Highly Structured Coalgebras and Comodules

by

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SUMMARY

Chapter 2 sets the definitions of our objects of studies: coalgebras and their comodules, both in the ordinary sense and in the ∞-categorical sense. Our main result here is Proposition 2.2.6 which shows that coalgebras in presentably symmetric monoidal ∞-categories are also presentable. As a consequence, we show in Theorem 2.3.16 that higher algebras are enriched over higher coalgebras in presentably symmetric monoidal ∞-categories. Although this result is not needed for the rest of the thesis, it can serve as a motivation on why to study coalgebras in the first place: they are part of the structure of algebras.

In Chapter 3, our main result is Theorem 3.3.2 which shows that weak monoidal Quillen equivalences of monoidal model categories lift to strong monoidal equivalences of symmetric monoidal ∞-categories. We apply the theorem to the Dold-Kan equivalence.

Chapter 4 presents the statement of the problem in full details: comparing homotopy coherent coassociative and cocommutative coalgebras with their strict analogue. We provide an example where rigidification does not hold in Example 4.1.2, and we show that rigidification does hold in the Cartesian case.

We explore the case of spectra in Chapter 5 and we show in Corollary 5.2.3 that rigidification of coassociative and cocommutative coalgebras does not hold in the current symmetric monoidal model categories of spectra.

We study rigidification for differentially graded comodules in Chapter 6. Our main result is Theorem 6.3.3 which shows that rigidification holds for simply connected coalgebras in non-negative chain complexes over a finite product of fields. We also observe in Theorem 6.4.7 that when a coalgebra C is equivalent to its dual algebra C^*, then rigidification of comodules also holds as comodules over C are equivalent to modules over C^*.

Chapter 7 shows that we can derive the cotensor product of comodules in the simply connected case in Theorem 7.5.2.

The Appendices are crucial in the arguments of Chapters 5 and 6. We essentially show that we can provide an inductive fibrant replacement of comodules in a very similar way as a Postnikov tower for a space does as seen in Corollary B.3.15.

CHAPTER 1

INTRODUCTION

Any $\mathbb{A}_\infty$-ring spectrum is homotopic to a strictly unital and associative ring spectrum, in some monoidal model category representing spectra, say symmetric spectra, as in (Hovey et al., 2000). Similarly, any $\mathbb{E}_\infty$-ring spectrum is homotopic to a strictly unital, associative and commutative ring spectrum. We are interested in this thesis in the dual question: can $\mathbb{A}_\infty$-coalgebras and $\mathbb{E}_\infty$-coalgebras be homotopic to strictly counital, coassociative and cocommutative coalgebras over the sphere spectrum? In other words, can we *rigidify* the comultiplication in spectra? We show in Corollary 5.2.3 and Corollary 5.2.4 that it is not the case. This follows from a previous result in (Péroux and Shipley, 2019).

We instead focus our attention to module spectra over a discrete commutative ring R, shift our rigidification question towards coalgebras and comodules in the derived category of R, and work instead with the model category of unbounded chain complexes of R-modules. Unfortunately, Example 4.1.2 hints that rigidification of coassociative coalgebras does not hold in the differential graded context. The main result of our paper, in Theorem 6.3.3, shows that we can always rigidify the coaction of comodules over any *simply connected* differential graded coalgebra over a finite product of fields, in the non-negative context.

Rigidification of algebras and modules usually holds in a good combinatorial monoidal model category, as seen in (Lurie, 2017, 4.1.8.4). Thus one could expect a good situation if we were working with coalgebras and comodules in "cocombinatorial" model categories. However, we still want to work with presentable categories and not "copresentable" categories. Instead, we investigate why the case of algebra works for a combinatorial model category. The key idea is that one can argue inductively cell by cell just as one can when studying CW-complexes of spaces. In model categories this is encoded in the small object argument. A dual theory would be instead of generalizing CW-complexes which present any space as a filtered colimit, we should now generalize Postnikov towers of spaces that present any space as a tower of spaces whose layers are easy computable. As we are not working with a copresentable model category, we do not have a "cosmall object argument" (see statement in Proposition A.1.7). Nevertheless we can provide an explicit ad-hoc Postnikov tower for certain types of comodules in chain complexes, see Corollary B.3.15. We introduce the notions of fibrantly generated model categories and Postnikov presentations of a model category following the work of (Hess, 2009) and (Bayeh et al., 2015) in Appendix A. This allows us to compute very explicitly homotopy limits of comodules in chain complexes.

Another formal consequence of our ad-hoc Postnikov towers is that we can now define a derived cotensor product of homotopy coherent comodules in the differential graded context.

CHAPTER 2

COALGEBRAS AND COMODULES IN HIGHER CATEGORY

We present here the formal definitions of coalgebras and comodules. Our main result in this chapter is that coalgebras of a presentably symmetric monoidal ∞-category form also a presentable ∞-category, see Proposition 2.2.6. We also observe that algebras are enriched over coalgebras in Theorem 2.3.16 in any presentably symmetric monoidal ∞-category, which generalizes the result in ordinary categories.

2.1 Classical Definitions and Results in Ordinary Category

Throughout this section, we let $(\mathsf{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category.

Definition 2.1.1. A *comonoid* (C, Δ, ε) in C consists of an object C in C together with a coassociative comultiplication $\Delta : C \to C \otimes C$, such that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id}_C \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}_C"] & C \otimes C \otimes C,
\end{tikzcd}$$

and admits a counit morphism $\varepsilon : C \to \mathbb{I}$ such that we have the following commutative diagram:

$$\begin{tikzcd}
C \otimes C \arrow[r, "\mathrm{id}_C \otimes \varepsilon"] & C \otimes \mathbb{I} \cong C \cong \mathbb{I} \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}_C"'] \\
& C. \arrow[ul, bend left, "\Delta"] \arrow[u, equal] \arrow[ur, bend right, "\Delta"'] &
\end{tikzcd}$$

The comonoid is *cocommutative* if the following diagram commutes:

$$\begin{tikzcd} C \otimes C \arrow[rr, "\tau"] & & C \otimes C \\ & C, \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

where τ is the twist isomorphism from the symmetric monoidal structure of C. A morphism of comonoids $f : (C, \Delta, \varepsilon) \to (C', \Delta', \varepsilon')$ is a morphism $f : C \to C'$ in C such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes C \arrow[r, "f \otimes f"] & C' \otimes C', \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ & \mathbb{I}. \end{tikzcd}$$

We denote $\mathsf{CoMon}(\mathsf{C})$ the category of comonoids in C. We denote $\mathsf{CoCMon}(\mathsf{C})$ the category of cocommutative comonoids in C.

Remark 2.1.2. Notice that we could have defined the category of comonoids with the help of the category of monoids by taking opposites: $\mathsf{CoMon}(\mathsf{C}) = (\mathsf{Mon}(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}}$.

Proposition 2.1.3 ((Porst, 2008, 2.6))**.** *Suppose the symmetric monoidal category* $(\mathsf{C}, \otimes, \mathbb{I})$ *is cocomplete. Then the category* $\mathsf{CoMon}(\mathsf{C})$ *is cocomplete and its associated forgetful functor* $U : \mathsf{CoMon}(\mathsf{C}) \to \mathsf{C}$ *is cocontinuous. Similarly, the category* $\mathsf{CoCMon}(\mathsf{C})$ *is cocomplete and its associated forgetful functor* $U : \mathsf{CoCMon}(\mathsf{C}) \to \mathsf{C}$ *is cocontinuous.*

We say that a category is *presentable* in the sense of *locally presentable* as in (Adámek and Rosický, 1994)

Proposition 2.1.4 ((Porst, 2008, 2.7)). *Let* $(\mathsf{C}, \otimes, \mathbb{I})$ *be a symmetric monoidal category. Suppose* C *is presentable and the tensor product* $\otimes$ *preserves filtered colimits in each variable. Then the categories* $\mathsf{CoMon}(\mathsf{C})$ *and* $\mathsf{CoCMon}(\mathsf{C})$ *are presentable.*

Combining the above results, we get the following.

Proposition 2.1.5. *Let* $(\mathsf{C}, \otimes, \mathbb{I})$ *be a symmetric monoidal category. Suppose* C *is presentable and the tensor product* $\mathsf{C} \otimes \mathsf{C} \to \mathsf{C}$ *preserves colimits in both variables. Then there exists a functor* $\mathsf{T}^{\vee} : \mathsf{C} \to \mathsf{CoMon}(\mathsf{C})$ *which forms the adjoint pair of functors:*

$$U : \mathsf{CoMon}(\mathsf{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{C} : \mathsf{T}^{\vee}.$$

Similarly, there exists a functor $\mathsf{S}^{\vee} : \mathsf{C} \to \mathsf{CoCMon}(\mathsf{C})$ *which forms the adjoint pair of functors:*

$$U : \mathsf{CoCMon}(\mathsf{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{C} : \mathsf{S}^{\vee}.$$

Definition 2.1.6. From Proposition 2.1.5, for any object X in C, we say that $\mathsf{T}^{\vee}(X)$ is the *cofree comonoid generated by* X, and $\mathsf{S}^{\vee}(X)$ is the *cofree cocommutative comonoid generated by* X.

Remark 2.1.7. Very little is known about these cofree functors in general. For explicit formulas in particular cases, we refer the interested reader to (Michaelis, 2003), (Getlzer and Goerss, 1999), and (Anel and Joyal, 2013).

Definition 2.1.8. Let (C, Δ, ε) be a comonoid in C. A *right comodule* (X, ρ) *over* C, or a *right* C*-comodule*, is an object X in C together with a coassociative and counital right coaction morphism $\rho : X \to X \otimes C$ in C, i.e., the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\rho"'] & X \otimes C \arrow[d, "\rho\otimes \mathsf{id}_C"] \\ X \otimes C \arrow[r, "\mathsf{id}_X \otimes \Delta"'] & X \otimes C \otimes C, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\rho"] \arrow[ddr, equal] & X \otimes C \arrow[d, "\mathsf{id}_X \otimes \varepsilon"] \\ & X \otimes \mathbb{I} \arrow[d, "\cong"] \\ & X. \end{tikzcd}$$

The category of right C-comodules in C is denoted $\mathsf{CoMod}_C(\mathsf{C})$. Similarly, we can define the category of left C-comodules where objects are endowed with a left coassociative counital coaction $X \to C \otimes X$ and we denote the category by ${}_C\mathsf{CoMod}(\mathsf{C})$.

Remark 2.1.9. If C is a cocommutative comonoid in C the categories of left and right comodules over C are naturally isomorphic: ${}_C\mathsf{CoMod}(\mathsf{C}) \cong \mathsf{CoMod}_C(\mathsf{C})$. In this case, we omit to mention if the coaction is left or right.

Remark 2.1.10. Since a comonoid in C is a monoid in C^{op}, then we can define the category of right comodules as modules in the opposite category: $\mathsf{CoMod}_C(\mathsf{C}) = (\mathsf{Mod}_C(\mathsf{C}^{\mathsf{op}}))^{\mathsf{op}}$, and similarly for the left case.

Proposition 2.1.11. *Let* $(\mathsf{C}, \otimes, \mathbb{I})$ *be symmetric monoidal category. Suppose that* C *is presentable and the tensor product* $\otimes$ *preserves filtered colimits in each variable. Then for any* *choice of comonoid in C in C, the category of right C-comodules (or left C-comodules) in C is presentable, and we have an adjunction:*

$$\begin{tikzcd} \mathsf{CoMod}_C(\mathsf{C}) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{C}. \arrow[l, shift left, "-\otimes C"] \end{tikzcd}$$

Proof. Notice that $\mathsf{CoMod}_C(\mathsf{C})$ is the category of coalgebras over the comonad $-\otimes C : \mathsf{C} \to \mathsf{C}$. Apply (Adámek and Rosický, 1994, 2.78, 2.j). □

Definition 2.1.12. Following Proposition 2.1.11, for any object X in C, we say that $X \otimes C$ is the *cofree right C-comodule generated by X*. Similarly, we can define the *cofree left C-comodule generated by X as $C \otimes X$.*

Recall that given a commutative monoid R in C, the category of (right) modules over R in C, denoted $\mathsf{Mod}_R(\mathsf{C})$ is a symmetric monoidal category, where the unit is R and the monoidal product is denoted $\otimes_R$ and is defined as the coequalizer:

$$\begin{tikzcd} M \otimes R \otimes N \arrow[r, shift left, "\mathsf{id}_M \otimes (\alpha_N \circ \tau)"] \arrow[r, shift right, "\alpha_M \otimes \mathsf{id}_N"'] & M \otimes N, \end{tikzcd}$$

where $\alpha_M : M \otimes R \to M$ and $\alpha_N : N \otimes R \to N$ are the (right) R-actions on M and N respectively. This leads to the following definition.

Definition 2.1.13. Let R be a commutative monoid in C. A *coalgebra* (C, Δ, ε) *over R in* C, or an *R-coalgebra in* C, is a comonoid (C, Δ, ε) in the symmetric monoidal category

$(\mathsf{Mod}_R(\mathsf{C}), \otimes_R, R)$. We denote the category of R-coalgebras by $\mathsf{CoAlg}_R(\mathsf{C})$. We denote the category of cocommutative R-coalgebras by $\mathsf{CoCAlg}_R(\mathsf{C})$.

Remark 2.1.14. Notice that $\mathsf{CoAlg}_R(\mathsf{C})$ is simply the category $\mathsf{CoMon}(\mathsf{Mod}_R(\mathsf{C}))$.

2.2 Definitions and Preliminary Results in Higher Categories

The following definitions and results are generalizations of Section 3.1 of (Lurie, 2018a), which was focused on the case of $\mathbb{E}_\infty$-coalgebras. We define and extend the results for coalgebras over any ∞-operad. Let Fin_* denote the category of all finite pointed sets, as in (Lurie, 2017, 2.0.0.2, 2.0.0.3). Recall the definition of a coCartesian fibration of simplicial sets in (Lurie, 2009, 2.4.2.1).

Definition 2.2.1 ((Lurie, 2017, 2.0.0.7))**.** A *symmetric monoidal ∞-category* $\mathcal{C}^\otimes$ is a coCartesian fibration of simplicial sets: $p : \mathcal{C}^\otimes \longrightarrow \mathcal{N}(\mathsf{Fin}_*)$, such that, for each $n \geq 0$, the maps in $\rho^i : \langle n \rangle \to \langle 1 \rangle$ induce a equivalences $(\rho^i_!)_{i=1}^n : \mathcal{C}^\otimes_{\langle n \rangle} \xrightarrow{\simeq} \prod_{i=1}^n \mathcal{C}^\otimes_{\langle 1 \rangle}$. We denote its underlying ∞-category by $\mathcal{C}$ which is equivalent to the fiber $\mathcal{C}^\otimes_{\langle 1 \rangle}$, as in (Lurie, 2017, 2.1.2.20).

The above definition can be generalized, where instead of working with the commutative operad $\mathcal{N}(\mathsf{Fin}_*)$, one can replace it by an ∞-operad $\mathcal{O}^\otimes$ as in (Lurie, 2017, 2.1.1.10). Then we define $\mathcal{C}$ to be an *$\mathcal{O}$-monoidal ∞-category* as in (Lurie, 2017, 2.1.2.15).

Definition 2.2.2. Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal ∞-category. An *$\mathcal{O}$-coalgebra object in $\mathcal{C}$* is an $\mathcal{O}$-algebra object in $\mathcal{C}^{\mathsf{op}}$. The ∞-category of $\mathcal{O}$-coalgebra objects in $\mathcal{C}$ is defined as the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) := (\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}$. More generally, given any map $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ of ∞-operads , we define the ∞-category of $\mathcal{O}'$-coalgebra in $\mathcal{C}$ as $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = (\mathcal{A}\ell g_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}$.

If we pick the associative operad $\mathcal{O} = \mathbb{A}_\infty$ or the commutative operad $\mathcal{O} = \mathbb{E}_\infty$, we have generalized the definition of coassociative and cocommutative coalgebras in ordinary categories. See more details in Chapter 3.

Proposition 2.2.3 ((Lurie, 2017, 3.2.4.4))**.** *If $\mathcal{C}$ is a symmetric monoidal ∞-category and $\mathcal{O}$ is any ∞-operad, then the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ inherits a symmetric monoidal structure, given by pointwise tensor product. Dually, the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ inherits a symmetric monoidal structure, given by pointwise tensor product.*

Remark 2.2.4. If $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞-category, then $\mathcal{C}^{\mathsf{op}}$ can be given an $\mathcal{O}$-monoidal structure uniquely up to contractible choice, as in (Lurie, 2017, 2.4.2.7). One can use the work of (Barwick et al., 2018) to give an explicit choice of the coCartesian fibration for $\mathcal{C}^{\mathsf{op}}$. For instance, let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be the coCartesian fibration associated to the symmetric monoidal structure of $\mathcal{C}$. Then straightening of the coCartesian fibration gives a functor $F : \mathcal{O}^{\otimes} \longrightarrow \widehat{\mathcal{C}at}_\infty$, where $\widehat{\mathcal{C}at}_\infty$ is the ∞-category of ∞-categories, as in (Lurie, 2017, 3.0.0.5). Then, by (Barwick et al., 2018, 1.5) the functor F also classifies a Cartesian fibration $p^{\vee} : (\mathcal{C}^{\otimes})^{\vee} \longrightarrow (\mathcal{O}^{\otimes})^{\mathsf{op}}$. An explicit construction is given in (Barwick et al., 2018, 1.7). The opposite map $(p^{\vee})^{\mathsf{op}} : ((\mathcal{C}^{\otimes})^{\vee})^{\mathsf{op}} \longrightarrow \mathcal{O}^{\otimes}$ is a coCartesian fibration that is classified by: $\mathcal{O}^{\otimes} \xrightarrow{F} \widehat{\mathcal{C}at}_\infty \xrightarrow{\mathsf{op}} \widehat{\mathcal{C}at}_\infty$. One can check that the fiber of $(p^{\vee})^{\mathsf{op}}$ over X in $\mathcal{O}$ is equivalent to $(\mathcal{C}_X)^{\mathsf{op}}$, and thus gives $\mathcal{C}^{\mathsf{op}}$ a $\mathcal{O}$-monoidal structure. We see that $\mathcal{O}$-coalgebras are sections of the Cartesian fibration $p^{\vee} : \mathcal{C}^{\otimes} \to (\mathcal{O}^{\otimes})^{\mathsf{op}}$ that sends inert morphisms in $(\mathcal{O}^{\otimes})^{\mathsf{op}}$ to $p^{\vee}$-Cartesian morphisms in $\mathcal{C}^{\otimes}$.

Proposition 2.2.5. *Let $\mathcal{C}$ be a $\mathcal{O}$-monoidal ∞-category and let K be a simplicial set. If, for each X in $\mathcal{O}$, the fiber $\mathcal{C}_X$ admits K-indexed colimits, then the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$*

admits K-indexed colimits, and the forgetful functor $U : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ preserves K-indexed colimits.

Proof. Apply (Lurie, 2017, 3.2.2.5) to the coCartesian $(p^{\vee})^{\mathsf{op}} : ((\mathcal{C}^{\otimes})^{\vee})^{\mathsf{op}} \longrightarrow \mathcal{O}^{\otimes}$ defined in Remark 2.2.4. □

The following dualizes the result on algebras in (Lurie, 2017, 3.2.3.5).

Proposition 2.2.6. *Let $\mathcal{O}^{\otimes}$ be an essentially small ∞-operad. Let $\mathcal{C}$ be an $\mathcal{O}$-monoidal ∞-category defined via a coCartesian fibration $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. Assume that, for each X in $\mathcal{O}$, the fiber $\mathcal{C}_X$ is presentable. Assume further that p is compatible with small colimits. Then $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is a presentable ∞-category.*

Proof. We apply (Lurie, 2009, 5.4.7.11) to the Cartesian fibration $p^{\vee} : (\mathcal{C}^{\otimes})^{\vee} \to \mathcal{O}^{\mathsf{op}}$ described in Remark 2.2.4. For any object X in $\mathcal{O}^{\otimes}$, the fiber of $p^{\vee}$ over X is equivalent to the fiber $\mathcal{C}_X$ of p over X. By (Lurie, 2017, 3.2.3.4), these fibers are accessible and $\mathcal{C}_X \to \mathcal{C}_{X'}$ are accessible maps. Thus the induced maps $\mathcal{C}_{X'}^{\vee} \to \mathcal{C}_X^{\vee}$ are also accessible by (Barwick et al., 2018, 1.3). □

Remark 2.2.7. In general, if $\mathcal{C}$ is compactly generated, there is no guarantee that $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is compactly generated. However, the *fundamental theorem of coalgebras* (see (Sweedler, 1969, II.2.2.1) or (Getlzer and Goerss, 1999, 1.6)) states that if $\mathcal{C}$ is (the nerve of) vector spaces, or chain complexes over a field, then $\mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}(\mathcal{C})$ is compactly generated and the forgetful functor $U : \mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}(\mathcal{C}) \to \mathcal{C}$ preserves and reflects compact objects. From (Adámek and Porst, 2004, 4.2), if κ is an uncountable regular cardinal, we conjecture that the fundamental theorem of coalgebra can be expended in the following sense. If $\mathcal{C}$ is κ-compactly generated then $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is κ-compactly generated and the forgetful functor preserves and reflects κ-compact objects.

In some cases, the ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is not mysterious. We recall the following result from Lurie. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, and denote by $\mathcal{C}_{\mathsf{fd}}$ the full subcategory spanned by the dualizable objects, see (Lurie, 2017, 4.6.1). It inherits a symmetric monoidal structure. For each dualizable object X, we denote $X^{\vee}$ its dual and this defines a contravariant endofunctor on $\mathcal{C}_{\mathsf{fd}}$.

Proposition 2.2.8 ((Lurie, 2018a, 3.2.4))**.** *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then taking dual objects assigns an equivalence of symmetric monoidal ∞-categories $(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \xrightarrow{\simeq} \mathcal{C}_{\mathsf{fd}}$. In particular, for any ∞-operad $\mathcal{O}$, we obtain an equivalence $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})$ of symmetric monoidal ∞-categories.*

One particular choice of ∞-operad can be the operad of left modules $\mathcal{LM}$ and right modules $\mathcal{RM}$, as in (Lurie, 2017, 4.2.1.13, 4.2.1.36). In particular, given $\mathcal{C}$ a monoidal ∞-category, and A an $\mathbb{A}_\infty$-algebra, we denote ${}_A\mathcal{M}od(\mathcal{C})$ the ∞-category of left A-modules, instead of $\mathcal{LM}od_A(\mathcal{C})$ as Lurie does. We similarly denote $\mathcal{M}od_A(\mathcal{C})$ the ∞-category of right A-modules.

Definition 2.2.9. Let $\mathcal{C}$ be a monoidal ∞-category. Let C be an $\mathbb{A}_\infty$-coalgebra in $\mathcal{C}$. Then define the category of right C-comodules in $\mathcal{C}$ as:

$$\mathcal{C}o\mathcal{M}od_C(\mathcal{C}) := (\mathcal{M}od_C(\mathcal{C}^{\mathsf{op}}))^{\mathsf{op}}.$$

We define the ∞-category of left C-comodules ${}_C\mathcal{C}o\mathcal{M}od(\mathcal{C})$ similarly.

2.3 Higher Algebras Enrichment in Higher Coalgebras

Classically, in any presentable symmetric monoidal closed ordinary category, the category of monoids is enriched, tensored and cotensored in the symmetric monoidal category of comonoids. This was proven in (Hyland et al., 2017, 5.2) and (Vasilakopoulou, 2019, 2.18). See also the example of the differential graded case in (Anel and Joyal, 2013). We show here in Theorem 2.3.16 an equivalent statement in ∞-categories.

An ∞-category shall be defined to be *enriched* over a symmetric monoidal ∞-category in the sense of (Hinich, 2018, 3.1.2), or in the sense of (Gepner and Haugseng, 2015). By (Hinich, 2018, 3.4.4) they are equivalent. An ∞-category is *tensored* or *cotensored* over a monoidal ∞-category in the classical sense of (Lurie, 2017, 4.2.1.19) or (Lurie, 2017, 4.2.1.28) respectively. Our desired enrichment in Theorem 2.3.16 will also be enriched in the sense of (Lurie, 2017, 4.2.1.28), see Remark 2.3.17 below. It is conjectured in (Gepner and Haugseng, 2015) that the definitions of enrichment of Lurie and Gepner-Haugseng are equivalent.

Throughout this section, let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. It is in particular closed, and thus the strong symmetric monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces a lax symmetric monoidal functor $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ characterized by the universal mapping property $\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(X, [Y, Z])$, for all X, Y, and Z in $\mathcal{C}$. In other words, the functor $- \otimes Y : \mathcal{C} \to \mathcal{C}$ is a left adjoint to $[Y, -] : \mathcal{C} \to \mathcal{C}$.

Let $\mathcal{O}^{\otimes}$ be an essentially small ∞-operad. From the lax symmetric monoidal structure of $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$, we obtain a functor $[-,-] : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}}) \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$. By definition of $\mathcal{O}$-coalgebras, we identify $\mathcal{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathsf{op}})$ simply as $\mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$, and thus obtain the following definition.

Definition 2.3.1. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$[-,-] : \mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{Alg}_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{Alg}_{\mathcal{O}}(\mathcal{C}),$$

the *Sweedler cotensor*. In the literature, it is sometimes called the *convolution algebra* or the *convolution product*, see (Sweedler, 1969, 4.0) and (Anel and Joyal, 2013).

Remark 2.3.2. The term convolution product stems from the algebra structure that generalizes the usual convolution product in representation theory. See (Hazewinkel et al., 2010, 2.12.3). It also generalizes the classical convolutions of real functions of compact support, see (Hazewinkel et al., 2010, 2.14.4).

Example 2.3.3. The Sweedler cotensor in the case where $\mathcal{O} = \mathbb{E}_\infty$ and $\mathcal{C}$ is the ∞-category of R-modules in a symmetric monoidal ∞-category, where R is an $\mathbb{E}_\infty$-algebra, was presented in (Lurie, 2018b, Section 1.3.1).

Example 2.3.4. Let $\mathbb{I}$ be the unit of the symmetric monoidal structure of $\mathcal{C}$. Let C be any $\mathcal{O}$-coalgebra, then the Sweedler cotensor $[C, \mathbb{I}]$ is simply the *linear dual* C^*, which is always an $\mathcal{O}$-algebra. Thus the linear dual functor $(-)^* : \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$ lifts to the particular Sweedler cotensor $(-)^* = [-, \mathbb{I}] : \mathcal{CoAlg}_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{Alg}_{\mathcal{O}}(\mathcal{C})$. Here we recover the classical result that the dual of a coalgebra is always an algebra, see (Sweedler, 1969, 1.1.1).

Remark 2.3.5. In a presentably symmetric monoidal ∞-category $\mathcal{C}$, an object X is dualizable precisely if X is equivalent to its linear dual X^*. Thus, the above defined functor $(-)^* : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ coincides with the equivalence of Proposition 2.2.8 $(-)^\vee : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}} \xrightarrow{\simeq} \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})$, when we restrict $(-)^*$ to the subcategory $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}_{\mathsf{fd}})^{\mathsf{op}}$.

Since $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ is a continuous functor, and limits in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ are computed in $\mathcal{C}$, we get that the Sweedler cotensor is a continuous functor. Fix C an $\mathcal{O}$-coalgebra in $\mathcal{C}$. Then the continuous functor $[C,-] : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is accessible (as filtered colimits in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ are computed in $\mathcal{C}$) and is between presentable ∞-categories. Therefore, by the adjoint functor theorem (Lurie, 2009, 5.5.2.9), the functor $[C,-]$ admits a left adjoint denoted $C \triangleright - : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Definition 2.3.6. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$- \triangleright - : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}),$$

the *Sweedler tensor.* Previously, it was called the *Sweedler product* in (Anel and Joyal, 2013) and later in (Vasilakopoulou, 2019). For C a fixed $\mathcal{O}$-coalgebra, the functor $C \triangleright -$ is left adjoint to $[C,-]$ and we have in particular the equivalence of spaces:

$$\mathcal{A}\ell g_{\mathcal{O}}(C \triangleright A, B) \simeq \mathcal{A}\ell g_{\mathcal{O}}(A, [C,B]),$$

for any $\mathcal{O}$-algebras A and B.

Example 2.3.7. In (Anel and Joyal, 2013, 3.4.1), an explicit formula of the Sweedler tensor was given in the discrete differential graded case.

Fix now A an $\mathcal{O}$-algebra in $\mathcal{C}$. The continuous functor $[-,A] : (\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}))^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ induces a cocontinuous functor on its opposites $[-,A]^{\mathsf{op}} : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to (\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}))^{\mathsf{op}}$. The cocontinuous functor is from a presentable ∞-category to an essentially locally small ∞-category: as the opposite of an essentially locally small ∞-category is also essentially locally small, and presentable ∞-category are always essentially locally small. Thus, by the adjoint functor theorem (Lurie, 2009, 5.5.2.9, 5.5.2.10), the functor $[-,A]^{\mathsf{op}}$ admits a right adjoint $\{-,A\} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Definition 2.3.8. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. We call the induced functor:

$$\{-,-\} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$$

the *Sweedler hom*. For A and B any $\mathcal{O}$-algebra in $\mathcal{C}$, the $\mathcal{O}$-coalgebra $\{A,B\}$ is called the *universal measuring coalgebra in $\mathcal{C}$ of A and B.* See (Sweedler, 1969, 7.0) for the discrete case in vector spaces. In particular, if we fix A, we obtain that $\{-,A\}$ is the right adjoint of $[-,A]^{\mathsf{op}}$ and we have the equivalence of spaces:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C,\{A,B\}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(B,[C,A]),$$

for any $\mathcal{O}$-coalgebra C.

