau)\left\|\widehat{u\otimes b}(\tau)\right\|_{\frac{3}{\beta}}\left\|\widehat{\nabla^\beta b}(\tau)\right\|_2\left\||\xi|^{1-\beta}(\psi^2-1)\right\|_{\frac{6}{3-2\beta}}\mathrm{d}\tau\end{aligned}$$

Noticing that $\psi^2-1=-2e^{-|\xi|^2}+e^{-2|\xi|^2}$ and $\||\xi|^{1-\beta}(\psi^2-1)\|_p$ is finite for any $p>1$ and $\beta\le 1$, we continue the estimate with the help of Sobolev inequaity as

$$\begin{aligned}\int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla b}(\tau), \psi^2\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau &\lesssim \int_s^t E(\tau)\|u(\tau)\|_2\|b(\tau)\|_{\frac{6}{3-2\beta}}\|\nabla^\beta b(\tau)\|_2\mathrm{d}\tau\\ &\lesssim \int_s^t E(\tau)\|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau.\end{aligned}$$

Utilizing the cancellation $\langle\nabla\times((\nabla\times b)\times b), b\rangle = 0$ and the fact that $\left\||\xi|^{2-\beta}(\psi^2-1)\right\|_p$ is finite for $p>1$, we have the following estimate for J_5.

$$\begin{aligned}&d_i\int_s^t E(\tau)\left|\left\langle \mathscr{F}\left(\nabla\times(\nabla\cdot(b\otimes b))\right)(\tau), \psi^2\hat{b}(\tau)\right\rangle\right|\mathrm{d}\tau\\ =&d_i\int_s^t E(\tau)\left|\left\langle \mathscr{F}\left(\nabla\times(\nabla\cdot(b\otimes b))\right)(\tau), (\psi^2-1)\hat{b}\right\rangle(\tau)\right|\mathrm{d}\tau\\ \le&d_i\int_s^t E(\tau)\left|\left\langle (\psi^2-1)|\xi|^{-\beta}\mathscr{F}\left(\nabla\times(\nabla\cdot(b\otimes b))\right)(\tau), |\xi|^\beta\hat{b}\right\rangle(\tau)\right|\mathrm{d}\tau\\ \lesssim&\int_s^t E(\tau)\left\||\xi|^{2-\beta}(\psi^2-1)\right\|_{\frac{6}{3-2\beta}}\left\|\widehat{(b\otimes b)}(\tau)\right\|_{\frac{3}{\beta}}\left\|\widehat{\nabla^\beta b}(\tau)\right\|_2\mathrm{d}\tau\\ \lesssim&\int_s^t E(\tau)\|b(\tau)\|_2\|b(\tau)\|_{\frac{6}{3-2\beta}}\|\nabla^\beta b(\tau)\|_2\mathrm{d}\tau\\ \lesssim&\int_s^t E(\tau)\|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau.\end{aligned}$$

Under condition (i), combining (3.4), (3.7) and the last three inequalities yields

$$\begin{aligned}\|\psi\hat{b}(t)\|_2^2 \lesssim & \frac{E(s)}{E(t)}\|\psi\hat{b}(s)\|_2^2 - 2\mu\int_s^t \frac{E(\tau)}{E(t)}\left\||\xi|^\beta\psi\hat{b}(\tau)\right\|_2^2\mathrm{d}\tau \\ & + \int_s^t \frac{E'(\tau)}{E(t)}\|\psi\hat{b}(\tau)\|_2^2\mathrm{d}\tau + \int_s^t \frac{E(\tau)}{E(t)}\|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau;\end{aligned}$$

Under condition (ii), combining (3.4), (3.8) and the estimates for J_3, J_4 and J_5 yields

$$\begin{aligned}\|\psi\hat{b}(t)\|_2^2 \lesssim & \frac{E(s)}{E(t)}\|\psi\hat{b}(s)\|_2^2 - 2\mu\int_s^t \frac{E(\tau)}{E(t)}\left\||\xi|^\beta\psi\hat{b}(\tau)\right\|_2^2\mathrm{d}\tau + \int_s^t \frac{E'(\tau)}{E(t)}\left\|\psi\hat{b}(\tau)\right\|_2^2\mathrm{d}\tau \\ & + \int_s^t \frac{E(\tau)}{E(t)}\|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau + \frac{1}{E(t)}\Big(\int_s^t E^2(\tau)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_s^t \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.9}$$

The above equation remains to be taken care of. We apply Fourier splitting technique to handle the second and third terms on the right hand side of (3.9). Defining the ball $B(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le G(t)\}$ with the radius $G(t)$ to be specified later, we infer

$$\begin{aligned}& -2\mu\int_s^t \frac{E(\tau)}{E(t)}\left\||\xi|^\beta\psi\hat{b}(\tau)\right\|_2^2\mathrm{d}\tau + \int_s^t \frac{E'(\tau)}{E(t)}\|\psi\hat{b}(\tau)\|_2^2\mathrm{d}\tau \\ \le & -2\mu\int_s^t \frac{E(\tau)}{E(t)}\int_{B^c(t)}\left||\xi|^\beta\psi\hat{b}(\tau)\right|^2\mathrm{d}\xi\mathrm{d}\tau + \int_s^t \frac{E'(\tau)}{E(t)}\int_{B^c(t)}|\psi\hat{b}(\tau)|^2\mathrm{d}\xi\mathrm{d}\tau \\ & + \int_s^t \frac{E'(\tau)}{E(t)}\int_{B(t)}|\psi\hat{b}(\tau)|^2\mathrm{d}\xi\mathrm{d}\tau \\ \le & \int_s^t \frac{E'(\tau) - 2\mu E(\tau)G^{2\beta}(\tau)}{E(t)}\int_{B^c(t)}|\psi\hat{b}(\tau)|^2\mathrm{d}\xi\mathrm{d}\tau + \int_s^t \frac{E'(\tau)}{E(t)}\int_{B(t)}|\psi\hat{b}(\tau)|^2\mathrm{d}\xi\mathrm{d}\tau.\end{aligned}$$

Setting

$$E(t) = e^{\varepsilon t}, \text{ and } G(t) = \left(\frac{\varepsilon}{2\mu}\right)^{\frac{1}{2\beta}},$$

which indicates that $E'(t) - 2\mu E(t) G^{2\beta}(t) = 0$, we estimate (3.9) as

$$\begin{aligned}\|\psi\hat{b}(t)\|_2^2 \leq & \frac{E(s)}{E(t)}\|\psi\hat{b}(s)\|_2^2 + \int_s^t \frac{E'(\tau)}{E(t)} \int_{B(t)} |\psi\hat{b}(\tau)|^2 \mathrm{d}\xi \mathrm{d}\tau \\ & + \int_s^t \frac{E(\tau)}{E(t)} \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau + \frac{1}{E(t)} \left(\int_s^t E^2(\tau) \mathrm{d}\tau\right)^{\frac{1}{2}} \left(\int_s^t \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\psi^2 \leq 1$, we can bound the second term with the help of Lemma 3.1.2,

$$\begin{aligned}\int_{B(t)} \left|\psi\hat{b}(\tau)\right|^2 \mathrm{d}\xi \lesssim \int_{B(t)} \left(1 + \frac{1}{|\xi|^{2\beta-1}}\right)^2 \mathrm{d}\xi \lesssim \int_{B(t)} \left(1 + \frac{1}{|\xi|^{4\beta-2}}\right) \mathrm{d}\xi \\ \lesssim \int_0^{G(t)} (1 + r^{2-4\beta}) r^2 \mathrm{d}r \lesssim \varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\psi\hat{b}(t)\|_2^2 \leq & \frac{E(s)}{E(t)}\|\psi\hat{b}(s)\|_2^2 + \frac{C}{e^{\varepsilon t}} \left(\frac{e^{2\varepsilon t}}{2\varepsilon}\right)^{\frac{1}{2}} \left(\int_s^t \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\ & + \frac{C}{E(t)} \int_s^t E(\tau) \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau + C\left(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}\right)\end{aligned}$$

with various constants C which independent of t, s and ε.

Now, we first pass the limit $t \to \infty$,

$$\begin{aligned}
\lim_{t\to\infty}\|\psi\hat{b}(t)\|_2^2 \le & \lim_{t\to\infty}\frac{E(s)}{E(t)}\|\psi\hat{b}(s)\|_2^2 + \lim_{t\to\infty}\frac{C}{\sqrt{\varepsilon}}\left(\int_s^t \|\nabla^\beta b(\tau)\|_2^2 \mathrm{d}\tau\right)^{\frac{1}{2}} \\
& + \lim_{t\to\infty}\frac{C}{E(t)}\int_s^t E(\tau)\|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau + C\left(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}\right) \\
\le & \lim_{t\to\infty} e^{\varepsilon(s-t)}\|b_0\|_2^2 + \frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\
& + C\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau + C(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}) \\
\le & C\left(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}\right) + \frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{1}{2}} + C\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau;
\end{aligned}$$

and then pass the limit $s \to \infty$,

$$\begin{aligned}
& \lim_{t\to\infty}\|\psi\hat{b}(t)\|_2^2 \\
\le & \lim_{s\to\infty}\left(C\left(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}\right) + \frac{C}{\sqrt{\varepsilon}}\left(\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau\right)^{\frac{1}{2}} + C\int_s^\infty \|\nabla^\beta b(\tau)\|_2^2\mathrm{d}\tau\right) \\
\le & C\left(\varepsilon^{\frac{3}{2\beta}} + \varepsilon^{\frac{5-4\beta}{2\beta}}\right).
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows that $\lim_{t\to\infty}\|\psi\hat{b}(t)\|_2 = 0$.

□

Proof of Theorem 3.2.3: Combining Proposition 3.2.4 and Proposition 3.2.5 yields

$$\lim_{t\to\infty}\|b(t)\|_2^2 = 0.$$

This convergence along with the basic energy law implies that $\lim_{t\to\infty}\|u(t)\|_2^2 = C$ for some constant C.

3.2.2 The viscous non-resistive case

In the viscous, non-resistive setting $\nu > 0, \mu = 0$ and $\beta > 0$, the kinetic energy $\|u(t)\|_2$ decays to zero while the magnetic energy converges to a certain constant, provided that b is bounded in $W^{1-\alpha,\infty}(\mathbb{R}^3)$.

Theorem 3.2.6. *Let $u_0 \in L^1 \cap L^2(\mathbb{R}^3)$, $b_0 \in L^2(\mathbb{R}^3)$ and (u, b) be a global strong solution to System 1.6 - 1.8 with $\mu = 0$, $\nu > 0$ and $0 < \alpha \leq 1$. Assume additionally that $b \in L^\infty(0, \infty; W^{1-\alpha,\infty}(\mathbb{R}^3))$. Then, we have*

$$\lim_{t\to\infty} \|u(t)\|_2^2 = 0, \quad \lim_{t\to\infty} \|b(t)\|_2^2 = C$$

for some absolute constant C.

We estimate the low frequency part $\|\varphi\hat{u}(t)\|_2$ and high frequency part $\|(1-\varphi)\hat{u}(t)\|_2$ separately by taking $\varphi(t,\xi) = e^{-|\xi|^{2\alpha}t}$. The following proposition concerns the decay of the low frequency part.

Proposition 3.2.7. *Let $(u_0, b_0) \in \left(L^2(\mathbb{R}^3)\right)^2$ and (u, b) be a strong solution to System 1.6 - 1.8 with $\nu > 0$ and $\mu = 0$. Let $\varphi = e^{-|\xi|^{2\alpha}t}$. Then, it holds that*

$$\lim_{t\to\infty} \|\varphi(t)\hat{u}(t)\|_2 = 0.$$

Proof: The generalized energy inequality (3.5) implies

$$
\begin{aligned}
\left\|\mathscr{F}^{-1}\varphi * u(t)\right\|_2^2 \leq & \left\|e^{\nu(t-s)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * u(t)\right\|_2^2 \\
& + 2\int_s^t \left|\left\langle u\cdot\nabla u(\tau), e^{2\nu(\tau-s)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * u(s)\right\rangle\right| \mathrm{d}\tau \\
& + 2\int_s^t \left|\left\langle b\cdot\nabla b(\tau), e^{2\nu(\tau-s)(-\Delta)^\alpha}\mathscr{F}^{-1}\varphi * \mathscr{F}^{-1}\varphi * u(s)\right\rangle\right| \mathrm{d}\tau \\
:= & I + II + III.
\end{aligned}
$$

It is clear that

$$
\limsup_{t\to\infty} I = \limsup_{t\to\infty} \|e^{-\nu|\xi|^{2\alpha}(t-s)}\varphi(s)\hat{u}(s)\|_2^2 = 0.
$$

As $\alpha \in [0,1]$, the fact that φ^2 is a rapidly decreasing function of $|\xi|$ along with Hölder's inequality leads to

$$
\begin{aligned}
& \left|\left\langle u\cdot\nabla u(\tau), e^{2\nu(-\Delta)^\alpha(\tau-s)}\mathscr{F}^{-1}(\varphi^2) * u(s)\right\rangle\right| \\
\leq & \left|\left\langle |\xi|\widehat{(u\otimes u)}(\tau), e^{-2\nu|\xi|^{2\alpha}(\tau-s)}\varphi^2\hat{u}(s)\right\rangle\right| \\
\leq & \left|\left\langle \widehat{(u\otimes u)}(\tau), |\xi|^{1-\alpha}e^{-2\nu|\xi|^{2\alpha}(\tau-s)}\varphi^2|\xi|^\alpha\hat{u}(s)\right\rangle\right| \\
\leq & \left\|\widehat{(u\otimes u)}(\tau)\right\|_{\frac{3}{\alpha}} \left\||\xi|^{1-\alpha}\varphi^2\right\|_{\frac{6}{3-2\alpha}} \left\||\xi|^\alpha e^{-2\nu|\xi|^{2\alpha}(\tau-s)}\hat{u}(s)\right\|_2.
\end{aligned}
$$

By Hausdorff-Young inequality, the boundedness of $\|u(t)\|_2^2$ and Sobolev inequality, we have

$$
\begin{aligned}
\left|\left\langle u\cdot\nabla u(\tau), e^{2\nu(-\Delta)^\alpha(\tau-s)}\mathscr{F}^{-1}(\varphi^2) * u(s)\right\rangle\right| \lesssim & \|(u\otimes u)(\tau)\|_{\frac{3}{3-\alpha}} \||\xi|^\alpha\hat{u}(\tau)\|_2 \\
\lesssim & \|u(\tau)\|_2\|u(\tau)\|_{\frac{6}{3-2\alpha}}\|\nabla^\alpha u(\tau)\|_2 \\
\lesssim & \|u(\tau)\|_2\|\nabla^\alpha u(\tau)\|_2^2 \lesssim \|\nabla^\alpha u(\tau)\|_2^2.
\end{aligned}
$$

