Curriculum Embedded Assessment:

An Examination of the Array Model for Observing Unitizing

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THESIS

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SUMMARY

Recognizing that a student's ability to think about a quantity in flexible ways is desirable, a formative assessment tool for typical curricular array multiplication problems was hypothesized based on unitizing (Lamon, 1996, Sophian, 2008), number sequences (Olive, 2001) and unit coordination schemes (Hackenberg & Tillema, 2009). Students' actual work, explanations, and gestures are compared to the anticipated actions based on a progression.

Video of 77 fourth grade students from a major metropolitan area solving array multiplication tasks was classified by unit coordination levels using the above-mentioned scoring guide in an exploratory approach to determine how gesture, written work, and explanations might support inferences about multiplicative concepts (MC). Data analysis revealed that an array diagram with a covered section can help distinguish between MC levels, indicating the importance of visual support for Pre-MC and MC1 and the impact of the diagram on student thinking. Break-apart multiplication (BAM) solutions were consistent, with 83% agreement for MC ratings where all items could be scored (n= 64). BAM can visually represent the distributive property.

Specific gestures consistently matched different unit coordination schemes as determined by explanation and written work. For example, students who counted squares with more of a hopping motion were less likely to be able to coordinate as many units as a student who counted squares by moving a finger across the line of squares. Students' reasoning for strategy choices provided more evidence that explaining steps or written work alone.

1. INTRODUCTION

The role of *unit* ideas in students' array multiplication thinking is explored as a source information about students' mathematical reasoning. In the 21st century, researchers recognize that observing children's thinking matters – what the teacher or researcher is thinking is not likely to be the same as the child (Copeland, 1974). This study seeks to document how students' mathematical knowledge of unit might be observed in array multiplication settings, thus adding knowledge regarding students' understanding of the relationships between quantities along a trajectory. Units ideas are explicitly identified in fraction instruction, but prior conceptual understanding of *unit* within whole numbers lays an important foundation for understanding numerical relationships in rational number, algebra and place value. Multiplication array models provide significant opportunities for students to explore ideas about the distributive property in a geometric model. Understanding how students perceive *units* while modeling the distributive property in arrays may make it possible to predict which students need more experiences to understand relationships between the parts and the whole before it is critical to understand those relationships, such as in fraction use or solving equations activity.

In this, the introductory chapter, the reader is introduced to the idea of *units* for describing students' mathematical reasoning growth over time, to fundamental aspects of learning theory and to the research questions. Chapter 2 supports the reader's background knowledge on unitizing, multiplication, and related stages of development from a constructivist perspective. The means for making inferences about student thinking is described in Chapter 3, including conceptual analysis, think-aloud protocol, and the multiple means for observing reasoning and thinking, highlighting the importance of gestures. Chapter 4 communicates the data resources. Chapter 5 explains the steps for developing the measurement instrument and the procedures for collecting data. The data analysis and primary results are articulated in Chapter 6. Analysis of unitizing performance on additional mathematical tasks is described in Chapter 7. Chapter 8 discusses the analytic results in light of the research questions, with the conclusions and implications offered in Chapter 9.

Researchers recognize that observing students' thinking matters – what the teacher or researcher identifies as a mathematical structure is not likely to be the same in the child's thinking (Carpenter, 1999; Copeland, 1974). Consequently, revealing *the child's* reasoning about quantities in formative assessments is an essential link between the child's cognitive processing, conceptual understanding, and teaching-learning processes.

Students develop their mathematical understanding by working out relationships between quantities (Sophian, 2008; Steffe & Olive, 2010). Sophian's *The Origins of Mathematical Knowledge in Childhood* (2008) describes the role of *unit* in students' developing understanding of the relationships between quantities. A child's cognitive conception of unit, as a structure that supports relational reasoning, impacts making sense of number (Sophian, 2008). A focus on quantity is different ontologically than focusing on counting because a quantity is *measureable* property of something and counting refers to *how many*. Sophian's (2008) definition is that "*quantities* are physical properties of things we can measure; and numbers are symbols used to represent the measured values of quantities." In determining a quantity, or in counting a collection, developing the concept of unit means recognizing the numerical amount will vary given the unit of measure.

Students' abilities to think about a quantity in flexible ways vary both developmentally (Sophian, 2008) and from access to learning opportunities (Dougherty, 2007; Tzur et. al., 2013). As such, *reasoning about quantities* is both an ability and a skill. This reasoning ability/skill is desirable, given the focus on STEM in K-12, the need for data scientists (Columbus, 2017), and the role thinking about quantity plays in general tasks like budgeting (Ludwick, 2015).

Many K-5 mathematics curricula include opportunities for students to practice working out relationships with quantities, but these opportunities are usually not highlighted as working out relationships. The opportunities are viewed as part of learning arithmetic calculation such as how to add, subtract, multiply or divide. The instructional focus for arithmetic operations is often on procedural aspects of the calculation with much less (if any) opportunities for students to think about how these operations relate quantities. Yet, it is understanding the way operations relate quantities that will become essential understanding in computational literacy, sciences, or mathematics, such as working with equations or calculating with fractions.

Additional focus on unitizing as students learn to multiply may reduce the number of students having difficulties later. By increasing students' opportunities to acquire number knowledge with an emphasis on constructing quantities for meaning-making versus only numerical digit manipulations more students may think about making sense with a quantity instead of randomly applying a procedure. Consequently, the potential to infer if students are constructing numbers when they are learning multiplication and division may help identify which students and what kinds of instructional supports are needed to strengthen whole number multiplicative thinking ability before students are expected to build on that knowledge.

