Convex Integration and the Navier–Stokes Equations

by

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THESIS

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Dedicated to my families.

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CONTRIBUTIONS OF AUTHORS

Chapter 1 is an general overview of the incompressible Navier-Stokes equations and the method of convex integration. Chapter 2 is from a published paper (58) that I am of the sole author while Chapter 3 is from a preprint (57) that I co-authored with my advisor Alexey Cheskidov.

TABLE OF CONTENTS

SUMMARY

This work is devoted to applying the convex integration technique that has been recently developed in fluid dynamics to the incompressible Navier-Stoke equations in dimensions $d \geq 3$. The main results include the existence of stationary weak solutions in dimensions $d \geq 4$ proved in Chapter 2 and in $3D$ which is proved in Chapter 3 where we also construct weak solutions in $3D$ whose energy profiles are discontinuous on some dense sets of positive Lebesgue measure in time.

CHAPTER 1

INTRODUCTION

1.1 The incompressible Navier-Stokes equations

The incompressible Navier-Stokes equations describe the motion of viscous and incompressible flows. It is one of the most fundamental and prominent equations in fluid dynamics. In this work, we consider the incompressible Navier-Stokes equations on the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$

$$\begin{aligned} \partial_t u - \nu \Delta u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0, \end{aligned} \tag{NSE}$$

where $u(x,t) : \mathbb{T}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is the unknown vector field, the scalar $p : \mathbb{T}^d \to \mathbb{R}$ is the pressure and ν is the viscosity. The system is supplemented by the periodic boundary condition $u(x+k,t) = u(x,t)$ for any $k \in \mathbb{Z}^d$. In this work, we consider the weak solutions of (Equation NSE) in dimensions $d \geq 3$.

The following weak formulation of (Equation NSE) will be used throughout the work and the term "weak solution" refers to this definition in the sequel.

Definition 1.1.1. *A vector field $u \in C_w(0,T; L^2(\mathbb{T}^d))$ is a weak solution of* (Equation NSE) *if it satisfies:*

1. *$u(t)$ has zero-mean on $\mathbb{T}^d$ and is weakly divergence-free for all $t \in [0,T]$, namely for any $\phi(x) \in C_0^\infty(\mathbb{T}^d)$*

$$\int_{\mathbb{T}^d} \nabla \phi u(x,t) dx = 0 \quad \textit{for all } t \in [0,T]. \tag{1.1}$$

2. *For any test function $\varphi(x,t) \in C_c^\infty([0,T] \times \mathbb{T}^d)$ such that $\varphi(x,t)$ is divergence-free in x for all $t \in [0,T]$ we have*

$$\int_{\mathbb{T}^d} u(x,0) \cdot \varphi(x,0)\, dx + \int_0^T \int_{\mathbb{T}^d} u \cdot (\partial_t \varphi + (u \cdot \nabla)\varphi + \Delta \varphi)\, dx d\tau = 0,$$

The vector field $u_0(\cdot) = u(\cdot, 0)$, which is also the weak L^2 limit of $u(\cdot, t)$ as $t \to 0^+$, is called the initial data. Often weak solutions with finite energy dissipation, i.e., $u \in L^2(0,T; H^1)$, are studied in the literature. Besides Definition 1.1.1, there are numerous equivalent ways to define such solutions, e.g., using alternative spaces of test functions (see (64)).

1.2 Leray-Hopf weak solutions

The system (Equation NSE) has been extensively studied by many researchers for a long time. In (52) Leray constructed weak solutions $u \in L_t^\infty L^2 \cap L_t^2 H^1$ for divergence-free initial data $u_0 \in L^2(\mathbb{R}^3)$. These solutions are termed Leray-Hopf weak solutions and also satisfy the energy inequality. A similar result was later obtained by Hopf in (44) for the bounded domain with Dirichlet boundary condition. It is also worth noting that even though the global existence results of Leray and Hopf are best known for $d = 2, 3$, they can be carried over to $\mathbb{R}^d$ or $\mathbb{T}^d$ for $d \geq 4$ without much trouble. Let us recall the relevant facts in the following.

Theorem 1.2.1 (Leray-Hopf weak solutions). *Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ for $d \geq 3$. Given any divergence-free $u_0 \in L^2(\Omega)$ and $0 < T \leq \infty$, there exists at least one weak solution $u \in C_w(0,T;L^2) \cap L^2(0,T;H^1)$ to* (Equation NSE) *and u verifies additionally the energy inequality:*

$$\|u(t)\|_2^2 + 2\nu \int_{t_0}^{t} \|\nabla u(s)\|_2^2 ds \leq \|u(t_0)\|_2^2 \tag{E.I.}$$

for any $t > 0$ and a.e. $t_0 \in [0,t)$ including 0.

In the literature, such solutions are referred to as the Leray-Hopf weak solutions. There has been a long history of extensive studies of these solutions (52; 44; 63; 66; 51; 22; 71; 30), however, the global regularity and uniqueness of Leray-Hopf weak solutions remain among the most important unsolved questions in mathematical fluid dynamics. What is more related to the present work, is the validity of energy equality (also known as Onsager's conjecture in the case of the Euler equations (21)). In the recent groundbreaking work (8) Buckmaster and Vicol proved nonuniqueness and anomalous dissipation in the class of weak solutions, but this is still an open question for Leray-Hopf solutions.

Even though the existence of weak solutions has been know for quite some time, the questions of uniqueness and regularity of Leray-Hopf weak solutions in $d \geq 3$ remain unknown to date and are considered one of the most important issues in mathematical fluid mechanics. More specifically the following questions are still open.

1. **Global regularity:** For any divergence-free initial data $u_0 \in L^2(\Omega)$, does there exist a global Leray-Hopf solution $u \in C^\infty((0,\infty)\times\Omega)$[1] ?

2. **Uniqueness:** For any divergence-free initial data $u_0 \in L^2(\Omega)$, is the Leray-Hopf weak solution $u(t)$ with initial data u_0 unique among all Leray-Hopf weak solutions with initial data u_0?

3. **Anomalous dissipation:** Is there a Leray-Hopf weak solution $u(t)$ satisfying (Equation E.I.) with a strict sign?

One of the motivations of our work is to obtain some insights into the above question of uniqueness. We are not yet able to obtain nonuniqueness results for the Leray-Hopf weak solutions. Instead, we consider the following pathway:

A possible approach: Find the "smoothest" function space $X \subset C_w(0,T;L^2)$ so that there exist two weak solutions of (Equation NSE) $u, v \in X$ such that $u(0) \neq v(0)$.

Let

$$\mathcal{E} = \{u \in C_w(0,T;L^2) : \|u\|_{L^\infty_t L^2} + \|u\|_{L^2_t H^1} < \infty\}.$$

The nonuniqueness problem of Leray-Hopf weak solutions can then be formulated as finding a space X so that $X \subset \mathcal{E}$ and weak solutions of (Equation NSE) in X satisfy (Equation E.I.).

Following such approach, we are able to obtain a nonuniqueness statement for the function space $X = H^\beta$ for $\beta < \frac{1}{200}$ in dimensions $d \geq 4$ and $X = L^2$ in 3D.

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[1]Here we mean that for any $k, m \in \mathbb{N}$ $\partial_t^k \nabla_x^m u$ is bounded on $[\epsilon,\infty)\times\Omega$.

1.3 Main results

The focus of this work is to prove the existence of weak solutions to the (Equation NSE) with very pathological energy behaviors. On one hand, we construct finite energy stationary solutions. We first do this in dimensions $d \geq 4$ in Chapter 2 and then extend the construction to 3D in Chapter 3

Theorem 1.3.1 (Existence of stationary weak solutions $d \geq 4$)**.** *Suppose $d \geq 4$. There exists a nontrivial stationary weak solution $u \in L^2(\mathbb{T}^d)$ of* (Equation NSE) *.*

Theorem 1.3.2 (Existence of stationary weak solutions in $3D$)**.** *Suppose $d = 3$. There exists a nontrivial stationary weak solution $u \in L^2(\mathbb{T}^3)$ of* (Equation NSE)*.*

On the other hand, we construct weak solutions with energy profiles discontinuous on a dense set of positive Lebesgue measure. So the set of discontinuities of the energy can be very large at least in the class of weak solutions.

Theorem 1.3.3. *For any $\varepsilon, T > 0$, there exists a weak solution $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ to the 3D NSE, which is discontinuous in L^2 on a set $E \subset [0,T]$, such that*

1. *E in dense in $[0,T]$.*
2. *The Lebesgue measure of E^c is less than ε.*

1.4 Convex integration technique

This work is based on the technique of convex integration. Although this method has been around since the work of Nash (62), its application to fluid dynamics was brought to attention only in recent years by the pioneering work of De Lellis and Szekelyhidi Jr. (26), where the authors obtained a bounded solution to the 3D Euler equations with compact support in space-time generalizing the results of Scheffer (65) and Shnirelman (68).. Since (26), it was developed over a series of works in the resolution of the Onsager's conjecture for the 3D Euler equations (26; 27; 28; 6; 7; 45; 3). Its extension to the NSE was done only very recently by Buckmaster-Vicol (8), where non-unique weak solutions of the 3D Navier-Stokes equations in the sense of Definition 1.1.1 are constructed. For a more detailed account of applications of convex integration in fluid dynamics, we refer to the survey (29) by De Lellis and Székelyhidi, Jr. and other interesting papers by different authors such as (4; 3; 20; 46; 48; 47).

In the context of fluid dynamics, the essence of convex integration is to construct a sequence of approximate solutions that converges to a desire exact solution in the limit. This is typically done by an iteration scheme. At each step, one specifically design a perturbation so that the nonlinear interaction of the perturbation cancels the previous error and thus produce a the new solution that is "closer" to the exact solution in a suitable functional space. More specifically, we will construct a sequence of solution (u_n, p_n, R_n) to the approximate system

$$\begin{cases} \partial_t u_n - \Delta u_n + \operatorname{div}(u_n \otimes u_n) + \nabla p_n = \operatorname{div} R_n \\ \operatorname{div} u_n = 0. \end{cases} \tag{1.2}$$

where R_n is a stress tensor measuring the distance of u_n to the exact solutions. The heart of the argument is then to design carefully at each step a perturbation w_n so that the new velocity $u_{n+1} := u_n + w_n$ verifies (Equation 1.2) with a much smaller stress error R_{n+1}. This is typically done by using the high-high to low integration of the velocity perturbation so that modulo a suitable pressure gradient, the term $\operatorname{div}(w_n \otimes w_n + R_n)$ has only very high frequencies. There are other restriction in this process, and we shall discuss this in the next section below.

So far, the focus of the convex integration method has been to produce wild solutions that are as regular as possible. For instance, the regularity of wild solutions of the Euler equations was pushed to the critical Onsager's exponent $1/3$ by Isett (45). Also, the extension of (8) to the fractional NSE $(-\Delta)^\alpha$ setting for $1 \leq \alpha < \frac{5}{4}$ was done in (56). Using the smoothing effect of the Stokes semigroup, Buckmaster-Colombo-Vicol (5) were able to construct non-unique weak solutions whose singular sets have Hausdorff dimension less than 1. Nonuniqueness of Leray-Hopf solutions has also been obtained for ipodissipative NSE and Hall-MHD (20; 24). However, it is not clear whether a convex integration scheme could ever produce non-unique wild solutions in a class where the Leray structure theorem would hold[1], except perhaps one very specific scenario.

1.5 The role of intermittency

The effect of intermittency on the regularity properties of solutions to the (Equation NSE) and toy models has been also studied in the past decade (18; 15; 16). Compared with other inviscid or ipodissipative models, such as the Euler equations, the Muskat problem, the Surface Geostrophic equations or the ipodissipative Navier-Stokes equations (4; 48; 33; 20) where results of nonuniqueness-type have been obtained, the Navier-Stokes system has a dissipation

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[1]Note that the solutions in (20; 24) do not obey the Leray structure theorem.

term $\nu\Delta u$ with two derivatives, making it much harder to find suitable building blocks in the convex integration scheme. To resolve this issue one has to start with building blocks that are intermittent, which means that Bernstein's inequality is highly saturated. The concept of intermittency is crucial to the theory of turbulence in hydrodynamics (37) and it is both instructive and interesting to understand its role from a mathematical point of view, see (59; 13; 15; 12). Besides the work of Buckmaster-Vicol (8), the idea of using building blocks that are intermittent has been used for other systems with diffusions, such as the transport-diffusion equations (61; 60).

Let us briefly discuss the concept of intermittency using the Littlewood-Paley decomposition as follows. Suppose $u_q = \Delta_q u$ is a Littlewood-Paley projection at frequency of size $\sim 2^q$, then the intermittency $D \in [0, d]$ of u is measured by

$$\|u_q\|_\infty \sim 2^{q\frac{d-D}{2}} \|u_q\|_2 \quad \text{for all } q \text{ sufficiently large.} \tag{1.3}$$

When $D = d$ the function u is homogeneous spatially and all L^p norms are of the same order. When $D = 0$ the function u has extreme intermittency and Bernstein's inequality is fully saturated. In view of the behavior in physical space, D roughly measures the concentration level of u in the sense that u is concentrated on some set of dimension D. The Beltrami flows used in (27; 6; 28; 7) for the 3D Euler equations and the Mikado flows used in the resolution of the Onsager's conjecture (45) all have intermittency $D = 3$. In contrast, the intermittent Beltrami flows constructed by Buckmaster-Vicol (8) can be made to have intermittency $D = 0$.

Heuristically speaking if a weak solution of (Equation NSE) is of intermittency $D \geq d-2$ then it is regular. So it appears impossible to have stationary weak solutions of Equation NSE with intermittency bigger than $d-2$. This is due to the fact that linear term dominates the nonlinear term in this regime and the problem becomes subcritical. We show this heuristics by considering the energy flux through each Littlewood-Paley shell as follows. Assuming only local interactions between scales, i.e. $\operatorname{div}(u \otimes u) \cdot u_q \sim \operatorname{div}(u_q \otimes u_q) \cdot u_q$ for simplicity. Then consider the energy flux equation obtained by multiplying (Equation NSE) with $\Delta_q u_q$ and then integrating in space:

$$\frac{d}{dt}\|u_q\|_2^2 + \text{Linear term} = \text{Nonlinear term}$$

with

$$\text{Linear term} = (\Delta u_q, u_q) \sim 2^{2q}\|u_q\|_2^2,$$

and

$$\text{Nonlinear term} = (\operatorname{div}(u \otimes u) \cdot u_q) \sim (\operatorname{div}(u_q \otimes u_q) \cdot u_q) \lesssim \|u_q\|_2^2 \|u_q\|_\infty \sim 2^{q\frac{d-D}{2}}\|u_q\|_2^3.$$

Due to the fact that $\|u_q\|_2 \to 0$ as $q \to \infty$ we have Linear term $\geq$ Nonlinear term when $D \geq d-2$.

Finally we remark that the building blocks that we are using have intermittency dimension 1, which is the limitation forcing us to work in $d \geq 4$. More discussions on the intermittency and its role in the construction can be found in Section 2.3.2.

1.5.1 Motivation from a energy balance viewpoint

If a solution of the NSE is regular enough, then the energy equality is satisfied. This can be seen by formally multiplying the equation (Equation NSE) by u and then integrating by parts thanks to the incompressiblity. However, such a formal computation can not be justified for weak solutions as there is not enough regularity to perform integration by parts.

It is known that Leray-Hopf weak solutions satisfy the energy inequality, but the continuity of the energy in time or the validity of energy equality is not known. If the energy has a jump discontinuity from the right, this immediately implies non-uniqueness since the solution can be restarted at that time to remove the jump and infinitely many solutions can be obtained via interpolation (50). However, the existence of such Leray-Hopf weak solutions that violets the energy equality is still unknown to date.

Nontrivial stationary weak solutions do not lose any energy over time even though the enstrophy is positive (in fact, infinite). As a result, the energy inequality is not satisfied. These solutions exhibit a interesting phenomenon, what we call the *anomalous energy influx*, where the backward energy cascade balances precisely the energy dissipation at each scale. Nontrivial stationary solutions are also known to exist for the dyadic model of the NSE (2), but the existence of such solutions was an open question for the NSE.

Another motivation comes from the continuity of the energy of the solution. Weak solutions (in the sense of Definition 1.1.1) are only lower semi-continuous in L^2. Therefore, it is natural to conjecture that there exist weak solutions that exhibit jumps in the energy. This is closely related to the anomalous dissipation in the Section 1.2. In fact, one can ask the following questions regarding the behavior of the energy:

Can energy $\|u(t)\|_2^2$ *have jumps? Can it be discontinuous on a dense subset of* $[0,T]$*? Can it be discontinuous almost everywhere? Can it be discontinuous everywhere?*

The answer to the last question is No. Indeed, the energy of a weak solution $\|u(t)\|_2^2$ is lower semi-continuous. Hence, by Baire's theorem, the energy is of the first Baire class and therefore the points of continuity are dense. Nevertheless, we believe that all the previous questions have positive answers. Theorem 1.3.3 can be seen as a first step in solving this conjecture.

1.6 Notations

Throughout this work we use the following standard notations.

- $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{T}^d)}$ is the Lebesgue norm (in space) for any $1 \leq p \leq \infty$ and $\|\cdot\|_{C^m} := \sum_{0\leq i\leq m} \|\nabla^i \cdot\|_\infty$ for any m is the Hölder norm. For uniform in time bounds we will use standard notations $\|\cdot\|_{L^\infty_t L^p}$ and $\|\cdot\|_{L^\infty_t C^m}$.
- For any $\mathbb{T}^d$-periodic function $f \in L^p(\mathbb{T}^d)$ and $\sigma > 0$, the notation $f(\sigma\cdot)$ is the scaled $\sigma^{-1}\mathbb{T}^d$-periodic function $f(\sigma x)$ so that $\|f(\sigma\cdot)\|_p = \|f\|_p$ for L^p norms.

- We say a function f is $\lambda^{-1}\mathbb{T}^d$-periodic if $f(x) = f(x+m)$ for any $m \in \lambda^{-1}\mathbb{Z}^d$. The space $C_0^\infty(\mathbb{T}^d)$ is the set of smooth functions with zero-mean on $\mathbb{T}^d$. $\fint_{\mathbb{T}^d} = \frac{1}{|\mathbb{T}|^d}\int_{\mathbb{T}^d}$ is the average integral any function $f \in L^1(\mathbb{T}^d)$.

- The gradient ∇ always refers to differentiation in space only. Sometimes we use $\nabla_{t,x}$ to indicate that the differentiation is for space-time.

- We say a function $f(x) : \mathbb{T}^d \to \mathbb{R}$ (or $\mathbb{T}^d \to \mathbb{R}^d$ for the vector case) is smooth if f has continuous derivative or any order and we denote $f \in C^\infty(\mathbb{T}^d)$. The space $C_0^\infty(\mathbb{T}^d)$ consists of all smooth functions with zero-mean.

- $x \lesssim y$ stands for the bound $x \leq Cy$ with some constant C which is independent of x and y but may change from line to line. Then $x \sim y$ means $x \lesssim y$ and $y \lesssim x$ at the same time. We use $x \ll y$ to indicate $x \leq cy$ for some small constant $0 < c < 1$.

- The space $C_0^\infty(\mathbb{T}^d)$ is the set of smooth functions with zero-mean on $\mathbb{T}^d$. $\fint_{\mathbb{T}^d} = \frac{1}{|\mathbb{T}|^d}\int_{\mathbb{T}^d}$ is the average integral and for any function $f \in L^1(\mathbb{T}^d)$, its average is denoted by $\overline{f} = \fint_{\mathbb{T}^d} f$.

- For vectors $a, b \in \mathbb{R}^d$, $a \otimes b$ is the matrix with $(a \otimes b)_{ij} = a_i b_j$ and $a \mathring{\otimes} b = a_i b_j(1 - \delta_{ij})$ is the trace-less product. For matrix-value functions $f = f_{ij}$ and $g = g_{ij}$, $\operatorname{div} f = \partial_i f_{ij}$ and $f : g = f_{ij} g_{ij}$.

- Δ_q is the standard periodic Littlewood-Paley projections on to the dyadic frequency shell $2^{q-1} \leq |\xi| \leq 2^{q+1}$ for any $q \geq -1$ and $\Delta_{\leq q} = \sum_{r \leq q} \Delta_r$ and $\Delta_{\geq q} = \sum_{r \geq q} \Delta_r$.

- We also use wavenumber projections to simplify notations. For any $\lambda \in \mathbb{N}$ define $\mathbb{P}_{\leq \lambda} = \sum_{q : 2^q \leq \lambda} \Delta_q$ and $\mathbb{P}_{\geq \lambda} = \operatorname{Id} - \mathbb{P}_{\leq \lambda}$.

- $\mathcal{S}_+^{n\times n}$ denotes the set of positive definite symmetric $n \times n$ matrices and $Q^d = [0,1]^d$ is the d-dimensional box.

CHAPTER 2

STATIONARY WEAK SOLUTIONS IN DIMENSION $D \geq 4$

The content of this chapter has been previously published as X. Luo, Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions, Arch. Ration. Mech. Anal., 233(2):701–747, 2019. Permission to reuse the materials has been obtained and attached in the appendix.

2.1 Background

In this section, let us review the some of the related works for the Navier-Stokes equations in different settings. We first discuss the progress towards proving the global regularity and uniqueness for the 3D Navier-Stokes equations. Then a brief summary of nonuniqueness results and the method of convex integration is given. Lastly, we describe some existence and regularity results on the forced stationary problem in high dimensions.

2.1.0.1 Regularity and uniqueness results in 3D

Since the seminal work of Leray, there have been a substantial amount of conditional regularity and uniqueness results for (Equation NSE) with $d = 3$, the most physical relevant case. Notably the classical Ladyzhenskaya-Prodi-Serrin criterion (51; 66; 63) says that if additionally a Leray-Hopf weak solution also belongs to $L^q_t L^p_x$ for some $\frac{2}{q} + \frac{3}{p} \leq 1$ with $p > 3$ then the solution is regular and unique among all Leray-Hopf solutions with the same initial data. The endpoint case $L^\infty_t L^3_x$ was solved by Escauriaza-Seregin-Šverák in (30) and extensive studies have been devoted to generalize and refine the classical Ladyzhenskaya-Prodi-Serrin criterion, see (43; 55; 9; 14; 10; 41) and reference therein for more conditional regularity and uniqueness results for the 3D NSE.

2.1.1 Nonuniqueness results and the method of convex integration

In the construction of Buckmaster-Vicol (8) weak solutions are allowed to have any prescribed non-negative smooth functions as the energy profiles and hence 0 is not the only weak solution with finite energy by taking a nontrivial compact energy profile. It is worth noting that the solutions constructed in (8) are not known to be Leray-Hopf nor do they have finite dissipation $L^2_t H^1$. So these solutions do not obey the Leray structure theorem of the 3D Navier-Stokes equations on the interval of regularity for Leray-Hopf weak solutions, see for instance the original paper by Leray (52) or the notes by Galdi (40). It would be very interesting to extend the results of Buckmaster-Vicol to the Leray-Hopf weak solutions.

2.1.2 The forced stationary problem

Unlike the case when $d = 3$, fewer results are available for the Navier-Stokes equations in high dimensions $d \geq 4$. We mention here a few studies on the forced stationary problem of NSE with the presence of external force as in a sense they are closely related to the main results in this chapter. The forced stationary problem consists of the following equations:

$$\begin{cases} -\nu\Delta u + \operatorname{div}(u \otimes u) + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \qquad \text{for all } x \in \Omega. \tag{2.1}$$

Remark 2.1.1. *In parallel with Definition 1.1.1 for the unforced NSE, we search for the weak solution to the above system that is $L^2(\Omega)$ and verifies* (Equation 2.1) *in the sense of distribution for all divergence-free test functions $\varphi \in C_c^\infty(\Omega)$ (or $C_0^\infty(\mathbb{T}^d)$ if $\Omega = \mathbb{T}^d$). Different types of formulations of the problem* (Equation 2.1) *have been considered in the literature, cf. (39; 49; 38).*

The existence of regular solutions to (Equation 2.1) has been known under various assumptions on the force f and the domain Ω. In the seventies, Gerhardt studied the four-dimensional case in (42), where he proved that if $f \in L^p$ then if there exists a solution, then $u \in W^{2,p}$. Since then there have been a considerable amount of studies on the forced stationary problem in high dimensions. Frehse-Růžička (35) and Struwe (69) showed the existence and regularity of the solutions in five dimensions. Later Frehse-Růžička obtained existence of regular solutions in bounded six-dimensional domain in (34) and on torus in dimensions up to $d = 15$ in (36). Recently, Farwig and Sohr (32) considered the general d-dimensional case where in particular a uniqueness result was obtained for small force f. We refer readers to (49; 38; 39) and reference therein for more interesting results on the forced stationary problem.

Despite the existence result on the regular solutions for the forced stationary case, the question of uniqueness of regular solutions to (Equation 2.1) remains mostly open, particularly when the data f is large. We offer here a partial result in this direction showing that at least weak solutions are not unique to the stationary problem (Equation 2.1) when $f = 0$.

2.2 Main theorems

Before giving the main results of the chapter, let us state some of the motivations of this work. In the hope of better understanding of the nonuniqueness issue of the Navier-Stokes system in both forced stationary case and unforced time-dependent case, we study the following Liouville-type problem:

(Q) Consider (Equation NSE) for $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$. Does there exist nontrivial stationary weak solution u, i.e. $\partial_t u = 0$ so that $u \in L^p(\Omega)$ for some $2 \leq p < d$ (or $H^s(\Omega)$ for some $0 \leq s < \frac{d-2}{2}$)?

From an energy balancing point of view, one can think of **(Q)** as investigating how strong the nonlinear term is in producing nontrivial energy flux to balance linear dissipation. Such phenomenon is closely related to the Onsager's conjecture and the concept of anomalous dissipation for the Navier-Stokes equations (31; 19; 11). If **(Q)** has a positive answer, then it might be possible to use this mechanism of nontrivial energy flux to construct Leray-Hopf solutions satisfying (Equation E.I.) with strict inequality. In terms of $L^q_t L^p_x$ norms, for the Navier-Stokes equations in dimension $d \geq 3$ the scaling of the conditions implying uniqueness (51; 66; 63; 55) corresponds to $\frac{2}{q} + \frac{d}{p} = 1$ while the one implying energy equality is $\frac{2}{q} + \frac{2}{p} = 1$ (67). So in view of such scaling gap, finding a Leray-Hopf solution with strict energy inequality could be the first step towards obtaining the nonuniqueness. We plan to address these issues in our future studies.

From a uniqueness point of view, a positive answer to the above question **(Q)** immediately would imply the nonuniqueness of weak solutions of (Equation NSE) in the class L^p. Indeed, as initial data, u gives rise to two different weak solutions: a stationary solution $u(t) = u$ itself and the other one $v(t)$ is Leray-Hopf. Then v as a Leary-Hopf weak solution satisfies the energy inequality but u does not and hence they are different. Moreover such existence result would also imply the nonuniqueness of weak solutions of the forced stationary problem (Equation 2.1) for a particular force: there exists a force $f = 0 \in C_0^\infty(\mathbb{T}^d)$ such that the system (Equation 2.1) admits two different solutions, with one trivial solution being regular and the other nontrivial one in L^p. It would be very interesting obtain the same result for other nontrivial forces.

The main aim of this chapter is to prove the following theorems.

Theorem 2.2.1 (Existence of stationary weak solutions)**.** *Suppose $d \geq 4$. There exists non-trivial steady-state weak solution $u \in L^2(\mathbb{T}^d)$ of* (Equation NSE) *.*

Remark 2.2.2. *In fact, we proved a slightly stronger result that the solution lies in $H^\beta(\mathbb{T}^d)$ for every $\beta < \frac{1}{200}$.*

As discussed in the paragraph preceding the statement of our main results, nonuniqueness of weak solutions of (Equation NSE) and of the stationary problem (Equation 2.1) in $d \geq 4$ both follow from Theorem 1.3.1.

Theorem 2.2.3 (Nonuniqueness of the NSE in $d \geq 4$)**.** *Suppose $d \geq 4$. There exists divergence-free initial data $u_0 \in L^2(\mathbb{T}^d)$ so that u_0 admits at least two different weak solutions of* (Equation NSE) *in the sense of Definition 1.1.1.*

Theorem 2.2.4 (Nonuniqueness of the stationary problem)**.** *Suppose $d \geq 4$. There exists a force $f \in C_0^\infty(\mathbb{T}^d)$ so that the system* (Equation 2.1) *admits at least two different weak solutions* *of* (Equation 2.1) *denoted as u and v so that $u = 0 \in C_0^\infty(\mathbb{T}^d)$ is trivial while $v \in L^2(\mathbb{T}^d)$ is nontrivial.*

To the author's knowledge, Theorem 1.3.1 is the first result showing the existence nontrivial stationary weak solutions for the system (Equation NSE) . So it provides a positive answer to **(Q)** for $p = 2$ (or any $p > 2$ sufficiently close to 2). It is interesting that even without the presence of external force, the nonlinear term itself can produce enough energy flux to balance the linear dissipation.

We remark that even though adaptations of the convex integration scheme have already been used for other partial differential equations in fluid dynamics, it is the first time that such method is used for the Navier-Stokes equations in dimensions $d \geq 4$. Moreover, it is also worth noting that the scheme used by Buckmaster-Vicol does not generate stationary weak solutions for the 3D NSE even if one takes a constant energy profile. The reason is that the building blocks of their construction are time-dependent by default. It is not clear whether one can adapt their scheme to obtain stationary weak solutions in 3D NSE. The existence of nontrivial stationary weak solutions of the 3D NSE still remains open. Another benefit of considering stationary weak solutions is that without the time-dependence of the solution our proof is much more streamlined.

It also appears that our current scheme is compatible with the time-dependent case, and it is likely that we can also obtain weak solutions with any given energy profile as in (27; 3; 8). However, it is unlikely that one is able to obtain nonuniqueness of Leray-Hopf weak solutions using current techniques without incorporating substantially new ideas.

2.3 Outline of the construction

In this section, we briefly introduce the main idea of the construction. The proof of Theorem 1.3.1 is based on an iteration scheme. We construct by induction a sequence of smooth solutions to the Navier-Stokes-Reynolds system verifying a certain set of estimates which guarantees the convergence to a stationary weak solution to (Equation NSE) in $L^2(\mathbb{T}^d)$. The iteration process is then summarized in Proposition 2.3.1. After stating the main proposition, we give a proof of Theorem 1.3.1. Lastly, we outline the explicit form of the velocity perturbation and explain its important role in the induction process.

2.3.1 Navier-Stokes-Reynolds system

Let us first recall the Navier-Stokes-Reynolds system[1] in the time-dependent case:

$$\begin{cases} \partial_t u - \Delta u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} R \\ \operatorname{div} u = 0. \end{cases} \tag{2.2}$$

where R is a trace-less symmetric matrix usually termed Reynolds stress in the literature.

The system (Equation NSR) arises naturally in the study of weak solutions of the 3D Navier-Stokes equations and the 3D Euler equations. The tensor R measures the distance to the (Equation NSE) . In fact every weak solution to the original Navier-Stokes equations

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[1] We normalize ν to 1 without loss of generality.

(Equation NSE) can generate a family of solutions v_l of (Equation 2.2). Let u be a weak solution of (Equation NSE) and define

$$v_l = u * \eta_l$$

where $*\eta_l$ is some kind of averaging process in space (for example frequency localization to wavenumber $\lesssim l^{-1}$ or standard smoothing mollifier at length scale $\sim l$) , then v_l is a solution to

$$\begin{cases} \partial_t v_l - \Delta v_l + \operatorname{div}(v_l \otimes v_l) + \nabla p_l = \operatorname{div} R_l \\ \operatorname{div} v_l = 0. \end{cases} \tag{2.3}$$

for some suitable pressure p_l where the symmetric trace-less matrix R_l is defined by

$$R_l = (u \mathring{\otimes} u) * \eta_l - (u * \eta_l) \mathring{\otimes} (u * \eta_l).$$

Since we are constructing stationary weak solutions to (Equation NSE) , it is convenient to consider the following stationary Navier-Stoke-Reynolds system:

$$\begin{cases} -\Delta u + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} R \\ \operatorname{div} u = 0. \end{cases} \tag{NSR}$$

For our consideration here all u, p and R are assumed to be $C_0^\infty(\mathbb{T}^d)$, i.e. smooth and zero-mean.

A sequence of solution triplet $\{(u_n, p_n, R_n)\}_{n\geq 1}$ to (Equation NSR) will be constructed in the proof, and we measure the solutions (u_n, p_n, R_n) by two parameters, a frequency λ_n and a amplitude δ_n. These two parameters are explicitly defined by

$$\begin{aligned} \lambda_n &= \left\lceil a^{b^n} \right\rceil \\ \delta_n &= \lambda_n^{-2\beta} \end{aligned} \tag{2.4}$$

where $\lceil x \rceil$ denotes the smallest integer $n \geq x$, the parameters $a > 0$ and the exponential frequency gap $b > 1$ are large depending on β, which is the L^2 regularity of the solution u. The double exponential growth of λ_n is critical for our proof, see Proposition 2.4.4 in Section 2.4.

Starting with zero solution $(u_1, p_1, R_1) = (0, 0, 0)$ we will construct a sequence of solutions (u_n, p_n, R_n) of the system (Equation NSR) so that the following set of estimates is verified:

$$\|R_n\|_1 \leq \delta_{n+1}\lambda_n^{-2\alpha} \tag{H1}$$

$$\|u_n\|_2 \leq 1 - \delta_n^{1/2} \tag{H2}$$

$$\|\nabla u_n\|_2 \leq \lambda_n \delta_n^{1/2} \tag{H3}$$

where $0 < a < \beta$ is another small parameter depending on β and b. The exact values of all the parameters will be given in Section 2.5.

We remark that unlike the schemes used for the 3D Euler equations, cf. (3; 45; 27), here the Reynolds stress R_n is measured in L^1 norm rather than L^∞ norm. The reason is that L^d is the critical norm for (Equation NSE) in d dimension. So as pointed out in the introduction, no stationary solution exists in $L^\infty(\mathbb{T}^d)$[1] regardless of the dimensions and hence R_n can not have any decay in L^∞.

With these in mind we state the main proposition.

Proposition 2.3.1. *There exists a sufficiently small $0 < \beta \ll 1$ such that we can find $b > 1$, $0 < \alpha \ll \beta$ and $a \gg 1$ so that there exists a sequence of smooth solution triplets (u_n, p_n, R_n) to the system* (Equation NSR) *for $n \in \mathbb{N}$ starting from $(u_1, p_1, R_1) = (0, 0, 0)$ verifying* (Equation H1), (Equation H2) *and* (Equation H3)*. Moreover each velocity increment $u_n - u_{n-1}$ is nontrivial and we have the estimate:*

$$\|u_n - u_{n-1}\|_2 + \frac{1}{\lambda_n}\|\nabla u_n - \nabla u_{n-1}\|_2 \le \lambda_n^{-\beta}. \tag{2.5}$$

Remark 2.3.2. *For example we can take $\beta = \frac{1}{200}$, $b = 5$, $\alpha = 10^{-6}$ independent of dimension d and a sufficiently large depending on some implicit constants from the computation. Such choice of β and b is definitely not optimal. By optimizing one can take larger β as the dimension d increases. However one is unlikely to get close to the critical space $H^{\frac{d-2}{2}}$ or L^d without substantially new ideas. So we do not pursue additional improvement in the regularity using the current scheme in this direction.*

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[1]Here and in what follows, weak solutions refer to Definition 1.1.1.