Applying a similar strategy as above, we have

$$\begin{aligned}\left|\left\langle b\cdot\nabla b(\tau), e^{2\nu(-\Delta)^\alpha(\tau-s)}\mathscr{F}^{-1}(\varphi^2)*u(s)\right\rangle\right| \le& \left|\left\langle |\xi|^{1-\alpha}\varphi^2\widehat{(b\otimes b)}(\tau), e^{-2\nu|\xi|^{2\alpha}(\tau-s)}|\xi|^\alpha\hat{u}(s)\right\rangle\right|\\ \le& \left\|\widehat{(b\otimes b)}(\tau)\right\|_\infty \left\||\xi|\varphi^2(\tau)\right\|_2 \|e^{-2\nu|\xi|^{2\alpha}(\tau-s)}\hat{u}(s)\|_2\\ \lesssim& \|(b\otimes b)(\tau)\|_1 \left\||\xi|\varphi^2(\tau)\right\|_2 \|\hat{u}(\tau)\|_2\\ \lesssim& \|b(\tau)\|_2^2 \left\||\xi|\varphi^2(\tau)\right\|_2 \|u(\tau)\|_2 \lesssim \||\xi|\varphi^2(\tau)\|_2.\end{aligned}$$

Combining the last two inequalities yields

$$II+III \lesssim \int_s^t \|\nabla^\alpha u(\tau)\|_2^2 \mathrm{d}\tau + \int_s^t \left\||\xi|\varphi^2(\tau)\right\|_2 \mathrm{d}\tau.$$

Straightforward computation shows that

$$\begin{aligned}\|\xi\varphi^2(\tau)\|_2^2 =& \int_{\mathbb{R}^3} |\xi|^2 e^{-4|\xi|^{2\alpha}\tau}\mathrm{d}\xi \lesssim \int_0^\infty r^4 e^{-4r^{2\alpha}\tau}\mathrm{d}r\\ =&\tau^{-\frac{5}{2\alpha}}\int_0^\infty w^4 e^{-4w^{2\alpha}}\,\mathrm{d}w \lesssim \tau^{-\frac{5}{2\alpha}}.\end{aligned}$$

which, together with the fact that $\nabla^\alpha u \in L^\infty(0,\infty; L^2(\mathbb{R}^3))$ and the basic energy law, yields

$$\lim_{t\to\infty}(II+III) \le \lim_{s\to\infty}\lim_{t\to\infty}\int_s^t \left(\|\nabla^\alpha u(\tau)\|_2^2 + \tau^{-\frac{5}{4\alpha}}\right)\mathrm{d}\tau = 0, \quad \text{for } 0<\alpha\le 1.$$

Therefore, we conclude

$$\lim_{t\to\infty}\|\varphi(t)\hat{u}(t)\|_2 = \lim_{t\to\infty}\left\|\mathscr{F}^{-1}\varphi(t)*u(t)\right\|_2 = 0.$$

$\square$

The decay of high frequency part is given by the following proposition.

Proposition 3.2.8. *Let $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $b_0 \in L^2(\mathbb{R}^3)$ and (u, b) be a strong solution to System 1.6 - 1.8 with $\mu = 0$ and $\nu > 0$. Assume that $b \in L^\infty(0, \infty; W^{1-\alpha,\infty}(\mathbb{R}^3))$. Then, for $\psi = 1 - e^{-|\xi|^{2\alpha}t}$ it holds that*

$$\lim_{t\to 0} \|\psi(t)\hat{u}(t)\|_2 = 0.$$

Proof: We start with estimating the last three integrals on the right hand side of the generalized energy inequality (3.6), recalled here,

$$\begin{aligned}
E(t)\|\psi(t)\hat{u}(t)\|_2^2 \leq & E(s)\,\|\psi(s)\hat{u}(s)\|_2^2 - 2\nu \int_s^t E(\tau)\||\xi|^\alpha\psi(\tau)\hat{u}(\tau)\|_2^2 \mathrm{d}\tau + \int_s^t E'(\tau)\|\psi(\tau)\hat{u}(\tau)\|_2^2 \mathrm{d}\tau \\
& - 2\nu \int_s^t E(\tau)\left\langle \psi'(\tau)\hat{u}(\tau), \psi(\tau)\widehat{u}(\tau)\right\rangle \mathrm{d}\tau + 2\int_s^t E(\tau)\left\langle \widehat{b\cdot\nabla b}(\tau), \psi^2(\tau)\hat{u}(\tau)\right\rangle \mathrm{d}\tau \\
& - 2\int_s^t E(\tau)\left\langle \widehat{u\cdot\nabla u}(\tau), \psi^2(\tau)\hat{u}(\tau)\right\rangle \mathrm{d}\tau \\
:= & J_0 + J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Recalling that $\psi = 1 - e^{-|\xi|^{2\alpha}t}$ and $\psi' = |\xi|^{2\alpha}\varphi$, J_3 can be estimated as

$$\begin{aligned}
\int_s^t E(\tau)\left|\langle \psi'(\tau)\hat{u}(\tau), \psi(\tau)\widehat{u}(\tau)\rangle\right| \mathrm{d}\tau &= \int_s^t E(\tau)\left|\langle |\xi|^{2\alpha}\varphi(\tau)\hat{u}(\tau), \psi(\tau)\hat{u}(\tau)\rangle\right| \mathrm{d}\tau \\
&\leq \int_s^t E(\tau)\|\nabla^\alpha u(\tau)\|_2^2 \mathrm{d}\tau.
\end{aligned} \tag{3.10}$$

In order to estimate J_4 where no cancellation is present, we need the additional assumption $b \in L^\infty(0,\infty; W^{1-\alpha,\infty})$. Using Hölder's inequality and Plancherel's theorem, we have

$$\begin{aligned}
\int_s^t E(\tau)\left|\left\langle \widehat{b\cdot\nabla b}(\tau), \psi^2\hat{u}(\tau)\right\rangle\right| \mathrm{d}\tau &\le \int_s^t E(\tau)\left|\left\langle |\xi|^{-\alpha}|\xi|\cdot\widehat{b\otimes b}(\tau), \psi^2|\xi|^\alpha\hat{u}(\tau)\right\rangle\right| \mathrm{d}\tau \\
&\le \int_s^t E(\tau)\left\||\xi|^{1-\alpha}\widehat{b\otimes b}(\tau)\right\|_2 \left\|\psi^2\xi^\alpha\hat{u}(\tau)\right\|_2 \mathrm{d}\tau \\
&\lesssim \int_s^t E(\tau)\|b\nabla^{1-\alpha}b(\tau)\|_2\|\nabla^\alpha u(\tau)\|_2 \mathrm{d}\tau \\
&\lesssim \int_s^t E(\tau)\|b\|_2\|\nabla^{1-\alpha}b(\tau)\|_\infty\|\nabla^\alpha u(\tau)\|_2 \mathrm{d}\tau \\
&\lesssim \int_s^t E(\tau)\|\nabla^\alpha u(\tau)\|_2 \mathrm{d}\tau.
\end{aligned} \tag{3.11}$$

To deal with J_5, we observe the cancellation $\langle u\cdot\nabla u, u\rangle = 0$ and obtain

$$\begin{aligned}
\int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla u}(\tau), \psi^2\hat{u}(\tau)\right\rangle\right| \mathrm{d}\tau &= \int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla u}(\tau), (\psi^2-1)\hat{u}(\tau)\right\rangle\right| \mathrm{d}\tau \\
&= \int_s^t E(\tau)\left|\langle \widehat{u\otimes u}(\tau), (\psi^2-1)\widehat{\nabla u}(\tau)\rangle\right| \mathrm{d}\tau \\
&\le \int_s^t E(\tau)\left\|\widehat{u\otimes u}(\tau)\right\|_{\frac{3}{\alpha}} \left\|\widehat{\nabla^\alpha u}(\tau)\right\|_2 \left\||\xi|^{1-\alpha}(\psi^2-1)\right\|_{\frac{6}{3-2\alpha}} \mathrm{d}\tau
\end{aligned}$$

Noticing that $\psi^2 - 1 = -2e^{-|\xi|^{2\alpha}t} + e^{-2|\xi|^{2\alpha}t}$ and $\||\xi|^{1-\alpha}(\psi^2-1)\|_p$ is finite for any $p > 1$ and $\alpha \le 1$, we continue the estimate with the help of Sobolev inequality as

$$\begin{aligned}
\int_s^t E(\tau)\left|\left\langle \widehat{u\cdot\nabla u}(\tau), \psi^2\hat{u}(\tau)\right\rangle\right| \mathrm{d}\tau &\lesssim \int_s^t E(\tau)\|u(\tau)\|_2\|u(\tau)\|_{\frac{6}{3-2\alpha}}\|\nabla^\alpha u(\tau)\|_2 \mathrm{d}\tau \\
&\lesssim \int_s^t E(\tau)\|\nabla^\alpha u(\tau)\|_2^2 \mathrm{d}\tau.
\end{aligned} \tag{3.12}$$

Combining (3.6) and (3.10) – (3.12) yields

$$
\begin{aligned}
\|\psi\hat{u}(t)\|_2^2 \leq & \frac{E(s)}{E(t)}\|\psi\hat{u}(s)\|_2^2 - 2\nu\int_s^t \frac{E(\tau)}{E(t)}\||\xi|^\alpha\psi\hat{u}(\tau)\|_2^2\mathrm{d}\tau + \int_s^t \frac{E'(\tau)}{E(t)}\|\psi\hat{u}(\tau)\|_2^2\mathrm{d}\tau \\
& + \int_s^t \frac{E(\tau)}{E(t)}\|\nabla^\alpha u(\tau)\|_2^2\mathrm{d}\tau + \frac{1}{E(t)}\Big(\int_s^t E^2(\tau)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_s^t \|\nabla^\alpha u(\tau)\|_2^2\mathrm{d}\tau\Big)^{\frac{1}{2}}
\end{aligned}
$$

which has the same form as (3.9). Therefore, we can apply the same Fourier splitting strategy as that in the proof of Proposition 3.2.5 to obtain that $\lim_{t\to\infty}\|\psi\hat{u}(t)\|_2 = 0$.

□

The statement of Theorem 3.2.6 follows from the two lemmas above and the basic energy equality.

3.3 Determining wavenumbers for the Hall-MHD system

A recurring idea in this thesis is to separate the high frequency and low frequency components of the solutions. The Fourier splitting technique used in the previous section has already revealed that low frequencies play a crucial role in the temporal decay of the solutions. Naturally, we can study the long-time behaviour of the solutions within the framework of frequency localization via Littlewood-Paley theory. For the Navier-Stokes equations, (Cheskidov et al., 2018) devised a determining wavenumber, which bounds from above the low frequencies that determine the long-time behaviour of the solutions; we adapt this notion to System 1.1 - 1.3.

Definition 3.3.1. Let u_q and b_q denote the q-th dyadic blocks of u and b on $\mathbb{T}^3$, respectively. Let $\kappa := \min\{\mu, \nu, {d_i}^{-1}\mu\}$ and $\delta > 1$. Let $r \in (2, 3)$ and c_r be a constant depending only on r.

We define the determining wavenumbers corresponding to a weak solution to System 1.1 - 1.3 (u, b) as

$$\Lambda_u(t) =: \min\left\{\lambda_q : \lambda_p^{-1+\frac{3}{r}}\|u_p\|_r < c_r\kappa, \forall p > q;\ \lambda_q^{-1+\frac{3}{r}}\|u_{\leq q}\|_r < c_r\kappa, q \in \mathbb{N}\right\},$$
$$\Lambda_b(t) =: \min\left\{\lambda_q : \lambda_{p-q}^{\delta}\|b_p\|_\infty < c_r\kappa, \forall p > q;\ \|b_{\leq q}\|_\infty < c_r\kappa, q \in \mathbb{N}\right\}.$$

Given two weak solutions to System 1.1 - 1.3 (u, b) and (v, h), we define

$$\Lambda_{u,v}(t) := \max\{\Lambda_u(t), \Lambda_v(t)\} \text{ and } \Lambda_{b,h}(t) := \max\{\Lambda_b(t), \Lambda_h(t)\}.$$

The integers $Q_{u,v}(t)$ and $Q_{b,h}(t)$ shall be such that $\lambda_{Q_{u,v}(t)} = \Lambda_{u,v}(t)$ and $\lambda_{Q_{b,h}(t)} = \Lambda_{b,h}(t)$.

In the above definition, the conditions on the low frequency parts are reminiscent of those for the dissipation wavenumbers in Definition 2.1.3.

In view of the Galilean invariance of the equations, we assume throughout this section that the two weak solutions (u, b) and (v, h) are such that the velocities are of zero mean and the magnetic fields have the same mean, i.e.,

$$\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} u(t,x)\mathrm{d}x = \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} v(t,x)\mathrm{d}x = 0 \text{ and } \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} (b(t,x) - h(t,x))\,\mathrm{d}x = 0.$$

The observation that the long time behaviour of the solutions are largely governed by the low frequency parts is reinforced by the following theorem, which states that two solutions coinciding on Fourier modes lower than the determining wavenumbers share the same long time behaviour.

Theorem 3.3.2. *If two weak solutions* (u, b) *and* (v, h) *satisfy*

$$\left(u_{\leq Q_{u,v}(t)}(t),\ b_{\leq Q_{b,h}(t)}(t)\right) = \left(v_{\leq Q_{u,v}(t)}(t), h_{\leq Q_{b,h}(t)}(t)\right),\ \forall t > 0,$$

in addition to aforementioned assumptions, then

$$\lim_{t\to\infty} \left(\|u(t) - v(t)\|_{L^2} + \|b(t) - h(t)\|_{L^2}\right) = 0.$$

3.3.1 An analysis of the electron-MHD system

As a part of the proof of Theorem 3.3.2, we first focus on the Hall term and prove an analogous result for the EMHD equations. Given two weak solutions b and h to System 1.4 - 1.5, we can show that their difference $m := b - h$ formally satisfies

$$m_t - \mu\Delta m = -d_i\nabla \times ((\nabla \times m) \times h) - d_i\nabla \times ((\nabla \times b) \times m). \tag{3.13}$$

Analyzing the above equation using harmonic analysis tools yields the following theorem.