One reason for the less explicit instruction regarding the relationships between quantities is visibility of student's reasoning: Without an instructional tool to recognize growth, reasoning about quantities is hard to measure and consequently it is hard to set expectations for growth in student's thinking. Traditionally reasoning about quantities is not part of the targeted assessment, even though flexible thinking about number relationships is part of highly valued *number sense* (Boaler, 2016; Courtney-Koestler, 2018; Devlin, 2017, Fennell, 2008; Howden, 1989). This study examines students' flexible thinking about number relationships within array multiplication to inform classroom tool development at the introductory level of multiplicative thinking. This work is part of a broader goal to increase educators' understanding about the role of *unit* in students' understanding of the relationships between quantities through training in the use of an appropriate classroom tool.

Formative assessment of students' reasoning about quantities has been elusive - in part because exposing student thinking is time-consuming, where students' words and actions not only must be gathered for data but also analyzed and then interpreted for next steps. Fortunately, now is a good time to revisit identifying students' reasoning more explicitly because the increased use of technology in classrooms is minimizing some of the hurdles to assessing students' reasoning (Moursund, 2016). Also, researchers have articulated more about quantitative reasoning progressions (Confrey & Maloney, 2009; Daro, Mosher & Corcoran, 2011). So, although there are hurdles for delivering feedback on reasoning efficiently and effectively to inform students and instruction, they are not insurmountable and the likelihood of strong learning gains from identifying a student's reasoning motivates using assessments that will match related instruction.

This study is designed to answer this question: (1) How might a continuum of students' unit coordination abilities be revealed through students' array multiplication solution processes? (2) To support formative assessment, a unit coordination scheme-based scoring guide based on measurement principles is developed for use with common classroom array multiplication problem types. (3) To the extent possible with the available data, the predictive validity of unit coordination scores for 2-digit by 1-digit multiplication is investigated.

The Context for Reasoning about *Units*

Although there is a more explicit focus on multiplication concepts in curriculum materials more is needed. For example, multiplication instructional materials have made an important shift to highlighting multiplication as “n” groups with “m” in a group (Burns, 2006; Fuson, 2009; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Van de Walle, Karp, & Bay-Williams, 2019; Wagreich et al., 2013), but teacher expectations and assessment of student understanding is still more often focused on procedural knowledge and less on conceptual schemes (Niemi, D., Valone, J., & Vendlinski, T., 2006; Wiggins, 2014). Understanding how to determine a student’s level of multiplicative reasoning along a continuum is less common.

In the *Second Handbook of Research on Mathematics Learning and Teaching* (Lester, 2007), Lamon promotes the multiplicative conceptual field as a framework to develop a needed understanding for rational number, in contrast to a more traditional approach of emphasizing targeted concepts. Within her chapter, she references the work of Steffe and others, supporting Steffe’s theory regarding the development and use of units in number and operations. While referencing her own work that demonstrates “how ratios and rates can be viewed as complex kinds of units,” she again suggests “that unit building may be an important mechanism in accounting for the development of increasingly sophisticated mathematical ideas” (p. 643). *Unit* as a means of thinking about quantity begins well in advance of fraction instruction (Olive, 2001), but base ten place value and grouping practices often established in multiplication instruction may have essential, but less explicit, *unit* instruction than observed in fraction instruction. More knowledge of how students structure quantities into units is needed in order to improve classroom learning (Beckmann & Izsak, 2015, Thompson, 2011). This study is designed to explore students' thinking about grouping and units to increase the foundation for pedagogical and curricular changes to improve students' number sense at the level of learning multiplication.

Awareness of how students think flexibly with regard to different unit sizes in multiplication and how students make connections between partitioning, splitting and iterating a unit is described along a trajectory by connecting research studies (Hackenberg & Tillema, 2009; Lamon, 1994, Olive, 2001). Array multiplication problems as situations to support inferences regarding student understanding of *unit* can add to the body of research on unitizing and unit coordination to build up the important foundation for understanding how students develop numerical relationships within whole numbers, something which continues to be valued but underutilized in the classroom. One aim of this study is to develop a tool for classroom teachers in the hope that the tool's use will prompt more discussion about unitizing and unit coordination in the context of their students' responses within 3rd-5th-grade math curricula.

Students' understanding of mathematical relationships develops with age and experiences. For this study, the idea of *unit* is key to describing students' developing understanding of the relationships between quantities. The role of *unit* in grouping and multiplicative concepts provides support for students' number sense development. Without an assessment tool to recognize growth, reasoning about quantities is hard to measure and consequently it is hard to set expectations for it. Although assessment of student reasoning can be time-consuming, recent developments in learning sciences, cognitive development, math education, and technology are helping to make an analysis of student reasoning more manageable. For these reasons, it is pertinent to develop tools within a curriculum to identify student's unitizing processes. Students may be able to get accurate answers presently, but without evidence of having established conceptual understanding, students may stumble at the next step in learning because they don't have the necessary conceptual framework. Curriculum designers, educators and others' awareness of students' cognitive understanding represented by unitizing ability may foster new ways to design instruction to help effectively construct students' mathematical structures.