Proof of Theorem 1.3.1. Let (u_n, p_n, R_n) be the sequence obtained from Proposition 2.3.1 and let $0 < \beta' < \beta$ First, by the Sobolev interpolation

$$\sum_{k\geq n} \|u_k - u_{k-1}\|_{H^{\beta'}(\mathbb{T}^d)} \leq \sum_{k\geq n} \|u_k - u_{k-1}\|_{H^1(\mathbb{T}^d)}^{\beta'} \|u_k - u_{k-1}\|_{L^2(\mathbb{T}^d)}^{1-\beta'}$$
$$\lesssim \sum_{k\geq n} \|\nabla u_k - \nabla u_{k-1}\|_2^{\beta'} \|u_k - u_{k-1}\|_2^{1-\beta'}.$$

Then directly from the estimate (Equation 2.5) we find that

$$\sum_{k\geq n} \|u_k - u_{k-1}\|_{H^{\beta'}(\mathbb{T}^d)} \lesssim \sum_{k\geq n} (\lambda_n \delta_n)^{\beta'}$$
$$\lesssim \sum_{k\geq n} \lambda_n^{\beta'-\beta} \lesssim \lambda_k^{-\beta'}$$

which means u_n is uniformly bounded in $H^{\beta'}$. So by the compactness of the embedding $H^{\beta'}(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d)$ after possibly passing to a subsequence which we still denote as u_n, we find that there exists a $u \in L^2(\mathbb{T})$ so that

$$u_n \to u \quad \text{strongly in } L^2.$$

Now we need to show that u is a weak solution of (Equation NSE) . This is done by a standard argument. Let $\varphi(x) \in C_0^\infty(\mathbb{T}^d)$. Multiplying (Equation NSR) by φ and integrating in space give

$$\int_{\mathbb{T}^d} -\varphi \cdot \Delta u_n + \varphi \cdot \operatorname{div}(u_n \otimes u_n) + \varphi \cdot \nabla p_n = \int_{\mathbb{T}^d} \varphi \cdot \operatorname{div} R_n.$$

Using the fact that u_n is divergence-free and integrating by parts we find that

$$\int_{\mathbb{T}^d} u_n \cdot \Delta\varphi + \int_{\mathbb{T}^d} u_n \cdot (u_n \cdot \nabla)\varphi - \int_{\mathbb{T}^d} \nabla\varphi : R_n = 0.$$

Due to the strong convergence of u_n in L^2 the first two terms converge to their natural limit:

$$\Big| \int_{\mathbb{T}^d} u_n \cdot \Delta\varphi - \int_{\mathbb{T}^d} u \cdot \Delta\varphi \Big| \leq \|u_n - u\|_2 \|\Delta\varphi\|_2 \to 0 \quad \text{as } n \to \infty;$$

and

$$\begin{aligned}
\Big| \int_{\mathbb{T}^d} u_n \cdot (u_n \cdot \nabla)\varphi - \int_{\mathbb{T}^d} u \cdot (u \cdot \nabla)\varphi \Big| &\leq \Big| \int_{\mathbb{T}^d} (u_n - u) \cdot (u_n \cdot \nabla)\varphi - u \cdot ((u - u_n) \cdot \nabla)\varphi \Big| \\
&\leq \|u - u_n\|_2 \|u_n\|_2 \|\nabla\varphi\|_\infty + \|u - u_n\|_2 \|u\|_2 \|\nabla\varphi\|_\infty \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

By the estimate (Equation H1) it follows that

$$R_n \to 0 \quad \text{strongly in } L^1,$$

and then

$$\Big| \int_{\mathbb{T}^d} \nabla\varphi : R_n \Big| \le \|R_n\|_1 \|\nabla\varphi\|_\infty \to 0 \quad \text{as } n \to \infty.$$

So $u \in H^{\beta'}$ for some $\beta' > 0$ and verifies the weak formulation of (Equation NSE) . To recover the pressure p associated to the solution u we can use the formula

$$p_n = \Delta^{-1} \operatorname{div}\operatorname{div}(u_n \otimes u_n + R_n).$$

To ensure p_n converges to some p in L^r for some $r > 1$ we need better convergence of u_n and R_n. This can be done by using the fact that $u_n \to u$ in $H^{\beta'}$ to obtain that there is some small $\epsilon > 0$ so that after possible relabeling

$$u_n \to u \quad \text{strongly in } L^{2+2\epsilon}$$

$$R_n \to 0 \quad \text{strongly in } L^{1+\epsilon}.$$

Then by the L^p boundedness of the Riesz transform for $p > 1$ we know

$$p_n \to p \quad \text{strongly in } L^{1+\epsilon}$$

for some $p \in L^{1+\epsilon}$. $\square$

2.3.2 The perturbation w_n

The main task is to construct w_n given $(u_{n-1}, p_{n-1}, R_{n-1})$. The exact definition for the precise w_n will be given later in Section 2.5. It should be noted that the exact scheme is more complicated than what we describe here.

We aim to design the w_n so that it gives rise to a new solution triplet (u_n, p_n, R_n) verifying (Equation H1), (Equation H2) and (Equation H3). To the leading order w_n will be of the form

$$w_n(x) = \sum_{i,k} a_{i,k}(R_{n-1})\psi_{i,k}^{\mu_n}(\sigma_n x)$$

where $a_{i,k}$ are the coefficients for the concentrated Mikado flow that will be defined by R_{n-1} and have low frequency $\sim \lambda_{n-1}$, the variable μ_n is concentration parameter so that each $\psi_{i,k}^{\mu_n}(x)$ is supported in some cylinder with radius μ_n and σ_n is oscillation parameter so that $\psi_{i,k}^{\mu_n}(\sigma_n \cdot)$ is $\sigma_n^{-1}\mathbb{T}^d$-periodic and supported on cylinders of radius λ_n^{-1} on $\mathbb{T}^d$. Thus w_n has frequency λ_n in the sense that $\lambda_n = \sigma_n \mu_n$.

To explain the role of σ_n and μ_n let us recall that for the new Reynolds stress R_n we need to solve the divergence equation:

$$\begin{aligned}\operatorname{div} R_n = \underbrace{\operatorname{div} \Delta w_n}_{\text{Linear error}} + \underbrace{\operatorname{div}(w_n \otimes w_n + R_{n-1})}_{\text{Oscillation error}} \\ + \underbrace{\operatorname{div}(u_{n-1} \otimes w_n + w_n \otimes u_{n-1})}_{\text{Quadratic error}} + \nabla(p_n - p_{n-1}).\end{aligned}$$

The idea of convex integration is to use the interaction $w_n \otimes w_n$ to balance the previous Reynolds stress R_n in the sense that

$$\operatorname{div}(w_n \otimes w_n + R_n) + \nabla p_n = \text{High frequency error term.} \tag{2.6}$$

And more importantly we need to do this while keeping the "Linear error" under control. So it is required for w_n to have small intermittent dimension $D < d - 2$. On the other hand in order to control the "Oscillation error" we need to have large spacing between each Fourier mode of w_n which is parametrized by σ_n.

The role of σ_n is to ensure $|\psi_{i,k}^{\mu_n}(\sigma_n \cdot)|^2$ only has Fourier modes of multiples of σ_n, namely

$$|\psi_{i,k}^{\mu_n}(\sigma_n \cdot)|^2 = \sum_{m \in \mathbb{Z}^d} \widehat{|\psi_{i,k}^{\mu_n}|^2}(m) e^{2\pi \sigma m \cdot x}$$

such that the "High frequency error term" obeys the right inductive estimate. To this end we will invoke a commutator-type estimate that takes advantage of the fast oscillation of $|\psi_{i,k}^{\mu_n}(\sigma_n \cdot)|^2$, for which it is required that

$$\lambda_{n-1} \ll \sigma_n. \tag{2.7}$$

On the other hand since $\psi_{i,k}^{\mu_n}$ is designed to be supported on small cylinders of radius μ_n, one expects the saturation of the Bernstein inequality up to the exponent $d-1$:

$$\|\psi_{i,k}^{\mu_n}\|_p \sim \mu_n^{\frac{d-1}{2}-\frac{d-1}{p}} \tag{2.8}$$

namely $\psi_{i,k}^{\mu_n}$ is of intermittency dimension $D = 1$. This is also the reason our construction only works in dimensions $d \geq 4$ since for the 3D Navier-Stokes equations the solution is regular if the intermittency is equal of greater than $3-2=1$. And as we shall see in the following discussion, the intermittent dimension of the perturbation w_n is strictly bigger than 1.

Taking the fast oscillation parameter σ_n into account the "Linear error" verifies

$$\begin{cases} \|\nabla w_n\|_1 \lesssim \delta_{n+1}^{\frac{1}{2}} \mu_n^{-\frac{d-1}{2}} \lambda_n \\ \|\nabla w_n\|_2 \lesssim \delta_{n+1}^{\frac{1}{2}} \lambda_n \end{cases} \qquad \text{with } \lambda_n = \sigma_n \mu_n. \tag{2.9}$$

To see what is the intermittency dimension of w_n, let $\mu_n = \lambda_n^{\mu}$ and $\sigma_n = \lambda_n^{\sigma}$ for some $\mu + \sigma = 1$. Then by simple algebra and the definition of intermittency in term of Littlewood-Paley decomposition (cf. Section 1.5) we find that

$$d - D = \mu(d-1).$$

Since $\mu < 1$ due to the requirement that $\lambda_{n-1} \ll \sigma_n$, we can infer that $D > 1$, which is the reason that the construction breaks down in 3D. Furthermore, to make sure the "Linear error" is small in L^1 we need

$$\sigma_n \ll \mu_n \tag{2.10}$$

which will ensure that $D < d - 2$ when $d \geq 4$, cf. Section 1.5. Then (Equation 2.10) and (Equation 2.7) together imply that $\lambda_{n-1} \ll \lambda_n$, namely the frequency gap $b \gg 1$. In view of the quadratic relation

$$\|w_n \otimes w_n\|_1 \sim \|R_{n-1}\|_1,$$

the regularity of w_n is determined by R_{n-1} so the large gap $b \gg 1$ results in a very small amount of regularity of w_n.

2.4 Concentrated Mikado flows

In this section, the building blocks of the solution sequence are constructed. Based on a variation of the Mikado flows introduced by Daneri and Székelyhidi Jr. in (25), we called these building blocks concentrated Mikado flows. As pointed out in the introduction, the idea of increasing the concentration of the flows is not new. Very recently, we learned that Modena and Székelyhidi Jr. had used a similar idea to tackle the nonuniqueness problem for transport equation and continuity equation, see (60; 61). However, being a systems of equations, the Navier-Stokes equations are fundamentally different than the transport equation and the continuity equation, which are both scalar equations. Here we point out some of the major differences between our construction and the "Mikado densities" and "Mikado fields" constructed by Modena-Székelyhidi Jr. in (60; 61).

1. The error terms in (60; 61) (counterparts of the Reynolds stress in our setting, see Section 2.3.1) are not matrices but vectors, therefore one does not need the geometric lemma for the space of symmetric traceless matrices, i.e. Lemma 2.4.1.

2. To resort the "leakage" when partitioning the Reynolds stress, we use another index i so that there are multiple flows in the same direction but at different locations.

3. The estimates for the perturbations in (60; 61) are scaling invariant, while in our case scalings are much more complicated since we will take advantage of the superexponetial nature of (Equation 2.4) using a commutator estimate, Proposition 2.4.4.

2.4.1 The velocity profile $\psi^{\mu}_{i,k}$

We first choose the velocity profile of the flow in this subsection. We fix a profile function $\psi : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp}\psi \subset [1/2, 1]$ so that we have

$$\int_{\mathbb{R}} \psi \, dx = 0. \tag{2.11}$$

Let $\mathbb{K} \subset \mathbb{Z}^d$ be a given finite set of lattice vectors and $N \in \mathbb{N}$. Since $d \geq 4$ we can then choose a collection of points $p_{i,k}$ for any $k \in \mathbb{K}$, $0 \leq i \leq N$ and a number $\mu_0 > 0$ with the following properties: Let

$$l_{i,k} := \{p_{i,k} + tk + m : t \in \mathbb{R}, m \in \mathbb{Z}^d\} \subset \mathbb{T}^d$$

be the $\mathbb{T}^d$-periodization of the line passing through $p_{i,k}$ in the k direction. Since $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $k \in \mathbb{K} \subset \mathbb{Z}^d$, the line only goes around the box $Q^d = [0, 1]^d$ finitely many times. So if we let

$$N_\delta(l_{i,k}) := \{x + h : x \in l_{i,k},\ |h| \leq \delta\}$$

denote the closed δ-neighborhood of $l_{i,k}$, then

$$N_{\mu_0^{-1}}(l_{i,k}) \cap N_{\mu_0^{-1}}(l_{j,k'}) = \emptyset \quad \text{if } k \neq k' \text{ or } i \neq j.$$

For each direction $k \in \mathbb{K}$, any $0 \leq i \leq N$ and any $\mu \geq \mu_0$ we define a profile function $\psi^{\mu}_{i,k}$ as

$$\psi^{\mu}_{i,k}(x) = c_{k,p}\mu^{\frac{d-1}{2}}\psi(\mu \mathrm{dist}(x, l_{i,k})) \tag{2.12}$$

where $c_{k,p}$ are normalizing constants so that

$$\fint_{\mathbb{T}^d} |\psi^{\mu}_{i,k}(x)|^2 dx = 1. \tag{2.13}$$

It is easy to see that due to (Equation 2.11), we have

$$\int_{\mathbb{T}^d} \psi^{\mu}_{i,k}(x) dx = 0. \tag{2.14}$$

Indeed, to show (Equation 2.14) one can use cylindrical coordinates along the direction k as $(z, r, \theta_1, \ldots, \theta_{d-2})$. Then

$$\begin{aligned}\int_{\mathbb{T}^d} \psi(\mu \mathrm{dist}(x, l_{i,k}))dx &= \int \prod_j f_j(\theta_1, \ldots, \theta_{d-2})^{\alpha_j} \psi(\mu r) dz dr d\theta_1 \ldots d\theta_{d-2} \\ &= 0.\end{aligned}$$

2.4.2 Definition of concentrated Mikado flows

It is clear that $\nabla \psi^{\mu}_{i,k} \cdot k = 0$ since $\psi^{\mu}_{i,k}$ is a smooth function on $\mathbb{T}^d$ whose level sets are concentric periodic cylinders with axis l_k. Immediately we have the following properties.

1. Every $\psi^{\mu}_{i,k} k$ is divergence free: $\nabla \cdot \psi^{\mu}_{i,k} k = 0$.

2. $\psi^{\mu}_{i,k}k$ solves d-dimensional Euler equations: $\mathrm{div}(\psi^{\mu}_{i,k}k \otimes \psi^{\mu}_{i,k}k) = \psi^{\mu}_{i,k}k \cdot \nabla\psi^{\mu}_{i,k}k = 0$.

3. These vector fields $\psi^{\mu}_{i,k}k$ have disjoint support: $\operatorname{supp}\psi^{\mu}_{i,k}k \cap \operatorname{supp}\psi^{\lambda}_{j,k'} = \emptyset$ if $k \neq k'$ or $i \neq j$.

4. $\psi^{\mu}_{i,k}k$ has intermittency dimension $D = 1$, namely

$$\|\nabla^m \psi^{\mu}_{i,k}\|_p \lesssim \mu^m \mu^{\frac{d-1}{2} - \frac{d-1}{p}}. \tag{2.15}$$

Recall that $\mathcal{S}^{d\times d}_{+}$ is the set of positive definite symmetric $n \times n$. The next geometric lemma allows us to form any R in a compact subset of $\mathcal{S}^{d\times d}_{+}$. A proof can be found in (70) or (62).

Lemma 2.4.1. *For any compact subset $\mathcal{N} \subset\subset \mathcal{S}^{d\times d}_{+}$, there exists $\lambda_0 \geq 1$ and smooth functions $\Gamma_k \in C^\infty(\mathcal{N}; [0,1])$ for any $k \in \mathbb{Z}^d$ with $|k| \leq \lambda_0$ such that*

$$R = \sum_{k\in\mathbb{Z}^d, |k|\leq\lambda_0} \Gamma_k^2(R) k \otimes k \qquad \textit{for all } R \in \mathcal{N}.$$

Finally we can define the concentrated Mikado flows $W^{\mu}_{i}(R,x) : \mathcal{B} \times \mathbb{T}^d \to \mathbb{R}^d$ as follows. We first apply Lemma 2.4.1 with $\mathcal{N} = \mathcal{B}$, where

$$\mathcal{B} = \{R \in \mathcal{S}^{d\times d}_{+} : |\operatorname{Id} - R| \leq \frac{1}{2}\}$$

is the ball of radius 1/2 centering at Id in $\mathcal{S}_+^{d\times d}$. This fixes the direction set $\mathbb{K} = \{k \in \mathbb{Z}^d : |k| \leq \lambda_0\}$, where λ_0 is obtained from Lemma 2.4.1. Then given $N \in \mathbb{N}$, by Section 2.4.1 there is a $\mu_0 > 1$ depending on N, d and $\mathbb{K}$ so that for any $\mu \geq \mu_0$ we let

$$W_i^{\mu}(R,x) = \sum_{k\in\mathbb{Z}^d} \Gamma_k(R)\psi_{i,k}^{\mu} k \quad \text{for any } R \in \mathcal{B} \text{ and } 0 \leq i \leq N. \tag{2.16}$$

Remark 2.4.2. *The first lower index i in $\psi_{i,k}^{\mu}k$ is to have multiple flows in the same direction k. So each flow $W_i^{\mu}(R,x)$ can only interact with itself to recover the matrix R as shown in the lemma below. This ensures the proper separation properties and is needed to control the "leakage" when partitioning the Reynolds stress in Section 2.5.*

Let us summarize the properties in the lemma below for future reference.

Lemma 2.4.3. *Suppose $d \geq 4$ and let $\mathcal{B}$ be the ball of radius $1/2$ centering at Id in $\mathcal{S}_+^{d\times d}$. For any $N \in \mathbb{N}$ there exists $\mu_0 > 1$ depending on N and d so that the divergence-free smooth vector fields $W_i^{\mu}(R,x) : \mathcal{B} \times \mathbb{T}^d \to \mathbb{R}^d$ defined above for $\mu \geq \mu_0$ and indexed by $0 \leq i \leq N$ have the following properties.*

$$W_i^{\mu}(R,x) \otimes W_i^{\mu}(R,x) = R + \sum_{k\in\mathbb{Z}^d} \Gamma_k^p(R)\big[\big(\psi_{i,k}^{\mu}\big)^2 - 1\big]\ k\otimes k \quad \textit{for all } R \in \mathcal{N}. \tag{2.17}$$

$$\operatorname{supp} W_i^{\mu}(R,x) \cap \operatorname{supp} W_j(R,x) = \emptyset \quad \textit{if } i \neq j. \tag{2.18}$$

Moreover the profile function $\psi^{\mu}_{i,k}$ obeys the bound:

$$\|\nabla^m \psi^{\mu}_{i,k}\|_p \lesssim \mu^m \mu^{\frac{d-1}{2}-\frac{d-1}{p}} \quad \text{for any } 1 \le p \le \infty \text{ and } m \in \mathbb{N}. \tag{2.19}$$

where the implicit constant depends on i, k, p and m but is dependent of μ.

2.4.3 A commutator estimate

Next, we need a commutator-type estimate involving functions with fast oscillation, which is crucial in obtaining the L^2 decay of the perturbation w_n. It should be noted that a similar result for $p = 1$ and 2 has been established in (8) using a different method. However, our result here requires a weaker assumption.

Proposition 2.4.4 (Commutator for fast oscillation)**.** *For any small $\theta > 0$ and any large $N > 0$ there exists $M \in \mathbb{N}$ and $\lambda_0 \in \mathbb{N}$ so that for any $\mu, \sigma \in \mathbb{N}$ satisfying $\lambda_0 \le \mu \le \sigma^{1-\theta}$ the following holds. Suppose $a \in C^\infty(\mathbb{T}^d)$ and let $C_a > 0$ be such that*

$$\|\nabla^i a\|_\infty \le C_a \mu^i \quad \text{for any } 0 \le i \le M.$$

Then for any $\sigma^{-1}\mathbb{T}^d$ periodic function $f \in L^p(\mathbb{T}^d)$, $1 < p < \infty$, the following estimates are satisfied.

- *If $p \ge 2$ is even, then*

$$\left\|af\right\|_p \lesssim_{p,d,\theta,N} \|a\|_p \|f\|_p + C_a \|f\|_p \sigma^{-N}. \tag{2.20}$$

- *If $\fint_{\mathbb{T}^d} f = 0$ then for $0 \leq s \leq 1$:*

$$\left\||\nabla|^{-1}(af)\right\|_p \lesssim_{p,s,d,\theta,N} \sigma^{-1+s}\left\||\nabla|^{-s}(af)\right\|_p + C_a\|f\|_p\sigma^{-N}. \tag{2.21}$$

All the implicit constants appeared in the statement are independent of a, μ and σ.

Remark 2.4.5. *In fact,* (Equation 2.21) *also holds for other -1 degree homogeneous Fourier multipliers, for example the inverse divergence operator $\mathcal{R}$ defined by* (Equation 2.60) *in Section 2.5.*

The proof of Proposition 2.4.4 is not difficult and we give one using the Littlewood-Paley decomposition in the Appendix A.3.

The significance of Proposition 2.4.4 is clearer upon recalling the Ansatz for the velocity increment:

$$w_n(x) = \sum_{i,k} a_{i,k}(R_{n-1})\psi^{\mu_n}_{i,k}(\sigma_n x). \tag{2.22}$$

So by the usual Hölder's inequality we can get the trivial estimate

$$\|w_n\|_2 \leq \sum_{i,k} \left\|a_{i,k}(R_{n-1})\right\|_\infty \left\|\psi^{\mu_n}_{i,k}(\sigma_n \cdot)\right\|_2$$

which is too big for the final solution u to be in L^2.

In contrast, if taking into account the fast oscillation $\lambda_{n-1} \ll \sigma_n$ we would apply Proposition 2.4.4, and then (Equation 2.22) can be estimated as

$$\|w_n\|_2 \lesssim \big\|a_{i,k}(R_{n-1})\big\|_2 \big\|\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\big\|_2,$$

and since

$$\big\|a_{i,k}(R_{n-1})\big\|_2 \ll \big\|a_{i,k}(R_{n-1})\big\|_\infty$$

we shall see this approach indeed gives the desired bound for final solution u to be in L^2.

2.5 Proof of Proposition 2.3.1

Let us give the main steps of the proof here. We prove by induction. It is obvious that $(0,0,0)$ verifies all the inductive estimates (Equation H1), (Equation H2) and (Equation H3). Given a solution triplet $(u_{n-1}, p_{n-1}, R_{n-1})$ verifying (Equation H1), (Equation H2), and (Equation H3) with $n-1$ in place of n we aim to construct a new triplet (u_n, p_n, R_n) verifying the same set of estimates so that for the velocity increment $u_n - u_{n-1}$ it holds (Equation 2.5). The main part of the proof is to find the proper velocity increment, which consists of the following several steps. We first set up all the constants except a in the beginning to convince the reader there is no loophole. Then we mollify the solution triplet $(u_{n-1}, p_{n-1}, R_{n-1})$ to obtain a mollified solution $(\overline{u}_l, \overline{p}_l, \overline{R}_l)$ and we derive several standard estimates for $(\overline{u}_l, \overline{p}_l, \overline{R}_l)$. The goal is then to find the perturbation w_n so that $u_n = \overline{u}_l + w_n$ verifies the inductive hypothesis (Equation H2) and (Equation H3). After mollification, we introduce a partition to properly decompose the Reynolds stress $\overline{R}_l$ so that Lemma 2.4.3 applies. Once the decomposition of the

Reynolds stress $\overline{R}_l$ is done, the velocity perturbations w_n^p and w_n^c can be defined, where w_n^p is the principle part and w_n^c is a lower order correction to ensure the divergence-free condition of w_n. We then derive a number of estimates for w_n^p and w_n^c from their definitions. After all the preparations, the new Reynolds stress R_n can be solved from a divergence equation, and the last inductive hypothesis (Equation H1) follows by using all the established estimates for $\overline{u}_l$, w_n^p and w_n^c.

Step 1: Set up constants.

We fix all the constants β, b, α appeared in the statement except a. The value of a will be required to be larger several times in the following, mainly to absorbed various implicit constants from computations.

- First, let $\beta = \frac{1}{200}$, $b = 5$ and $\alpha = 10^{-6}$ regardless of the dimensions $d \geq 4$.
- Second, we define respectively the concentration and oscillation parameters μ_n, σ_n as

$$\mu_n = \lambda_n^{\frac{3}{4}} \quad \text{and} \quad \sigma_n = \lambda_n^{\frac{1}{4}}. \tag{2.23}$$

- Third, let $l > 0$ be as

$$l = \frac{\delta_n^{1/2}}{\delta_{n-1}^{1/2} \lambda_{n-1}^{1+\alpha}}. \tag{2.24}$$

Then $\lambda_{n-1} \leq l^{-1} \leq \lambda_{n-1}^{1+\frac{1}{40}}$ and there exists $\theta > 0$ so that $l^{-1-\theta} \leq \lambda_n$.

Let us explain the role of each parameter. The parameter β is the resulting regularity of the final solution u, i.e. $u \in H^\beta$. Since we aim to prove the existence of stationary weak solutions in L^2, we need the compactness of some embedding $H^\beta \hookrightarrow L^2$ to obtain a convergent sequence.

The parameter b is the exponential frequency gap between each step of the iteration. Recall that the frequency is given by $\lambda_n = a^{(b^n)}$. So $\lambda_{n-1}^b = \lambda_n$ and hence $\lambda_{n-1} \ll \lambda_n$.

Compared with β, the parameter $\alpha > 0$ is very small which will be used for absorbing lower order factors in conjunction with a large $a \gg 1$. More precisely, in the sequel we often use the following fact: for any constant $C > 1$ there is a sufficiently large $a > 0$ so that $C\lambda_{n-1}^{-\alpha} \leq 1$ (recall that $n \geq 2$ in the proof so in view of (Equation 2.4) this is possible).

Step 2: Mollification.

This step is to fix the problem of possible loss of derivative which is usual in convex integration. Since by mollifying we inevitably introduce new errors from the nature of noncommutativity between (Equation NSR) and mollification, the length scale that we use for the mollification is expected to be larger than λ_{n-1} so that the new errors are not too large. This is the reason that we choose the length scale l as in (Equation 2.24). Fix a standard mollifying kernel η in space and define the mollifications of the velocity field, pressure scaler and Reynolds stress

$$\overline{u}_l = \eta_l * u_{n-1}$$

$$\overline{p}_l = \eta_l * p_{n-1} + \eta_l * (u_{n-1}^2) - |\eta_l * u_{n-1}|^2$$

$$\overline{R}_l = \eta_l * R_{n-1} + \eta_l * (u_{n-1} \mathring{\otimes} u_{n-1}) - (\eta_l * u_{n-1} \mathring{\otimes} \eta_l * u_{n-1})$$

where l is the length parameter defined by (Equation 2.24) and $\mathring{\otimes}$ is the trace-less tensor product $f\mathring{\otimes}g := f_i g_j - \delta_{ij} f_i g_j$.

And hence we have a mollified solution triplet $(u_l, p_l, \overline{R}_l)$ verifying the following mollified system:

$$\begin{cases} -\Delta\overline{u}_l + \operatorname{div}(\overline{u}_l \otimes \overline{u}_l) + \nabla\overline{p}_l = \operatorname{div}\overline{R}_l \\ \operatorname{div}\overline{u}_l = 0 \end{cases} \tag{Mollified-NSR}$$

We can derive the following set of estimates by standard properties of mollifier.

Lemma 2.5.1. *For any $m \in \mathbb{N}$ we have*

$$\|\overline{u}_l - u_{n-1}\|_2 \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} \tag{2.25}$$

$$\|\nabla^{m+1}\overline{u}_l\|_2 \lesssim l^{-m} \delta_{n-1}^{\frac{1}{2}} \lambda_{n-1} \tag{2.26}$$

$$\|\nabla^m \overline{R}_l\|_1 \lesssim l^{-m} \delta_n \lambda_{n-1}^{-2\alpha}, \tag{2.27}$$

where all implicit constants are independent of n and l.

Proof. By the hypothesis (Equation H3) and the obvious estimate for mollifier, we have that

$$\|\overline{u}_l - u_{n-1}\|_2 \leq l\|\nabla u_{n-1}\|_2 \lesssim \delta_{n-1}^{\frac{1}{2}} \lambda_{n-1} l \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}$$

where we have used (Equation 2.24) to get the last inequality.

Again by the standard property of mollifier, we obtain that

$$\|\nabla^{m+1}\overline{u}_l\|_2 \lesssim l^{-m}\|\nabla u_{n-1}\|_2 \lesssim \delta_{n-1}^{\frac{1}{2}}\lambda_{n-1}l^{-m}.$$

The last inequality for $\overline{R}_l$ follows from the Constantin-E-Titi commutator estimate, Proposition A.2.1 in the Appendix A.2:

$$\|\nabla^m\overline{R}_l\|_1 \lesssim l^{-m}\|R_{n-1}\|_1 + \|\nabla u_{n-1}\|_2^2 l^{2-m}.$$

Using the definition of the length scale (Equation 2.24) and (Equation H3) we have

$$\|\nabla u_{n-1}\|_2^2 l^2 = l^2\delta_{n-1}\lambda_{n-1}^2 \leq l^{-m}\delta_n\lambda_{n-1}^{-2\alpha}.$$

So it follows that

$$\|\nabla^m\overline{R}_l\|_1 \lesssim l^{-m}\|R_{n-1}\|_1 + l^{-m}\delta_n\lambda_{n-1}^{-2\alpha} \lesssim l^{-m}\delta_n\lambda_{n-1}^{-2\alpha}.$$

□

Remark 2.5.2. *It is worth noting that the constants in Lemma 2.5.1 depend on m. This type of dependence will appear in the sequel as well. However, we will only require these higher order Sobolev norms up to some fixed order throughout the proof.*

Step 3: Decompose the Reynolds stress

Recall that Lemma 2.4.3 is for symmetric matrices in a given compact subset $\mathcal{B} \subset \mathcal{S}_+^{d\times d}$. Since $\overline{R}_l$ is measured in L^1, we need to properly decompose $\overline{R}_l$ so that we are able to use the concentrated Mikado flows.

Choose a smooth cutoff function $\chi : \mathbb{R}^{d\times d} \to [0,1]$ so that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \in [0, \frac{3}{4}] \\ \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{2.28}$$

Now let $\chi_i(x) = \chi(4^{-i}x)$ for any $i \geq 0$ and define positive cutoff functions $\phi_i : \mathbb{R}^{d\times d} \to [0,1]$ as

$$(\phi_i)^{\frac{1}{2}}(x) = \begin{cases} \chi_i - \chi_{i-1} & \text{if } i \geq 1 \\ \\ \chi_0 & \text{if } i = 0 \end{cases} \tag{2.29}$$

so that we have by telescoping

$$\phi_0^2(x) + \sum_{i\geq 1} \phi_i^2(x) = 1.$$

Then define the partition for the Reynolds stress

$$\chi_{i,n}(x) := \phi_i\Big(\frac{\overline{R}_l}{\delta_n \lambda_{n-1}^{-2\alpha}}\Big) \quad \text{for any } i \in \mathbb{N}. \tag{2.30}$$

We are ready to define the velocity increment. First apply Lemma 2.4.3 with $N = 1$ and then let the principle part of the perturbation be:

$$w_n^p(x) = \sum_{i \geq 0} 2^{i+1} \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} \chi_{i,n} W_{[i]}^{\mu_n}\left(\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n \lambda_{n-1}^{-2\alpha}}, \sigma_n x \right) \tag{2.31}$$

where $[i] = i \mod 2$. Since

$$4^{i-1}\delta_n \lambda_{n-1}^{-2\alpha} \leq |\overline{R}_l| \leq 4^i \delta_n \lambda_{n-1}^{-2\alpha} \quad \text{for all } x \in \operatorname{supp} \chi_{i,n}$$

we know that

$$\left| \frac{\overline{R}_l}{4^{i+1}\delta_n \lambda_{n-1}^{-2\alpha}} \right| \leq \frac{1}{4}$$

and thus

$$\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n \lambda_{n-1}^{-2\alpha}} \in \mathcal{B} \subset \mathcal{S}_+^{d \times d} \quad \text{for all } x \in \mathbb{T}^d.$$

So w_n^p is well-defined in view of Lemma 2.4.3. In what follows, with a slight abuse of notations we often use the shorthand

$$w_n^p = \sum_i a_{i,k,n} \psi_{i,k}^{\mu_n}(\sigma_n \cdot) \tag{2.32}$$

where it is understood that in the functions $\psi_{i,k}^{\mu_n}$ the lower index $i = 0$ when even and $i = 1$ when odd and the short notation $a_{i,k,n}$ is defined by

$$a_{i,k,n} = 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\chi_{i,n}k\Gamma_k\Big(\,\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big). \tag{2.33}$$

Let us show that w_n^p is non-trivial in the following. Recall that $\mathcal{B} \subset \mathcal{S}_+^{d\times d}$ is the ball of radius $\frac{1}{2}$ centered at Id. So $0 \notin \mathcal{B}$, which means that

$$\text{for any } R \in \mathcal{B} \text{ there exists at least a } k \in \mathbb{Z}^d \text{ so that } \Gamma_k(R) \neq 0. \tag{2.34}$$

And then it follows that for any $x \in \mathbb{T}^d$ there exists at least one Γ_k so that

$$\Gamma_k\Big(\,\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big) \neq 0. \tag{2.35}$$

Thus due to the partition (Equation 2.30) for this particular k there exists $i \in \mathbb{N}$ so that

$$|a_{i,k,n}| > 0 \quad \text{for some } x \in \mathbb{T}^d \tag{2.36}$$

which makes sure the non-triviality of w_n^p.

One notices that w_n^p is not divergence-free. However we can fix this by introducing a corrector:

$$w_n^c = -|\nabla|^{-1}\mathcal{R}_j\Big[\sum_i \big(\operatorname{div} a_{i,k,n}\big)\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\Big] \tag{2.37}$$

where $\mathcal{R}_j$ is the Riesz transform with symbol $\frac{k_j}{|k|}$ for $1 \leq j \leq d$. Again for better exposition we will use the short notation

$$w_n^c = -|\nabla|^{-1}\mathcal{R}_j \sum_i b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot) \tag{2.38}$$

where

$$b_{i,k,n} = \operatorname{div} a_{i,k,n} = 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\operatorname{div}\Big[\chi_{i,n}k\Gamma_k\Big].$$

Considering the fact that to the leading order w_n^p is divergence-free, the corrector is expected to be much smaller and will not be of any trouble. It is easy to check

$$\operatorname{div} w_n = \operatorname{div} w_n^p + \operatorname{div} w_n^c = 0$$

and thanks to (Equation 2.35), (Equation 2.36), and (Equation 2.37), w_n is not identically 0.

Now that we have successfully define the velocity perturbation w_n, the velocity at step n is then given by

$$u_n = \overline{u}_l + w_n \tag{2.39}$$

Note that we perturb $\overline{u}_l$ not u_{n-1} since the mollified velocity field verifies much nicer estimates than u_{n-1}. If one instead uses $u_{n-1} + w_n$ then the typical problem of losing derivative appears.

The new Reynolds stress R_n will be computed later via (Equation Mollified-NSR).

Step 4: Estimate the coefficients $a_{i,k,n}$ and $b_{i,k,n}$

We show that the coefficients $a_{i,k,n}$ and $b_{i,k,n}$ have frequency $\sim l^{-1}$ and are of the correct sizes for proving regularity of w_n in the next step. This requires a result on the Hölder norm of composition of functions, Proposition A.1.1.