Theorem 3.3.3. *Let* $\Lambda_b(t)$, $\Lambda_h(t)$, $\Lambda_{b,h}(t)$ *and* $Q_{b,h}(t)$ *be as in Definition 3.3.1. If* b *and* h *have the same mean and*

$$b_{\leq Q_{b,h}(t)}(t) = h_{\leq Q_{b,h}(t)}(t),\ \forall t > 0,$$

then

$$\lim_{t\to\infty} \|b(t) - h(t)\|_{L^2} = 0.$$

Proof: Multiplying Equation 3.13 by $\Delta_q^2 m$, integrating by parts and summing over q lead to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\|m_q\|_2^2 + \mu\sum_{q\geq -1}\lambda_q^2\|m_q\|_2^2 =& d_i\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q((\nabla\times m)\times h)\cdot(\nabla\times m_q)\mathrm{d}x\\ &+ d_i\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q((\nabla\times b)\times m)\cdot(\nabla\times m_q)\mathrm{d}x\\ =:& I + J.\end{aligned}$$

We further apply Bony's paraproduct decomposition to I and J –

$$\begin{aligned}I =& d_i\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q((\nabla\times m_p)\times h_{\leq p-2})\cdot(\nabla\times m_q)\mathrm{d}x\\ &+ d_i\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q((\nabla\times m_{\leq p-2})\times h_p)\cdot(\nabla\times m_q)\mathrm{d}x\\ &+ d_i\sum_{q\geq -1}\sum_{p\geq q-2}\int_{\mathbb{T}^3}\Delta_q((\nabla\times \tilde{m}_p)\times h_p)\cdot(\nabla\times m_q)\mathrm{d}x\\ =:& I_1 + I_2 + I_3;\end{aligned}$$

$$
\begin{aligned}
J =& d_i \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \Delta_q(m_{\leq p-2} \times (\nabla \times b_p)) \cdot (\nabla \times m_q) \mathrm{d}x \\
&+ d_i \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \Delta_q(m_p \times (\nabla \times b_{\leq p-2})) \cdot (\nabla \times m_q) \mathrm{d}x \\
&+ d_i \sum_{q\geq -1} \sum_{p\geq q-2} \int_{\mathbb{T}^3} \Delta_q(\tilde{m}_p \times (\nabla \times b_p)) \cdot (\nabla \times m_q) \mathrm{d}x \\
=:& J_1 + J_2 + J_3.
\end{aligned}
$$

We then proceed to estimate the terms I_1, I_2, I_3 and J_1, J_2, J_3. As for I_1, we rewrite it using the commutator (1.22) and notice that I_{12} in the following expression vanishes.

$$
\begin{aligned}
I_1 =& d_i \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \big([\Delta_q, h_{\leq p-2} \times \nabla\times] m_p\big) \cdot (\nabla \times m_q) \mathrm{d}x \\
&- d_i \sum_{q\geq -1} \int_{\mathbb{T}^3} (h_{\leq q-2} \times (\nabla \times m_q)) \cdot (\nabla \times m_q) \mathrm{d}x \\
&+ d_i \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} ((h_{\leq q-2} - h_{\leq p-2}) \times (\nabla \times (m_p)_q)) \cdot (\nabla \times m_q) \mathrm{d}x \\
=:& I_{11} + I_{12} + I_{13}.
\end{aligned}
$$

Taking into account that $m_{\leq Q_{b,h}} = 0$, we split I_{11} by the wavenumber.

$$
\begin{aligned}
I_{11} =& d_i \sum_{q> Q_{b,h}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \big([\Delta_q, h_{\leq Q_{b,h}} \times \nabla\times] m_p\big) \cdot (\nabla \times m_q) \mathrm{d}x \\
&+ d_i \sum_{q> Q_{b,h}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \big([\Delta_q, h_{(Q_{b,h}, p-2]} \times \nabla\times] m_p\big) \cdot (\nabla \times m_q) \mathrm{d}x \\
=:& I_{111} + I_{112}.
\end{aligned}
$$

By Lemma 1.3.4, Hölder's inequality, Definition 3.3.1, Young's inequality, we estimate I_{111} as follows.

$$
\begin{aligned}
|I_{111}| \leq & d_i \|\nabla h_{\leq Q_{b,h}}\|_\infty \sum_{p>Q_h-2} \|m_p\|_2 \sum_{|q-p|\leq 2} \|\nabla \times m_q\|_2 \\
\lesssim & d_i \|h_{\leq Q_{b,h}}\|_\infty \sum_{p>Q_h-2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \|\nabla \times m_q\|_2 \\
\lesssim & c_r \mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

We estimate I_{112} using Lemma 1.3.4, Hölder's inequality, Definition 3.3.1, Young's and Jensen's inequalities.

$$
\begin{aligned}
|I_{112}| \leq & d_i \sum_{q>Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \|m_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'} \|h_{p'}\|_\infty \\
\leq & d_i \sum_{q>Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \|\nabla m_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \|h_{p'}\|_\infty \lambda_{p'-p} \\
\leq & c_r \mu \sum_{q>Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \|\nabla m_p\|_2 \sum_{Q_{b,h}<p'\leq q} \lambda_{q-p'}^{-1} \\
\lesssim & c_r \mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

For $p, q \in \mathbb{Z}$ satisfying $|p-q| \leq 2$, it is true that $|h_{\leq q-2} - h_{\leq p-2}| \leq \sum_{i=0}^{3} |h_{q-i}|$. Since $m_q = 0, \forall q \leq Q_{b,h}$, the following generic bound is true -

$$
|I_{13}| \lesssim d_i \sum_{q>Q_{b,h}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} |h_{q-2}| |\nabla \times (m_p)_q| |\nabla \times m_q| \mathrm{d}x.
$$

The sum is then split by the wavenumber $Q_{b,h}$.

$$\begin{aligned}
|I_{13}| \lesssim & d_i \sum_{Q_{b,h}<q\le Q_{b,h}+2} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} |h_{q-2}||\nabla\times (m_p)_q||\nabla\times m_q|\mathrm{d}x \\
& + d_i \sum_{q>Q_{b,h}+2} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} |h_{q-2}||\nabla\times (m_p)_q||\nabla\times m_q|\mathrm{d}x \\
=: & I_{131} + I_{132}.
\end{aligned}$$

I_{131} is estimated as follows.

$$\begin{aligned}
I_{131} \le & d_i \|h_{\le Q_{b,h}}\|_\infty \sum_{Q_{b,h}<q\le Q_{b,h}+2} \|\nabla\times m_q\|_2 \sum_{|p-q|\le 2} \|\nabla\times m_p\|_2 \\
\le & c_r\mu \sum_{Q_{b,h}<q\le Q_{b,h}+2} \|\nabla\times m_q\|_2 \sum_{|p-q|\le 2} \|\nabla\times m_p\|_2 \\
\lesssim & c_r\mu \sum_{Q_{b,h}-2<q\le Q_{b,h}+2} \|\nabla m_q\|_2^2.
\end{aligned}$$

I_{132} is estimated with Hölder's inequality, Definition 3.3.1 and Young's inequality.

$$\begin{aligned}
I_{132} \le & d_i \sum_{q>Q_{b,h}+2} \|h_{q-2}\|_\infty \|\nabla\times m_q\|_2 \sum_{|p-q|\le 2} \|\nabla\times m_p\|_2 \\
\le & c_r\mu \sum_{q>Q_{b,h}+2} \|\nabla\times m_q\|_2 \sum_{|p-q|\le 2} \|\nabla\times m_p\|_2 \\
\lesssim & c_r\mu \sum_{q\ge -1} \|\nabla m_q\|_2^2.
\end{aligned}$$

As $m_{\le Q_{b,h}} = 0$, it is perceivable that I_2 consists of only high frequency parts and can be written as follows.

$$I_2 = d_i \sum_{p>Q_{b,h}+2} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} \Delta_q\big(h_p \times (\nabla \times m_{(Q_{b,h},p-2]})\big) \cdot (\nabla \times m_q)\mathrm{d}x.$$

Let $\delta > 0$. Hölder's inequality, Definition 3.3.1, Young's and Jensen's inequalities lead to

$$\begin{aligned}
|I_2| \le & d_i \sum_{p>Q_{b,h}} \|h_p\|_\infty \sum_{|p-q|\le 2} \|\nabla \times m_q\|_2 \sum_{Q_{b,h}<p'\le p-2} \|\nabla \times m_{p'}\|_2 \\
\lesssim & d_i \sum_{q>Q_{b,h}-2} \|\nabla \times m_q\|_2 \sum_{|p-q|\le 2} \sum_{Q_{b,h}<p'\le p-2} \lambda^\delta_{p'-p} \|\nabla \times m_{p'}\|_2 \lambda^\delta_{p-Q_{b,h}} \|h_p\|_\infty \\
\lesssim & c_r\mu \sum_{q>Q_{b,h}-2} \|\nabla \times m_q\|_2 \sum_{|p-q|\le 2} \sum_{Q_{b,h}<p'\le p-2} \lambda^\delta_{p'-p} \|\nabla \times m_{p'}\|_2 \\
\lesssim & c_r\mu \sum_{q>Q_{b,h}-2} \|\nabla \times m_q\|_2 \sum_{Q_{b,h}<p'\le q} \|\nabla \times m_{p'}\|_2 \lambda^\delta_{p'-q} \\
\lesssim & c_r\mu \sum_{q\ge -1} \|\nabla m_q\|_2^2.
\end{aligned}$$

I_3 is split into three terms as follows.

$$\begin{aligned}
I_3 = & d_i \sum_{Q_{b,h}<q\le Q_{b,h}+2} \sum_{q-2\le p\le Q_{b,h}} \int_{\mathbb{T}^3} \Delta_q(h_p \times (\nabla \times \tilde{m}_p)) \cdot (\nabla \times m_q)\mathrm{d}x \\
& + d_i \sum_{Q_{b,h}<q\le Q_{b,h}+2} \sum_{p>Q_{b,h}} \int_{\mathbb{T}^3} \Delta_q(h_p \times (\nabla \times \tilde{m}_p)) \cdot (\nabla \times m_q)\mathrm{d}x \\
& + d_i \sum_{q>Q_{b,h}+2} \sum_{p\ge q-2} \int_{\mathbb{T}^3} \Delta_q(h_p \times (\nabla \times \tilde{m}_p)) \cdot (\nabla \times m_q)\mathrm{d}x \\
=: & I_{31} + I_{32} + I_{33}.
\end{aligned}$$

Invoking Definition 3.3.1 and applying Hölder's, Young's and Jensen's inequalities, we can estimate I_{31}, I_{32} and I_{33} as follows.

$$\begin{aligned}
|I_{31}| \leq & d_i \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{q-2\leq p\leq Q_{b,h}} \|h_p\|_\infty \|\nabla \times \tilde{m}_p\|_2 \\
\lesssim & c_r\mu \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{q-3\leq p\leq Q_{b,h}+1} \|\nabla \times m_p\|_2 \\
\lesssim & c_r\mu \sum_{Q_{b,h}-3\leq q\leq Q_{b,h}+2} \|\nabla m_q\|_2^2,
\end{aligned}$$

$$\begin{aligned}
|I_{32}| \leq & d_i \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{p>Q_{b,h}} \|h_p\|_\infty \|\nabla \times \tilde{m}_p\|_2 \\
\lesssim & c_r\mu \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{p>Q_{b,h}} \|\nabla \times m_p\|_2 \lambda_{Q_{b,h}-p}^\delta \\
\lesssim & c_r\mu \sum_{q\geq -1} \|\nabla m_q\|_2^2,
\end{aligned}$$

$$\begin{aligned}
|I_{33}| \leq & d_i \sum_{q>Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{p\geq q-2} \|h_p\|_\infty \|\nabla \times \tilde{m}_p\|_2 \\
\lesssim & c_r\mu \sum_{q>Q_{b,h}+2} \|\nabla \times m_q\|_2 \sum_{p\geq q-2} \|\nabla \times m_p\|_2 \lambda_{Q_{b,h}-p}^\delta \\
\lesssim & c_r\mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.
\end{aligned}$$

Thus, the estimation for I is completed.

J_1, J_2 and J_3 remain to be estimated. We can write J_1, whose low frequency parts vanish due to $m_{\le Q_{b,h}} = 0$, as

$$J_1 = d_i \sum_{p > Q_{b,h}+2} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} \Delta_q\big(m_{(Q_{b,h}, p-2]} \times (\nabla \times b_p)\big) \cdot (\nabla \times m_q) \mathrm{d}x.$$

Recalling Definition 3.3.1, we can estimate J_1 using Hölder's, Young's and Jensen's inequalities, provided that $\delta > 1$.

$$\begin{aligned}
|J_1| \le & d_i \sum_{p > Q_{b,h}+2} \lambda_p \|b_p\|_\infty \sum_{|p-q|\le 2} \lambda_q^{2s} \|\nabla \times m_q\|_2 \sum_{Q_{b,h} < p' \le p-2} \|m_{p'}\|_2 \\
\le & d_i \sum_{q > Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\le 2} \sum_{Q_{b,h} < p' \le p-2} \lambda_{p'}^{\delta} \|m_{p'}\|_2 \lambda_{p-Q_{b,h}}^{\delta} \|b_p\|_\infty \lambda_p^{1-\delta} \\
\le & c_r \mu \sum_{q > Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{Q_{b,h} < p' \le q} \lambda_{p'} \|m_{p'}\|_2 \lambda_{p'-q}^{\delta-1} \\
\lesssim & c_r \mu \sum_{q \ge -1} \|\nabla m_q\|_2^2.
\end{aligned}$$

J_2 can be partitioned into two terms by $Q_{b,h}$.

$$\begin{aligned}
J_2 = & d_i \sum_{q > Q_{b,h}} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} \Delta_q\big(m_p \times (\nabla \times b_{\le Q_{b,h}})\big) \cdot (\nabla \times m_q) \mathrm{d}x \\
& + d_i \sum_{q > Q_{b,h}} \sum_{|p-q|\le 2} \int_{\mathbb{T}^3} \Delta_q\big(m_p \times (\nabla \times b_{(Q_{b,h}, p-2]})\big) \cdot (\nabla \times m_q) \mathrm{d}x \\
=: & J_{21} + J_{22}.
\end{aligned}$$

To estimate J_{21}, we apply Hölder's and Young's inequalities.

$$\begin{aligned}|J_{21}| \leq & d_i \|\nabla b_{\leq Q_{b,h}}\|_\infty \sum_{q>Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \|m_p\|_2 \\ \lesssim & d_i \Lambda_{b,h} \|b_{\leq Q_{b,h}}\|_\infty \sum_{q>Q_{b,h}} \|\nabla m_q\|_2 \sum_{|p-q|\leq 2} \|m_p\|_2 \\ \lesssim & c_r \mu \sum_{q>Q_{b,h}} \|\nabla m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|m_p\|_2 \\ \lesssim & c_r \mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.\end{aligned}$$

For J_{22}, Hölder's inequality, Definition 3.3.1, Young's and Jensen's inequalities yield

$$\begin{aligned}|J_{22}| \leq & d_i \sum_{q>Q_{b,h}} \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \|m_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'} \|b_{p'}\|_\infty \\ \leq & c_r \mu \sum_{q>Q_{b,h}} \lambda_q \|\nabla \times m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|m_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p-Q_{b,h}}^{-1} \lambda_{p'-Q_{b,h}}^{1-\delta} \\ \lesssim & c_r \mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.\end{aligned}$$

Taking advantage of $m_{\leq Q_{b,h}} = 0$, we write J_3 as

$$|J_3| = d_i \sum_{q>Q_{b,h}} \sum_{p\geq q+2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q\big(\tilde{m}_p \times (\nabla \times b_p)\big) \cdot (\nabla \times m_q) \mathrm{d}x,$$

which can then be estimated as follows.

$$\begin{aligned}|J_3| \leq & d_i \sum_{p\geq Q_{b,h}+2} \lambda_p \|b_p\|_\infty \|m_p\|_2 \sum_{q\leq p-2} \|\nabla\times m_q\|_2 \\ \leq & c_r\mu \sum_{q>Q_{b,h}} \|\nabla\times m_q\|_2 \sum_{p\geq q+2} \lambda_p^{1-\delta}\lambda_{Q_{b,h}}^\delta \|m_p\|_2 \\ \leq & c_r\mu \sum_{q>Q_{b,h}} \|\nabla\times m_q\|_2 \sum_{p\geq q+2} \lambda_p \|m_p\|_2 \lambda_{Q_{b,h}-p}^\delta \\ \lesssim & c_r\mu \sum_{q\geq -1} \|\nabla m_q\|_2^2.\end{aligned}$$

Let $c_r = 1-(2\mu)^{-1}$. Assembling all the estimates above leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\|m_q\|_2^2 \lesssim -\sum_{q\geq -1}\lambda_q^2\|m_q\|_2^2 \lesssim -\Lambda_{b,h}^2 \sum_{q>Q_{b,h}}\|m_q\|_2^2,$$

since we have assumed that

$$\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}(b(t,x)-h(t,x))\,\mathrm{d}x = 0.$$

Therefore, the desired outcome in Theorem 3.3.3 follows from Grönwall's inequality.