In order to inform teaching and assessing at a point where students begin to develop multiplicative reasoning structures, this study is designed to answer these questions: Q1. Given what is known about unit coordination, how might a continuum of students' unit coordination abilities be revealed through students' array multiplication solution processes performed in the context of an elementary math curriculum? Q2. To what extent might the identified unit coordination schemes provide a model for use in teachers' formative assessment processes? and Q3. Can students' unit coordination scores accurately predict students' future performances on multiplication problems?

2. BACKGROUND

Unitizing and unit coordination concepts support understanding student's thinking about quantities. After a general overview of unitizing and unit coordination, array multiplication and children's thinking regarding multiplication is examined to provide the context for observing students' reasoning while solving problems in array settings.

Unitizing

As an essential but less-explored topic for improving school mathematics, unitizing is a foundational structure for much of number and operations learning across time. It undergirds acquiring number sense as defined by the National Council of Teachers of Mathematics: number sense is "the ability to decompose numbers naturally, use particular numbers like 100 or ½ as referents, use the relationships among arithmetic operations to solve problems, understand the base-ten number system, estimate, make sense of numbers, and recognize the relative and absolute magnitude of numbers" (National Council of Teachers of Mathematics, 2000, p 32). Unitizing helps students make sense of measuring quantity as well as counting. A unitizing continuum has been identified across studies of learning whole numbers (Olive, 2001; Lamon, 1994, Ulrich, 2016), multiplicative thinking (Steffe, 1994; Hackenberg, 2010; Tzur et. al., 2013), conceptualizing fractions (Steffe & Olive, 2010; Hackenberg & Tillema, 2009; Norton & Boyce, 2013), adding integers (Ulrich, 2012), and reasoning algebraically (Ellis, 2007).

A child's cognitive conception of unit, as a structure that supports relational reasoning, impacts making sense of number (Sophian, 2008). A unit can be a singleton item, or a unit can be composed of multiple items that form the unit – a unit is "what counts as 'one'" (Sophian, 2008, p. 7) In particular, Sophian (2008) argues that understanding of numbers depends on quantitative comparisons, citing the work of Davydov (1975) and his colleagues. Comprehending there are potential relationships between quantities and within a unit quantity can inform discussions about the relational reasoning that impacts making sense of number. Pat Thompsons's definition of a quantity is "a measurable quality of something. A magnitude of a quantity is the quantity's measure in some unit" (Thompson, P. 1988). A student's choice of a unit to portray the quantity is related to number and operations ideas (Bass, 2015; Dougherty & Simon, 2014; Dougherty & Venenciano, 2007, Lamon, 1996; Simon & Placa, 2007; Sophian, 2008) and to mathematical reasoning about a quantity (Behr, Harel, Post & Lesh, 1994). Taken together, the variation from counting to multiplicative grouping involves students' ability to conceptualize units that accompany the number to represent the quantity (Sophian, 2008), which in turn impacts how a student can think about various number and operations ideas.

A Historical View for Defining *Unitizing*

Unitizing has multiple, but related definitions. Historically it can refer to a mental structure for describing a quantity in terms of a given piece size (Lamon, 1996), but many researchers today describe unitizing as an operation, referring to different ways that students think as they seek to achieve a goal (Steffe & Olive, 2010). Lamon defines unitizing as "the cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (Lamon, 1996, p 170). Following this line of reasoning, quantity is described with two parts, the *number* of the particular "chunks" described by the unit of measurement, and the unit of measurement that describes the chunk size. (Behr, Harel, Post, & Lesh, 1992; Kaput, 1985; Schwartz, 1988; Thompson, 1988). To explain the quantity involved, *number* references *how many* of a unit of the measure and the combination of the *number* and the *unit* describes the quantity. These researchers called unitizing a mental structure. Many researchers today, however, describe unitizing as an *operation* which students use in a recursive fashion for developing and coordinating their mathematical knowledge of unit by operating on the quantity with increasingly more sophisticated units over time (Hackenberg & Tillema, 2009; Norton & Boyce, 2013; Steffe, 1994, 2013; Ulrich, 2012; von Glaserfeld, 1981).

This view of unitizing as operating reflects Steffe and Olive's use of von Glaserfeld's "model on the conceptual construction of unitizing" (Steffe & Olive, 2010, p. 27). Von Glaserfeld articulated types of items including the abstract unit item that is related to Piaget's arithmetical unit based in Piaget's logico-mathematical theories about how students learn mathematics. In Piaget's theory of intellectual development, to know something is "to act on it, to modify it or transform it, and, in the process, to understand the way the object is constructed" (Copeland, 1974, p. 35), and this is called a psychological *operation*, "an interiorized (mental) action that modifies the object" (Copeland, 1974 p. 35).

As students *operate* with units, their reflections are combined, sequenced or compared to their initial mental actions. Mental models are conjectures for students' mathematical thinking activity. For the purposes of this research, unitizing is the interiorized action that modified units understanding. For the purposes of this research, unitizing is an *operation* which students use in a recursive fashion for developing and coordinating their mathematical knowledge of unit by operating on the quantity with increasingly more sophisticated units over time.