Proposition 2.5.3. *There exists an index $i_{\max} \lesssim \ln \lambda_n^{-1}$ so that*

$$a_{i,k,n} = b_{i,k,n} = 0 \quad \textit{for all } i \geq i_{\max}.$$

Moreover we have for any $m \in \mathbb{N}$ that

$$\begin{aligned}
\|\nabla^m a_{i,k,n}\|_{L^\infty} &\lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-2} \\
\|\nabla^m (a_{i,k,n}^2)\|_{L^\infty} &\lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-2} \\
\|\nabla^m b_{i,k,n}\|_{L^\infty} &\lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-3}
\end{aligned}$$

and

$$\|a_{i,k,n}\|_{L^1} + l\|\nabla a_{i,k,n}\|_{L^1} \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}$$

$$\|a_{i,k,n}\|_{L^2} \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}$$

$$\|b_{i,k,n}\|_{L^1} \lesssim l^{-1} \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}.$$

Proof. Let $i_{\max}$ be defined as the smallest integer so that

$$4^{i_{\max}+2} \delta_n \lambda_{n-1}^{-2\alpha} \geq \|\overline{R}_l\|_\infty. \tag{2.40}$$

Then, from the definition of $a_{i,k,n}$ and $b_{i,k,n}$ we know that

$$a_{i,k,n} = b_{i,k,n} = 0 \quad \text{for all } i \geq i_{\max}.$$

By the Sobolev embedding $W^{d+1,1}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ and the estimate from Lemma 2.5.1 that

$$\|\nabla^i \overline{R}_l\|_1 \lesssim l^{-i} \delta_n \lambda_{n-1}^{-2\alpha}$$

we can conclude that

$$\|\overline{R}_l\|_\infty \lesssim \|\overline{R}_l\|_{W^{d+1,1}} \lesssim l^{-(d+1)} \delta_n \lambda_{n-1}^{-2\alpha}$$

which together with (Equation 2.40) implies

$$i_{\max} \lesssim \ln \lambda_n^{-1}.$$

We then estimate the Hölder semi-norms. Let $m \in \mathbb{N}$. By a crude use of product rule the following pointwise bound holds

$$\left|\nabla^m a_{i,k,n}\right| \lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sum_{0\le j\le m}\left|\nabla^j \chi_{i,n}\right|\left|\nabla^{m-j}\Gamma_k\Big(\operatorname{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\right|.$$

Let $E_{i,n} = \operatorname{supp}\chi_{i,n}$. We will estimate each summand on the support set $E_{i,n}$. Using the Hölder estimate for composition of functions, i.e. Proposition A.1.1 we have

$$\left\|\nabla^{m-j}\Gamma_k\Big(\operatorname{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\right\|_{L^\infty(E_{i,n})} \lesssim \left\|\nabla^{m-j}\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\right\|_{L^\infty(E_{i,n})}\sum_{i\le m-j}\left\|\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\right\|_{L^\infty(E_{i,n})}^{i-1}, \tag{2.41}$$

where we have put the Hölder norms of Γ_k into the implicit constant. Since there are only finitely many Γ_k (depending only on the dimension d), this is allowable. And then we notice that on $E_{i,n}$ it holds that

$$\overline{R}_l \sim 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha} \quad \text{for all } x \in E_{i,n}\ .$$

Taking this into account we obtain

$$\Big\|\nabla^{m-j}\Gamma_k\Big(\mathrm{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\Big\|_{L^\infty(E_{i,n})}\lesssim\Big\|\nabla^{m-j}\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{L^\infty(E_{i,n})}. \tag{2.42}$$

We can proceed similarly for $\nabla^j\chi_{i,n}$ to obtain the bound:

$$\begin{aligned}\Big\|\nabla^{j}\chi_{i,n}\Big\|_{L^\infty(E_{i,n})}&\lesssim\Big\|\nabla^{j}\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{L^\infty(E_{i,n})}\sum_{i\le j}\Big\|\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{L^\infty(E_{i,n})}^{i-1}\\&\lesssim\Big\|\nabla^{j}\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{L^\infty(\mathbb{T}^d)}.\end{aligned} \tag{2.43}$$

Moreover thanks to the Sobolev embedding $W^{d+1,1}(\mathbb{T}^d)\hookrightarrow L^\infty(\mathbb{T}^d)$ it follows from (Equation 2.42) and Lemma 2.5.1 that

$$\begin{aligned}\Big\|\nabla^{m-j}\Gamma_k\Big(\mathrm{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\Big\|_{L^\infty(E_{i,n})}&\lesssim\Big\|\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{W^{m-j+d+1,1}(\mathbb{T}^d)}\\&\lesssim 4^{-i-1}l^{-m+j-d-1}\end{aligned} \tag{2.44}$$

and from (Equation 2.43) Lemma 2.5.1 that

$$\begin{aligned}\Big\|\nabla^{j}\chi_{i,n}\Big\|_{L^\infty(E_{i,n})}&\lesssim\Big\|\frac{\overline{R}_l}{4^{i}\delta_n\lambda_{n-1}^{-2\alpha}}\Big\|_{W^{j+d+1,1}(\mathbb{T}^d)}\\&\lesssim 4^{-i-1}l^{-j-d-1}.\end{aligned} \tag{2.45}$$

Inserting these bounds into the estimate for $\nabla^m a_{i,k,n}$ we have

$$\|\nabla^m a_{i,k,n}\|_{L^\infty(\mathbb{T}^d)} \lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sum_{0\le j\le m}\left\|\nabla^j\chi_{i,n}\right\|_{L^\infty(E_{i,n})}\left\|\nabla^{m-j}\Gamma_k\Big(\,\mathrm{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\right\|_{L^\infty(E_{i,n})}$$
$$\lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-m-2d-2}.$$

Observing that

$$\nabla^m(a_{i,k,n})^2 \lesssim 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}\sum_{0\le j\le m}\left|\nabla^j\chi_{i,n}^2\right|\left|\nabla^{m-j}\Gamma_k^2\Big(\,\mathrm{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\right|,$$

and that

$$\nabla^m b_{i,k,n} \lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sum_{0\le j\le m+1}\left|\nabla^j\chi_{i,n}^2\right|\left|\nabla^{m-j}\Gamma_k^2\Big(\,\mathrm{Id}-\frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)\right|,$$

where the Hölder norms of all factors in the summation have been estimated, we can conclude without proof that

$$\|\nabla^m(a_{i,k,n})^2\|_{L^\infty(\mathbb{T}^d)} \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-m-2d-2},$$

and

$$\|\nabla^m b_{i,k,n}\|_{L^\infty(\mathbb{T}^d)} \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-m-2d-3}.$$

It remains to show L^1 and L^2 bounds. Since these bounds are more delicate than the ones in L^∞-based norms, we estimate them in a more precise manner. By the definition of $\chi_{i,n}$ we have

$$\begin{aligned}\|a_{i,k,n}\|_1 &\leq 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\int_{\mathbb{T}^d}\chi_{i,k,n}dx\\ &\lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}|\operatorname{supp}\chi_{i,n}|.\end{aligned}$$

and

$$\begin{aligned}\|a_{i,k,n}\|_2^2 &\leq 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}\int_{\mathbb{T}^d}\chi_{i,k,n}^2dx\\ &\lesssim 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}|\operatorname{supp}\chi_{i,n}|.\end{aligned}$$

By the decomposition of the Reynolds stress $\overline{R}_l$ we know that

$$|\operatorname{supp}\chi_{i,n}| \leq \left|\{x\in\mathbb{T}^d : \left|\frac{\overline{R}_l}{\delta_n\lambda_{n-1}^{-2\alpha}}\right| \geq 4^{i-1}\}\right|$$

from which it follows by the Chebyshev inequality that

$$|\operatorname{supp}\chi_{i,n}| \leq 4^{-i+1}.$$

So

$$\|a_{i,k,n}\|_1 \lesssim 2^{-i}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}.$$

and respectively

$$\|a_{i,k,n}\|_2^2 \lesssim \delta_n \lambda_{n-1}^{-2\alpha}.$$

To bound $\nabla a_{i,k,n}$ we obtain first by the product rule and chain rule:

$$|\nabla a_{i,k,n}| \lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\bigg(|\chi_{i,n}||\nabla\Gamma_k|\Big|\frac{\nabla\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big| + |\Gamma_k||\nabla\phi_i|\Big|\frac{\nabla\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big|\bigg).$$

Then using the obvious bounds

$$\begin{aligned} &|\chi_{i,n}| \leq 1, \quad |\nabla\phi_i| \lesssim 4^{-i} \\ &|\Gamma_k| \lesssim 1, \quad |\nabla\Gamma_k| \lesssim 1 \end{aligned}$$

we can estimate L^1 norm as follows:

$$\begin{aligned} \|\nabla a_{i,k,n}\|_1 &\lesssim 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\int_{\mathbb{T}^d}\Big|\frac{\nabla\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big|dx \\ &\lesssim 2^{-i}\delta_n^{-\frac{1}{2}}\lambda_{n-1}^{\alpha}\|\nabla\overline{R}_l\|_1 \\ &\lesssim l^{-1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}, \end{aligned}$$

where we have used $\|\nabla\overline{R}_l\|_1 \lesssim l^{-1}\delta_n\lambda_{n-1}^{-2\alpha}$ from Lemma 2.5.1.

From the definition of $b_{i,k,n}$ it is clear that $b_{i,k,n}$ also verifies bound:

$$\|b_{i,k,n}\|_1 \lesssim l^{-1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}.$$

□

Step 5: Estimate the velocity perturbation

We summarize the regularity properties of w_n^p in the below proposition.

Proposition 2.5.4 (Regularity of w_n^p)**.** *There exists* $a_0 > 0$ *sufficiently large so that for any* $a \geq a_0$ *the principle part of velocity increment defined by* (Equation 2.31) *verifies*

$$\|w_n^p\|_2 + \frac{1}{\lambda_n}\|\nabla w_n^p\|_2 \leq \frac{1}{8}\delta_n^{\frac{1}{2}} \tag{2.46}$$

$$\|w_n^p\|_1 + \frac{1}{\lambda_n}\|\nabla w_n^p\|_1 \leq \frac{1}{8}\delta_n^{\frac{1}{2}}\mu_n^{\frac{-d+1}{2}}. \tag{2.47}$$

Proof. Thanks to Proposition 2.5.3, we know that w_n^p consists of finitely many concentrated Mikado flows:

$$w_n^p = \sum_{0\leq i\leq i_{\max}}\sum_k a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot).$$

To show the bound for $\|w_n^p\|_2$ it suffices to show that

$$\|a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\|_2 \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} \tag{2.48}$$

The reason is that the extra factor $\lambda_{n-1}^{-\alpha}$ can be used to absorbed the logarithmic error causing by $i_{\max}$ and any constant factors provided that a is sufficiently large.

Since $\psi_{i,k}^{\mu_n}(\sigma_n \cdot)$ is $\sigma_n^{-1}\mathbb{T}^d$-periodic and by Proposition 2.5.3

$$\|\nabla^m a_{i,k,n}\|_{L^\infty} \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-2}$$

we can apply the first part of Proposition 2.4.4 with $C_a = \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-2d-2}$, $\mu = l^{-1}$, $\sigma = \sigma_n$ to obtain that

$$\|a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_2 \lesssim \|a_{i,k,n}\|_2 \|\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_2 + C_a \sigma_n^{-100d}.$$

The second term appeared on the right is essentially a small error term. Indeed since $C_a \sigma_n^{-100d} \ll \lambda_n^{-10d}$ due to the fact that $l^{-1} \leq \sigma_n^2$ we obtain

$$\|a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_2 \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}.$$

And hence by taking a sufficiently large, (Equation 2.46) can be obtained:

$$\|w_n^p\|_2 \leq \frac{1}{16}\delta_n^{\frac{1}{2}}$$

To show the bound for $\|w_n^p\|_1$ we simply observe that

$$\left|\operatorname{supp} a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right| \le \left|\operatorname{supp} \psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right| \lesssim \mu_n^{-(d-1)}$$

and thus by Jensen's inequality

$$\|a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\|_1 \lesssim \|a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\|_2 \mu_n^{-\frac{d-1}{2}}$$

Again by taking a sufficiently large, (Equation 2.47) can be obtained:

$$\|w_n^p\|_1 \le \frac{1}{16}\delta_n^{\frac{1}{2}}\mu^{\frac{-d+1}{2}}.$$

Now we turn to estimate $\|\nabla w_n^p\|_p$ for $p = 1$ or 2. By the same argument of using small support set and Jensen's inequality, it suffices to only show the bound for $\|\nabla w_n^p\|_2$. Taking derivative on w_n^p we have

$$\nabla w_n^p = \sum_{i,k} \nabla a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot) + \sigma_n \sum_{i,k} a_{i,k,n}\nabla\psi_{i,k}^{\mu_n}(\sigma_n\cdot).$$

Following the same argument we apply Proposition 2.4.4 to the two above summands with $C_a = \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-2d-2}$, $\mu = l^{-1}$, $\sigma = \sigma_n$ and then obtain that

$$\|\nabla a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\|_2 \lesssim \delta_n^{\frac{1}{2}}l^{-1}\lambda_{n-1}^{-\alpha}$$

and

$$\|a_{i,k,n} \nabla \psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_2 \lesssim \delta_n^{\frac{1}{2}} \mu_n \lambda_{n-1}^{-\alpha}.$$

Hence by the relationship $\lambda_n = \sigma_n \mu_n$ it is obtained that

$$\frac{1}{\lambda_n} \|\nabla w_n^p\|_2 \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha}$$

By choosing a sufficiently large it holds that

$$\frac{1}{\lambda_n} \|\nabla w_n^p\|_2 \leq \frac{1}{16} \delta_n^{\frac{1}{2}}.$$

□

Next, we turn to estimate the correction part of the velocity w_n^c. As expected w_n^c is much smaller than w_n^p.

Proposition 2.5.5 (Regularity of w_n^c)**.** *There exists $a_0 > 0$ sufficiently large so that for any $a \geq a_0$ the correction part of velocity increment defined by* (Equation 2.31) *verifies*

$$\|w_n^c\|_2 + \frac{1}{\lambda_n} \|\nabla w_n^c\|_2 \leq \frac{1}{8} \delta_n^{\frac{1}{2}} l^{-1} \sigma_n^{-1} \tag{2.49}$$

$$\|w_n^c\|_1 + \frac{1}{\lambda_n} \|\nabla w_n^c\|_1 \leq \frac{1}{8} \delta_n^{\frac{1}{2}} \mu_n^{\frac{-d+1}{2}} l^{-1} \sigma_n^{-1}. \tag{2.50}$$

Proof. Observe that the definition of w_n^c involves Riesz transform which is only bounded $L^p \to L^p$ when $1 < p < \infty$. So to resolve this issue let us fix a parameter $s > 1$ sufficiently close to 1 such that

$$\mu_n^{\frac{d-1}{1} - \frac{d-1}{s}} \leq \lambda_{n-1}^{\frac{1}{2}\alpha}. \tag{2.51}$$

And we instead estimate the L^s norm rather than L^1 norm. Let us first prove the bounds for ∇w_n^c as in this case we can follow along the lines of Proposition 2.5.4. By the $L^p \to L^p$ boundedness of the Riesz transform for any $1 < p < \infty$ we notice that

$$\|\nabla w_n^c\|_{L^p(\mathbb{T}^d)} \lesssim \left\| \sum_{i \leq i_{\max}} b_{i,k,n} \psi_{i,k}^{\mu_n}(\sigma_n \cdot) \right\|_{L^p(\mathbb{T}^d)}. \tag{2.52}$$

Since by Proposition 2.5.3

$$\|\nabla^m b_{i,k,n}\|_{L^\infty} \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-3}$$

so applying the first part of Proposition 2.4.4 with $C_a = \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-2d-3}$, $\mu = l^{-1}$, $\sigma = \sigma_n$ and after simplifying one obtains

$$\left\| b_{i,k,n} \psi_{i,k}^{\mu_n}(\sigma_n \cdot) \right\|_2 \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-1} \tag{2.53}$$

and then by small support of $\psi_{i,k}^{\mu_n}(\sigma_n x)$, namely

$$\left| \operatorname{supp} b_{i,k,n} \psi_{i,k}^{\mu_n}(\sigma_n x) \right| \leq \left| \operatorname{supp} \psi_{i,k}^{\mu_n}(\sigma_n x) \right| \lesssim \mu_n^{-(d-1)}$$

and Jensen's inequality we have

$$\left\|b_{i,k,n}\psi^{\mu_n}_{i,k}(\sigma_n\cdot)\right\|_s \lesssim \mu_n^{\frac{d-1}{1}-\frac{d-1}{s}}\left\|b_{i,k,n}\psi^{\mu_n}_{i,k}(\sigma_n\cdot)\right\|_2$$
$$\lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-1}\mu_n^{\frac{-d+1}{2}}\lambda_{n-1}^{-\frac{1}{2}\alpha} \tag{2.54}$$

where we have used the fact that $\mu_n^{\frac{d-1}{1}-\frac{d-1}{s}} \leq \lambda_{n-1}^{\frac{1}{2}\alpha}$.

Since $\sigma_n = \lambda_n^{\frac{1}{4}} \leq \lambda_n$ from (Equation 2.53) and (Equation 2.52) we get

$$\frac{1}{\lambda_n}\|\nabla w_n^c\|_2 \leq \frac{1}{16}\delta_n^{\frac{1}{2}}l^{-1}\sigma_n^{-1}$$

as long as a is sufficiently large.

To recover the L^1 bound for ∇w_n^c we simply first bound L^1 norm by its L^s norm:

$$\|\nabla w_n^c\|_1 \leq \|\nabla w_n^c\|_s$$

and it follows that

$$\|\nabla w_n^c\|_1 \lesssim \sum_i \left\|b_{i,k,n}\psi^{\mu_n}_{i,k}(\sigma_n\cdot)\right\|_s.$$

Then taking a sufficient large and using (Equation 2.54) we can ensure that

$$\|\nabla w_n^c\|_1 \leq \frac{1}{16}\delta_n^{\frac{1}{2}}\mu_n^{\frac{-d+1}{2}}l^{-1}$$

which implies that

$$\frac{1}{\lambda_n}\|\nabla w_n^c\|_1 \leq \frac{1}{16}\delta_n^{\frac{1}{2}}\mu_n^{\frac{-d+1}{2}} l^{-1}\lambda_n^{-1}.$$

It remains to prove the estimates of $\|w_n^c\|_p$ for $p = 1, 2$. It follows from the L^p boundedness of Riesz transform that

$$\|w_n^c\|_2 \lesssim \sum_{i,k} \big\||\nabla|^{-1}\big[b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\big]\big\|_2$$

$$\|w_n^c\|_s \lesssim \sum_{i,k} \big\||\nabla|^{-1}\big[b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\big]\big\|_s.$$

Therefore it suffices to derive suitable estimates for the functions

$$|\nabla|^{-1}\big[b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\big]$$

where we note that $b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)$ has zero-mean since

$$\int_{\mathbb{T}^d} b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n x)\ dx = \int_{\mathbb{T}^d} \operatorname{div} a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n x)\ dx = 0.$$

So, thanks to Proposition 2.5.3 the assumptions in Proposition 2.4.4 are fulfilled:

$$\|\nabla^m b_{i,k,n}\|_{L^\infty} \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} l^{-m-2d-3}$$

and then we can obtain from the second part of Proposition 2.4.4

$$\left\| |\nabla|^{-1}\left[b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right]\right\|_2 \lesssim \sigma_n^{-1}\left\|b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right\|_2 + \sigma_n^{-10d} \tag{2.55}$$

and

$$\left\| |\nabla|^{-1}\left[b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right]\right\|_s \lesssim \sigma_n^{-1}\left\|b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right\|_s + \sigma_n^{-10d}. \tag{2.56}$$

where we have used the fact that here $\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-2d-3}\sigma_n^{-100d} \ll \sigma_n^{-10d}$. Then it follows again from the first part of Proposition 2.4.4 that

$$\left\|b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right\|_2 \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sigma_n^{-1}l^{-1} + \sigma_n^{-10d}, \tag{2.57}$$

which by Jensen's inequality and the small support of $\psi_{i,k}^{\mu_n}(\sigma_n x)$ also implies that

$$\left\|b_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\right\|_s \lesssim \mu_n^{\frac{d-1}{s}-\frac{d-1}{2}}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sigma_n^{-1}l^{-1} + \sigma_n^{-10d}, \tag{2.58}$$

So putting together (Equation 2.51), (Equation 2.55), (Equation 2.56), and (Equation 2.57) we have

$$\begin{aligned}
\|w_n^c\|_2 &\lesssim \sum_{i,k}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\sigma_n^{-1}l^{-1} \\
\|w_n^c\|_s &\lesssim \sum_{i,k}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\frac{1}{2}\alpha}\mu_n^{\frac{-d+1}{2}}\sigma_n^{-1}l^{-1}.
\end{aligned}$$

Finally using the extra factors $\lambda_{n-1}^{-\alpha}$ and $\lambda_{n-1}^{-\frac{1}{2}\alpha}$ to absorb any logarithmic and constant factors we can get rid of the summation in i, k to obtain that

$$\|w_n^c\|_2 \leq \frac{1}{16}\delta_n^{\frac{1}{2}} l^{-1}\sigma_n^{-1}$$

and that

$$\|w_n^c\|_1 \leq \|w_n^c\|_s \leq \frac{1}{16}\delta_n^{\frac{1}{2}} \mu_n^{\frac{-d+1}{2}} l^{-1}\sigma_n^{-1}.$$

□

Remark 2.5.6. *It seems that one should be able to gain a factor of* $l^{-1}\lambda_n^{-1}$ *rather than* $l^{-1}\sigma_n^{-1}$ *since* $\psi_{i,k}^{\mu_n}(\sigma_n \cdot)$ *has frequency* λ_n*. Such improvement can be obtained by carefully choosing the profile function* ψ *in Section 2.4 with vanishing moments up to a sufficiently high order, which may be useful in the future study of constructing solutions with better regularity.*

Step 6: Check (Equation 2.5) **and hypothesis** (Equation H2) **and** (Equation H3)

Let us first check (Equation 2.5). Since by the definition of u_n, namely (Equation 2.39) we have

$$\|u_n - u_{n-1}\|_2 \leq \|w_n\|_2 + \|\overline{u}_l - u_{n-1}\|_2$$

it suffices to estimate

$$\|u_n - u_{n-1}\|_2 \leq \|w_n^p\|_2 + \|w_n^c\|_2 + \|\overline{u}_l - u_{n-1}\|_2.$$

From the estimates (Equation 2.46) and (Equation 2.49) and Lemma 2.5.1 we know that

$$\begin{aligned} \|w_n^p\|_2 + \|w_n^c\|_2 &\leq \frac{1}{4}\delta_n^{\frac{1}{2}} \\ \|\overline{u}_l - u_{n-1}\|_2 &\leq C\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}. \end{aligned}$$

Taking a sufficiently large a, we can arrange that

$$\|u_n - u_{n-1}\|_2 \leq \frac{1}{4}\delta_n^{\frac{1}{2}}.$$

and therefore

$$\|u_n - u_{n-1}\|_2 \leq \frac{1}{2}\delta_n^{\frac{1}{2}}. \tag{2.59}$$

We will bound the term $\nabla(u_n - u_{n-1})$ in almost the same way. As before we first obtain from the definitions of u_n and w_n that

$$\begin{aligned} \|\nabla u_n - \nabla u_{n-1}\|_2 &\leq \|\nabla w_n\|_2 + \|\nabla \overline{u}_l\|_2 + \|\nabla u_{n-1}\|_2 \\ &\leq \|\nabla w_n^p\|_2 + \|\nabla w_n^c\|_2 + \|\nabla \overline{u}_l\|_2 + \|\nabla u_{n-1}\|_2 \end{aligned}$$

Thanks to estimates (Equation 2.46), (Equation 2.49) and Lemma 2.5.1, we obtain

$$\begin{aligned}\|\nabla u_n - \nabla u_{n-1}\|_2 &\le \|\nabla w_n^p\|_2 + \|\nabla w_n^c\|_2 + \|\nabla \overline{u}_l\|_2 + \|\nabla u_{n-1}\|_2 \\ &\le \frac{1}{4}\lambda_n \delta_n^{\frac{1}{2}} + C\delta_{n-1}^{\frac{1}{2}}\lambda_{n-1}\end{aligned}$$

where C is a constant depending only on the mollifier η. Again by choose a sufficiently large we can guarantee that

$$\|\nabla u_n - \nabla u_{n-1}\|_2 \le \frac{1}{2}\lambda_n \delta_n^{\frac{1}{2}},$$

which together with (Equation 2.59) means (Equation 2.5) is satisfied.

Now we show (Equation H2). First, we obtain the obvious bound

$$\|u_n\|_2 = \|\overline{u}_l + w_n\|_2 \le \|u_{n-1}\|_2 + \|\overline{u}_l - u_{n-1}\|_2 + \|w_n\|_2.$$

Then, from Proposition 2.5.4 and 2.5.5, and Lemma 2.5.1 we see that

$$\begin{aligned}\|u_n\|_2 &\le \|u_{n-1}\|_2 + \|\overline{u}_l - u_{n-1}\|_2 + \|w_n\|_2 \\ &\le 1 - \delta_{n-1}^{\frac{1}{2}} + C\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} + \frac{1}{2}\delta_n^{\frac{1}{2}}.\end{aligned}$$

where again C is some constant depending only on the mollifier η. Now choosing a sufficiently large depending on b so that

$$3\delta_n^{\frac{1}{2}} \leq \delta_{n-1}^{\frac{1}{2}}$$

$$C\lambda_{n-1}^{-\alpha} \leq 1$$

we are able to find

$$\begin{aligned} 1 - \delta_{n-1}^{\frac{1}{2}} + C\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} + \frac{1}{2}\delta_n^{\frac{1}{2}} &\leq 1 - 2\delta_n^{\frac{1}{2}} + \delta_n^{\frac{1}{2}} + \frac{1}{2}\delta_n^{\frac{1}{2}} \\ &\leq 1 - \delta_n^{\frac{1}{2}} \end{aligned}$$

and hence we obtain the desire bound (Equation H2):

$$\|u_n\|_2 \leq 1 - \delta_n^{\frac{1}{2}}.$$

As for (Equation H3), the proof is very similar. We first obtain

$$\|\nabla u_n\|_2 \leq \|\nabla \overline{u}_l\|_2 + \|\nabla w_n\|_2$$

and then using Lemma 2.5.1, estimates (Equation 2.46) and (Equation 2.49) we find that

$$
\begin{aligned}
\|\nabla u_n\|_2 &\leq \|\nabla \overline{u}_l\|_2 + \|\nabla w_n^p\|_2 + \|\nabla w_n^c\|_2 \\
&\leq C\delta_{n-1}^{\frac{1}{2}}\lambda_{n-1} + \frac{1}{2}\delta_n^{\frac{1}{2}}\lambda_n
\end{aligned}
$$

where the constant C depends only on the mollifier η. Letting a sufficiently large it can be arranged that

$$
C\lambda_{n-1}^{1-\beta} \leq \frac{1}{2}\lambda_n^{1-\beta}.
$$

And then we have

$$
C\delta_{n-1}^{\frac{1}{2}}\lambda_{n-1} + \frac{1}{2}\delta_n^{\frac{1}{2}}\lambda_n \leq \delta_n^{\frac{1}{2}}\lambda_n,
$$

which implies

$$
\|\nabla u_n\|_2 \leq \delta_n^{\frac{1}{2}}\lambda_n.
$$

So (Equation H3) is also fulfilled.

Step 7: Estimate the new Reynolds stress

Thanks to (Equation Mollified-NSR), the new Reynolds stress is defined by the divergence equation:

$$
\begin{aligned}
\operatorname{div} R_n + \nabla P_n = & \operatorname{div} \overline{R}_l + \Delta w_n + \operatorname{div} w_n^p \otimes w_n^p \\
& + \operatorname{div}(w_n \otimes \overline{u}_l + \overline{u}_l \otimes w_n) \\
& + \operatorname{div}(w_n^c \otimes w_n^p + w_n^p \otimes w_n^c + w_n^c \otimes w_n^c).
\end{aligned}
$$

To estimate the L^1 norm of R_n, one needs to somehow invert the divergence. For this purpose we follow the construction given in (3). The operator $\mathcal{R} : C^\infty(\mathbb{T}^d, \mathbb{R}^d) \to \mathbb{R}^{d\times d}$ is defined as

$$
\begin{aligned}
(\mathcal{R}f)_{ij} &= \mathcal{R}_{ijk} f_k \\
\mathcal{R}_{ijk} &= \frac{2-d}{d-1}\Delta^{-2}\partial_i\partial_j\partial_k + \frac{-1}{d-1}\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.
\end{aligned}
\tag{2.60}
$$

It is clear that for any $f \in C^\infty(\mathbb{T}^d)$ the matrix $(\mathcal{R}f)_{ij}$ is symmetric. Taking the trace we have

$$
\begin{aligned}
\operatorname{Tr} \mathcal{R}f &= \frac{2-d}{d-1}\Delta^{-1}\partial_k f_k + \frac{-d}{d-1}\Delta^{-1}\partial_k f_k + \Delta^{-1}\partial_k f_k + \Delta^{-1}\partial_k f_k \\
&= (\frac{2-d}{d-1} + \frac{-d}{d-1} + 2)\Delta^{-1}\partial_k f_k = 0
\end{aligned}
$$

which means that $\mathcal{R}f$ is also trace-less.

And lastly, we have

$$
\operatorname{div} \mathcal{R}f = \partial_j (\mathcal{R}f)_{ij} = \partial_j \mathcal{R}_{ijk} f_k
$$

so by direct computation we can check that

$$\operatorname{div} \mathcal{R} f = \frac{2-d}{d-1} \Delta^{-1} \partial_i \partial_k f_k + \frac{-1}{d-1} \Delta^{-1} \partial_k \partial_i f_k + \Delta^{-1} \partial_i \partial_k f_k + f_i = f_i = f.$$

Lemma 2.5.7. *The operator $\mathcal{R}$ defined by* (Equation 2.60) *has the following properties. For any $f \in C_0^\infty(\mathbb{T}^d)$ the matrix $\mathcal{R} f$ is symmetric trace-free and we have*

$$\operatorname{div} \mathcal{R} f = f. \tag{2.61}$$

If additionally $\operatorname{div} f = 0$ *then*

$$\mathcal{R} \Delta f = \nabla f + (\nabla f)^T. \tag{2.62}$$

Thanks to Lemma 2.5.7 we need to estimate the following new Reynolds stress defined by using the inverse divergence operator $\mathcal{R}$.

$$\begin{aligned} R_n = & \underbrace{\mathcal{R}(\operatorname{div} w_n \otimes u_{n-1} + u_{n-1} \otimes w_n)}_{\text{quadratic error}} \\ & + \underbrace{\mathcal{R}(\Delta w_n)}_{\text{linear error}} + \underbrace{\mathcal{R}(\operatorname{div} w_n^p \otimes w_n^p + \overline{R}_l)}_{\text{oscillation error}} + \underbrace{\mathcal{R} \operatorname{div}(w_n^c \otimes w_n^p + w_n^p \otimes w_n^c + w_n^c \otimes w_n^c)}_{\text{correction error}} \end{aligned} \tag{2.63}$$

which is well-defined since all terms involved have zero-mean, and we simply denote the equation as

$$R_n = E_q + E_l + E_o + E_c. \tag{2.64}$$

It suffices to check that for each part we have

$$\max\{\|E_q\|_1, \|E_l\|_1, \|E_o\|_1, \|E_c\|_1\} \leq \frac{1}{4}\delta_{n+1}.$$

Oscillation error

Due to the fact that each $\chi_{i,n}W_{[i]}$ has disjoint support we can compute the nonlinear term as

$$w_n^p \otimes w_n^p = \sum_i 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}\chi_{i,n}^2 W_{[i]} \otimes W_{[i]}^{\mu_n}\Big(\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}, \sigma_n x\Big).$$

From Lemma 2.4.3 it follows

$$w_n^p \otimes w_n^p = -\overline{R}_l + \sum_i 4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}\chi_{i,n}^2 \,\mathrm{Id} + \sum_{i,k} \rho_{i,k,n}^2 \phi_{i,k}^{\mu_n}(\sigma_n \cdot)\; k \otimes k \tag{2.65}$$

where scaler functions $\rho_{i,k,n} \in C^\infty(\mathbb{T}^d)$ and $\phi_{i,k}^{\mu_n} \in C_0^\infty(\mathbb{T}^d)$ are defined respectively as

$$\rho_{i,k,n} = 2^{i+1}\delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}\chi_{i,n}\Gamma_k\Big(\mathrm{Id} - \frac{\overline{R}_l}{4^{i+1}\delta_n\lambda_{n-1}^{-2\alpha}}\Big)$$

and

$$\phi_{i,k}^{\mu_n} = \big(\psi_{i,k}^{\mu_n}\big)^2 - \fint \big(\psi_{i,k}^{\mu_n}\big)^2.$$

Upon taking divergence, we can find a pressure P to absorb the second term in (Equation 2.65) so that

$$\operatorname{div}(w_n^p \otimes w_n^p) + \nabla P = -\operatorname{div} \overline{R}_l + \sum_{i,k} \operatorname{div}\left(\rho_{i,k,n}^2 \phi_{i,k}^{\mu_n}(\sigma_n \cdot)\ k \otimes k\right)$$

Noticing the fact that

$$\operatorname{div}\left(\phi_{i,k}^{\mu_n}(\sigma_n x)\ k \otimes k\right) = 0$$

we get

$$\begin{aligned} E_o &= \mathcal{R}(\operatorname{div} w_n^p \otimes w_n^p + \operatorname{div} \overline{R}_l) \\ &= \sum_{i,k} \mathcal{R}\big[\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\ k \otimes k\big]. \end{aligned}$$

For the remainder of this part, we fix some $p > 1$ (depending on d and α) sufficiently close to 1 such that $L^p(\mathbb{T}^d) \hookrightarrow W^{-\alpha,1}(\mathbb{T}^d)$, and will estimate $\|E_o\|_p$.

Since $a_{i,k,n} = \rho_{i,k,n}\ k$, we get from Proposition 2.5.3 that

$$\|\nabla^m \rho_{i,k,n}^2\|_\infty \lesssim \delta_n^{\frac{1}{2}} \lambda_{n-1}^{-\alpha} l^{-m-2d-2} \quad \text{for all } m \in \mathbb{N}$$

and by definition that

$$\fint_{\mathbb{T}^d} \phi_{[i],k}^{\mu_n}(\sigma_n \cdot) = 0.$$

Hence we have the following estimate for E_o by invoking the second part of Proposition 2.4.4 with $C_a = \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-2d-2}$, $\mu = l^{-1}$, $\sigma = \sigma_n$ and p:

$$\|E_o\|_p \lesssim \sum_{i,k} \sigma_n^{-1+\alpha} \big\| |\nabla|^{-\alpha} \big(\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big) \big\|_p + \sigma_n^{-100d} \tag{2.66}$$

where we have used the bound

$$\big\|\phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big\|_1 \lesssim 1. \tag{2.67}$$

Then the embedding $L^p(\mathbb{T}^d) \hookrightarrow W^{-\alpha,1}(\mathbb{T}^d)$ implies that

$$\|E_o\|_p \lesssim \sum_{i,k} \sigma_n^{-1+\alpha} \big\|\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big\|_1 + \sigma_n^{-100d} \tag{2.68}$$

Since it is easy to see that

$$\sigma_n^{-10d} \ll \delta_{n+1},$$

to bound $\|E_o\|$, it suffices to bound $\big\|\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big\|_1$. We attempt to apply the first part of Proposition 2.4.4 with the same parameters, but (Equation 2.66) is in L^1 rather than L^2 and if one uses the small support argument as in the proof of Proposition 2.5.4 and 2.5.5, one has to bound $\|\nabla \rho_{i,k,n}\|_2$, which will be too big and have no decay in view of Proposition 2.5.3. To resolve this issue, we appeal to the following heuristics:

$$\big\|\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big\|_1 \lesssim l^{-1} \big\|\rho_{i,k,n} \psi_{[i],k}^{\mu_n}(\sigma_n \cdot)\big\|_2^2 + \text{Error terms}$$

We will show a slightly weaker bound in the following. Firstly, Let

$$\rho_l = \mathbb{P}_{\leq l^{-1-\alpha}} \rho_{i,k,n}^2$$

where the extra factor α will allow us to exploit the derivative bounds for $a_{i,k,n}$. More precisely, applying the same integration by parts argument as in the proof of Proposition 2.4.4, one can show that

$$\|\nabla(\rho_{i,k,n}^2 - \rho_l)\|_\infty + \|\rho_{i,k,n}^2 - \rho_l\|_\infty \lesssim l^{1000d} \tag{2.69}$$

where the implicit constant depends on α and d. Using this and (Equation 2.67), we have

$$\left\|\nabla \rho_{i,k,n}^2 \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_1 \lesssim \left\|\nabla \rho_l \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_1 + l^{1000d}. \tag{2.70}$$

Thus it suffices to get rid of the derivative on ρ_l and then bound $\left\|\rho_l \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_1$. Using the convolution representation of ρ_l by the Littlewood-Paley theory

$$\rho_l = \mathbb{P}_{\leq 2l^{-1-\alpha}} \rho_l$$

we have

$$\left\|\nabla \rho_l \phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_1 = \int \left|\phi_{[i],k}^{\mu_n}(\sigma_n x)\right| \left| \int \rho_l(x-y) \nabla \widetilde{\varphi}_{l^{-1-\alpha}}(y) dy \right| dx$$

where $\widetilde{\varphi}_{l^{-1-\alpha}}$ is the Fourier inverse for the frequency cut-off $\mathbb{P}_{\leq 2l^{-1-\alpha}}$. So by the Fubini's theorem and the bound

$$\left\|\nabla\widetilde{\varphi}_{l^{-1-\alpha}}\right\|_1 \lesssim l^{-1-\alpha}$$

we find that

$$\begin{aligned}
\left\|\nabla\rho_l\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 &= \int\int\left|\rho_l(x-y)\phi^{\mu_n}_{[i],k}(\sigma_n x)\right|dx\left|\nabla\widetilde{\varphi}_{l^{-1-\alpha}}(y)\right|dy \\
&\leq \left\|\nabla\widetilde{\varphi}_{l^{-1-\alpha}}\right\|_1 \sup_y \left\|\rho_l(\cdot-y)\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 \\
&\lesssim l^{-1-\alpha}\sup_y \left\|\rho_l(\cdot-y)\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1. \qquad (2.71)
\end{aligned}$$

Thanks to (Equation 2.69) and (Equation 2.67) again, we get

$$\left\|\left[\rho^2_{i,k,n}(\cdot-y)-\rho_l(\cdot-y)\right]\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 \lesssim \left\|\rho^2_{i,k,n}-\rho_l\right\|_\infty\left\|\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 \lesssim l^{1000d}$$

where the implicit constant is independent of y. Then (Equation 2.71) becomes

$$\left\|\nabla\rho_l\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 \lesssim l^{-1-\alpha}\sup_y\left\|\rho^2_{i,k,n}(\cdot-y)\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 + l^{1000d}. \qquad (2.72)$$

Putting together (Equation 2.66), (Equation 2.70), and (Equation 2.72) and using the fact that $\sigma_n \leq l^{-10}$ we have

$$\|E_o\|_p \lesssim \sum_{i,k}\sigma_n^{-1+\alpha}l^{-1-\alpha}\sup_y\left\|\rho^2_{i,k,n}(\cdot-y)\phi^{\mu_n}_{[i],k}(\sigma_n\cdot)\right\|_1 + \sigma_n^{-10d}. \qquad (2.73)$$

For each fix $y \in \mathbb{T}^d$ we compute that

$$\left\|\rho_{i,k,n}^2(\cdot - y)\phi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_1 \leq \left\|\rho_{i,k,n}(\cdot - y)\psi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_2^2 + \left\|\rho_{i,k,n}\right\|_2^2 \tag{2.74}$$

and now we can apply the first part of Proposition 2.4.4 with the parameters $C_a = \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha}l^{-2d-2}$, $\mu = l^{-1}$, and $\sigma = \sigma_n$ to obtain

$$\begin{aligned}\left\|\rho_{i,k,n}(\cdot - y)\psi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_2 &\lesssim \left\|\rho_{i,k,n}(\cdot - y)\right\|_2 \left\|\psi_{[i],k}^{\mu_n}(\sigma_n \cdot)\right\|_2 + \sigma_n^{-10d} \\ &\lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} + \sigma_n^{-10d} \end{aligned} \tag{2.75}$$

where we have used Proposition 2.5.3 to get the bound of $\|\rho_{i,k,n}\|_2 \sim \|a_{i,k,n}\|_2$. Therefore, from (Equation 2.73), (Equation 2.74), and (Equation 2.75) it follows that

$$\|E_o\|_p \lesssim \sum_{i,k} \sigma_n^{-1+\alpha} l^{-1-\alpha} \delta_n \lambda_{n-1}^{-2\alpha}.$$

Again by taking a sufficiently large a and using $\lambda_{n-1}^{-2\alpha}$ to absorb the constant and the logarithmic factor causing by the summation in i, we can ensure that

$$\|E_o\|_1 \leq \|E_o\|_p \leq \frac{1}{4}\sigma_n^{-1+\alpha} l^{-1-\alpha}\delta_n.$$

In view of the choice of constants $d \geq 4$, $\alpha = 10^{-6}$, $\beta = \frac{1}{200}$ and $b = 5$, we have the following numerical inequality

$$-\frac{1+\alpha}{4} + \frac{(1-\beta+\alpha)(1+\alpha)}{b} - \beta < -2b\beta$$

which implies that

$$\|E_o\|_1 \leq \frac{1}{4}\sigma_n^{-1} l^{-1-\alpha}\delta_n \leq \frac{1}{4}\delta_{n+1}. \tag{2.76}$$

Linear error

For the linear error, we first use Lemma 2.5.7 to obtain

$$\|E_l\|_1 = \|\mathcal{R}\Delta w_n\|_1 \leq 2\|\nabla w_n\|_1.$$

Then we can simply use the estimates (Equation 2.47) and (Equation 2.50) to get

$$\begin{aligned}\|E_l\|_1 &\leq 2\|\nabla w_n^p\|_1 + 2\|\nabla w_n^c\|_1 \\ &\leq \frac{1}{4}\delta_n^{\frac{1}{2}}\lambda_n \mu_n^{\frac{-d+1}{2}}.\end{aligned}$$

To check the validity of $\|E_l\|_1 \leq \frac{1}{4}\delta_{n+1}$, we need to make sure that

$$\delta_n^{\frac{1}{2}}\lambda_n \mu_n^{\frac{-d+1}{2}} \leq \delta_{n+1} \tag{2.77}$$

which after taking logarithm and using the definitions of various constants is equivalent to

$$-\beta + 1 + \frac{3}{4}\frac{1-d}{2} \leq -2b\beta. \tag{2.78}$$

Since $d \geq 4$, $\beta = \frac{1}{200}$ and $b = 5$ the above inequality holds trivially. So we can conclude that

$$\|E_l\|_1 \leq \frac{1}{4}\delta_{n+1}. \tag{2.79}$$

Quadratic error

Thanks to Lemma 2.5.7, we need to estimate the terms

$$\begin{aligned}\|E_q\|_1 &\leq \|\mathcal{R}\,\mathrm{div}(\overline{u}_l \otimes w_n^p)\|_1 + \|\mathcal{R}\,\mathrm{div}(\overline{u}_l \otimes w_n^p)\|_1 \\ &:= \|E_{q1}\|_1 + \|E_{q2}\|_1.\end{aligned}$$

Let us show that $\|E_{qj}\| \leq \frac{1}{8}\delta_{n+1}$ for $j = 1, 2$ in the following.