Example 2.3.9. Let $\mathbb{I}$ be the unit of the symmetric monoidal structure of $\mathcal{C}$. Then, for any $\mathcal{O}$-algebra A in $\mathcal{C}$, define A° to be the measuring coalgebra $\{A, \mathbb{I}\}$. It is called the *Sweedler dual* or *finite dual* of the $\mathcal{O}$-algebra A in $\mathcal{C}$. In particular, we obtain a functor $(-)^\circ = \{-, \mathbb{I}\}^{\mathsf{op}} : \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$, which is the left adjoint of the linear dual functor $(-)^* : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ defined in Example 2.3.4. In particular, we have the equivalence of spaces: $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(A, C^*) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C, A^\circ)$, for any $\mathcal{O}$-coalgebra C and any $\mathcal{O}$-algebra A. This was proven in the discrete classical case of vector spaces in (Sweedler, 1969, 6.0.5). By Remark 2.3.5, when the $\mathcal{O}$-algebra A is dualizable in $\mathcal{C}$, then $A^\circ \simeq A^*$ as an object in $\mathcal{C}$.

Example 2.3.10. The origin of the term *measure* could be due to the following example. Let X be a compact Hausdorff space. Then the Sweedler dual of the algebra of continuous real functions $\mathsf{Map}(X, \mathbb{R})$ is equivalent to finitely supported measures on X, see (Hazewinkel et al., 2010, 2.12.10).

We shall explain where the term *universal* measuring is coming from. Recall that the internal hom property of $\mathcal{C}$ implies that, for any X, Y and Z objects in $\mathcal{C}$, there is an equivalence of spaces: $\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(Y, [X, Z])$. The Sweedler cotensor guarantees conditions for an $\mathcal{O}$-algebra structure on $[X, Z]$. The following is a generalization of (Sweedler, 1969, 7.0.1) and (Anel and Joyal, 2013, 3.3.1).

Definition 2.3.11. Let $\mathcal{C}$ and $\mathcal{O}$ be as above. Let C be an $\mathcal{O}$-coalgebra in $\mathcal{C}$, and A and B be $\mathcal{O}$-algebras in $\mathcal{C}$. Let $\psi : C \otimes A \to B$ be a map in $\mathcal{C}$. We say that (C, ψ) *measures A to B* (or *(C, ψ) is a measuring of A to B*) if the adjoint map $A \to [C, B]$ is a map of $\mathcal{O}$-algebras in $\mathcal{C}$.

We give examples generalized from (Anel and Joyal, 2013).

Example 2.3.12 ((Anel and Joyal, 2013, 3.3.3))**.** If $\mathbb{I}$ is the unit of the symmetric monoidal structure of $\mathcal{C}$, then a map $\mathbb{I} \otimes A \to B$ in $\mathcal{C}$ is a measuring of A to B if and only if it is a map in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.

Example 2.3.13 ((Anel and Joyal, 2013, 3.3.4))**.** The adjoint of the identity map on $[C, A]$ is a map $C \otimes [C, A] \to A$ and is always a measuring. In particular, the evaluation $C \otimes C^* \to \mathbb{I}$ is always a measuring of C^* to $\mathbb{I}$. Similarly $A^\circ \otimes A \to \mathbb{I}$ is a measuring of A to $\mathbb{I}$.

By definition of the Sweedler hom, as we have $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(C, \{A, B\}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})(B, [C, A])$, we see that the $\mathcal{O}$-coalgebra $\{A, B\}$, together with the natural map $\{A, B\} \otimes A \to B$ (adjoint of the identity over $\{A, B\}$), is indeed the universal measuring algebra of A to B, in the following sense. Given any other measuring (C, ψ) of A to B, there exists a unique (up to contractible choice) map $C \to \{A, B\}$ of $\mathcal{O}$-coalgebras in $\mathcal{C}$ such that the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} C \otimes A \arrow[d, dashed] \arrow[dr, "\psi"] & \\ \{A, B\} \otimes A \arrow[r] & B. \end{tikzcd}$$

Remark 2.3.14. Following (Anel and Joyal, 2013, 3.3.6), we see that, given maps $A' \to A$ and $B \to B'$ in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, a map $C' \to C$ in $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, together with a map $A \to [C, B]$ in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, we obtain the following map in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$: $A' \longrightarrow A \longrightarrow [C, B] \longrightarrow [C', B']$. This shows that the space of measurings provides a functor:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}} \times \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \longrightarrow \mathcal{S},$$

that is representable in each variable with respect to the Sweedler hom, tensor and cotensor.

Let $\mathcal{D}^{\otimes}$ be a monoidal ∞-category. Its *reverse*, denoted $(\mathcal{D}^{\otimes})^{\mathsf{rev}}$ or simply $\mathcal{D}^{\mathsf{rev}}$, is defined in (Hinich, 2018, 2.13.1). Essentially, $\mathcal{D}$ and $\mathcal{D}^{\mathsf{rev}}$ have the same underlying ∞-category but the tensor $X \otimes Y$ in $\mathcal{D}^{\mathsf{rev}}$ corresponds precisely to $Y \otimes X$ in $\mathcal{D}$. Left modules over $\mathcal{D}$ corresponds to right modules over $\mathcal{D}^{\mathsf{rev}}$. If $\mathcal{D}$ is symmetric, then $\mathcal{D}^{\mathsf{rev}} = \mathcal{D}$ by (Hinich, 2018, 2.13.4). We shall be interested with the *reverse opposite*, denoted $\mathcal{D}^{\mathsf{rop}} = (\mathcal{D}^{\mathsf{op}})^{\mathsf{rev}}$, of a monoidal ∞-category $\mathcal{D}$. The following is a generalization of the discrete ordinary case (Hyland et al., 2017, 5.1).

Lemma 2.3.15. *Let $\mathcal{C}$ and $\mathcal{O}$ be as above. Then the Sweedler cotensor endows the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ the structure of a right module over the reverse opposite of the (symmetric) monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$.*

Proof. Notice first that $\mathcal{C}$ is a right module over its reverse opposite $\mathcal{C}^{\mathsf{rop}}$ via its internal hom $[-,-] : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$, as it is lax symmetric monoidal. Therefore, by Proposition 2.2.3, the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is a right module over $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rop}})$ via the Sweedler cotensor. Since $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rev}}) \simeq \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{rev}}$, then $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}^{\mathsf{rop}}) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{rop}}$. $\square$

Since $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is a presentably symmetric monoidal ∞-category, it is enriched over itself by (Gepner and Haugseng, 2015, 7.4.10). We denote $\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}(D, E)$ the $\mathcal{O}$-coalgebra in $\mathcal{C}$ which classifies coalgebra maps from D to E, characterized by the universal mapping property:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})\Big(C \otimes D, E\Big) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \left(C, \underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}(D, E)\right).$$

Theorem 2.3.16. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $\mathcal{O}$ be an essentially small ∞-operad. The ∞-category of $\mathcal{O}$-algebras $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over the symmetric monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, via the Sweedler hom. Moreover it is tensored and cotensored respectively using the Sweedler tensor and Sweedler cotensor. In particular, we have an equivalence of $\mathcal{O}$-coalgebras:*

$$\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}\Big(C, \{A, B\}\Big) \simeq \Big\{A, [C, B]\Big\} \simeq \Big\{C \triangleright A, B\Big\},$$

for any $\mathcal{O}$-coalgebra C in $\mathcal{C}$ and any $\mathcal{O}$-algebras A and B in $\mathcal{C}$.

Proof. By Lemma 2.3.15, the ∞-category $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ is a left module over the symmetric monoidal ∞-category $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, via $[-,-]^{\mathsf{op}}$ the opposite of the Sweedler cotensor, such that $[-, A]^{\mathsf{op}} : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \to \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ admits a right adjoint $\{-, A\}$ for all A in $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$. By (Hinich, 2018, 6.3.1, 7.2.1) (see also (Gepner and Haugseng, 2015, 7.4.9)) this shows that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})^{\mathsf{op}}$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, with tensor $[-,-]^{\mathsf{op}}$. Thus, by (Hinich, 2018, 6.2.1), we get that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$, with cotensor $[-,-]$. □

Remark 2.3.17. We could have applied (Lurie, 2017, 4.2.1.33) in the proof of Theorem 2.3.16 to show that $\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ is enriched over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$ in the sense of Lurie, see (Lurie, 2017, 4.2.1.28). It is conjectured that the definitions of enrichment are equivalent in (Gepner and Haugseng, 2015).

Remark 2.3.18. The previous theorem shows that we can enrich the equivalence in Example 2.3.9 to an equivalence of $\mathcal{O}$-coalgebras in $\mathcal{C}$:

$$\underline{\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})}\Big(C, A^{\circ}\Big) \simeq \Big\{A, C^{*}\Big\} \simeq \Big(C \triangleright A\Big)^{\circ},$$

for any $\mathcal{O}$-coalgebra C and any $\mathcal{O}$-algebra A.

A particular consequence of the theorem gives the following adjunction which was shown in (Anel and Joyal, 2013, 5.3.14) to generalize the algebraic cobar-bar adjunction.

Corollary 2.3.19. *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞-category. Let $\mathcal{O}$ be an essentially small ∞-category. Let A be an $\mathcal{O}$-algebra in $\mathcal{C}$. Then there is an adjunction of enriched ∞-categories over $\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C})$:*

$$- \triangleright A : \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{A}\ell g_{\mathcal{O}}(\mathcal{C}) : \{A, -\}.$$

CHAPTER 3

THE DWYER-KAN LOCALIZATION OF A MODEL CATEGORY

Let M be a model category and W its morphism class of weak equivalences. Recall that the homotopy category $\mathsf{Ho}(\mathsf{M})$, associated to M, is an ordinary category obtained by inverting all weak equivalences, and can also be denoted $\mathsf{M}[\mathsf{W}^{-1}]$, see (Hovey, 1999, 1.2.1, 1.2.10). However, the higher homotopy information is lost in $\mathsf{Ho}(\mathsf{M})$. Dwyer and Kan, in (Dwyer and Kan, 1980), suggested instead a simplicial category $\mathsf{L}^H(\mathsf{M}, \mathsf{W})$ sometimes called the hammock localization of M, that retains the higher information. We will not define the hammock localization $\mathsf{L}^H(\mathsf{M}, \mathsf{W})$, but invite the reader to read the explicit definition in (Dwyer and Kan, 1980, 2.1). The idea is translated into ∞-categories by Lurie in (Lurie, 2017) as we see below. Following (Hinich, 2016), we shall prefer the less confusing term of *Dwyer-Kan localization* instead of *underlying ∞-category* of a model category, motivated by Remark 3.1.3.

3.1 The General Definition

We first start by some generality.

Definition 3.1.1 ((Lurie, 2017, 1.3.4.1)). Let $\mathcal{C}$ be an ∞-category and fix a collection $\mathcal{W} \subseteq \mathsf{Hom}_{\mathsf{sSet}}(\Delta^1, \mathcal{C})$ of morphisms in $\mathcal{C}$. The *Dwyer-Kan localization of $\mathcal{C}$ with respect to the collection $\mathcal{W}$* is an ∞-category, denoted $\mathcal{C}[\mathcal{W}^{-1}]$, together with a functor $f : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ that respects the following universal property.

(U) For any other ∞-category $\mathscr{D}$, the functor f induces an equivalence of ∞-categories:

$$\mathscr{F}un(\mathscr{C}[\mathscr{W}^{-1}],\mathscr{D}) \xrightarrow{\simeq} \mathscr{F}un^{\mathscr{W}}(\mathscr{C},\mathscr{D}),$$

where $\mathscr{F}un^{\mathscr{W}}(\mathscr{C},\mathscr{D})$ is the full subcategory of functors $\mathscr{C} \to \mathscr{D}$ that sends morphisms in $\mathscr{W}$ to equivalences in $\mathscr{D}$.

The Dywer-Kan localization $\mathscr{C}[\mathscr{W}^{-1}]$ always exists, for any choice of $\mathscr{C}$ and $\mathscr{W}$, see (Lurie, 2017, 1.3.4.2), and is unique up to contractible choice. We shall be more interested in the case when $\mathscr{C} = \mathscr{N}(\mathsf{M})$ for some model category M.

Definition 3.1.2 ((Lurie, 2017, 1.3.4.15))**.** Let M be a model category and W its class of weak equivalences. We call $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$ the *Dwyer-Kan localization of* M *with respect to* W as in Definition 3.1.1, where we abuse notation and let W denote the induced class of morphisms in $\mathscr{N}(\mathsf{M})$.

Notice that the homotopy category of $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$ is precisely the category $\mathsf{Ho}(\mathsf{M})$.

Remark 3.1.3. Since simplicial categories represents ∞-categories, the hammock localisation simplicial category $\mathsf{L}^H(\mathsf{M},\mathsf{W})$ is a model for the Dwyer-Kan localization $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$. More presicely, by (Lurie, 2009, 2.2.5.1), there is a Quillen equivalence between the category of simplicial sets sSet endowed with the Joyal model structure and the category of simplicial categories sCat endowed with the Bergner model structure:

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, shift left=1.5, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{sCat}. \arrow[l, shift left=1.5, "\mathfrak{N}"] \end{tikzcd}$$

The functor $\mathfrak{N} : \mathsf{sCat} \to \mathsf{sSet}$ is the homotopy coherent nerve, or the simplicial nerve, as in (Lurie, 2009, 1.1.5.5). After a fibrant replacement, the functor $\mathfrak{N}$ sends $\mathsf{L}^H(\mathsf{M}, \mathsf{W})$ to the equivalence class of $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$, as seen in (Hinich, 2016, 1.3.1).

Remark 3.1.4. As noted in (Lurie, 2017, 1.3.4.16), (Hinich, 2016, 1.3.4), and (Dwyer and Kan, 1980, 8.4), if the model category M admits *functorial* fibrant and cofibrant replacement, in the sense of (Hovey, 1999, 1.1.1. 1.1.3), then the following ∞-categories are equivalent:

$$\mathscr{N}(\mathsf{M}_c)[\mathsf{W}^{-1}] \simeq \mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}] \simeq \mathscr{N}(\mathsf{M}_f)[\mathsf{W}^{-1}],$$

where $\mathsf{M}_c \subseteq \mathsf{M}$ is the full subcategory of cofibrant objects, and $\mathsf{M}_f \subseteq \mathsf{M}$ is the full subcategory of fibrant objects.

3.2 Symmetric Monoidal Dwyer-Kan Localization

We now construct the symmetric monoidal structure on the Dwyer-Kan localization of a symmetric monoidal model category M. This is a recollection of Appendix A in (Nikolaus and Scholze, 2018) and Section 4.1.7 on monoidal model categories in (Lurie, 2017).

Definition 3.2.1 ((Lurie, 2017, 4.1.7.4), (Nikolaus and Scholze, 2018, A.4, A.5))**.** Let $\mathscr{C}^\otimes$ be a symmetric monoidal ∞-category. Let $\mathscr{W} \subseteq \mathsf{Hom}_{\mathsf{sSet}}(\Delta^1, \mathscr{C})$ be a class of edges in $\mathscr{C}$ that is stable under homotopy, composition and contains all equivalences. Suppose further that $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ preserves the class $\mathscr{W}$ seperately in each variable. The *symmetric monoidal Dywer-Kan localization of* $\mathscr{C}^\otimes$ *with respect to* $\mathscr{W}$ is a symmetric monoidal ∞-category, denoted

$\mathcal{C}[\mathcal{W}^{-1}]^{\otimes}$, together with a symmetric monoidal functor $i : \mathcal{C}^{\otimes} \to \mathcal{C}[\mathcal{W}^{-1}]^{\otimes}$ which is characterized by the following universal property.

(U) For any other symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$, the functor i induces an equivalence of ∞-categories:

$$\mathcal{F}un_{\otimes}(\mathcal{C}[\mathcal{W}^{-1}]^{\otimes}, \mathcal{D}^{\otimes}) \xrightarrow{\simeq} \mathcal{F}un_{\otimes}^{\mathcal{W}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}),$$

where $\mathcal{F}un_{\otimes}^{\mathcal{W}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ is the full subcategory of symmetric monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ that sends $\mathcal{W}$ to equivalences.

As noticed in (Nikolaus and Scholze, 2018, A.5), the underlying ∞-category of the symmetric monoidal category $\mathcal{C}[\mathcal{W}^{-1}]^{\otimes}$ is precisely the Dwyer-Kan localization of $\mathcal{C}$ with $\mathcal{W}$ in the sense of Definition 3.1.1, i.e.:

$$\left(\mathcal{C}[\mathcal{W}^{-1}]^{\otimes}\right)_{\langle 1 \rangle} \simeq \mathcal{C}[\mathcal{W}^{-1}].$$

Remark 3.2.2. Let $\mathcal{C}^{\otimes}$ and $\mathcal{W}$ be as in Definition 3.2.1. Given the symmetric monoidal structure $\mathcal{C}^{\otimes} \to \mathcal{N}(\mathsf{Fin}_*)$, products of n edges in $\mathcal{W}$ in $\mathcal{C}$ correspond precisely, under the equivalence:

$$\mathcal{C}^{\times n} \simeq \mathcal{C}^{\otimes}_{\langle n \rangle},$$

to morphisms lying over $\mathsf{id}_{\langle n \rangle}$ in $\mathcal{N}(\mathsf{Fin}_*)$. This defines a class of edges $\mathcal{W}^{\otimes} \subseteq \mathsf{Hom}_{\mathsf{sSet}}(\Delta^1, \mathcal{C}^{\otimes})$. Then the Dwyer-Kan localization of $\mathcal{C}^{\otimes}$ with respect to $\mathcal{W}^{\otimes}$, in the sense of Definition 3.1.1, denoted $\mathcal{C}^{\otimes}\left[(\mathcal{W}^{\otimes})^{-1}\right]$, is equivalent to $\mathcal{C}[\mathcal{W}^{-1}]^{\otimes}$ defined above.

We would like to study the case where the underlying ∞-category of $C^{\otimes}$ is the Dwyer-Kan localization $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$ of a model category M. We first recall the induced symmetric monoidal structure on the nerve of a symmetric monoidal category.

Definition 3.2.3 ((Lurie, 2017, 2.0.0.1))**.** Let $(\mathsf{C}, \otimes, \mathbb{I})$ be a symmetric monoidal category. Define a new category $\mathsf{C}^{\otimes}$ as follows.

- Objects are sequences $(C_1, \ldots, C_n)$ where each C_i is an object in C, for all $1 \leq i \leq n$, for some $n \geq 1$. We allow the case $n = 0$ and thus the empty set $\emptyset$ as a sequence.
- A morphism $(C_1, \ldots, C_n) \to (C'_1, \ldots, C'_m)$ in $\mathsf{C}^{\otimes}$ is a pair $(\alpha, \{f_j\})$, where α is a map of finite sets $\alpha : \langle n \rangle \to \langle m \rangle$ and $\{f_j\}$ is a collection of m-morphisms in C:

  $$f_j : \bigotimes_{i \in \alpha^{-1}(j)} C_i \longrightarrow C'_j,$$

  for all $1 \leq j \leq m$. If $\alpha^{-1}(j) = \emptyset$, then f_j is a morphism $\mathbb{I} \to C'_j$.
- The composition of morphisms in $\mathsf{C}^{\otimes}$ is defined using the compositions in Fin_* and C together with the associativity of the symmetric monoidal structure of C.
- The identity morphism on an object $(C_1, \ldots, C_n)$ is given by the identites in Fin_* and C: $(\mathsf{id}_{\langle n \rangle}, \{\mathsf{id}_{C_j}\})$.

We obtain a functor:

$$\mathsf{C}^{\otimes} \longrightarrow \mathsf{Fin}_*,$$

that sends $(C_1, \dots C_n)$ to $\langle n \rangle$. The induced functor $\mathscr{N}(\mathsf{C}^{\otimes}) \to \mathscr{N}(\mathsf{Fin}_*)$ in ∞-categories is coCartesian and defines a symmetric monoidal structure.

Proposition 3.2.4 ((Lurie, 2017, 2.1.2.21))**.** *Given any symmetric monoidal category* $(\mathsf{C}, \otimes, \mathbb{I})$*, let* $\mathsf{C}^{\otimes}$ *be as* Definition 3.2.3*. Then the nerve* $\mathscr{N}(\mathsf{C}^{\otimes})$ *is a symmetric monoidal* ∞*-category whose underlying* ∞*-category is* $\mathscr{N}(\mathsf{C})$*.*

Remark 3.2.5. In particular, given $(C_1, \dots, C_n)$ in $\mathsf{C}^{\otimes}$, and $\alpha : \langle n \rangle \to \langle m \rangle$ a map in Fin_*, the associated coCartesian lift is induced by defining C'_j as follows:

$$C'_j := \bigotimes_{i \in \alpha^{-1}(j)} C_i,$$

for each $1 \leq j \leq m$. Define $C'_j = \mathbb{I}$ if j is such that $\alpha^{-1}(j) = \emptyset$. This defines a morphism $(C_1, \dots, C_n) \to (C'_1, \dots, C'_m)$ in $\mathsf{C}^{\otimes}$ as desired.

If the symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{I})$ happens to be endowed with a model structure, the bifunctor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ need not preserve weak equivalences in either variable. We need to restrict to the following type of model category.

Definition 3.2.6 ((Hovey, 1999, 4.2.6))**.** A *(symmetric) monoidal model category* M is a category endowed with both a model structure and a (symmetric) monoidal structure $(\mathsf{M}, \otimes, \mathbb{I})$, such that the tensor product $\otimes : \mathsf{M} \times \mathsf{M} \to \mathsf{M}$ is a Quillen bifunctor (see (Hovey, 1999, 4.2.1)), and for any cofibrant replacement $c\mathbb{I} \to \mathbb{I}$ of the unit, the induced morphism $c\mathbb{I} \otimes X \to \mathbb{I} \otimes X \cong X$ is a weak equivalence, for any cofibrant object X of M. The latter requirement is automatic if $\mathbb{I}$ is already cofibrant.

Therefore, in any monoidal model category $(\mathsf{M}, \otimes, \mathbb{I})$, the tensor $\otimes : \mathsf{M} \times \mathsf{M} \to \mathsf{M}$ preserves weak equivalences in each variable, if we restrict to cofibrant objects $\mathsf{M}_c \subseteq \mathsf{M}$. Moreover. the tensor product of cofibrant objects is again cofibrant. In model categories, this allows us to define a *derived tensor product* for the homotopy category $\mathsf{Ho}(\mathsf{M}) = \mathsf{M}[\mathsf{W}^{-1}]$, see (Hovey, 1999, 4.3.2). In higher category, the transition between the tensor product and the derived tensor product is exactly through the Dwyer-Kan localization of a symmetric monoidal ∞-category as in Definition 3.2.1. If we suppose in addition that $\mathbb{I}$ is cofibrant, then, as in Definition 3.2.3, we can define $\mathsf{M}_c^{\otimes} \subseteq \mathsf{M}^{\otimes}$ from the full subcategory of cofibrant objects $\mathsf{M}_c \subseteq \mathsf{M}$, since $(\mathsf{M}_c, \otimes, \mathbb{I})$ is symmetric monoidal.

Proposition 3.2.7 ((Lurie, 2017, 4.1.7.6), (Nikolaus and Scholze, 2018, A.7))**.** *Let* $(\mathsf{M}, \otimes, \mathbb{I})$ *be a symmetric monoidal model category. Suppose that* $\mathbb{I}$ *is cofibrant. Then the Dwyer-Kan localization* $\mathscr{N}(\mathsf{M}_c)[\mathsf{W}^{-1}]$ *of* M *can be given the structure of symmetric monoidal* ∞*-category via the symmetric monoidal Dwyer-Kan localization of* $\mathscr{N}(\mathsf{M}_c^{\otimes})$ *in the sense of* Definition 3.2.1*,*

$$\mathscr{N}(\mathsf{M}_c^{\otimes}) \longrightarrow \mathscr{N}(\mathsf{M}_c)[\mathsf{W}^{-1}]^{\otimes},$$

where W *is the class of weak equivalences restricted to cofibrant objects in* M*.*

Remark 3.2.8. The inclusion of cofibrant objects $\mathsf{M}_c \subseteq \mathsf{M}$ induces a *lax* symmetric monoidal functor $\mathscr{N}(\mathsf{M}_c^{\otimes}) \to \mathscr{N}(\mathsf{M}^{\otimes})$. From Remark 3.1.4, Proposition 3.2.7 implies we can also construct a symmetric monoidal ∞-category $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]^{\otimes}$ whose fiber over $\langle 1 \rangle$ is precisely $\mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]$. However, cofibrant replacement induces a functor $\mathscr{N}(\mathsf{M}^{\otimes}) \to \mathscr{N}(\mathsf{M})[\mathsf{W}^{-1}]^{\otimes}$ that is only lax symmetric monoidal and does not share the same properties of universality as in Definition 3.2.1. We invite the interested reader to look at (Nikolaus and Scholze, 2018, A.7) for more details.

If C is a left proper cellular simplicial symmetric monoidal model category, then its category of symmetric spectra $\mathsf{Sp}^{\Sigma}(\mathsf{C})$ is also a symmetric monoidal model category, when endowed with its projective stable model structure, see (Hovey, 2001, 7.3). If $\mathscr{C}^{\otimes}$ is a symmetric monoidal ∞-category, then so is its stabilization $\mathscr{S}p(\mathscr{C}^{\otimes})$. These are compatible with each other with respect to the symmetric monoidal Dwyer-Kan localization.

Proposition 3.2.9 ((Ando et al., 2018, B.3))**.** *Let C be a left proper cellular simplicial symmetric monoidal model category. Then there is an equivalence of symmetric monoidal ∞-categories:*

$$\mathscr{N}\left(\mathsf{Sp}^{\Sigma}(\mathsf{C})_c\right)\left[\mathsf{W}_{\mathsf{st}}^{-1}\right]^{\otimes} \simeq \mathscr{S}p\left(\mathscr{N}\left(\mathsf{C}_c\right)\left[\mathsf{W}^{-1}\right]^{\otimes}\right),$$

where W denotes the class of weak equivalences in C and W_{st} are the induced stable weak equivalences in $\mathsf{Sp}^{\Sigma}(\mathsf{C})$.

3.3 Weak Monoidal Quillen Equivalence

Given C and D model categories, denote W_{C} and W_{D} their respective class of weak equivalences. Let:

$$L : \mathsf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{D} : R,$$

be a Quillen adjunction. Then as the left adjoint functor L preserves weak equivalences between cofibrant objects and the right adjoint functor R preserves weak equivalences between fibrant objects, we get, by (Hinich, 2016, 1.5.1), a pair of adjoint functors in ∞-categories between the Dwyer-Kan localizations of C and D:

$$\mathbb{L} : \mathscr{N}(\mathsf{C})[\mathsf{W}_{\mathsf{C}}^{-1}] \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{N}(\mathsf{D})[\mathsf{W}_{\mathsf{D}}^{-1}] : \mathbb{R},$$

where $\mathbb{L}$ and $\mathbb{R}$ represent the derived functors of L and R. If C and D are symmetric monoidal model categories, we investigate when the derived functors are symmetric monoidal functors of ∞-categories.

Definition 3.3.1 ((Schwede and Shipley, 2003, 3.6))**.** Let $(\mathsf{C}, \otimes, \mathbb{I})$ and $(\mathsf{D}, \wedge, \mathbb{J})$ be symmetric monoidal model categories. A *weak monoidal Quillen pair* consists of a Quillen adjunction:

$$L : (\mathsf{C}, \otimes, \mathbb{I}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathsf{D}, \wedge, \mathbb{J}) : R,$$

where L is lax comonoidal such that the following two conditions hold.

(i) For all cofibrant objects X and Y in C, the comonoidal map:

$$L(X \otimes Y) \longrightarrow L(X) \wedge L(Y),$$

is a weak equivalence in D.

(ii) For some (hence any) cofibrant replacement $\lambda : c\mathbb{I} \xrightarrow{\sim} \mathbb{I}$ in C, the composite map:

$$L(c\mathbb{I}) \xrightarrow{L(\lambda)} L(\mathbb{I}) \longrightarrow \mathbb{J},$$

is a weak equivalence in D, where the unlabeled map is the natural comonoidal structure of L.

A weak monoidal Quillen pair is a *weak monoidal Quillen equivalence* if the underlying Quillen pair is a Quillen equivalence.