Unitizing capacity builds upon prior units understanding. For example, four countable items can be seen as one unit of 4 as well as four countable units. Then a quantity, say six, of the *unit of 4*s can become one unit. This implies that students' ability to describe quantity with more complicated relationships (e.g., *units* within *units*) will impact their ability to manipulate the more complex number and operations. Students begin learning about units by comparing quantities, counting and using units as a measuring tool. Perceptual units are counted at an earlier age than abstract ones (Sophian, 2008; Wright, Martland, & Stafford, 2006). Units are understood initially as objects, then students comprehend continuous quantities along a continuum of increased structure, perhaps as represented on a number line. Whole number concepts can be situated in the context of unit-coordination as well as quantity (Steffe, 1994, 2010, 2013).

Developing and coordinating the mathematical knowledge of unit is recursive.   Prior to multiplicative thinking, unitizing occurs with the construction of quantities as a number of items both as a collection, as distinct items and at the same time have a perception of relative number quantities. *Three* is one more than *two*, such that *3* refers to three perceptual units (of 1) and *2* refers to two perceptual units. In addition, the students' ability to comprehend both the numbers being added and the sum at the same time i.e., *1* unit (of 1s) and *2* units (of 1s) is the same as the 1 and 2 combined to make *3* units (of 1s) and at the same time, a *3 unit,* is considered for composite units. Students come to recognize a quantity as a *composite* unit (CU) (Olive, 2001; Steffe, 1994, 2013), where amounts are nested: (1+ 1+1+1) is 4 leading to 7 = [4 + (1+1+1)] = 4 + 3. Numbers can be decomposed, where 7 minus 3 will be 4, or 4 + ○ = 7. Building on this knowledge, in multiplication instead of a unit *of 1s,* the unit is the group size, for example in *3 units* (of 4s) the group size, (multiplicand) is *4s* instead of *1s*.

Students' flexible use of number is based in part on understanding that a quantity can be represented in multiple ways, with varying unit measures. Units have more complexity in multiplication than addition. In addition, the units stay the same, with like things, but in multiplication, the unit label can document changes – i.e., *units* (of 4s) or when finding area, side lengths multiplied together produce *square units*. Students are not likely to be thinking in terms of the relationship of *length times width* measures to determine the number of squares, because this is a more complex relationship than noticing squares (Nunes, Light, & Mason, 1993; Battista, 1998). In the present research, students are considered to be counting squares; the idea of side lengths creating square units is not considered without explicit evidence.

Since numbers relate to the unit to describe the quantity, when the quantity is more than one, more coordination of related numbers is necessary as described earlier. From teaching experiment research, students have varied ways of thinking about a given quantity. For counting and addition/subtraction, the unit of measure is the same for the *addends* and *sum* or the *subtrahend*, *minuend* and *difference*, so the role of units in describing a quantity is more implicit because the unit stays constant. For instance, at the intersection of addition and subtraction, 12 cookies can be described as 12 items, as in counting, or 5 cookies here and 7 cookies there to make a total of 12 cookies. For each number, in this example the *5* and the *7*, the unit, one cookie/object is the same, but there are five cookie/objects or seven cookie/objects. Attention is not focused on cookie/object as the unit for calculation purposes. However, moving into multiplication, 12 cookies can also be *1 dozen* or one of a *12-pack*. Twelve cookies is also 1/12 of a gross, or 1/12 of a *144-pack*. A quantity of twelve can be considered as one unit, i.e. one dozen, or, part of one unit, i.e. one-twelfth of a gross, or, 12 units, where the unit is a cookie. Yet in each description, there is the same quantity, but it is described with a different referent.

Table 1
Cookie Example

Number	Unit	Quantity
12	1 cookie	12 cookies
1	1 dozen cookies	12 cookies
1/12	1 gross of cookies	12 cookies

As researchers continue to describe students' reasoning about quantities in an activity, increasingly complex unitizing schemes can expand upon earlier research. Various studies have identified stages of growth incorporating how students conceptualize quantity across mathematical domains (Tzur & Lambert, 2011; Norton & McCloskey, 2008; Hackenberg and Lee, 2015; Steffe, 1994; Steffe & Olive, 2010). This project seeks to identify how the rectangular array constructed response problem type may reveal students' unitizing schemes and consequently provide opportunities to make inferences for instructional next steps regarding multiplicative thinking development and related number sense. The progression of unitizing complexity at the level of multiplication will be described next.

Unit Coordination Progression

For the purposes of this study, unitizing levels with counting objects are included. These levels are pre-cursors to Multiplicative Concepts (Hackenberg & Tillema, 2009; Hackenberg, 2010). After the student can interiorize an item's *unitary wholeness*, the student can view a group of these items as countable. If the items must be visible to coordinate the counting act with the picture or items, the students are considered perceptual counters. If the student can visualize the items to coordinate the counting act, they are figurative counters. (Steffe & Olive, 2010 p. 31-33). For instance, the student finds the number of squares in the array by considering each square to be a separate unit, and touching or marking each one, accounts for the quantity as singleton units. At some point, a student can establish a quantity, and then continue to add to that quantity. The established quantity is both a set of singletons and an established collection of that amount. For instance, if the student established there are six squares in the first section, in addition to seeing 6 individual squares, the student also sees a collection of six as a unit. The student can recognize *six* and then continue to count on from there.

Thinking in terms of multiplication requires students to think along two number sequences. This type of thinking might look like adding on an amount, say three, while keeping track of the total – for example, six is *one*, seven is *two*, eight is *three*. When a student can think this way, it opens the possibility of inserting a singleton unit amount into another for coordinating two levels of units.