□

3.3.2 Analysis of the full Hall-MHD system

Let (u, b) and (v, h) be two weak solutions to System 1.1 - 1.3. Let π be the difference between the pressure terms. Straightforward calculations show that the difference $(w, m) := (u - v, b - h)$ formally satisfies the following system of equations.

$$w_t - \nu\Delta w = -(u \cdot \nabla)w - (w \cdot \nabla)v + (b \cdot \nabla)m + (m \cdot \nabla)h - \nabla\pi, \tag{3.14}$$

$$\begin{aligned} m_t - \mu\Delta m = &-(v \cdot \nabla)m - (w \cdot \nabla)b + (b \cdot \nabla)w + (m \cdot \nabla)v \\ &- d_i\nabla \times (\nabla \times m) \times h) - d_i\nabla \times ((\nabla \times b) \times m). \end{aligned} \tag{3.15}$$

Via the same strategy as that in the proof of Theorem 3.3.3, we shall eventually prove the following inequality –

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w\|_{L^2}^2 + \|m\|_{L^2}^2\right) \lesssim -\left(\|\nabla w\|_{L^2}^2 + \|\nabla m\|_{L^2}^2\right). \tag{3.16}$$

To this end, we consider System 3.14 - 3.15 localized in the frequency space. We multiply the equations by $\Delta_q^2 w$ and $\Delta_q^2 m$, respectively and integrate by parts. Summing over q, we obtain the following inequalities –

For Equation 3.14, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\|w_q\|_2^2+\nu\sum_{q\geq -1}\lambda_q^2\|w_q\|_2^2 \leq &-\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(u\cdot\nabla w)\cdot w_q\mathrm{d}x-\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(w\cdot\nabla v)\cdot w_q\mathrm{d}x\\ &+\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(b\cdot\nabla m)\cdot w_q\mathrm{d}x+\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(m\cdot\nabla h)\cdot w_q\mathrm{d}x\\ =&:A+B+C+D,\end{aligned}$$

and for Equation 3.15,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{q\geq -1}\|m_q\|_2^2+\mu\sum_{q\geq -1}\lambda_q^2\|m_q\|_2^2\\ \leq&-\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(v\cdot\nabla m)\cdot m_q\mathrm{d}x-\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(w\cdot\nabla b)\cdot m_q\mathrm{d}x\\ &+\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(b\cdot\nabla w)\cdot m_q\mathrm{d}x+\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q(m\cdot\nabla v)\cdot m_q\mathrm{d}x\\ &-d_i\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q((\nabla\times m)\times h)\cdot(\nabla\times m_q)\mathrm{d}x\\ &-d_i\sum_{q\geq -1}\int_{\mathbb{T}^3}\Delta_q((\nabla\times b)\times m)\cdot(\nabla\times m_q)\mathrm{d}x\\ =&:E+F+G+H+I+J.\end{aligned}$$

Since the estimates for I and J are as those in the proof of Theorem 3.3.3, our tasks are then to control the remaining terms $A, B, ..., H$.

Proposition 3.3.4. *For the term A, it holds that*

$$|A|\lesssim c_r\nu\sum_{q\geq -1}\|\nabla w_q\|_2^2.$$

Proof: Bony's paraproduct decomposition leads to the following -

$$
\begin{aligned}
A=&-\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q(u_{\leq p-2}\cdot\nabla w_p)\cdot w_q\mathrm{d}x\\
&-\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q(u_p\cdot\nabla w_{\leq p-2})\cdot w_q\mathrm{d}x\\
&-\sum_{q\geq -1}\sum_{p\geq q-2}\int_{\mathbb{T}^3}\Delta_q(u_p\cdot\nabla \tilde{w}_p)\cdot w_q\mathrm{d}x\\
=:&A_1+A_2+A_3.
\end{aligned}
$$

According to Definition 3.3.1, we then separate the low and high modes of A_1.

$$
\begin{aligned}
|A_1|\leq&\sum_{p>Q_{u,v}}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{T}^3}|\Delta_q(u_{\leq p-2}\cdot\nabla w_p)\cdot w_q|\mathrm{d}x\\
\leq&\sum_{p>Q_{u,v}}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{T}^3}|\Delta_q(u_{\leq Q_{u,v}}\cdot\nabla w_p)\cdot w_q|\mathrm{d}x\\
&+\sum_{p'>Q_{u,v}}\sum_{p\geq p'+2}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{T}^3}|\Delta_q(u_{p'}\cdot\nabla w_p)\cdot w_q|\mathrm{d}x\\
=:&A_{11}+A_{12}.
\end{aligned}
$$

To control the low frequency parts, we use Definition 3.3.1, Lemma 1.3.1, Hölder's and Young's inequalities.

$$
\begin{aligned}
A_{11}\lesssim&\|u_{\leq Q_{u,v}}\|_r\sum_{p>Q_{u,v}}\lambda_p\|w_p\|_2\sum_{|q-p|\leq 2}\|w_q\|_{\frac{2r}{r-2}}\\
\lesssim&c_r\nu\sum_{p>Q_{u,v}}\lambda_p\|w_p\|_2\sum_{|q-p|\leq 2}\lambda_{Q_{u,v}}^{1-\frac{3}{r}}\lambda_q^{\frac{3}{r}}\|w_q\|_2\\
\lesssim&c_r\nu\sum_{q\geq -1}\|\nabla w_q\|_2^2.
\end{aligned}
$$

The high modes are estimated as follows.

$$
\begin{aligned}
A_{12} \lesssim & \sum_{p'>Q_{u,v}} \|u_{p'}\|_r \sum_{p>p'+2} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \|w_q\|_{\frac{2r}{r-2}} \\
\lesssim & c_r \nu \sum_{p'>Q_{u,v}} \lambda_{p'}^{1-\frac{3}{r}} \sum_{p>p'+2} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q^{\frac{3}{r}} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{p'>Q_{u,v}} \sum_{p>p'+2} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q \|w_q\|_2 \lambda_{p'-q}^{1-\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\ge -1} \|\nabla w_q\|_2^2.
\end{aligned}
$$

It follows from the condition $w_{\le Q_{u,v}} = 0$ that

$$
A_2 = -\sum_{p>Q_{u,v}+2} \sum_{|q-p|\le 2} \int_{\mathbb{T}^3} \Delta_q (u_p \cdot \nabla w_{\le p-2}) \cdot w_q \mathrm{d}x.
$$

Recalling Definition 3.3.1, we then estimate A_2 using Hölder's and Young's inequalities.

$$
\begin{aligned}
|A_2| \le & \sum_{p>Q_{u,v}+2} \|u_p\|_r \sum_{Q_{u,v}<p'\le p+2} \lambda_{p'} \|w_{p'}\|_{\frac{2r}{r-2}} \sum_{|q-p|\le 2} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+2} \lambda_p^{1-\frac{3}{r}} \sum_{Q_{u,v}<p'\le p+2} \lambda_{p'}^{1+\frac{3}{r}} \|w_{p'}\|_2 \sum_{|q-p|\le 2} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{q>Q_{u,v}} \lambda_q \|w_q\|_2 \sum_{Q_{u,v}<p'\le q} \lambda_{p'} \|w_{p'}\|_2 \lambda_{p'-q}^{\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\ge -1} \|\nabla w_q\|_2^2.
\end{aligned}
$$

Separating the low and high modes of A_3 with the wavenumber $Q_{u,v}$ results in

$$\begin{aligned}A_3 =& -\sum_{p=Q_{u,v}}\sum_{q\le p+2}\lambda_q^{2s}\int_{\mathbb{T}^3}\Delta_q(u_p\cdot\nabla\tilde{w}_p)\cdot w_q\mathrm{d}x\\&-\sum_{p>Q_{u,v}}\sum_{q\le p+2}\lambda_q^{2s}\int_{\mathbb{T}^3}\Delta_q(u_p\cdot\nabla\tilde{w}_p)\cdot w_q\mathrm{d}x\\=:&A_{31}+A_{32}.\end{aligned}$$

We have no difficulty in controlling the low modes, which are rather meager.

$$\begin{aligned}|A_{31}| \lesssim&\Lambda_{u,v}\|u_{Q_{u,v}}\|_r\|w_{Q_{u,v}+1}\|_2\sum_{Q_{u,v}<q\le Q_{u,v}+2}\|w_q\|_{\frac{2r}{r-2}}\\\lesssim&c_r\nu\Lambda_{u,v}^{2-\frac{3}{r}}\|w_{Q_{u,v}}\|_2\sum_{Q_{u,v}<q\le Q_{u,v}+2}\lambda_q^{\frac{3}{r}}\|w_q\|_2\\\lesssim&c_r\nu\sum_{Q_{u,v}<q\le Q_{u,v}+2}\|\nabla w_q\|_2^2.\end{aligned}$$

Let $r<3$. The high modes are estimated using Hölder's, Young's and Jensen's inequalities.

$$\begin{aligned}|A_{32}| \le&\sum_{p>Q_{u,v}}\|u_p\|_r\|\nabla\tilde{w}_p\|_2\sum_{q\le p+2}\|w_q\|_{\frac{2r}{r-2}}\\\lesssim&c_r\nu\sum_{p>Q_{u,v}}\lambda_p^{2-\frac{3}{r}}\|w_p\|_2\sum_{q\le p+2}\lambda_q^{\frac{3}{r}}\|w_q\|_2\\\lesssim&c_r\nu\sum_{p>Q_{u,v}}\lambda_p\|w_p\|_2\sum_{q\le p+2}\lambda_q\|w_q\|_2\lambda_{q-p}^{-1+\frac{3}{r}}\\\lesssim&c_r\nu\sum_{q\ge -1}\|\nabla w_q\|_2^2.\end{aligned}$$

□

Proposition 3.3.5. *For the term B, it holds that*

$$|B| \lesssim c_r \nu \sum_{q\geq -1} \|\nabla w_q\|_2^2.$$

Proof: As a result of Bony's paraproduct decomposition

$$\begin{aligned}
B =& -\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3} \mathit{\Delta}_q(w_{\leq p-2}\cdot\nabla v_p)\cdot w_q \mathrm{d}x \\
& -\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3} \mathit{\Delta}_q(w_p\cdot\nabla v_{\leq p-2})\cdot w_q \mathrm{d}x \\
& -\sum_{q\geq -1}\sum_{p\geq q-2}\int_{\mathbb{T}^3} \mathit{\Delta}_q(w_p\cdot\nabla \tilde{v}_p)\cdot w_q \mathrm{d}x \\
=:& B_1 + B_2 + B_3.
\end{aligned}$$

Since $w_{\leq Q_{u,v}} = 0$, B_1 consists of only high frequencies.

$$B_1 = -\sum_{p>Q_{u,v}+2}\sum_{|q-p|\leq 2}\lambda_q^{2s}\int_{\mathbb{T}^3} \mathit{\Delta}_q(w_{\leq p-2}\cdot\nabla v_p)\cdot w_q \mathrm{d}x.$$

Let $1-\frac{3}{r}<0$. We can estimate B_1 using Definition 3.3.1, Hölder's, Young's and Jensen's inequalities.

$$\begin{aligned}
|B_1| \lesssim & \sum_{p>Q_{u,v}+2} \lambda_p \|v_p\|_r \sum_{Q_{u,v}<p'\le p-2} \|w_{p'}\|_{\frac{2r}{r-2}} \sum_{|q-p|\le 2} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+2} \lambda_p^{2-\frac{3}{r}} \sum_{Q_{u,v}<p'\le p-2} \lambda_{p'}^{\frac{3}{r}} \|w_{p'}\|_2 \sum_{|q-p|\le 2} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+2} \lambda_p^{2-\frac{3}{r}} \|w_p\|_2 \sum_{Q_{u,v}<p'\le p-2} \lambda_{p'}^{\frac{3}{r}} \|w_{p'}\|_2 \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+2} \lambda_p \|w_p\|_2 \sum_{Q_{u,v}<p'\le p-2} \lambda_{p'} \|w_{p'}\|_2 \lambda_{p-p'}^{1-\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\ge -1} \|\nabla w_q\|_2^2.
\end{aligned}$$

Splitting B_2 with the wavenumber $Q_{u,v}$ results in

$$\begin{aligned}
B_2 = & -\sum_{Q_{u,v}<p\le Q_{u,v}+2} \sum_{|q-p|\le 2} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla v_{\le p-2}) \cdot w_q \mathrm{d}x \\
& - \sum_{p>Q_{u,v}+2} \sum_{|q-p|\le 2} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla v_{\le Q_{u,v}}) \cdot w_q \mathrm{d}x \\
& - \sum_{p>Q_{u,v}+2} \sum_{|q-p|\le 2} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla v_{(Q_{u,v},p-2]}) \cdot w_q \mathrm{d}x \\
=: & B_{21} + B_{22} + B_{23}.
\end{aligned}$$

Let $r > 2$. The estimate for the low modes $|B_{21}| + |B_{22}|$ are as follows.

$$\begin{aligned}
|B_{21}| + |B_{22}| \lesssim & \sum_{p>Q_{u,v}} \|w_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\le 2} \|w_q\|_2 \sum_{p'<Q_{u,v}} \lambda_{p'} \|v_{p'}\|_r \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}} \lambda_p^{\frac{3}{r}} \|w_p\|_2 \sum_{|q-p|\le 2} \|w_q\|_2 \Lambda_{u,v}^{2-\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q \|w_q\|_2 \lambda_{Q_{u,v}-p}^{2-\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\ge -1} \|\nabla w_q\|_2^2.
\end{aligned}$$

The estimate for B_{23} follows from Definition 3.3.1 and Hölder's inequality.

$$\begin{aligned}
|B_{23}| \le & \sum_{p>Q_{u,v}+2} \|w_p\|_2 \sum_{Q_{u,v}<p'\le p-2} \|v_{p'}\|_r \sum_{|q-p|\le 2} \|w_q\|_{\frac{2r}{r-2}} \\
\le & c_r \nu \sum_{p>Q_{u,v}+2} \|w_p\|_2 \sum_{Q_{u,v}<p'\le p-2} \lambda_{p'}^{2-\frac{3}{r}} \sum_{|q-p|\le 2} \lambda_q^{\frac{3}{r}} \|w_q\|_2 \\
\le & c_r \nu \sum_{p>Q_{u,v}+2} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q \|w_q\|_2 \sum_{Q_{u,v}<p'\le p-2} \lambda_{p'-p}^{2-\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\ge -1} \|\nabla w_q\|_2^2.
\end{aligned}$$