Theorem 3.3.2. *Let* $(\mathsf{C}, \otimes, \mathbb{I})$ *and* $(\mathsf{D}, \wedge, \mathbb{J})$ *be symmetric monoidal model categories with cofibrant units. Let* W_{C} *and* W_{D} *be the classes of weak equivalence in* C *and* D *respectively. Let:*

$$L : (\mathsf{C}, \otimes, \mathbb{I}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathsf{D}, \wedge, \mathbb{J}) : R,$$

be a weak monoidal Quillen pair. Then the derived functor of $L : \mathsf{C} \to \mathsf{D}$ *induces a symmetric monoidal functor between the Dwyer-Kan localizations:*

$$\mathbb{L} : \mathscr{N}(\mathsf{C}_c)\left[\mathsf{W}_{\mathsf{C}}^{-1}\right] \longrightarrow \mathscr{N}(\mathsf{D}_c)\left[\mathsf{W}_{\mathsf{D}}^{-1}\right],$$

where $\mathsf{C}_c \subseteq \mathsf{C}$ *and* $\mathsf{D}_c \subseteq \mathsf{D}$ *are the full subcategories of cofibrant objects. If* L *and* R *form a weak monoidal Quillen equivalence, then* $\mathbb{L}$ *is a symmetric monoidal equivalence of* ∞*-categories.*

Proof. Let $\mathsf{C}_c^{\otimes}$ and $\mathsf{D}_c^{\otimes}$ be as Definition 3.2.3. Denote the symmetric monoidal Dwyer-Kan localizations (Definition 3.2.1) by:

$$i_{\mathsf{C}} : \mathscr{N}(\mathsf{C}_c^{\otimes}) \longrightarrow \mathscr{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}]^{\otimes}, \qquad i_{\mathsf{D}} : \mathscr{N}(\mathsf{D}_c^{\otimes}) \longrightarrow \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes},$$

and denote their coCartesian fibrations by:

$$p : \mathscr{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}]^{\otimes} \longrightarrow \mathscr{N}(\mathsf{Fin}_*), \qquad q : \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes} \longrightarrow \mathscr{N}(\mathsf{Fin}_*).$$

The functor $L : \mathsf{C} \to \mathsf{D}$, as a left Quillen functor, defines $\mathscr{N}(\mathsf{C}_c) \to \mathscr{N}(\mathsf{D}_c)$, and hence a functor $L^{\otimes} : \mathscr{N}(\mathsf{C}_c^{\otimes}) \to \mathscr{N}(\mathsf{D}_c^{\otimes})$ that is compatible with the coCartesian structures:

$$\begin{tikzcd}
\mathscr{N}(\mathsf{C}_c^{\otimes}) \arrow[r, "L^{\otimes}"] \arrow[dr] & \mathscr{N}(\mathsf{D}_c^{\otimes}) \arrow[r, "i_{\mathsf{D}}"] \arrow[d] & \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes} \arrow[dl, "q"] \\
& \mathscr{N}(\mathsf{Fin}_*). &
\end{tikzcd}$$

We show that the composite:

$$\mathscr{N}(\mathsf{C}_c^{\otimes}) \xrightarrow{L^{\otimes}} \mathscr{N}(\mathsf{D}_c^{\otimes}) \xrightarrow{i_{\mathsf{D}}} \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes}$$

is a symmetric monoidal functor that sends W_{C} to equivalences, i.e., belongs to the ∞-category $\mathcal{F}un_{\otimes}^{\mathsf{W}_{\mathsf{C}}}(\mathscr{N}(\mathsf{C}_c^{\otimes}), \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes})$, as in Definition 3.2.1. The latter is clear as L is a left Quillen functor. We are left to show that the composite sends p-coCartesian lifts to q-coCartesian lifts. Let $(C_1, \ldots, C_n)$ be an object of $\mathsf{C}_c^{\otimes}$, and let $\alpha : \langle n \rangle \to \langle m \rangle$ be a morphism of finite sets. A

p-lift of $(\alpha, (C_1, \dots, C_n))$ is given in Remark 3.2.5 by a certain sequence $(C'_1, \dots, C'_m)$ in $\mathsf{C}_c^{\otimes}$, i.e., the induced map $(C_1, \dots, C_n) \to (C'_1, \dots, C'_m)$ is sent to α via the coCartesian functor p. Since L is weak monoidal functor, from (i) of Definition 3.3.1, we get that:

$$\bigwedge_{i \in \alpha^{-1}(j)} L(C_i) \xleftarrow{\sim} L\left(\bigotimes_{i \in \alpha^{-1}(j)} C_i\right) = L(C'_j),$$

is a weak equivalence in D, for all $1 \leq j \leq m$. In the case $\alpha^{-1}(j) = \emptyset$, we apply (ii) of Definition 3.3.1 to obtain a weak equivalence:

$$\mathbb{J} \xleftarrow{\sim} L(\mathbb{I}) = L(C_j).$$

Applying the localization i_D and Remark 3.2.2, we get that $(L(C'_1), \dots, L(C'_m))$ defines the desired q-coCartesian lift.

By the universal property (U) of the symmetric monoidal Dwyer-Kan localization in Definition 3.2.1, the composite functor $i_{\mathsf{D}} \circ L^{\otimes}$ represents a symmetric monoidal ∞-functor:

$$\mathbb{L}^{\otimes} : \mathscr{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}]^{\otimes} \longrightarrow \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]^{\otimes}.$$

Fiberwise over $\mathscr{N}(\mathsf{Fin}_*)$, the functor $\mathbb{L}^{\otimes}$ is precisely the product of the derived left adjoint functor $\mathbb{L} : \mathscr{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}] \to \mathscr{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]$. In particular, if L is a Quillen equivalence, then $\mathbb{L}$ is an equivalence of ∞-category, and hence $\mathbb{L}^{\otimes}$ is an equivalence of symmetric monoidal ∞-categories. $\square$

Remark 3.3.3. In (Schwede and Shipley, 2003, 3.12), Schwede and Shipley show that given a weak monoidal Quillen pair $L : (\mathsf{C}, \otimes, \mathbb{I}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathsf{D}, \wedge, \mathbb{J}) : R$, with cofibrant units, then the right adjoint R induces Quillen equivalences between the category of monoids $\mathsf{Mon}(\mathsf{D})$ and $\mathsf{Mon}(\mathsf{C})$, and also their categories of modules. Our Theorem 3.3.2 strenghten the results when we worked with ∞-categories. In particular, given any ∞-operad $\mathcal{O}^{\otimes}$, we get an equivalence of ∞-categories $\mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}]\right) \simeq \mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]\right)$, which has been challenging to prove in the case of $\mathcal{O} = \mathbb{E}_\infty$ in the past, see for instance (Richter and Shipley, 2017) and (Mandell, 2003, 1.3, 1.4) for $\mathcal{O} = \mathbb{E}_\infty$. We also obtain an equivalence on the coalgebras:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}(\mathsf{C}_c)[\mathsf{W}_{\mathsf{C}}^{-1}]\right) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}(\mathsf{D}_c)[\mathsf{W}_{\mathsf{D}}^{-1}]\right).$$

Such a result on coalgebras has been showed to be challenging in model categories, see for instance (Soré, 2017), (Soré, 2019) and Remark 3.4.3 below.

3.4 The Derived Dold-Kan Equivalence

We now apply our Theorem 3.3.2 to the weak monoidal Quillen equivalence appearing in (Schwede and Shipley, 2003), all missing details can be found there. Let R be a commutative discrete ring subsequently. Let sMod_R denote the category of simplicial R-modules, and let $\mathsf{Ch}_R^{\geq 0}$ denote the category of non-negative chain complexes. The *Dold-Kan equivalence* says that the *normalization functor*:

$$\mathsf{N} : \mathsf{sMod}_R \xrightarrow{\cong} \mathsf{Ch}_R^{\geq 0}, \tag{3.4.1}$$

is an equivalence of categories. Its inverse functor is denoted $\Gamma : \mathsf{Ch}_R^{\geq 0} \to \mathsf{sMod}_R$.

We can endow each category with a model structure. For sMod_R, the weak equivalences and fibrations are the underlying weak equivalences and fibrations in simplicial sets, i.e., they are weak homotopy equivalences and Kan fibrations. In other words, the model structure of sMod_R is right-induced from sSet via the forgetful functor, in the sense of (Hess et al., 2017). For $\mathsf{Ch}_R^{\geq 0}$, we use the usual projective model structure. The weak equivalences are the quasi-isomorphisms, and the fibrations are the positive levelwise epimorphisms. The isomorphism of categories from (Equation 3.4.1) can be regarded now as two Quillen equivalences, depending on the choice of left and right adjoints:

$$\begin{tikzcd} \mathsf{Ch}_R^{\geq 0} \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathsf{sMod}_R, \arrow[l, shift left, "\mathsf{N}"] \end{tikzcd} \tag{3.4.2}$$

and:

$$\begin{tikzcd} \mathsf{sMod}_R \arrow[r, shift left, "\mathsf{N}"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_R^{\geq 0}. \arrow[l, shift left, "\Gamma"] \end{tikzcd} \tag{3.4.3}$$

Both categories can be endowed with their usual symmetric monoidal structure induced by the tensor product of R-modules. However, the Dold-Kan equivalence (Equation 3.4.1) does *not* preserve the monoidal structure. Nonetheless, with respect to the above choice of model structures, the categories sMod_R and $\mathsf{Ch}_R^{\geq 0}$ are both symmetric monoidal model categories with cofibrant units. If we choose the normalization functor N to be the right adjoint as in (Equation 3.4.2), then it can be considered as lax symmetric monoidal via the shuffle map:

$$\nabla : \mathsf{N}(A) \otimes \mathsf{N}(B) \longrightarrow \mathsf{N}(A \otimes B).$$

If we choose N to be the left adjoint as in (Equation 3.4.3), then the Alexander-Whitney formula gives a lax comonoidal structure:

$$\mathsf{AW} : \mathsf{N}(A \otimes B) \longrightarrow \mathsf{N}(A) \otimes \mathsf{N}(B),$$

which is *not* symmetric. Nevertheless, this shows that both Quillen equivalences form a weak monoidal Quillen equivalence with cofibrant units, which is symmetric in the case where N is a right adjoint (Equation 3.4.2). We can therefore apply our Theorem 3.3.2 to obtain the following.

Corollary 3.4.1 (The Derived Dold-Kan Equivalence)**.** *Let R be a commutative discrete ring. Then the Dwyer-Kan localizations of sMod_R and $\mathsf{Ch}_R^{\geq 0}$ are equivalent as symmetric monoidal ∞-categories:*

$$\mathscr{N}\left(\mathsf{sMod}_R\right)\left[\mathsf{W}_\Delta^{-1}\right] \simeq \mathscr{N}\left(\mathsf{Ch}_R^{\geq 0}\right)\left[\mathsf{W}_{\mathsf{dg}}^{-1}\right],$$

via the right Quillen derived functor of $\mathsf{N} : \mathsf{sMod}_R \to \mathsf{Ch}_R^{\geq 0}$ from the Quillen equivalence of (Equation 3.4.2)*, where W_Δ is the class of weak homotopy equivalences between simplicial R-*

modules, and W_{dg} *is the class of quasi-isomorphisms between non-negative chain complexes over* R.

In particular, applying our Remark 3.3.3, we get the following result.

Corollary 3.4.2. *For any* ∞*-operad* $\mathcal{O}^{\otimes}$*, there is an equivalence of* ∞*-categories:*

$$\mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}\left(\mathsf{sMod}_R\right)\left[\mathsf{W}_{\Delta}^{-1}\right]\right) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathcal{O}}\left(\mathcal{N}\left(\mathsf{Ch}_R^{\geq 0}\right)\left[\mathsf{W}_{\mathsf{dg}}^{-1}\right]\right).$$

Remark 3.4.3. The above result bypasses a difficulty on the level of model categories and strict coalgebras. If we choose the second adjunction (Equation 3.4.3) as a weak Quillen monoidal pair, then the normalization functor, being lax comonoidal, lifts to coalgebras $\mathsf{N} : \mathsf{CoAlg}_R\left(\mathsf{sMod}_R\right) \to \mathsf{CoAlg}_R(\mathsf{Ch}_R^{\geq 0})$, but its inverse Γ, being only lax monoidal, does not lift to coalgebras. Nevertheless, a right adjoint exists on the level of R-coalgebras, either by presentability, or using dual methods as in section 3.3 of (Schwede and Shipley, 2003). We shall denote it by Γ_{CoAlg}. Then, using left-induced methods, we can endow model structures such that we get a Quillen adjunction:

$$\begin{tikzcd} \mathsf{CoAlg}_R\left(\mathsf{sMod}_R\right) \arrow[r, shift left, "\mathsf{N}"] \arrow[r, phantom, "\perp"] & \mathsf{CoAlg}_R\left(\mathsf{Ch}_R^{\geq 0}\right) \arrow[l, shift left, "\Gamma_{\mathsf{CoAlg}}"] \end{tikzcd}.$$

The weak equivalences are the underlying weak equivalences and every object is cofibrant, in both model categories. However, it was shown in (Soré, 2019, 4.16) that the above Quillen pair is *not* a Quillen equivalence, at least when R is a field. It was shown that for a particular choice of fibrant object C in $\mathsf{CoAlg}_R(\mathsf{Ch}_R^{\geq 0})$, the counit $\mathsf{N}\left(\Gamma_{\mathsf{CoAlg}}\left(C\right)\right) \longrightarrow C$ is not a weak equivalence (i.e. not a quasi-isomorphism). This will have a very important consequence for rigidification results, see Example 4.1.2.

Our approach also gives a new proof of the *stable Dold-Kan equivalence*. This well-known result was formalized with ∞-categories in (Lurie, 2017, 7.1.2.13) as follows. Let R be a commutative discrete ring. Then the ∞-category of HR-modules $\mathcal{Mod}_{HR}$ is equivalent to ∞-category of derived R-modules $\mathcal{D}(R)$ as symmetric monoidal ∞-categories: $\mathcal{Mod}_{HR} \simeq \mathcal{D}(R)$. However, the equivalence was not described explicitly in Lurie. In (Shipley, 2007, 2.10), Shipley provided an explicit zig-zag of (weak monoidal) Quillen equivalences between the standard model category Mod_{HR} of HR-modules in symmetric spectra and the projective model category of chain complexes over R:

$$\begin{tikzcd} \mathsf{Mod}_{HR} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^{\Sigma}\left(\mathsf{sMod}_R\right) \arrow[l, shift left] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \\ & \mathsf{Sp}^{\Sigma}\left(\mathsf{Ch}_R^{\geq 0}\right) \arrow[u, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_R. \arrow[l, shift left] \end{tikzcd}$$

Notice that the Dwyer-Kan localizations of Mod_{HR} and Ch_R are precisely the ∞-categories $\mathcal{Mod}_{HR}$ and $\mathcal{D}(R)$ respectively. If we derive and combine the Quillen functors above, we obtain an explicit functor of ∞-categories $\Theta : \mathcal{Mod}_{HR} \to \mathcal{D}(R)$. Recall that both sMod_R and $\mathsf{Ch}_R^{\geq 0}$ are left proper cellular symmetric monoidal model categories. Combining Corollary 3.4.1 with Proposition 3.2.9, and applying Theorem 3.3.2 yields the following.

Corollary 3.4.4 (The Stable Dold-Kan Equivalence)**.** *Let R be a commutative discrete ring. Then the ∞-category of HR-modules is equivalent to ∞-category of derived R-modules as symmetric monoidal ∞-categories via the functor $\Theta : \mathcal{Mod}_{HR} \xrightarrow{\simeq} \mathcal{D}(R)$.*

CHAPTER 4

THE RIGIDIFICATION PROBLEM

In this chapter, we want to compare homotopy coherent coalgebras and comodules with their strict analogue. One one hand, given a nice enough symmetric monoidal model category M, we can obtain its Dwyer-Kan localization which is a symmetric monoidal ∞-category. We can then apply Definitions 2.2.2 and 2.2.9, and define $\mathbb{A}_\infty$ or $\mathbb{E}_\infty$-coalgebras and their comodules. Alternatively, we can consider comonoids and comodules in M as in Definitions 2.1.1 and 2.1.8, and then take their Dwyer-Kan localization as in Definition 3.1.1.

There are classical rigidification results that compare $\mathbb{A}_\infty$-algebras with their strict associative analogue, see (Lurie, 2017, 4.1.8.4). There is also a comparison between the $\mathbb{E}_\infty$-case with the commutative case in (Lurie, 2017, 4.5.4.7). However, there is no reason to expect that these results dualize in general. In particular, if $\mathbb{A}_\infty$-algebras correspond to strict associative algebras in a model category M, there is no reason to expect that $\mathbb{A}_\infty$-coalgebras correspond to strict coassociative coalgebras in M, see for instance our Example 4.1.2 below.

4.1 Rigidification Properties

Let C be a symmetric monoidal category. Let $\mathsf{C}^\otimes$ be as in Definition 3.2.3. Let $p : \mathsf{C}^\otimes \to \Delta^{\mathsf{op}}$ be its associated Grothendieck opfibration (see (Groth, 2015, 4.5)) that determines the monoidal structure of C, and induces the coCartesian fibration $\mathscr{N}(\mathsf{C}^\otimes) \to \mathscr{N}(\Delta^{\mathsf{op}})$. There is a correspondance between monoids in C and sections of p that sends convex morphisms to p-coCartesian arrows (see (Groth, 2015, 4.21)). In particular, we obtain the following identification in ∞-categories:

$$\mathscr{N}(\mathsf{Mon}(\mathsf{C})) \longrightarrow \mathscr{A}\ell g_{\mathbb{A}_\infty}(\mathscr{N}(\mathsf{C})).$$

By using opposite categories, we obtain therefore an identification:

$$\mathscr{N}(\mathsf{CoMon}(\mathsf{C})) \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}(\mathscr{N}(\mathsf{C})).$$

Let M be a symmetric monoidal model category with cofibrant unit. Consider $\mathsf{M}_c \subseteq \mathsf{M}$ the full subcategory of cofibrant objects. Apply the above identification to $\mathsf{C} = \mathsf{M}_c$ to obtain the following functor in ∞-categories:

$$\mathscr{N}\Big(\mathsf{CoMon}(\mathsf{M}_c)\Big) \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}(\mathscr{N}(\mathsf{M}_c))$$

Let W be the class of weak equivalences in M. By Proposition 3.2.7, there is a symmetric monoidal functor $\mathscr{N}(\mathsf{M}_c^{\otimes}) \to \mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]^{\otimes}$, which thus provides a map of ∞-categories:

$$\mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}(\mathscr{N}(\mathsf{M}_c)) \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right),$$

and therefore we obtain a functor of ∞-categories:

$$\alpha : \mathscr{N}\Big(\mathsf{CoMon}(\mathsf{M}_c)\Big) \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right).$$

Denote $\mathsf{W}_{\mathsf{CoMon}}$ the class of morphisms in $\mathsf{CoMon}(\mathsf{M}_c)$ that are weak equivalences as underlying morphisms in M. Notice that the above functor α sends $\mathsf{W}_{\mathsf{CoMon}}$ to equivalences. By the universal property of Dwyer-Kan localizations as in Definition 3.1.1, we obtain the following natural functor of ∞-categories:

$$\alpha : \mathscr{N}\Big(\mathsf{CoMon}(\mathsf{M}_c)\Big)\left[\mathsf{W}^{-1}_{\mathsf{CoMon}}\right] \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right).$$

Similarly, for the cocommutative case we obtain the natural functor of ∞-categories:

$$\beta : \mathscr{N}\Big(\mathsf{CoCMon}(\mathsf{M}_c)\Big)\left[\mathsf{W}^{-1}_{\mathsf{CoCMon}}\right] \longrightarrow \mathscr{C}o\mathscr{A}\ell g_{\mathbb{E}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right).$$

Definition 4.1.1. Let M be a symmetric monoidal model category with cofibrant unit. Let α and β be the functors described above. If α is an equivalence of ∞-categories, we say that the model category M (or its Dwyer-Kan localization) *satisfies coassociative rigidification*. If β is an equivalence of ∞-categories, we say that M (or its Dwyer-Kan localization) *satisfies cocommutative rigidification.*

In general there is no reason to expect that if a model category M respects the associative rigidification then it also respects the coassociative rigidification, as we see from the following counter-example.

Example 4.1.2. We saw in Corollary 3.4.2 that the normalization functor $\mathsf{N} : \mathsf{sMod}_R \to \mathsf{Ch}_R^{\geq 0}$ induces an equivalence between the $\mathbb{A}_\infty$-coalgebras:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}\left(\mathsf{sMod}_R\right)\left[\mathsf{W}_\Delta^{-1}\right]\right) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}\left(\mathsf{Ch}_R^{\geq 0}\right)\left[\mathsf{W}_{\mathsf{dg}}^{-1}\right]\right).$$

But on the level of model categories, we saw that the normalization does not induce a Quillen equivalence in Remark 3.4.3:

$$\mathsf{N} : \mathscr{N}\Big(\mathsf{CoMon}(\mathsf{sMod_R})\Big)\left[\mathsf{W}_{\Delta,\mathsf{Comon}}^{-1}\right] \xrightarrow{\not\simeq} \mathscr{N}\Big(\mathsf{CoMon}(\mathsf{Ch}_R^{\geq 0})\Big)\left[\mathsf{W}_{\mathsf{dg},\mathsf{Comon}}^{-1}\right].$$

Here $\mathsf{W}_{\Delta,\mathsf{Comon}} \subseteq \mathsf{W}_\Delta$ and $\mathsf{W}_{\mathsf{dg},\mathsf{Comon}} \subseteq \mathsf{W}_{\mathsf{dg}}$ denote the subclasses of their respective weak equivalences between comonoid objects. This shows that either sMod_R or $\mathsf{Ch}_R^{\geq 0}$ (or both) does *not* satisfy the coassociative rigidification.

Remark 4.1.3. If we inspect the dual case of algebras (Lurie, 2017, 4.1.8.4, 4.5.4.7), we see that we should have considered the ∞-category $\mathscr{N}(\mathsf{CoMon}(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right]$ and not the ∞-category $\mathscr{N}(\mathsf{CoMon}(\mathsf{M}_c))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right]$. There are several issues with that.

- In general, these ∞-categories are not equivalent unless for instance M admits a functorial lax comonoidal cofibrant replacement. This means there is a functor $Q : \mathsf{M} \to \mathsf{M}_c$ such that there is a natural map $Q(X \otimes Y) \to Q(X) \otimes Q(Y)$ for any X and Y in M. The main issue is that in general the functor $\mathscr{N}(\mathsf{M}_c^{\otimes}) \to \mathscr{N}(\mathsf{M}^{\otimes})$ is only lax symmetric monoidal, see Remark 3.2.8. Of course, if all objects in M are cofibrant, no such issues appear.

- There is no good guarantee to have a model structure on $\mathsf{CoMon}(\mathsf{M})$ whose weak equivalences are $\mathsf{W}_{\mathsf{CoMon}}$, even when using the dual methods from Appendix A. Even though we do not need a model category to define $\mathscr{N}(\mathsf{CoMon}(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right]$, this would help relate if there was some kind of compatibility with M. For instance, if we suppose M is combinatorial monoidal model category and there exists a model category on comonoids so that the forgetful-cofree adjunction (Proposition 2.1.5):

$$U : \mathsf{CoMon}(\mathsf{M}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{M} : \mathsf{T}^{\vee},$$

  is a Quillen adjunction, then there exists a functorial cofibrant replacement $\mathsf{CoMon}(\mathsf{M}) \to \mathsf{CoMon}(\mathsf{M}_c)$ that induces an equivalence of ∞-categories:

$$\mathscr{N}(\mathsf{CoMon}(\mathsf{M}_c))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right] \simeq \mathscr{N}(\mathsf{CoMon}(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right].$$

- In the cases where $\mathsf{CoMon}(\mathsf{M})$ does admit a model structure it is in general left-induced by a model category that is not a monoidal model category. Indeed the lifting often uses the injective model structures instead of the projective ones.

All the above also applies to the cocommutative case.

For any comonoid C in M that is cofibrant in M, we obtain the natural functor of ∞-categories:

$$\mathscr{N}(\mathsf{CoMod}_C(\mathsf{M}_c))\left[\mathsf{W}_{\mathsf{CoMod}}^{-1}\right] \longrightarrow \mathscr{CoMod}_C\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right),$$

just as in the comonoid case. If we further assume that $X \otimes - : \mathsf{M} \to M$ preserves all weak equivalences for any cofibrant object X, we obtain a map of ∞-categories:

$$\gamma_C : \mathscr{N}(\mathsf{CoMod}_C(\mathsf{M}))\left[\mathsf{W}^{-1}_{\mathsf{CoMod}}\right] \longrightarrow \mathscr{CoMod}_C\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right)$$

that factors the above functor, and is defined via the assignement:

$$\widetilde{X} \xrightarrow{\sim} X \longrightarrow X \otimes C \xleftarrow{\sim} \widetilde{X} \otimes^{\mathbb{L}} C,$$

where $\widetilde{X} \xrightarrow{\sim} X$ is a cofibrant replacement of C-comodule X in M.

Definition 4.1.4. Let M be a symmetric monoidal model category as above. Let γ_C be the functor described above. If γ_C is an equivalence of ∞-categories, we say that the model category M (and its Dwyer-Kan localization) *satisfies rigidification of comodules over C.* If γ_C is an equivalence for all comonoids C that are cofibrant in M, then we say that M (and its Dwyer-Kan localization) *satisfies the comodular rigidification.*

4.2 The Cartesian Case

We provide here a simple case of model categories satisfying the coassociative, cocommutative and comodular rigidification in the sense of Definitions 4.1.1 and 4.1.4. Let $(\mathsf{M}, \times, *)$ be a symmetric monoidal model category with respect to its Cartesian monoidal structure. Let W be the class of weak equivalences in M. Suppose it respects the monoid axiom and that the terminal object $*$ is cofibrant. Suppose also that M admits a functorial cofibrant replacement.

Proposition 4.2.1. *Let* $(\mathsf{M}, \times, *)$ *be as above. Then,* M *satisfies the coassociative and cocommutative rigidification, i.e. the following natural maps are equivalences of* ∞*-categories:*

$$\mathscr{N}(\mathsf{CoMon}(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoMon}}^{-1}\right] \xrightarrow{\simeq} \mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right),$$

$$\mathscr{N}(\mathsf{CoCMon}(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoCMon}}^{-1}\right] \xrightarrow{\simeq} \mathcal{C}o\mathcal{A}\ell g_{\mathbb{E}_\infty}\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right),$$

and all four of the ∞*-categories above are equivalent to the Dwyer-Kan localization* $\mathscr{N}(\mathsf{M})\left[\mathsf{W}^{-1}\right]$*. Moreover, the model category* M *also satisfies the comodular rigidification: for any cofibrant object* X *in* M*, we have the following equivalence of* ∞*-categories:*

$$\mathscr{N}(\mathsf{CoMod}_X(\mathsf{M}))\left[\mathsf{W}_{\mathsf{CoMod}}^{-1}\right] \longrightarrow \mathcal{C}o\mathcal{M}o\mathcal{d}_X\left(\mathscr{N}(\mathsf{M}_c)\left[\mathsf{W}^{-1}\right]\right),$$

where both ∞*-categories are equivalent to* $\mathscr{N}(\mathsf{M}_{/X})\left[\mathsf{W}_X^{-1}\right]$*. Here* W_X *is the class of morphisms in* M *over* X *that are weak equivalences.*

Proof. For any Cartesian monoidal ∞-category $\mathcal{C}$, we have the equivalence:

$$\mathcal{C}o\mathcal{A}\ell g_{\mathbb{A}_\infty}(\mathcal{C}) \simeq \mathcal{C}o\mathcal{A}\ell g_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathcal{C},$$

see (Lurie, 2017, 2.4.3.10). Moreover, for any choice of object C in $\mathcal{C}$, we have:

$$\mathcal{C}o\mathcal{M}o\mathcal{d}_C(\mathcal{C}) \simeq \mathcal{C}_{/C},$$

see (Beardsley and Péroux, 2019, 3.14). For any Cartesian monoidal (ordinary) category C, we have the isomorphism of categories:

$$\mathsf{CoMon}(\mathsf{C}) \cong \mathsf{CoCMon}(\mathsf{C}) \cong \mathsf{C},$$

and for any C in C, we have the isomorphism:

$$\mathsf{CoMod}_X(\mathsf{C}) \cong \mathsf{C}_{/C},$$

see (Aguiar and Mahajan, 2010, 1.19). Apply Remark 3.1.4 to conclude. $\square$

CHAPTER 5

COALGEBRAS IN SPECTRA

Based on the main result of (Péroux and Shipley, 2019), we prove here (in Corollaries 5.2.3 and 5.2.4) that the monoidal model categories of symmetric spectra (see (Hovey et al., 2000)), orthogonal spectra (see (Mandell et al., 2001) (Mandell and May, 2002)), Γ-spaces (see (Segal, 1974) (Bousfield and Friedlander, 1978)), $\mathscr{W}$-spaces (see (Anderson, 1974)) and $\mathbb{S}$-modules (in the sense of (Elmendorf et al., 1997)), do not respect the coassociative nor cocommutative rigidification, in the sense of Definition 4.1.1. In other words, the strictly (possibly cocommutative) coassociative counital coalgebras in these monoidal categories of spectra do *not* have the correct homotopy type.

We work with the symmetric monoidal model category of symmetric spectra, denoted Sp^Σ (see (Hovey et al., 2000)), and claim that similar results can be obtained with the other categories mentioned above, following (Péroux and Shipley, 2019). Notice that we have the equivalence of ∞-categories:

$$\mathscr{N}(\mathsf{Sp}^\Sigma)[\mathsf{W}^{-1}] \simeq \mathcal{S}p,$$

where W is the class of stable equivalences of symmetric spectra, and $\mathcal{S}p$ is the ∞-category of spectra as in (Lurie, 2017, 1.4.3.1).

5.1 Model Structures for Coalgebras

Although not necessary to show the non-rigidification, as seen in Remark 4.1.3, we provide here a model category for coalgebras and cocommutative coalgebra in symmetric spectra. We shall use the left-induced methods from Appendix A. We follow here the approach of Section 5 of (Hess et al., 2017). In (Hovey et al., 2000, Section 5) there is a simplicial, combinatorial model structure on Sp^Σ with all objects cofibrant called the *(absolute) injective stable model stucture*, see also (Schwede, , Remark III.4.13). The fibrant objects are the injective Ω-spectra.

Proposition 5.1.1 ((Hess et al., 2017, 5.0.1, 5.0.2))**.** *For any $\mathbb{S}$-algebra A in Sp^Σ, there exists an injective model structure on $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$ left-induced from the injective stable model structure on Sp^Σ:*

$$\begin{tikzcd} \mathsf{Mod}_A(\mathsf{Sp}^\Sigma) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^\Sigma, \arrow[l, shift left, "\mathsf{Hom}_{\mathsf{Sp}^\Sigma}(A,-)"] \end{tikzcd}$$

with cofibrations the monomorphisms and weak equivalences the stable equivalences. This model structure on $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$ is simplicial and combinatorial.