Whole number multiplicative concepts are described in terms of unit coordination by Hackenberg & Tillema (2009). They articulate Multiplicative Concept (MC) levels where unit coordination is described as “inserting a composite unit into the units of another composite unit” (p. 3). MC levels align with the definition for unitizing as the interiorized action that modifies units understanding, and, the body of research on unit coordination.

The “*first multiplicative concept* (MC1) is based on the coordination of two levels of units in activity” (p. 3) where *in activity* refers to the actions of finding a solution, perhaps the use of a diagram or story. In other words, in the actions of finding the solution, one unit is inserted into another. For example, there are *5* columns on the grid, and in each of them, there are *4* squares. The unit *4* is inserted into each column, to create *5* columns of *4* squares on the paper. Students are considered to be at Multiplicative Concept 1 (MC1) when they can keep track of how many groups, say columns, and the total amount of singletons. For example, a student can see a column of the array as 4 squares as well as one column and uses the array diagram to help her keep track of the count for the number of squares by column counting each by one. At this level, students are actively using the diagram or are otherwise engaged in ways that help keep track of inserting a unit amount into another unit.

The *second multiplicative concept* (MC2) is making groups where each quantity in the group is more than one, such as *4* groups of *5* stars, without using a diagram or story. For the second MC, students can coordinate two levels of units in their head, called an *interiorized* action. In addition, a student who is trying to add two rows of six might recognize each row as six, but to find the sum, take 4 from one row of six and add it to the other row of six to make ten, and then add on two more, while still keeping track that the 4 is still part of the 6 squares row. The student might also view a row in the array as a single unit to operate with it in her head. For a six by nine array, the student can think one row is six so two rows make twelve, and 4 rows is 24, so eight rows will be 48 and with one more row will be 54. These students recognize a singleton unit, or one, as an abstract quantity that can be iterated. Students at this level are said to have constructed part to whole operations because they can view a quantity as containing parts that can be pulled out or put back in the quantity while still seeing that amount as illustrated in the 2 rows of six example.

With this interiorized understanding, an additional layer of unit complexity can be constructed within the activity of solving the problem. Students who can think about coordinating two levels of units in their head can use the diagram to coordinate three levels of units. Students might be able to determine a quantity for a given number of rows and columns and then iterate the amount for that array. For example, a rectangular array with 12 rows of five is seen as containing a group of 20 items in *4* rows, with *5* in each row, and continue to make another group of 4 rows of five and then make another group of four groups of five to determine there are 6o squares in the rectangle, where the rectangular array has three groups of four rows with five in a row. The student accomplishes this thinking with the help of a diagram or within the activity of solving the problem. The diagram, representation, story, or other activity helped support the student to recognize the quantity *as y groups with x in a group* while manipulating the groups as well as thinking about the number of singletons.

The *third multiplicative concept* (MC3) is being able to think about the level of singletons, the grouping of groups and be able to manipulate the number of groups to gain new information within the mind, that is, in an *interiorized* fashion. A student who is solving for the number of squares in 14 rows of 4 could recognize an array with six columns of four as 24 within the larger array as well as think about the eight columns of four that are left within the fourteen columns of four and then add the amounts of the smaller arrays to determine there are 56 squares.

The MC levels describe a unidimensional progression of multiplicative reasoning. MC1 students have interiorized one level of unit and can coordinate two levels of units in activity. MC2 students have interiorized two levels of units, with the potential of coordinating three levels of units in activity. MC3 student have interiorized three levels of units (Hackenberg & Tillema, 2009; Tillema, 2013).

When students use the rectangular array, tallies, fingers or other drawings, diagrams or story elements to keep track of their counting, they may be coordinating units *in activity*. It is important to note that some students choose to do this because they anticipate this is expected and not because they need to do so to determine amounts. When the student does not need the diagram or tallies to keep track, the coordination of units is *interiorized*.

Unit Coordination Contexts

In this study, the researcher hypothesizes that as students describe their solution for finding the number of squares in a rectangular array, they will provide evidence for unit coordination schemes that can be interpreted to indicate additive or multiplicative thinking along a continuum. The following sections build support for the array multiplication item type's potential to identify students' unitizing schemes in a constructivist learning environment.

Multiplication

This section provides general background on multiplication research and specific information about the array diagram that is the focus tool for identifying unitizing in classroom settings.

Multiplication Research Findings. Studies of students' ability to move from additive strategies to multiplication strategies for problem solving or calculation have identified stages in a transition from addition to multiplication understanding: counting, skip counting, additive strategies, multiplicative strategies (Anghileri, 1989; Carrier, 2014; Ell, Irwin, & McNaughton, 2004; Jacob & Willis, 2001, 2003; Kouba, 1989; Mulligan & Mitchelmore, 1997). For example, Mulligan & Mitchelmore (1997) and Kouba (1989) used various story problems with second and third graders for their research. Anghileri (1989) documented students' strategy choices during interviews for a range of problems using manipulatives with students between four and twelve.

Jacob and Willis (2001) used tasks with concrete materials with seven to nine-year old students. Ell, Irwin & McNaughton (2004) observed 10-year old students doing 2-digit by 1-digit calculations. Carrier (2014) analyzed data where fourth grade students answered questions regarding quantities of fish, pizzas and objects such as paper clips. Sherin & Fuson (2005) analyzed data from word problems and straight calculation problems to posit that the affordances of *number specific computational resources* used in learning single digit multiplication were key components and from this research they also developed an accompanying invented *strategies* taxonomy.