For the first term E_{q1}, we have by the L^p boundedness of the Riesz transform, $p > 1$ that

$$\|E_{q1}\|_1 \leq \|E_{q1}\|_p \lesssim_p \sum_{i,k} \|\overline{u}_l \otimes a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_p.$$

Then by Hölder's inequality and the fact that $\left|\mathrm{supp}\,\varphi_{i,k,n}\right| \lesssim \mu_n^{-d+1}$

$$\|\overline{u}_l \otimes a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n \cdot)\|_p \lesssim \|\overline{u}_l\|_\infty \|a_{i,k,n}\varphi_{i,k,n}\|_2 \mu_n^{(d-1)(\frac{1}{p}-\frac{1}{2})} \tag{2.80}$$

From (Equation 2.48) in Proposition 2.5.4 we know that

$$\|a_{i,k,n}\varphi_{i,k,n}\|_2 \lesssim \delta_n^{\frac{1}{2}}\lambda_{n-1}^{-\alpha} \tag{2.81}$$

and from Lemma 2.5.1 and the Sobolev embedding $H^{\frac{d+1}{2}} \hookrightarrow L^\infty$ we get

$$\|\overline{u}_l\|_\infty \lesssim \|\overline{u}_l\|_{H^{\frac{d+1}{2}}} \lesssim l^{-\frac{d+1}{2}} \tag{2.82}$$

Now choosing $p > 1$ sufficiently close to 1 such that

$$\mu_n^{(d-1)(\frac{1}{p}-\frac{1}{2})} \leq \mu_n^{-\frac{d-1}{2}}\lambda_{n-1}^{\alpha}$$

and combining (Equation 2.80), (Equation 2.81) and (Equation 2.82) we have

$$\|\overline{u}_l \otimes a_{i,k,n}\psi_{i,k}^{\mu_n}(\sigma_n\cdot)\|_p \lesssim l^{-\frac{d+1}{2}}\delta_n^{\frac{1}{2}}\mu_n^{-\frac{d-1}{2}}$$

Since $d \geq 4$, $b = 5$ and $\beta = \frac{1}{200}$, by taking a sufficiently large a we have

$$l^{-\frac{d+1}{2}}\delta_n^{\frac{1}{2}}\mu_n^{-\frac{d-1}{2}} \ll \lambda_{n+1}^{-2\beta} = \delta_{n+1}.$$

So it follows that

$$\|E_{q1}\|_1 \leq \frac{1}{8}\delta_{n+1}. \tag{2.83}$$

For E_{q2} we will use a simple argument to get a very crude bound that suffices for our purpose. We first obtain by the L^p boundedness of the Riesz transform, $p > 1$ and Hölder's inequality that

$$\|E_{q2}\|_1 \leq \|E_{q2}\|_p \lesssim_p \|\overline{u}_l \otimes w_n^c\|_p \leq \|\overline{u}_l\|_2 \|w_n^c\|_{\frac{2p}{2-p}}.$$

Then choosing $p > 1$ sufficiently close to 1 such that in view of Proposition 2.5.5 we have

$$\|w_n^c\|_{\frac{2p}{2-p}} \leq \frac{1}{8} \delta_n^{\frac{1}{2}} \mu_n^{\frac{-d+1}{2}} l^{-1} \sigma_n^{-1} \lambda_{n-1}^{\alpha}$$

Then we get

$$\|E_{q2}\|_1 \lesssim \delta_n^{\frac{1}{2}} l^{-1} \sigma_n^{-1} i_{\max} \lambda_{n-1}^{\alpha}.$$

Again using $d \geq 4$, $b = 5$ and $\beta = \frac{1}{200}$ we find that

$$\frac{1 - \beta + 2\alpha}{b} - \frac{1}{4} < -2b\beta.$$

Since for any $\epsilon > 0$, there exists a sufficiently large so that $i_{\max} \leq \lambda_n^{\epsilon}$. Then by taking a sufficiently large we can ensure that

$$\|E_{q2}\|_1 \lesssim \delta_n^{\frac{1}{2}} l^{-1} \sigma_n^{-1} i_{\max} = \lambda_n^{\frac{1-\beta+2\alpha}{b} - \frac{1}{4}} i_{\max} \ll \delta_{n+1},$$

So provided that a is large enough, we can conclude that

$$\|E_{q2}\|_1 \leq \frac{1}{8}\delta_{n+1}. \tag{2.84}$$

Remark 2.5.8. *One may notice that the bound we obtained for E_{q2} is worse than that of E_{q1}. In fact E_{q2} should be much smaller than E_{q1} since w_n^c is much smaller than w_n^p. As we do not plan to obtain the optimal regularity of the final solution u, a rougher bound for E_{q2} still suffices for our purpose.*

Correction error

It follows directly from Lemma 2.5.7 and Hölder's inequality that

$$\begin{aligned}
\|E_c\|_1 &= \|\mathcal{R}\,\mathrm{div}(w_n^c \otimes w_n^p + w_n^p \otimes w_n^c + w_n^c \otimes w_n^c)\|_1 \\
&\leq \|w_n^c \otimes w_n^p + w_n^p \otimes w_n^c + w_n^c \otimes w_n^c\|_1 \\
&\leq \|w_n^c\|_2\|w_n^p\|_2 + \|w_n^c\|_2^2.
\end{aligned}$$

From (Equation 2.46) and (Equation 2.49) we get

$$\begin{aligned}
\|w_n^c\|_2^2 &\leq \frac{1}{16}\delta_n\lambda_n^{-2}l^{-2} \\
\|w_n^c\|_2\|w_n^p\|_2 &\leq \frac{1}{16}\delta_n\lambda_n^{-1}l^{-1}.
\end{aligned}$$

So

$$\|E_c\|_1 \leq \frac{1}{4}\delta_n \lambda_n^{-1} l^{-1}$$

which is smaller than the final estimate for E_{q2} and thus we conclude that

$$\|E_c\|_1 \leq \frac{1}{4}\delta_{n+1}. \tag{2.85}$$

Step 8: Check inductive hypothesis (Equation H1)

Adding up estimates (Equation 2.76), (Equation 2.79), (Equation 2.83), (Equation 2.84) ,and (Equation 2.85) we obtain the bound for the new Reynolds stress

$$\|R_n\|_1 \leq \|E_o\|_1 + \|E_l\|_1 + \|E_{q1}\|_1 + \|E_{q2}\|_1 + \|E_c\|_1 \leq \delta_{n+1}. \tag{2.86}$$

So (Equation H1) is verified and the proof of Proposition 2.3.1 is completed.

CHAPTER 3

STATIONARY AND DISCONTINUOUS WEAK SOLUTIONS IN 3D

The content of this chapter is based on a joint work that I co-authored with my advisor A. Cheskidov and has been previously appeared on arXiv.org (see (57)).

3.1 Main theorems

We state the main results of this chapter. In particular, Theorem 1.3.2 and 1.3.3 listed in Chapter 1are simpler versions of Theorem 3.1.1 and 3.1.3 accordingly.

The first theorem concerns the existence of stationary weak solutions for the 3D Navier-Stokes equations, which extends the previous work (58) of the second author in dimension $d \geq 4$.

Theorem 3.1.1 (Finite energy stationary weak solution)**.** *Given any divergence-free $f \in C^\infty(\mathbb{T}^3)$ with zero mean, there is $M_f > 0$ such that for any $M \geq M_f$, there exists a weak solution $u \in L^2(\mathbb{T}^3)$ to* (Equation NSE) *with forcing term f satisfying $\|u\|_2^2 = M$.*

The next two theorems are about weak solutions with discontinuous energy profiles.

Theorem 3.1.2 (Energy with dense discontinuities)**.** *Let $\varepsilon, T > 0$ and $a \in C^\infty(\mathbb{T}^3 \times [0,T])$ be a smooth divergence-free vector field with zero mean for all $t \in [0,T]$. There exists a dense subset $E \subset [0,T]$ and a constant $M_a > 0$, such that for any $M \geq M_a$ there exists a weak solution $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ to* (Equation NSE) *so that the following holds:*

1. *The energy* $\|u(t)\|_2^2$ *is bounded by* $2M$*:*

$$\|u(t)\|_2^2 \leq 2M \quad \textit{for any } t \in [0,T] \ , \tag{3.1}$$

and has jump discontinuities on set E*:*

$$\lim_{s \to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \textit{for any } t \in E \ . \tag{3.2}$$

2. $u(t)$ *coincides with* $a(t)$ *at* $t = 0, T$*:*

$$u(x,0) = a(x,0) \quad \textit{and} \quad u(x,T) = a(x,T), \tag{3.3}$$

but the energy jump is of size M*:*

$$\lim_{s \to 0+} \|u(s)\|_2^2 - \|u(0)\|_2^2 = \lim_{s \to T-} \|u(s)\|_2^2 - \|u(T)\|_2^2 = M. \tag{3.4}$$

3. u *is smooth on* E*:*

$$u(t) \in C^\infty(\mathbb{T}^3) \quad \textit{for all } t \in E, \tag{3.5}$$

and uniformly ε*-close to* a *in* $W^{1,1}(\mathbb{T}^3)$*:*

$$\|u - a\|_{L_t^\infty W^{1,1}} < \varepsilon. \tag{3.6}$$

The set E in Theorem 3.1.2 is dense in $[0,T]$ and, in fact, countable. Using a gluing argument, we are also able to construct weak solutions whose energy discontinuities are dense and of positive measure.

Theorem 3.1.3 (Energy with dense and positive measure discontinuities)**.** *Let* $\varepsilon > 0$ *and* $0 < \alpha \leq T$. *There exist a set* $E_\alpha \subset [0,T]$ *with* $E_\alpha = \mathcal{C}_\alpha \cup F_\alpha$ *where* $\mathcal{C}_\alpha$ *is a fat Cantor set on* $[0,T]$ *such that* $|[0,T] \setminus \mathcal{C}_\alpha| \leq \alpha$ *and* F_α *is a countable dense subset of* $[0,T]$*, and a weak solution* $u \in C_w([0,T]; L^2(\mathbb{T}^3))$ *of* (Equation NSE) *so that the following holds:*

1. *The energy profile* $\|u(t)\|_2^2$ *is discontinuous at every* $t \in E_\alpha$*. In fact,*

$$\limsup_{s\to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \text{for all } t \in \mathcal{C}_\alpha, \tag{3.7}$$

   and

$$\lim_{s\to t} \|u(s)\|_2^2 > \|u(t)\|_2^2 \quad \text{for all } t \in F_\alpha. \tag{3.8}$$

2. $u(t)$ *is uniformly* ε*-small in* $W^{1,1}(\mathbb{T}^3)$*:*

$$\|u\|_{L_t^\infty W^{1,1}} < \varepsilon, \tag{3.9}$$

   smooth on F_α*:*

$$u(t) \in C^\infty(\mathbb{T}^3) \quad \text{for all } t \in F_\alpha, \tag{3.10}$$

and vanishes on $\mathcal{C}_\alpha$:

$$u(t) = 0 \quad \text{for all } t \in \mathcal{C}_\alpha. \tag{3.11}$$

3.1.1 Some remarks on the results

Remark 3.1.4. *It is known that for any smooth force f* (Equation NSE) *on torus $\mathbb{T}^3$ admits at least one smooth stationary solution (22). Theorem 3.1.1 shows that there are infinitely many finite energy stationary weak solutions.*

Remark 3.1.5. *As our building blocks are compactly supported, it seems likely that there also exist finite energy stationary weak solutions in $\mathbb{R}^3$ with compact supports. We plan to address this problem in future works.*

Remark 3.1.6. *We note that weak solutions constructed in (8; 56; 5) can not be stationary as the building blocks are time-dependent and their schemes rely on fast time oscillations.*

Remark 3.1.7. *The smoothness of the vector field a in Theorem 3.1.2 and the force f in Theorem 3.1.1 can definitely be lower, but we are not interested in this direction here. Also, Theorem 3.1.2 shows that any smooth initial data u_0 admits infinitely many weak solutions with discontinuous energy.*

Remark 3.1.8. *It is possible to construct a weak solution with discontinuous energy by gluing the solutions in (8), see Appendix B.3. However, those discontinuities are not jumps. More importantly, such an argument can not generate dense discontinuities.*

Remark 3.1.9. *In view of the theory of Baire category, the set of discontinuities of a semi lower-continuous function is of Baire-1, which still can have full measure in* $[0, T]$*. At the moment, our method is not able to produce such examples.*

Remark 3.1.10. *Very recently, Luo and Titi (56) have extended the nonuniqueness result of (8) to factional NSE with* $(-\Delta)^{\alpha}$ *for any* $\alpha < \frac{5}{4}$*, which is sharp in view of Lion's wellposedness result (53; 54). Even though our method seems to work for factional NSE for some* $\alpha > 1$*, extensions to the full range of* $\alpha < \frac{5}{4}$ *are unavailable at this point.*

3.1.2 Effect of intermittency

As discussed in Chapter 2, in order for the d-dimensional Navier-Stokes equations to develop singularities, the intermittency dimension D of the flows should be less than $d - 2$, so that the Bernstein's inequality is highly saturated. So $D = 1$ is critical for the 3D NSE. It was also confirmed in (8; 58) that the main difficulty of conducting convex integration for the Navier-Stokes equations is the intermittency of the flow. Such a constraint, however, is not presented in the 3D Euler equations: Beltrami flows and Mikado flows used in the constructions of wild solutions for the 3D Euler equations are essentially homogeneous in space, namely the the intermittency dimension $D = 3$. This is also reflected in the difference between L^3 based norm in the best known energy conservation condition $L^3_t B^{\frac{1}{3}}_{3,c_0(\mathbb{N})}$ in (11) and L^∞ based norm of the counterexamples (CC^α for $\alpha < \frac{1}{3}$ in (45)) for the 3D Euler equations (45; 3).

To resolve the issue of intermittency when applying convex integration, Buckmaster-Vicol introduced *intermittent Beltrami flows* in (8) and *intermittent jets* in (5) as building blocks with arbitrary small intermittency dimension $D > 0$, allowing them to successfully implement convex integration scheme in the presence of the dissipative term Δu. This was done by introducing a Dirichlet type kernel to the classical Beltrami flows in (8) or using a space-time cutoff in (5) respectively, rendering the linear term manageable. Even though such modifications produce unwanted interactions that are too large for the convex integration scheme to go through, they were handled with an additional "convex integration in time" with a help of very fast temporal oscillations. We note that even though it was possible to take advantage of all the interactions between Dirichlet kernels in (13; 17), this is out of reach in the convex integration scheme at this point.

In this chapter, we will design new building blocks specifically for the NSE. These vector fields, that we call *viscous eddies*, will be both stationary and compactly supported in $\mathbb{R}^3$. The construction is partly motivated by the geometric Lemma 3.3.1 used for the Mikado flows which were introduced in (25) and have been successfully used for the Euler equations on the torus $\mathbb{T}^n$ for $n \geq 3$. The Mikado flows can also be rescaled so that its intermittency dimension becomes $D = 1$ as demonstrated in Chapter 2, (see also (60; 61) for the setting in transport equation). This just misses the $D < 1$ requirement for the 3D NSE (see discussions in Chapter 2 and heuristics in Section 2 of (12)).

In order to increase concentration that decreases the intermittency dimension, we start with a pipe flow in $\mathbb{R}^3$, use a lower order cutoff only in space along the direction of the flow, and add a correction profile to the existing one so that it will take advantage of the Laplacian to balance some of the unwanted interactions. This is possible due to the fact that the error introduced by the space cutoff along the major axis of the eddies is not a general stress term, but basically one-dimensional. By design, *viscous eddies* are divergence-free up to the leading term. Moreover, they are compactly supported approximate stationary solutions of the NSE (not the Euler equations). See Theorem 3.3.13 for a precise statement. Compared with the previously used building blocks for the NSE, such an approach mainly has two advantages. First, the new flows are time-independent and hence can be used to construct nontrivial stationary weak solutions, which was an open question for the 3D NSE. Second, they are compactly supported and can be used in the case of the whole space $\mathbb{R}^3$ in the future, whereas Beltrami flows, Mikado flows, intermittent Beltrami flows, and intermittent jets only exist on the torus $\mathbb{T}^d$.

3.1.3 Energy pumping mechanism

In order to produce discontinuous energy we introduce a new energy pumping mechanism that uses more energy than needed to cancel the stress error term in the convex integration scheme. In previous works, there is a correspondence between the growth of the frequency and the decay of the energy so that the energy is not changed much along the iteration process. In other words, the high frequency part of the solution is very small uniformly in time. This is typical and desirable in order to improve the regularity of the wild solutions.

In contrast, to produce discontinuities in the energy, one can not adhere to such a uniformity in time in the scheme. We need to allow high frequencies to carry sizable energy on some time intervals, so that there is energy coming from/escaping to infinite wavenumber[1]. Consider the following toy model. Suppose $u(t)$ is a function with Fourier support in a shell of size $\lambda(t)$, and

[1] Such possible scenarios are closely related to the energy balance equation for the Navier-Stokes equations. See for instance (12)

$\lambda(t) \to \infty$ as $t \to T$. Then the energy remains constant for $t < T$, but at $t = T$, the solution is zero, as all the energy has escaped to the infinite wavenumber. To reproduce this toy model in the convex integration scheme, one needs to construct an approximate sequence of solutions with temporal supports away from time T and sizable energy near T, such that the weak limit is 0 at $t = T$. Generalizing this example, one can construct a wild solution of the Navier-Stokes equations whose energy is constant on $(0, T)$ but vanishes at 0 and T.

However, if one uses solutions of such type with disjoint temporal support and glues them together, the resulting solution will only have finitely or countably many discontinuities. The next goal is to achieve the density of jumps. An exercise in real analysis shows that there exist unbounded L^2 functions that blow up on a dense subset of $[0, 1]$. Roughly speaking, we will construct solutions whose energy mimics the behavior of such functions. More precisely, there will be infinitely many blowing-up wavenumbers $\lambda(t)$ with smaller and smaller lifespan and energy. This is also consistent with the fact that the jumps decrease to zero along the iterations, which is anticipated as the energy, which we want to be bounded, needs some time to be transferred to lower/higher modes. We refer to Section 3.2 for more technical details in this regard.

3.2 The main proposition

The main objective of this section is to prove Theorems 3.1.1, 3.1.2, and 3.1.3 using Proposition 3.2.1, which we will refer to as the main proposition.

3.2.1 Generalized Navier-Stokes system

Let $a, f \in C^\infty(\mathbb{T}^3 \times [0,T])$ be smooth divergence-free vector fields with zero mean for all $t \in [0,T]$. We consider the following generalized Navier-Stokes system:

$$\begin{cases} \partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla p = f \\ \operatorname{div} v = 0, \end{cases} \tag{gNSE}$$

where

$$L_a v = -\Delta v + \operatorname{div}(v \otimes a) + \operatorname{div}(a \otimes v).$$

The reason to consider such a generalization is as follows. Suppose v is a weak solution to (Equation gNSE) with given vector field a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$. Then $u := v + a$ solves (Equation NSE). We note that the added terms are of lower order compared to the nonlinearity $\operatorname{div}(v \otimes v)$, and thus will not be of any trouble in the proof.

To construct weak solutions to (Equation gNSE), let us consider the approximate equations

$$\begin{cases} \partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} R + f \\ \operatorname{div} v = 0, \end{cases} \tag{gNSR}$$

where R is a symmetric traceless matrix. If (v, p, R, f) is a solution to (Equation gNSR), then we say (v, R) is a solution to (Equation gNSR) with data a and f. The above system is reminiscent to the so-called Navier-Stokes-Reynolds system used in the previous works (5; 8; 58). Our main proposition is to construct weak solutions to (Equation gNSE) using a sequence of solutions

(v_n, R_n) of the approximate system (Equation gNSR) so that the stress term $R_n \to 0$ as $n \to \infty$ in a suitable sense.

3.2.2 Main proposition

In this subsection, we will introduce the main proposition of the chapter, which will enable us to prove all the main theorems listed in the introduction.

Throughout the chapter we use the following notations. For any $r > 0$ and any finite set $F \subset [0, T]$, let

$$\begin{aligned} B_r(F) &= \{t \in [0,T] : \operatorname{dist}(t, F) < r\}, \\ I_r(F) &= [0,T] \setminus B_r(F). \end{aligned} \tag{3.12}$$

Proposition 3.2.1. *Let $c_0 = 10^{-2}$, $T > 0$.*[1] *Consider the system* (Equation gNSR) *with given $a, f \in C^\infty(\mathbb{T}^3 \times [0,T])$ smooth vector fields with zero mean. There exists a small universal constant C such that the following holds.*

Let $\varepsilon, r > 0$, $0 < e_0 < e_1 < \infty$, and $\mathcal{F}_0, \mathcal{F}_1 \subset [0,T]$ be two finite sets such that $\mathcal{F}_0 \subset \mathcal{F}_1$. If (v_0, R_0) is a smooth solution to (Equation gNSR) *on $[0,T]$ with data a and f so that*

1. *the energy $\|v_0(t)\|_2^2 \le e_0$ for all t, and is almost constant e_0 away from the set $\mathcal{F}_0$:*

$$\big| \|v_0(t)\|_2^2 - e_0 \big| \le c_0(e_1 - e_0) \quad \text{for all } t \in I_r(\mathcal{F}_0) \ ,$$

---

[1] *Since we only use c_0 to measure the approximate level of the energy to a constant, the exact value of c_0 is not important.*

2. (v_0, R_0) *is close to a solution of* (Equation gNSE) *in the sense that*

$$\delta_0 \leq C(e_1 - e_0),$$

where $\delta_0 = \|R_0\|_{L^\infty_t L^1_x(\mathbb{T}^3 \times [0,T])}$,

then there is another smooth solution (v, R) *to* (Equation gNSE) *with data* a *and* f *such that*

1. The energy $\|v(t)\|_2^2 \leq e_1$ *for all* t*, and is almost constant* e_1 *away from the set* $\mathcal{F}_1$*:*

$$\big|\|v(t)\|_2^2 - e_1\big| \leq \frac{c_0}{2}(e_1 - e_0) \quad \textit{for all } t \in I_{4^{-1}r}(\mathcal{F}_1)\ .$$

2. The new stress R *verifies*

$$\|R(t)\|_1 \leq \begin{cases} \varepsilon & \textit{for } t \in I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 + \varepsilon & \textit{for } t \in I_{4^{-2}r}(\mathcal{F}_1) \setminus I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 & \textit{for } t \in [0,T] \setminus I_{4^{-2}r}(\mathcal{F}_1). \end{cases} \tag{3.13}$$

.

Moreover, the velocity increment $w = v - v_0$ *verifies*

$$\operatorname{supp}_t w \subset I_{4^{-2}r}(\mathcal{F}_1) \quad \textit{and} \quad \|w\|_{L^\infty_t W^{1,1}} \leq \varepsilon, \tag{3.14}$$

and if $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$ and v_0 is stationary[1]*, i.e. $\partial_t v_0 = 0$, then w is also stationary: $\partial_t w = 0$.*

3.2.3 Proof of main theorems

We first prove Theorem 3.1.2, it suffices to prove the following result for (Equation gNSE):

Theorem 3.2.2. *Let $\varepsilon > 0$ and $a \in C^\infty(\mathbb{T}^3 \times [0,T])$, $T > 0$ be a smooth divergence-free function with zero mean for all $t \in [0,T]$. Consider the associated generalized Navier-Stokes system* (Equation gNSE) *with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$. There exists a dense subset $E \subset [0,T]$, a constant $M_a > 0$ such that for any $M \geq M_a$ there exists weak solution $v \in C_w(0,T; L^2(\mathbb{T}^3))$* (Equation NSE) *so that the followings hold:*

1. *The energy $\|v(t)\|_2^2$ is bounded by M:*

$$\|v(t)\|_2^2 \leq M \quad \textit{for any } t \in [0,T] \ , \tag{3.15}$$

   and has jump discontinuities on set E:

$$\lim_{s \to t} \|v(s)\|_2^2 > \|v(t)\|_2^2 \quad \textit{for any } t \in E \ . \tag{3.16}$$

2. *$v(t)$ vanishes at $t = 0, T$:*

$$v(x,0) = v(x,T) = 0, \tag{3.17}$$

---

[1] *In this case, we of course require both a and f to be time-independent.*

but the energy jump is of size M:

$$\lim_{s\to 0+} \|v(s)\|_2^2 - \|v(0)\|_2^2 = \lim_{s\to T-} \|v(s)\|_2^2 - \|v(T)\|_2^2 = M. \tag{3.18}$$

3. $v(x,t)$ is smooth on E:

$$v(t) \in C^\infty(\mathbb{T}^3) \quad \text{for all } t \in E, \tag{3.19}$$

and is ε-small in $L_t^\infty W_x^{1,1}$:

$$\|v\|_{L_t^\infty W^{,1}} < \varepsilon. \tag{3.20}$$

The implication from Theorem 3.2.2 to Theorem 3.1.2 can be obtained simply by shifting $u = v + a$ since the vector field a is smooth. Now we prove Theorem 3.2.2 with the help of Proposition 3.2.1.

Proof of Theorem 3.2.2 assuming Proposition 3.2.1. We first construct the set E, then a sequence of approximate solution v_n such that v_n converges to the desire solution v in a suitable sense. Without loss of generality, we assume $T = 1$.

Step 1: Constructing the set E. Consider the binary representation of $x \in [0,1]$:

$$x = \sum_{j=0}^{\infty} x_j 2^{-j}.$$

Now let F_n be the collection of all real numbers in $[0,1]$ whose binary representation has at most n digits, namely $x \in F_n \subset [0,1]$ if and only if $x_j = 0$ for all $j > n$. Assuming $F_{-1} = \emptyset$, let also $E_n = F_{n+1} \setminus F_n$, $n \geq -1$. For instance, $E_{-1} = \{0,1\}$, $E_0 = \{1/2\}$, $E_1 = \{1/4, 3/4\}$. Let

$$E = \lim_{n\to\infty} F_n = \bigcup_{n\geq -1} E_n,$$

which is a dense subset of $[0,1]$.

Denoting $r_n = 4^{-n-1}$, let us show the following important property of the set E for later use:

$$\liminf_{n\to\infty} B_{r_n}(F_{n-1}) \subset E. \tag{3.21}$$

Suppose $t \in \liminf B_{r_n}(F_{n-1})$, which means that there exist N and $t_n \in F_{n-1}$ for every $n \geq N$, such that

$$|t - t_n| = \operatorname{dist}(t, F_{n-1}) < r_n. \tag{3.22}$$

We claim that $t_{n+1} = t_n$ for all $n \geq N$. Otherwise, for some $n \geq N$ there must be

$$|t - t_n| \geq |t_{n+1} - t_n| - |t - t_{n+1}| \geq 2^{-n} - r_{n+1} \geq 2^{-n-1},$$

which contradicts (Equation 3.22):

$$2^{-n-1} < r_n = 2^{-2n-2}.$$

Hence, it follows from (Equation 3.22) that $t = t_N \in F_{N-1}$ which implies that $t \in E$.

Step 2: Constructing approximate solutions v_n. Given smooth vector field a, we set $v_0 = 0$ and $R_0 = \mathcal{R}\big(\partial_t a - \Delta a + \operatorname{div}(a \otimes a)\big)$, where $\mathcal{R}$ is defined in Definition Equation 3.112. Then (v_0, R_0) is a smooth solution of (Equation gNSR) with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$ on $[0, 1]$. We choose

$$M_a = \frac{4}{C} \|R_0\|_{L^\infty_t L^1}, \tag{3.23}$$

where C is the constant in Proposition 3.2.1.

Let $r_n = 4^{-n-1}$ and $M \geq M_a$ and choose the energy level $e_n = (1 - 2^{-n})M$ for $n \in \mathbb{N}$. Note that the choice of e_n is admissible in view of (Equation 3.23).

Starting with (v_0, R_0), we apply Proposition 3.2.1 with data a and f on $[0, 1]$ to obtain a sequence (v_n, R_n) of smooth solutions of (Equation gNSR). More precisely, (v_{n+1}, R_{n+1}) is obtained by applying Proposition 3.2.1 to the previous solution (v_n, R_n) with parameters

$$(r, e_0, e_1, \varepsilon, \mathcal{F}_0, \mathcal{F}_1) := (r_n, e_n, e_{n+1}, \varepsilon_n, F_{n-1}, F_n),$$

where the small parameters ε_n are defined inductively by

$$\varepsilon_n = \frac{2^{-n-1}\varepsilon}{1 + \sum_{j \leq n-1} \sup_t \|w_j\|_\infty}, \tag{3.24}$$

and $w_j := v_j - v_{j-1}$ is the j-th velocity perturbation for $j \geq 1$.

Clearly, each (v_n, R_n) in the obtained sequence is a smooth solution of (Equation gNSR) on $[0,1]$ with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$, and by Proposition 3.2.1 we have the following properties:

1. For any $n \in \mathbb{N}$

$$\begin{aligned} \Big| \|v_n(t)\|_2^2 - e_n \Big| &\le c_0 2^{-n} M \\ \|R_n(t)\|_1 &\le \varepsilon_n \end{aligned} \qquad \text{for all } t \in I_{r_n}(F_{n-1})\ , \tag{3.25}$$

   and

$$\begin{aligned} \|v_n(t)\|_2^2 &\le e_n \le M, \\ \|R_n(t)\|_1 &\le \|R_0\|_{L_t^\infty L^1} + \varepsilon. \end{aligned} \qquad \text{for all } t \in [0,1]. \tag{3.26}$$

2. The velocity increment $w_n = v_n - v_{n-1}$ verifies that

$$\big\| w_n \big\|_{L_t^\infty W^{1,1}} \le \varepsilon_n. \tag{3.27}$$

3. If $t \in F_n$ for some $n \in \mathbb{N}$, then

$$v_k(t) = v_n(t) \quad \text{for all } k \ge n. \tag{3.28}$$

Step 3: L^2 convergence of v_n. The solution $v(t)$ is constructed as a strong L^2 limit of approximate smooth solutions $v_n(t)$,

$$v(t) = \lim_{n\to\infty} v_n(t) = \sum_{j=1}^{\infty} w_j, \qquad t \in [0,1].$$

We first prove that v is well-defined, i.e. v_n converges pointwise in L^2. Indeed, thanks to (Equation 3.24) and (Equation 3.27) the velocity perturbations w_k are almost orthogonal in L^2:

$$\sup_t |\langle w_j, w_k\rangle| \leq 2^{-j-1}\varepsilon \quad \text{for all } j > k. \tag{3.29}$$

As a result, due to (Equation 3.26)

$$\sum_{j=1}^{n} \|w_j\|_2^2 \leq \|v_n\|_2^2 + 2 \sum_{1\leq j<k\leq n} |\langle w_j, w_k\rangle| < M + 2\varepsilon \quad \text{for all } n.$$

So, for $0 \leq n < m$ we have

$$\begin{aligned}\|v_m - v_n\|_2^2 &= \sum_{n<j\leq m} \|w_j\|_2^2 + 2 \sum_{n<j<k\leq m} |\langle w_j, w_k\rangle| \\ &< \sum_{j>n} \|w_j\|_2^2 + 2^{-n+1}\varepsilon \to 0 \quad \text{as } n, m \to \infty,\end{aligned}$$

i.e., $v_n(t)$ is Cauchy in L^2 for every $t \in [0,1]$.

Next, we show that v is a weak solution of (Equation gNSE). Let test function $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,1))$ be mean-free and divergence-free for all $t \in [0,1]$. Using the weak formulation for the solution (v_n, R_n)) of (Equation gNSR) with data a and $f = -\partial_t a + \Delta a - \operatorname{div}(a \otimes a)$, we get

$$\begin{aligned} \int_{\mathbb{T}^3} v_n(\cdot,0)\cdot\varphi(\cdot,0) + &\int_{\mathbb{T}^3\times[0,1]} v_n\cdot\partial_t\varphi + v_n\cdot(v_n\cdot\nabla)\varphi + v_n\cdot\Delta\varphi \\ &+ \int_{\mathbb{T}^3\times[0,1]} a\cdot(v_n\cdot\nabla)\varphi + v_n\cdot(a\cdot\nabla)\varphi = \int_{\mathbb{T}^3\times[0,1]} R_n : \nabla\varphi + f\cdot\varphi. \end{aligned} \tag{3.30}$$

For simplicity of notation, let

$$I_n = \bigcap_{k\geq n} I_{r_k}(F_{k-1}).$$

Immediately

$$\big|[0,1]\setminus I_n\big| \lesssim 2^{-n}. \tag{3.31}$$

From (Equation 3.25) and (Equation 3.29) it follows that

$$\|v - v_n\|^2_{L^\infty L^2(\mathbb{T}^3\times I_n)} \leq \sup_{I_n}\big(\|v(t)\|_2^2 - \|v_n(t)\|_2^2 - 2\langle v - v_n, v_n\rangle\big) \lesssim 2^{-n}, \tag{3.32}$$

and

$$\|R_n\|_{L_t^\infty L^1(\mathbb{T}^3\times I_n)} \lesssim 2^{-n}. \tag{3.33}$$

Using the bounds (Equation 3.31), (Equation 3.32), and (Equation 3.33) together with (Equation 3.26), it is easy to check the convergence of all the terms in (Equation 3.30) to their natural limits by splitting the domain of integrals into $\mathbb{T}^3 \times I_n$ and $\mathbb{T}^3 \times I_n^c$.

Next, let us show that as the pointwise L^2 limit of v_n, the solution v is weakly continuous. Let $\varphi \in L^2(\mathbb{T}^3)$ and $t_0 \in [0,1]$. Consider the following split:

$$\big|\langle v(t) - v(t_0), \varphi\rangle\big| \leq \big|\langle v(t) - v_n(t), \varphi\rangle\big| + \big|\langle v_n(t) - v_n(t_0), \varphi\rangle\big| + \big|\langle v_n(t_0) - v(t_0), \varphi\rangle\big|.$$

The first and last terms go to zero as $n \to \infty$ by the uniform $W^{1,1}$ convergence of v_n. For the second term, since $v_n \in C_0^\infty(\mathbb{T}^3 \times [0,1])$, we get

$$\big|\langle v_n(t) - v_n(t_0), \varphi\rangle\big| \to 0 \quad \text{as } t \to t_0.$$

So we may conclude that $\langle v(t) - v(t_0), \varphi\rangle \to 0$ as $t \to t_0$.