Let A be a commutative ring spectrum (i.e. a commutative $\mathbb{S}$-algebra). The symmetric monoidal category $(\mathsf{Mod}_A(\mathsf{Sp}^\Sigma), \wedge_A, A)$ is presentable and the smash product $\wedge_A$ preserves colimits in both variables. Thus we can apply Proposition 2.1.5 and we obtain the (forgetful-cofree)-adjunction between A-coalgebras and A-modules in Sp^Σ:

$$\begin{tikzcd} \mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{Mod}_A(\mathsf{Sp}^\Sigma). \arrow[l, shift left, "\mathsf{T}^\vee"] \end{tikzcd}$$

Proposition 5.1.2. *Let A be any commutative $\mathbb{S}$-algebra in symmetric spectra Sp^Σ. There exists a model structure on A-coalgebras $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma)$ left-induced by the (forgetful-cofree) adjunction from the injective stable model structure on $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$. In particular, the weak equivalences in $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma)$ are the underlying stable equivalences, and the cofibrations are the underlying monomorphisms.*

Proof. We mimic the proof of (Hess et al., 2017, Theorem 5.0.3). We apply Proposition A.3.2. Tensoring with a simplicial set lifts to A-coalgebras. Indeed, let K be a simplicial set and $(C, \Delta_C, \varepsilon_C)$ be an A-coalgebra. Then the free $\mathbb{S}$-module $\Sigma^\infty_+ K$ is endowed with a unique (co-commutative) $\mathbb{S}$-coalgebra structure $(\Sigma^\infty_+ K, \Delta_K, \varepsilon_K)$, see (Péroux and Shipley, 2019, Lemma 2.4), where the comultiplication Δ_K is induced by the diagonal $K_+ \to K_+ \wedge K_+$ and the counit ε_K is induced by the non-trivial map $K_+ \to S^0$. Then the tensor $K \otimes C := \Sigma^\infty_+ K \wedge_{\mathbb{S}} C$ is an A-coalgebra with comultiplication:

$$\begin{aligned} \Sigma^\infty_+ K \wedge_{\mathbb{S}} C \quad & \xrightarrow{\Delta_K \wedge \Delta_C} \quad (\Sigma^\infty_+ K \wedge_{\mathbb{S}} \Sigma^\infty_+ K) \wedge_{\mathbb{S}} (C \wedge_A C) \\ & \qquad \cong (\Sigma^\infty_+ K \wedge_{\mathbb{S}} C) \wedge_A (\Sigma^\infty_+ K \wedge_{\mathbb{S}} C), \end{aligned}$$

and counit:

$$\Sigma^\infty_+ K \wedge_{\mathbb{S}} C \xrightarrow{\varepsilon_K \wedge \varepsilon_C} \mathbb{S} \wedge_{\mathbb{S}} A \cong A.$$

There is a good cylinder object in sSet given by the factorization:

$$S^0 \coprod S^0 \rightarrowtail \Delta[1]_+ = I \xrightarrow{\sim} S^0.$$

Since $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$ is simplicial, all objects are cofibrant, and that the smash product of an A-coalgebra with this factorization in sSet lifts to $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma)$, this defines a good cylinder object in $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma)$ for any A-coalgebra C:

$$C \coprod C \rightarrowtail C \otimes I \xrightarrow{\sim} C,$$

as $C \otimes S^0 \cong C$, and colimits in $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma)$ are computed in $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$ by Proposition 2.1.3. □

We can also easily extends the results to cocommutative A-coalgebras.

Proposition 5.1.3. *Let A be any commutative $\mathbb{S}$-algebra in symmetric spectra Sp^Σ. There exists a model structure on cocommutative A-coalgebras $\mathsf{CoCAlg}_A(\mathsf{Sp}^\Sigma)$ left-induced by the (forgetful-cofree) adjunction from the injective stable model structure on $\mathsf{Mod}_A(\mathsf{Sp}^\Sigma)$. In particular, the weak equivalences in $\mathsf{CoCAlg}_A(\mathsf{Sp}^\Sigma)$ are the underlying stable equivalences, and the cofibrations are the underlying monomorphisms.*

5.2 The Failure of Rigidification

We show here the failure of rigidification. Let A and B be commutative $\mathbb{S}$-coalgebras. A map $A \to B$ is defined to be a *positive flat cofibration of commutative $\mathbb{S}$-algebras* if it is a cofibration in the model category of commutative $\mathbb{S}$-algebras defined in (Shipley, 2004, 3.2) (or the positive flat stable model structure defined in (Schwede, , III.6.1)). As noted in (Péroux and Shipley, 2019, 2.4), every comonoid in $(\mathsf{sSet}_*, \wedge, S^0)$ is of the form Y_+ and the comultiplication is given by the diagonal $Y_+ \to (Y \times Y)_+ \cong Y_+ \wedge Y_+$.

Theorem 5.2.1 ((Péroux and Shipley, 2019, 3.4, 3.6))**.** *Let A be a positive flat cofibrant commutative $\mathbb{S}$-algebra in Sp^Σ. Then, given any counital coassociative A-coalgebra C in Sp^Σ, the comultiplication is cocommutative and induced by the following epimorphism of A-coalgebras:*

$$A \wedge C_0 \longrightarrow C,$$

where $A \wedge C_0$ is given an A-coalgebra structure via the diagonal on the pointed space $C_0 \to C_0 \wedge C_0$.

Remark 5.2.2. As noted in (Péroux and Shipley, 2019, 3.6), any $\mathbb{E}_\infty$-ring spectrum is equivalent (as an $\mathbb{E}_\infty$-ring spectrum) to a positive flat cofibrant commutative $\mathbb{S}$-algebra in Sp^Σ.

Let A be any commutative $\mathbb{S}$-algebra. Let $\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma_c)$ denote the comonoid in the cofibrant objects of A-modules in Sp^Σ endowed with the absolute projective stable model structure (as in (Schwede, , IV.6.1)). There is a natural map of ∞-categories:

$$\alpha : \mathscr{N}(\mathsf{CoAlg}_A(\mathsf{Sp}^\Sigma_c)) \left[\mathsf{W}^{-1}\right] \longrightarrow \mathscr{CoAlg}_{\mathbb{A}_\infty}(\mathscr{Mod}_A(\mathscr{Sp})),$$

where W is the class of stable equivalences between A-coalgebras.

Corollary 5.2.3. *Let A be a positive flat cofibrant commutative $\mathbb{S}$-algebra in Sp^Σ. Then the ∞-category of A-modules $\mathscr{Mod}_A(\mathscr{Sp})$ does not satisfy the coassociative rigidification. In particular, for $A = \mathbb{S}$ we have:*

$$\mathscr{CoAlg}_{\mathbb{A}_\infty}(\mathscr{Sp}) \not\simeq \mathscr{N}(\mathsf{CoAlg}_{\mathbb{S}}(\mathsf{Sp}^\Sigma)) \left[\mathsf{W}^{-1}\right].$$

Proof. Let (C, Δ, ε) be an A-coalgebra in Sp^{Σ} that is cofibrant as an A-modules in the (absolute) projective stable model structure. Suppose the functor:

$$\alpha : \mathscr{N}(\mathsf{CoAlg}_A(\mathsf{Sp}^{\Sigma}_c)) \left[\mathsf{W}^{-1}\right] \longrightarrow \mathcal{CoAlg}_{\mathbb{A}_\infty}(\mathcal{Mod}_A(\mathcal{Sp})),$$

is an equivalence of ∞-category. By Theorem 5.2.1, we see that $\alpha(C)$ is automatically an $\mathbb{E}_\infty$-coalgebra. But there exist $\mathbb{A}_\infty$-coalgebras in $\mathcal{Sp}$ that are not $\mathbb{E}_\infty$-coalgebras. Indeed, take any compact topological group that is not Abelian (say $O(2)$), then $A \wedge O(2)_+$ is an $\mathbb{A}_\infty$-algebra in $\mathcal{Mod}_A(\mathcal{Sp})$ that is not commutative and is a compact spectrum. By Spanier-Whitehead duality, we obtain an $\mathbb{A}_\infty$-coalgebra that is not $\mathbb{E}_\infty$ in spectra. □

Similarly, as there are examples of $\mathbb{E}_\infty$-coalgebras that are not the diagonal in spectra by Spanier-Whitehead duality, we also obtain the following.

Corollary 5.2.4. *Let A be a positive flat cofibrant commutative $\mathbb{S}$-algebra in Sp^{Σ}. Then the ∞-category of A-modules $\mathcal{Mod}_A(\mathcal{Sp})$ does not satisfy the cocommutative rigidification. In particular, for $A = \mathbb{S}$ we have:*

$$\mathcal{CoAlg}_{\mathbb{E}_\infty}(\mathcal{Sp}) \not\simeq \mathscr{N}(\mathsf{CoCAlg}_{\mathbb{S}}(\mathsf{Sp}^{\Sigma})) \left[\mathsf{W}^{-1}\right].$$

CHAPTER 6

COMODULES IN CHAIN COMPLEXES

Let Ch_R be the category of unbounded chain complexes of R-modules. It is a symmetric monoidal category $(\mathsf{Ch}_R, \otimes_R, R)$. Subsequently, we may write $\otimes_R$ simply as $\otimes$. A *differential graded coalgebra* is a comonoid in Ch_R. We show, in Theorem 6.3.3, that when R is a finite product of fields, the projective model structure on non-negative chain complexes $\mathsf{Ch}_R^{\geq 0}$ satisfies the comodular rigidification when it is over a simply connected differential graded coalgebra. We also show in Theorem 6.4.7 that Ch_R satisfies rigidification for any comodules over a differential graded coalgebra that is perfect as a chain complex.

We saw in Example 4.1.2 that rigidification of coassociative coalgebras in non-negative chain complexes $\mathsf{Ch}_R^{\geq 0}$ over a commutative ring R might not be satisfied (when we endow it with the projective model structure). Therefore our results in this chapter show that comodules are less pathological than coalgebras in chain complexes.

6.1 Barr-Beck-Lurie Comonadicity Theorem

We invite the reader to look at the definition of monadicity in ∞-categories in (Lurie, 2017, 4.7.3.4). A functor $\mathcal{C} \to \mathcal{D}$ of ∞-categories will be called *comonadic* if its opposite $\mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ is monadic. More precisely, a left adjoint functor $\mathcal{C} \to \mathcal{D}$ in ∞-categories exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$ if $\mathcal{C}$ is equivalent to coalgebras over the comonad over $\mathcal{D}$ determined by the adjunction.

We recall a necessary and sufficent condition for a left adjoint functor to be comonadic. This is an analogue to the situation in ordinary categories where a left adjoint L is comonadic if and only it preserves L-split equalizers. The ∞-categorical notion of L-split coaugmented cosimplical objects is entirely dual to the simplicial analogue described in (Lurie, 2017, 4.7.2.2).

Theorem 6.1.1 ((Lurie, 2017, 4.7.3.5) Barr-Beck-Lurie Comonadicity Theorem)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ in ∞-categories exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$ if and only if it admits a right adjoint, is conservative, and preserves all limits of F-split coaugmented cosimplicial objects.*

Theorem 6.1.2 ((Lurie, 2017, 4.7.3.16))**.** *A functor $V : \mathcal{C} \to \mathcal{C}'$ is an equivalence of ∞-categories if there is a left adjoint functor $F' : \mathcal{C}' \to \mathcal{D}$ such that F' and $F' \circ V$ exhibit both $\mathcal{C}$ and $\mathcal{C}'$ as comonadic over $\mathcal{D}$ over the same comonad. More precisely, given the following diagram of ∞-categories where V commutes with the left adjoints:*

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, shift left, "F"] \arrow[dr, phantom, "\perp"] \arrow[dd, "V"'] & \\
& \mathcal{D} \arrow[ul, shift left, "G"] \arrow[dl, shift left, "G'"] \arrow[dl, phantom, "\perp"] \\
\mathcal{C}' \arrow[ur, shift left, "F'"] &
\end{tikzcd}$$

the functor $V : \mathcal{C} \to \mathcal{C}'$ is an equivalence of ∞-categories if:

- *the functor $F : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$;*
- *the functor $F' : \mathcal{C}' \to \mathcal{D}$ exhibits $\mathcal{C}'$ as comonadic over $\mathcal{D}$;*
- *the canonical map $(F \circ G) \to (F' \circ G')$ is an equivalence of functors.*

Proposition 6.1.3 ((Lurie, 2017, 4.7.2.5))**.** *Let $\mathcal{C}$ be a monoidal ∞-category. Given any $\mathbb{A}_\infty$-coalgebra C in $\mathcal{C}$, the forgetful functor exhibits the ∞-category of (right) C-comodule $\mathcal{CoMod}_C(\mathcal{C})$ as comonadic over $\mathcal{C}$.*

The following argument appeared in the proof of Theorem 0.3 in (Heuts, 2018). Given a pair of adjoint functors $L : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : R$, we define the *canonical RL-resolution* which is the following L-split coaugmented cosimplicial object in $\mathcal{C}$, induced by the comonad LR on $\mathcal{D}$:

$$X \longrightarrow RL(X) \rightrightarrows RLRL(X) \, \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \, RLRLRL(X) \cdots .$$

We shall denote the L-split coaugmented cosimplicial object by $X \to RL^{\bullet+1}(X)$.

Proposition 6.1.4. *Given a pair of adjoint functors $L : \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D} : R$ in ∞-categories, such that L is conservative. Then L is comonadic if and only if the map $X \xrightarrow{\simeq} \ell im^{\mathcal{C}}_{\Delta}(RL^{\bullet+1}(X))$ is an equivalence for all objects X in $\mathcal{C}$.*

Proof. We show the sufficient condition. Let $X^{-1} \to X^\bullet$ be an L-split cosimplicial object of $\mathcal{C}$. We have the following square:

$$\begin{tikzcd}
X^{-1} \arrow[r] \arrow[d] & \ell im^{\mathcal{C}}_{\Delta}(X^\bullet) \arrow[d] \\
\ell im^{\mathcal{C}}_{\Delta}\left(RL^{\bullet+1}(X^{-1})\right) \arrow[r] & \ell im^{\mathcal{C}}_{\Delta\times\Delta}\left(RL^{\bullet+1}(X^\bullet)\right).
\end{tikzcd}$$

The vertical maps are equivalences by assumption. The bottom horizontal map is an equivalence as $X^{-1} \to X^\bullet$ is L-split. Indeed, we have $L(X^{-1}) \simeq \ell im^{\mathcal{D}}_{\Delta} L(X^\bullet)$, and since R preserves limits, we get $RL(X^{-1}) \simeq \ell im^{\mathcal{C}}_{\Delta} RL(X^\bullet)$. Since the coaugmented cosimplicial object

$RL(X^{-1}) \to RL(X^{\bullet})$ remains L-split, we can reiterate our argument and thus show that the bottom horizontal map is an equivalence in $\mathcal{C}$. Therefore the top horizontal map is an equivalence, as desired. □

6.2 Model Category for Comodules

Recall there exist two model categories on chain complexes. The first one is called the *projective model structure*, denoted $(\mathsf{Ch}_R)_{\mathsf{proj}}$, where its weak equivalences are the quasi-isomorphisms and the fibrations are the levelwise epimorphisms. All objects are fibrant. It is cofibrantly generated by a pair of sets, see (Hovey, 1999, 2.3.11). It is a symmetric monoidal model category. The second one is called the *injective model structure*, denoted $(\mathsf{Ch}_R)_{\mathsf{inj}}$, where its weak equivalences are the quasi-isomorphisms and the cofibrations are the levelwise monomorphisms. All objects are cofibrant. It is cofibrantly generated, see (Hovey, 1999, 2.3.13). It is *not* in general a monoidal model category. The identity functor on Ch_R gives the following Quillen equivalences:

$$(\mathsf{Ch}_R)_{\mathsf{proj}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathsf{Ch}_R)_{\mathsf{inj}}.$$

We shall also be interested in the particular case where $R = \Bbbk$ is a *finite product of fields.* It is a commutative ring $\Bbbk$ such that it is a product in commutative rings: $\Bbbk = \Bbbk_1 \times \cdots \times \Bbbk_n$, where each $\Bbbk_i$ is a field, for some $1 \leq n < \infty$. In the literature, such rings are referred to as *commutative semisimple Artinian rings.* For instance, if the integer n is the product of distinct prime numbers $p_1 \cdots p_n$, then the commutative ring $\mathbb{Z}/n\mathbb{Z}$ is a finite product of fields. In the case where $\Bbbk$ is a finite product of fields, then the model structures above are equal: $(\mathsf{Ch}_{\Bbbk})_{\mathsf{proj}} = (\mathsf{Ch}_{\Bbbk})_{\mathsf{inj}}$. In particular, its fibrations and cofibrations are levelwise epimorphisms and monomorphisms respectively. All objects are cofibrant and fibrant. If we restrict to the full subcategory $\mathsf{Ch}_{\Bbbk}^{\geq 0} \subseteq \mathsf{Ch}_{\Bbbk}$ of non-negative chain complexes, we obtain a model category for $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ where the weak-equivalences are the quasi-isomorphisms, the fibrations are positive levelwise epimorphisms, and the cofibrations are levelwise monomorphisms. In fact, this model structure is left-induced from the adjunction $\mathsf{Ch}_{\Bbbk}^{\geq 0} \underset{\longleftarrow}{\overset{\hookrightarrow}{\perp}} \mathsf{Ch}_{\Bbbk}$, where the right adjoint is the 0-th truncation functor $\tau_{\geq 0} : \mathsf{Ch}_{\Bbbk} \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ defined in (Weibel, 1994, 1.2.7).

A differential graded R-coalgebra is a comonoid in Ch_R. Let us describe model structures for right comodules over a differential graded R-coalgebra.

Proposition 6.2.1 ((Hess et al., 2017, 6.3.7))**.** *Let R be any commutative ring. Let C be a differential graded R-coalgebra. Then the category of right C-comodules in Ch_R admits a model structure left induced from the injective model structure $(\mathsf{Ch}_R)_{\mathsf{inj}}$, via the forgetful-cofree adjunction:*

$$\begin{tikzcd} \mathsf{CoMod}_C(\mathsf{Ch}_R) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_R. \arrow[l, shift left, "-\otimes C"] \end{tikzcd}$$

In particular U preserves and reflects cofibrations and weak equivalences.

Definition 6.2.2. We denote $(\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{inj}}$ the model structure constructed in Proposition 6.2.1 and call it the *injective model structure* on the category of right C-comodules in Ch_R.

In general, it is not possible to induce a model structure on $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ from the projective model structure on chain complexes unless R is a finite product of fields. However, we shall see in Proposition 6.4.9 it is possible to induce a model structure for certain choices of differential graded coalgebras.

Proposition 6.2.3. *The injective model structure $(\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{inj}}$ is combinatorial.*

Proof. Apply Proposition A.3.3. ☐

We can adapt the arguments to the non-negative case.

Proposition 6.2.4. *Let $\Bbbk$ be a finite product of fields. Let C be a non-negative differential graded R-coalgebra. Then the category of right C-comodules in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ admits a combinatorial model category left induced from the forgetful-cofree adjunction:*

$$\begin{tikzcd} \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_{\Bbbk}^{\geq 0}. \arrow[l, shift left, "-\otimes C"] \end{tikzcd}$$

In particular U preserves and reflects cofibrations and weak equivalences.

Notice that $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is enriched, tensored and cotensored over $\mathsf{Ch}_{\Bbbk}^{\geq 0}$, the tensor product $\mathsf{Ch}_{\Bbbk}^{\geq 0} \times \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \xrightarrow{\otimes} \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is given by $(M, X) \mapsto M \otimes X$ where the right C-coaction is induced on X. It is then elementary to show that $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is a $(\mathsf{Ch}_{\Bbbk}^{\geq 0})$-model category in the sense of (Hovey, 1999, 4.2.18). In particular, this shows that $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is a simplicial model category.

6.3 The Simply Connected Case

We state and show here our main theorem. We shall make use of the results in the appendices, in particular we shall need Corollary B.3.15. We first start by a definition.

Definition 6.3.1. Let R be any commutative ring. A differential graded R-coalgebra C is 1-*connected* or *simply connected* if: $C_0 = R$, $C_1 = 0$ and $C_i = 0$ for all $i < 0$.

Remark 6.3.2. Any simply connected differential graded R-coalgebra C is naturally coaugmented, i.e., there is a map of coalgebras $\eta : R \to C$ which is trivial in every non-zero degree, and in degree zero is the identity id_R.

Theorem 6.3.3. *Let $\Bbbk$ be a finite product of fields. Let C be a simply connected differential graded $\Bbbk$-coalgebra. Then $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ satisfies rigidification of comodules over C: we have the following equivalence of ∞-categories: $\mathscr{N}\left(\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})\right)\left[\mathsf{W}^{-1}\right] \simeq \mathscr{C}o\mathscr{M}o\mathscr{d}_C\left(\mathscr{D}^{\geq 0}(\Bbbk)\right)$.*

The canonical $((- \otimes C) \circ U)$-resolution is the following U-split coaugmented cosimplicial object in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$X \longrightarrow U(X) \otimes C \rightrightarrows U\Big(U(X) \otimes C\Big) \otimes C \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots .$$

We shall denote it simply by $X \to \Omega^\bullet(X, C, C)$ and refer to it as *the cobar resolution of the C-comodules X*. Since $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is a simplicial model category, homotopy limits over cosimplicial diagrams are computed as in (Hirschhorn, 2003, 18.1.8). We denote the homotopy limit of the cosimplicial diagram $\Omega^\bullet(X, C, C)$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ by $\Omega(X, C, C)$. Notice that each object in the cosimplicial diagram $\Omega^\bullet(X, C, C)$ is a right cofree C-comodule, hence fibrant. Thus $\Omega(X, C, C)$ is a fibrant right C-comodule by (Hirschhorn, 2003, 18.5.2).

Remark 6.3.4. We warn the reader that in the literature $\Omega(X, C, C)$ denotes the homotopy limit in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ (which is obviously quasi-isomorphic to X since $\Omega^\bullet(X, C, C)$ is U-split) and not in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$. But as we will show in Lemma 6.3.6, this distinction won't matter.

Lemma 6.3.5. *Let $\mathbb{k}$ be a finite product of fields. Let C be a simply connected differential graded $\mathbb{k}$-coalgebra. Let $X = M \otimes C$ be a cofree right C-comodule. Then the cobar resolution of X induces a weak equivalence $X \xrightarrow{\simeq} \Omega(X, C, C)$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$, i.e., a quasi-isomorphism.*

Proof. Regard C as a right C-comodule via its comultiplication $\Delta : C \to C \otimes C$. Then the coaugmented cosimplicial diagram $C \to \Omega^\bullet(C, C, C)$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$:

$$C \longrightarrow C^{\otimes 2} \rightrightarrows C^{\otimes 3} \cdots,$$

splits in the Dwyer-Kan localization $\mathscr{N}\left(\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})\right)\left[\mathsf{W}^{-1}\right]$ via the map of C-comodules $\varepsilon \otimes \mathsf{id}_C : C^{\otimes 2} \to C$, where $\varepsilon : C \to S^0$ is the counit of C.

As we are working over a finite product of fields, tensoring with a chain complex M preserves monomorphisms and quasi-isomorphisms, hence the functor $M\otimes : \mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0}) \to \mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ is left Quillen, and thus induces a derived functor on the Dwyer-Kan localizations $M \otimes - : \mathscr{N}\left(\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})\right)\left[\mathsf{W}^{-1}\right] \to \mathscr{N}\left(\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})\right)\left[\mathsf{W}^{-1}\right]$, thus it preserves split cosimplicial objects. From the isomorphism of cosimplicial diagrams: $M \otimes \Omega^\bullet(C, C, C) \cong \Omega^\bullet(M \otimes C, C, C)$, we get the quasi-isomorphism $M \otimes C \simeq M \otimes \Omega(C, C, C) \simeq \Omega(M \otimes C, C, C)$. $\square$

Lemma 6.3.6. *Let $\mathbb{k}$ be a finite product of fields. Let C be a simply connected differential graded $\mathbb{k}$-coalgebra. Let X be any right C-comodule in $\mathsf{Ch}_{\mathbb{k}}^{\geq 0}$. Then the cobar resolution of X induces a weak equivalence $X \xrightarrow{\simeq} \Omega(X, C, C)$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$, i.e., a quasi-isomorphism.*

Proof. We make use of Corollary B.3.15 and Definition B.3.16. Let $\{X(n)\}$ be the Postnikov tower of X, and denote by $\widetilde{X}$ the (homotopy) limit of the tower in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$. Then the acyclic cofibration $X \overset{\simeq}{\hookrightarrow} \widetilde{X}$ induces an objectwise weak equivalence $\Omega^\bullet(X, C, C) \to \Omega^\bullet(\widetilde{X}, C, C)$ between objectwise fibrant cosimplicial diagrams. Thus $\Omega(X, C, C) \to \Omega(\widetilde{X}, C, C)$ is a weak equivalence by (Hirschhorn, 2003, 18.5.3). Therefore it suffices to show that $\widetilde{X} \to \Omega(\widetilde{X}, C, C)$ is a weak equivalence.

Since the Postnikov tower $\{X(n)\}$ stabilizes in each degree, we have the weak equivalence $U(\mathsf{holim}^C_n X(n)) \simeq \mathsf{holim}_n U(X(n))$. Since the functor $- \otimes C : \mathsf{Ch}^{\geq 0}_{\mathbb{k}} \to \mathsf{CoMod}_C(\mathsf{Ch}^{\geq 0}_{\mathbb{k}})$ is right Quillen, we also obtain the weak equivalence $(U(\mathsf{holim}^C_n X(n))) \otimes C \simeq \mathsf{holim}^C_n (U(X(n)) \otimes C)$. Notice that the tower $\Big\{U(X(n)) \otimes C\Big\}$ also stabilizes in each degree by Lemma B.3.5. The maps in that tower are fibrations in $\mathsf{CoMod}_C(\mathsf{Ch}^{\geq 0}_{\mathbb{k}})$ and in $\mathsf{Ch}^{\geq 0}_{\mathbb{k}}$. Thus the homotopy limit can also be computed in $\mathsf{Ch}^{\geq 0}_{\mathbb{k}}$. Therefore:

$$\Omega(\widetilde{X}, C, C) \simeq \mathsf{holim}^C_n \Big(\Omega(X(n), C, C)\Big).$$

Hence it is enough to show that for all $n \geq 0$, the canonical maps $X(n) \to \Omega(X(n), C, C)$ are weak equivalences in $\mathsf{CoMod}_C(\mathsf{Ch}^{\geq 0}_{\mathbb{k}})$, i.e., quasi-isomorphisms.

We shall prove it inductively. For $n = 0$, we have $X(0) = 0$ and the map is trivial and hence a quasi-isomorphim. For $n = 1$, we know that $X(1)$ is a cofree right C-comodule, and hence, by Lemma 6.3.5, we have $X(1) \to \Omega(X(1), C, C)$ is a quasi-isomorphism. Suppose now that we have shown $X(n) \to \Omega(X(n), C, C)$ is a quasi-isomorphism for some $n \geq 1$. Then by construction, the comodule $X(n+1)$ is the following homotopy pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd} X(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \otimes C \arrow[d] \\ X(n) \arrow[r] & S^n(V) \otimes C. \end{tikzcd}$$

By (Hirschhorn, 2003, 18.5.2), it induces a homotopy pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd} \Omega(X(n+1), C, C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega(D^n(V) \otimes C, C, C) \arrow[d] \\ \Omega(X(n), C, C) \arrow[r] & \Omega(S^n(V) \otimes C, C, C). \end{tikzcd}$$

Since $X(n)$, $S^n(V) \otimes C$ and $D^n(V) \otimes C$ are weakly equivalent to their respective homotopy limits of their cobar cosimplicial resolutions, either by induction or by Lemma 6.3.5, we get then that $X(n+1) \to \Omega(X(n+1), C, C)$ is a weak equivalence since homotopy pullbacks preserve weak equivalences. $\square$

Proof of Theorem 6.3.3. Since the forgetful functor $U : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ preserves and reflects weak equivalences by definition of the model structures, we immediately get that the left Quillen derived functor $\mathscr{N}(\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}))\left[\mathsf{W}^{-1}\right] \to \mathscr{D}^{\geq 0}(\Bbbk)$ is conservative. By (Lurie, 2017, 1.3.4.23, 1.3.4.25), homotopy limits over cosimplicial diagrams in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ correspond exactly to limits over cosimplicial diagrams in the ∞-categorical sense. Hence the left Quillen derived forgetful functor is comonadic by Lemma 6.3.6 and Proposition 6.1.4. We can conclude by Theorem 6.1.2 as Proposition 6.1.3 shows that $\mathcal{CoMod}_C(\mathcal{D}^{\geq 0}(\Bbbk))$ is also comonadic over the same comonad $-\otimes C : \mathcal{D}^{\geq 0}(\Bbbk) \to \mathcal{D}^{\geq 0}(\Bbbk)$. $\square$

6.4 The Perfect Case

We let R be any commutative ring. In general, we have the forgetful-cofree adjunction:

$$\begin{tikzcd} \mathsf{CoMod}_C(\mathsf{Ch}_R) \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{Ch}_R. \arrow[l, shift left, "-\otimes C"] \end{tikzcd}$$

We are interested here to investigate when the forgetful functor U is a *right adjoint*. We begin by recalling the following classical results.

Definition 6.4.1. A chain complex X in Ch_R is said to be *flat over R* if the induced functor $-\otimes X : \mathsf{Ch}_R \to \mathsf{Ch}_R$ preserves monomorphisms. In other words, the chain complex X is flat if it is a chain complex of flat R-modules.

The next lemma is a classical result.

Lemma 6.4.2. *Let X be any chain complex over R. The following are equivalent.*

- **(i)** *The functor $-\otimes X : \mathsf{Ch}_R \to \mathsf{Ch}_R$ preserves equalizers (i.e. is left exact).*
- **(ii)** *The chain complex X is flat over R.*

We obtain the following result since $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ is the category of coalgebras over the comonad $-\otimes C : \mathsf{Ch}_R \to \mathsf{Ch}_R$.

Lemma 6.4.3. *Let C be any differential graded coalgebra over R. The following are equivalent:*

(i) *The forgetful functor* $U : \mathsf{CoMod}_C(\mathsf{Ch}_R) \to \mathsf{Ch}_R$ *preserves equalizers.*

(ii) *The chain complex* C *is flat over* R*.*

Similarly we have the following result, perhaps less well known.

Lemma 6.4.4. *Let* X *be a chain complex over* R*. The following are equivalent.*

(i) *The functor* $- \otimes X : \mathsf{Ch}_R \to \mathsf{Ch}_R$ *preserves infinite products.*

(ii) X *is a bounded chain complex of finitely presented* R*-modules.*

Proof. This follows directly from the fact that, for any R module M, the functor $- \otimes M$ in R-modules preserves infinite products if and only if M is finitely presented as an R-module (see (Brzezinski and Wisbauer, 2003, 40.17)). $\square$

Definition 6.4.5. A *perfect chain complex* in Ch_R is a bounded chain complex of finitely generated projective R-modules.