Others have specifically focused on conceptual schema based on students' interaction with sticks or towers of 3-D cubes and related computer software (Norton & Wilkins, 2012; Steffe, 1988; Steffe and Olive, 2010; Tzur et al., 2013). Students' conceptual schemes are described as student work or actions during specific tasks with a given goal. The array model, specifically included in the Common Core State Standards in multiple grades, is considered useful to scaffold students' multiplication learning and thus becomes a setting for observing students' conceptual schemes.

Representing Multiplication with Arrays. The *Principles and Standards for School Mathematics (*National Council of Teachers of Mathematics, 2000) includes *representation* as one of the ten Standards. "The term *representation* refers both to process and to the product—in other words, the act of capturing a mathematical concept or relationship in some form and to the form itself." (p. 67). Examples of representations include symbolic expressions, diagrams, and graphical displays. Arrays are also listed among the representations for creating solutions to multiplication problems within the many dimensions of problem structures that are described in the Ongoing Assessment Project (OGAP) Multiplicative Framework, updated in January, 2017

(Ongoing Assessment Project, 2017). The OGAP Multiplicative Framework is a derivative product of the Vermont Mathematics Partnership Ongoing Assessment Project, designed to help teachers make inferences about their students' understanding based on evidence within formative assessment tasks and consequently to make instructional decisions that move students forward as much as possible.

The rectangular array is a valuable representation because is supports students' understanding of mathematical concepts and relationships especially related to multiplication, helping communication between students, student to teacher and student to self. The array promotes the idea of "*times as many*" with *n* times a given *m* unit – where *m* is a unit of units, but students can also find the number of squares by counting or by a repeated addition of a row (or column) or use another approach, thus making it a versatile model. Consequently, the rectangular array as a model allows the child to reason about quantity of squares at various levels of sophistication.

The Common Core State Standards for Mathematics (CCSSM) includes the use of array models for multiplication and division learning over grades 3-6, indicating the acceptance and prevalence of array models. The CCSSM explicitly relates arrays and area to the operations of multiplication and addition in third grade multiplication, fourth grade whole number multiplication calculations and explanations for word problem operation choices (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 29). Arrays help model decimal operations and the model helps connect finding the area of a rectangle with fractional side lengths by tiling squares with finding area using side lengths in grade 5 and arrays help model division of fractions by fractions in grade 6. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). These standards demonstrate that the CCSSM includes the use of array models for multiplication and division learning over grades 3-6, indicating the acceptance and prevalence of array models.

Arrays are considered useful teaching tools for multiplication (National Council of Teachers of Mathematics, 2000; Izsak, 2005; Young-Loveridge & Mills, 2009; Van de Walle, Karp & Bay-Williams, 2019) and are used in various elementary curricula (e.g.,; Bell, M., 2007; Fuson, K., 2009; University of Chicago, 2015; Wagreich et al., 2013) to highlight the structure and pattern of the multiplication process without word problem contexts for which students may find the context helpful or may need support to decipher the text..

Using Arrays to Understand Quantity. How people relate area or arrays with multiplication varies a great deal based on perceptions of rows by columns to produce square units. With a focus on measurement or geometry as well as multiplication, some studies indicate individuals' constructions based on conceptions of units and unit coordination impact accuracy (Battista, 1998; Izsak, 2005; Huang, 2014; Rathouz, 2011a, 2011b). Finding square units from side lengths is cognitively more challenging than thinking about rows and columns of singleton units. Students may not make the connection between side lengths and area (Battista, 1998). When using dot paper, students may have trouble distinguishing counting dot arrays or the square formed by dots in each corner when finding area based on past experiences. Explicitly recognizing students' past experiences with selected or related representations to solve problems when designing or evaluating new tasks is recommended (Izsak, 2005).

In addition to helping students establish that multiplication includes a consistent amount in a *group* times the number of *groups* to find the total quantity, the array helps support the shift from thinking about individual square units to groups where a group is composed of these square units that are in the same row or column of an array. Thinking about individual square units is more indicative of additive thinking, and *times as many* as a *group* of these individual square units is more indicative of multiplicative thinking, leading into relational thinking. The way students conceptualize units is part of this shift, which will receive more attention in the next section.

Initial experiences with arrays involve perceptual materials like square tiles or stacks of cubes forming rows and columns. The array provides squares to count, but also a physical pattern in which to represent grouping with equal amounts in more than one way, such as rows, columns or rectangles within the larger rectangle. Using manipulative objects like tiles, students build rectangles from tiles and then receive opportunities to learn that "breaking apart" one side length into a sum provides a way to split the array into two parts, find the amount for each and adding them together to find the entire amount, thus demonstrating the distributive property. The same process can be accomplished by drawing on grids. Later array experiences use base ten blocks to show arrays with a single-digit number of rows with a two-digit number of columns, where the ten-stick plus some *ones* in each row can demonstrate that multiplication of a tens-place number will result in ten times more than multiplying by a ones-place number. Selecting a place-value split for one side of an array as a *break-apart* selection is suggested in many texts (Wagreich et al., 2013; Bell, M. et al., 2007; Fuson, K., 2009). Break-apart refers to partitioning a rectangle into smaller rectangles and after determining the area of the smaller rectangles, add the amounts together to find the area of the larger rectangle. The place-value split can provide a representation for standard algorithm work. As noted in the CCSSM, the rectangular array/area model also provides the basis for multiplying with fractional amounts or numbers larger than a single digit in both decimals and fractions. Consequently, array/area model is used for multiplication learning from early multiplication through rational number multiplication and the break-apart solution methods includes a continuum of multiplicative reasoning stages, including pre-multiplication. This study is designed to connect array multiplication solution strategies to unit coordination as part of the cognitive basis for number sense about quantities. The *times as many* aspect of multiplicative thinking (Devlin, 2008, 2011) requires operating with multiple levels of units that the array's focus on two-dimensional representations supports.