Step 4: Verifying properties of v. Finally, we show that v is a weak solution satisfying all the properties (1), (2) and (3) stated in Theorem 3.2.2. First, $\|v(t)\|_2^2 \leq M$ for all $t \in [0,1]$ due to (Equation 3.26). Therefore, to show (1), it remains to prove that E consists of jump discontinuities.

Indeed, given $t \in E$, there exists n such that $t \in E_n$, which implies $t \in I_{r_{n+1}}(F_n)$ and $v(t) = v_{n+1}(t)$. Using (Equation 3.25) we get

$$M - \|v(t)\|_2^2 \geq M - e_{n+1} - c_0 M 2^{-n-1}$$
$$\gtrsim M 2^{-n}.$$

We will show that $\lim_{s \to t} \|v(s)\|_2^2 = M$. To this end, let

$$I_\varepsilon = \{s \in [0,1] : t - \varepsilon < s < t \text{ or } t < s < t + \varepsilon\},$$

and

$$N_\varepsilon = \max\{j \in \mathbb{N} : I_\varepsilon \cap F_j = \emptyset\}.$$

By definitions of the sets F_n we have $N_\varepsilon > n$ provided $\varepsilon \leq 2^{-n-1}$, which implies that $\lim_{\varepsilon \to 0+} N_\varepsilon = \infty$. Moreover, from (Equation 3.21) it follows that

$$E^c = [0,1] \setminus E \subset \limsup I_{r_j}(F_{j-1}),$$

which by (Equation 3.25) and the pointwise L^2 convergence of v_n implies that

$$\|v(s)\|_2^2 = M \qquad \text{for all } s \in E^c.$$

Thus we only need to consider $s \in I_\epsilon \cap E$. In this case $s \notin F_{N_\varepsilon}$, however, $s \in E_m$ for some $m \geq N_\varepsilon$ and $v(s) = v_{m+1}(s)$. Then $s \in I_{r_{m+1}}(F_m)$, and therefore, (Equation 3.25) implies that

$$||\|v(s)\|_2^2 - M| \lesssim 2^{-N_\epsilon}.$$

Taking a limit $\epsilon \to 0$ we obtain $\lim_{s\to t} \|v(s)\|_2^2 = M$. Thus statement (1) is proved. As a special case of the jump discontinuities, statement (2) follows as well.

The smoothness of v on the set E and the uniform smallness of v in $W^{1,1}$ follow directly from (Equation 3.28) and (Equation 3.27) respectively. So, statement (3) has been obtained as well.

□

Next, we use a gluing technique to glue pieces of weak solutions given by Theorem 3.1.2 to obtain Theorem 3.1.3.

Proof of Theorem 3.1.3. It is clear that Theorem 3.1.2 works for any interval $[t_0, t_1]$. Also, the energy level M_a depends only on the vector field a and M_a can be any positive number when $a = 0$. Without loss of generality, we assume $T = 1$.

Step 1: Constructing approximate sequence u_n. Let $\mathcal{C}_\alpha$ be a fat Cantor set on $[0, 1]$ with measure $(1 - \alpha)$ (each time remove the middle interval of length $(\frac{\alpha}{1+2\alpha})^n$). In other words,

$$\mathcal{C}_\alpha = [0,1] \setminus \bigcup_{n\geq 1} \bigcup_{1\leq j\leq 2^{n-1}} I^\alpha_{j,n},$$

where $I^{\alpha}_{j,n}$ are the open intervals removed from the fat Cantor set $\mathcal{C}_{\alpha}$ at step n.

Let us first construct a sequence of weak solutions of (Equation NSE) that are supported on $\overline{I^{\alpha}_{j,n}}$. Applying Theorem 3.1.2 on each interval $I^{\alpha}_{j,n}$ with $(\varepsilon, a, M_a) := (\varepsilon 4^{-n}, 0, 1)$, we obtain a weak solution $u_{j,n}$, which we then extend trivially to the whole interval $[0,1]$. The resulting sequence of weak solutions $u_{j,n}$ satisfy

1. $u_{j,n}$ is supported on $\overline{I^{\alpha}_{j,n}}$. Moreover,

$$u_{j,n}(t) = 0, \quad \text{for } t \notin I^{\alpha}_{j,n}.$$

2. $u_{j,n}$ is small in $W^{1,1}$:

$$\|u_{j,n}\|_{L^\infty W^{1,1}} \leq \varepsilon 4^{-n}. \tag{3.34}$$

3. $\|u_{j,n}\|_2^2$ is discontinuous on a dense subset $F^{\alpha}_{j,n} \subset \overline{I^{\alpha}_{j,n}}$.

Since $\overline{I^{\alpha}_{j,n}} \cap \overline{I^{\alpha}_{j',n'}} = \emptyset$ if $j \neq j'$ or $n \neq n'$, namely $u_{j,n}$ have disjoint temporal supports, we can construct another sequence of weak solutions of (Equation NSE) by defining

$$u_n = \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq 2^{n-1}} u_{j,k}.$$

As both summations are finite, u_n are weakly continuous in L^2 and are indeed weak solutions on $\mathbb{T}^3 \times [0,1]$.

Step 2: Convergence and weak continuity of u_n. We claim that $u_n(t)$ pointwise converges in L^2 and define

$$u(t) = \lim_{n\to\infty} u_n(t), \qquad t \in [0,1].$$

To prove this claim, consider two sub-cases.

(a) If $t \in \mathcal{C}_\alpha$, then $u_n(t) = \sum_{k\le n}\sum_j u_{j,k}(t) = 0$ for all n. So, in particular, $u_n(t) \to 0$ in L^2.

(b) If $t \in [0,T]\setminus\mathcal{C}_\alpha$, then there exist $j, n \in \mathbb{N}$ such that $t \in I^\alpha_{j,n}$. Thus $u_m(t) = u_n(t)$ for any $m \ge n$, and consequently $u(t) = u_n(t)$.

Combining this with (Equation 3.34), it is also clear that statement (2) holds.

Next, we show that $u \in C_w([0,1]; L^2)$, i.e., $u(t)$ is weakly continuous. Let $\varphi \in L^2(\mathbb{T}^3)$ and $t_0 \in [0,1]$. As usual, we consider the split

$$\big|\langle u(t) - u(t_0), \varphi\rangle\big| \le \big|\langle u(t) - u_n(t), \varphi\rangle\big| + \big|\langle u_n(t) - u_n(t_0), \varphi\rangle\big| + \big|\langle u_n(t_0) - u(t_0), \varphi\rangle\big|. \quad (3.35)$$

Thanks to (Equation 3.34), for any $t \in [0,1]$ we have

$$\big|\langle u(t) - u_n(t), \varphi\rangle\big| \le \big\|u - u_n\big\|_{L^\infty W^{1,1}} \|\varphi\|_\infty \le \|\varphi\|_\infty \sum_{k>n}\sum_{1\le j\le 2^{n-1}} \big\|u_{j,k}\big\|_{L^\infty W^{1,1}} \le \varepsilon 2^{-n}\|\varphi\|_\infty.$$

So the first and the last terms in (Equation 3.35) go to zero as $n \to \infty$, which together with the weak continuity of u_n implies the weak continuity of u in L^2.

Finally, we show that u is a weak solution of (Equation NSE). Let test function $\varphi \in C_c^\infty(\mathbb{T}^3 \times [0,1))$ be mean-free and divergence-free for all $t \in [0,1]$. By the weak formulation of (Equation NSE) for u_n we get

$$\int_{\mathbb{T}^3} u_n(x,0) \cdot \varphi(x,0)\, dx + \int_0^1 \int_{\mathbb{T}^3} u_n \cdot \partial_t \varphi + u_n \cdot (u_n \cdot \nabla)\varphi + u_n \cdot \Delta \varphi \, dx d\tau = 0. \tag{3.36}$$

Since $u_n(0) = u(0) = 0$, the first term is zero. For the rest of the terms it suffices to show that

$$u_n \to u \quad \text{in } L^2_{t,x} \quad \text{as } n \to \infty.$$

Consider a remainder set

$$I_n = \bigcup_{m>n} \bigcup_{1 \le j \le 2^{n-1}} I^\alpha_{j,m}.$$

Since $\operatorname{supp}_t u_{j,m} \subset I^\alpha_{j,m}$ we know that

$$u(t) = u_n(t) \quad \text{for all } t \in [0,1] \setminus I_n.$$

Moreover, the set I_n is small by direct computation:

$$|I_n| \lesssim \Big(\frac{2\alpha}{1+2\alpha}\Big)^n.$$

Thanks to the above, we have

$$\left\|u_n - u\right\|_{L^2_{t,x}(\mathbb{T}^3\times[0,1])} = \left\|u_n - u\right\|_{L^2_{t,x}(\mathbb{T}^3\times I_n)} \leq \left\|u_n - u\right\|_{L^\infty_t L^2_x}\left|I_n\right|^{\frac{1}{2}} \to 0$$

as $n \to \infty$. So, we have proved that $u \in C_w(0,1;L^2)$ is a weak solution of (Equation NSE) satisfying statement (2).

Step 3: Discontinuities of $\|u\|_2^2$ on E_α. We first define the countable set F_α:

$$F_\alpha = \bigcup_{j,m} F^\alpha_{j,m}$$

where recall that $F^\alpha_{j,m}$ is the set of jump discontinuities of $\|u_{j,m}\|_2^2$. From the definition of $F^\alpha_{j,m}$ it follows that $F_\alpha \cap \mathcal{C}_\alpha = \emptyset$. Moreover, it is clear that F_α is a dense subset of $[0,1]$.

Let us show the discontinuity on $E_\alpha = \mathcal{C}_\alpha \cup F_\alpha$. Suppose $t_0 \in F_\alpha$, then $t_0 \in I^\alpha_{j,m}$ for some j,m. Moreover, this implies that

$$u(s) = u_{j,m}(s) \quad \text{for all } s \in I^\alpha_{j,m}.$$

Since $u_{j,m}$ is a weak solution given by Theorem 3.1.2, $\|u\|_2^2$ is discontinuous at t_0:

$$\lim_{s\to t_0} \|u(s)\|_2^2 > \|u(t_0)\|_2^2\ . \tag{3.37}$$

Next, suppose $t_0 \in \mathcal{C}_\alpha$, then $\|u(t_0)\|_2^2 = 0$. Let t_k be a sequence such that $t_k \to t_0$ as $k \to \infty$ and each t_k is the endpoint of $I^\alpha_{j,k}$ for some $j = j(k)$. Then from Theorem 3.1.2 we get

$$\limsup_{s \to t_k} \|u(s)\|_2^2 \geq \limsup_{s \to t_k} \|u_k(s)\|_2^2 = 1.$$

So, for any $t_0 \in \mathcal{C}_\alpha$ we have

$$\limsup_{s \to t_0} \|u(s)\|_2^2 > \|u(t_0)\|_2^2.$$

Statement (1) is now proved.

□

We finish this section by proving Theorem 3.1.1.

Proof of Theorem 3.1.1 assuming Proposition 3.2.1. Given any smooth force term f, let $v_0 = 0$ and $R_0 = -\mathcal{R}f$. So (v_0, R_0) solves (Equation gNSR) with data $a = 0$ and f. Then define

$$M_f = \frac{4}{C}\|R_0\|_{L^1}.$$

For any $M \geq M_f$ we can construct the solution as follows. Let the energy level $e_n = (1-2^{-n})M$ for $n \in \mathbb{N}$. Again, the choice of e_n is admissible due to $M \geq M_f$.

Starting with (v_0, R_0), we apply Proposition 3.2.1 to (v_n, R_n) with the same parameters as in the proof of Theorem 3.2.2:

$$(r, e_0, e_1, \varepsilon, \mathcal{F}_0, \mathcal{F}_1) = (4^{-n-1}, e_n, e_{n+1}, \varepsilon_n, \emptyset, \emptyset),$$

where ε_n is the same as (Equation 3.24). It should be noted that the value of r does not matter here as all v_n are stationary and $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$. Clearly, (v_n, R_n) are smooth solutions of (Equation gNSR) with data $a = 0$ and f such that

$$\left| \|v_n\|_2^2 - e_n \right| \leq c_0 M 2^{-n-1},$$

$$\|R_n\|_1 \leq 2^{-n-1}\varepsilon.$$

Using the same argument as in the proof of Theorem 3.2.2, one can show that v_n converges to a stationary weak solution $v \in L^2$ of (Equation gNSE) with data $a = 0$ and f such that $\|v\|_2^2 = M$. So v is a stationary weak solution of (Equation NSE) with forcing term f. $\square$

3.3 Stationary viscous eddies

In this section, the building blocks of the solution sequence are constructed. The entire construction is done in the whole space $\mathbb{R}^3$ not on torus $\mathbb{T}^3$. Recall the standard stationary Mikado flows can be rescaled so that the intermittency dimension $D = 1$ (58), which is insufficiently intermittent to be the building blocks for the 3D Navier-Stokes equations. Being also stationary, our *viscous eddies* are in the intermittency regime $D < 1$, but the full range $0 < D < 1$ is unattainable.

There are two main major differences between our new building blocks and previous ones used for the NSE, *intermittent jets* in (8). First, existing building blocks for the NSE are exact or approximate solutions of the Euler equations. As a result, the linear term is purely a useless error in those convex integration schemes. In contrast, *viscous eddies* are a family of approximate stationary solutions to the NSE, not Euler equations, see Theorem 3.3.13. The Laplacian is essential as it balances the leading term in the equations. Second, *viscous eddies* are time-independent, which enables us to obtain stationary weak solutions with time-independent (or zero) external force. In other words, our scheme does not require time oscillations, which might be of interest in improving the temporal regularity of wild solutions.

3.3.1 A geometric lemma

We start with a geometric lemma that dates back to the work of Nash (62). A proof of the following version, which is essentially due to De Lellis and Székelyhidi Jr., can be found in (70, Lemma 3.3). This lemma allows us to reconstruct any stress tensor R in a compact subset of $\mathcal{S}^{3\times 3}_{+}$, the set of positive definite symmetric 3×3 matrices.

Lemma 3.3.1. *For any compact subset $\mathcal{N} \subset \mathcal{S}^{3\times 3}_{+}$, there exists $\lambda_0 \geq 1$ and smooth functions $\Gamma_k \in C^\infty(\mathcal{N};[0,1])$ for any $k \in \mathbb{Z}^3$ with $|k| \leq \lambda_0$ such that*

$$R = \sum_{k\in\mathbb{Z}^3, |k|\leq\lambda_0} \Gamma_k^2(R) \frac{k}{|k|} \otimes \frac{k}{|k|} \qquad \textit{for all } R \in \mathcal{N}.$$

Lemma 3.3.1 is one of the reasons we choose to construct *viscous eddies*, which will be nonisotropic, closed to pipe flows, and divergence-free up to the leading order terms.

Fix a compact subset $\mathcal{N} \subset \mathcal{S}_+^{3\times 3}$ and let $\mathbb{K} \subset \mathbb{R}^3$ be the finite set of vectors given by Lemma 3.3.1[1], the directions of the major axis of *viscous eddies*. We can then choose a collection of points $p_k \in [0,1]^3$ for $k \in \mathbb{K}$ and a number $\mu_0 > 0$ such that

$$\bigcup_k B_{\mu_0^{-1}}(p_k) \subset [0,1]^3,$$

and

$$B_{2\mu_0^{-1}}(p_k) \cap B_{2\mu_0^{-1}}(p_{k'}) = \emptyset \quad \text{if } k \neq k'.$$

These points p_k will be the centers of our eddies and the balls $B_{\mu_0^{-1}}(p_k)$ will contain the supports of the eddies. Let

$$l_k := \{p_k + tk : t \in \mathbb{R}\} \subset \mathbb{R}^3$$

be the line passing through the point p_k in the k direction.

3.3.2 Velocity profiles

Let $\psi \in C_c^\infty(\mathbb{R}^+)$ be a smooth non-negative non-increasing function so that $\operatorname{supp}\psi \subset [0,1]$. Then let

$$\phi(r) := -\frac{1}{r}\int_r^\infty \psi(s)s\,ds. \tag{3.38}$$

---

[1]For applications in this chapter, the set $\mathcal{N} \subset \mathcal{S}_+^{3\times 3}$ is fixed. See Section 3.4.5.

Note that $\phi \in C^\infty((0,\infty))$, $\phi(r) = 0$ for $r > 1$, and ϕ has a singularity r^{-1} near the origin due to the monotonicity of ψ.

At this time we also assume

$$\int_0^\infty (\psi^2 - \phi\psi')r\,dr = 0, \tag{3.39}$$

which will be verified in the next lemma.

Lemma 3.3.2. *There exits a smooth non-negative non-increasing* $\psi \in C_c^\infty([0,1])$ *such that* (Equation 3.39) *holds and* $\psi' = 0$ *in a neighborhood of* 0.

Proof. Integrating by parts we obtain

$$\begin{aligned}\int_0^\infty (\psi^2 - \phi\psi')\,r dr &= \int_0^\infty r\psi^2 dr + \int_0^\infty \int_r^\infty \psi(s)s\,ds\,\psi'(r)\,dr \\ &= 2\int_0^\infty r\psi^2 dr - \psi(0)\int_0^\infty r\psi\,dr.\end{aligned}$$

We first fix a non-negative non-increasing $\psi \in C_c^\infty([0,1])$ such that

$$\psi(r) = 1 \quad \text{for all } r \in [0,1/2] \quad \text{and} \quad 2\int_0^\infty r\psi^2 dr - \int_0^\infty r\psi dr > 0.$$

Note that the existence of such functions can be seen by taking mollification on the characteristic function $\chi_{[0,1]}$

Let us consider $\psi_a = \psi + a\psi(ar)$, $a \geq 1$ to be determined, for which we need the solve the equation

$$F(a) := 2\int_0^\infty r\psi_a^2\, dr - \psi_a(0)\int_0^\infty r\psi_a\, dr = 0.$$

It is clear that once a solution $F(a) = 0$ is found, the lemma is proven.

A direct computation yields that

$$F(a) = 4\left(\int_0^\infty r\psi^2\, dr + \int_0^\infty ra\psi(r)\psi(ar)\, dr\right) - (1+a)\left(\int_0^\infty r\psi\, dr + \int_0^\infty ra\psi(ar)\, dr\right). \tag{3.40}$$

In particular, our assumption on ψ implies

$$F(1) = 8\int_0^\infty r\psi^2\, dr - 4\int_0^\infty r\psi\, dr > 0.$$

As $a \to \infty$ we notice in (Equation 3.40) that

$$\int_0^\infty ra\psi(r)\psi(ar)dr \leq \int_0^\infty ra\psi(ar)dr = a^{-1}\int_0^\infty r\psi dr \to 0,$$

and thus there exist some $c_0, c_1 > 0$ depending of ψ such that

$$F(a) \leq c_0 - c_1(1+a) \quad \text{for all sufficiently large } a,$$

which implies that there exits $1 < a < \infty$ such that $F(a) = 0$. $\square$

Throughout this section we will work in cylindrical coordinates to simply notations. Let

$$z_k =(x - p_k) \cdot \frac{k}{|k|}, \tag{3.41}$$

$$r_k =\mathrm{dist}(x, l_k) \tag{3.42}$$

be the cylindrical coordinates with respect to the basis$\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ centered at p_k, with $\mathbf{e}_z = \frac{k}{|k|}$.

It would also be convenient to introduce the following decomposition

$$\mathbb{R}^3 = \Omega_k \oplus l_k, \tag{3.43}$$

where $\Omega_k = \{x \in \mathbb{R}^3 : x \cdot k = 0\}$ is the plane orthogonal to l_k.

Finally, let us fix a smooth nontrivial function $\eta \in C_c^\infty(\mathbb{R})$ such that $\int \eta = 0$ and $\eta = 0$ for $|x| \geq 1$.

Definition 3.3.3 (Principle profiles ψ_k and η_k)**.** *For $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$ let $\eta_k, \psi_k \in C^\infty(\mathbb{R}^3)$ and $\phi_k \in C^\infty(\mathbb{R}^3 \setminus l_k)$ be defined by*

$$\begin{aligned}
\eta_k &= c\tau^{1/2}\eta(\tau z_k), \\
\psi_k &= \mu\psi(\mu r_k), \\
\phi_k &= \phi(\mu r_k),
\end{aligned} \tag{3.44}$$

where c is a normalizing constant such that $\int_{\mathbb{R}^3} |\eta_k \psi_k|^2 dx = 1$.

Remark 3.3.4. *Note that η_k and ψ_k are smooth and compactly supported in Ω_k, but not ϕ_k which still has a compact support in Ω_k but also a singularity $1/r$ at the origin. We can use a mollification to smear out the singularity thanks to Proposition 3.3.9.*

Using cylindrical coordinates we can easily prove the following simple lemma regarding the profiles η_k and ψ_k.

Lemma 3.3.5. *For any $k \in \mathbb{K}$, the rescaled functions ψ_k and ϕ_k verify the identities*

$$\frac{\partial(r_k\phi_k)}{\partial r_k} = r_k\psi_k \quad \textit{and} \quad \int_0^\infty \left(\psi_k^2 - \phi_k\frac{\partial\psi_k}{\partial r_k}\right) r_k\, dr_k = 0. \tag{3.45}$$

For any $1 \le p \le \infty$, there hold

$$\begin{aligned} \|\eta_k\|_{L^p(l_k)} &\lesssim \tau^{1/2-1/p}, \\ \|\psi_k\|_{L^p(\Omega_k)} &\lesssim \mu^{1-2/p}, \end{aligned} \tag{3.46}$$

and

$$\|\phi_k\|_{L^p(\Omega_k)} \lesssim_p \mu^{-2/p} \quad \textit{if } 1 \le p < 2. \tag{3.47}$$

Proof. The first two identities (Equation 3.45) follow from the rescalings (Equation 3.44), (Equation 3.38) as well as the zero-mean condition (Equation 3.39).

The first two estimates (Equation 3.46) follow from rescaling and the the fact that $\eta, \psi \in C_c^\infty(\mathbb{R}^+)$ while (Equation 3.47) follows from rescaling and the fact that $\phi \in L^p(rdr)$ for any $1 \le p < 2$.

□

Next, we introduce another family of profiles that will be used to form the Laplacian corrector part of the eddies.

Thanks to the zero-mean condition (Equation 3.45) and the vanishing of ψ' near the origin obtained in Lemma 3.3.2, Lemma B.2.1 implies that there exists $h \in C^\infty(\mathbb{R}^+)$, such that $h(|\cdot|) \in C^\infty(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$ for $1 < p \leq \infty$, and

$$\Delta h(|x|) = (\psi(|x|))^2 - \phi(|x|)\psi'(|x|). \tag{3.48}$$

Then define $\Psi_k \in C^\infty(\mathbb{R}^3)$ by

$$\Psi_k := h(\mu r_k), \tag{3.49}$$

for which we have

$$\Delta\big(\Psi_k\big) = \psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}. \tag{3.50}$$

.

Let us fix some nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^+)$, such that $\phi(r) = 1$ for $r \leq 1$, $\operatorname{supp}\varphi \in [0,2]$, and $\int_0^\infty \varphi\, r\, dr = 1$. This function will be used as a cutoff in Definition 3.3.6 below and a radial mollification in Definition 3.3.7.

Now we define another two profile functions, $\widetilde{\psi}_k$ and $\widetilde{\eta}_k$, which will constitute an important part of our eddies.

Definition 3.3.6 (Viscous profiles $\widetilde{\psi}_k$ and $\widetilde{\eta}_k$)**.** *For $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$, define*

$$\widetilde{\psi}_k = \varphi(\tau r_k)\Psi_k,$$

and

$$\widetilde{\eta}_k = \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}.$$

Note that the extra mild cutoff $\phi(\tau r_k)$ is to make sure the support of $\widetilde{\psi}_k$ is contained in a cylinder centered at the line l_k in $\mathbb{R}^3$ so that $\widetilde{\eta}_k\widetilde{\psi}_k$ is compactly supported.

3.3.3 Vector fields $\mathbb{W}_k$ and $\mathbb{V}_k$.

Let us first introduce vector fields $\mathbb{W}_k$ and $\mathbb{V}_k$, which corresponds to the principle part and respectively the Laplacian correction part of the eddies.

Definition 3.3.7. *Let $\mathbb{K} \subset \mathbb{R}^3$ be a finite set and $\gamma > 0$ be a small constant. For each $k \in \mathbb{K}$ and $\mu \geq \tau \geq \mu_0$, the vector fields $\mathbb{W}_k : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbb{V}_k : \mathbb{R}^3 \to \mathbb{R}^3$ are defined by*

$$\mathbb{W}_k = (W_z + W_r)_\gamma \quad \text{and} \quad \mathbb{V}_k = \widetilde{\eta}_k\widetilde{\psi}_k\mathbf{e}_z, \tag{3.51}$$

where the vector fields W_z and W_r are respectively defined by

$$W_z = \eta_k\psi_k\mathbf{e}_z, \qquad W_r = -\frac{\partial\eta_k}{\partial z_k}\phi_k\mathbf{e}_r. \tag{3.52}$$

Here $(\cdot)_\gamma := \varphi_\gamma *$ indicates a radial mollification at scale $\mu^{-1-\gamma}$ in the Ω_k-plane via the kernel

$$\varphi_\gamma = \tfrac{1}{2\pi}\mu^{2+2\gamma}\varphi(\mu^{1+\gamma}r_k).$$

In addition, let W_k be the non-smooth counterpart of $\mathbb{W}_k$ defined by

$$W_k = W_z + W_r. \tag{3.53}$$

The role of each parameter is as follows.

- μ^{-1} parametrizes the concentration level of eddies.
- τ^{-1} measures the closeness of eddies to the pipe flows
- γ is a small constant that we use to achieve the smoothness of the eddies.

We will choose the parameters so that $\|\mathbb{V}_k\|_2 \ll \|\mathbb{W}_k\|_2$ and $\|W_r\|_2 \ll \|W_z\|_2$. Hence, viscous eddies are quantitatively determined by W_z.

Note that $\mathbb{W}_k$ is divergence-free. Indeed, using standard vector calculus (see Appendix B.1) we compute

$$\begin{aligned}
\operatorname{div}(\mathbb{W}_k) &= \operatorname{div}\left(\eta_k\psi_k\mathbf{e}_z - \frac{\partial\eta_k}{\partial z_k}\phi_k\mathbf{e}_r\right)_\gamma \\
&= \left(\frac{\partial\eta_k}{\partial z_k}\psi_k - \frac{\partial\eta_k}{\partial z_k}\frac{1}{r}\frac{\partial(r\phi_k)}{\partial r_k}\right)_\gamma \\
&= 0,
\end{aligned}$$

thanks to (Equation 3.45).

Note that for $\mathbb{W}_k$ we can choose $\gamma \ll 1$ and $\tau \ll \mu$ so that it has any small intermittency $D > 0$:

$$\|\nabla^m \mathbb{W}_k\|_p \lesssim_m \mu^{m(1+\gamma)} \mu^{1-2/p} \tau^{1/2-1/p}, \tag{3.54}$$

however, besides being much smaller than $\mathbb{W}_k$, the viscous part $\mathbb{V}_k$ will impose other restrictions on admissible choices of τ, μ, as indicated by Proposition 3.3.11.

As a direct consequence of Definition 3.3.3 and 3.3.6 we obtain

Lemma 3.3.8 (Compact support of $\mathbb{W}_k$ and $\mathbb{V}_k$)**.** *For any $\mu \geq \tau \geq \mu_0$, the supports set of $\mathbb{W}_k$ and $\mathbb{V}_k$ verify*

$$\begin{aligned}
&\operatorname{supp} \mathbb{W}_k \cup \operatorname{supp} \mathbb{V}_k \subset [0,1]^3 \quad \textit{for any } k \in \mathbb{K}, \\
&\operatorname{supp} \mathbb{W}_k \cap \operatorname{supp} \mathbb{W}_{k'} = \emptyset \quad \textit{and} \quad \operatorname{supp} \mathbb{V}_k \cap \operatorname{supp} \mathbb{V}_{k'} = \emptyset \quad \textit{if } k \neq k',
\end{aligned}$$

and the estimate

$$|\operatorname{supp} \mathbb{W}_k| \lesssim \tau^{-1} \mu^{-2}.$$

Moreover, the vector fields $\mathbb{W}_k$ have zero mean

$$\int_{\mathbb{R}^3} \mathbb{W}_k = 0. \tag{3.55}$$

Proof. The compactness and disjointness of the support follow from the definitions. The estimate of the support set follows from the fact that μ^{-1}-mollification only alter the diameter of the support set by μ^{-1} and $\tau \leq \mu$.

The zero-mean property (Equation 3.55) follows from integrating in cylindrical coordinates with basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and the fact that the profile function $\eta \in C_c^\infty(\mathbb{R})$ used in (Equation 3.44) has zero mean. $\square$

3.3.4 Definition of viscous eddies

We will show that $\mathbb{W}_k$ and $\mathbb{V}_k$ can be used to form stationary solutions of the Navier-Stokes equations. The choice of $\mathbb{V}_k$ is inspired by the following results.

The first estimate shows that the leading order term in $\operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k)$ is $\operatorname{div}(W_k \otimes W_z)$.

Proposition 3.3.9. *Suppose $\tau \leq \mu^{1-\gamma}$. Then the following estimate holds*

$$\left\|\mathbb{W}_k \otimes \mathbb{W}_k - W_k \otimes W_z\right\|_p \lesssim_p \mu^{-\gamma}\left[\mu^{2-2/p}\tau^{1-1/p}\right],$$

for all $1 \leq p < 2$.

The next two results show a precise structure of the error term $\operatorname{div}(W_k \otimes W_z)$. In particular, it has a fixed direction $\mathbf{e}_z$ and zero mean over the Ω_k-plane thanks to Lemma 3.3.5. Hence, it can be balanced by adding a Laplacian term.

Lemma 3.3.10. *There holds*

$$\operatorname{div}(W_k \otimes W_z) = \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}\left(\psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}\right)\mathbf{e}_z. \tag{3.56}$$

Proof. Since $W_k = W_z + W_r$ is divergence-free, by a direct computation using cylindrical coordinates (cf. Appendix B.1) we conclude

$$
\begin{aligned}
\operatorname{div}(W_k \otimes W_z) &= \big((W_z + W_r) \cdot \nabla\big) W_z \\
&= -\frac{\partial \eta_k}{\partial z_k} \phi_k \eta_k \frac{\partial \psi_k}{\partial r_k} \, \mathbf{e}_z + \eta_k \psi_k \frac{\partial \eta_k}{\partial z_k} \psi_k \, \mathbf{e}_z \\
&= \frac{1}{2} \frac{\partial (\eta_k^2)}{\partial z_k} \left(\psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k} \right) \mathbf{e}_z .
\end{aligned}
$$

□

Proposition 3.3.11. *Suppose $\tau \le \mu$. Then the following important estimate holds:*

$$
\big\| \operatorname{div}(W_k \otimes W_z) - \Delta \mathbb{V}_k \big\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2 \mu^{-1} \Big[\mu^{2-2/p} \tau^{1-1/p} \Big], \tag{3.57}
$$

for all $1 < p \le \infty$.

While Lemma 3.3.10 follows from a direct computation using cylindrical coordinates, we postpone the proofs of Proposition 3.3.9 and Proposition 3.3.11 to the end of this section. With these results at hand, it is natural to consider the following family of vector fields.

Definition 3.3.12 (Viscous eddies)**.** *Viscous eddies are vector fields of the form*

$$
u = \sum_k a_k \mathbb{W}_k - a_k^2 \mathbb{V}_k, \tag{3.58}
$$

where coefficients $a_k \in \mathbb{R}$ for each $k \in \mathbb{K}$.

One of the advantages of *viscous eddies* is that they are approximate solutions of the stationary Navier-Stokes equations.

Theorem 3.3.13 (Approximate stationary solutions in $\mathbb{R}^3$). *Let $\mathbb{K} \subset \mathbb{R}^3$ be finite and u be a viscous eddy:*

$$u = \sum_k a_k \mathbb{W}_k - a_k^2 \mathbb{V}_k,$$

where constants $a_k \in \mathbb{R}$ for each $k \in \mathbb{K}$.

Then $u \in C_c^\infty(\mathbb{R}^3)$ is an approximate solution of the stationary Navier-Stokes equations in the following sense. There exist a stress $R \in C_c^\infty(\mathbb{R}^{3\times3})$ and a vector field $r \in C_c^\infty(\mathbb{R}^3)$ so that

$$\Delta u + \operatorname{div}(u \otimes u) = \operatorname{div} R + r.$$

Moreover, for any $\varepsilon > 0$, one can choose $\tau, \mu > 0$ such that

$$\|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^1(\mathbb{R}^3)} \leq \varepsilon.$$

For simplicity of presentation we include the pressure in the stress term R and do not assume R is symmetric traceless. It might be possible to write the vector field r in the divergence form, gaining an additional one derivative. Such a method will require the use of inverse divergence operator on $\mathbb{R}^3$. However, the inverse divergence $\mathcal{R}$ in defined in Equation 3.112 does not preserve compact support on $\mathbb{R}^3$.

As one can see, u is an approximate stationary solution to the NSE for an arbitrary direction k, whereas both intermittent jets in (5) and Mikado flows in (58) must have lattice directions to be periodic.

Proof of Theorem 3.3.13. Denote $u_1 = \sum_k a_k \mathbb{W}_k$ and $u_2 = -\sum_k a_k^2 \mathbb{V}_k$ then define the stress term R by

$$R = \nabla u_1 + u_1 \otimes u_2 + u_2 \otimes u_1 + u_2 \otimes u_2.$$

and the vector field r as

$$r = \Delta u_2 + \operatorname{div}(u_1 \otimes u_1).$$

Immediately, by direct computation

$$\Delta u + \operatorname{div}(u \otimes u) = \operatorname{div} R + r.$$

As a result,

$$\|R\|_{L^1(\mathbb{R}^3)} \lesssim \|\nabla u_1\|_1 + \|u_1\|_2 \|u_2\|_2 + \|u_2\|_2^2, \tag{3.59}$$

and

$$\|r\|_{L^p(\mathbb{R}^3)} \lesssim \sum_k \big\| \operatorname{div}(W_k \otimes W_z) - \Delta \mathbb{V}_k \big\|_{L^p(\mathbb{R}^3)} + \big\| \operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k - W_k \otimes W_z) \big\|_{L^p(\mathbb{R}^3)}.$$

By Propositions 3.3.9 and 3.3.11, it is easy to choose $p > 1$ sufficiently close to 1 and τ, μ sufficiently large depending on a_k such that

$$\|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^1(\mathbb{R}^3)} \le \|R\|_{L^1(\mathbb{R}^3)} + \|r\|_{L^p(\mathbb{R}^3)} \le \varepsilon.$$

□

3.3.5 Estimates for the *viscous eddies*

Proposition 3.3.14. *For any $\tau \le \mu^{1-\gamma}$ and μ sufficiently large, the following estimates hold:*

$$\mu^{-m(1+\gamma)} \big\|\nabla^m \mathbb{W}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_m \mu^{1-2/p}\tau^{1/2-1/p}, \quad 1 \le p \le \infty,$$
$$\mu^{-m(1+\gamma)} \big\|\nabla^m \mathbb{V}_k\big\|_{L^p(\mathbb{R}^3)} \lesssim_{m,p} \mu^{-1}\tau^{3/2}\Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big], \quad 1 < p \le \infty.$$

Proof. By a dimensional analysis and smoothness of $\mathbb{W}_k$ and $\mathbb{V}_k$, it suffices to prove the estimates for $m = 0$.

Let us first estimate $\mathbb{W}_k$. Definitions 3.3.3, 3.3.7 and Lemma 3.3.5 immediately imply that

$$\|W_z\|_{L^p} \lesssim \mu^{1-2/p}\tau^{1/2-1/p}, \qquad 1 \le p \le \infty, \tag{3.60}$$

and

$$\|W_r\|_{L^p} \lesssim_p \mu^{-2/p}\tau^{3/2-1/p}, \qquad 1 \le p < 2. \tag{3.61}$$

Note that $W_r \notin L^2$, and hence the implicit constant in (Equation 3.61) blows up as $p \to 2-$. Now we will show that the mollified radial component of the eddy satisfies

$$\|(W_r)_\gamma\|_{L^p} \lesssim \mu^{\gamma - 2/p} \tau^{3/2 - 1/p}, \qquad 1 \le p \le \infty, \tag{3.62}$$

provided μ is large enough.