Lemma 6.4.6. *Let* C *be any differential graded coalgebra over* R*. Then the following are equivalent.*

(i) *The forgetful functor* $U : \mathsf{CoMod}_C(\mathsf{Ch}_R) \to \mathsf{Ch}_R$ *is a right adjoint.*

(ii) *The coalgebra* C *is a perfect chain complex.*

Proof. This is a combination of the previous lemmas. Notice that any flat bounded chain complex of finitely presented R-modules is precisely a perfect chain complex over R. $\square$

Theorem 6.4.7. *Let R be any commutative ring. Let C be a differential graded-R coalgebra that is a perfect as a chain complex. Then the projective model structure on Ch_R satisfies rigidification of comodules over C. In particular, we obtain the following equivalence of ∞-categories:*

$$\mathscr{N}\Big(\mathsf{CoMod}_C(\mathsf{Ch}_R)\Big)\left[\mathsf{W}^{-1}\right] \simeq \mathscr{CoMod}_C(\mathscr{D}(R)),$$

where W is the class of quasi-isomorphisms between C-comodules in Ch_R.

We shall prove the above theorem later in the section. We first make the following observation.

Remark 6.4.8. The result of Theorem 6.4.7 is perhaps not surprising as we have the following. For any chain complex X in Ch_R, let us denote $X^* = \mathsf{Hom}_{\mathsf{Ch}_R}(X, R)$ its linear dual. For any differential graded coalgebra C in Ch_R, we have a faithful functor towards the category of left C^*-modules:

$$\mathsf{CoMod}_C(\mathsf{Ch}_R) \longrightarrow {}_{C^*}\mathsf{Mod}(\mathsf{Ch}_R).$$

Indeed, to any right C-comodule $\rho : X \to X \otimes C$, we associate a left C^*-modules by:

$$C^* \otimes X \xrightarrow{\mathsf{id}_{C^*}\otimes\rho} C^* \otimes X \otimes C \xrightarrow{\text{evaluation}} R \otimes X \cong X.$$

One can easily check that the functor is an equivalence of categories whenever C is a perfect chain complex. Therefore, rigidification of C-comodules is equivalent to rigidification of C^*-modules, which is already known.

From the identification of right C-comodule with left C^*-modules, we can view the differential graded algebra C^* as a right C-comodule. Then since the free module functor $-\otimes C^* : \mathsf{Ch}_R \to {}_{C^*}\mathsf{Mod}(\mathsf{Ch}_R)$ is the left adjoint of the forgetful functor $U : {}_{C^*}\mathsf{Mod}(\mathsf{Ch}_R) \to \mathsf{Ch}_R$, it is also the left adjoint of the forgetful functor on comodules $U : \mathsf{CoMod}_C(\mathsf{Ch}_R) \to \mathsf{Ch}_R$. In particular we get the following result.

Proposition 6.4.9. *Let R be any commutative ring. Let C be a differential graded coalgebra over R such that it is a perfect chain complex. Then the category of right C-comodules in Ch_R admits model categories right induced from the projective model structure $(\mathsf{Ch}_R)_{\mathsf{proj}}$, via the free-forgetful adjunction:*

$$\begin{tikzcd} \mathsf{Ch}_R \arrow[r, shift left, "-\otimes C^*"] \arrow[r, phantom, "\perp"] & \mathsf{CoMod}_C(\mathsf{Ch}_R), \arrow[l, shift left, "U"] \end{tikzcd}$$

were C^* is regarded as a right C-comodule. In particular, the weak equivalences and fibrations in $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ are precisely the underlying quasi-isomorphisms and projective fibrations. The generating cofibrations and acyclic cofibrations are the sets $\{S^n \otimes C^* \hookrightarrow D^{n+1} \otimes C^*\}_{n\in\mathbb{Z}}$ and $\{0 \to D^n \otimes C^*\}_{n\in\mathbb{Z}}$ respectively, and the model category is combinatorial.

Proof. Recall that both categories are presentable (see Proposition 2.1.11). The projective model structure $(\mathsf{Ch}_R)_{\mathsf{proj}}$ is cofibrantly generated by the pair of sets $I = \{S^n \hookrightarrow D^{n+1}\}_{n\in\mathbb{Z}}$ and $J = \{0 \to D^n\}_{n\in\mathbb{Z}}$. Thus the sets $I \otimes C^*$ and $J \otimes C^*$ permit the small object argument. The functor U takes relative $(J \otimes C^*)$-cell complexes to weak equivalences as U preserves all colimits. We conclude by (Hirschhorn, 2003, 11.3.2). $\square$

Definition 6.4.10. We denote $(\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{proj}}$ the model structure constructed in Proposition 6.4.9 and call it the *projective model structure* on the category of right C-comodules in Ch_R.

Proposition 6.4.11. *Let R be any commutative ring. Let C be a differential graded R-coalgebra that is perfect as a chain complex. Then the projective and injective model structures are Quillen equivalent:*

$$(\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{proj}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{inj}}.$$

Proof. The generating projective acyclic cofibrations $0 \hookrightarrow D^n \otimes C^*$ are clearly injective acyclic cofibrations, i.e. are levelwise monomorphisms and quasi-isomorphisms. Let $X \to Y$ be an injective fibration of right C-comodules. Then in the diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook, "\simeq"] & X \arrow[d] \\ D^n \otimes C^* \arrow[r] & Y. \end{tikzcd}$$

there is always a lift $D^n \otimes C^* \to X$ as $0 \hookrightarrow D^n \otimes C^*$ is also an injective acyclic cofibration. $\square$

Proof of Thoerem 6.4.7. We apply Theorem 6.1.2. Since C is a perfect chain complex, it is cofibrant in the projective model structure of Ch_R. Thus the natural functor:

$$\gamma_C : \mathscr{N}\Big(\mathsf{CoMod}_C(\mathsf{Ch}_R)\Big)\left[\mathsf{W}^{-1}\right] \longrightarrow \mathscr{C}o\mathscr{M}od_C(\mathscr{D}(R)),$$

induces an obvious equivalence of the comonads $-\otimes C : \mathscr{D}(R) \to \mathscr{D}(R)$. We are only left to show that the (derived) forgetful functor:

$$\mathscr{N}\Big(\mathsf{CoMod}_C(\mathsf{Ch}_R)\Big)\left[\mathsf{W}^{-1}\right] \longrightarrow \mathscr{D}(R),$$

exhibits the Dwyer-Kan localization as comonadic over $\mathscr{D}(R)$. This follows directly from (Lurie, 2017, 1.3.4.23, 1.3.4.25) and the fact that $U : (\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{proj}} \to (\mathsf{Ch}_R)_{\mathsf{proj}}$ is a right Quillen functor and thus preserves all homotopy limits. $\square$

We shall show in Theorem B.4.1 that $(\mathsf{CoMod}_C(\mathsf{Ch}_R))_{\mathsf{inj}}$ also admits a Postnikov presentation and therefore also allows inductive arguments to compute limits in $\mathscr{C}o\mathscr{M}o\mathscr{d}_C(\mathscr{D}(R))$ whenevever C is a perfect chain complex.

Remark 6.4.12. Our argument in Theorem 6.4.7 can be generalized to any closed symmetric monoidal combinatorial model category $(\mathsf{M}, \otimes, \mathbb{I})$. A sufficient condition on an object X such that the functor $-\otimes X : \mathsf{M} \to \mathsf{M}$ preserves all limits is to require the object X to be *strong dualizable* in the monoidal category (see (Dold and Puppe, 1980, 1.2) for a definition). In that case, just as in Proposition 6.4.9, we can right induced a model category from M to the category of right C-comodule in M, if C is a strong dualizable object in M. Then, just as in Theorem 6.4.7, we obtain:

$$\mathscr{N}\Big(\mathsf{CoMod}_C(\mathsf{M})\Big)\left[\mathsf{W}_{\mathsf{CoMod}}^{-1}\right] \simeq \mathscr{C}o\mathscr{M}o\mathscr{d}_C\Big(\mathscr{N}(\mathsf{M})\left[\mathsf{W}^{-1}\right]\Big),$$

for any coalgebra C in M that is strong dualizable. When $\mathsf{M} = \mathsf{Ch}_R$, a chain complex is strong dualizable if and only if it is a perfect chain complex (see (Dold and Puppe, 1980, 1.6)). In practice though, strong dualizable objects are rare in a non algebraic context. For instance, a free pointed space X_+ is strong dualizable in Top_*, the category of pointed spaces together with the smash product, if and only if $X = *$. Therefore, the category $\mathsf{CoMod}_{X_+}(\mathsf{Top}_*)$ is isomorphic to Top_* and the case is vacuous. We can argue similarly that, in symmetric spectra, the symmetric spectrum $\Sigma^\infty X_+$ is strong dualizable if and only if X is a point.

CHAPTER 7

DERIVED COTENSOR OF COMODULES

We saw in Theorem 6.3.3 that there is a correspondance between (left or) right strict C-comodules and (left or) right homotopically coherent C-comodules in non-differentially graded context, over a finite product of fields, whenever C is simply connected. We show here, in Theorem 7.5.2, that we can also lift a symmetric monoidal structure via the cotensor product of comodules. This shows that the ∞-category of comodules over a simply connected coalgebra in connective $H\Bbbk$-modules is endowed with a symmetric monoidal structure given by the derived cotensor product, which is equivalent to the cobar resolution.

Throughout this chapter, let $\Bbbk$ be a finite product of fields and let C be a simply connected differential graded $\Bbbk$-coalgebra. We shall always assume C to be cocommutative so that $\mathcal{C}o\mathcal{M}o\mathcal{d}_C(\mathcal{D}^{\geq 0}(\Bbbk))$ represents both left and right C-comodules. All the results in this chapter would remain true if we consider the ∞-category of bicomodules over a non-cocommutative C, but we choose C to be cocommutative for simplicity. We shall write CoMod_C instead of $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ and $\mathcal{C}o\mathcal{M}o\mathcal{d}_C$ instead of $\mathcal{C}o\mathcal{M}o\mathcal{d}_C(\mathcal{D}^{\geq 0}(\Bbbk))$.

7.1 Definition and Properties

We begin by introducing the main construction of this chapter which is the cotensor product of C-comodules.

Definition 7.1.1. Let X and Y be C-comodules in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. Define their *cotensor product* $X\Box_C Y$ to be the following equalizer in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$:

$$X\Box_C Y \longrightarrow X \otimes Y \rightrightarrows X \otimes C \otimes Y,$$

where the two parallel morphisms are induced by the coactions $X \to X \otimes C$ and $Y \to C \otimes Y$.

Lemma 7.1.2. *The cotensor $X\Box_C Y$ is endowed with a C-comodule structure.*

Proof. Since $- \otimes C : \mathsf{Ch}_{\Bbbk}^{\geq 0} \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ preserves equalizers, we obtain the following dashed map below by universality of equalizers:

$$\begin{tikzcd} X\Box_C Y \arrow[r] \arrow[d, dashed] & X \otimes Y \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \otimes C \otimes Y \arrow[d] \\ (X\Box_C Y) \otimes C \arrow[r] & X \otimes Y \otimes C \arrow[r, shift left] \arrow[r, shift right] & X \otimes C \otimes Y \otimes C. \end{tikzcd}$$

We can check easily that the map is a coaction of a C-comodule. $\Box$

Lemma 7.1.3 ((Eilenberg and Moore, 1966, 2.2))**.** *For any C-comodule X, we have $X\Box_C C \cong X \cong C\Box_C X$.*

Lemma 7.1.4 ((Eilenberg and Moore, 1966, 2.1))**.** *Let M be a non-negative chain complex. Then for any cofree comodule $M \otimes C$ we have: $(M \otimes C)\Box_C X \cong M \otimes X$ and $X\Box_C(C \otimes M) \cong X \otimes M$.*

Proposition 7.1.5. *The cotensor product defines a symmetric monoidal structure on C-comodules and we shall denote it $(\mathsf{CoMod}_C, \Box_C, C)$.*

Proposition 7.1.6. *Let X be a C-comodule. Then $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is a left exact functor that preserves finite limits and filtered colimits.*

Proof. This follows directly from the fact that, when over a finite product of fields, any chain complex M induces a functor $M \otimes - : \mathsf{Ch}_{\Bbbk}^{\geq 0} \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ that preserves finite limits and all colimits. The cotensor product preserves filtered colimits as equalizers in presentable categories commute with filtered colimits. $\square$

Remark 7.1.7. For a general C-comodule X, there is no reason to expect that the functor $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is a left nor a right adjoint. We shall see in Propositions 7.3.2 and 7.3.7 that when X is fibrant, then $X\square_C -$ is a left adjoint. So up to weak equivalence, we can always have $X\square_C -$ being a left adjoint. In (Takeuchi, 1977), the author introduced the notion of *quasi-finite* C-comodules. Essentially, a C-comodule X is quasi-finite if and only if $X\square_C -$ is a right adjoint. However, it is easy to see that a C-comodule is not weakly equivalent to a quasi-finite one. For instance, if we choose $C = \Bbbk$, then X is quasi-finite if and only if X is a perfect chain complex.

In order to produce a derived cotensor product of C-comodule, we shall show that the cotensor product almost defines a *co-monoidal model category*. Indeed, we will prove that when X is a fibrant C-comodule, then $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is a functor that preserves fibrant objects (Proposition 7.5.1) and weak equivalences (Corollary 7.4.2). It won't be a co-monoidal model category as $X\square_C -$ does not preserves all fibrations, see Remark 7.5.3. Surprisingly, when X is fibrant, we will have $X\Box_C-$ is a left Quillen functor (Corollary 7.4.3) but this fact alone won't allow us to derive it.

One core issue with the cotensor product is that it does not behave well with non-finite limits. However, the key point in this chapter is that the cotensor product does behave well with respect to Postnikov towers.

Lemma 7.1.8. *Let $\{X(n)\}$ be a Postnikov tower of a C-comodule X. Let Y be any C-comodule. Then $\{X(n)\Box_C Y\}$ stabilizes in each degree and $(\mathsf{lim}^C_n X(n))\Box_C Y \cong \mathsf{lim}^C_n(X(n)\Box_C Y)$. In particular, if we denote $\widetilde{X}$ the limit of $\{X(n)\}$, then the Postnikov tower of $\widetilde{X}\Box Y$ is given by $\{X(n)\Box_C Y\}$.*

Proof. Equalizers of towers that stabilize in each degree also stabilize in each degree. Then the result follows from Lemma B.3.5. $\square$

7.2 The CoTor Functor

The category of C-comodules CoMod_C is (Grothendieck) Abelian and has enough injective objects. More specifically, any injective chain complex M in $\mathsf{Ch}^{\geq 0}_{\Bbbk}$ induces an injective C-comodule $M\otimes C$, and thus we easily see that any C-comodule X can be embedded into an injective C-comodule. Thus we can derive the cotensor product in the sense of Abelian categories.

Remark 7.2.1. Notice that a C-comodule is injective if and only if it acyclic fibrant in CoMod_C. We precisely used the fact that CoMod_C has enough injective objects in Lemma B.3.7.

Definition 7.2.2. The functor $X\square_C- : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is left exact by Proposition 7.1.6, for any C-comodule X. Define $\mathsf{CoTor}^i_C(X,-) : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ to be the i-th right derived functor of $X\square_C-$, for $i \geq 0$. More specifically, given an injective resolution of a C-comodule Y:

$$0 \longrightarrow Y \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots,$$

then $\mathsf{CoTor}^i_C(X,Y)$ is given by the i-th cohomology $H^i(X\square_C I^\bullet)$.

As usual, we have that $\mathsf{CoTor}^0_C(X,Y) = X\square_C Y$ for any comodules X and Y. If Y is an injective C-comodule, then $\mathsf{CoTor}^i_C(X,Y) = 0$ for any comodule X and $i > 0$.

Following (Eilenberg and Moore, 1966) and (Ravenel, 1986), we shall not use injective resolutions but relative injective resolutions to compute CoTor.

Definition 7.2.3 ((Ravenel, 1986, A1.2.7, A1.2.10))**.** A *relative injective C-comodule* is the direct summand of a cofree C-comodule. A *resolution by relative injectives* of a C-comodule Y is a long exact sequence in CoMod_C:

$$0 \longrightarrow Y \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots,$$

in which each J^i is a relatively injective C-comodule and the images of the maps $J^i \to J^{i+1}$ is a direct summand in $\mathsf{Ch}^{\geq 0}_{\Bbbk}$.

Proposition 7.2.4 ((Ravenel, 1986, A1.2.4, A1.2.8))**.** *Given a resolution by relative injectives $Y \to J^\bullet$ for a C-comodule Y, then for any C-comodule X, $\mathsf{CoTor}^*_C(X,Y)$ is given precisely by the cohomology of the induced cochain complex:*

$$0 \longrightarrow X \square_C J^0 \longrightarrow X \square_C J^1 \longrightarrow \cdots .$$

We shall now show that the cobar resolution induces a resolution by relative injectives. Recall from previous chapter that the cobar resolution of a C-comodule Y is the cosimplicial object $\Omega^\bullet(C,C,Y)$ in CoMod_C defined as:

$$C \otimes Y \rightleftarrows C^{\otimes 2} \otimes Y \quad \cdots ,$$

where the first coface maps are given by the coaction $Y \to C \otimes Y$ and the other cofaces maps are induced by the comultiplication $C \to C \otimes C$. The codegeneracies are induced by the counit $\varepsilon : C \to \Bbbk$.

Given any Abelian category M, recall that the conormalization functor provides an equivalence of categories $\mathsf{N}^\bullet : \mathsf{M}^\Delta \xrightarrow{\cong} \mathsf{CoCh}^{\geq 0}(\mathsf{M})$, between cosimplicial objects in M and non-negative cochain complexes of M. Given Φ a cosimplicial object in M, we have that $\mathsf{N}^i(\Phi)$ is given by Φ^0 if $i = 0$, and by the kernel of the codegeneracies:

$$\bigcap_{j=0}^{i-1} \mathsf{ker}(\Phi^j \to \Phi^{j-1}),$$

for $i \geq 1$. The differentials are given by the alternating sum of the coface maps of Φ.

If we apply the conormalization functor on $\Omega^\bullet(C, C, Y)$ then we obtain a cochain complex of C-comodules that we denote $\underline{\Omega^\bullet}(C, C, Y)$.

Definition 7.2.5. Let X and Y be C-comodules. Define the *normalized cobar resolution* of X and Y to be the cochain complex $X \Box_C \underline{\Omega^\bullet}(C, C, Y)$ in CoMod_C, which is denoted $\underline{\Omega^\bullet}(X, C, Y)$. If we denote $\underline{C}$ the unit coideal, i.e. the kernel of the counit $\varepsilon : C \to \Bbbk$, then $\underline{\Omega^n}(X, C, Y)$ is given by $X \otimes \underline{C}^{\otimes n} \otimes Y$.

Proposition 7.2.6 ((Ravenel, 1986, A1.2.12))**.** *Let X and Y be any C-comodules. Then $\underline{\Omega^\bullet}(C, C, Y)$ is a resolution by relative injectives for Y, and: $\mathsf{CoTor}^i_C(X, Y) \cong H^i(\underline{\Omega^\bullet}(X, C, Y))$, for all $i \geq 0$.*

7.3 Coflat Comodules

We introduce here a new class of C-comodules that behaves well with respect to the cotensor product. We shall see that this class includes all fibrant C-comodules.

Definition 7.3.1. A C-comodule X is said to be *coflat* if $X \Box_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is (right) exact.

The following proposition is an immidate consequence of Proposition 7.1.6.

Proposition 7.3.2. *Let X be a C-comodule. The following are equivalent:*

(i) *the C-comodule X is coflat;*

(ii) *the functor $X \Box_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ preserves all colimits;*

(iii) *the functor* $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ *is a left adjoint;*

(iv) *for any* C*-comodule* Y*, we have* $\mathsf{CoTor}^i_C(X,Y) = 0$ *for all* $i \geq 1$*.*

We see that every injective C-comodule is automatically coflat. More generally, we shall show that any fibrant C-comodule is coflat in Proposition 7.3.7 below. We first observe the following result.

Proposition 7.3.3. *Let* X *and* Y *be coflat* C*-comodules. Then* $X\square_C Y$ *is coflat. In particular, the full subcategory of coflat* C*-comodules form a symmetric monoidal category when endowed with the cotensor product.*

Proof. We consider the following exact sequence in CoMod_C:

$$0 \longrightarrow Z' \longrightarrow Z \longrightarrow Z'' \longrightarrow 0.$$

Since Y is coflat, we obtain the following exact sequence:

$$0 \longrightarrow Y\square_C Z' \longrightarrow Y\square_C Z \longrightarrow Y\square_C Z'' \longrightarrow 0.$$

Since X is coflat, we then obtain the following exact sequence:

$$0 \longrightarrow X\square_C(Y\square_C Z') \longrightarrow X\square_C(Y\square_C Z) \longrightarrow X\square_C(Y\square_C Z'') \longrightarrow 0.$$

By associativity of cotensor product, this exact sequence is equivalent to the following one:

$$0 \longrightarrow (X\Box_C Y)\Box_C Z' \longrightarrow (X\Box_C Y)\Box_C Z \longrightarrow (X\Box_C Y)\Box_C Z'' \longrightarrow 0.$$

Thus $X\Box_C Y$ is coflat by definition. $\square$

Lemma 7.3.4. *Any cofree C-comodule is coflat.*

Proof. Let $M \otimes C$ be a cofree C-comodule. The functor $(M \otimes C)\Box_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is equivalent to the functor $M \otimes - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ by Lemma 7.1.4, hence we get that it preserves exactness, as M is flat in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ (since we are working over a finite product of fields). $\square$

Lemma 7.3.5. *Coflat C-comodules are closed under extensions.*

Proof. Given a short exact sequence in CoMod_C:

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

where X and Z are coflat, let us show that Y is coflat. Let W be any C-comodule. We obtain a long exact sequence of C-comodules:

$$0 \to X\Box_C W \to Y\Box_C W \to Z\Box_C W \to \mathsf{CoTor}^1_C(X, W) \to \mathsf{CoTor}^1_C(Y, W) \to \cdots .$$

In particular, for any $i \geq 1$, we get that the following sequence is exact:

$$\mathsf{CoTor}^i_C(X,W) \longrightarrow \mathsf{CoTor}^i_C(Y,W) \longrightarrow \mathsf{CoTor}^i_C(Z,W).$$

Since X and Z are coflat, then $\mathsf{CoTor}^i_C(X,W) = 0 = \mathsf{CoTor}^i_C(Z,W)$. Thus for any C-comodule W, we have $\mathsf{CoTor}^i_C(Y,W) = 0$ for all $i \geq 1$. Hence Y is coflat. $\square$

Lemma 7.3.6. *Coflat C-comodules are closed under retracts.*

Proof. Suppose a C-comodule X is a retract of a coflat comodule Y. Then for any C-comodule Z, and any $i \geq 1$, we have that $\mathsf{CoTor}^i_C(X,Z)$ is a retract of $\mathsf{CoTor}^i_C(Y,Z) = 0$. Thus $\mathsf{CoTor}^i_C(X,Z) = 0$, hence X is coflat. $\square$

Proposition 7.3.7. *Every fibrant C-comodule is a coflat C-comodule.*

Proof. Let X be a fibrant C-comodule. By Corollary B.3.15, X is a retract of the limit $\widetilde{X}$ of its Postnikov tower $\{X(n)\}$. By Lemma 7.3.6, it is enough to show that $\widetilde{X}$ is coflat.

We first argue by induction on n that $X(n)$ is a coflat comodule. It is trivial for the case $n = 0$. The case $n = 1$ follows from Lemma 7.3.4. Suppose now that $X(n)$ is coflat, and let us show that $X(n+1)$ is coflat. Since it is given by the pullback in CoMod_C:

$$\begin{tikzcd}
X(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D^n(V) \otimes C \arrow[d] \\
X(n) \arrow[r] & S^n(V) \otimes C,
\end{tikzcd}$$

and that pullbacks conserve kernels, we obtain the short exact sequence in CoMod_C:

$$0 \longrightarrow S^{n-1}(V) \otimes C \longrightarrow X(n+1) \longrightarrow X(n) \longrightarrow 0.$$

By induction and Lemma 7.3.4, we get that $X(n+1)$ is coflat by Lemma 7.3.5.

For any short exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$, we therefore obtain a short exact sequence of towers in CoMod_C:

$$0 \longrightarrow \{X(n)\Box_C Y'\} \longrightarrow \{X(n)\Box_C Y\} \longrightarrow \{X(n)\Box_C Y''\} \longrightarrow 0.$$

Each of these towers has the Mittag-Leffler condition as we have the pullback:

$$\begin{tikzcd}
X(n+1)\Box_C Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \otimes Y \arrow[d] \\
X(n)\Box_C Y \arrow[r] & S^n(V) \otimes Y.
\end{tikzcd}$$

Thus, by Proposition 7.1.8, we obtain the following short exact sequence:

$$0 \longrightarrow \widetilde{X}\Box_C Y' \longrightarrow \widetilde{X}\Box_C Y \longrightarrow \widetilde{X}\Box_C Y'' \longrightarrow 0.$$

Thus $\widetilde{X}$ is coflat. $\square$

7.4 An Eilenberg-Moore Spectral Sequence

We would like to compute the homology of the cotensor product $X\Box_C Y$ given the homologies of $H_*(C)$, $H_*(X)$ and $H_*(Y)$. These are computed in an Eilenberg-Moore spectral sequence similar to (Eilenberg and Moore, 1966) if we require X to be coflat.

Recall that to any chain complex M, we can regard its homology $H_*(M)$ as a chain complex with trivial differentials. Then since C is simply connected, we easily verify that $H_*(C)$ is also a simply connected cocommutative differential graded coalgebra. Moreover, for any C-comodule X, we can check that $H_*(X)$ is a $H_*(C)$-comodule.

Theorem 7.4.1 (Eilenberg-Moore Spectral Sequence)**.** *Let X be a coflat C-comodule. Let Y be any C-comodule. Then there is a convergent spectral sequence:*

$$E^2_{\bullet,q} = \mathsf{CoTor}^q_{H_*(C)}(H_*(X), H_*(Y)) \Rightarrow E^\infty_{\bullet,0} = H_*(X\Box_C Y).$$

Proof. The normalized cobar resolution $\underline{\Omega^\bullet}(X,C,Y)$ of X and Y is a cochain complex of a chain complex and thus defines a second quadrant double chain complex $(\underline{\Omega^\bullet}(X,C,Y))_\bullet$, where we grade the row cohomologically, but the columns homologically. For any $p,q \geq 0$, we have:

$$(\underline{\Omega^q}(X,C,Y))_p = (X \otimes \underline{C}^{\otimes q} \otimes Y)_p.$$

Since C is simply connected, its unit coideal $\underline{C}$ is trivial in degrees 0 and 1. Therefore we obtain $(\underline{\Omega^q}(X,C,Y))_p = 0$, for $0 \leq p \leq 2q-1$. Hence the two associated spectral sequences to the double complex converge, see (McCleary, 2001, 2.15).

The first spectral sequence has its E^1-page induced by the cohomology of the rows, and therefore:

$$E^1_{\bullet,q} = H^q(\underline{\Omega^\bullet}(X,C,Y) \cong \mathsf{CoTor}^q_C(X,Y).$$

Since X is coflat, then $E^1_{\bullet,q} = 0$ for all $q \geq 1$, and we have $E^1_{\bullet,0} = X\square_C Y$. Thus the spectral sequence collapses onto its second page $E^2_{\bullet,0} = H_*(X\square_C Y)$. The second spectral sequence has its E^1-page induced by the homology of the columns, and therefore:

$$E^1_{\bullet,q} = H_*(\underline{\Omega^q}(X,C,Y)) = \underline{\Omega^q}(H_*(X), H_*(C), H_*(Y)).$$

Thus, as its E^2-page is given by the cohomology of the induced cochain complex, we obtain:

$$E^2_{\bullet,q} = \mathsf{CoTor}^q_{H_*(C)}(H_*(X), H_*(Y)).$$

It converges to the page with trivial columns except $H_*(X\square_C Y)$ as its 0-th column. $\square$

Corollary 7.4.2. *Let X be a coflat C-comodule. Let $Y \xrightarrow{\simeq} Y'$ be a weak equivalence of C-comodules. Then $X\square_C Y \xrightarrow{\simeq} X\square_C Y'$ is a weak equivalence of C-comodules.*

Proof. The weak equivalence induces an isomorphism $H_*(Y) \cong H_*(Y')$ of $H_*(C)$-comodules. Therefore we obtain:

$$\mathsf{CoTor}^q_{H_*(C)}(H_*(X), H_*(Y)) \cong \mathsf{CoTor}^q_{H_*(C)}(H_*(X), H_*(Y')),$$

for all $q \geq 0$. By Theorem 7.4.1, we obtain $H_*(X\square_C Y) \cong H_*(X\square_C Y')$ via the map $Y \to Y'$. $\square$

Corollary 7.4.3. *Let X be coflat C-comodule. Then $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ is a left Quillen functor that preserves all weak equivalences.*

7.5 Cotensor Product Closed On Fibrant Objects

We shall prove at the end of this section the following result.

Proposition 7.5.1. *If X and Y are fibrant C-comodules, then so is $X\square_C Y$. In particular, the full subcategory of fibrant C-comodules is a symmetric monoidal category when endowed with the cotensor product.*

Combining with Corollary 7.4.2, the above proposition allows us to apply the symmetric monoidal Dwyer-Kan localization of Definition 3.2.1 to get the following theorem.

Theorem 7.5.2. *The ∞-category $\mathcal{CoMod}_C(\mathcal{D}^{\geq 0}(\Bbbk))$ of C-comodules in $\mathcal{D}^{\geq 0}(\Bbbk)$ is endowed with a symmetric monoidal structure defined by the derived cotensor product.*

The main idea of the proof is that the tensor product $\mathsf{Ch}_{\Bbbk}^{\geq 0} \times \mathsf{CoMod}_C \xrightarrow{\otimes} \mathsf{CoMod}_C$ is almost a "co-Quillen bifunctor", i.e. there is somekind of compatibility with certain fibrations on $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ and on CoMod_C.