Similar to previous terms, $|B_3|$ is bounded above by the estimates for the low modes and for the high modes.

$$
\begin{aligned}
|B_3| \leq & \sum_{p=Q_{u,v}+1} \sum_{q\leq p+2} \int_{\mathbb{T}^3} |\Delta_q(w_p \cdot \nabla v_{p-1}) \cdot w_q| \, \mathrm{d}x \\
& + \sum_{p>Q_{u,v}+1} \sum_{q\leq p+2} \int_{\mathbb{T}^3} |\Delta_q(w_p \cdot \nabla v_{p-1}) \cdot w_q| \, \mathrm{d}x \\
& =: B_{31} + B_{32}.
\end{aligned}
$$

The term B_{31}, consisting of scarce low modes, can be controlled with ease.

$$
\begin{aligned}
B_{31} \lesssim & \Lambda_{u,v} \|w_{Q_{u,v}}\|_2 \|v_{Q_{u,v}}\|_r \sum_{Q_{u,v}<q\leq Q_{u,v}+3} \|w_q\|_{\frac{2r}{r-2}} \\
\lesssim & c_r \nu \Lambda_{u,v}^{2-\frac{3}{r}} \|w_{Q_{u,v}}\|_2 \sum_{Q_{u,v}<q\leq Q_{u,v}+3} \lambda_q^{\frac{3}{r}} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{Q_{u,v}<q\leq Q_{u,v}+3} \|\nabla w_q\|_2^2.
\end{aligned}
$$

Let $-1+\frac{3}{r}>0$. We can estimate B_{32} using Hölder's, Young's and Jensen's inequalities.

$$
\begin{aligned}
B_{32} \lesssim & \sum_{p>Q_{u,v}+1} \lambda_p \|v_p\|_r \|w_p\|_2 \sum_{q\leq p+2} \|w_q\|_{\frac{2r}{r-2}} \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+1} \lambda_p^{2-\frac{3}{r}} \|w_p\|_2 \sum_{q\leq p+2} \lambda_q^{\frac{3}{r}} \|w_q\|_2 \\
\lesssim & c_r \nu \sum_{p>Q_{u,v}+1} \lambda_p \|w_p\|_2 \sum_{q\leq p+2} \lambda_q \|w_q\|_2 \lambda_{q-p}^{-1+\frac{3}{r}} \\
\lesssim & c_r \nu \sum_{q\geq -1} \|\nabla w_q\|_2^2.
\end{aligned}
$$

□

Proposition 3.3.6. *For the term C, it holds that*

$$|C| \lesssim c_r \kappa \sum_{q \geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).$$

Proof: Bony's paraproduct decomposition yields

$$\begin{aligned} C =& \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \Delta_q (b_{\leq p-2} \cdot \nabla m_p) \cdot w_q \mathrm{d}x \\ &+ \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \Delta_q (b_p \cdot \nabla m_{\leq p-2}) \cdot w_q \mathrm{d}x \\ &+ \sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{T}^3} \Delta_q (b_p \cdot \nabla \tilde{m}_p) \cdot w_q \mathrm{d}x \\ =:& C_1 + C_2 + C_3. \end{aligned}$$

Moreover, we rewrite C_1 using the commutator (1.21) as

$$\begin{aligned} C_1 =& \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} [\Delta_q, b_{\leq p-2} \cdot \nabla] m_p \cdot w_q \mathrm{d}x \\ &+ \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} (b_{\leq q-2} \cdot \nabla \Delta_q m_p) \cdot w_q \mathrm{d}x \\ &+ \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} ((b_{\leq p-2} - b_{\leq q-2}) \cdot \nabla \Delta_q m_p) \cdot w_q \mathrm{d}x \\ =:& C_{11} + C_{12} + C_{13}. \end{aligned}$$

As we shall see later, C_{12} cancels a part of the term G.

Taking into account that $m_{\leq Q_{b,h}} = 0$, we split C_{11} using the wavenumber $Q_{b,h}$.

$$
\begin{aligned}
C_{11} = & \sum_{Q_{b,h} < p \leq Q_{b,h}+2} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} [\Delta_q, b_{\leq p-2} \cdot \nabla] m_p \cdot w_q \mathrm{d}x \\
& + \sum_{p > Q_{b,h}+2} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} [\Delta_q, b_{\leq Q_{b,h}} \cdot \nabla] m_p \cdot w_q \mathrm{d}x \\
& + \sum_{p > Q_{b,h}+2} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} [\Delta_q, b_{(Q_{b,h}, p-2]} \cdot \nabla] m_p \cdot w_q \mathrm{d}x \\
=: & C_{111} + C_{112} + C_{113}.
\end{aligned}
$$

By Definition 3.3.1, Lemma 1.3.4 and Young's inequality, the following estimate holds.

$$
\begin{aligned}
|C_{111}| + |C_{112}| \leq & \|\nabla b_{\leq Q_{b,h}}\|_\infty \sum_{p > Q_{b,h}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|w_q\|_2 \\
\leq & \Lambda_{b,h} \|b_{\leq Q_{b,h}}\|_\infty \sum_{p > Q_{b,h}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|w_q\|_2 \\
\lesssim & c_r \kappa \sum_{p \geq -1} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|w_q\|_2 \\
\lesssim & c_r \kappa \sum_{q \geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

As a result of Definition 3.3.1, Lemma 1.3.4, Hölder's and Young's inequalities, the following estimate for C_{113} holds true.

$$
\begin{aligned}
|C_{113}| \leq & \sum_{p>Q_{b,h}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'} \|b_{p'}\|_\infty \\
\leq & c_r \kappa \sum_{p>Q_{b,h}} \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^2 \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{Q_{b,h}-p}^{\delta} \lambda_p^{-1} \\
\leq & c_r \kappa \sum_{p>Q_{b,h}} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{Q_{b,h}-p}^{\delta} \lambda_p^{-1} \\
\leq & c_r \kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

$|C_{13}|$ is bounded above by two terms as follows.

$$
\begin{aligned}
|C_{13}| \lesssim & \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} (|b_{q-3}| + |b_{q-2}| + |b_{q-1}| + |b_q|) |\nabla \Delta_q m_p w_q| \mathrm{d}x \\
\lesssim & \sum_{-1\leq q\leq Q_{b,h}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} |b_q| |\nabla \Delta_q m_p w_q| \mathrm{d}x \\
& + \sum_{q>Q_{b,h}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} |b_q| |\nabla \Delta_q m_p w_q| \mathrm{d}x \\
=: & C_{131} + C_{132}.
\end{aligned}
$$

We estimate C_{131} in the following fashion.

$$\begin{aligned}
C_{131} \leq & \sum_{-1\leq q\leq Q_{b,h}} \|b_q\|_\infty \|w_q\|_2 \sum_{|p-q|\leq 2} \|\nabla m_p\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \lambda_q \|w_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}$$

C_{132} enjoys the following estimate, thanks to Definition 3.3.1.

$$\begin{aligned}
C_{132} \leq & \sum_{q> Q_{b,h}} \|b_q\|_\infty \|w_q\|_2 \sum_{|p-q|\leq 2} \|\nabla m_p\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \lambda_q \|w_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}$$

Since $\nabla m_{\leq Q_{b,h}} = 0$, the low frequency part of C_2 vanishes and it can be seen that

$$C_2 = \sum_{p>Q_{b,h}+2} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} \Delta_q(b_p \cdot \nabla m_{(Q_{b,h},p-2]}) \cdot w_q \mathrm{d}x,$$

which is estimated using Hölder's, Young's and Jensen's inequalities as

$$\begin{aligned}|C_2| \leq& \sum_{p>Q_{b,h}+2} \|b_p\|_\infty \sum_{|p-q|\leq 2} \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'}\|m_{p'}\|_2\\ \lesssim& \sum_{p>Q_{b,h}+2} \|b_p\|_\infty \sum_{|p-q|\leq 2} \lambda_q\|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'}\|m_{p'}\|_2\lambda_q^{-1}\lambda_{p-Q_{b,h}}^{-\delta}\\ \lesssim& c_r\kappa \sum_{q>Q_{b,h}} \lambda_q\|w_q\|_2 \sum_{|p-q|\leq 2}\sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'}\|m_{p'}\|_2\lambda_{p-p'}^{-\delta}\\ \lesssim& c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).\end{aligned}$$

We split C_3 into low and high modes.

$$\begin{aligned}C_3 =& \sum_{p=Q_{b,h}}\sum_{q\leq p+2}\int_{\mathbb{T}^3} \varDelta_q(b_p\cdot\nabla\tilde{m}_p)\cdot w_q \mathrm{d}x\\ &+ \sum_{p>Q_{b,h}}\sum_{q\leq p+2}\int_{\mathbb{T}^3} \varDelta_q(b_p\cdot\nabla\tilde{m}_p)\cdot w_q \mathrm{d}x\\ =:& C_{31} + C_{32}.\end{aligned}$$

C_{31}, made up of the scarce low frequencies, is estimated as follows.

$$\begin{aligned}|C_{31}| \leq& \|b_{Q_{b,h}}\|_\infty \|\nabla\tilde{m}_{Q_{b,h}+1}\|_2 \sum_{q\leq p+2}\|w_q\|_2\\ \leq& c_r\kappa\lambda_{Q_{b,h}+1}\|m_{Q_{b,h}+1}\|_2 \sum_{q\leq Q_{b,h}+2}\lambda_q\|w_q\|_2\lambda_{-q}\\ \lesssim& c_r\kappa \sum_{q\geq -1}\left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).\end{aligned}$$

For C_{32}, we recall Definition 3.3.1 and apply Hölder's, Young's and Jensen's inequalities.

$$\begin{aligned}|C_{32}| \leq & \sum_{p>Q_{b,h}} \|b_p\|_\infty \|\nabla \tilde{m}_p\|_2 \sum_{q\leq p+2} \|w_q\|_2 \\ \leq & c_r\kappa \sum_{p>Q_{b,h}} \lambda_p \|m_p\|_2 \sum_{q\leq p+2} \lambda_q \|w_q\|_2 \lambda_q^{-1} \lambda_{p-Q_{b,h}}^{-\delta} \\ \leq & c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).\end{aligned}$$

□

Proposition 3.3.7. *For the term D, it holds that*

$$|D| \lesssim c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).$$

Proof: Bony's paraproduct decomposition yields

$$\begin{aligned}D = & \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q(m_p \cdot \nabla h_{\leq p-2}) \cdot w_q \mathrm{d}x \\ & + \sum_{q\geq -1} \sum_{|p-q|\leq 2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q(m_{\leq p-2} \cdot \nabla h_p) \cdot w_q \mathrm{d}x \\ & + \sum_{q\geq -1} \sum_{p\geq q-2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q(\tilde{m}_p \cdot \nabla h_p) \cdot w_q \mathrm{d}x \\ =: & D_1 + D_2 + D_3.\end{aligned}$$

Utilizing the wavenumber $Q_{b,h}$, we split D_1 into three terms.

$$\begin{aligned}
D_1 = & \sum_{Q_{b,h}<p\le Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla h_{\le Q_{b,h}})\cdot w_q \mathrm{d}x\\
& + \sum_{p>Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla h_{\le Q_{b,h}})\cdot w_q \mathrm{d}x\\
& + \sum_{p>Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla h_{(Q_{b,h},p-2]})\cdot w_q \mathrm{d}x\\
=: & D_{11}+D_{12}+D_{13}
\end{aligned}$$

We can estimate $|D_{11}|+|D_{12}|$ without difficulties.

$$\begin{aligned}
|D_{11}|+|D_{12}| \le & \|\nabla h_{\le Q_{b,h}}\|_\infty \sum_{p>Q_{b,h}}\|m_p\|_2\sum_{|q-p|\le 2}\|w_q\|_2\\
\le & c_r\kappa\sum_{p>Q_{b,h}}\lambda_p\|m_p\|_2\sum_{|q-p|\le 2}\|w_q\|_2\\
\lesssim & c_r\kappa\sum_{p>Q_{b,h}}\lambda_p\|m_p\|_2\sum_{|q-p|\le 2}\lambda_q\|w_q\|_2\\
\lesssim & c_r\kappa\sum_{q\ge -1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).
\end{aligned}$$

By Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, we have

$$
\begin{aligned}
|D_{13}| \leq & \sum_{p>Q_{b,h}+2} \|m_p\|_2 \sum_{|q-p|\leq 2} \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'} \|h_{p'}\|_\infty \\
\leq & c_r \kappa \sum_{p>Q_{b,h}+2} \lambda_p^2 \|m_p\|_2 \sum_{|q-p|\leq 2} \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'-Q_{b,h}}^{-\delta} \lambda_{p-p'}^{-1} \lambda_p^{-1} \\
\lesssim & c_r \kappa \sum_{p>Q_{b,h}+2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'-p}^2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

It turns out that D_2 consists of only high modes, as $m_{\leq Q_{b,h}} = 0$.

$$
D_2 = \sum_{p>Q_{b,h}+2} \sum_{|q-p|\leq 2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q (m_{(Q_{b,h},p-2]} \cdot \nabla h_p) \cdot w_q \mathrm{d}x.
$$

Using Hölder's Young's and Jensen's inequalities, we estimate D_2 as

$$
\begin{aligned}
|D_2| \leq & \sum_{p>Q_{b,h}+2} \lambda_p \|h_p\|_\infty \sum_{|q-p|\leq 2} \|w_q\|_2 \sum_{Q_{b,h}<p'\leq p-2} \|m_{p'}\|_2 \\
\lesssim & c_r \kappa \sum_{q>Q_{b,h}} \lambda_q \|w_q\|_2 \sum_{Q_{b,h}<p'\leq q} \|m_{p'}\|_2 \lambda_{q-Q_{b,h}}^{-\delta} \\
\lesssim & c_r \kappa \sum_{q>Q_{b,h}} \lambda_q \|w_q\|_2 \sum_{Q_{b,h}<p'\leq q} \lambda_{p'} \|m_{p'}\|_2 \lambda_{p'-q}^{\delta-1} \lambda_q^{-1} \\
\lesssim & c_r \kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

We divide D_3 into the low modes, which are rather few, and high modes.

$$\begin{aligned}D_3 =& \sum_{q\le Q_{b,h}+2} \int_{\mathbb{T}^3} \Delta_q(m_{Q_{b,h}+1}\cdot\nabla h_{Q_{b,h}})\cdot w_q \mathrm{d}x\\&+\sum_{p>Q_{b,h}}\sum_{q\le p+2}\int_{\mathbb{T}^3}\Delta_q(\tilde m_p\cdot\nabla h_p)\cdot w_q\mathrm{d}x\\=:&D_{31}+D_{32}.\end{aligned}$$

D_{31} satisfies the following estimate.

$$\begin{aligned}D_{31}\le&\|\nabla h_{Q_{b,h}}\|_\infty\sum_{-1\le q\le Q_{b,h}+2}\|w_q\|_2\|m_{Q_{b,h}+1}\|_2\\\lesssim& c_r\kappa\lambda_{Q_{b,h}+1}\|m_{Q_{b,h}+1}\|_2\sum_{-1\le q\le Q_{b,h}+2}\lambda_q\|w_q\|_2\\\lesssim& c_r\kappa\sum_{q\ge -1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).\end{aligned}$$