Indeed, due to Lemma 3.3.2, there exist constants $c_1 \in \mathbb{R}$ and $0 < \alpha_0 < 1$, such that $\psi(r) = c_1$ for all $r \le \alpha_0$. By definition (Equation 3.38), for all $r \le \alpha_0$ we have

$$\begin{aligned}
\phi(r) &= -\frac{1}{r}\int_r^\infty \psi(s) s\, ds \\
&= -\frac{1}{r_k}\left(\int_r^\alpha c_1 s\, ds + \int_\alpha^\infty \psi(s) s\, ds\right) \\
&= c_1 \frac{r}{2} + c_2 \frac{1}{r},
\end{aligned}$$

for some constant $c_2 \in \mathbb{R}$. Clearly there exists $\alpha \le \alpha_0$, such that $|\phi(r)|$ is decreasing for all $r \le \alpha$, and $|\phi(\alpha)| \ge |\phi(r)|$ for all $r \ge \alpha$. Therefore, $\left|(\phi_k)_\gamma\right|$ attains a global maximum at $r_k = 0$, provided $2\mu^{-\gamma} \le \alpha$. A direct computation shows that

$$\begin{aligned}
\left|(\phi_k)_\gamma(0)\right| &= \left|\int_0^{\mu^{-1-\gamma}} \varphi_\gamma(r)\phi_k(r) r\, dr\right| \\
&= \left|\int_0^{\mu^{-1-\gamma}} \mu^{2+2\gamma}\varphi(\mu^{1+\gamma} r)\frac{1}{\mu r} r\, dr\right| \\
&\lesssim \mu^\gamma.
\end{aligned}$$

Now using the fact that $|\operatorname{supp}(W_r)_\gamma| \lesssim \mu^{-2}\tau^{-1}$, we can conclude that

$$\begin{aligned}\|(W_r)_\gamma\|_{L^p} &\lesssim \mu^{-2/p}\tau^{-1/p}\|(W_r)_\gamma\|_{L^\infty} \\ &\lesssim \mu^{-2/p}\tau^{-1/p}\left\|\frac{\partial\eta_k}{\partial z_k}\right\|_{L^\infty}\|(\phi_k)_\gamma\|_{L^\infty} \\ &\lesssim \mu^{-2/p}\tau^{-1/p}\tau^{3/2}\mu^\gamma,\end{aligned}$$

provided μ is large enough (so that $2\mu^{-\gamma} \leq \alpha$).

Now we can easily estimate viscous eddies using (Equation 3.60) and (Equation 3.62):

$$\|\mathbb{W}_k\|_{L^p} \leq \|(W_z)_\gamma\|_{L^p} + \|(W_r)_\gamma\|_{L^p} \lesssim (1+\tau\mu^{\gamma-1})\left[\mu^{1-2/p}\tau^{1/2-1/p}\right] \lesssim \left[\mu^{1-2/p}\tau^{1/2-1/p}\right],$$

due to the assumption $\tau \leq \mu^{1-\gamma}$.

Next, we estimate $\|\mathbb{V}_k\|_{L^p}$ in cylindrical coordinates. Since $\mathbb{V}_k$ is axisymetric, using the decomposition $\mathbb{R}^3 = \Omega_k \oplus l_k$, we obtain

$$\|\mathbb{V}_k\|_{L^p(\mathbb{R}^3)} \lesssim \|\widetilde{\eta}_k\|_{L^p(l_k)}\|\widetilde{\psi}_k\|_{L^p(\Omega_k)}.$$

By Definitions 3.3.3 snd 3.3.6,

$$\|\widetilde{\eta}_k\|_{L^p(l_k)} \lesssim \left\|\frac{\partial(\eta_k^2)}{\partial z_k}\right\|_{L^p(l_k)} \lesssim \tau^{2-\frac{1}{p}}. \tag{3.63}$$

Then for $p > 1$ we have

$$\begin{aligned} \|\widetilde{\psi}_k\|_{L^p(\Omega_k)} &\le \|\varphi\|_{L^\infty(\Omega_k)} \big\|\Psi_k\big\|_{L^p(\Omega_k)} \\ &\lesssim_p \Big(\int \big|h(\mu r_k)\big|^p r_k dr_k\Big)^{\frac{1}{p}} \\ &\lesssim \mu^{-2/p}, \end{aligned} \tag{3.64}$$

where in the last estimate we have used the fact that $h \in L^p(\mathbb{R}^2)$ for any $1 < p \le \infty$.

Putting together (Equation 3.63) and (Equation 3.64) we obtain the desired estimate

$$\|\mathbb{V}_k\|_{L^p} \lesssim_p \tau^{3/2}\mu^{-1}\Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big] \quad \text{for any } 1 < p \le \infty.$$

□

Using the above estimates, we prove Proposition 3.3.9 and Proposition 3.3.11.

Proof of Proposition 3.3.9. We start with the decomposition

$$\mathbb{W}_k \otimes \mathbb{W}_k = W_k \otimes W_z + \big(\mathbb{W}_k - W_k\big) \otimes W_z + \mathbb{W}_k \otimes \big((W_z)_\gamma - W_z\big) + \mathbb{W}_k \otimes (W_r)_\gamma.$$

So by Hölder's inequality we will focus on the following

$$\begin{aligned} \big\|\mathbb{W}_k \otimes \mathbb{W}_k - W_k \otimes W_z\big\|_p &\lesssim \big\|\mathbb{W}_k - W_k\big\|_p \|W_z\|_\infty + \|\mathbb{W}_k\|_\infty \big\|(W_z)_\gamma - W_z\big\|_p + \|\mathbb{W}_k\|_\infty \big\|(W_r)_\gamma\big\|_p \\ &\lesssim X_1 + X_2 + X_3 \end{aligned} \tag{3.65}$$

Let us first estimate X_1. We start with the definition of $\mathbb{W}_k$ and obtain

$$\begin{aligned} X_1 &\lesssim \big(\|W_z - (W_z)_\gamma\|_p + \|W_r - (W_r)_\gamma\|_p\big)\|W_z\|_{L^\infty} \\ &\lesssim \big(\|W_z - (W_z)_\gamma\|_p + \|W_r\|_p\big)\|W_z\|_{L^\infty}. \end{aligned} \tag{3.66}$$

To estimate the above terms, we first notice that by a standard approach to mollification,

$$\|W_z - (W_z)_\gamma\|_p \lesssim \|W_z\|_{W^{1,p}}\mu^{-1-\gamma}. \tag{3.67}$$

Moreover, by Lemma 3.3.5 (cf. (Equation 3.60) and (Equation 3.61)), we have

$$\|W_z\|_{W^{1,p}} \lesssim \mu\mu^{1-2/p}\tau^{1/2-1/p}, \qquad \|W_z\|_{L^\infty} \lesssim \mu\tau^{1/2}, \tag{3.68}$$

and, since $1 \le p < 2$,

$$\|W_r\|_{L^p} \lesssim_p \mu^{-1}\tau\mu^{1-2/p}\tau^{1/2-1/p}. \tag{3.69}$$

Substituting bounds (Equation 3.67), (Equation 3.68), and (Equation 3.69) into (Equation 3.66) gives

$$X_1 \lesssim (\mu^{-\gamma} + \mu^{-1}\tau)\big[\mu^{2-2/p}\tau^{1-1/p}\big], \tag{3.70}$$

which is the desired estimate since $\tau \le \mu^{1-\gamma}$.

Next, we estimate X_2. By Proposition 3.3.14 we have

$$\|\mathbb{W}_k\|_\infty \lesssim \mu\tau^{1/2}, \tag{3.71}$$

which together with (Equation 3.67) and (Equation 3.68) implies that

$$X_2 \lesssim \mu^{-\gamma}\big[\mu^{2-2/p}\tau^{1-1/p}\big]. \tag{3.72}$$

Finally, we need to bound X_3. All the estimates for X_3 have been obtained before. In particular, since $1 \le p < 2$, (Equation 3.69) and (Equation 3.71) imply

$$\begin{aligned} X_3 &\lesssim \|\mathbb{W}_k\|_\infty \|W_r\|_p \\ &\lesssim \mu^{-1}\tau\big[\mu^{2-2/p}\tau^{1-1/p}\big], \end{aligned}$$

which is what we need due to the assumption $\tau \le \mu^{1-\gamma}$.

□

Proof of Proposition 3.3.11. By a direct computation,

$$\Delta\mathbb{V}_k = \Delta(\widetilde{\eta}_k\varphi)\Psi_k\mathbf{e}_z + 2\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\mathbf{e}_z + \widetilde{\eta}_k\varphi\Delta\Psi_k\mathbf{e}_z, \tag{3.73}$$

where we write $\varphi = \varphi(\tau r_k)$ for short. Recall from (Equation 3.50) that

$$\Delta\big(\Psi_k\big) = \psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k},$$

and, in particular, $\Delta\Psi_k = 0$ for $r_k \geq \mu^{-1}$. Since $\tau \leq \mu$, we have that $\varphi(\tau r_k) = 1$ on $\operatorname{supp} \Delta\Psi_k$. Then using Definition 3.3.6 and Lemma 3.3.10, we obtain

$$\begin{aligned} \widetilde{\eta}_k \varphi \Delta \Psi_k \mathbf{e}_z &= \frac{1}{2}\frac{\partial(\eta_k^2)}{\partial z_k}\Big(\psi_k^2 - \phi_k \frac{\partial \psi_k}{\partial r_k}\Big)\mathbf{e}_z \\ &= \operatorname{div}(W_k \otimes W_z). \end{aligned}$$

Combining this with (Equation 3.73), we get

$$\big\| \operatorname{div}(W_k \otimes W_z) - \Delta \mathbb{V}_k \big\|_{L^p(\mathbb{R}^3)} \lesssim \big\| \Delta(\widetilde{\eta}_k \varphi) \Psi_k \big\|_{L^p(\mathbb{R}^3)} + \big\| \nabla(\widetilde{\eta}_k \varphi) \nabla \Psi_k \big\|_{L^p(\mathbb{R}^3)}. \tag{3.74}$$

Since $\tau \leq \mu$, it suffices to bound the second term in (Equation 3.74). By Definition 3.3.6, we have a pointwise bound

$$\big|\nabla(\widetilde{\eta}_k \varphi)\big| \lesssim \big|\nabla^2(\eta_k^2)\big| + \tau \big|\nabla(\eta_k^2)\big|.$$

Thus for the second term in (Equation 3.74) we have

$$\big\|\nabla(\widetilde{\eta}_k \varphi)\nabla\Psi_k \mathbf{e}_z\big\|_{L^p(\mathbb{R}^3)} \lesssim \big\|\nabla^2(\eta_k^2)\big\|_{L^p(l_k)} \big\|\nabla \Psi_k\big\|_{L^p(\Omega_k)} + \tau \big\|\nabla(\eta_k^2)\big\|_{L^p(l_k)} \big\|\nabla\Psi_k\big\|_{L^p(\Omega_k)}. \tag{3.75}$$

Now by rescaling (Equation 3.49) and Definition 3.3.3,

$$\left\|\nabla\Psi_k\right\|_{L^p(\Omega_k)} \lesssim_p \mu^{1-2/p} \quad \text{for } 1<p\le\infty, \quad \text{and} \quad \left\|\nabla^n(\eta_k^2)\right\|_{L^p(l_k)} \lesssim_n \tau^n\tau^{1-1/p} \quad \text{for } 1\le p\le\infty,$$

so we get

$$\left\|\nabla(\widetilde{\eta}_k\varphi)\nabla\Psi_k\mathbf{e}_z\right\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2\mu^{-1}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big],$$

which implies the desired bound:

$$\left\|\operatorname{div}(W_k\otimes W_z)-\Delta\mathbb{V}_k\right\|_{L^p(\mathbb{R}^3)} \lesssim_p \tau^2\mu^{-1}\Big[\mu^{2-2/p}\tau^{1-1/p}\Big] \quad \text{for } 1<p\le\infty. \tag{3.76}$$

□

3.4 Proof of main proposition: velocity perturbation

In this section, we start proving Proposition 3.2.1. The main objective of the section is to define and estimate the velocity perturbation. More specifically, we will carefully design the velocity perturbation w so that the new solution $v = v_0 + w$ has the desired properties listed in Proposition 3.2.1. The key is to reduce the size of the stress error term and make sure w carries a precise amount of energy on the intervals $I_{4^{-1}r}(\mathcal{F}_1)$ at the same time.

The rest of this section is organized as follows. We first give a general introduction of the proof, and then introduce all the necessary preparation work to define w, namely, fix constants τ and μ appeared in the *viscous eddies*, choose suitable cutoff functions in space and time, and introduce the Leray projection and a fast oscillation operator $\mathbf{P}_\sigma$. Finally, we define the velocity perturbation w and derive various estimates needed in the next two sections.

3.4.1 General introduction

To better illustrate the idea, we provide some heuristics and try to outline the general idea of the proof here. To the leading order, the velocity perturbation w consists of finitely many highly oscillating *viscous eddies*:

$$w = \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k + a_k^2 \mathbf{P}_\sigma \mathbb{V}_k := w^{(p)} + w^{(l)},$$

where coefficients a_k are determined by the old Reynolds stress R_0, and $\mathbf{P}_\sigma$ is a fast oscillation operator (see Definition 3.4.4).

On one hand, we nee to control the new stress term, which, according to (Equation gNSR), is implicitly defined by

$$\operatorname{div} R = \partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div}(R_0 + w \otimes w) - \nabla p_1.$$

The old Reynolds stress R_0 will be canceled by the interaction $w^{(p)} \otimes w^{(p)}$ together with $w^{(l)}$. More precisely,

$$\operatorname{div}(w^{(p)} \otimes w^{(p)}) + \operatorname{div} R_0 + \Delta w^{(l)} = \text{High frequency errors} + \text{Lower order terms}.$$

On the left hand side, R_0 will be canceled by the high-high interaction of $w^{(p)} \otimes w^{(p)}$, and $\Delta w^{(l)}$ will balance the error essentially introduced by the unwanted $\operatorname{div}(\mathbb{W}_k \otimes \mathbb{W}_k)$ as shown in Theorem 3.3.13. On the right hand side, lower order terms are automatically small, but high frequency errors will gain a factor of σ^{-1} after inverting the divergence. This will be shown in Lemma 3.5.8, Section 3.5.

On the other hand, we need to make sure the new solution v has the desired energy profile. This is in fact mostly compatible with the above effort of controlling the new stress error. Heuristically, to balance the stress term R_0, one must spend the energy of size at least $\sim \|R_0\|_1$. In other words,

$$\|w(t)\|_2^2 \gtrsim \|R_0(t)\|_1 \quad \text{for all } t.$$

There is a lot of flexibility in choosing the size of w though, as one can use more energy than needed to balance the old stress term R_0. In our scheme, the size of $\|w\|_2$ is determined by the given energy levels e_0 and e_1 on the intervals $I_{4^{-1}r}(\mathcal{F}_1)$, where the old stress error term is already quite small (the second condition for (v_0, R_0) in Proposition 3.2.1). This makes control of the stress and pumping of the energy compatibility. See (Equation 3.79) and Section 3.6 for more details.

3.4.2 Setup of constants

First, we set up the constants appeared in the definition of the vector fields $\mathbb{W}_k^{\tau,\mu}$ and the *viscous eddies*.

The major parameter λ, the (spacial) frequency of the perturbation, will be a sufficiently large. The parameters μ and τ in the *viscous eddies* are defined explicitly as powers of λ while

γ is taken to be small. Moreover, we also define an integer σ to parametrize the oscillations of the eddies.

In the sequel, we fix

$$\begin{cases} \sigma = \lambda^{1/30} \\ \mu = \lambda^{14/15} \\ \tau = \lambda^{2/5} \\ \gamma = \frac{1}{28} \end{cases} \tag{3.77}$$

Clearly, it holds that $\mu^\gamma = \sigma$ and $\sigma\mu^{1+\gamma} = \lambda$. We also have the following hierarchy of constants:

$$\sigma \ll \tau \ll \mu \ll \lambda.$$

For periodicity, we also require σ to be an integer. Let us briefly discuss the scales involved in the definition of w. In essence, the choice of parameters ensures that by raising the value of λ, the new stress term R_0 introduced by w on $I_{4^{-1}r}(\mathcal{F}_1)$ can be as small as we want, and, at the same time, the energy of new solution $\|v(t)\|_2^2$ can be controlled precisely.

There are mainly four constraints in choosing the scales:

- The first constraint is due to the small intermittency requirement. Since λ is the frequency of w which consists of oscillation σ and concentration τ and μ, then for w to be small in $W^{1,1}$ it requires (see (Equation 3.46))

$$\lambda\tau^{-\frac{1}{2}}\mu^{-1} \ll 1.$$

- The second constraint is needed to achieve the correct energy level. Since $\|w^{(p)}\|_2$ controls the energy level of the new solution v, we need $\|w^{(l)}\|_2 \ll \|w^{(p)}\|_2$ and $\|w^{(c)}\|_2 \ll \|w^{(p)}\|_2$. According to definitions of $w^{(l)}$ and $w^{(c)}$, i.e. (Equation 3.88) and (Equation 3.89), this implies

$$\tau^{\frac{3}{2}} \ll \mu.$$

- The previous two constraints are due to the viscous part $w^{(l)}$. There is a new error introduced by Δ, namely R_{low} in Lemma 3.5.8. To make sure R_{low} is small, we need

$$\tau^2 \ll \mu.$$

- We use a mollification in the scale $\mu^{-1-\gamma}$ to remove $1/r$ singularity of a viscous eddy in the radial direction. This singularity is needed so that we can take advantage of the Laplacian. In order to control norms of the viscous eddy, we need an upper bound on γ. More precisely, as we have seen in the previous section, we need the following condition:

$$\tau \leq \mu^{1-\gamma}.$$

It is easy to verify that our choice of constants (Equation 3.77) satisfies all the above constraints.

Next, we introduce a constant M, whose role is to limit the order of the derivative that we will be taking so that the implicit constants stay bounded.

Definition 3.4.1 (The constant M)**.** *Let $N = 300$ and $\theta = 1/2$. We define M to be the constant obtained from applying Proposition 2.4.4 with such θ and N.*

3.4.3 Cut-offs in space and time

Let $\chi : \mathbb{R}^{3\times 3} \to \mathbb{R}^+$ be a positive smooth function so that it is monotone increasing with respect to $|x|$ and

$$\chi^2(x) = \begin{cases} 1, & 0 \le |x| \le 1 \\ |x|, & |x| \ge 2 \end{cases} \tag{3.78}$$

where $|\cdot|$ denotes the Euclidean matrix norm. Note that by definition

$$\|\nabla^m \chi\|_\infty \lesssim_m 1.$$

Now we choose a proper threshold $\rho_0(t)$ to control how much energy is added. Given an solution (v_0, R_0) and energy level e_1 as in the statement of Proposition 3.2.1, let

$$\rho_0(t) = \frac{1}{12}(\widetilde{e}_1 - \|v_0(t)\|_2^2), \tag{3.79}$$

where $\widetilde{e}_1 = e_1 - 10^{-6}(e_1 - e_0)$ is to leave room for future corrections. Note that ρ_0 is bounded from below:

$$\rho_0(t) \gtrsim e_1 - e_0 \gtrsim C^{-1}\delta_0, \tag{3.80}$$

due to the assumptions (1) and (2) in Proposition 3.2.1, where $\delta_0 = \|R_0\|_{L_t^\infty L_x^1(\mathbb{T}^3\times[0,T])}$ and the universal constant C in Proposition 3.2.1 will be specified in Section 3.6.

To deal with the issue of the Reynolds stress R_0 having large magnitudes, we introduce a divisor as follows. Define $\rho : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^+$ to be

$$\rho(x,t) = 4\rho_0 \chi^2(\rho_0^{-1} R_0). \tag{3.81}$$

It follows from the above definitions that

$$\frac{|R_0|}{\rho} = \frac{|R_0|}{4\rho_0 \chi^2(\rho_0^{-1} R_0)} \leq 1/2 \quad \text{for all } (x,t) \in \mathbb{T}^3 \times [0,T].$$

Next, we introduce a cutoff in time so that the energy profile of the new solution satisfies all the required properties. For the exceptional set $\mathcal{F}_1$ (cf. (Equation 3.12)), let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a smooth cut-off function such that

$$\theta(t) = \begin{cases} 1, & t \in I_{4^{-1}r}(\mathcal{F}_1) \\ 0, & t \notin I_{4^{-2}r}(\mathcal{F}_1), \end{cases} \tag{3.82}$$

and

$$\|\theta^{(n)}\|_\infty \lesssim_n r^{-n} \quad \text{for all } n \in \mathbb{N}. \tag{3.83}$$

Remark 3.4.2. *When $\mathcal{F}_1 = \emptyset$, we take $\theta = 1$, so there is no cutoff in time. This will ensure that if $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$ and the solution v_0 is stationary, then the velocity perturbation w is also stationary.*

3.4.4 Leray projection and fast periodization operator

To define the velocity perturbation, we recall the definition of Leray projection.

Definition 3.4.3 (Leray projection)**.** *Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ be a smooth vector field. Define the operator $\mathcal{Q}$ as*

$$\mathcal{Q}v := \nabla f + \fint_{\mathbb{T}^3} v,$$

where $f \in C^\infty(\mathbb{T}^3)$ is the smooth zero-mean solution of

$$\Delta f = \operatorname{div} v, \qquad x \in \mathbb{T}^3.$$

Furthermore, let $\mathcal{P} = \operatorname{Id} - \mathcal{Q}$ be the Leray projection onto divergence-free vector fields with zero mean.

To avoid potential abuse of notation, we will utilize the following fast periodization operator $\mathbf{P}_\sigma$ for functions whose support sets are contained in $[0,1]^3$. We will apply $\mathbf{P}_\sigma$ to the *viscous eddies* so that they oscillate at a frequency much higher than that of the solution (v_0, R_0).

Definition 3.4.4 (Fast periodization operator $\mathbf{P}_\sigma$)**.** *Let $\sigma \in \mathbb{N}$. Suppose $f \in C_c^\infty(\mathbb{R}^3)$ and* $\operatorname{supp} f \subset [0,1]^3$, *define the fast periodization operator $\mathbf{P}_\sigma$ by*

$$\mathbf{P}_\sigma f(x) = \sum_{m \in \mathbb{Z}^3} f(\sigma x + m). \tag{3.84}$$

By definition $\mathbf{P}_\sigma f$ is $\sigma^{-1}\mathbb{T}^3$-periodic, and for any differentiation ∇^n, we have

$$\nabla^n \mathbf{P}_\sigma f = \sigma^n \mathbf{P}_\sigma \nabla^n f \tag{3.85}$$

which will be used without mentioning in the future.

3.4.5 Definitions of the perturbation

With all the preparations in hand, we can define the velocity perturbation w.

We first apply Lemma 3.3.1 for $\mathcal{B} = \{R \in \mathcal{S}^{3\times 3}_+ : |\operatorname{Id} - R| \leq 1/2\}$ to obtain smooth functions $\Gamma_k : \mathcal{B} \to \mathbb{R}$ for $k \in \mathbb{Z}^3$, $|k| \leq \lambda_0$. Then the coefficients for the *viscous eddies* are defined by

$$a_k(x,t) = \rho^{1/2}(x,t)\Gamma_k\Big(\operatorname{Id} - \frac{R_0}{\rho}\Big) \quad \text{for } k \in \mathbb{Z}^3, |k| \leq \lambda_0. \tag{3.86}$$

In view of Theorem 3.3.13, define vector fields

$$w^{(p)} = \theta \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k = w^{(p)}_z + w^{(p)}_r, \tag{3.87}$$

where

$$w^{(p)}_z = \theta \sum_k a_k \mathbf{P}_\sigma (W_z)_\gamma, \qquad \text{and} \qquad w^{(p)}_r = \theta \sum_k a_k \mathbf{P}_\sigma (W_r)_\gamma,$$

and

$$w^{(l)} = -\theta^2 \sigma^{-1} \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k. \tag{3.88}$$

Also define a divergence-free correction term

$$w^{(c)} = -\mathcal{Q}w^{(p)} - \mathcal{Q}w^{(l)}. \tag{3.89}$$

Finally, the velocity increment w is defined by

$$w = \theta \sum_k a_k \mathbf{P}_\sigma \mathbb{W}_k - \theta^2 \sigma^{-1} \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k + w^{(c)}. \tag{3.90}$$

which also reads

$$w = w^{(p)} + w^{(l)} + w^{(c)}. \tag{3.91}$$

Thanks to Lemma 3.3.8, $\mathbf{P}_\sigma$ may be applied and w is well-defined. It is clear that w is periodic due to the periodicity of coefficients a_k and the periodization operator $\mathbf{P}_\sigma$. By design w is divergence-free. Also since the operator $\mathcal{P}$ removes the mean, w has zero mean as well.

Next, we show the smoothness of w, for which it suffices to show the following simple result for the coefficients a_k.

Lemma 3.4.5 (Properties of coefficients a_k)**.** *The coefficients a_k defined by* (Equation 3.86) *are smooth on $\mathbb{T}^3 \times [0,T]$. There exist a number $\kappa = \kappa(e_1, v_0, R_0) \geq r^{-1}$ such that*

$$\max_k \|a_k\|_{C^m_{t,x}} \leq \kappa^{m+1}, \quad \textit{for any integer} \quad 0 \leq m \leq 4M;$$

the following bounds hold

$$\begin{aligned} \|\rho(t)\|_{L^1} &\lesssim \rho_0(t), \\ \|a_k(t)\|_{L^2} &\lesssim \rho_0(t)^{1/2}; \end{aligned} \tag{3.92}$$

and we have the identity

$$\sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma \left(W_k \otimes W_z\right) = \rho \operatorname{Id} - R_0. \tag{3.93}$$

Proof. Recall that

$$a_k = 2\rho_0^{1/2} \chi(\rho_0^{-1} R_0) \Gamma_k \Big(\operatorname{Id} - \frac{R_0}{\rho} \Big). \tag{3.94}$$

To show that a_k has bounded space-time Hölder norms of order $4M$, it suffices to check that each factor above is smooth as the domain $\mathbb{T}^3 \times [0, T]$ is compact. Since

$$\rho_0^{1/2} = \frac{1}{2\sqrt{3}} (\widetilde{e}_1 - \|v_0(t)\|_2^2)^{1/2},$$

which is bounded from below by (Equation 3.80), the function $\rho_0^{1/2}$ is smooth on $[0, T]$. By the same argument and the definition of χ in (Equation 3.78), we may also conclude that $\chi(\rho_0^{-1} R_0) \in C^\infty_{x,t}(\mathbb{T}^3 \times [0, T])$. Since $\Gamma_k \in C^\infty(\mathcal{B})$, the last term in (Equation 3.94) is also in $C^\infty_{t,x}$.

Next, let us prove (Equation 3.92). Since $0 \le \theta \le 1$, by definition of ρ in (Equation 3.81), we have

$$\begin{aligned}\|\rho(t)\|_{L^1} &\le \int_{|R_0|\le\rho_0} \rho(x,t)dx + \int_{|R_0|\ge\rho_0} \rho(x,t)dx \\ &\lesssim \rho_0\Big(\int_{|R_0|\le\rho_0} 1dx + \int_{|R_0|\ge\rho_0} |R_0|dx\Big) \lesssim \rho_0,\end{aligned}$$

where we have used $\|R_0\|_{L^\infty_t L^1} = \delta_0 \lesssim \rho_0$ due to (Equation 3.80).

For the second bound in (Equation 3.92), we can directly compute to obtain:

$$\|a_k(t)\|_2^2 \lesssim \rho_0\theta^2 \int_{\mathbb{T}^3} \chi^2(\rho_0^{-1}R_0)dx \lesssim \rho_0\theta^2.$$

To show the last identity, thanks to Lemma 3.3.1, it suffices to show

$$\fint_{\mathbb{T}^3} \mathbf{P}_\sigma\left(W_k \otimes W_z\right) = \frac{k}{|k|} \otimes \frac{k}{|k|}.$$

Since

$$W_r \otimes W_z = \frac{\partial \eta_k}{\partial z_k}\phi_k\eta_k\psi_k\, \mathbf{e}_r \otimes \mathbf{e}_z.$$

where profile function $\frac{\partial \eta_k}{\partial z_k}\phi_k\eta_k\psi_k$ is axisymmetric, we have

$$\fint_{\mathbb{T}^3} \mathbf{P}_\sigma\left(W_r \otimes W_z\right) = 0.$$

Then by Definitions 3.4.4 and 3.3.3,

$$\begin{aligned}\fint_{\mathbb{T}^3} \mathbf{P}_\sigma \left(W_k \otimes W_z\right) = \fint_{\mathbb{T}^3} \mathbf{P}_\sigma \left(W_z \otimes W_z\right) &= \int_{\mathbb{R}^3} W_z \otimes W_z \\ &= \int_{\mathbb{R}^3} |\eta_k \psi_k|^2 \mathbf{e}_z \otimes \mathbf{e}_z \\ &= \frac{k}{|k|} \otimes \frac{k}{|k|}.\end{aligned}$$

Hence, the identity (Equation 3.93) follows from (Equation 3.86) and Lemma 3.3.1. □

3.4.6 Estimates for the perturbations

This subsection is devoted to various estimates for the perturbation w. We start with decomposing the corrector $w^{(c)}$ using standard vector calculus. Here the inverse Laplacian Δ^{-1} on torus $\mathbb{T}^3$ is defined via a multiplier with symbol $-|k|^{-2}$ for $k \neq 0$ and 0 for $k = 0$.

Lemma 3.4.6 (Structure of the corrector)**.** *The corrector $w^{(c)}$ verifies*

$$w^{(c)} = w^{(cp)} + w^{(cl)}$$

where $w^{(cp)}$ and $w^{(cl)}$ are respectively

$$w^{(cp)} = \theta \sum_k \nabla \Delta^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big) - \fint_{\mathbb{T}^3} w^{(p)},$$

and

$$w^{(cl)} = \theta^2 \sigma^{-1} \mathcal{Q}\Big(\sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k\Big).$$

Proof. Noticing that $\operatorname{div} \mathbb{W}_k = 0$, these formulae immediately follow from Definition 3.4.3. $\square$

We recall the following improved Hölder's inequality for functions with fast oscillation proven in (58), which is crucial in obtaining the L^2 decay of the perturbation w.

Proposition 3.4.7. *For any small $\theta > 0$ and any large $N > 0$ there exist $M \in \mathbb{N}$ and $\lambda_0 \in \mathbb{N}$ so that for any $\mu > 0$, $\sigma \in \mathbb{N}$ satisfying $\lambda_0 \leq \sigma$ and $\mu \leq \sigma^{1-\theta}$ the following holds. Suppose $a \in C^\infty(\mathbb{T}^3)$ and let $C_a > 0$ be such that*

$$\|\nabla^i a\|_\infty \leq C_a \mu^i \quad \textit{for any } 0 \leq i \leq M.$$

Then for any $\sigma^{-1}\mathbb{T}^3$ periodic function $f \in L^p(\mathbb{T}^3)$, $1 < p < \infty$, the following estimates are satisfied.

- *If $p \geq 2$ is even, then*

$$\|af\|_p \lesssim_{p,\theta,N} \|a\|_p \|f\|_p + C_a \|f\|_p \sigma^{-N}. \tag{3.95}$$

- *If $\fint_{\mathbb{T}^d} f = 0$ then for $0 \leq s \leq 1$*

$$\big\||\nabla|^{-1}(af)\big\|_p \lesssim_{p,s,\theta,N} \sigma^{-1+s} \big\||\nabla|^{-s}(af)\big\|_p + C_a \|f\|_p \sigma^{-N}. \tag{3.96}$$

All the implicit constants appeared in the statement are independent of a, μ and σ.

Remark 3.4.8. *Throughout the chapter, we will always apply Proposition 2.4.4 for* $\theta = \frac{1}{2}$ *and* $N = 300$*. These two fixed constants determine the constant* M*.*

With the help of Proposition 2.4.4, we are in the position to derive useful estimates for the velocity perturbation w.

Proposition 3.4.9 (Spacial frequency estimates)**.** *For any* λ *sufficiently large and integer* $0 \leq m \leq M$ *the following estimates hold:*

$$\lambda^{-m} \|\nabla^m w^{(p)}(t)\|_p \lesssim \rho_0^{1/2}(t) \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2, \tag{3.97}$$

$$\lambda^{-m} \|\nabla^m w_r^{(p)}(t)\|_p \lesssim \rho_0^{1/2}(t) \tau \mu^{2\gamma-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2, \tag{3.98}$$

$$\lambda^{-m} \|\nabla^m w^{(l)}(t)\|_p \lesssim_p \tau^{3/2} \mu^{-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 < p \leq 2, \tag{3.99}$$

$$\lambda^{-m} \|\nabla^m w^{(c)}(t)\|_p \lesssim \sigma^{-1} \Big[\mu^{1-2/p} \tau^{1/2-1/p}\Big], \qquad 1 \leq p \leq 2. \tag{3.100}$$

Proof. *Bounds for* $w^{(p)}$*:*

Since by Lemma 3.3.8

$$\left|\mathbb{T}^3 \cap \operatorname{supp} \mathbf{P}_\sigma \mathbb{W}_k\right| \lesssim \tau^{-1} \mu^{-2}, \tag{3.101}$$

it suffices to show (Equation 3.97) for $p = 2$.

By product rule,

$$\left|\nabla^m w^{(p)}\right| \lesssim_m \sum_k \sum_{0 \leq i \leq m} \sigma^{m-i} \left|\nabla^i a_k\right| \left|\nabla^{m-i} \mathbf{P}_\sigma \mathbb{W}_k\right|. \tag{3.102}$$

As $\mathbf{P}_\sigma \mathbb{W}_k$ is $\sigma^{-1}\mathbb{T}^3$-periodic and, thanks to Lemma 3.4.5,

$$\|\nabla^i a_k\|_{C^m_x} \leq \|a_k\|_{C^{m+i}_x} \leq \kappa^{i+1+m} \quad \text{for all } 0 \leq m \leq M.$$

Since for large enough λ we have $\kappa^2 < \sigma \in \mathbb{N}$, we can apply Proposition 2.4.4 with $\theta = \frac{1}{2}$, $N = 300$, and $C_a = \kappa^{i+1}$ (cf. Definition 3.4.1) to obtain that

$$\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2 \lesssim \|\nabla^i a_k\|_2 \|\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k\|_2 + \kappa^{i+1}\|\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k\|_2 \sigma^{-N}. \tag{3.103}$$

Let us consider two sub-cases: $m = 0$ and $m \geq 1$. When $m = 0$, it follows that

$$\|a_k \mathbf{P}_\sigma \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2} + \kappa\sigma^{-N}.$$

As $\sigma^{-N} = \lambda^{-10}$ and $\rho_0 \gtrsim e_1 - e_0 > 0$, we can make sure for any sufficiently large $\lambda(e_0, e_1, \kappa)$ that

$$\|a_k \mathbf{P}_\sigma \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2},$$

from which we immediately get

$$\|w^{(p)}(t)\|_2 \lesssim \rho_0^{1/2}.$$

When $m \geq 1$, we consider the split:

$$\sum_{0\leq i\leq m} \sigma^{m-i}\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2 \leq \sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 + \sum_{1\leq i\leq m} \sigma^{m-i}\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2. \tag{3.104}$$

We will bound these two terms separately. For the first term in (Equation 3.104), we use (Equation 3.103), Lemma 3.4.5, and Proposition 3.3.14 to obtain

$$\begin{aligned}\sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 &\lesssim \sigma^m \Big(\rho_0^{1/2}\|\mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 + \sigma^{-N}\kappa \|\mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2\Big) \\ &\lesssim \sigma^m \mu^{m(1+\gamma)}\Big(\rho_0^{1/2} + \sigma^{-N}\kappa\Big).\end{aligned}$$

Since $\sigma^{-N} = \lambda^{-10}$, $\sigma\mu^{1+\gamma} = \lambda$, and $\rho_0 \gtrsim e_1 - e_0$, for λ sufficiently large we get

$$\sigma^m \|a_k \mathbf{P}_\sigma \nabla^m \mathbb{W}_k\|_2 \lesssim \rho_0^{1/2}\lambda^m. \tag{3.105}$$

For the second term in (Equation 3.104), we simply use Hölder's inequality, Lemma 3.4.5, and Proposition 3.3.14 to obtain

$$\begin{aligned}\sum_{1\leq i\leq m} \sigma^{m-i}\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2 &\leq \sum_{1\leq i\leq m} \sigma^{m-i}\|\nabla^i a_k\|_{L^\infty_{x,t}}\|\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k\|_2 \\ &\lesssim \sum_{1\leq i\leq m} \sigma^{m-i}\kappa^{i+1}\mu^{(m-i)(1+\gamma)} \lesssim \kappa^2\sigma^{m-1}\mu^{(m-1)(1+\gamma)},\end{aligned}$$

where we have also used $\kappa \ll \mu$ in the last inequality. Then again, for λ sufficiently large, we get

$$\sum_{1\le i\le m} \sigma^{m-i}\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2 \lesssim \rho_0^{1/2}\lambda^m. \tag{3.106}$$

So for $\lambda(\rho_0, \kappa, e_1, e_0)$ sufficiently large, putting together (Equation 3.105) and (Equation 3.106), we can bound (Equation 3.104) as

$$\sum_{0\le i\le m} \sigma^{m-i}\big\||\nabla^i a_k||\mathbf{P}_\sigma \nabla^{m-i}\mathbb{W}_k|\big\|_2 \lesssim \rho_0^{1/2}\lambda^m,$$

which implies that

$$\|\nabla^m w^{(p)}(t)\|_2 \lesssim \rho_0^{1/2}\lambda^m, \quad \text{for any } 1 \le m \le M\ .$$

Since for any integer $0 \le m \le M$ the desire estimate holds for $p = 2$, by Hölder's inequality and (Equation 3.101), for $1 \le p \le 2$ we have

$$\lambda^{-m}\|\nabla^m w^{(p)}(t)\|_p \lesssim \rho_0^{1/2}\mu^{1-2/p}\tau^{1/2-1/p}.$$

Bounds for $w_r^{(p)}$:

In light of estimate (Equation 3.62), the above argument also gives the desired bound for $w_r^{(p)}$. In particular, for $m = 0$, thanks to Proposition 2.4.4 we have

$$\begin{aligned}\|a_k \mathbf{P}_\sigma (W_r)_\sigma\|_2 &\lesssim \|\nabla^i a_k\|_2 \|\mathbf{P}_\sigma (W_r)_\sigma\|_2 + \kappa \|\mathbf{P}_\sigma (W_r)_\sigma\|_2 \sigma^{-N} \\ &\lesssim (\rho_0^{1/2} + \kappa\sigma^{-N})\tau\mu^{2\gamma-1} \\ &\lesssim \rho_0^{1/2}\tau\mu^{2\gamma-1}.\end{aligned}$$

Bounds for $w^{(l)}$:

Without loss of generality, we prove this bound for $m = 0$ as well, since general cases for $0 \le m \le M$ follow from applying an additional product rule, which can be seen in the estimates for $w^{(p)}$.