Remark 7.5.3. In general $X\square_C - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ does not preserve fibrations, even if X is fibrant. A simple example is given by applying the functor to the generating fibration $0 \to S^0(V) \otimes C$. If we choose $V = \Bbbk$, then we obtain a map $0 \to X$ which is clearly not a fibration (consider the case $C = \Bbbk$). Similarly, for any chain complex M, we see that $M \otimes - : \mathsf{CoMod}_C \to \mathsf{CoMod}_C$ does not preserve fibrations.

We observe the following characterizations of fibrations in the injective model structure setting. We are grateful for Pete Bousfield to have pointed out this result.

Proposition 7.5.4. *Let* M *be an Abelian category endowed with a model structure where acyclic cofibrations are precisely monomorphisms with acyclic cokernels. Let* $f : X \to Y$ *be an epimorphism in* M*. Let* F *be its kernel. Then* f *is a fibration if and only if* F *is fibrant.*

Proof. A fibration always has fibrant kernel, regardless of being an epimorphism. This is because pullbacks preserve fibrations and the the kernel F is given by the pullback:

$$\begin{tikzcd} F \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "f"] \\ 0 \arrow[r] & Y. \end{tikzcd}$$

Now suppose F is fibrant, let us show that f is a fibration. Since M is a model category, we can factor f as follows:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"', "\simeq"] & & Y \\ & X', \arrow[ur, two heads, "f'"'] & \end{tikzcd}$$

where i is an acylic cofibration and f' is a fibration. Denote F' the kernel of f'. We obtain the following morphism of short exact sequences in M:

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, hook, "i", "\simeq"'] & Y \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & F' \arrow[r] & X' \arrow[r, "f'"] & Y \arrow[r] & 0. \end{tikzcd}$$

We have used the fact that since f is an epimorphism and $f = f' \circ i$, then f' must also be an epimorphism. Since i is a monomorphism, the snake lemma guarantees that the induced map $F \to F'$ is also a monomorphism. Therefore we can take the cokernels of the vertical maps:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d, hook] & X \arrow[r, "f"] \arrow[d, hook, "i", "\simeq"'] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & X' \arrow[r, "f'"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The 9-lemma guarantees that the third row is exact, and thus K is acylic. Therefore $F \to F'$ is an acylic cofibration. Since F is fibrant, then we obtain the following section of $F \to F'$:

$$
\begin{tikzcd}
F \arrow[r, equal] \arrow[d, hook, "\simeq"'] & F \arrow[d, two heads] \\
F' \arrow[r] \arrow[ur, dashed, "\ell"] & 0.
\end{tikzcd}
$$

We define then P to be the following pushout in M:

$$
\begin{tikzcd}
F' \arrow[r, hook] \arrow[d, "\ell"'] & X' \arrow[d] \\
F \arrow[r] & P. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

In an Abelian category, pushouts preserve monomorphisms so $F \to P$ is a monomorphism. Pushouts also preserve cokernels, thus Y is the cokernel of $F \to P$. Therefore we obtain the following composite of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, hook, "\simeq"'] & X \arrow[r, "f"] \arrow[d, hook, "\simeq"', "i"] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d, "\ell"] & X' \arrow[r, "f'"] \arrow[d] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r] & P \arrow[r] & Y \arrow[r] & 0.
\end{tikzcd}$$

The composite of the left vertical arrows is the identity on F by construction of ℓ. By the 5-lemma, we get that P is isomorphic to Y. Therefore, we have just shown that f is a retract of f' which is a fibration. Hence f is also a fibration. □

Lemma 7.5.5. *Let $M \in \mathsf{Ch}_{\Bbbk}^{\geq 0}$ and Y a fibrant C-comodule. Then $M \otimes Y$ is a fibrant C-comodule.*

Proof. Let $\widetilde{Y}$ be the (homotopy) limit of the Postnikov tower $\{Y(n)\}$ of Y. Since Y is a retract of $\widetilde{Y}$, then $M \otimes Y$ is a retract of $M \otimes \widetilde{Y}$, and thus it suffices to show $M \otimes \widetilde{Y}$ is fibrant. Notice that as $\{M \otimes Y(n)\}$ stabilizes in each degree, then $M \otimes \widetilde{Y} \cong \mathsf{lim}_n^C(M \otimes Y(n))$.

We show that $\{M \otimes Y(n)\}$ is a fibrant tower in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$, in the sense of Proposition A.1.13. For $n = 0$, then $M \otimes Y(0) = 0$ is trivially fibrant. For $n = 1$, since every cofree comodule is fibrant, then $M \otimes Y(1)$ is fibrant. Since $M \otimes - : \mathsf{Ch}_{\Bbbk}^{\geq 0} \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ preserves epimorphisms, then $M \otimes D^n(V) \to M \otimes S^n(V)$ is a fibration in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$, and so $M \otimes D^n(V) \otimes C \to M \otimes S^n(V) \otimes C$ is a fibration of C-comodule (alternatively, apply Proposition 7.5.4 as its kernel $M \otimes S^{n-1}(V) \otimes C$

is clearly fibrant). Since $\mathsf{M}\otimes - : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ preserves pullbacks, then from the pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd} M\otimes Y(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M\otimes D^n(V)\otimes C \arrow[d] \\ M\otimes Y(n) \arrow[r] & M\otimes S^n(V)\otimes C, \end{tikzcd}$$

we get that $M\otimes Y(n+1)\to M\otimes Y(n)$ is a fibration. $\square$

Lemma 7.5.6. *Let V be a $\Bbbk$-module. Let Y be a fibrant C-comodule. Let $L\to M$ be an epimorphism in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. Then $L\otimes Y\to M\otimes Y$ is a fibration of C-comodules. In particular, for any $n\geq 1$, the map $D^n(V)\otimes Y\to S^n(V)\otimes Y$ is a fibration of C-comodule.*

Proof. Let F be the kernel of $L\to M$. Since $-\otimes Y : \mathsf{Ch}_{\Bbbk}^{\geq 0}\to \mathsf{CoMod}_C$ preserves short exact sequences, we obtain the following short exact sequence in CoMod_C:

$$0 \longrightarrow F\otimes Y \longrightarrow L\otimes Y \longrightarrow M\otimes Y \longrightarrow 0.$$

By Lemma 7.5.5 and Proposition 7.5.4, we can conclude. $\square$

Proof of Proposition 7.5.1. Let $\{X(n)\}$ be the Postnikov tower of X and $\widetilde{X}$ its (homotopy) limit. Then since X is a retract of $\widetilde{X}$, then $X\square_C Y$ is a retract of $\widetilde{X}\square_C Y$. Whence it suffices to check that $\widetilde{X}\square_C Y = \mathsf{lim}_n^C(X(n)\square_C Y)$ is a fibrant C-comodule. This will follow from the fact that $\{X(n)\square_C Y\}$ is a fibrant tower of C-comodules. For $n=0$, we trivially have that $X(0)\square_C Y = 0$ is fibrant. For $n=1$, since $X(1)=U(X)\otimes C$, then $X(1)\square_C Y\cong U(X)\otimes Y$

which is fibrant by Lemma 7.5.5. For $n \geq 1$, since $X(n+1)$ is defined as the pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}0)$:

$$\begin{tikzcd} X(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \otimes C \arrow[d] \\ X(n) \arrow[r] & S^n(V) \otimes C, \end{tikzcd}$$

and $-\Box_C Y : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ preserves pullbacks, then we have the following pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd} X(n+1)\Box_C Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \otimes Y \arrow[d] \\ X(n)\Box_C Y \arrow[r] & S^n(V) \otimes Y, \end{tikzcd}$$

where we have used Lemma 7.1.4 to identify the right vertical map. Since this map is a fibration of C-comodules by previous lemma, then we get $X(n+1)\Box_C Y \to X(n)\Box_C Y$ is a fibration of C-comodules. $\square$

7.6 Change of Coalgebras

We observe here a direct consequence from Corollary 7.4.2. Let $f : C \to D$ be a map of simply connected cocommutative differential graded $\Bbbk$-coalgebras. The map endows the coalgebra C with a D-comodule structure:

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{\mathsf{id}_C \otimes f} C \otimes D,$$

such that $f : C \to D$ is a map of D-comodules. We obtain a functor $f^* : \mathsf{CoMod}_C \to \mathsf{CoMod}_D$ where each C-comodule (X, ρ) is sent to the D-comodule $(X, (\mathsf{id}_X \otimes f) \circ \rho)$. We shall often write $f^*(X)$ simply as X.

Given any D-comodule X, we can form the cotensor of D-comodules $X \Box_D C$, which can be endowed with the structure of C-comodule as follows. The C-coaction is induced by the natural map of equalizers:

$$\begin{tikzcd}
X\Box_D C \arrow[r] \arrow[d, dashed] & X \otimes C \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X \otimes D \otimes C \arrow[d] \\
(X\Box_D C) \otimes C \arrow[r] & X \otimes C \otimes C \arrow[r, shift left] \arrow[r, shift right] & X \otimes D \otimes C \otimes C,
\end{tikzcd}$$

where the vertical arrows are induced by the comultiplication on C. One can easily check that we obtain a functor $-\Box_D C : \mathsf{CoMod}_D \to \mathsf{CoMod}_C$ which is right adjoint to f^*.

Proposition 7.6.1. *Let $f : C \to D$ be a map of cocommutative simply connected differential graded $\Bbbk$-coalgebras. Then the adjunction:*

$$\begin{tikzcd}
\mathsf{CoMod}_C \arrow[r, shift left, "f^*"] \arrow[r, phantom, "\perp"] & \mathsf{CoMod}_D, \arrow[l, shift left, "-\Box_D C"]
\end{tikzcd}$$

is a Quillen pair. Moreover, the adjunction is a Quillen equivalence if and only if the map f is a quasi-isomorphism.

Proof. The first statement follows directly from the fact that the functor f^* preserves monomorphisms and quasi-isomorphisms. For the second statement, we shall apply (Hovey, 1999, 1.3.16).

Notice that f^* reflects weak equivalences. Suppose first that f is a quasi-isomorphism. Now let X be any fibrant D-comodule, the counit of the adjunction:

$$X \Box_D C \xrightarrow{\simeq} X \Box_D D \cong X,$$

is a quasi-isomorphism by Corollary 7.4.2. Conversely, if we suppose the adjunction to be a Quillen equivalence, then the map:

$$f : C \cong D \Box_D C \longrightarrow D \Box_D D \cong D,$$

must be a weak equivalence, as D is always fibrant as a D-comodule. $\Box$

APPENDICES

Appendix A

DUAL TERMINOLOGY IN MODEL CATEGORY

One of the main tool of model categories is to assume the structure is *cofibrantly generated* by a pair of sets (see definition in (Hovey, 1999, 2.1.17)). If in addition the category is presentable, we say it is *combinatorial.* In such a case, cofibrations and acyclic cofibrations are retracts of maps built out of pushouts and transfinite compositions, and we can inductively construct a cofibrant replacement.

Simply dualizing the notions would be a fine method if one were working with *copresentable* categories. However, if we still want to work with presentable categories, then naively dualizing the notion of cofibrantly generated to fibrantly generated causes issues, as a model category is rarely this way, see (Adámek and Rosický, 1994, 1.64). We instead weaken the definition (as in Definitions A.1.9 and A.2.1). Unfortunately we cannot apply the *cosmall object argument* and thus showing that a model category has an interesting Postnikov presentation will be challenging in general.

A.1 Postnikov Presentation

We present the definition of Postnikov presentations, introduced by Kathryn Hess, which is dual to cellular presentations and appeared in (Hess, 2009), (Hess and Shipley, 2014) and (Bayeh et al., 2015).

We first dualize the notion of relative cell complex (Hovey, 1999, 2.2.9).

Appendix A (Continued)

Definition A.1.1 ((Hess, 2009, 5.12))**.** Let P be a class of morphisms in a category closed under pullbacks C. Let λ be an ordinal. Given a functor $Y : \lambda^{\mathsf{op}} \to \mathsf{C}$ such that for all $\beta < \lambda$, the morphism $Y_{\beta+1} \to Y_\beta$ fits into the pullback diagram:

$$\begin{tikzcd} Y_{\beta+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X'_{\beta+1} \arrow[d] \\ Y_\beta \arrow[r] & X_{\beta+1}, \end{tikzcd}$$

where $X'_{\beta+1} \to X_{\beta+1}$ is some morphism in P, and $Y_\beta \to X_{\beta+1}$ is a morphism in C, and we denote:

$$Y_\gamma := \lim_{\beta<\gamma} Y_\beta,$$

for any limit ordinal $\gamma < \lambda$. We say that the composition of the tower Y:

$$\lim_{\lambda^{\mathsf{op}}} Y_\beta \longrightarrow Y_0,$$

if it exists, is a P-*Postnikov tower.* The class of all P-Postnikov towers is denoted $\mathsf{Post}_{\mathsf{P}}$.

Proposition A.1.2 ((Bayeh et al., 2015, 2.10))**.** *If* C *is a complete category, the class* $\mathsf{Post}_{\mathsf{P}}$ *is the smallest class of morphism in* C *containing* P *closed under composition, pullbacks and limits indexed by ordinals.*

Proof. See dual statements in (Hovey, 1999, 2.1.12, 2.1.13). $\square$

Proposition A.1.3. *Let* $R : \mathsf{C} \to \mathsf{D}$ *be a right adjoint between complete categories. Let* P *be a class of morphisms in* C*. Then we have:* $R(\mathsf{Post}_{\mathsf{P}}) \subseteq \mathsf{Post}_{R(\mathsf{P})}$.

Proof. Right adjoints preserve limits. □

We also recall the dual notion of small object in a category.

Definition A.1.4. Let D be a subcategory of a complete category C. We say an object A in C is *cosmall relative to* D if there is a cardinal κ such that for all κ-filtered ordinals λ (see (Hovey, 1999, 2.1.2)) and all λ-towers $Y : \lambda \to \mathsf{D}^{\mathsf{op}}$, the induced map of sets:

$$\operatorname*{\mathsf{colim}}_{\beta<\lambda} (\mathsf{Hom}_{\mathsf{C}}(Y_\beta, A)) \longrightarrow \mathsf{Hom}_{\mathsf{C}}\left(\lim_{\beta<\lambda} Y_\beta, A\right),$$

is a bijection. We say that A is *cosmall* if it is cosmall relative to C itself.

Example A.1.5. The terminal object, if it exists, is always cosmall. In procategories, every object is cosmall. Therefore in *copresentable* categories, every object is cosmall.

Example A.1.6. As noted after (Hovey, 1999, 2.1.18), the only cosmall objects in the category of sets are the empty set and the one-point set. In practice, objects in a presentable categories are rarely cosmall.

The dual of the small object argument (Hovey, 1999, 2.1.14) can be stated as follows.

Proposition A.1.7 (The cosmall object argument)**.** *Let* C *be a complete category and* P *be a set of morphisms in* C*. If the codomains of maps in* P *are cosmall relative to* $\mathsf{Post}_{\mathsf{P}}$*, then every morphism* f *of* C *can be factored functorially as:*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\gamma(f)"'] & & B \\ & C^f, \arrow[ur, "\delta(f)"'] & \end{tikzcd}$$

where $\delta(f)$ is a P-Postnikov tower and $\gamma(f)$ admits the left lifting property with respect to all maps in P.

Notation A.1.8. Given a class of morphisms A in C, we denote $\widehat{\mathsf{A}}$ its closure under formation of retracts.

Definition A.1.9. A *Postnikov presentation* (P, Q) *of a model category* M is a pair of classes of morphisms P and Q such that the class of fibrations is $\widehat{\mathsf{Post}_{\mathsf{P}}}$, the class of acyclic fibrations is $\widehat{\mathsf{Post}_{\mathsf{Q}}}$, and for any morphism $f : X \to Y$ in M:

(a) the morphism f factors as:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & V \arrow[ur, "q"'] & \end{tikzcd}$$

where i is a cofibration and q is a Q-Postnikov tower;

(b) the morphism f factors as:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & W \arrow[ur, "p"'] & \end{tikzcd}$$

where j is an acyclic cofibration and p is a P-Postnikov tower.

We say in this case that the model category M *is Postnikov presented by* (P, Q).

Remark A.1.10. Since we do not require sets, every model category is trivially Postnikov presented by the classes of all fibrations and acyclic fibrations. Although it was noted in (Bayeh et al., 2015, 2.13, 2.14) that this trivial presentation can occasionally be useful (as we will see in Theorem B.4.1), we use more interesting subclasses in this paper, see Theorems B.2.1 and B.3.3.

Appendix A (Continued)

Definition A.1.11. Let M be a complete model category that admits a Postnikov presentation (P, Q). Given any object X in M, we can provide an *inductive fibrant replacement* FX as follows. Let $*$ be the terminal object of M. There is an object FX in M that factors the trivial map:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "j"'] & & * \\ & FX, \arrow[ur, "p"'] & \end{tikzcd}$$

where $j : X \overset{\sim}{\hookrightarrow} FX$ is an acyclic cofibration in M, and p is a P-Postnikov tower. This means that FX can be defined as iterated maps of pullbacks along P, starting with $(FX)_0 = *$.

We shall sometimes make us of homotopy limits of countable towers, we record here some notation and an explicit formula.

Notation A.1.12. Denote $\mathbb{N}$ the poset $\{0 < 1 < 2 < \cdots\}$. Let C be any complete category. Objects in $\mathsf{C}^{\mathbb{N}}$ are diagrams of shape $\mathbb{N}$ and can be represented as (countable) towers in C:

$$\cdots \xrightarrow{f_3} X(2) \xrightarrow{f_2} X(1) \xrightarrow{f_1} X(0).$$

We denote such object by $\{X(n)\} = (X(n), f_n)_{n \in \mathbb{N}}$. The limit of the tower is denote $\mathsf{lim}_n X(n)$.

Proposition A.1.13 ((Goerss and Jardine, 1999, VI.1.1))**.** *Let M be a model category. Then the category of towers $\mathsf{M}^{\mathbb{N}}$ can be endowed with the Reedy model structure, where a map $\{X(n)\} \to \{Y(n)\}$ is a weak equivalence (respectively a cofibration), if each map $X(n) \to Y(n)$ is a weak equivalence (respectively a cofibration) in M, for all $n \geq 0$. An object $\{X(n)\}$ is fibrant if and only if $X(0)$ is fibrant and all the maps $X(n+1) \to X(n)$ in the tower are fibrations in M.*

Moreoever, if we denote $\iota : \mathsf{M} \to \mathsf{M}^{\mathbb{N}}$ *the functor induced by the constant diagram, then we obtain a Quillen adjunction* $\iota : M \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{M}^{\mathbb{N}} : \mathsf{lim}_n$.

A.2 Fibrantly Generated Model Categories

We introduce the notion of fibrantly generated as in (Bayeh et al., 2015).

Definition A.2.1. A model category is *fibrantly generated by* (P, Q) if the cofibrations are precisely the morphisms that have the left lifting property with respect to Q, and the acyclic cofibrations are precisely the morphisms that have the left lifting property with respect to P. We call P and Q the *generating fibrations* and *generating acyclic fibrations* respectively.

Remark A.2.2. Our definition of fibrantly generated makes **no assumption of cosmallness** and is not the usual definition one can find in the literature.

Remark A.2.3. Just as for Remark A.1.10, since we allow P and Q to be classes, any model category is trivially fibrantly generated by its fibrations and acyclic fibrations.

Proposition A.2.4. *If a model category is Postnikov presented by a pair of classes* (P, Q)*, then it is fibrantly generated by* (P, Q)*.*

Proof. Direct consequence of the retract argument (see (Hovey, 1999, 1.1.9)). □

Remark A.2.5. The converse of Proposition A.2.4 is true if P and Q are sets that permit the cosmall object argument. However, this rarely happens in context of interest as seen in Example A.1.6.

A.3 Left-Induced Model Categories

Definition A.3.1. Let M be a model category and A be any category, such that there is a pair of adjoint functors: $\mathsf{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{M}$. We say that the left adjoint $L : \mathsf{A} \to \mathsf{M}$ *left-induces* a model structure on A if the category A can be endowed with a model structure where a morphism f in A is defined to be a cofibration (respectively a weak equivalence) if $L(f)$ is a cofibration (respectively a weak equivalence) in M. This model structure on A, if it exists, is called *the left-induced model structure from* M.

The next result is the dual of the Quillen path object argument and is in practice the way we verify left-induced model structures exist.

Proposition A.3.2 ((Hess et al., 2017, 2.2.1))**.** *Let* M *and* A *be presentable categories. Suppose we have an adjunction:*

$$\begin{tikzcd} \mathsf{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathsf{M}. \arrow[l, shift left, "R"] \end{tikzcd}$$

Suppose M *is endowed with a cofibrantly generated model structure where all objects are cofibrant. If, for every object* A *in* A*, there is a factorization in* A*:*

$$A \coprod A \xrightarrow{j} \mathsf{Cyl}(A) \xrightarrow{p} A,$$

such that, after applying the left adjoint L*, we obtain a good cylinder object in* M *(i.e.* $L(j)$ *is a cofibration and* $L(p)$ *is a weak equivalence in* M*), then the left-induced model structure from* M *on* A *exists.*

Appendix A (Continued)

The following result guarantees that left-inducing from a combinatorial model category gives back a combinatorial model category.

Proposition A.3.3 ((Bayeh et al., 2015, 2.23),(Hess et al., 2017, 3.3.4)). *Suppose* A *is a model category left-induced by a model category* M*. Suppose both* A *and* M *are presentable. If* M *is cofibrantly generated by a pair of sets, then* A *is cofibrantly generated by a pair of sets.*

Proposition A.3.4 ((Bayeh et al., 2015, 2.18)). *Suppose* A *is a model category left-induced by a model category* M*, via an adjunction* $L : \mathsf{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{M} : R$*. If* M *is fibrantly generated by* (P, Q)*, then* A *is fibrantly generated by* $(R(\mathsf{P}), R(\mathsf{Q}))$*.*

Remark A.3.5. If M is Postnikov presented by (P, Q), there is no reason to expect that A is Postnikov presented by $(R(\mathsf{P}), R(\mathsf{Q}))$, as we made no assumption of cosmallness in general.

Appendix B

POSTNIKOV PRESENTATIONS OF DIFFERENTIAL GRADED COMODULES

In this appendix, we present the explicit ad-hoc Postnikov presentations. We first show, in Theorem B.1.8, that chain complexes over a finite product of fields $\Bbbk$ are fibrantly generated in the sense of Definition A.2.1. Then we show that comodules over a simply connected differential graded $\Bbbk$-coalgebra also admit a Postnikov presentation, generalizing the presentation defined in (Hess, 2009). The induced explicit Postnikov tower of comodules defined in Corollary B.3.15 will be crucial to prove rigidification result in Theorem 6.3.3. We also observe in Theorem B.4.1 that we can produce a Postnikov presentation for comodules over a coalgebra that is a perfect chain complex. However, it will not be used in this paper, but remains useful for any explicit homotopy limit computations.

B.1 The Generating Fibrations

We start with the following definition.

Definition B.1.1. Let R be any commutative ring. Let V be an R-module. Let n be any integer. Denote $S^n(V)$, the n-sphere over V, the chain complex that is V concentrated in degree n and zero elsewhere. Denote $D^n(V)$, the n-disk over V, the chain complex that is V concentrated in degree $n-1$ and n, with differential the identity. As noted in (Bayeh et al., 2015, 3.1), it is enlightening to regard $S^n(V)$ as the Eilenberg-Mac Lane space $K(V,n)$ and $D^n(V)$ as the based path of $K(V,n)$. We obtain the obvious map $D^n(V) \to S^n(V)$:

$$
\begin{array}{cc}
D^n(V) & \cdots \longleftarrow 0 \longleftarrow V = V \longleftarrow 0 \longleftarrow \cdots \\
\downarrow & \quad\;\; \| \qquad\;\; \downarrow \quad\;\; \| \qquad\; \| \\
S^n(V) & \cdots \longleftarrow 0 \longleftarrow 0 \longleftarrow V \longleftarrow 0 \longleftarrow \cdots .
\end{array}
$$

This defines functors:

$$S^n(-) : \mathsf{Mod}_R \longrightarrow \mathsf{Ch}_R, \qquad D^n(-) : \mathsf{Mod}_R \longrightarrow \mathsf{Ch}_R.$$

The map defined above is natural, i.e. we have a natural transformation $D^n(-) \Rightarrow S^n(-)$, for all $n \in \mathbb{Z}$. When $V = R$, we simply write D^n and S^n.

Chain complexes over a field are advantageous as they are all split. In general, if R is a unital domain ring, and if all short exact sequences of R-modules are split, then R must be a field. In fact, a direct consequence of Wedderburn-Artin theorem gives the following result.

Proposition B.1.2. *Let $\Bbbk$ be a commutative (unital) ring. The following are equivalent.*

(i) *Every $\Bbbk$-module is projective.*

(ii) *Every $\Bbbk$-module is injective.*

(iii) *Ever short exact sequence of $\Bbbk$-modules splits.*

(iv) *$\Bbbk$ is a finite product of fields.*

Notation B.1.3. Given any chain complex, we denote $B_n(X)$ the n-boundaries of X and $Z_n(X)$ the n-cycles of X.

Proposition B.1.4. *Let $\Bbbk$ be a finite product of fields. Let X be a chain complex in $\mathsf{Ch}_{\Bbbk}$. Then X is split as a chain complex and we have a non-canonical decomposition:*

$$X_n \cong H_n(X) \oplus B_n(X) \oplus B_{n-1}(X).$$

In particular any chain complex X can be decomposed non-canonically as product of disks and spheres:

$$X \cong \prod_{n \in \mathbb{Z}} S^n(V_n) \oplus D^n(W_n),$$

where $V_n = H_n(X)$ and $W_n = B_{n-1}(X)$.

Proof. We have the following short exact exact sequences of $\Bbbk$-modules:

$$\begin{tikzcd} 0 \arrow[r] & B_n(X) \arrow[r, hook] & Z_n(X) \arrow[r, two heads] & H_n(X) \arrow[r] \arrow[l, dashed, bend right] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & Z_n(X) \arrow[r, hook] & X_n \arrow[r, "d_n"'] & B_{n-1}(X) \arrow[r] \arrow[l, dashed, bend right] & 0. \end{tikzcd}$$

Since any short exact sequence splits (Proposition B.1.2), we can choose sections (the dashed maps denoted above), such that we obtain the following isomorphism of $\Bbbk$-modules:

$$\begin{aligned} X_n &\cong Z_n(X) \oplus B_{n-1}(X) \\ &\cong H_n(X) \oplus B_n(X) \oplus B_{n-1}(X). \quad \square \end{aligned}$$

Appendix B (Continued)

We introduce now our generating fibrations and acyclic fibrations in $\mathsf{Ch}_{\Bbbk}$.

Definition B.1.5. Let $\Bbbk$ be a finite product of fields. Define $\mathscr{P}$ and $\mathscr{Q}$ to be the following *sets* of maps in $\mathsf{Ch}_{\Bbbk}$:

$$\mathscr{P} = \{D^n \longrightarrow S^n\}_{n\in\mathbb{Z}}, \qquad \mathscr{Q} = \{D^n \longrightarrow 0\}_{n\in\mathbb{Z}}.$$

We thicken the sets $\mathscr{P}$ and $\mathscr{Q}$ to *classes* $\mathscr{P}_{\oplus}$ and $\mathscr{Q}_{\oplus}$ of morphisms in $\mathsf{Ch}_{\Bbbk}$:

$$\mathscr{P}_{\oplus} := \Big\{D^n(V) \longrightarrow S^n(V) \mid V \text{ any } \Bbbk\text{-module}\Big\}_{n\in\mathbb{Z}},$$

$$\mathscr{Q}_{\oplus} := \Big\{D^n(V) \longrightarrow 0 \mid V \text{ any } \Bbbk\text{-module}\Big\}_{n\in\mathbb{Z}}.$$

Clearly, the maps in $\mathscr{P}$ and $\mathscr{P}_{\oplus}$ are fibrations in $\mathsf{Ch}_{\Bbbk}$ and the maps in $\mathscr{Q}$ and $\mathscr{Q}_{\oplus}$ are acyclic fibrations in $\mathsf{Ch}_{\Bbbk}$.

Remark B.1.6. When $\Bbbk$ is a field, as every $\Bbbk$-module is free, we get:

$$\mathscr{P}_{\oplus} = \left\{\bigoplus_{\lambda} D^n \longrightarrow \bigoplus_{\lambda} S^n \mid \lambda \text{ any ordinal}\right\}_{n\in\mathbb{Z}},$$

$$\mathscr{Q}_{\oplus} = \left\{\bigoplus_{\lambda} D^n \longrightarrow 0 \mid \lambda \text{ any ordinal}\right\}_{n\in\mathbb{Z}}.$$

Lemma B.1.7. *Let R be any commutative ring. Given a split exact sequence of R-modules:*

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, hook] & V \oplus W \arrow[r] \arrow[l, bend right, dashed] & W \arrow[r] & 0, \end{tikzcd}$$

Appendix B (Continued)

it induces the following diagram of split exact sequences in Ch_R *for all* $n \in \mathbb{Z}$ *with compatible retracts:*

$$\begin{tikzcd} 0 \arrow[r] & D^n(V) \arrow[r, hook] \arrow[d] & D^n(V) \oplus D^n(W) \arrow[r] \arrow[d] \arrow[l, dashed, bend right] & D^n(W) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S^n(V) \arrow[r, hook] & S^n(V) \oplus S^n(W) \arrow[r] \arrow[l, dashed, bend right] & S^n(W) \arrow[r] & 0. \end{tikzcd}$$

Proof. Notice first that we have the equalities $S^n(V \oplus W) = S^n(V) \oplus S^n(W)$ and $D^n(V \oplus W) = D^n(V) \oplus D^n(W)$. The choice of a splitting $V \oplus W \to V$ provides a coherent choice of chain maps $D^n(V \oplus W) \to D^n(V)$ and $S^n(V \oplus W) \to S^n(V)$. $\square$

Theorem B.1.8. *Let* $\Bbbk$ *be a finite product of fields. Let* C *be any differential graded coalgebra over* $\Bbbk$*.*

- **(i)** *The model category of unbounded chain complex* $\mathsf{Ch}_{\Bbbk}$ *is fibrantly generated by the pair of sets* $(\mathscr{P}, \mathscr{Q})$*.*
- **(ii)** *The injective model category* $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk})$ *of comodules over* $\Bbbk$ *is fibrantly generated by the pair of sets* $(\mathscr{P} \otimes C, \mathscr{Q} \otimes C)$*.*

A similar result as of **(i)** above was proved in early unpublished versions of (Soré, 2016) in (Soré, 2010, 3.1.11, 3.1.12) for non-negative chain complexes over a field. We extend the results for the unbounded case and show that **(i)** of Theorem B.1.8 follows from Lemma B.1.11 and Lemma B.1.12 below. Notice that **(ii)** above is a direct consequence of **(i)** by Propositions A.3.4.