The estimate for D_{32} follows from Definition 3.3.1, Hölder's, Young's and Jensen's inequalities.

$$\begin{aligned}D_{32}\le&\sum_{p>Q_{b,h}}\|\nabla h_p\|_\infty\|\tilde m_p\|_2\sum_{q\le p+2}\|w_q\|_2\\\lesssim& c_r\kappa\sum_{p>Q_{b,h}}\lambda_p\|\tilde m_p\|_2\sum_{q\le p+2}\|w_q\|_2\lambda^\delta_{Q_{b,h}-p}\\\lesssim& c_r\kappa\sum_{p>Q_{b,h}}\lambda_p\|m_p\|_2\sum_{q\le p+2}\lambda_q\|w_q\|_2\lambda_q^{-1}\lambda^\delta_{Q_{b,h}-p}\\\lesssim& c_r\kappa\sum_{q\ge -1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).\end{aligned}$$

$\square$

Proposition 3.3.8. *For the term E, it holds that*

$$|E| \lesssim c_r \kappa \sum_{q \geq -1} \|\nabla m_q\|_2^2.$$

Proof: By Bony's paraproduct decomposition,

$$\begin{aligned}
E =& - \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (v_p \cdot \nabla m_{\leq p-2}) \cdot m_q \mathrm{d}x \\
& - \sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (v_{\leq p-2} \cdot \nabla m_p) \cdot m_q \mathrm{d}x \\
& - \sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (v_p \cdot \nabla \tilde{m}_p) \cdot m_q \mathrm{d}x \\
=: & E_1 + E_2 + E_3.
\end{aligned}$$

Utilizing the wavenumber $Q_{u,v}$, E_1 is split into two.

$$\begin{aligned}
E_1 =& - \sum_{p \leq Q_{u,v}} \sum_{|q-p| \leq 2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (v_p \cdot \nabla m_{\leq p-2}) m_q \mathrm{d}x \\
& - \sum_{p > Q_{u,v}} \sum_{|q-p| \leq 2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (v_p \cdot \nabla m_{\leq p-2}) m_q \mathrm{d}x \\
=: & E_{11} + E_{12}.
\end{aligned}$$

By Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, E_{11} and E_{12} are estimated in the following ways.

$$\begin{aligned}
|E_{11}| \le & \sum_{p\le Q_{u,v}} \|v_p\|_r \|\nabla m_{\le p-2}\|_{\frac{2r}{r-2}} \sum_{|q-p|\le 2} \|m_q\|_2 \\
\le & \sum_{p\le Q_{u,v}} \lambda_p^{-1+\frac{3}{r}} \|v_p\|_r \sum_{p'\le p-2} \lambda_{p'}^{1+\frac{3}{r}} \|m_{p'}\|_2 \sum_{|q-p|\le 2} \lambda_q^{1-\frac{3}{r}} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{q\le Q_{u,v}+2} \lambda_q \|m_q\|_2 \sum_{p'\le q} \lambda_{p'}^{s+1} \|m_{p'}\|_2 \lambda_{q-p'}^{-\frac{3}{r}} \\
\lesssim & c_r\kappa \sum_{q\ge -1} \|\nabla m_q\|_2^2;
\end{aligned}$$

$$\begin{aligned}
|E_{12}| \le & \sum_{p> Q_{u,v}} \|v_p\|_r \|\nabla m_{\le p-2}\|_{\frac{2r}{r-2}} \sum_{|q-p|\le 2} \|m_q\|_2 \\
\le & \sum_{p> Q_{u,v}} \lambda_p^{-1+\frac{3}{r}} \|v_p\|_r \sum_{p'\le p-2} \lambda_{p'}^{1+\frac{3}{r}} \|m_{p'}\|_2 \sum_{|q-p|\le 2} \lambda_q^{1-\frac{3}{r}} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{q> Q_{u,v}-2} \lambda_q \|m_q\|_2 \sum_{p'\le q} \lambda_{p'} \|m_{p'}\|_2 \lambda_{q-p'}^{-\frac{3}{r}} \\
\lesssim & c_r\kappa \sum_{q\ge -1} \|\nabla m_q\|_2^2.
\end{aligned}$$

Rewriting E_2 using the commutator in 1.21, we have

$$\begin{aligned}
E_2 = &\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}[\Delta_q, v_{\leq p-2}\cdot\nabla]m_p\cdot m_q \mathrm{d}x \\
&+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3} v_{\leq q-2}\cdot\nabla\Delta_q m_p\cdot m_q \mathrm{d}x \\
&+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}((v_{\leq p-2}-v_{\leq q-2})\cdot\nabla\Delta_q m_p)\cdot m_q \mathrm{d}x \\
=:&E_{21}+E_{22}+E_{23},
\end{aligned}$$

where E_{22} vanishes as $\nabla\cdot v_{\leq q-2}=0$.

Splitting E_{21} by the wavenumber $Q_{u,v}$, we have

$$\begin{aligned}
E_{21} = &\sum_{-1\leq p\leq Q_{u,v}+2}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}[\Delta_q, v_{\leq p-2}\cdot\nabla]m_p\cdot m_q \mathrm{d}x \\
&+\sum_{p>Q_{u,v}+2}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}[\Delta_q, v_{\leq Q_{u,v}}\cdot\nabla]m_p\cdot m_q \mathrm{d}x \\
&+\sum_{p>Q_{u,v}+2}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}[\Delta_q, v_{(Q_{u,v},p-2]}\cdot\nabla]m_p\cdot m_q \mathrm{d}x \\
=:&E_{211}+E_{212}+E_{213}.
\end{aligned}$$

Using Definition 3.3.1, Lemma 1.3.4, Hölder's and Young's inequalities, we can estimate E_{211}.

$$
\begin{aligned}
|E_{211}| \leq & \sum_{-1 \leq p \leq Q_{u,v}+2} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p' \leq p-2} \lambda_{p'} \|v_{p'}\|_r \\
\lesssim & c_r \kappa \sum_{-1 \leq p \leq Q_{u,v}+2} \lambda_p^{\frac{3}{r}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|m_q\|_2 \lambda_p^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{-1 \leq p \leq Q_{u,v}+2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{q \geq -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

The term E_{212} can be estimated in a similar fashion.

$$
\begin{aligned}
|E_{212}| \leq & \sum_{p > Q_{u,v}+2} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p' \leq Q_{u,v}} \lambda_{p'} \|v_{p'}\|_r \\
\lesssim & c_r \kappa \sum_{p > Q_{u,v}+2} \lambda_p^{\frac{3}{r}} \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q^{2s} \|m_q\|_2 \lambda_p^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{p > Q_{u,v}+2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{q \geq -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

The estimate for E_{213} follows from Definition 3.3.1, Lemma 1.3.4, Hölder's and Young's inequalities.

$$
\begin{aligned}
|E_{213}| \leq & \sum_{p>Q_{u,v}+2} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{Q_{u,v}<p'\leq p-2} \lambda_{p'} \|v_{p'}\|_r \\
\leq & c_r \kappa \sum_{p>Q_{u,v}+2} \lambda_p^{\frac{3}{r}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{Q_{u,v}<p'\leq p-2} \lambda_{p'}^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{p>Q_{u,v}+2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \sum_{Q_{u,v}<p'\leq p-2} \lambda_{p'-p}^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

Explicitly writing out $(v_{\leq p-2} - v_{\leq q-2})$ leads to

$$
\begin{aligned}
|E_{23}| \lesssim & \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} (|v_{q-3}| + |v_{q-2}| + |v_{q-1}| + |v_q|) |\nabla \mathit{\Delta}_q m_p| |m_q| \mathrm{d}x \\
\lesssim & \sum_{-1\leq q\leq Q_{u,v}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} |v_q| |\nabla \mathit{\Delta}_q m_p| |m_q| \mathrm{d}x \\
& + \sum_{q>Q_{u,v}} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} |v_q| |\nabla \Delta_q m_p| |m_q| \mathrm{d}x \\
=: & E_{231} + E_{232}.
\end{aligned}
$$

The estimate for E_{231} is as follows.

$$
\begin{aligned}
E_{231} \lesssim & \sum_{-1\le q\le Q_{u,v}} \|v_q\|_r \|m_q\|_{\frac{2r}{r-2}} \sum_{|p-q|\le 2} \|\nabla m_p\|_2 \\
\lesssim & c_r\kappa \sum_{-1\le q\le Q_{u,v}} \lambda_q \|m_q\|_2 \sum_{|p-q|\le 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r\kappa \sum_{-1\le q\le Q_{u,v}} \lambda_q \|m_q\|_2 \sum_{|p-q|\le 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r\kappa \sum_{q\ge -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

By Definition 3.3.1, Hölder's and Young's inequalities, E_{232} can be estimated as

$$
\begin{aligned}
E_{232} \le & \sum_{q>Q_{u,v}} \|v_q\|_r \|m_q\|_{\frac{2r}{r-2}} \sum_{|p-q|\le 2} \|\nabla m_p\|_2 \\
\lesssim & c_r\kappa \sum_{q>Q_{u,v}} \lambda_q \|m_q\|_2 \sum_{|p-q|\le 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r\kappa \sum_{q>Q_{u,v}} \lambda_q \|m_q\|_2 \sum_{|p-q|\le 2} \lambda_p \|m_p\|_2 \\
\lesssim & c_r\kappa \sum_{q\ge -1} \|\nabla m_q\|_2^2.
\end{aligned}
$$

We separate the low modes and high modes of E_3 using the wavenumber $Q_{u,v}$.

$$
\begin{aligned}
E_3 = & -\sum_{-1\le p\le Q_{u,v}} \sum_{q\le p-2} \int_{\mathbb{T}^3} \Delta_q(v_p\cdot\nabla\tilde{m}_p)\cdot m_q \mathrm{d}x \\
& -\sum_{p>Q_{u,v}} \sum_{q\le p-2} \int_{\mathbb{T}^3} \Delta_q(v_p\cdot\nabla\tilde{m}_p)\cdot m_q \mathrm{d}x \\
=: & E_{31} + E_{32}.
\end{aligned}
$$

With the help of Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, we can estimate the terms E_{31} and E_{32} as follows.

$$
\begin{aligned}
|E_{31}| \leq & \sum_{-1\leq p\leq Q_{u,v}} \|v_p\|_r \|\nabla m_p\|_2 \sum_{q\leq p-2} \|m_q\|_{\frac{2r}{r-2}} \\
\leq & c_r\kappa \sum_{-1\leq p\leq Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p-2} \|m_q\|_{\frac{2r}{r-2}} \\
\leq & c_r\kappa \sum_{-1\leq p\leq Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p-2} \lambda_q^{\frac{3}{r}} \|m_q\|_2 \\
\leq & c_r\kappa \sum_{-1\leq p\leq Q_{u,v}} \lambda_p \|m_p\|_2 \sum_{q\leq p-2} \lambda_q \|m_q\|_2 \lambda_{q-p}^{\frac{3}{r}-1} \\
\leq & c_r\kappa \sum_{q\geq -1} \|\nabla m_q\|_2^2;
\end{aligned}
$$

$$
\begin{aligned}
|E_{32}| \leq & \sum_{p> Q_{u,v}} \|v_p\|_r \|\nabla m_p\|_2 \sum_{q\leq p-2} \|m_q\|_{\frac{2r}{r-2}} \\
\leq & c_r\kappa \sum_{p> Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p-2} \|m_q\|_{\frac{2r}{r-2}} \\
\leq & c_r\kappa \sum_{p> Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p-2} \lambda_q^{\frac{3}{r}} \|m_q\|_2 \\
\leq & c_r\kappa \sum_{p> Q_{u,v}} \lambda_p \|m_p\|_2 \sum_{q\leq p-2} \lambda_q \|m_q\|_2 \lambda_{q-p}^{\frac{3}{r}-1} \\
\leq & c_r\kappa \sum_{q\geq -1} \lambda_q^{2s+2} \|\nabla m_q\|_2^2.
\end{aligned}
$$

□

Proposition 3.3.9. *For the term F, it holds that*

$$|F| \lesssim c_r \kappa \sum_{q \geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).$$

Proof: By Bony's paraproduct decomposition, we have

$$\begin{aligned} F = & -\sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla b_{\leq p-2}) \cdot m_q \mathrm{d}x \\ & -\sum_{q \geq -1} \sum_{|p-q| \leq 2} \int_{\mathbb{T}^3} \Delta_q (w_{\leq p-2} \cdot \nabla b_p) \cdot m_q \mathrm{d}x \\ & -\sum_{q \geq -1} \sum_{p \geq q-2} \int_{\mathbb{T}^3} \Delta_q (\tilde{w}_p \cdot \nabla b_p) \cdot m_q \mathrm{d}x \\ =: & F_1 + F_2 + F_3. \end{aligned}$$

Using the fact that $m_{\leq Q_{b,h}} = 0$, we split F_1 into two terms.

$$\begin{aligned} F_1 = & -\sum_{p > Q_{b,h}+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla b_{\leq Q_{b,h}}) \cdot m_q \mathrm{d}x \\ & -\sum_{p > Q_{b,h}+2} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{T}^3} \Delta_q (w_p \cdot \nabla b_{(Q_{b,h}, p-2]}) \cdot m_q \mathrm{d}x \\ =: & F_{11} + F_{12}. \end{aligned}$$

To estimate F_{11}, we use Definition 3.3.1, Hölder's and Young's inequalities.

$$\begin{aligned}
|F_{11}| \leq &\|\nabla b_{\leq Q_{b,h}}\|_\infty \sum_{q>Q_{b,h}} \|m_q\|_2 \sum_{|p-q|\leq 2} \|w_p\|_2 \\
\leq &\|b_{\leq Q_{b,h}}\|_\infty \sum_{q>Q_{b,h}} \lambda_q \|m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|w_p\|_2 \\
\leq &c_r\kappa \sum_{q>Q_{b,h}} \lambda_q \|m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|w_p\|_2 \\
\leq &c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}$$

By Definition 3.3.1, Hölder's and Young's inequalities, F_{12} satisfies the following.

$$\begin{aligned}
|F_{12}| \leq &\sum_{q>Q_{b,h}} \|m_q\|_2 \sum_{|p-q|\leq 2} \|w_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'} \|b_{p'}\|_\infty \\
\leq &\sum_{q>Q_{b,h}} \lambda_q \|m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|w_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'-p} \lambda_p^{-1} \|b_{p'}\|_\infty \\
\leq &c_r\kappa \sum_{q>Q_{b,h}} \lambda_q \|m_q\|_2 \sum_{|p-q|\leq 2} \lambda_p \|w_p\|_2 \sum_{Q_{b,h}<p'\leq p-2} \lambda_{p'-p}^2 \\
\leq &c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right)
\end{aligned}$$

F_2 is split into two parts based on the wavenumber $Q_{b,h}$ as well as the fact that $m_{\leq Q_{b,h}} = 0$.