Break Apart Multiplication

In the break-apart multiplication process, students split a rectangle into smaller rectangles, find the quantity in the smaller rectangles (preferably by multiplying the number of squares in a row by the number of rows) and then add the amounts of the smaller rectangles. Break-apart multiplication presents a model for the distributive property, although typically, the curriculum focus is on multiplication. Developing the framework for how students account for the quantity of squares in the break-apart array holds the potential to build an assessment tool for observing students' thinking about quantity in many classrooms.

The data for this study was collected while students completed array multiplication problems, including break-apart, from the 4th grade *Math Trailblazers* Field Test edition (TIMS Project 2010, 2011). This was beneficial to the study because activities frequently include measurement and describing thinking thus increasing the likelihood that students would describe how they determine a quantity of squares more fluently because it is likely that they have had more practice talking about their thinking given the focus on explaining thinking.

Building on the rectangular array model, *Math Trailblazers* 4th grade field test authors provide teachers with multiple opportunities to teach the break-apart strategy for multiplication prior to where the rectangular array item of interest is located. (TIMS Project, *2010* p 107). The "break-apart" strategy (see Figures 1 and 2) is of particular interest because eventually it extends

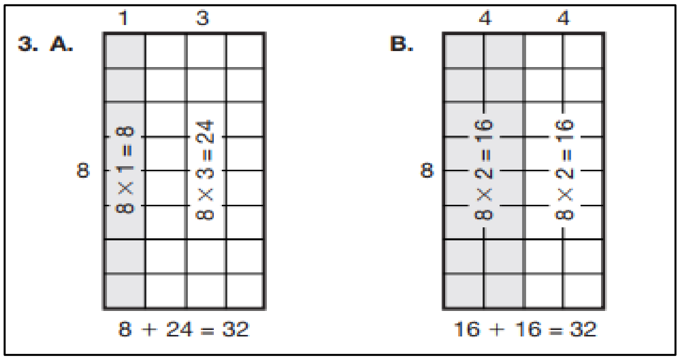


Figure 1. Example of Break Apart Multiplication for 8 x 4..Adapted from "Figure 1: Responses to Questions 2 and 3 on the Exploring Break-Apart Products Activity" by TIMS Project, 2010, *Lesson 5 in MTB4 Unit Resource Guide Unit 3 Field Test Edition.* Chicago, IL: Kendall Hunt and University of Illinois at Chicago, p 111. Copyright 2010 by TIMS Project.

the array to model more complex situations, such as multi-digit or rational number multiplication. Typically, the strategy is initially presented within a number range of single-digit factors in various curricula. Students are explicitly taught to break the rectangular array into parts, find the quantity in a part using multiplication strategies, and then add the quantities together to get the total quantity. Formally, the distributive property is being applied with this sequence, but the curricular materials suggest only making informal reference to the distributive property.

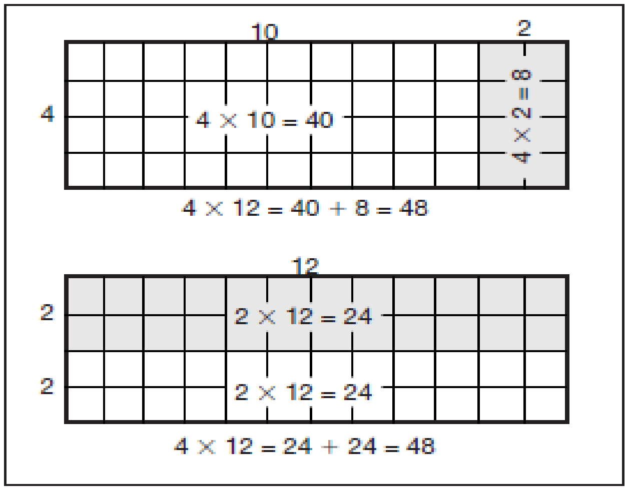


Figure 2. Example of Break Apart Multiplication for 4 x 12. Adapted from "Figure 2: Sample student solutions for Questions 1A and 1B on the More Break-Apart Products Activity Page" by TIMS Project, 2010, *Lesson 5 in MTB4 Unit Resource Guide Unit 3 Field Test Edition.* Chicago, IL: Kendall Hunt and University of Illinois at Chicago, p 113. Copyright 2010 by TIMS Project.

The break apart activity selected for student protocols provides opportunities to observe the student's strategies in the process of using arrays in order to make inferences about the student's schemes. There are multiple opportunities, as the larger rectangle is potentially split into smaller ones, each of which requires finding the array quantity. Finding the area of smaller rectangles, the break-apart method increases the opportunities for students to show calculation strategies, including multiplication. Breaking the larger rectangle into smaller ones and adding the quantities together provides a way to observe students unit conceptions with both adding and multiplying tasks. Thinking additively or multiplicatively can be inferred from these observations.