Recall the definition (Equation 3.88) that

$$w^{(l)} = -\sigma^{-1}\theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \mathbb{V}_k.$$

By Hölder's inequality, Lemma 3.4.5, and Proposition 3.3.14, we have

$$\begin{aligned}\|w^{(l)}\|_p &\lesssim \sigma^{-1} \sum_k \|a_k^2\|_{L^\infty_{t,x}} \|\mathbf{P}_\sigma \mathbb{V}_k\|_p \\ &\lesssim \kappa^2 \sigma^{-1} \tau\mu^{-2} \tau^{1-1/p} \mu^{2-2/p}.\end{aligned}$$

Therefore, for sufficiently large $\lambda(\kappa)$, we can use σ^{-1} to absorb the factor with κ to obtain

$$\|w^{(l)}\|_p \lesssim \tau^{3/2}\mu^{-1}\Big[\mu^{1-2/p}\tau^{1/2-1/p}\Big]. \tag{3.107}$$

Bounds for $w^{(c)}$:

Again, we only prove the bound for $m = 0$.Thanks to Lemma 3.4.6, we need to estimate $\|w^{(cp)}\|_p$ and $\|w^{(cl)}\|_p$. It suffices to estimate the following term:

$$\begin{aligned}\Big\|\sum_k \nabla\Delta^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p &= \Big\|\sum_k \mathcal{R}|\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p \\ &\lesssim \Big\|\sum_k |\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p,\end{aligned}$$

where $\mathcal{R}$ is the Riesz transform. $\mathcal{R}$ and $|\nabla|^{-1}$ are defined via multipliers with symbols $-i\frac{k}{|k|}$ and $|k|^{-1}$ respectively for $k \neq 0$, and zero for $k = 0$. Recall that $\mathbf{P}_\sigma \mathbb{W}_k$ is $\sigma^{-1}\mathbb{T}^3$-periodic and of zero mean. Moreover, due to Lemma 3.4.5,

$$\|\nabla a_k\|_{C^m_x} \leq \|a_k\|_{C^{m+1}_x} \leq \kappa^{m+2} \quad \text{for all } 0 \leq m \leq M.$$

Once again we can apply Proposition 2.4.4 with $C_a = \kappa^2$ to obtain the bound

$$\begin{aligned}\Big\||\nabla|^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p &\lesssim \sigma^{-1}\Big\|\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p + \kappa^2\Big\|\Big(\mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p \sigma^{-N} \\ &\lesssim \left(\sigma^{-1}\|\nabla a_k\|_\infty + \kappa^2\sigma^{-N}\right)\|\mathbf{P}_\sigma \mathbb{W}\|_p \\ &\lesssim (\sigma^{-1}\kappa^2 + \kappa^2\sigma^{-300})\mu^{1-2/p}\tau^{1/2-1/p} \\ &\lesssim \sigma^{-1}\kappa^2\big[\mu^{1-2/p}\tau^{1/2-1/p}\big].\end{aligned}$$

Finally, since

$$\left|\fint_{\mathbb{T}^3} |w^{(p)}|\right| \lesssim \rho_0^{1/2}\lambda^{-17/15},$$

we have

$$\|w^{(cp)}\|_p \lesssim \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big],$$

provided $\lambda(\kappa, e_1)$ is large enough.

To estimate the term $w^{(cl)}$, let us introduce $p_\varepsilon = p + \varepsilon$, for $\varepsilon \geq 0$, such that $1 < p_\varepsilon \leq 2$ and

$$\tau^{3/2}\mu^{-1}\big[\mu^{1-2/p_\varepsilon}\tau^{1/2-1/p_\varepsilon}\big] \leq \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big].$$

Note that the operator $\mathcal{Q}$ is bounded on $L^{p_\varepsilon}(\mathbb{T}^3)$, and hence we have

$$\|w^{(cl)}\|_p \leq \|w^{(cl)}\|_{p_\varepsilon} \lesssim \|w^{(l)}\|_p \lesssim \tau^{3/2}\mu^{-1}\big[\mu^{1-2/p_\varepsilon}\tau^{1/2-1/p_\varepsilon}\big] \leq \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big], \tag{3.108}$$

due to the choice of constants (Equation 3.77).

$\square$

Using the choice of constants (Equation 3.77) and the established bounds (Equation 3.97), (Equation 3.100), and (Equation 3.99), we get the next useful corollary.

Corollary 3.4.10 (Estimates with explicit exponents)**.** *For any λ sufficiently large we have*

$$\begin{aligned}
\|w^{(p)}\|_p + \lambda^{-1}\|\nabla w^{(p)}\|_p &\lesssim \rho_0^{1/2}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \le p \le 2 \\
\|w_r^{(p)}\|_p + \lambda^{-1}\|\nabla w_r^{(p)}\|_p &\lesssim \rho_0^{1/2}\lambda^{-\frac{7}{15}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \le p \le 2 \\
\|w^{(l)}\|_p + \lambda^{-1}\|\nabla w^{(l)}\|_p &\lesssim_p \lambda^{-\frac{1}{3}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 < p \le 2, \\
\|w^{(c)}\|_p + \lambda^{-1}\|\nabla w^{(c)}\|_p &\lesssim \lambda^{-\frac{1}{30}}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \le p \le 2,
\end{aligned}$$

and consequently

$$\|w\|_p + \lambda^{-1}\|\nabla w\|_p \lesssim \rho_0^{1/2}\lambda^{\frac{17}{15}(1-\frac{2}{p})}, \qquad 1 \le p \le 2. \tag{3.109}$$

In particular, given any $\varepsilon > 0$, for λ sufficiently large,

$$\|w\|_{L_t^\infty W_x^{1,1}} \le \varepsilon. \tag{3.110}$$

The last estimate concerns the time derivative of the perturbation w. Since the velocity profiles in $\mathbb{W}_k$ and $\mathbb{V}_k$ are stationary, time derivative only falls on the slow variables a_k and θ.

Proposition 3.4.11 (Temporal frequency estimates)**.** *For any λ sufficiently large, $1 \le p \le 2$, and integer $0 \le m \le M$, the following estimate holds:*

$$\kappa^{-m-1}\|\partial_t^m w\|_{L_t^\infty L_x^p} \lesssim \mu^{1-2/p}\tau^{1/2-1/p}. \tag{3.111}$$

Moreover, if (v_0, R_0) is stationary and $\mathcal{F}_0 = \mathcal{F}_1 = \emptyset$, then $v = v_0 + w$ is also stationary.

Proof. The last statement follows from (Equation 3.82) and (Equation 3.86). Let us show (Equation 3.111). In view of Lemma 3.4.5, it suffices to prove the bound for $m = 1$. Thanks to Lemma 3.4.6, we can use the decomposition

$$\partial_t w = \partial_t w^{(p)} + \partial_t w^{(cp)} + \partial_t w^{(cl)} + \partial_t \mathcal{P} w^{(l)}.$$

We first bound the term $\partial_t w^{(p)}$. By its definition, Lemma 3.4.5, Hölder's inequality and Proposition 3.3.14 we have that

$$\begin{aligned}\big\|\partial_t w^{(p)}\big\|_p &\lesssim \sum_k \big\|\theta a_k\big\|_{C^1_{t,x}}\big\|\mathbf{P}_\sigma \mathbb{W}_k\big\|_p \\ &\lesssim \kappa^2\big[\mu^{1-2/p}\tau^{1/2-1/p}\big],\end{aligned}$$

which is exactly the bound that we need.

Next, we show the same estimate holds for the term $\partial_t \mathcal{P} w^{(l)}$. As done in the proof of Proposition 3.4.9, let $p_\varepsilon = p + \varepsilon$ with $\varepsilon \geq 0$ chosen small enough such that $1 < p_\epsilon \leq 2$ and

$$\mu^{1-2/p_\varepsilon} \tau^{1/2-1/p_\varepsilon} \leq \mu^{1-2/p} \tau^{1/2-1/p} \sigma^{1/2},$$

which is possible thanks to (Equation 3.77). Then, using the L^{p_ε} boundedness of the Leray projection, Hölder's inequality, Proposition 3.3.14 and the above choice of p_ε, for any $1 \leq p \leq 2$ it follows that

$$\begin{aligned} \left\|\partial_t \mathcal{P} w^{(l)}\right\|_p &\leq \left\|\mathcal{P} \partial_t w^{(l)}\right\|_{p_\varepsilon} \lesssim \left\|\partial_t w^{(l)}\right\|_{p_\varepsilon} \lesssim \sigma^{-1} \sum_k \left\|\theta^2 a_k^2\right\|_{C^1_{t,x}} \left\|\mathbf{P}_\sigma \mathbb{V}_k\right\|_{p_\varepsilon} \\ &\lesssim \kappa^3 \sigma^{-1} \tau^{3/2} \mu^{-1} \mu^{1-2/p_\varepsilon} \tau^{1/2-1/p_\varepsilon} \lesssim \kappa^3 \sigma^{-1/2} \tau^{3/2} \mu^{-1} \left[\mu^{1-2/p} \tau^{1/2-1/p}\right]. \end{aligned}$$

Due to our choice of constants, (Equation 3.77), for any sufficiently large $\lambda(\kappa)$ we have

$$\kappa^3 \sigma^{-1/2} \tau^{3/2} \mu^{-1} \leq \kappa^2$$

and hence

$$\left\|\partial_t \mathcal{P} w^{(l)}\right\|_p \lesssim \kappa^2 \left[\mu^{1-2/p} \tau^{1/2-1/p}\right].$$

Finally, it remains to bound the terms $\partial_t w^{(cp)}$ and $\partial_t w^{(cl)}$. As in the proof of Proposition 3.4.9, we have the following estimates:

$$\begin{aligned}
\Big\| \sum_k \partial_t(\theta a_k) \nabla \Delta^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p &\lesssim \|\theta a_k\|_{C^1_{t,x}} \Big\| \nabla \Delta^{-1}\Big(\nabla a_k \cdot \mathbf{P}_\sigma \mathbb{W}_k\Big)\Big\|_p \\
&\lesssim \kappa^2 \sigma^{-1}\big[\mu^{1-2/p}\tau^{1/2-1/p}\big] \\
&\lesssim \kappa^2 \big[\mu^{1-2/p}\tau^{1/2-1/p}\big],
\end{aligned}$$

which is the desired bound. □

3.5 Proof of main proposition: new Reynolds stress

In this section, we construct a new Reynolds stress R such that (Equation 3.13) holds. The majority of this section is devoted to obtaining bounds on the new Reynolds stress R using the established estimates for the velocity perturbations in Section 3.4. We split R into four parts and then estimate them separately.

To do this, one needs to obtain a symmetric traceless matrix R as the new stress term. Since the underdetermined system (Equation gNSR) only provides an implicit definition of R, i.e. its divergence, the divergence has to be "inverted". This is a standard technique in elliptic PDEs. Here, we follow the one used in (3).

Definition 3.5.1 (Inverse divergence). *Let $f \in C^\infty(\mathbb{T}^3)$ be a smooth vector field. The inverse divergence operator $\mathcal{R} : C^\infty(\mathbb{T}^3, \mathbb{R}^3) \to \mathbb{R}^{3\times 3}$ is defined by*

$$
\begin{aligned}
&(\mathcal{R}f)_{ij} = \mathcal{R}_{ijk} f_k, \\
&\mathcal{R}_{ijk} = -\frac{1}{2}\Delta^{-2}\partial_i\partial_j\partial_k - \frac{1}{2}\Delta^{-1}\partial_k\delta_{ij} + \Delta^{-1}\partial_i\delta_{jk} + \Delta^{-1}\partial_j\delta_{ik}.
\end{aligned}
\tag{3.112}
$$

Remark 3.5.2. *We note that in the definition, the inverse Laplacian Δ^{-1} is defined on $\mathbb{T}^3$ and gives functions with zero mean. So $\mathcal{R}f$ is always well-defined and mean free.*

With the above definition, a simple exercise leads to the following.

Lemma 3.5.3. *The operator $\mathcal{R}$ defined by* (Equation 3.112) *has the following properties. For any vector field $f \in C^\infty(\mathbb{T}^3)$ the matrix $\mathcal{R}f$ is symmetric trace-free, and*

$$
\operatorname{div} \mathcal{R}f = f. \tag{3.113}
$$

If additionally $\operatorname{div} f = 0$*, then*

$$
\mathcal{R}\Delta f = \nabla f + (\nabla f)^T. \tag{3.114}
$$

With this inverse divergence operator, we are ready to give the definition of the new Reynolds stress.

Definition 3.5.4 (New Reynolds stress R). *Define the new Reynolds stress by*

$$
R = \mathcal{R}\Big(\partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div}(\theta^2 R_0 + w \otimes w) - \nabla p_1\Big) + (1-\theta^2)R_0 \tag{3.115}
$$

where the pressure term $p_1 = \theta^2 \rho$ *and* ρ *is defined in* (Equation 3.81)*.*

It is immediate that the new Reynold stress R verifies the following equation thanks to Lemma 3.5.3

$$\operatorname{div} R = \partial_t w + L_a w + \operatorname{div}(w \otimes v_0 + v_0 \otimes w) + \operatorname{div} R_0 + \operatorname{div}(w \otimes w) - \nabla p_1.$$

Consequently, since (v_0, R_0) is a solution of (Equation gNSR), there exists a uniquely determined zero-mean pressure P such that the new solution $v = v_0 + w$ verifies

$$\partial_t v + L_a v + \operatorname{div}(v \otimes v) + \nabla P = \operatorname{div} R.$$

In view of $w = w^{(p)} + w^{(l)} + w^{(c)}$, the new Reynolds stress can be rewritten as

$$R = R_{\text{lin}} + R_{\text{cor}} + R_{\text{osc}} + R_{\text{rem}}, \tag{3.116}$$

where the linear part R_{lin}, the correction part R_{cor}, oscillation part R_{osc} and the reminder part R_{rem} are respectively defined by

$$\begin{aligned}
R_{\text{lin}} &= \mathcal{R}\big(\partial_t w + L_a w - \Delta w^{(l)} + \operatorname{div}(w \otimes v_0 + v_0 \otimes w)\big), \\
R_{\text{cor}} &= \mathcal{R}\big(\operatorname{div} \big((w^{(c)} + w^{(l)}) \otimes w + w^{(p)} \otimes (w^{(c)} + w^{(l)})\big)\big), \\
R_{\text{osc}} &= \mathcal{R}\big(\operatorname{div}(\theta^2 R_0 + w^{(p)} \otimes w^{(p)}) + \Delta w^{(l)} - \nabla p_1\big), \\
R_{\text{rem}} &= (1 - \theta^2) R_0.
\end{aligned}$$

In the remainder of this section, we will estimate R via the decomposition $\|R\|_1 \le \|R_{\mathrm{lin}}\|_1 + \|R_{\mathrm{cor}}\|_1 + \|R_{\mathrm{osc}}\|_1 + \|R_{\mathrm{rem}}\|_1$ and show the following.

Lemma 3.5.5 (Estimates for R)**.** *The new Reynolds stress R obeys the estimates:*

$$\|R(t)\|_1 \le \begin{cases} \varepsilon & \text{for } t \in I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 + \varepsilon & \text{for } t \in I_{4^{-2}r}(\mathcal{F}_1) \setminus I_{4^{-1}r}(\mathcal{F}_1) \\ \delta_0 & \text{for } t \in [0,T] \setminus I_{4^{-2}r}(\mathcal{F}_1). \end{cases} \tag{3.117}$$

Since $\operatorname{supp}_t w \subset I_{4^{-2}r}(\mathcal{F}_1)$, it is sufficient to show that

$$\left\|R_{\mathrm{lin}}\right\|_{L^\infty_t L^1_x} + \left\|R_{\mathrm{cor}}\right\|_{L^\infty_t L^1_x} + \|R_{\mathrm{osc}}\|_{L^\infty_t L^1_x} \le \varepsilon.$$

We first estimate the linear part. For this term, the smallness of the intermittency plays a key role.

Lemma 3.5.6 (Linear error)**.** *For any λ sufficiently large,*

$$\left\|R_{\mathrm{lin}}\right\|_{L^\infty_t L^1_x} \le \frac{\varepsilon}{4}. \tag{3.118}$$

Proof. Considering the fact that

$$\|\mathcal{R}\|_{L^p(\mathbb{T}^3) \to L^p(\mathbb{T}^3)} \lesssim 1 \quad \text{for any } 1 < p < \infty \tag{3.119}$$

due to the Hardy-Littlewood-Sobolev inequality, and that

$$\|\mathcal{R}\operatorname{div}\cdot\|_{L^p(\mathbb{T}^3)\to L^p(\mathbb{T}^3)} \lesssim 1 \quad \text{for any } 1 < p < \infty \tag{3.120}$$

due to the boundedness of the Riesz transform, throughout the proof we fix $p > 1$ close to 1 such that

$$\mu^{1-2/p}\tau^{1/2-1/p} = \lambda^{\frac{17}{15}(1-2/p)} \leq \lambda^{-16/15}. \tag{3.121}$$

Split the linear error $R_{\text{lin}} = R_t + R_d$, where the first part R_t is the error caused by time derivative $R_t = \mathcal{R}\partial_t w$, and the second part R_d consists of the dissipative and drifts errors

$$R_d = \mathcal{R}\Delta(w^{(p)} + w^{(c)}) + \mathcal{R}\operatorname{div}\big(w \otimes (a + v_0)\big) + \mathcal{R}\operatorname{div}\big((a + v_0) \otimes w\big).$$

For the liner error caused by time derivative, by (Equation 3.119) and Proposition 3.4.11 we have

$$\big\|R_t\big\|_1 \leq \big\|\mathcal{R}\partial_t w\big\|_p \lesssim \big\|\partial_t w\big\|_p \lesssim \kappa^2\mu^{1-2/p}\tau^{1/2-1/p} \leq \kappa^2\lambda^{-\frac{16}{15}}. \tag{3.122}$$

We turn to estimate the liner error caused by drifts and the Laplacian. So using Lemma 3.5.3, (Equation 3.120) and Hölder's inequality we get

$$\begin{aligned}\big\|R_d\big\|_1 &\le \|\mathcal{R}\Delta(w^{(p)}+w^{(c)})\|_1 + \big\|\mathcal{R}\,\mathrm{div}\,\big(w\otimes(a+v_0)\big)\big\|_p + \big\|\mathcal{R}\,\mathrm{div}\,\big((a+v_0)\otimes w\big)\big\|_p \\ &\lesssim \|\nabla(w^{(p)}+w^{(c)})\|_1 + \|w\|_p\big[\|a\|_\infty + \|v_0\|_\infty\big]. \end{aligned} \tag{3.123}$$

By Corollary 3.4.10 and using (Equation 3.121) we have

$$\begin{aligned}\|\nabla(w^{(p)}+w^{(c)})\|_1 &\lesssim \big[\rho_0^{1/2} + \lambda^{-1/3}\big]\lambda^{-2/15} \\ \|w\|_p &\lesssim \rho_0^{1/2}\lambda^{-16/15}.\end{aligned}$$

It follows from the above and (Equation 3.123) that

$$\big\|R_d\big\|_1 \lesssim \rho_0^{1/2}\lambda^{-2/15} + \rho_0^{1/2}\lambda^{-16/15}\big(\|a\|_\infty + \|v_0\|_\infty\big). \tag{3.124}$$

Combining (Equation 3.122) and (Equation 3.124), for any sufficiently large $\lambda(a,\varepsilon,e_1,\kappa,v_0)$ it holds

$$\big\|R_{\mathrm{lin}}\big\|_1 \le \big\|R_t\big\|_1 + \big\|R_d\big\|_1 \le \frac{\varepsilon}{4}. \tag{3.125}$$

□

Next, we turn to estimating the correction part of the new Reynolds stress R. This part is essentially caused by $w^{(c)}$ and $w^{(l)}$ which are both much smaller than $w^{(p)}$.

Lemma 3.5.7 (Correction error)**.** *For any λ sufficiently large,*

$$\left\|R_{\mathrm{cor}}\right\|_{L_t^\infty L_x^1} \leq \frac{\varepsilon}{8}. \tag{3.126}$$

Proof. In view of Corollary 3.4.10, fix a $p>1$ close to 1 such that

$$\begin{aligned}
&\|w^{(c)}\|_{\frac{2p}{p-2}} \lesssim \lambda^{-\frac{1}{30}},\\
&\|w^{(l)}\|_{\frac{2p}{p-2}} \lesssim \lambda^{-\frac{1}{30}}.
\end{aligned}$$

By the L^p boundedness of $\mathcal{R}\,\mathrm{div}$ and Hölder's inequality, we have

$$\begin{aligned}
\|R_{\mathrm{cor}}\|_1 \lesssim \|R_{\mathrm{cor}}\|_p &\lesssim_p \|\big((w^{(c)}+w^{(l)})\otimes w\|_p + \|w^{(p)}\otimes(w^{(c)}+w^{(l)})\|_p && (3.127)\\
&\lesssim \big(\|w^{(c)}\|_{\frac{2p}{p-2}} + \|w^{(l)}\|_{\frac{2p}{p-2}}\big)\|w\|_2 && (3.128)\\
&\lesssim \lambda^{-\frac{1}{30}}\big(\rho_0^{1/2} + \lambda^{-\frac{1}{3}} + \lambda^{-\frac{1}{30}}\big). && (3.129)
\end{aligned}$$

Due to the negative exponent in λ on the right hand side, for any sufficiently large $\lambda(\varepsilon, e_0, e_1, \kappa)$ we have

$$\|R_{\mathrm{cor}}\|_1 \leq \frac{\varepsilon}{8}.$$

□

Finally, we turn to estimating the oscillation error $R_{\rm osc}$, where we will utilize the fact that *viscous eddies* are approximate stationary solutions of the NSE.

Lemma 3.5.8 (Decomposition of $R_{\rm osc}$). *The oscillation error $R_{\rm osc}$ can be decomposed into two parts:*

$$R_{\rm osc} = R_{\rm high} + R_{\rm low} + R_{\rm err}, \tag{3.130}$$

where $R_{\rm high}$ is the high frequency part

$$R_{\rm high} = \theta^2 \mathcal{R} \sum_k \nabla (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \big(W_k \otimes W_z \big), \tag{3.131}$$

$R_{\rm low}$ consists of lower order terms

$$\begin{aligned} R_{\rm low} =& \sigma \theta^2 \mathcal{R} \sum_k a_k^2 \mathbf{P}_\sigma \big(\operatorname{div}(W_k \otimes W_z) - \Delta \mathbb{V}_k \big) \\ & - \sigma^{-1} \theta^2 \mathcal{R} \sum_k \Big[\Delta a_k^2 \mathbf{P}_\sigma \mathbb{V}_k + 2 \nabla a_k^2 \cdot \mathbf{P}_\sigma \nabla \mathbb{V}_k \Big], \end{aligned} \tag{3.132}$$

and $R_{\rm err}$ is the symmetry breaking error

$$R_{\rm err} = \theta^2 \mathcal{R} \operatorname{div} \sum_k a_k^2 \Big(\mathbf{P}_\sigma \big(\mathbb{W}_k \otimes \mathbb{W}_k \big) - \mathbf{P}_\sigma \big(W_k \otimes W_z \big) \Big).$$

Proof. Since $\mathbb{W}_k$ have disjoint supports in space, we have

$$w^{(p)} \otimes w^{(p)} = \theta^2 \sum_k (a_k)^2 \mathbf{P}_\sigma\big(\mathbb{W}_k \otimes \mathbb{W}_k\big),$$

which in view of Lemma 3.4.5 gives

$$\begin{aligned} w^{(p)} \otimes w^{(p)} &- \theta^2 \sum_k a_k^2 \Big(\mathbf{P}_\sigma\big(\mathbb{W}_k \otimes \mathbb{W}_k\big) - \mathbf{P}_\sigma\big(W_k \otimes W_z\big)\Big) \\ &= \theta^2(t) \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_k \otimes W_z\big) + \theta^2 \sum_k a_k^2 \Big(\mathbf{P}_\sigma\big(W_k \otimes W_z\big) - \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_k \otimes W_z\big)\Big) \\ &= \theta^2 \rho \operatorname{Id} - \theta^2 R_0 + \theta^2 \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma\big(W_k \otimes W_z\big). \end{aligned} \tag{3.133}$$

Upon taking the divergence on both sides of (Equation 3.133) we have for the oscillation error

$$\begin{aligned} R_{\mathrm{osc}} &= \mathcal{R}\big(\operatorname{div} \theta^2 R_0 + \operatorname{div}(w^{(p)} \otimes w^{(p)}) - \nabla p_1 + \Delta w^{(l)}\big) \\ &= R_{\mathrm{err}} + \mathcal{R}\Big(\theta^2 \operatorname{div} \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma\big(W_k \otimes W_z\big) + \Delta w^{(l)}\Big). \end{aligned}$$

By the product rule we may obtain

$$R_{\mathrm{osc}} = R_{\mathrm{err}} + R_{\mathrm{high}} + \mathcal{R}\Big(\sigma\theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \operatorname{div}\big(W_k \otimes W_z\big) + \Delta w^{(l)}\Big). \tag{3.134}$$

It remains to compute the second term in (Equation 3.134). Using the definition of $w^{(l)}$, a routine computation gives

$$\Delta w^{(l)} = -\sigma\theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \Delta \mathbb{V}_k - \theta^2 \sum_k \Big[\sigma^{-1} \Delta a_k^2 \mathbf{P}_\sigma \mathbb{V}_k + 2\nabla a_k^2 \mathbf{P}_\sigma \nabla \mathbb{V}_k\Big],$$

which implies exactly

$$\mathcal{R}\Big(\sigma\theta^2 \sum_k a_k^2 \mathbf{P}_\sigma \operatorname{div}\big(W_k \otimes W_z\big) + \Delta w^{(l)}\Big) = R_{\text{low}}.$$

Hence the oscillation error verifies the identity $R_{\text{osc}} = R_{\text{high}} + R_{\text{low}} + R_{\text{err}}$.

□

Remark 3.5.9. *The term R_{high} is typical in convex integration, where the derivative falls on "slow variable" a_k and the term $\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(\mathbb{W}_k \otimes \mathbb{W}_k\big)$ has fast oscillation and zero mean. The presence of R_{low} and R_{err} is one the fundamental differences between our scheme and previous ones.*

We are ready to estimate the oscillation error. The term R_{high} will be able to gain a factor of σ^{-1} via the inverse divergence $\mathcal{R}$, while the term R_{low} is already quite small thanks to the inverse Laplacian. In other words, R_{high} is of high frequency, while R_{low} is not of high frequency but instead lower order.

Lemma 3.5.10 (Oscillation error: R_{high})**.** *For any λ sufficiently large,*

$$\left\|R_{\text{high}}\right\|_{L_t^\infty L_x^1} \leq \frac{\varepsilon}{4}. \tag{3.135}$$

Proof. Throughout the proof, let us fix two parameters $0 < \alpha < 1$ and $1 < p < 2$, such that the Sobolev embedding $W^{\alpha,1}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^3)$ holds.

It follows from the L^p boundedness of the Riezs transform that

$$\|R_{\text{high}}\|_{L^1(\mathbb{T}^3)} \leq \|R_{\text{high}}\|_{L^p(\mathbb{T}^3)} \lesssim \sum_k \left\||\nabla|^{-1}\big(\nabla(a_k^2)\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\big)\right\|_p. \tag{3.136}$$

Obviously $\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big)$ is $\sigma^{-1}\mathbb{T}^3$-periodic and has zero mean, and by Lemma 3.4.5

$$\left\|\nabla a_k^2\right\|_{C_x^m} \leq \left\|a_k^2\right\|_{C_x^{m+1}} \leq \kappa^{m+3} \quad \text{for all } 0 \leq m \leq M.$$

Thus we may apply Proposition 2.4.4 with $C_a = \kappa^3$ to obtain that

$$\begin{aligned}\left\||\nabla|^{-1}\big(\nabla(a_k^2)\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\big)\right\|_p \lesssim & \sigma^{-1+\alpha}\left\||\nabla|^{-\alpha}\big(\nabla(a_k)^2\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\big)\right\|_p \\ & + \kappa^3\sigma^{-N}\left\|\mathbf{P}_\sigma\big(W_k \otimes W_z\big)\right\|_p.\end{aligned} \tag{3.137}$$

The first term in (Equation 3.137) can be estimated by the Sobolev embedding $W^{\alpha,1}(\mathbb{T}^3) \hookrightarrow L^p(\mathbb{T}^3)$, and Lemma 3.4.5 as follows:

$$\begin{aligned}
\sigma^{-1+\alpha}\Big\||\nabla|^{-\alpha}\big(\nabla(a_k)^2\mathbb{P}_{\neq 0}\mathbf{P}_\sigma\big(W_k\otimes W_z\big)\big)\Big\|_p &\lesssim \sigma^{-1+\alpha}\big\|a_k^2\big\|_{C^1_{t,x}}\big\|\mathbf{P}_\sigma(W_k\otimes W_z)\big\|_1 \\
&\lesssim \sigma^{-1+\alpha}\kappa^4\big\|\mathbf{P}_\sigma\big((W_z+W_r)\otimes W_z\big)\big\|_1.
\end{aligned} \tag{3.138}$$

Now recall that $W_r \notin L^2$ due to the $1/r$ singularity on the Ω_k-plane, but $W_r \in L^p$ since $1 \le p < 2$ (see (Equation 3.61)). Hence, Hölder's inequality, (Equation 3.60), and (Equation 3.61) imply

$$\begin{aligned}
\big\|\mathbf{P}_\sigma\big((W_z+W_r)\otimes W_z\big)\big\|_1 &\lesssim \big(\|W_z\|_{L^p(\mathbb{R}^3)} + \|W_r\|_{L^p(\mathbb{R}^3)}\big)\|W_z\|_{L^{1-1/p}(\mathbb{R}^3)} \\
&\lesssim_p \big(\mu^{1-2/p}\tau^{1/2-1/p} + \mu^{-2/p}\tau^{3/2-1/p}\big)\mu^{-1+2/p}\tau^{-1/2+1/p} \\
&= \mu^0\tau^0 + \mu^{-1}\tau^1 \\
&\lesssim 1.
\end{aligned} \tag{3.139}$$

The second term in (Equation 3.137) can be handled easily using Proposition 3.3.14 and $N = 300$,

$$\kappa^3\sigma^{-N}\big\|\mathbf{P}_\sigma\big(W_k\otimes W_z\big)\big\|_p \lesssim \kappa^3\lambda^{-10}\big(\|W_z\|_{L^p(\mathbb{R}^3)} + \|W_r\|_{L^p(\mathbb{R}^3)}\big)\|W_z\|_{L^\infty(\mathbb{R}^3)} \lesssim \kappa^3\lambda^{-1}. \tag{3.140}$$

Collecting (Equation 3.136), (Equation 3.137), (Equation 3.138), (Equation 3.139), and (Equation 3.140) we arrive at

$$\big\|R_{\text{high}}\big\|_1 \lesssim (\kappa^3+\kappa^4)\sigma^{-1+\alpha}.$$

As $0 < \alpha < 1$, for all $\lambda(\varepsilon, \kappa)$ sufficiently large we can conclude that

$$\left\|R_{\text{high}}\right\|_{L_t^\infty L_x^1} \leq \frac{\varepsilon}{8}.$$

□

Lemma 3.5.11 (Oscillation error: R_{low})**.** *For any λ sufficiently large*

$$\left\|R_{\text{low}}\right\|_{L_t^\infty L_x^1} \leq \frac{\varepsilon}{8}. \tag{3.141}$$

Proof. Let us fix $p > 1$ such that

$$\sigma\tau^2\mu^{-1}\big(\tau^{1-1/p}\mu^{2-2/p}\big) \leq \lambda^{-\frac{1}{30}}. \tag{3.142}$$

So by the boundedness of $\mathcal{R}$ on L^p and Hölder's inequality, we have

$$\begin{aligned}\left\|R_{\text{low}}\right\|_{L^1(\mathbb{T}^3)} \leq \left\|R_{\text{low}}\right\|_{L^p(\mathbb{T}^3)} \lesssim \sum_k \sigma\left\|a_k^2\right\|_{L^\infty_{t,x}}\Big\|\mathbf{P}_\sigma\big(\operatorname{div}(W_k \otimes W_z) - \Delta\mathbb{V}_k\big)\Big\|_p \\ + \sigma^{-1}\left\|a_k^2\right\|_{C^2_{t,x}}\left\|\mathbf{P}_\sigma\mathbb{V}_k\right\|_p + \sigma^{-1}\left\|a_k^2\right\|_{C^1_{t,x}}\left\|\mathbf{P}_\sigma\nabla\mathbb{V}_k\right\|_p\end{aligned}$$

Thanks to Proposition 3.3.11,

$$\Big\|\mathbf{P}_\sigma\big(\operatorname{div}(W_k \otimes W_z) - \Delta\mathbb{V}_k\big)\Big\|_p \lesssim \tau^2\mu^{-1}\big(\tau^{1-1/p}\mu^{2-2/p}\big).$$

Combining this with the estimates in Proposition 3.3.14 and Lemma 3.4.5, it follows that

$$\begin{aligned} \left\|R_{\text{low}}\right\|_1 &\lesssim \left(\kappa^2\sigma\tau^2\mu^{-1} + \kappa^4\sigma^{-1}\tau\mu^{-2} + \kappa^3\sigma^{-1}\tau\mu^{\gamma-1}\right)\left(\tau^{1-1/p}\mu^{2-2/p}\right) \\ &\lesssim \left(\kappa^2 + \kappa^3 + \kappa^4\right)\sigma\tau^2\mu^{-1}\left(\tau^{1-1/p}\mu^{2-2/p}\right), \end{aligned} \tag{3.143}$$

where we used $\mu^\gamma = \sigma \leq \sigma^2\tau$ for the third term.

Using (Equation 3.142) and taking $\lambda(\kappa, \varepsilon)$ sufficiently large, the desired bound follows:

$$\left\|R_{\text{low}}\right\|_1 \leq \frac{\varepsilon}{8}.$$

□

Lemma 3.5.12 (Symmetry breaking error: R_{err})**.**

$$\left\|R_{\text{err}}\right\|_{L^\infty_t L^1_x} \leq \frac{\varepsilon}{8}.$$

Proof. We fix $1 < p < 2$ so that $\mu^{-\gamma}\mu^{2-2/p}\tau^{1-1/p} \leq \mu^{-\gamma/2}$. Recall that $\mathcal{R}\,\mathrm{div}$ is bounded on L^p. Then

$$\left\|R_{\text{err}}\right\|_{L^1(\mathbb{T}^3)} \leq \left\|R_{\text{err}}\right\|_{L^p(\mathbb{T}^3)} \lesssim \sum_k \left\|a_k^2\right\|_{L^\infty_{t,x}} \left\|\mathbf{P}_\sigma\left(\mathbb{W}_k \otimes \mathbb{W}_k\right) - \mathbf{P}_\sigma\left(W_k \otimes W_z\right)\right\|_p.$$

Now using Lemma 3.4.5 and Proposition 3.3.9, we obtain

$$\begin{aligned}\left\|R_{\mathrm{err}}\right\|_1 &\lesssim \kappa^2\mu^{-\gamma}\mu^{2-2/p}\tau^{1-1/p}\\ &\lesssim \kappa^2\mu^{-\gamma/2}\\ &\leq \frac{\varepsilon}{8},\end{aligned}$$

for $\lambda(\varepsilon,\kappa)$ large enough. □

Note that Lemma 3.5.5 is proved, as it follows directly from Lemma 3.5.6, 3.5.7, 3.5.10, 3.5.11, and 3.5.12.

3.6 Proof of main proposition: energy level

In this section, we prove properties related to the energy in the main proposition. To show the correct energy level of the solution v, let us first show that the energy in the perturbation w is dominated by $w_z^{(p)}$, which is anticipated in view of the estimates in Proposition 3.4.9.