$$\begin{aligned}
F_2 = &- \sum_{Q_{b,h}-2<p\leq Q_{b,h}} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} \Delta_q (w_{\leq p-2} \cdot \nabla b_p) \cdot m_q \mathrm{d}x \\
&- \sum_{p>Q_{b,h}} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} \Delta_q (w_{\leq p-2} \cdot \nabla b_p) \cdot m_q \mathrm{d}x \\
=: &F_{21} + F_{22}.
\end{aligned}$$

It follows from Definition 3.3.1, Hölder's, Young's and Jensen's inequalities that

$$\begin{aligned}
|F_{21}| \leq & \sum_{Q_{b,h}-2<p\leq Q_{b,h}} \lambda_p \|b_p\|_\infty \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p'\leq p-2} \|w_{p'}\|_2 \\
\lesssim & c_r\kappa \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \lambda_q \|m_q\|_2 \sum_{p'\leq q} \|w_{p'}\|_2 \\
\lesssim & c_r\kappa \sum_{Q_{b,h}<q\leq Q_{b,h}+2} \lambda_q \|m_q\|_2 \sum_{p'\leq q} \lambda_{p'} \|w_{p'}\|_2 \lambda_{p'}^{-1} \\
\lesssim & c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}$$

We estimate F_{22} with the help of Hölder's, Young's and Jensen's inequalities.

$$\begin{aligned}
|F_{22}| \leq & \sum_{p>Q_{b,h}} \|\nabla b_p\|_\infty \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p'\leq p-2} \|w_{p'}\|_2 \\
\leq & \sum_{q>Q_{b,h}} \|m_q\|_2 \lambda_p^{1-\delta} \Lambda_{b,h}^{\delta} \sum_{|p-q|\leq 2} \lambda_{p-Q_{b,h}}^{\delta} \|b_p\|_\infty \sum_{p'\leq p-2} \lambda_{p'} \|w_{p'}\|_2 \lambda_{p'}^{-1} \\
\leq & c_r\kappa \sum_{q>Q_{b,h}} \lambda_q \|m_q\|_2 \sum_{-1\leq p'\leq q} \lambda_{p'} \|w_{p'}\|_2 \lambda_{p'-q}^{\delta} \lambda_{p'}^{-1} \\
\leq & c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}$$

As $m_{\leq Q_{b,h}} = 0$, we split F_3 into two terms.

$$\begin{aligned}
F_3 = & -\sum_{p\leq Q_{b,h}} \sum_{Q_{b,h}<q\leq p+2} \int_{\mathbb{T}^3} \Delta_q(\tilde{w}_p \cdot \nabla b_p) m_q \mathrm{d}x \\
& -\sum_{p>Q_{b,h}} \sum_{Q_{b,h}<q\leq p+2} \int_{\mathbb{T}^3} \Delta_q(\tilde{w}_p \cdot \nabla b_p) m_q \mathrm{d}x \\
=: & F_{31} + F_{32}.
\end{aligned}$$

The estimate for F_{31} is as follows.

$$
\begin{aligned}
|F_{31}| \leq & \sum_{p\leq Q_{b,h}} \|\nabla b_p\|_\infty \|\tilde{w}_p\|_2 \sum_{Q_{b,h}<q\leq p+2} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{p\leq Q_{b,h}} \lambda_p \|w_p\|_2 \sum_{Q_{b,h}<q\leq p+2} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{p\leq Q_{b,h}} \lambda_p \|w_p\|_2 \sum_{Q_{b,h}<q\leq p+2} \lambda_q \|m_q\|_2 \lambda_q^{-1} \\
\lesssim & c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

We use Hölder's, Young's and Jensen's inequalities to estimate F_{32}.

$$
\begin{aligned}
|F_{32}| \leq & \sum_{p> Q_{b,h}} \|\nabla b_p\|_\infty \|\tilde{w}_p\|_2 \sum_{q\leq p+2} \|m_q\|_2 \\
\leq & c_r\kappa \sum_{p> Q_{b,h}} \lambda_p \|w_p\|_2 \sum_{q\leq p+2} \|m_q\|_2 \\
\leq & c_r\kappa \sum_{p> Q_{b,h}} \lambda_p \|w_p\|_2 \sum_{q\leq p+2} \lambda_q \|m_q\|_2 \lambda_q^{-1} \\
\leq & c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

□

Proposition 3.3.10. *For the term F, it holds that*

$$
|G| \lesssim c_r\kappa \sum_{q\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
$$

Proof: Using Bony's paraproduct decomposition, we have

$$\begin{aligned}G=&\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q(b_p\cdot\nabla w_{\leq p-2})\cdot m_q\mathrm{d}x\\&+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}\Delta_q(b_{\leq p-2}\cdot\nabla w_p)\cdot m_q\mathrm{d}x\\&+\sum_{q\geq -1}\sum_{p\geq q-2}\int_{\mathbb{T}^3}\Delta_q(b_p\cdot\nabla \tilde{w}_p)\cdot m_q\mathrm{d}x\\=:&G_1+G_2+G_3.\end{aligned}$$

Taking into account that $m_{\leq Q_{b,h}}=0$, we separate low modes and high modes of G_1 by the wavenumber $Q_{b,h}$.

$$\begin{aligned}G_1=&\sum_{Q_{b,h}-2\leq p\leq Q_{b,h}}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}\Delta_q(b_p\cdot\nabla w_{\leq p-2})\cdot m_q\mathrm{d}x\\&+\sum_{p>Q_{b,h}}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}\Delta_q(b_p\cdot\nabla w_{\leq p-2})\cdot m_q\mathrm{d}x\\=:&G_{11}+G_{12}.\end{aligned}$$

Thanks to the fact that $q=Q_{b,h}+1$ or $Q_{b,h}+2$, we can control G_{11}.

$$\begin{aligned}|G_{11}|\leq&\sum_{Q_{b,h}-2<p\leq Q_{b,h}}\|b_p\|_\infty\sum_{-1\leq p'\leq p-2}\lambda_{p'}\|w_{p'}\|_2\sum_{Q_{b,h}<q\leq Q_{b,h}+2}\|m_q\|_2\\\leq& c_r\kappa\sum_{Q_{b,h}<q\leq Q_{b,h}+2}\lambda_q\|m_q\|_2\sum_{-1\leq p'\leq q}\lambda_{p'}\|w_{p'}\|_2\lambda_q^{-1}\\\lesssim& c_r\kappa\sum_{q>-1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).\end{aligned}$$

Using Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, we have

$$
\begin{aligned}
|G_{12}| \leq & \sum_{p>Q_{b,h}} \|b_p\|_\infty \sum_{-1\leq p'\leq p-2} \lambda_{p'}\|w_{p'}\|_2 \sum_{|q-p|\leq 2} \|m_q\|_2 \\
\leq & c_r\kappa \sum_{p>Q_{b,h}} \sum_{-1\leq p'\leq p-2} \lambda_{p'}\|w_{p'}\|_2 \sum_{|q-p|\leq 2} \lambda_q\|m_q\|_2\lambda_q^{-1} \\
\lesssim & c_r\kappa \sum_{p>Q_{b,h}} \lambda_p\|m_p\|_2 \sum_{-1\leq p'\leq p-2} \lambda_{p'}\|w_{p'}\|_2\lambda_p^{-1}\lambda_{Q_{b,h}-p}^{\delta} \\
\lesssim & c_r\kappa \sum_{q>-1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2\right).
\end{aligned}
$$

Rewriting G_2 using the commutator notation yields

$$
\begin{aligned}
G_2 = & \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} [\Delta_q, b_{\leq p-2}\cdot\nabla] w_p \cdot m_q \mathrm{d}x \\
& + \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} ((b_{\leq q-2}\cdot\nabla)\Delta_q w_p) \cdot m_q \mathrm{d}x \\
& + \sum_{q\geq -1} \sum_{|p-q|\leq 2} \int_{\mathbb{T}^3} (b_{\leq p-2} - b_{\leq q-2}) \cdot \nabla\Delta_q w_p \cdot m_q \mathrm{d}x \\
=: & G_{21} + G_{22} + G_{23}.
\end{aligned}
$$

We further split G_{21} into three parts by the wavenumber $Q_{b,h}$.

$$\begin{aligned}
G_{21}=&\sum_{Q_{b,h}-2<p\le Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}[\Delta_q, b_{\le p-2}\cdot\nabla]w_p\cdot m_q\mathrm{d}x\\
&+\sum_{p>Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}[\Delta_q, b_{\le Q_{b,h}}\cdot\nabla]w_p\cdot m_q\mathrm{d}x\\
&+\sum_{p>Q_{b,h}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}[\Delta_q, b_{(Q_{b,h},p-2]}\cdot\nabla]w_p\cdot m_q\mathrm{d}x\\
=:&G_{211}+G_{212}+G_{213}.
\end{aligned}$$

Using Definition 3.3.1, Hölder's and Young's inequalities, we can estimate $|G_{211}|+|G_{212}|$.

$$\begin{aligned}
|G_{211}|+|G_{212}|\le&\|\nabla b_{\le Q_{b,h}}\|_\infty\sum_{p>Q_{b,h}-2}\|w_p\|_2\sum_{|q-p|\le 2}\|m_q\|_2\\
\lesssim&\|b_{\le Q_{b,h}}\|_\infty\sum_{p>Q_{b,h}-2}\lambda_p\|w_p\|_2\sum_{|q-p|\le 2}\|m_q\|_2\\
\lesssim&c_r\kappa\sum_{p\ge -1}\lambda_p\|w_p\|_2\sum_{|q-p|\le 2}\lambda_q\|m_q\|_2\\
\lesssim&c_r\kappa\sum_{p\ge -1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).
\end{aligned}$$

The estimate for G_{213} is as follows.

$$\begin{aligned}
|G_{213}|\le&\sum_{p>Q_{b,h}+2}\|w_p\|_2\sum_{Q_{h,b}<p'\le p-2}\lambda_{p'}\|b_{p'}\|_\infty\sum_{|q-p|\le 2}\|m_q\|_2\\
\lesssim&c_r\kappa\sum_{q\ge -1}\lambda_p\|w_p\|_2\sum_{|q-p|\le 2}\lambda_q\|m_q\|_2\lambda_q^{-1}\sum_{Q_{h,b}<p'\le p-2}\lambda_{p'-p}\lambda^{\delta}_{Q_{b,h}-p'}\\
\lesssim&c_r\kappa\sum_{p\ge -1}\left(\|\nabla w_q\|_2^2+\|\nabla m_q\|_2^2\right).
\end{aligned}$$

As noted before, G_{22} and C_{12} cancel each other.

$$\begin{aligned}
C_{12}+G_{22}=&\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}(b_{\leq q-2}\cdot\nabla)(\Delta_q w_p\cdot m_q+\Delta_q m_p\cdot w_q)\mathrm{d}x\\
=&\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}(b_{\leq q-2}\cdot\nabla)\Delta_q w_p\cdot(m_q+w_q)\mathrm{d}x\\
&+\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}(b_{\leq q-2}\cdot\nabla)\Delta_q m_p(w_q+m_q)\mathrm{d}x\\
=&\sum_{q\geq -1}\int_{\mathbb{T}^3}(b_{\leq q-2}\cdot\nabla)(m_q+w_q)\cdot(m_q+w_q)\mathrm{d}x\\
=&0.
\end{aligned}$$

Since $m_{\leq Q_{b,h}}=0$, G_{23} consists of mostly high modes.

$$\begin{aligned}
|G_{23}|\lesssim&\sum_{q\geq -1}\sum_{|p-q|\leq 2}\int_{\mathbb{T}^3}|b_p||\nabla\Delta_q w_p m_q|\mathrm{d}x\\
\lesssim&\sum_{Q_{b,h}-2<p\leq Q_{b,h}}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}|b_p||\nabla\Delta_q w_p m_q|\mathrm{d}x\\
&+\sum_{p>Q_{b,h}}\sum_{|q-p|\leq 2}\int_{\mathbb{T}^3}|b_p||\nabla\Delta_q w_p m_q|\mathrm{d}x\\
=:&G_{231}+G_{232}.
\end{aligned}$$

By Definition 3.3.1, Hölder's and Young's inequalities, we have

$$
\begin{aligned}
G_{231} \lesssim & \sum_{-1\le p > Q_{b,h}} \|b_p\|_\infty \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p\ge -1} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p\ge -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

G_{232} is estimated as follows.

$$
\begin{aligned}
G_{232} \lesssim & \sum_{p > Q_{b,h}} \|b_p\|_\infty \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p\ge -1} \lambda_p \|w_p\|_2 \sum_{|q-p|\le 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p\ge -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}
$$

We divide G_3 into two terms using the wavenumber $Q_{b,h}$.

$$
\begin{aligned}
G_3 = & \sum_{Q_{b,h}-2 < p \le Q_{b,h}} \sum_{q\le p+2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (b_p \cdot \nabla \tilde{w}_p) \cdot m_q \mathrm{d}x \\
& + \sum_{p > Q_{b,h}} \sum_{q\le p+2} \int_{\mathbb{T}^3} \mathit{\Delta}_q (b_p \cdot \nabla \tilde{w}_p) \cdot m_q \mathrm{d}x \\
=: & G_{31} + G_{32}.
\end{aligned}
$$

We can estimate G_{31} in the following way.

$$\begin{aligned}
|G_{31}| \leq & \sum_{Q_{b,h}-2<p\leq Q_{b,h}} \|b_p\|_\infty \|\nabla \tilde{w}_p\|_2 \sum_{q\leq p+2} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{Q_{b,h}-2<p\leq Q_{b,h}+1} \lambda_p \|w_p\|_2 \sum_{q\leq p+2} \lambda_q \|m_q\|_2 \lambda_q^{-1} \\
\lesssim & c_r\kappa \sum_{p\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}$$

Meanwhile, by Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, it holds that

$$\begin{aligned}
|G_{32}| \leq & \sum_{p>Q_{b,h}} \|b_p\|_\infty \|\nabla \tilde{w}_p\|_2 \sum_{q\leq p+2} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{p>Q_{b,h}-1} \lambda_p \|w_p\|_2 \sum_{q\leq p+2} \lambda_q \|m_q\|_2 \lambda_{Q_{b,h}-p}^{\delta} \lambda_q^{-1} \\
\lesssim & c_r\kappa \sum_{p\geq -1} \left(\|\nabla w_q\|_2^2 + \|\nabla m_q\|_2^2 \right).
\end{aligned}$$

□

Proposition 3.3.11. *For the term H, it holds that*

$$|H| \lesssim c_r\kappa \sum_{q\geq -1} \|\nabla m_q\|_2^2.$$

Proof: By Bony's paraproduct decomposition, we have

$$
\begin{aligned}
H = & \sum_{q\ge -1}\sum_{|p-q|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla v_{\le p-2})\cdot m_q \mathrm{d}x \\
& + \sum_{q\ge -1}\sum_{|p-q|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_{\le p-2}\cdot\nabla v_p)\cdot m_q \mathrm{d}x \\
& + \sum_{q\ge -1}\sum_{p\ge q-2}\int_{\mathbb{T}^3}\Delta_q(\tilde{m}_p\cdot\nabla v_p)\cdot m_q \mathrm{d}x \\
=: & H_1 + H_2 + H_3.
\end{aligned}
$$

By the wavenumber $Q_{u,v}$, the term H_1 can be split into three parts.