Notation B.1.9. Given any class of maps $\mathscr{A}$ in a category C, we denote $\mathsf{Llp}(\mathscr{A})$ the class of maps in C having the left lifting property with respect to all maps in $\mathscr{A}$.

Appendix B (Continued)

Lemma B.1.10. *Let $\Bbbk$ be a finite product of fields. We have the equalities of classes:* $\mathsf{Llp}(\mathscr{P}_\oplus) = \mathsf{Llp}(\mathscr{P})$ *and* $\mathsf{Llp}(\mathscr{Q}_\oplus) = \mathsf{Llp}(\mathscr{Q})$ *in* $\mathsf{Ch}_\Bbbk$.

Proof. Since $\mathscr{P} \subseteq \mathscr{P}_\oplus$, we get $\mathsf{Llp}(\mathscr{P}_\oplus) \subseteq \mathsf{Llp}(\mathscr{P})$. Suppose now f is in $\mathsf{Llp}(\mathscr{P})$, let us argue it also belongs in $\mathsf{Llp}(\mathscr{P}_\oplus)$. Suppose we have a diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & D^n(V) \arrow[d] \\
Y \arrow[r] & S^n(V),
\end{tikzcd}$$

for some $\Bbbk$-module V. Since V is projective, there is another $\Bbbk$-module W such that $V \oplus W$ is free. Thus $V \oplus W \cong \bigoplus_\lambda \Bbbk$, for some basis λ. In particular, by Lemma B.1.7, we obtain the commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & D^n(V) \arrow[r, hook] \arrow[d] & \bigoplus\limits_{\alpha\in\lambda} D^n_\alpha \arrow[r, hook] \arrow[d] \arrow[l, dashed, bend right] & \prod\limits_{\alpha\in\lambda} D^n_\alpha \arrow[r] \arrow[d] \arrow[l, dashed, bend right] & D^n_\alpha \arrow[d] \\
Y \arrow[r] & S^n(V) \arrow[r, hook] & \bigoplus\limits_{\alpha\in\lambda} S^n_\alpha \arrow[r, hook] \arrow[l, dashed, bend right] & \prod\limits_{\alpha\in\lambda} S^n_\alpha \arrow[r] \arrow[l, dashed, bend right] & S^n_\alpha,
\end{tikzcd}$$

where D^n_α and S^n_α are a copies of D^n and S^n. Since f is in $\mathsf{Llp}(\mathscr{P})$, we obtain a lift $\ell_\alpha : Y \to D^n_\alpha$, for each α. It induces a lift $\ell : Y \to \prod_\alpha D^n_\alpha$ which restricts to $Y \to D^n(V)$ via the retracts (dashed maps in the diagram). $\square$

Lemma B.1.11. *Let $\Bbbk$ be a finite product of fields. Maps in the set $\mathscr{Q} = \{D^n \to 0\}_{n\in\mathbb{Z}}$ are the generating acyclic fibrations in* $\mathsf{Ch}_\Bbbk$.

Appendix B (Continued)

Proof. Let $f : X \to Y$ be a map in $\mathsf{Llp}(\mathscr{Q})$, let us show it is a cofibration in $\mathsf{Ch}_{\Bbbk}$, i.e. a monomorphism. Following Proposition B.1.4, we decompose X as:

$$X \cong \prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n).$$

Then the canonical inclusions $S^n(V_n) \hookrightarrow D^{n+1}(V_n)$ induce a monomorphism ι:

$$\iota : \prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n) \longhookrightarrow \prod_{n\in\mathbb{Z}} D^{n+1}(V_n) \oplus D^n(W_n).$$

Since f is in $\mathsf{Llp}(\mathscr{Q})$, then there is a map ℓ such that $\iota = \ell \circ f$. Hence f must be a monomorphism and thus a cofibration. □

Lemma B.1.12. *Let $\Bbbk$ be a finite product of fields. Maps in the set $\mathscr{P} = \{D^n \to S^n\}_{n\in\mathbb{Z}}$ are the generating fibrations in $\mathsf{Ch}_{\Bbbk}$.*

Proof. Notice that $\mathsf{Llp}(\mathscr{P}) \subseteq \mathsf{Llp}(\mathscr{Q})$ as any lift $Y \to D^n$ in the following commutative diagram induces the dashed lift:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & D^n \arrow[d] \\
Y \arrow[r] \arrow[ur] \arrow[d, equal] & S^n \arrow[d] \\
Y \arrow[r] \arrow[uur, dashed] & 0.
\end{tikzcd}$$

Appendix B (Continued)

In particular, Lemma B.1.11 shows that maps in $\mathsf{Llp}(\mathscr{P})$ are monomorphisms. Let $f : X \to Y$ be map in $\mathsf{Llp}(\mathscr{P})$ and let us show it is a quasi-isomorphism. Since f is a monomorphism, there is an induced short exact sequence in $\mathsf{Ch}_{\mathbb{k}}$:

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow K \longrightarrow 0,$$

where $K = \mathsf{coker}(f)$. It remains to show that K is acyclic. Notice first that K is defined as the pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r] \arrow[ur, phantom, "\ulcorner"] & K, \end{tikzcd}$$

and so, since f is in $\mathsf{Llp}(\mathscr{P})$, then $0 \to K$ is in $\mathsf{Llp}(\mathscr{P})$. Following Proposition B.1.4, decompose K as:

$$K \cong \prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n),$$

where $V_n = H_n(K)$. Then we obtain a map by projection:

$$\prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n) \longrightarrow \prod_{n\in\mathbb{Z}} S^n(V_n),$$

that factors through the non-trivial map:

$$\prod_{n\in\mathbb{Z}} D^n(V_n) \longrightarrow \prod_{n\in\mathbb{Z}} S^n(V_n)$$

as $0 \to K$ is in $\mathsf{Llp}(\mathscr{P}) = \mathsf{Llp}(\mathscr{P}_\oplus)$. But this is only possible when $V_n = 0$, hence K must be acyclic. Thus f is a quasi-isomorphism. □

B.2 Postnikov Presentation for Unbounded Chain Complexes

A Postnikov presentation was constructed in (Hess, 2009) and (Bayeh et al., 2015) for finitely generated non-negative chain complexes over a field. We extend here the argument to the unbounded non-finitely generated case, over a finite product of fields.

Theorem B.2.1. *Let $\Bbbk$ be a finite product of fields. The pair $(\mathscr{P}_\oplus, \mathscr{Q})$ is a Postnikov presentation of the model category of unbounded chain complex $\mathsf{Ch}_\Bbbk$.*

We shall prove Theorem B.2.1 with Lemmas B.2.4 and B.2.7 below. The theorem provides an inductive fibrant replacement for diagram categories in $\mathsf{Ch}_\Bbbk$ endowed with the injective model structure and thus provides inductive arguments to compute homotopy limits in $\mathsf{Ch}_\Bbbk$.

Remark B.2.2. We were not able to restrict ourselves to the set $\mathscr{P}$ and had to consider the class $\mathscr{P}_\oplus$. We note here a few basic results.

(i) As $\mathscr{P} \subseteq \mathscr{P}_\oplus$, we get $\mathsf{Post}_\mathscr{P} \subseteq \mathsf{Post}_{\mathscr{P}_\oplus}$.

(ii) The maps $S^n \to 0$ are in $\mathsf{Post}_\mathscr{P}$ as they are obtained as pullbacks:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^{n+1} \arrow[d] \\ 0 \arrow[r] & S^{n+1}. \end{tikzcd}$$

Similarly, for any $\Bbbk$-module V, the maps $S^n(V) \to 0$ are in $\mathsf{Post}_{\mathscr{P}_\oplus}$.

Appendix B (Continued)

(iii) Since $D^n \to 0$ is the composite $D^n \longrightarrow S^n \longrightarrow 0$, we see that $\mathcal{Q} \subseteq \mathsf{Post}_{\mathcal{P}}$, and thus $\mathsf{Post}_{\mathcal{Q}} \subseteq \mathsf{Post}_{\mathcal{P}} \subseteq \mathsf{Post}_{\mathcal{P}_\oplus}$ by Proposition A.1.2.

(iv) Although $\mathcal{P}_\oplus \nsubseteq \mathsf{Post}_{\mathcal{P}}$, we have $\mathcal{P}_\oplus \subseteq \widehat{\mathsf{Post}_{\mathcal{P}}}$ (see Notation A.1.8). Indeed, for any $\mathbb{k}$-module V, Lemma B.1.7 shows that any map $D^n(V) \to S^n(V)$ is the retract of a map $D^n(F) \to S^n(F)$ where F is a free $\mathbb{k}$-module. Then, for λ a basis of F, we have the retract in $\mathsf{Ch}_{\mathbb{k}}$:

$$\begin{array}{ccccc} \bigoplus_\lambda D^n & \longrightarrow & \prod_\lambda D^n & \longrightarrow & \bigoplus_\lambda D^n \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_\lambda S^n & \longrightarrow & \prod_\lambda S^n & \longrightarrow & \bigoplus_\lambda S^n, \end{array}$$

induced by the split short exact sequence in $\mathbb{k}$-modules:

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_\lambda \mathbb{k} \arrow[r, hook, "\iota"] & \prod_\lambda \mathbb{k} \arrow[l, bend right, dashed] \arrow[r] & \mathsf{coker}(\iota) \arrow[r] & 0, \end{tikzcd}$$

where $\iota : \bigoplus_\lambda \mathbb{k} \hookrightarrow \prod_\lambda \mathbb{k}$ is the natural monomorphism.

Lemma B.2.3. *Let $\mathbb{k}$ be a finite product of fields. Let X be any chain complex over $\mathbb{k}$. Then the trivial map $X \to 0$ is a $\mathcal{P}_\oplus$-Postnikov tower. If X is acyclic, the trivial map is a $\mathcal{Q}_\oplus$-Postnikov tower.*

Proof. Follows from Proposition B.1.4, **(ii)** of Remark B.2.2, and Proposition A.1.2. □

Lemma B.2.4. *Every acyclic fibration in $\mathsf{Ch}_{\mathbb{k}}$ is a retract of a $\mathcal{Q}$-Postnikov tower. Every map in $\mathsf{Ch}_{\mathbb{k}}$ factors as a cofibration followed by a $\mathcal{Q}$-Postnikov tower.*

Proof. We provide two proofs by presenting two different factorizations. The first one has the advantage to be functorial but harder to compute. The second is not functorial but is easier to compute. Let us do the first possible factorization. By Theorem B.1.8, the set $\mathcal{Q}$ of maps in $\mathsf{Ch}_{\Bbbk}$ is the set of generating acyclic fibrations. Their codomain is the terminal object in $\mathsf{Ch}_{\Bbbk}$ and is thus cosmall. We can then apply the cosmall object argument (Proposition A.1.7) to obtain the desired factorization.

For the second possible factorization, start with any morphism $f : X \to Y$ in $\mathsf{Ch}_{\Bbbk}$. Choose a decomposition of X by using Proposition B.1.4: $X \cong \prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n)$, for some collection of $\Bbbk$-modules V_n and W_n. The inclusions $S^n(V_n) \hookrightarrow D^{n+1}(V_n)$ define then a monomorphism in $\mathsf{Ch}_{\Bbbk}$:

$$X \cong \prod_{n\in\mathbb{Z}} S^n(V_n) \oplus D^n(W_n) \lhook\joinrel\longrightarrow \prod_{n\in\mathbb{Z}} D^{n+1}(V_n) \oplus D^n(W_n).$$

Since V_n and W_n are projective $\Bbbk$-modules, they can be embedded into free $\Bbbk$-modules, say F_n and G_n, with basis λ_n and γ_n. Then we obtain the following monomorphisms in $\mathsf{Ch}_{\Bbbk}$:

$$\prod_{n\in\mathbb{Z}} D^{n+1}(V_n) \oplus D^n(W_n) \lhook\joinrel\longrightarrow \prod_{n\in\mathbb{Z}} \bigoplus_{\lambda_n} D^{n+1} \oplus \bigoplus_{\gamma_n} D^n$$

Thus, we get the monomorphism in $\mathsf{Ch}_{\Bbbk}$: $X \lhook\joinrel\longrightarrow \prod\limits_{n\in\mathbb{Z}} \prod\limits_{\lambda_n,\gamma_n} D^{n+1} \oplus D^n$. Denote Z the acyclic chain complex $\prod_{n\in\mathbb{Z}} \prod_{\lambda_n,\gamma_n} D^{n+1} \oplus D^n$. We obtain the desired second factorization in $\mathsf{Ch}_{\Bbbk}$: $X \xhookrightarrow{\iota\oplus f} Z \oplus Y \xrightarrow{q} Y$, where the map q is the projection onto Y, which is indeed a $\mathcal{Q}$-Postnikov tower by Proposition A.1.2. $\square$

Appendix B (Continued)

The following arguments are based on the proof of (Bayeh et al., 2015, Lemma 3.3). We begin with preliminary results.

Lemma B.2.5. *Let $\Bbbk$ be a finite product of fields. Let X be any chain complex in $\mathsf{Ch}_{\Bbbk}$. Let V be any $\Bbbk$-module. Let n be any integer in $\mathbb{Z}$. Given a surjective linear map $f_n : X_n \to V$ non-trivial only on n-cycles, there is a map of chain complexes $f : X \to S^n(V)$, and the pullback chain complex P in the following diagram:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \arrow[d] \\ X \arrow[r] & S^n(V), \end{tikzcd}$$

has homology:

$$H_i(P) \cong \begin{cases} \mathsf{ker}\,(H_n(f)) & i = n, \\ H_i(X) & i \neq n, \end{cases}$$

and we have $P_i = X_i$ for $i \neq n-1$ and $P_{n-1} = X_{n-1} \oplus V$.

Proof. By construction, since pullbacks in $\mathsf{Ch}_{\Bbbk}$ are taken levelwise, for $i \neq n, n-1$, we have the pullbacks of $\Bbbk$-modules:

$$\begin{tikzcd} P_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\ X_{n-1} \arrow[r] & 0, \end{tikzcd} \qquad \begin{tikzcd} P_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, equal] \\ X_n \arrow[r, "f_n"] & V, \end{tikzcd} \qquad \begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d, equal] \\ X_i \arrow[r] & 0. \end{tikzcd}$$

Thus $P_{n-1} \cong X_{n-1} \oplus V$ and $P_i = X_i$ for any $i \neq n-1$. The differential $P_n \to P_{n-1}$ is the linear map $X_n \xrightarrow{d_n \oplus f} X_{n-1} \oplus V$, and the differential $P_{n-1} \to P_{n-2}$ is the linear map:

$$X_{n-1} \oplus V \longrightarrow X_{n-1} \xrightarrow{d_{n-1}} X_{n-2},$$

where the unlabeled map is the natural projection. All the differentials $P_i \to P_{i-1}$ for $i \neq n, n-1$ are the differentials $X_i \to X_{i-1}$ of the chain complex X. Clearly, we get $H_i(P) = H_i(X)$ for $i \neq n, n-1$. For $i = n-1$, by Proposition B.1.4, we can choose a decomposition:

$$X_n \cong H_n(X) \oplus B_{n-1}(X) \oplus B_n(X).$$

The differential $d_n : X_n \to X_{n-1}$ sends the factor $B_{n-1}(X)$ in X_n to itself, and the factor $H_n(X) \oplus B_n(X)$ to zero. By definition, the map $f_n : X_n \to V$ sends the factor $H_n(X)$ in X_n to the image of f_n, which is V since f_n is surjective, and the factor $B_{n-1}(X) \oplus B_n(X)$ to zero. Thus the image of the differential $P_n \longrightarrow P_{n-1}$, is precisely $B_{n-1}(X) \oplus V$. Therefore, we obtain:

$$\begin{aligned} H_{n-1}(P) &= \frac{\mathsf{ker}(P_{n-1} \to P_{n-2})}{\mathsf{im}(P_n \to P_{n-1})} \\ &\cong \frac{Z_{n-1}(X) \oplus V}{B_{n-1}(X) \oplus V} \\ &\cong \frac{Z_{n-1}(X)}{B_{n-1}(X)} \\ &= H_{n-1}(X). \end{aligned}$$

For $i = n$, notice that the n-boundaries of P are precisely the n-boundaries of X, the n-cycles of P are the n-cycles x in X such that $f_n(x) = 0$. Since $f_n : X_n \to V$ is entirely defined on the copy $H_n(X)$ in X_n, we get from the commutative diagram:

$$\begin{tikzcd} Z_n(X) \arrow[r, hook] \arrow[d] & X_n \arrow[r, "f_n"] & V \\ H_n(X), \arrow[urr, "H_n(f)"'] & & \end{tikzcd}$$

that $H_n(P) \cong \mathsf{ker}(H_n(f))$. □

Lemma B.2.6. *Let $\Bbbk$ be a finite product of fields. Let $j : X \to Y$ be a monomorphism in $\mathsf{Ch}_{\Bbbk}$, such that it induces a monomorphism in homology in each degree. Let n be a fixed integer in $\mathbb{Z}$. Then the map j factors in $\mathsf{Ch}_{\Bbbk}$ as:*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, hook, "F_n(j)"'] & & Y \\ & F_n(Y) \arrow[ur, "F_n(p_j)"'] & \end{tikzcd}$$

where $F_n(Y)$ is a chain complex built with the following properties.

- *The chain map $F_n(p_j) : F_n(Y) \to Y$ is a $\mathscr{P}_{\oplus}$-Postnikov tower.*
- *The chain map $F_n(j) : X \to F_n(Y)$ is a monomorphism (i.e. a cofibration in $\mathsf{Ch}_{\Bbbk}$).*
- *The $\Bbbk$-module $(F_n(Y))_i$ differs from Y_i only in degree $i = n - 1$.*
- *In degrees $i \neq n$ in homology, we have $H_i(F_n(Y)) \cong H_i(Y)$ and the maps:*

$$H_i(F_n(j)) : H_i(X) \longrightarrow H_i(F_n(Y)) \cong H_i(Y),$$

are precisely the maps $H_i(j) : H_i(X) \to H_i(Y)$*. In particular, the maps* $H_i(F_n(j))$ *are monomorphisms. Moreover, if the maps* $H_i(j)$ *are isomorphisms, then so are the maps* $H_i(F_n(j))$.

- *In degree* n *in homology, the map* $H_n(F_n(j)) : H_n(X) \xrightarrow{\cong} H_n(F_n(Y))$ *is an isomorphism.*

Proof. We construct below the chain complex $F_n(Y)$ explicitly using Lemma B.2.5. By Proposition B.1.4, we can decompose Y_n as:

$$Y_n \cong H_n(Y) \oplus \overline{Y_n} \cong \mathsf{im}\Big(H_n(j)\Big) \oplus \mathsf{coker}\Big(H_n(j)\Big) \oplus \overline{Y_n},$$

where $\overline{Y_n}$ is the direct sum of the copies of the boundaries. Denote the $\Bbbk$-module $V = \mathsf{coker}(H_n(j))$ and define the linear map $f_n : Y_n \to V$ to be the natural projection. In particular, the map f_n sends n-boundaries of Y to zero. This defines a chain map: $f : Y \longrightarrow S^n(V)$. Notice that since $j : X \to Y$ is a monomorphism, we get $j(\overline{X_n}) \subseteq \overline{Y_n}$, and so, by construction of f, we get that the composite: $X \xrightarrow{j} Y \xrightarrow{f} S^n(V)$, is the zero chain map. We obtain $F_n(Y)$ as the following pullback in $\mathsf{Ch}_{\Bbbk}$, with a chain map $F_n(j)$ induced by the universality of pullbacks:

$$\begin{tikzcd}
X \arrow[dr, dashed, "F_n(j)", "\exists!"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "j"'] & & \\
& F_n(Y) \arrow[r] \arrow[d, "F_n(p_j)"'] \arrow[dr, phantom, "\lrcorner"] & D^n(V) \arrow[d] \\
& Y \arrow[r, "f"'] & S^n(V).
\end{tikzcd}$$

By construction, the induced chain map $F_n(p_j) : F_n(Y) \to Y$ is in $\mathsf{Post}_{\mathscr{P}_\oplus}$. From the commutativity of the diagram:

$$\begin{tikzcd} X \arrow[r, "F_n(j)"] \arrow[dr, hook, "j"'] & F_n(Y) \arrow[d, "F_n(p_j)"] \\ & Y, \end{tikzcd}$$

since j is a monomorphism, so is $F_n(j)$. Since $H_i(j)$ is a monomorphism for $i \in \mathbb{Z}$, then so is $H_i(F_n(j))$. By Lemma B.2.5, we get $H_i(F_n(Y)) \cong H_i(Y)$ for all $i \neq n$. For $i = n$, we get:

$$H_n(F_n(Y)) \cong \mathsf{ker}\Big(H_n(f)\Big) \cong H_n(X),$$

as we have the short exact sequence of $\Bbbk$-vector spaces:

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(j)} H_n(Y) \xrightarrow{H_n(f)} V \longrightarrow 0,$$

since $V = \mathsf{coker}(H_n(j))$. Thus $H_n(F_n(j))$ is an isomorphism as desired. $\square$

Lemma B.2.7. *Every fibration in* $\mathsf{Ch}_\Bbbk$ *is a retract of a* $\mathscr{P}_\oplus$*-Postnikov tower and every map in* $\mathsf{Ch}_\Bbbk$ *factors as an acyclic cofibration followed by a* $\mathscr{P}_\oplus$*-Postnikov tower.*

Remark B.2.8. Unlike Lemma B.2.4, we cannot use the cosmall object argument in order to prove Lemma B.2.7. Indeed, as noted in (Soré, 2010), the codomains S^n of maps in the set $\mathscr{P}$

are not cosmall relative to $\mathsf{Post}_{\mathscr{P}}$. Indeed, let $Y_k = S^n$ for all $k \geq 0$ and $Y_{k+1} \to Y_k$ be the zero maps. Let $Y = \lim\limits_{k \geq 0} Y_k$ be the limit in $\mathsf{Ch}_{\Bbbk}$. The set map:

$$\operatorname*{colim}_{k \geq 0} \left(\mathsf{Hom}_{\mathsf{Ch}_{\Bbbk}}(Y_k, S^n)\right) \longrightarrow \mathsf{Hom}_{\mathsf{Ch}_{\Bbbk}}(Y, S^n)$$

is not a bijection. Indeed, the map is equivalent to the map:

$$\bigoplus_{k \geq 0} \Bbbk \longrightarrow \left(\prod_{k \geq 0} \Bbbk\right)^*,$$

which is never a bijection. A similar argument can be applied to show that the codomains $S^n(V)$ of the maps in the class $\mathscr{P}_{\oplus}$ are not cosmall relative to $\mathsf{Post}_{\mathscr{P}_{\oplus}}$, for any $\Bbbk$-module V.

Appendix B (Continued)

Proof of Lemma B.2.7. The first statement follows from the second using the retract argument. Given a chain map $f : X \to Y$, we build below a chain complex $\widetilde{W}$ as a tower in $\mathsf{Ch}_{\Bbbk}$ using Lemma B.2.6 repeatedly so that f factors as:

$$\begin{tikzcd}
 & \widetilde{W} \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & G_1^-(W^+) \arrow[d, "G_1^-(p)"] & \\
 & W^+ \arrow[d, "p"] & \\
 & \vdots \arrow[d] & \\
 & G_1^+(W) \arrow[d, "G_1^+(p)"] & \\
 & G_0^+(W) \arrow[d, "G_0^+(p)"] & \\
X \arrow[r, "j"] \arrow[uuuuuuur, bend left, "\widetilde{j}"] \arrow[uuuuur, bend left, "G_1^-(j^+)"] \arrow[uuuur, bend left, "j^+"] \arrow[uur, bend left, "G_1^+(j)"] \arrow[ur, bend left, "G_0^+(j)"] \arrow[rr, bend right, "f"'] & W \arrow[r, "p"] & Y
\end{tikzcd}$$

where $\widetilde{j}$ is a monomorphism and a quasi-isomorphism, and all the vertical maps and p are in $\mathsf{Post}_{\mathscr{P}_\oplus}$. The composition of all the vertical maps and p is a chain map $\widetilde{W} \to Y$ which is in $\mathsf{Post}_{\mathscr{P}_\oplus}$, by Proposition A.1.2.

Appendix B (Continued)

We first start by noticing the following factorization:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "j"'] & & Y, \\ & X \oplus Y \arrow[ur, "p"'] & \end{tikzcd}$$

induced by the following pullback in $\mathsf{Ch}_{\Bbbk}$:

$$\begin{tikzcd} X \arrow[drr, bend left, equal] \arrow[dr, hook, "j"] \arrow[ddr, bend right, "f"'] & & \\ & X \oplus Y \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ & Y \arrow[r] & 0. \end{tikzcd}$$

The map p is in $\mathsf{Post}_{\mathscr{P}_{\oplus}}$ by Lemma B.2.3 and Proposition A.1.2. By commutativity of the upper triangle, we see that the monomorphism j induces a monomorphism in homology. We denote $W = X \oplus Y$.

The second step is to replace the map $j : X \to W$ by a chain map $j^+ : X \to W^+$ that remains a cofibration, a monomorphism in homology in negative degrees, and an isomorphism in homology in non-negative degrees. We construct W^+ as the limit $\lim\limits_{n \geq 0}(G_n^+(W))$ in $\mathsf{Ch}_{\Bbbk}$ of the tower of maps:

$$\cdots \longrightarrow G_2^+(W) \xrightarrow{G_2^+(p)} G_1^+(W) \xrightarrow{G_1^+(p)} G_0^+(W) \xrightarrow{G_0^+(p)} W,$$

where each $G_n^+(p)$ is in $\mathsf{Post}_{\mathscr{P}_\oplus}$. The map $j^+ : X \to W^+$ is induced by the monomorphisms $G_n^+(j) : X \to G_n^+(W)$ which are compatible with the tower:

$$\begin{tikzcd} & G_n^+(W) \arrow[d, "G_n^+(p)"] \\ X \arrow[ur, "G_n^+(j)"] \arrow[r, "G_{n-1}^+(j)"'] & G_{n-1}^+(W), \end{tikzcd}$$

and $G_n^+(j)$ induces an isomorphism in homology in degrees i, for $0 \leq i \leq n$, and a monomorphism otherwise. We construct the chain complexes $G_n^+(W)$ of the tower inductively as follows.

- For the initial step, apply Lemma B.2.6 to the monomorphism $j : X \to W$, for $n = 0$. Denote $G_0^+(W) := F_0(W)$. The cofibration $G_0^+(j)$ defined as the chain map:

  $$F_0(j) : X \longrightarrow F_0(W) = G_0^+(W),$$

  is an isomorphism in homology in degree 0, and a monomorphism in other degrees. The chain map $G_0^+(p)$ defined as the map:

  $$F_0(p_j) : G_0^+(W) = F_0(W) \longrightarrow W,$$

  is a $\mathscr{P}_\oplus$-Postnikov tower.

- For the inductive step, suppose, for a fixed integer $n \geq 0$, the chain complex $G_n^+(W)$ is defined, together with a cofibration $G_n^+(j) : X \to G_n^+(W)$ inducing an isomorphism in homology for degrees i, where $0 \leq i \leq n$, and a monomorphsim in homology for other degrees. Apply Lemma B.2.6 to the monomorphism $G_n^+(j)$ for the degree $n+1$. Denote:

$$G_{n+1}^+(W) := F_{n+1}(G_n^+(W)).$$

The cofibration $G_{n+1}^+(j)$ defined as the chain map:

$$F_{n+1}(G_n(j)) : X \longrightarrow F_{n+1}(G_n^+(W)) = G_{n+1}^+(W),$$

is an isomorphism in homology in degrees i where $0 \leq i \leq n+1$, and a monomorphism in other degrees. We obtain a $\mathscr{P}_{\oplus}$-Postnikov tower $G_{n+1}^+(p)$ defined as the chain map:

$$F_{n+1}\left(p_{G_n^+(j)}\right) : G_{n+1}^+(W) = F_{n+1}(G_n^+(W)) \longrightarrow G_n^+(W),$$

such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, hook, "G_n^+(j)"] \arrow[dr, hook, "G_{n+1}^+(j)"'] & & G_n^+(W) \\
& G_{n+1}^+(W). \arrow[ur, "G_{n+1}^+(p)"'] &
\end{tikzcd}$$

The induced map $j^+ : X \to W^+$ is a monomorphism of chain complexes. Indeed, for any fixed $i \in \mathbb{Z}$, we have:

$$\left(G_{i+1}^+(W)\right)_i = \left(G_{i+2}^+(W)\right)_i = \left(G_{i+3}^+(W)\right)_i = \cdots .$$

Appendix B (Continued)

Thus $(W^+)_i = \big(G^+_{i+1}(W)\big)_i$. Therefore the linear map $(j^+)_i : X_i \to (W^+)_i$ is the linear map:

$$\big(G^+_{n+1}(j)\big)_i : X_i \longrightarrow \big(G^+_{i+1}(W)\big)_i,$$

which is a monomorphism. Similarly, we get: $H_i(W^+) \cong H_i(G^+_{i+1}(W))$, for all $i \in \mathbb{Z}$, and so j^+ is a monomorphism in negative degrees in homology, and an isomorphism in homology in non-negative degrees.