Lemma 3.6.1. *For any λ sufficiently large*

$$\left|\|v(t)\|_2^2 - \|v_0(t)\|_2^2 - \|w_z^{(p)}(t)\|_2^2\right| \leq 10^{-7}(e_1 - e_0) \quad \textit{for all } t \in [0,T]. \tag{3.144}$$

Proof. Since $w = w_z^{(p)} + w_r^{(p)} + w^{(l)} + w^{(c)}$, we have

$$\|v(t)\|_2^2 - \|v_0(t)\|_2^2 - \|w_z^{(p)}(t)\|_2^2 = E_{\mathrm{error}}$$

where the error term E_{error} is

$$E_{\text{error}} = 2\langle w, v_0\rangle + 2\langle w_z^{(p)}, w_r^{(p)} + w^{(c)} + w^{(l)}\rangle + \|w_r^{(p)} + w^{(c)} + w^{(l)}\|_2^2.$$

Fix any $1 < p < 2$. By Hölder's inequality, we have

$$\big|E_{\text{error}}\big| \lesssim \|w(t)\|_p \|v_0(t)\|_{\frac{p}{p-1}} + \big(\|w_r^{(p)}\|_2 + \|w^{(c)}\|_2 + \|w^{(l)}\|_2\big)\|w_z^{(p)}\|_2 + \|w_r^{(p)}\|_2^2 + \|w^{(c)}\|_2^2 + \|w^{(l)}\|_2^2.$$

Thanks to Corollary 3.4.10, for any sufficiently large $\lambda(e_1, \kappa, v_0)$ we have

$$\|w^{(c)}\|_2^2 + \|w^{(l)}\|_2^2 \lesssim \lambda^{-\frac{3}{10}},$$

$$\|w_r^{(p)}\|_2 \lesssim \rho_0^{1/2} \lambda^{-\frac{7}{15}} \lambda^{\frac{17}{15}(1-\frac{2}{p})},$$

$$\|w_z^{(p)}\|_2 \lesssim \|w^{(p)}\|_2 + \|w_r^{(p)}\|_2 \lesssim \rho_0^{1/2},$$

$$\|w\|_p \lesssim (\rho_0^{1/2} + \lambda^{-\frac{1}{30}}) \lambda^{\frac{17}{15}(1-\frac{2}{p})}.$$

Since $\rho_0(t) \lesssim e_1$, for any sufficiently large $\lambda(e_1, e_0, \kappa, v_0)$, we can make sure that

$$\big|E_{\text{error}}\big| \le 10^{-7}(e_1 - e_0).$$

$\square$

Next, we estimate the energy of $w^{(p)}$ more precisely than Proposition 3.4.9. Note that the choice of ρ_0, namely (Equation 3.79), is crucial in the proof. Recall that $\widetilde{e}_1 = e_1 - 10^{-6}(e_1 - e_0)$.

Lemma 3.6.2. *Suppose that the constant C in the statement of Proposition 3.2.1 is small enough. For any λ sufficiently large, the energy of $w^{(p)}$ verifies*

$$\big|\|w_z^{(p)}\|_2^2 - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2)\big| \leq 10^{-7}(e_1 - e_0) \quad \textit{for all } t \in [0, T].$$

Proof. First, note that as in (Equation 3.67) and (Equation 3.68),

$$\|W_z - (W_z)_\gamma\|_2 \lesssim \mu^{-1-\gamma}\|W_z\|_{H^1} \lesssim \mu^{-2-\gamma}\tau = \lambda^{-1/3}.$$

Hence, thanks to Lemma 3.4.5,

$$\left|\|w_z^{(p)}\|_2 - \Big\|\theta a_k \sum_k \mathbf{P}_\sigma W_z\Big\|_2\right| \lesssim \Big\|\theta a_k \sum_k \mathbf{P}_\sigma (W_z - (W_z)_\gamma)\Big\|_2 \lesssim \kappa\lambda^{-1/3},$$

and consequently

$$\left|\|w_z^{(p)}\|_2 - \Big\|\theta a_k \sum_k \mathbf{P}_\sigma W_z\Big\|_2\right| \leq 10^{-8}(e_1 - e_0) \tag{3.145}$$

for $\lambda(e_0, e_1, \kappa)$ large enough. Now recall that

$$\fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_k \otimes W_z\big) = \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_z \otimes W_z\big).$$

Thus, similarly to (Equation 3.133), we obtain

$$\begin{aligned}\theta^2 \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_z \otimes W_z\big) &= \theta^2 \sum_k a_k^2 \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_z \otimes W_z\big) \\ &\quad + \sum_k a_k^2 \Big(\mathbf{P}_\sigma\big(W_z \otimes W_z\big) - \fint_{\mathbb{T}^3} \mathbf{P}_\sigma\big(W_z \otimes W_z\big)\Big) \\ &= \theta^2 \rho \operatorname{Id} - \theta^2 R_0 + \theta^2 \sum_k (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \Big(W_z \otimes W_z\Big).\end{aligned}$$

Upon taking the trace and integrating in space, it follows that

$$\Big\|\theta a_k \sum_k \mathbf{P}_\sigma W_z\Big\|_2^2 = 3\theta^2 \int_{\mathbb{T}^3} \rho(x,t) + \theta^2 \sum_k \int_{\mathbb{T}^3} (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big),$$

Using the definition of ρ_0 (Equation 3.79), we can consider the split

$$\Big\|\theta a_k \sum_k \mathbf{P}_\sigma W_z\Big\|_2^2 - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2) = X_l + X_h, \tag{3.146}$$

where X_l is the low frequency error term

$$X_l = 3\theta^2 \int_{\mathbb{T}^3} \rho(x,t) - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2), \tag{3.147}$$

and X_h is the high frequency error term

$$X_h = \theta^2 \int_{\mathbb{T}^3} (a_k)^2 \mathbb{P}_{\neq 0} \mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big). \tag{3.148}$$

The goal is to show that $|X_l| + |X_h| \leq 10^{-7}(e_1 - e_0)$. Let us first estimate the term X_h. Using a standard integration by parts argument, we have[1]

$$\big|X_h\big| \lesssim \sum_k \|a_k^2\|_{C^M_{t,x}} \big\||\nabla|^{-M}\mathbb{P}_{\neq 0}\mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big)\big\|_2, \tag{3.149}$$

where M is as defined in Definition 3.4.1. Since $\mathbb{P}_{\neq 0}\mathbf{P}_\sigma \operatorname{Tr}\big(\mathbb{W}_k \otimes \mathbb{W}_k\big)$ is $\sigma^{-1}\mathbb{T}$-periodic and of zero mean, we have

$$\begin{aligned}
\big\||\nabla|^{-M}\mathbb{P}_{\neq 0}\mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big)\big\|_2 &\lesssim \sigma^{-M+3}\big\||\nabla|^{-3}\mathbb{P}_{\neq 0}\mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big)\big\|_2 \\
&\lesssim \sigma^{-M+3}\big\|\mathbb{P}_{\neq 0}\mathbf{P}_\sigma \operatorname{Tr}\big(W_z \otimes W_z\big)\big\|_1 \\
&\lesssim \sigma^{-M+3}\big\|W_z\big\|^2_{L^2(\mathbb{R}^3)} \\
&\lesssim \sigma^{-M+3},
\end{aligned}$$

where the second inequality follows from the Sobolev embedding $H^{-3}(\mathbb{T}^3) \hookrightarrow L^1(\mathbb{T}^3)$, and the last inequality follows from Proposition 3.3.14. Combining this with (Equation 3.149) and using Lemma 3.4.5, we get

$$\big|X_h\big| \lesssim \|a_k^2\|_{C^M_{t,x}}\sigma^{-M+3} \lesssim \kappa^{M+2}\sigma^{-M+3}. \tag{3.150}$$

---

[1]Recall that $\|a_k\|_{C^m_{t,x}} \leq \kappa^{m+1}$ is only valid for $0 \leq m \leq 4M$.

Hence for sufficiently large $\lambda(e_0, e_1, \kappa)$, we can ensure that

$$\left|X_h\right| \le 10^{-8}(e_1 - e_0). \tag{3.151}$$

On the other hand, for the term X_l using the definitions of ρ and ρ_0 (namely (Equation 3.81) and (Equation 3.79)) we get

$$X_l = -12\theta^2 \rho_0 \left(1 - \int \chi^2(\rho_0^{-1} R_0)\right)$$

First, Let us split the integral

$$\int \chi^2(\rho_0^{-1} R_0) = \left(\int_{|R_0| \le \rho_0} + \int_{|R_0| \ge \rho_0}\right) \chi^2(\rho_0^{-1} R_0).$$

Next, by the above split we have

$$\left|X_l\right| \lesssim \rho_0 \left|1 - \int_{|R_0| \le \rho_0} \chi^2(\rho_0^{-1} R_0)\right| + \rho_0 \left|\int_{|R_0| \ge \rho_0} \chi^2(\rho_0^{-1} R_0)\right|. \tag{3.152}$$

Since $\delta_0 = \|R_0\|_{L_t^\infty L_x^1}$, thanks to the Chebyshev inequality we have

$$\left|\{x \in \mathbb{T}^3 : |R_0| \ge \rho_0\}\right| \le \frac{\delta_0}{\rho_0},$$

which together with the definition of χ in (Equation 3.78) and the fact that $|\mathbb{T}^3| = 1$ implies that

$$\begin{aligned} \left|X_l\right| &\lesssim \rho_0 \left| 1 - \int_{|R_0| \le \rho_0} 1 dx \right| + \rho_0 \int_{|R_0| \ge \rho_0} \rho_0^{-1} |R_0| \\ &\lesssim \rho_0 \left| \int_{|R_0| > \rho_0} 1 dx \right| + \int_{|R_0| \ge \rho_0} |R_0| \\ &\lesssim \delta_0. \end{aligned}$$

Note that in the estimates for X_l, all implicit constants are universal. In view of the assumption $\delta_0 \le C(e_1 - e_0)$ in the statement of Proposition 3.2.1, we may choose the constant C small enough such that

$$\left|X_l\right| \le 10^{-8}(e_1 - e_0). \tag{3.153}$$

Combining (Equation 3.145), (Equation 3.151), and (Equation 3.153) with (Equation 3.146) we obtain

$$\left| \|w^{(p)}\|_2^2 - \theta^2(\widetilde{e}_1 - \|v_0\|_2^2) \right| \le 10^{-7}(e_1 - e_0). \tag{3.154}$$

$\square$

With the help of Lemma 3.6.1 and 3.6.2, we obtain the desire energy level of the new solution v as a corollary.

Corollary 3.6.3. *Suppose that the constant C in the statement of Proposition 3.2.1 is small enough. For any λ sufficiently large, the energy of new solution $v(t)$ verifies*

$$\sup_t \|v(t)\|_2^2 \le e_1,$$

and

$$\big|\|v(t)\|_2^2 - e_1\big| \le \frac{c_0}{2}(e_1 - e_0) \quad \text{for all } t \in I_{4^{-1}r}(\mathcal{F}_1).$$

Proof. Both bounds immediately follow from Lemma 3.6.1, 3.6.2 and the facts that $\widetilde{e}_1 = e_1 - 10^{-6}(e_1 - e_0)$ and $\theta = 1$ on $I_{4^{-1}r}(\mathcal{F}_1)$. □

APPENDICES

Appendix A

SUPPLEMENTARY MATERIALS FOR CHAPTER 2

A.1 An Estimate for Hölder norms

We collect here the following classical result on the Hölder norms of composition of functions. A proof using the multivariable chain rule can be found in (28).

Proposition A.1.1. *Let $F : \Omega \to \mathbb{R}$ be a smooth function with $\Omega \subset \mathbb{R}^d$. For any smooth function $u : \mathbb{R}^d \to \Omega$ and any $1 \leq m \in \mathbb{N}$ we have*

$$\|\nabla^m (F \circ u)\|_\infty \lesssim \|\nabla^m u\|_\infty \sum_{1 \leq i \leq m} \|\nabla^i F\|_\infty \|u\|_\infty^{i-1} \tag{A.1}$$

where the implicit constant depends on m, d.

A.2 Constantin-E-Titi commutator estimate

The following commutator-type estimate originates from the one proved in (21). Compared with other versions used in (3; 6) the one stated below is homogeneous, i.e. it only involves highest order derivative. For reader's continence we include a proof here following closely the argument of Lemma 1 in (23).

Appendix A (Continued)

Proposition A.2.1. *Let* $f, g \in C^\infty(\mathbb{T}^d)$ *and let* η_ϵ *be a family of mollifier. For any* $m \in \mathbb{N}$ *and* $1 \le p \le \infty$ *we have*

$$\left\| \nabla^m \big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon) \big] \right\|_p \lesssim \epsilon^{2-m} \|\nabla f\|_{2p} \|\nabla g\|_{2p} \tag{A.2}$$

Proof. It suffices to prove for any multi-index α with $|\alpha| = m$ the following estimate:

$$\left\| \partial_\alpha \big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon) \big] \right\|_p \lesssim \epsilon^{2-m} \|\nabla f\|_{2p} \|\nabla g\|_{2p}. \tag{A.3}$$

By the product rule and the fact that mollification commutes with differentiation we compute

$$\begin{aligned} &\partial_\alpha \big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon) \big] \\ &\quad = (fg) * \partial_\alpha \eta_\epsilon - \sum_\beta C^\alpha_\beta (f * \partial_{\alpha-\beta} \eta_\epsilon)(g * \partial_\beta \eta_\epsilon) \end{aligned}$$

where the summation is taking over all multi-index $0 \le \beta \le \alpha$. So

$$\begin{aligned} \partial_\alpha \big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon) \big] = (fg) * \partial_\alpha \eta_\epsilon - (f * \partial_\alpha \eta_\epsilon)(g * \eta_\epsilon) - (f * \eta_\epsilon)(g * \partial_\alpha \eta_\epsilon) \\ - \sum_{\beta \ne 0, \alpha} C^\alpha_\beta (f * \partial_{\alpha-\beta} \eta_\epsilon)(g * \partial_\beta \eta_\epsilon) \end{aligned}$$

The fact that $\int \eta_\epsilon = 1$ and $\int \partial \eta_\epsilon = 0$ implies

$$f * \eta_\epsilon = [f - f(x)] * \eta_\epsilon + f(x) \quad \text{and} \quad f * \partial_\beta \eta_\epsilon = [f - f(x)] * \partial_\beta \eta_\epsilon$$

Appendix A (Continued)

for any multi-index $\beta \neq 0$. Let

$$r_\alpha(f,g) = \int \big[f(x-y) - f(x)\big]\big[g(x-y) - g(x)\big]\partial_\alpha \eta_\epsilon(y)dy,$$

and it follows

$$\begin{aligned} r_\alpha(f,g) &= (fg) * \partial_\alpha\eta_\epsilon - (f * \eta_\epsilon)(g * \partial_\alpha\eta_\epsilon) - (f * \partial_\alpha\eta_\epsilon)(g * \eta_\epsilon) \\ &\quad + [f - f(x)] * \eta_\epsilon(g * \partial_\alpha\eta_\epsilon) + (f * \partial_\alpha\eta_\epsilon)[g - g(x)] * \eta_\epsilon \\ &= (fg) * \partial_\alpha\eta_\epsilon - (f * \eta_\epsilon)(g * \partial_\alpha\eta_\epsilon) - (f * \partial_\alpha\eta_\epsilon)(g * \eta_\epsilon) \\ &\quad + [f - f(x)] * \eta_\epsilon([g - g(x)] * \partial_\alpha\eta_\epsilon) + ([f - f(x)] * \partial_\alpha\eta_\epsilon)[g - g(x)] * \eta_\epsilon \end{aligned}$$

and

$$\sum_{\beta\neq 0,\alpha} C^\alpha_\beta (f * \partial_{\alpha-\beta}\eta_\epsilon)(g * \partial_\beta\eta_\epsilon) = \sum_{\beta\neq 0,\alpha} C^\alpha_\beta \Big[(f - f(x)) * \partial_{\alpha-\beta}\eta_\epsilon\Big]\Big[(g - g(x)) * \partial_\beta\eta_\epsilon\Big].$$

Putting together the preceding two equations we have

$$\partial_\alpha\big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon)\big] = r_\alpha(f,g) - \sum_\beta C^\alpha_\beta\big(f - f(x)\big) * \partial_{\alpha-\beta}\eta_\epsilon \cdot \big(g - g(x)\big) * \partial_\beta\eta_\epsilon.$$

Appendix A (Continued)

On the one hand by Minkowski's inequality we have

$$\Big\| \int \big[f(x-y) - f(x)\big]\big[g(x-y) - g(x)\big]\partial_\alpha \eta_\epsilon(y)dy\Big\|_p \lesssim \int \big\|f(\cdot - y) - f(\cdot)\big\|_{2p}\big\|g(\cdot - y) - g(\cdot)\big\|_{2p}\partial_\alpha \eta_\epsilon(y)dy.$$

From the integral form of Mean Value Theorem and Minkowski's inequality it follows that

$$\big\|f(\cdot - y) - f(\cdot)\big\|_{2p} \lesssim |y|\|\nabla f\|_{2p}$$

$$\big\|g(\cdot - y) - g(\cdot)\big\|_{2p} \lesssim |y|\|\nabla g\|_{2p}$$

which enables us to obtain

$$\begin{aligned}\Big\| \int \big[f(x-y) - f(x)\big]\big[g(x-y) - g(x)\big]\partial_\alpha \eta_\epsilon(y)dy\Big\|_p &\lesssim \|\nabla f\|_{2p}\|\nabla g\|_{2p}\int |y|^2\partial_\alpha \eta_\epsilon(y)dy \\ &\lesssim \epsilon^{2-m}\|\nabla f\|_{2p}\|\nabla g\|_{2p}.\end{aligned}$$

On the other hand by Hölder's inequality we have

$$\begin{aligned}&\Big\| \sum_\beta C_\beta^\alpha \Big[(f - f(x)) * \partial_{\alpha-\beta}\eta_\epsilon\Big]\Big[(g - g(x)) * \partial_\beta \eta_\epsilon\Big]\Big\|_p \\ &\qquad \lesssim \sum_\beta \big\|(f - f(x)) * \partial_{\alpha-\beta}\eta_\epsilon\big\|_{2p}\big\|(g - g(x)) * \partial_\beta \eta_\epsilon)\big\|_{2p} \\ &\qquad \lesssim \epsilon^{m-2}\|\nabla f\|_{2p}\|\nabla g\|_{2p}\end{aligned}$$

where we have used the fact that $\|(f - f(x)) * \partial_\beta \eta_\epsilon\|_{2p} \lesssim \epsilon^{\beta-1}\|\nabla f\|_{2p}$.

Therefore

$$\Big\|\partial_\alpha\big[(fg) * \eta_\epsilon - (f * \eta_\epsilon)(g * \eta_\epsilon)\big]\Big\|_p \lesssim \epsilon^{2-m}\|\nabla f\|_{2p}\|\nabla g\|_{2p}.$$

$\square$

A.3 Proof of Proposition 2.4.4

We include a proof of Proposition 2.4.4 in the d-dimensional case. By considering $\widetilde{a} := \frac{1}{C_a}a$ it suffices to prove both of the results for $C_a = 1$. Notice that since $p \geq 2$ is even, the function $|a|^p$, which is a composition of $a : \mathbb{T}^d \to [-1, 1]$ and x^p, is smooth. Therefore, applying Proposition A.1.1 we see that

$$\begin{aligned} \|\nabla^m |a|^p\|_\infty &\lesssim_p \|\nabla^m a\|_\infty + \sum_{i \leq m} \|\nabla a\|_\infty^{i-1} \qquad \text{for any } m \in \mathbb{N}. \\ &\lesssim_p \mu^m \end{aligned}$$

We can now introduce the split:

$$\|af\|_p^p = \int_{\mathbb{T}^d} (a^p - \overline{|a|^p})(|f|^p - \overline{|f|^p})\, dx + \|a\|_p^p \|f\|_p^p,$$

Appendix A (Continued)

where $\overline{\cdot}$ denotes the integral over $\mathbb{T}^d$. By Parseval's theorem, we get[1]

$$\|af\|_p^p \leq \left| \int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p}) |\nabla|^{-M} (|f|^p - \overline{|f|^p}) dx \right| + \|a\|_p^p \|f\|_p^p.$$

We need show the first term is very small. By Hölder's inequality:

$$\left| \int_{\mathbb{T}^d} |\nabla|^M (a^p - \overline{|a|^p}) |\nabla|^{-M} (|f|^p - \overline{|f|^p}) dx \right| \lesssim \big\| |\nabla|^M a^p \big\|_2 \big\| |\nabla|^{-M} (|f|^p - \overline{|f|^p}) \big\|_2. \tag{A.4}$$

By the L^2 boundedness of Riesz transform we can replace the nonlocal $|\nabla|^M$ by ∇^M to obtain

$$\begin{aligned} \big\| |\nabla|^M a^p \big\|_2 &\lesssim \big\| \nabla^M a^p \big\|_2 \\ &\leq \big\| \nabla^M a^p \big\|_\infty \\ &\lesssim \mu^M. \end{aligned} \tag{A.5}$$

---

[1] The nonlocal operators $|\nabla|^s$ and $|\nabla|^{-s}$ are defined respectively by multipliers with symbols $|k|^s$ and $|k|^{-s}$ for $k \neq 0$ and zero for $k = 0$.

Appendix A (Continued)

We turn to estimate the second factor in (Equation A.4). Considering the fact that the function $(|f|^p - \overline{|f|^p})$ is zero-mean and $\sigma^{-1}\mathbb{T}^d$-periodic we have

$$\begin{aligned}\Big\||\nabla|^{-M}(|f|^p - \overline{|f|^p})\Big\|_2 &\lesssim \sigma^{-M+d}\Big\||\nabla|^{-d}(|f|^p - \overline{|f|^p})\Big\|_2 \\ &\lesssim \sigma^{-M+d}\Big\|(|f|^p - \overline{|f|^p})\Big\|_1 \\ &\lesssim \sigma^{-M+d}\|f\|_p^p,\end{aligned}$$

where the first inequality is a direct consequence of the Littlewood-Paley theory and the second inequality follows from the Sobolev embedding $L^1(\mathbb{T}^d) \hookrightarrow H^d(\mathbb{T}^d)$.

Combining this with estimates (Equation A.4) and (Equation A.5) we find that

$$\left|\int_{\mathbb{T}^d} |\nabla|^M(a^p - \overline{|a|^p})|\nabla|^{-M}(|f|^p - \overline{|f|^p})\,dx\right| \lesssim \sigma^{-M+d}\mu^M\|f\|_p^p.$$

By the assumption $\mu \le \sigma^{1-\theta}$, there exists a number $M_{\theta,p,N,d} \in \mathbb{N}$ sufficiently large so that

$$\sigma^{-M+d}\mu^M \le \sigma^{-Np}. \tag{A.6}$$

Then we have

$$\left|\int_{\mathbb{T}^d} |\nabla|^M(a^p - \overline{|a|^p})|\nabla|^{-M}(|f|^p - \overline{|f|^p})\,dx\right| \lesssim \sigma^{-Np}\|f\|_p^p,$$

which finishes the proof of (Equation 2.20) due to the elementary inequality $(a^p+b^p) \le (a+b)^p$.

Appendix A (Continued)

To prove (Equation 2.21) let us first recall the wavenumber projection. For any $\lambda \in \mathbb{R}$ define $\mathbb{P}_{\leq \lambda} = \sum_{q:2^q \leq \lambda} \Delta_q$ and $\mathbb{P}_{\geq \lambda} = \mathrm{Id} - \mathbb{P}_{\leq \lambda}$, where Δ_q is the Littlewood-Paley projection. Consider the following decomposition:

$$
\begin{aligned}
|\nabla|^{-1}(af) &= |\nabla|^{-1+s}|\nabla|^{-s}\big(\mathbb{P}_{\leq 2^{-4}\sigma} a\big) f + |\nabla|^{-1+s}|\nabla|^{-s}\big(\mathbb{P}_{\geq 2^{-4}\sigma} a\big) f \\
&:= |\nabla|^{-1+s} A_1 + |\nabla|^{-1+s} A_2
\end{aligned}
$$

For the term A_1, since f is $\sigma^{-1}\mathbb{T}^d$-periodic and zero-mean, it follows that

$$
\mathbb{P}_{\geq 2^{-1}\sigma} f = f
$$

and then by the support of Fourier modes of $\big(\mathbb{P}_{\leq 2^{-4}\sigma} a\big) f$ we have

$$
\mathbb{P}_{\leq 2^{-2}\sigma}\Big[\mathbb{P}_{\leq 2^{-4}\sigma} a f\Big] = 0 \quad \text{and} \quad \fint_{\mathbb{T}^d} \mathbb{P}_{\leq 2^{-4}\sigma} a f = 0
$$

which implies that

$$
|\nabla|^{-1+s} A_1 = |\nabla|^{-1+s}\mathbb{P}_{\geq 2^{-2}\sigma} A_1.
$$

By the Littlewood-Paley theory, we have

$$
\Big\| |\nabla|^{-1+s}\mathbb{P}_{\geq 2^{-2}\sigma} \Big\|_{L^p \to L^p} \lesssim_p \sigma^{-1+s}, \qquad 1 < p < \infty.
$$

Appendix A (Continued)

So, we have

$$\left\||\nabla|^{-1+s}A_1\right\|_p \lesssim_p \sigma^{-1+s}\left\||\nabla|^{-s}\big(\mathbb{P}_{\le 2^{-4}\sigma}af\big)\right\|_p.$$

To get the exact form of the estimate, noticing that $|\nabla|^{-s}$ is bounded on L^p, $1 < p < \infty$, we conclude that

$$\begin{aligned}\left\||\nabla|^{-1+s}A_1\right\|_p &\le \sigma^{-1+s}\left\||\nabla|^{-s}(af)\right\|_p + \sigma^{-1+s}\left\||\nabla|^{-s}\big(\mathbb{P}_{\ge 2^{-4}\sigma}af\big)\right\|_p \\ &\lesssim \sigma^{-1+s}\left\||\nabla|^{-s}(af)\right\|_p + \sigma^{-1+s}\left\|\mathbb{P}_{\ge 2^{-4}\sigma}a\right\|_\infty \|f\|_p. \end{aligned} \tag{A.7}$$

Similarly for A_2, since $|\nabla|^{-1}$ is bounded on L^p, we have

$$\left\||\nabla|^{-1+s}A_2\right\|_p = \left\||\nabla|^{-1}\big(\mathbb{P}_{\ge 2^{-4}\sigma}a\big)f\right\|_p \lesssim \left\|\mathbb{P}_{\ge 2^{-4}\sigma}af\right\|_p \le \left\|\mathbb{P}_{\ge 2^{-4}\sigma}a\right\|_\infty \|f\|_p.$$

So it suffices to show $\|\Delta_q a\|_\infty \lesssim 2^{-Nq}$ for all $2^q \ge 2^{-4}\sigma$. Recall from the definition of the periodic Littlewood-Paley projection that

$$\Delta_q a = \int_{\mathbb{T}^d} \varphi_q(x-y)a(y)dy,$$

where the frequency cutoffs satisfy

$$\left\||\nabla|^{-M}\varphi_q\right\|_2 \lesssim 2^{-qM}\|\varphi_q\|_2 \lesssim 2^{-qM+qd}. \tag{A.8}$$

By Parseval's theorem and Young's inequality,

$$\begin{aligned}\|\Delta_q a\|_\infty &= \left\| \int_{\mathbb{T}^d} |\nabla|^{-M}\varphi_q(\cdot - y)|\nabla|^M a(y)\,dy \right\|_\infty \\ &\le \left\| |\nabla|^{-M}\varphi_q \right\|_2 \left\| |\nabla|^M a \right\|_2.\end{aligned}$$

From L^2 boundedness of Riesz transform and the assumption on a it follows

$$\||\nabla|^M a\|_2 \lesssim \|\nabla^M a\|_2 \lesssim \|\nabla^M a\|_\infty \le \mu^M, \tag{A.9}$$

where we used $C_a = 1$. Thus, combining estimates (Equation A.9) and (Equation A.8) we find

$$\begin{aligned}\|\Delta_q a\|_\infty &\lesssim 2^{qd}\mu^M 2^{-qM} \\ &\le 2^{qd}\sigma^{(1-\theta)M}2^{-qM},\end{aligned}$$

where we used the fact that $\mu \le \sigma^{1-\theta}$. Now choosing λ_0 large enough so that $\sigma^{\theta/2} \ge 2^{4(1-\theta/2)}$ for all $\sigma \ge \lambda_0$, we obtain

$$\begin{aligned}\|\Delta_q a\|_\infty &\le 2^{qd}\sigma^{(1-\theta/2)M}2^{-4(1-\theta/2)}2^{-qM} \\ &\le 2^{qd}2^{-q\theta M/2},\end{aligned} \tag{A.10}$$

provided $2^q \ge 2^{-4}\sigma$. Choosing any $M \ge 2(N-d)/\theta$, in view of (Equation A.10), we have

$$\|\Delta_q a\|_\infty \lesssim 2^{-Nq} \quad \text{for all } 2^q \ge 2^{-4}\sigma.$$

Appendix A (Continued)

After taking a summation in q for $2^q \geq 2^{-4}\sigma$ we obtain

$$\|\mathbb{P}_{\geq 2^{-4}\sigma} a\|_\infty \lesssim \sigma^{-N}.$$

Then collecting all the estimates, we have

$$\begin{aligned}
\big\||\nabla|^{-1+s}(af)\big\|_p &\leq \big\||\nabla|^{-1+s}A_1\big\|_p + \big\||\nabla|^{-1+s}A_2\big\|_p \\
&\lesssim \sigma^{-1+s}\||\nabla|^{-s}(af)\|_p + \sigma^{-N}\|f\|_p.
\end{aligned}$$

Appendix B

SUPPLEMENTARY MATERIALS FOR CHAPTER 3

B.1 Del formulae in cylindrical coordinates

In this appendix, we collect some useful vector calculus identities concerning the cylindrical coordinates (see for example (1)).

Let f be a scaler function. The gradient of f

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_r + \frac{\partial f}{\partial z}\mathbf{e}_z. \tag{B.1}$$

For vector field $A = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_z\mathbf{e}_z$, its divergence

$$\operatorname{div} A = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}, \tag{B.2}$$

and curl

$$\begin{aligned}\nabla \times A =& \Big(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\Big)\mathbf{e}_r \\ &+ \Big(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\Big)\mathbf{e}_\theta \\ &+ \frac{1}{r}\Big(\frac{\partial (rA_r)}{\partial r} - \frac{\partial A_r}{\partial \theta}\Big)\mathbf{e}_z.\end{aligned} \tag{B.3}$$

Appendix B (Continued)

For two vector field A and B, the material derivative

$$\begin{aligned}(A\cdot\nabla)B =&\Big(A_r\frac{\partial B_r}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta}+A_z\frac{\partial B_r}{\partial z}-\frac{A_\theta B_\theta}{r}\Big)\mathbf{e}_r\\ &+\Big(A_r\frac{\partial B_\theta}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta}+A_z\frac{\partial B_\theta}{\partial z}+\frac{A_\theta B_r}{r}\Big)\mathbf{e}_\theta\\ &+\Big(A_r\frac{\partial B_z}{\partial r}+\frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta}+A_z\frac{\partial B_z}{\partial z}\Big)\mathbf{e}_z.\end{aligned} \tag{B.4}$$

B.2 Decay estimates for the Possion equation

Here we derive some decay estimates for solutions of the planar Poisson equation. Let $f\in C_c^\infty(\mathbb{R}^2)$ be radially symmetric with zero mean

$$\int_{\mathbb{R}^2} f\,dx = 0. \tag{B.5}$$

We show that

Lemma B.2.1. *Let h be the solution of*

$$\Delta h = f \quad \text{on } \mathbb{R}^2, \tag{B.6}$$

such that $|h|\to 0$ as $x\to\infty$. Then h is radially symmetric and $h\in W^{1,p}(\mathbb{R}^2)$ for $1<p\le\infty$.

Proof. Since the solution h is given explicitly by the Newton potential

$$h(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-y|)f(y)\,dy, \tag{B.7}$$

we only need to verify the decay estimates.

Appendix B (Continued)

The first decay $|h| \to 0$ as $x \to \infty$ follows from removing the mean

$$h(x) = -\frac{1}{2\pi} \int (\ln(|x-y|) - \ln(x)) f(y)\, dy,$$

and the Mean Value Theorem. It is clear that h is smooth on $\mathbb{R}^2$.

To show that $h \in W^{1,p}(\mathbb{R}^2)$ for $1 < p \leq \infty$, let us consider the Taylor expansion of $\ln(|x-y|)$

$$\ln(|x-y|) = \ln(|x|) - \frac{x \cdot y}{|x|^2} + \sum_{|\beta|=2} R_\beta(x,y) y^\beta, \tag{B.8}$$

where the remainder is given by

$$R_\beta(x,y) = \int_0^1 (1-t) D^\beta g(x - ty)\, dt, \tag{B.9}$$

with $g(x) = \ln(|x|)$ and $|\nabla^2 g| \lesssim \frac{1}{|x|^2}$.

Let us show that $h \in L^p$ for $1 < p \leq \infty$. Since f has zero mean and zero first moment due to radial symmetry, combining (Equation B.8) and (Equation B.7) we have

$$h(x) = -\frac{1}{2\pi} \sum_{|\beta|=2} \int R_\beta(x,y) y^\beta f(y)\, dy. \tag{B.10}$$

Then by Minkowski's inequality, we have

$$\|h\|_{L^p(\mathbb{R}^2)} \lesssim \sum_{|\beta|=2} \int \Big(\int \Big| R_\beta(x,y) \Big|^p dx \Big)^{\frac{1}{p}} |f(y)||y|^2 dy. \tag{B.11}$$

To estimate $R_\beta(x,y)$, we use Minkowski's inequality once again

$$\int \Big|R_\beta(x,y)\Big|^p dx \lesssim \Big[\int_0^1 \Big(\int \Big|D^\beta g(x-ty)\Big|^p dx\Big)^{\frac{1}{p}} dt\Big]^p.$$

Note that from definition,

$$\Big|D^\beta g(x-ty)\Big| \lesssim_\beta \frac{1}{|x-ty|^2} \tag{B.12}$$

and we get

$$\int \Big|R_\beta(x,y)\Big|^p dx \lesssim_p 1, \quad \text{for } x > 2R \text{ and } 1 < p \leq \infty,$$

where $R > 0$ is chosen sufficiently large such that $\operatorname{supp} f \subset B_R$ which together with the smoothness of h on B_{2R} implies

$$\|h\|_{L^p(\mathbb{R}^2)} < \infty, \quad \text{for } 1 < p \leq \infty.$$

The claim that $\nabla h \in L^p$ for $1 < p \leq \infty$ is easier since differentiating (Equation B.7) already gives a decay of $1/|x|$ in the kernel, and in this case just removing the mean is sufficient. $\square$

B.3 Essential discontinuities by Buckmaster-Vicol solutions

In this section, we show that it is possible to use the weak solution constructed in (8) to obtained essential discontinuities of positive measure in the energy profile. First, recall

Appendix B (Continued)

Theorem B.3.1 (Theorem 1.2 of (8)). *There exists $\beta > 0$, such that for any nonnegative smooth function $e(t) : [0,T] \to \mathbb{R}^+$, there exists $v \in C([0,T]; H^\beta(\mathbb{T}^3))$ a weak solution of the Navier-Stokes equations, such that $\int_{\mathbb{T}^3} |v(x,t)|^2 dx = e(t)$ for all $t \in [0,T]$.*

Let $e(t)$ be a nonnegative bump function supported on $(1/2, 1)$ such that $\max_t e(t) = 1$. Consider a weak solution $u \in C((0,1]; L^2(\mathbb{T}^3))$ such that on each interval $[2^{-n-1}, 2^{-n}]$, $u(t)$ is the Buckmaster-Vicol solution with energy profile $e(2^n t)$. As a consequence, we have

$$\liminf_{t\to 0-} \|u(t)\|_2^2 = 0, \qquad \limsup_{t\to 0-} \|u(t)\|_2^2 = 1.$$

Such an example does not extend to the whole interval $[0,1]$ as Theorem B.3.1 on its own does not guarantee the existence of the weak limit as $t \to 0+$ since there are no other available bounds as opposed to in the proof of Theorem 3.1.3 where we used (Equation 3.35).

However, we can modify this construction in the following way. Consider a Buckmaster-Vicol solution $u_n(t)$ on $[1/2, 1]$ with the energy profile $e_n(t) = 2^{-2n} e(t)$ and define (on $\mathbb{T}^3$)

$$u(t) = \sum_{n=0}^{\infty} 2^n u_n(2^n x, 2^{2n} t).$$

Then $u(t)$ is weakly continuous at $t = 0$, as the weak limit is zero. And it is a weak solution on $[0,1]$ with energy bounded by 1. Moreover,

$$\liminf_{t\to 0+} \|u(t)\|_2 = 0, \qquad \limsup_{t\to 0+} \|u(t)\|_2 = 1.$$

Appendix B (Continued)

Using a similar argument in the proof of Theorem 3.1.3, one can also use Buckmaster-Vicol solutions to obtain weak solutions whose discontinuities have positive measure in time. Note that this method does not produce jump discontinuities nor density of the set of discontinuities since the resulting solution is "intermittent" on the time interval.

Appendix C

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As stated in Chapter 1, the work in Chapter 2 and Chapter 3 has respectively already appeared in (58) and (57). Permission was not required to reuse the material in Chapter 3 while the proof of permission to reuse the material in Chapter 2 was obtained and attached in the following.

Appendix C (Continued)

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Stationary Solutions and Nonuniqueness of Weak Solutions for the Navier–Stokes Equations in High Dimensions

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	X. Luo "Stationary solutions and nonuniqueness of weak solutions for the Navier-Stokes equations in high dimensions." *Arch. Ration. Mech. Anal.*, 233(2):701–747, 2019.
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