$$
\begin{aligned}
H_1 = & \sum_{-1\le p\le Q_{u,v}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla v_{\le p-2})\cdot m_q \mathrm{d}x \\
& + \sum_{p>Q_{u,v}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla v_{\le Q_{u,v}})\cdot m_q \mathrm{d}x \\
& + \sum_{p>Q_{u,v}+2}\sum_{|q-p|\le 2}\int_{\mathbb{T}^3}\Delta_q(m_p\cdot\nabla v_{(Q_{u,v},p-2]})\cdot m_q \mathrm{d}x \\
=: & H_{11} + H_{12} + H_{13}.
\end{aligned}
$$

We can estimate H_{11} with the help of Definition 3.3.1, Hölder's and Young's inequalities.

$$\begin{aligned}
|H_{11}| \leq & \sum_{-1\leq p\leq Q_{u,v}+2} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{-1\leq p'\leq p-2} \lambda_{p'} \|v_{p'}\|_r \\
\lesssim & c_r \kappa \sum_{-1\leq p\leq Q_{u,v}+2} \lambda_p^{\frac{3}{r}} \|m_p\|_2 \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{-1\leq p'\leq p-2} \lambda_{p'}^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{-1\leq p\leq Q_{u,v}+2} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \sum_{-1\leq p'\leq p-2} \lambda_{Q_{u,v}-p}^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}$$

To estimate H_{12}, we have Definition 3.3.1 and applies Hölder's and Young's inequalities.

$$\begin{aligned}
|H_{12}| \leq & \|\nabla v_{\leq Q_{u,v}}\|_r \sum_{p>Q_{u,v}} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \\
\lesssim & \Lambda_{u,v}^{-1+\frac{3}{r}} \|v_{\leq Q_{u,v}}\|_r \sum_{p>Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p\geq -1} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}$$

As a result of Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, we have

$$
\begin{aligned}
|H_{13}| \leq & \sum_{p>Q_{u,v}+2} \|m_p\|_{\frac{2r}{r-2}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{Q_{u,v}<p'\leq p-2} \lambda_{p'} \|v\|_r \\
\leq & c_r \kappa \sum_{p>Q_{u,v}+2} \lambda_p^2 \|m_p\|_2 \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{Q_{u,v}<p'\leq p-2} \lambda_{p'-p}^{2-\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{p\geq -1} \lambda_p \|m_p\|_2 \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}
$$

H_2 is split into low modes and high modes.

$$
\begin{aligned}
H_2 = & \sum_{-1\leq p\leq Q_{u,v}} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} \Delta_q(m_{\leq p-2} \cdot \nabla v_p) \cdot m_q \mathrm{d}x \\
& + \sum_{p>Q_{u,v}} \sum_{|q-p|\leq 2} \int_{\mathbb{T}^3} \Delta_q(m_{\leq p-2} \cdot \nabla v_p) \cdot m_q \mathrm{d}x \\
=: & H_{21} + H_{22},
\end{aligned}
$$

which are estimated by Definition 3.3.1, Hölder's, Young's and Jensen's inequalities.

$$
\begin{aligned}
|H_{21}| \leq & \sum_{-1\leq p\leq Q_{u,v}} \|\nabla v_p\|_r \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p'\leq p-2} \|m_{p'}\|_{\frac{2r}{r-2}} \\
\leq & c_r \kappa \sum_{-1\leq p\leq Q_{u,v}} \lambda_p^{2-\frac{3}{r}} \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}^{\frac{3}{r}} \|m_{p'}\|_2 \\
\leq & c_r \kappa \sum_{-1\leq p\leq Q_{u,v}} \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \sum_{p'\leq p-2} \lambda_{p'} \|m_{p'}\|_2 \lambda_{p'-q}^{\frac{3}{r}-1} \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2;
\end{aligned}
$$

$$
\begin{aligned}
|H_{22}| \leq & \sum_{p>Q_{u,v}} \|\nabla v_p\|_r \sum_{|q-p|\leq 2} \|m_q\|_2 \sum_{p'\leq p-2} \|m_{p'}\|_{\frac{2r}{r-2}} \\
\lesssim & \sum_{p>Q_{u,v}} \lambda_p^{-1+\frac{3}{r}} \|v_p\|_r \sum_{|q-p|\leq 2} \lambda_q^{2-\frac{3}{r}} \|m_q\|_2 \sum_{p'\leq p-2} \lambda_{p'}^{\frac{3}{r}} \|m_{p'}\|_2 \\
\lesssim & c_r\kappa \sum_{p>Q_{u,v}} \sum_{|q-p|\leq 2} \lambda_q \|m_q\|_2 \sum_{p'\leq p-2} \lambda_{p'} \|m_{p'}\|_2 \lambda_{q-p'}^{s+1-\frac{3}{r}} \\
\lesssim & c_r\kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}
$$

We also divide H_3 into two parts.

$$
\begin{aligned}
H_3 = & \sum_{-1\leq p\leq Q_{u,v}} \sum_{q\leq p+2} \int_{\mathbb{T}^3} \Delta_q(\tilde{m}_p \cdot \nabla v_p) \cdot m_q \mathrm{d}x \\
& + \sum_{p>Q_{u,v}} \sum_{q\leq p+2} \int_{\mathbb{T}^3} \Delta_q(\tilde{m}_p \cdot \nabla v_p) \cdot m_q \mathrm{d}x \\
=: & H_{31} + H_{32}.
\end{aligned}
$$

By Definition 3.3.1, Hölder's, Young's and Jensen's inequalities, we have

$$
\begin{aligned}
|H_{31}| \leq & \sum_{-1\leq p\leq Q_{u,v}} \|\nabla v_p\|_r \|\tilde{m}_p\|_2 \sum_{q\leq p+2} \|m_q\|_{\frac{2r}{r-2}} \\
\lesssim & c_r\kappa \sum_{-1\leq p\leq Q_{u,v}+1} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p+2} \lambda_q^{\frac{3}{r}} \|m_q\|_2 \\
\lesssim & c_r\kappa \sum_{-1\leq p\leq Q_{u,v}+1} \lambda_p \|m_p\|_2 \sum_{q\leq p+2} \lambda_q \|m_q\|_2 \lambda_{q-p}^{-1+\frac{3}{r}} \\
\lesssim & c_r\kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}
$$

For H_{32}, the following estimate holds.

$$\begin{aligned}
|H_{32}| \leq & \sum_{p>Q_{u,v}} \|\nabla v_p\|_r \|\tilde{m}_p\|_2 \sum_{q\leq p+2} \|m_q\|_{\frac{2r}{r-2}} \\
\lesssim & c_r \kappa \sum_{p>Q_{u,v}-1} \lambda_p^{2-\frac{3}{r}} \|m_p\|_2 \sum_{q\leq p+2} \lambda_q^{\frac{3}{r}} \|m_q\|_2 \\
\lesssim & c_r \kappa \sum_{p>Q_{u,v}-1} \lambda_p \|m_p\|_2 \sum_{q\leq p+2} \lambda_q \|m_q\|_2 \lambda_{q-p}^{-1+\frac{3}{r}} \\
\lesssim & c_r \kappa \sum_{q\geq -1} \|\nabla m_q\|_2.
\end{aligned}$$

□

Summing up all the previous estimates from the proof of Theorem 3.3.3 and from Proposition 3.3.4 – 3.3.11, we choose a suitable constant c_r to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{q\geq -1} \left(\|w_q\|_2^2 + \|m_q\|_2^2\right) \lesssim - \sum_{q\geq -1} \lambda_q^2 \left(\|w_q\|_2^2 + \|m_q\|_2^2\right) \lesssim \sum_{q\geq -1} \left(\|w_q\|_2^2 + \|m_q\|_2^2\right).$$

As a result of Grönwall's inequality, $\left(\|w\|_{L^2}^2 + \|m\|_{L^2}^2\right)$ decays exponentially as $t \to \infty$, which leads to Theorem 3.3.2.

3.3.3 Bounds on the averages of the wavenumbers

As alluded in (Kolmogorov, 1941), the degrees of freedom pertaining to turbulent flows should be finite. For the 2D Navier-Stokes equations, estimates of the number of the determining Fourier modes were obtained by (Foiaş et al., 1983) in terms of the Grashof number, and later improved by (Jones and Titi, 1993), whereas (Constantin et al., 1985) estimated the number of determining modes for the 3D Navier-Stokes equations assuming uniform boundedness of solutions in H^1. An incomplete list of references concerning the study of finite dimensionality of Navier-Stokes and MHD flows include (Eden and Libin, 1989; Constantin et al., 1988; Foiaş et al., 2012; Foiaş et al., 2001).

We denote by $\langle \Lambda_u \rangle$ the time average of the determining wavenumber $\Lambda_u(t)$ corresponding to the fluid component u of a Leray-Hopf solution to 1.1 – 1.3. Then, as shown in (Cheskidov et al., 2018) for the 3D Navier-Stokes equations, $\langle \Lambda_u \rangle$ can be bounded above by the average energy dissipation rate $\varepsilon := \langle \|\nabla u\|_{L^2}^2 \rangle$. Indeed, suppose $\Lambda_u(t) > \lambda_0$, then either

$$(\Lambda_u(t))^{-1+\frac{3}{r}} \|u_{Q_u(t)}\|_r \geq c_r\kappa, \text{ or } \left(\lambda_{\leq Q_u(t)-1}\right)^{-1+\frac{3}{r}} \|u_{\leq Q_u(t)-1}\|_r \geq c_r\kappa.$$

By Lemma 1.3.1 and the condition $(\Lambda_u(t))^{-1+\frac{3}{r}} \|u_{Q_u(t)}\|_r \geq c_r\kappa$, we have

$$c_r\kappa \leq \Lambda_u^{\frac{3}{2}-\frac{3}{r}} \Lambda^{-1+\frac{3}{r}} \|u_{Q_u}\|_2.$$

It follows that

$$\Lambda_u^{\frac{1}{2}} \leq (c_r\kappa)^{-1} \Lambda_u \|u_{Q_u}\|_2 \lesssim \|\nabla u\|_2,$$

which leads to

$$\Lambda_u(t) \lesssim \|\nabla u(t)\|_2^2.$$

Similarly, the condition $(\lambda_{\leq Q_u-1})^{-1+\frac{3}{r}} \|u_{\leq Q_u-1}\|_r \geq c_r\kappa$ yields

$$c_r\kappa \leq \frac{1}{2} \Lambda_{u,v}^{\frac{1}{2}} \|u_{\leq Q_u-1}\|_2.$$

It follows that

$$\Lambda_u^{\frac{1}{2}} \leq (c_r\kappa)^{-1}\lambda_{Q_u-1}\|u_{\leq Q_u-1}\|_2 \lesssim \|\nabla u\|_2.$$

Hence, in this case we also have

$$\Lambda_u(t) \lesssim \|\nabla u(t)\|_2^2.$$

Unlike solutions to the stationary Navier-Stokes equations, the steady-state solutions to the Hall-MHD system are only known to be partially regular, which hinders us from finding a satisfactory upper bound on the wavenumber $\Lambda_b(t)$ corresponding to the magnetic component of a Leray-Hopf type weak solution (u, b). In particular, it seems hopeless to bound $\Lambda_b(t)$ by the average magnetic energy dissipation rate $\langle\|\nabla b\|_{L^2}^2\rangle$. At this moment, we can only restrict our attentions to strong solutions, for which we can bound $\Lambda_{b,h}(t)$ in an average sense.

Indeed, whenever $\Lambda_b(t) > \lambda_0$, it must be that one of the conditions in Definition 3.3.1 is unfulfilled, i.e., $\|b_{Q_b(t)}\|_\infty > c_r\kappa$ or $\|b_{\leq Q_b(t)-1}\|_\infty > c_r\kappa$.

The inequality $\|b_{Q_b(t)}\|_\infty > c_r\kappa$ implies that

$$\Lambda_b(t)\|b_{Q_b(t)}\|_\infty > c_r\kappa\Lambda_b(t).$$

By Lemma 1.3.1, one has

$$\|\nabla b\|_\infty^2 \geq \|\nabla b_{Q_b(t)}\|_\infty^2 > \big(c_r\kappa\Lambda_b(t)\big)^2.$$

Meanwhile, if $\|b_{\leq Q_b(t)-1}\|_\infty > c_r\kappa$, then

$$\frac{1}{2}\Lambda_b(t)\|b_{\leq Q_b(t)-1}\|_\infty > c_r\kappa\Lambda_b(t),$$

which, by Lemma 1.3.1, results in

$$\|\nabla b\|_\infty^2 \geq \|\nabla b_{\leq Q_b(t)-1}\|_\infty^2 \gtrsim \big(\Lambda_b(t)\big)^2.$$

Summarizing the above inequalities, we conclude that if (u,b) is a Leray-Hopf type weak solution, then

$$\langle \Lambda_u \rangle \lesssim \big\langle \|\nabla u\|_{L^2}^2 \big\rangle < \infty,$$

while if $(u,b) \in L^\infty\big(0,\infty;(H^s(\mathbb{T}^3))^2\big)$ with $s > \frac{5}{2}$, we have by Theorem 2.1.1, the following bound –

$$\big(\big\langle \Lambda_{b,h}^2 \big\rangle\big)^{\frac{1}{2}} \lesssim \|\nabla b\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} < \infty.$$

As for the wavenumber $\Lambda_b(t)$, we are still seeking a more satisfactory bound. In $2\frac{1}{2}$-dimension, a better bound is expected; further studies along this line could be proven worthwhile.

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APPENDIX

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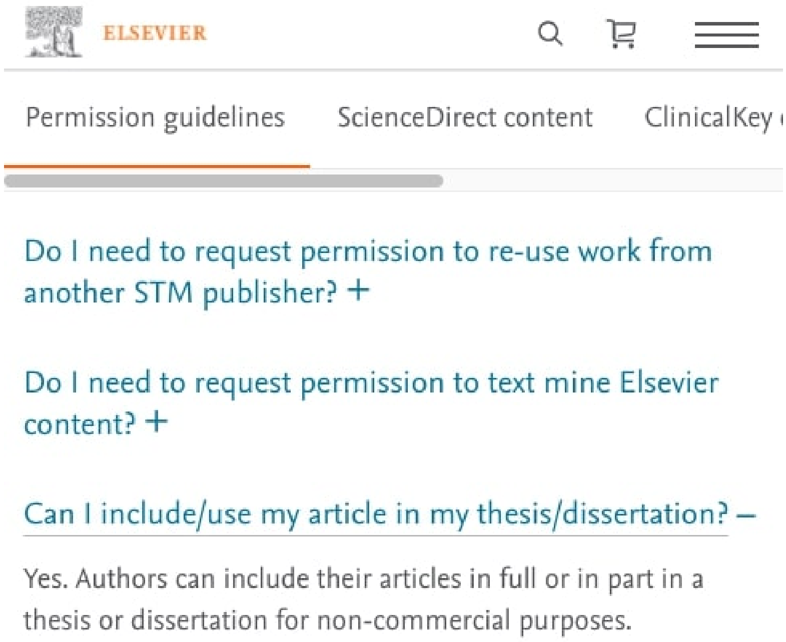

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