The last step is to replace the map $j^+ : X \to W^+$ by the desired chain map $\widetilde{j} : X \to \widetilde{W}$ that is an acyclic cofibration. We construct $\widetilde{W}$ similarly as W^+ (inductively applying Lemma B.2.6) but in negative degrees. We build $\widetilde{W}$ as the limit $\lim\limits_{n \geq 0}(G^-_n(W^+))$ in $\mathsf{Ch}_{\Bbbk}$ of the tower of maps:

$$\cdots \longrightarrow G^-_2(W^+) \xrightarrow{G^-_2(p)} G^-_1(W^+) \xrightarrow{G^-_1(p)} G^-_0(W^+) = W^+,$$

where each $G^-_n(p)$ is in $\mathsf{Post}_{\mathscr{P}_\oplus}$. The map $\widetilde{j} : X \to \widetilde{W}$ is induced by the monomorphisms $G^-_n(j) : X \to G^-_n(W^+)$ which are compatible with the tower:

$$\begin{tikzcd} & G^-_n(W) \arrow[d, "G^-_n(p)"] \\ X \arrow[ur, "G^-_n(j)"] \arrow[r, "G^-_{n-1}(j)"'] & G^-_{n-1}(W), \end{tikzcd}$$

and $G^-_n(j)$ induces an isomorphism in homology in degrees i, for $i \geq -n$, and a monomorphism otherwise. Similarly as the positive case, the map $\widetilde{j} : X \to \widetilde{W}$ can be shown to be a monomorphism and quasi-isomorphism, hence an acyclic cofibration, as desired. $\square$

Appendix B (Continued)

B.3 Postnikov Presentation for Comodules over Simply-Connected Coalgebras

Definition B.3.1. For all commutative ring R, we denote $\tau_{\geq 0} : \mathsf{Ch}_R \to \mathsf{Ch}_R^{\geq 0}$ the 0-th truncation (see (Weibel, 1994, 1.2.7)). Let $\Bbbk$ be a finite product of fields. From the sets and classes of Definition B.1.5, we denote their image under the truncation by:

$$\mathscr{P}^{\geq 0} = \{D^n \longrightarrow S^n\}_{n\geq 1} \cup \{0 \to S^0\}, \qquad \mathscr{Q}^{\geq 0} = \{D^n \longrightarrow 0\}_{n\geq 1},$$

and:

$$\begin{aligned} \mathscr{P}_{\oplus}^{\geq 0} \quad := \quad & \Big\{ D^n(V) \longrightarrow S^n(V) \mid V \text{ any } \Bbbk\text{-module} \Big\}_{n\geq 1} \\ & \bigcup \Big\{ 0 \longrightarrow S^0(V) \mid V \text{ any } \Bbbk\text{-module} \Big\}. \end{aligned}$$

Since $\tau_{\geq 0}(\mathsf{Post}_{\mathscr{P}_\oplus}) \subseteq \mathsf{Post}_{\mathscr{P}_\oplus^{\geq 0}}$ and $\tau_{\geq 0}(\mathsf{Post}_{\mathscr{Q}}) \subseteq \mathsf{Post}_{\mathscr{Q}^{\geq 0}}$ by Proposition A.1.3, we can easily adapt our arguments and show fibrant generation and cocellular presentation for $\mathsf{Ch}_{\Bbbk}^{\geq 0}$.

We can easily adapt our arguments of before to show the following (it also follows from Theorem B.2.1 and Proposition A.3.4).

Proposition B.3.2. *Let $\Bbbk$ be a finite product of fields. Let C be a non-negative differential graded $\Bbbk$-coalgebra. Then the model category $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ is fibrantly generated by $(\mathscr{P}^{\geq 0}, \mathscr{Q}^{\geq 0})$. The model category $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is fibrantly generated by $(\mathscr{P}^{\geq 0} \otimes C, \mathscr{Q}^{\geq 0} \otimes C)$.*

We shall focus in this section to show the following, which is a generalization of the result in (Hess, 2009).

Theorem B.3.3. *Let $\Bbbk$ be a finite product of fields. Let C be a simply connected differential graded $\Bbbk$-coalgebra. Then $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C, \mathscr{Q}^{\geq 0} \otimes C)$ is a Postnikov presentation of the model category $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ of right C-comodules in non-negative chain complexes.*

We shall prove Theorem B.3.3 with Lemmas B.3.7 and B.3.12 below. This will provide us with a very explicit inductive fibrant replacement for comodules as we will see in Corollary B.3.15.

In order to understand a Postnikov presentation of $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ we must be able to describe limits of towers and pullbacks. Recall that $U : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ preserves and reflects colimits and finite limits. Thus pullbacks in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ are computed in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. In general, limits of towers in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ are very different than limits of the underlying towers in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. If $\{X(n)\}$ is a tower of right C-comodules, we denote its limit by $\mathsf{lim}_n^C X(n)$, and if we forget the C-comodule coactions, we denote the limit in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ by $\mathsf{lim}_n U(X(n))$. Nevertheless, in good situations, we can describe those towers.

Definition B.3.4. Let R be a commutative ring. A tower $\{X(n)\}$ in $\mathsf{Ch}_R^{\geq 0}$ *stabilizes in each degree* if for each degree $i \geq 0$, the tower $\{X(n)_i\}$ of $\Bbbk$-modules stabilizes for $n \geq i+1$, i.e., for all $n \geq 0$, and all $0 \leq i \leq n$, we have: $X(n+1)_i = X(n+2)_i = X(n+3)_i = \cdots$. Let C be a non-negative differential graded R-coalgebra. A tower $\{X(n)\}$ in $\mathsf{CoMod}_C(\mathsf{Ch}_R^{\geq 0})$ *stablizes in each degree* if the underlying tower $\{U(X(n))\}$ in $\mathsf{Ch}_R^{\geq 0}$ stabilizes in each degree.

Appendix B (Continued)

Lemma B.3.5. *Let R be a commutative ring. Let $\{X(n)\}$ be a tower in $\mathsf{Ch}_R^{\geq 0}$ that stabilizes in each degree. Let C be any chain complex in $\mathsf{Ch}_R^{\geq 0}$. Then the tower $\Big\{X(n) \otimes C\Big\}$ in $\mathsf{Ch}_R^{\geq 0}$ also stabilizes in each degree and we have: $\Big(\mathsf{lim}_n X(n)\Big) \otimes C \cong \mathsf{lim}_n\Big(X(n) \otimes C\Big)$.*

Proof. For all $n \geq 0$, and all $0 \leq i \leq n$, we have:

$$\begin{aligned} \Big(X(n+1) \otimes C\Big)_i &= \bigoplus_{a+b=i} X(n+1)_a \otimes C_b \\ &= \bigoplus_{a+b=i} X(n+2)_a \otimes C_b \\ &= \Big(X(n+2) \otimes C\Big)_i, \end{aligned}$$

as $0 \leq a \leq i \leq n$. This argument generalizes in higher degrees and thus shows that the desired tower stabilizes in each degree. For all $i \geq 0$, notice that both $\Big(\Big(\mathsf{lim}_n X(n)\Big) \otimes C\Big)_i$ and $\Big(\mathsf{lim}_n\Big(X(n) \otimes C\Big)\Big)_i$ are equal to $\bigoplus_{a+b=i} X(i+1)_a \otimes C_b$. $\square$

Corollary B.3.6. *Let R be a commutative ring. Let C be a non-negative differential graded R-coalgebra. Let $\{X(n)\}$ be a tower in $\mathsf{CoMod}_C(\mathsf{Ch}_R^{\geq 0})$ that stabilizes in each degree. Then the natural map:*

$$U(\mathsf{lim}_n^C X(n)) \xrightarrow{\cong} \mathsf{lim}_n U(X(n))$$

is an isomorphism in $\mathsf{Ch}_R^{\geq 0}$.

Proof. This follows directly from Lemma B.3.5 as U preserves and reflects a limit precisely when the comonad $- \otimes C : \mathsf{Ch}_R^{\geq 0} \to \mathsf{Ch}_R^{\geq 0}$ preserves that limit. In detail, if we denote X the chain complex $\mathsf{lim}_n U(X(n))$, then the coaction $X \to X \otimes C$ is constructed as follows. For each degree $i \geq 0$, the map $X_i \to (X \otimes C)_i$ is entirely determined by the coaction $X(i+1) \to X(i+1) \otimes C$. $\square$

We now start proving Theorem B.3.3. The following lemma remains true for the unbounded case and actually follows from (Hess, 2009, 1.15).

Lemma B.3.7. *Let $\mathbb{k}$ be a finite product of fields. Let C be a non-negative differential graded $\mathbb{k}$-coalgebra. Every acyclic fibration in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ is a retract of a $(\mathcal{Q}^{\geq 0} \otimes C)$-Postnikov tower. Every map in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ factors as a cofibration followed by a $(\mathcal{Q}^{\geq 0} \otimes C)$-Postnikov tower.*

Proof. Just as in Lemma B.2.4, the proof follows either from the cosmall object argument, or given any map $X \to Y$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$, choose an acylic chain complex Z which is a product of 1-dimensional disks, such that $U(X) \hookrightarrow Z$, just as in Lemma B.2.4. By adjunction, we obtain a monomorphism $X \hookrightarrow Z \otimes C$ into an acyclic cofree C-comodule. Then the desired factorization is given by factoring through $(Z \otimes C) \oplus Y$. $\square$

Corollary B.3.8. *Let $\mathbb{k}$ be a finite product of fields. Let C be a non-negative simply connected differential graded $\mathbb{k}$-coalgebra. Then the forgetful functor $U : \mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0}) \to \mathsf{Ch}_{\mathbb{k}}^{\geq 0}$ preserves acyclic fibrations.*

Appendix B (Continued)

Proof. Every acyclic fibration $X \to Y$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ is a retract of the projection $(Z \otimes C) \oplus Y \to Y$ as constructed in the proof of Lemma B.3.7. Notice that we have the projection is the following pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ (and in $\mathsf{Ch}_{\mathbb{k}}^{\geq 0}$):

$$\begin{tikzcd}
(Z \otimes C) \oplus Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \otimes C \arrow[d] \\
Y \arrow[r] & 0.
\end{tikzcd}$$

Since $Z \otimes C \to 0$ is clearly an acyclic fibration in $\mathsf{Ch}_{\mathbb{k}}^{\geq 0}$, the result follows. $\square$

For any chain complex C and any $\mathbb{k}$-module V, we see that the i-th term of the chain complex $S^n(V) \otimes C$ is the $\mathbb{k}$-module $V \otimes C_{i-n}$. If we choose C to be a 1-connected differential graded $\mathbb{k}$-coalgebra, we get:

$$(S^n(V) \otimes C)_i = \begin{cases} 0 & i < n, \\ V & i = n, \\ 0 & i = n+1, \\ V \otimes C_{i-n} & i \geq n+2. \end{cases}$$

Thus, around the n-th term, the chain complex $S^n(V) \otimes C$ is similar to $S^n(V)$. We can therefore modify the homology of a C-comodule for a specific degree without modifying the lower degrees.

Lemma B.3.9. *Let $\mathbb{k}$ be a finite product of fields. Let C be a simply connected differential graded $\mathbb{k}$-coalgebra. Let X be any object in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$. Let V be any $\mathbb{k}$-module. Let $n \geq 1$ be any integer. Given a surjective linear map $f_n : (U(X))_n \to V$ non-trivial only on n-cycles,* *there is a comodule map* $f : X \to S^n(V) \otimes C$*, and the pullback comodule* P *in the following diagram in* $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$*:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D^n(V) \otimes C \arrow[d] \\ X \arrow[r] & S^n(V) \otimes C, \end{tikzcd}$$

has homology:

$$H_i(P) \cong \begin{cases} \mathsf{ker}\left(H_n(f)\right) & i = n, \\ H_i(X) & i < n, \end{cases}$$

and we have $P_i = X_i$ *for* $i < n-1$ *and* $i = n$*, and* $P_{n-1} = X_{n-1} \oplus V$*.*

Proof. The proof is similar to Lemma B.2.5, as we have:

$$(D^n(V) \otimes C)_i = \begin{cases} 0 & i < n-1, \\ V & i = n-1, n, \\ (V \otimes C_{i-n}) \oplus (V \otimes C_{i-(n-1)}) & i \geq n+1. \end{cases}$$

Notice that the differential $(D^n(V) \otimes C)_{n+1} \to (D^n(V) \otimes C)_n$ is trivial. Thus we can adapt our arguments. □

Lemma B.3.10. *Let* $\Bbbk$ *be a finite product of fields. Let* C *be a simply connected differential graded* $\Bbbk$*-coalgebra. Let* $j : X \to Y$ *be a monomorphism in* $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$*, such that it induces* *a monomorphism in homology in each degree. Let $n \geq 1$ be a fixed integer. Then the map j factors in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ as:*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, hook, "F_n(j)"'] & & Y \\ & F_n(Y) \arrow[ur, "F_n(p_j)"'] & \end{tikzcd}$$

where $F_n(Y)$ is a right C-comodule built with the following properties.

- *The map $F_n(p_j) : F_n(Y) \to Y$ is a $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C)$-Postnikov tower.*
- *The map $F_n(j) : X \to F_n(Y)$ is a monomorphism (i.e. a cofibration in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$).*
- *The $\Bbbk$-module $(F_n(Y))_i$ differs from Y_i in degrees $i = n-1$ and $i \geq n+1$.*
- *In degrees $i < n$ in homology, we have $H_i(F_n(Y)) \cong H_i(Y)$ and the maps:*

$$H_i(F_n(j)) : H_i(X) \longrightarrow H_i(F_n(Y)) \cong H_i(Y),$$

  are precisely the maps $H_i(j) : H_i(X) \to H_i(Y)$. For all degrees $i \geq 0$, the maps $H_i(F_n(j))$ are monomorphisms, such that, if the maps $H_i(j)$ are isomorphisms, then so are the maps $H_i(F_n(j))$.

- *In degree n in homology, we have $H_n(F_n(Y)) \cong H_n(X)$ and the map:*

$$H_n(F_n(j)) : H_n(X) \longrightarrow H_n(F_n(Y)) \cong H_n(X),$$

  is an isomorphism.

Proof. This is similar to the proof of Lemma B.2.6 and thus we shall omit some details. Define V as the cokernel of $H_n(j)$ and obtain a chain map $U(Y) \to S^n(V)$. By adjointness, obtain a C-comodule map $Y \to S^n(V) \otimes C$. Define $F_n(Y)$ as the following pullback $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd} F_n(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D^n(V) \otimes C \arrow[d] \\ Y \arrow[r] & S^n(V) \otimes C, \end{tikzcd}$$

and the argument follows from previous lemma. □

We state the case $n = 0$ carefully.

Lemma B.3.11. *Let $\Bbbk$ be a finite product of fields. Let C be a 1-connected differential graded $\Bbbk$-coalgebra. Let $j : X \to Y$ be a monomorphism in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$, such that it induces a monomorphism in homology in each degree. Then the map j factors in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ as:*

$$\begin{tikzcd} X \arrow[rr, hook, "j"] \arrow[dr, hook, "F_0(j)"'] & & Y \\ & F_0(Y) \arrow[ur, "F_0(p_0)"'] & \end{tikzcd}$$

where $F_0(Y)$ is a right C-comodule built with the following properties.

- *The map $F_0(p_0) : F_0(Y) \to Y$ is a $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C)$-Postnikov tower.*
- *The map $F_0(j) : X \to F_0(Y)$ is a monomorphism and a monomorphism in homology.*
- *In degree zero, the map $H_0(F_0(j)) : H_0(X) \to H_0(F_0(Y))$ is an isomorphism of $\Bbbk$-modules.*

Proof. Let $V = \mathsf{coker}(H_0(j))$ which defines a map $Y \to S^0(V) \otimes C$ of right C-comodules, such that, if we precompose with $j : X \to Y$, it is the zero map. Define the right C-comodule $F_0(Y)$ as follows:

$$\begin{tikzcd}
F_0(Y) \arrow[r] \arrow[d, "F_0(p_0)"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
Y \arrow[r, "f"'] & S^0(V) \otimes C.
\end{tikzcd}$$

One can easily check that $F_0(Y)$ has all the desired properties by the same arguments as before. $\square$

Lemma B.3.12. *Let $\Bbbk$ be a finite product of fields. Let C be a simply connected differential graded $\Bbbk$-coalgebra. Every fibration in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ is a retract of a $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C)$-Postnikov tower. Any morphism in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ factors as a cofibration followed by a $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C)$-Postnikov tower.*

Proof. We argue similarly as in the proof of Lemma B.2.7. Let $f : X \to Y$ be any morphism in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$. We can factor through the C-comodule $W := (U(X) \otimes C) \oplus Y$ via the following pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$:

$$\begin{tikzcd}
X \arrow[dr, hook] \arrow[drr, hook, bend left, "\rho"] \arrow[ddr, bend right, "f"'] & & \\
& W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U(X) \otimes C \arrow[d] \\
& Y \arrow[r] & 0,
\end{tikzcd}$$

where ρ is the C-coaction of X. Then define $W^+ = \mathsf{lim}_n^C F_n(W)$ inductively using previous lemmas. Notice that the tower stabilizes in each degree and thus, for each $i \geq 0$:

$$(W^+)_i = (\mathsf{lim}_n^C F_n(W))_i = (\mathsf{lim}_n F_n(W))_i = (F_{i+1}(W))_i = (F_{i+2}(W))_i = \cdots,$$

by Corollary B.3.6. Thus $H_i(W^+) = H_i(F_{i+1}(W)) \cong H_i(X)$ and we get the desired factorization. $\square$

Remark B.3.13. The forgetful functor $U : \mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0}) \to \mathsf{Ch}_{\Bbbk}^{\geq 0}$ does not preserve fibrations in general. Indeed, the generating fibration $0 \to S^0(V) \otimes C$ is not a positive levelwise epimorphism.

Remark B.3.14. Using the vocabulary of (Hess and Shipley, 2014), we have essentially shown that the comonad $- \otimes C$ on $\mathsf{Ch}_{\Bbbk}^{\geq 0}$ is *tractable* and *allows the inductive arguments* and thus by (Hess and Shipley, 2014, 5.8) we indeed have that $(\mathscr{P}_{\oplus}^{\geq 0} \otimes C, \mathscr{Q}^{\geq 0} \otimes C)$ is a Postnikov presentation of $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$.

The following crucial result follows directly from Lemma B.3.12 where we apply the factorization to a trivial map of right C-comodule $X \to 0$. We recall that we define homotopy limits of towers as limits of fibrant towers as in Proposition A.1.13.

Corollary B.3.15. *Let X be any right C-comodule in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. Then there exists a countable tower $\{X(n)\}$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ with limit $\widetilde{X} := \mathsf{lim}_n^C X(n)$ where the right C-comodules $X(n)$ are built inductively as follows.*

- *Define $X(0)$ to be the trivial C-comodule 0.*
- *Define $X(1)$ to be the cofree C-comodule $U(X)\otimes C$. The map $X(1)\to X(0)$ is trivial.*
- *Suppose $X(n)$ was constructed for a certain $n\geq 1$. Then there exists a certain $\mathbb{k}$-module V_n and a map of C-comodule $X(n)\to S^n(V_n)\otimes C$ such that $X(n+1)$ is defined as the following pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$:*

$$\begin{tikzcd} X(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D^n(V_n)\otimes C \arrow[d] \\ X(n) \arrow[r] & S^n(V_n)\otimes C. \end{tikzcd}$$

The tower $\{X(n)\}$ enjoys the following properties.

(i) *The map $\widetilde{X}\longrightarrow 0$ is a $(\mathscr{P}_{\oplus}\otimes C)$-Postnikov tower and there exists an acyclic cofibration of right C-comodules $X \overset{\simeq}{\hookrightarrow} \widetilde{X}$.*

(ii) *If X is a fibrant right C-comodule, then X is a retract of $\widetilde{X}$.*

(iii) *For all $n\geq 1$, we have $H_i(X(n))\cong H_i(X)$ for all $0\leq i\leq n-1$.*

(iv) *The tower $\{X(n)\}$ stabilizes in each degree. In particular $U(\widetilde{X}) = U(\mathsf{lim}^C_n X(n)) \cong \mathsf{lim}_n(U(X(n)))$.*

(v) *Each map $X(n+1)\to X(n)$ for $n\geq 0$ is a fibration in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$, and its underlying map $U(X(n+1))\to U(X(n))$ is also a fibration in $\mathsf{Ch}_{\mathbb{k}}^{\geq 0}$. In particular $\widetilde{X}$ is the homotopy limit of $\{X(n)\}$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\mathbb{k}}^{\geq 0})$ and we have: $U(\widetilde{X})\simeq U(\mathsf{holim}^C_n X(n))\simeq \mathsf{holim}_n(U(X(n)))$.*

Proof. Observe that we do not need to apply Lemma B.3.11 as $X(1) = U(X) \otimes C$ has already the correct homology: $H_0(X(1)) = H_0(X) \otimes H_0(C) \cong H_0(X)$ as C is 1-connected. Notice that the generating fibrations $D^n(V) \otimes C \to S^n(V) \otimes C$ are all levelwise positive epimorphisms as chain maps, and thus are fibrations in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. Since pullbacks in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ are computed in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$, we get that each $U(X(n+1)) \to U(X(n))$ is a fibration in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. $\square$

Definition B.3.16. Let X be a right C-comodule in $\mathsf{Ch}_{\Bbbk}^{\geq 0}$. The *Postnikov tower of* X is the tower $\{X(n)\}$ in $\mathsf{CoMod}_C(\mathsf{Ch}_{\Bbbk}^{\geq 0})$ built in Corollary B.3.15. The construction is not functorial.

B.4 Postnikov Presentation Over a Perfect Coalgebra

In the previous section, we followed the approach of (Hess, 2009). For comodules over a differential graded coalgebra C that is a perfect chain complex, we shall follow the approach of (Smith, 2011). Although not used for the arguments in this paper, this can help compute homotopy limits in $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ but also in ${}_{C^*}\mathsf{Mod}(\mathsf{Ch}_R)$, see Remark 6.4.8. The Postnikov towers will be functorial but not constructed degree by degree, unlike the case for finite product of fields.

Let R be a commutative ring. In this section, we shall always assume that the category of unbounded chain complexes Ch_R is endowed with its injective model structure and we shall always assume that $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ is endowed with its injective model structure too.

Let Fib and $\widetilde{\mathsf{Fib}}$ denote the classes of injective fibrations and acyclic injective fibrations respectively. Then Ch_R is (trivially) fibrantly generated and Postnikov presented by $(\mathsf{Fib}, \widetilde{\mathsf{Fib}})$. For any differential graded R-coalgebra C, we then get, by Proposition A.3.4, that $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ is fibrantly generated by $(\mathsf{Fib} \otimes C, \widetilde{\mathsf{Fib}} \otimes C)$. We shall show the following here.

Theorem B.4.1. *Let R be a commutative ring. Let C be a differential graded R-coalgebra that is perfect as a chain complex. Then $(\mathsf{Fib} \otimes C, \widetilde{\mathsf{Fib}} \otimes C)$ is a Postnikov presentation of the injective model structure of right C-comodules $\mathsf{CoMod}_C(\mathsf{Ch}_R)$.*

Proof. Let $f : X \to Y$ be a map of right C-comodules. We need to show first that f factors through an acyclic cofibration followed by $(\widetilde{\mathsf{Fib}} \otimes C)$-Postnikov tower. This follows from (Hess, 2009, 1.15). In more details, there is an acyclic chain complex Z and a (functorial) factorization in Ch_R:

$$\begin{tikzcd}
U(X) \arrow[rr] \arrow[dr, hook] & & 0 \\
& Z \arrow[ur, two heads, "\simeq"'] &
\end{tikzcd}$$

Since the functor $-\otimes C : \mathsf{Ch}_R \to \mathsf{CoMod}_C(\mathsf{Ch}_R)$ is right Quillen, then $Z \otimes C \to 0$ is in $\mathsf{Post}_{\widetilde{\mathsf{Fib}} \otimes C}$. The chain map $U(X) \to Z$ induces a comodule map $X \to Z \otimes C$ that remains a monomorphism (as C is a flat chain complex). We obtain the desired factorization via the following pullback in $\mathsf{CoMod}_C(\mathsf{Ch}_R)$:

$$\begin{tikzcd}
X \arrow[dr, hook, dashed] \arrow[drr, hook, bend left] \arrow[ddr, bend right, "f"'] & & \\
& (Z \otimes C) \oplus Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Z \otimes C \arrow[d, two heads, "\simeq"] \\
& Y \arrow[r] & 0.
\end{tikzcd}$$

Appendix B (Continued)

We now want to show the second factorization, i.e. we want to show that f factors through an acyclic cofibration followed by a $(\mathsf{Fib} \otimes C)$-Postnikov tower. For any chain complex M, let us fix a following (functorial in M) factorization in Ch_R:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, hook, "\simeq"'] & & M \\ & P(M). \arrow[ur, two heads] & \end{tikzcd}$$

Now we inductively define our desired factorization. Define $W(0) = (Z \otimes C) \oplus Y$ as above, and let $j_0 : X \hookrightarrow W(0)$ and $p_0 : W(0) \to Y$ be the cofibration and the $(\widetilde{\mathsf{Fib}} \otimes C)$-Postnikov tower respectively defined above. Notice that $\mathsf{Post}_{\widetilde{\mathsf{Fib}} \otimes C} \subseteq \mathsf{Post}_{\mathsf{Fib} \otimes C}$. Let $W(-1)$ denote Y and $j_{-1} = f$. Now, for $n \geq 0$, suppose we have defined a cofibration $j_n : X \hookrightarrow W(n)$ and a $(\mathsf{Fib} \otimes C)$-Postnikov tower $p_n : W(n) \to W(n-1)$ such that $p_n \circ j_n = j_{n-1}$. Define the right C-comodule $K(n)$ as the cokernel of j_n:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "j_n"'] & 0 \arrow[d] \\ W(n) \arrow[r, "k_n"] \arrow[ur, phantom, "\ulcorner"] & K(n). \end{tikzcd}$$

Then the comodule map k_n induces a map $\underline{k_n} : W(n) \to U(K(n)) \otimes C$ which is the adjoint of $U(k_n) : U(W(n)) \to U(K(n))$. It is easy to check that $\underline{k_n} \circ j_n = 0$. Let us denote $\underline{K(n)} :=$

$U(K(n))$. Then the fibration $P(\underline{K(n)}) \to \underline{K(n)}$ induces a $(\mathsf{Fib} \otimes C)$-Postnikov tower map $P(\underline{K(n)}) \otimes C \to \underline{K(n)} \otimes C$. Define $W(n+1)$ as the following pullback of right C-comodules:

$$\begin{tikzcd}
X \arrow[dr, dashed, "j_{n+1}"] \arrow[drr, bend left, "0"] \arrow[ddr, hook, bend right, "j_n"'] & & \\
& W(n+1) \arrow[r] \arrow[d, two heads, "p_{n+1}"] \arrow[dr, phantom, "\lrcorner", very near start] & P(\underline{K(n)}) \otimes C \arrow[d, two heads] \\
& W(n) \arrow[r, "\underline{k_n}"] & \underline{K(n)} \otimes C.
\end{tikzcd}$$

Define W as the limit of right C-comodules of the tower $\{W(n)\}$ (recall that since C is a perfect chain complex, the limit is computed in Ch_R). Notice that naturality of cokernels induces tower comodule maps $K(n+1) \to K(n)$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook, "j_{n+1}"'] \arrow[dd, bend right=60, hook, "j_n"'] & 0 \arrow[d] \\
W(n+1) \arrow[r, "k_{n+1}"] \arrow[d, "p_{n+1}"'] \arrow[ur, phantom, "\ulcorner", very near end] & K(n+1) \arrow[d, dashed, "\exists!"] \\
W(n) \arrow[r, "k_n"] & K(n).
\end{tikzcd}$$

This defines a tower $\{K(n)\}$ of right C-comodules, such that we obtain the following exact sequence of towers of right C-comodules:

$$0 \longrightarrow \{X\} \longrightarrow \{W(n)\} \longrightarrow \{K(n)\} \longrightarrow 0,$$

where $\{X\}$ denotes the constant tower, which trivially satisfies the Mittag-Leffler condition (see (Weibel, 1994, 3.5.6)) as a tower in Ch_R. Thus, since tower limits in $\mathsf{CoMod}_C(\mathsf{Ch}_R)$ are computed in Ch_R, we obtain the following exact sequence of right C-comodules:

$$0 \longrightarrow X \longrightarrow W \longrightarrow K \longrightarrow 0,$$

where K is the limit of the tower $\{K(n)\}$. Thus the map $f : X \to Y$ factors through W, the map $X \to W$ is a cofibration and $W \to Y$ is a $(\mathsf{Fib} \otimes C)$-Postnikov tower by construction. We are only left to show that K is an acyclic chain complex. This will follow from the fact that the maps $K(n+1) \to K(n)$ are trivial in homology. Indeed, the counit $\varepsilon : C \to \Bbbk$ induces the following commutative diagram in Ch_R (we have dropped U from some of the notations):

$$\begin{tikzcd}
W(n+1) \arrow[r] \arrow[d] & P(\underline{K(n)}) \otimes C \arrow[r] \arrow[d] & P(\underline{K(n)}) \arrow[d] \\
W(n) \arrow[r, "k_n"] \arrow[rr, bend right, "U(k_n)"'] & \underline{K(n)} \otimes C \arrow[r] & \underline{K(n)}.
\end{tikzcd}$$

Notice that the horizontal composite $W(n+1) \to P(\underline{K(n)})$ is trivial if we precompose it with $j_{n+1} : X \to W(n+1)$. Therefore by universality of the cokernel, we get that $K(n+1) \to K(n)$ factors in Ch_R through the chain complex $P(\underline{K(n)})$ which is acyclic. Thus the induced map in homology $H_i(K(n+1)) \to H_i(K(n))$ is trivial for all degrees i and all $n \geq 0$. Since $W(n+1) \to W(n)$ are levelwise epimorphisms then so is $K(n+1) \to K(n)$ (as pushouts preserves epimorphisms). Therefore the tower $\{K(n)\}$, considered in Ch_R, satisfies the Mittag-

Appendix B (Continued)

Leffler condition and the induced maps in homologies are trivial. Thus by (Weibel, 1994, 3.5.8), the homology of K is trivial. $\square$

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