10. Find the number of squares in the rectangle below using the break-apart method.

- Break the rectangle into parts to make it easier to multiply. Write number sentences to show the number of squares in each part.
- Write a number sentence to show how you found the total number of squares in the large rectangle.

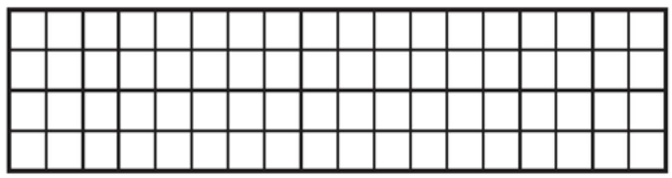


Figure 3. Focal Activity Item 10. Adapted from “Unit 3 Test” by TIMS Project, 2010, *MTB4 Unit Resource Guide Unit 3 Field Test Edition.* Chicago, IL: Kendall Hunt and University of Illinois at Chicago, p 210. Copyright 2010 by TIMS Project.

Within this break-apart problem structure, student choice accommodates variation in students’ multiplication fact knowledge, multiplication conceptual knowledge, and working memory. *MTB4* break-apart instruction gives students choices within a sequence of steps, allowing students to use numbers within the range they comprehend, using multiplication, skip-counting variations or singleton counting. Students are more likely to choose known facts, and be able to visualize rectangles with specific number context (Sherin and Fuson, 2005), especially factors related to highlighted instruction. The flexibility in this type of task provides an opportunity to identify students’ schema since varied approaches can produce correct answers, allowing students to keep working even when they could not use the most advanced strategies. As students explain their reasoning for the strategy choices, unit coordination evidence is gathered.

Developing knowledge of students' understanding of units has been a topic of interest for early numeracy (Wright, et al., 2006; Olive, 2001) and for developmental psychologists (Sophian, 2008) and well as for fraction and rational number (Behr, Harel, Post & Lesh, 1994; Hackenberg & Tillema, 2009; Lamon, 2005). Tzur, Johnson, McClintock, Kenney, Xin, Si, Woodward, Hord, & Jin's work (2013) with learning disabilities students identified how students use referents to encourage unit understanding and suggests supportive teaching strategies. Researchers have developed multiplication scheme terminology that encourages essential discussion about student thinking inferences (Hackenberg and Lee, 2015; Hackenberg & Tillema, 2009; Steffe 1994, 2010). Both are important steps for understanding unit coordination within multiplication and division. Students' unit coordination related to multiplication and division operations have not been explored with school rectangular array problem types. An analytical method to describe and keep track of students' actions and how these actions indicate students' reasoning about quantities in arrays follows.

3. EXPOSING REASONING

Conceptual Analysis and Schemes

Conceptual analysis is an analytical method for describing conceptual operations for related thinking and reasoning (von Glasersfeld, 1995). Conceptual analysis of measurement or multiplication is not the same as measuring or multiplying, which are activities. Conceptual analysis is designed to produce mental models for how someone is thinking about a particular idea. In this research, it is used to infer unit coordination operations a student is using to solve problems. The conceptual analysis can guide task development or other instructional moves for increasing student understanding because it helps both the researcher and the education professionals anticipate the student's thinking.

Conceptualizing is making mental images by doing the activity (Thompson & Saldanha, 2003). To help analyze students' mathematical reasoning, thus developing conceptual analysis tools, von Glasersfeld (1995) interpreted Piaget's psychological construct called a *scheme* to make principled inferences about thinking and learning. He identified three elements within what is called a *scheme*: (1) the situation as perceived or conceived by the learner (which determines the learning goal), (2) the activity is related and connected with the situation, and (3) anticipated results of the activity. A student's actual actions may fall into a range of expected choices.   Documenting and incorporating what a student notices at the juncture of the activity with a given goal can provide evidence for making judgments about students' conceptual understanding when students are engaged in open response problem situations designed to encourage the construction of units. After a researcher collects and analyzes data, the researcher's conception of what the child is thinking may be called a second-order model. This research is based on second-order models because access to student thinking is inferred not explicit.

Conceptual analysis using schemes is the method the researcher used to uncover students' reasoning about array quantities. This research uses the scheme trio of goal, activity, and results to make inferences about students' multiplicative concept use. The first part of the trio, the goal of the rectangular array task type, is determining a quantity, usually the total number of squares. Students' actions while finding the number of squares include gestures, words, and written work; these are the behaviors for making inferences about students' mental models. A model for how students' group and regroup the quantity of squares is connected to observing behaviors. Noticing what the student does is compared to predicted schemes. Schemes connect interactions with the diagrams such as touching squares or explanations about grouping to a continuum of mental models for student use of increasingly nested units, (Hackenberg, 2010, Steffe, 1992). An individual scheme has expected results, but multiple schemes are possible with the same array problem because there are a variety of ways to find a solution with arrays as noted earlier. In general, as students' reason about increasingly complex quantities, the resulting observable student actions are more likely to vary (Olive, 2001; Tzur & Lambert, 2011).

When students can determine the result of an action and then return to the initial situation, the scheme is labeled as an interiorized scheme. With interiorized schemes, students do not need actions to reconstruct, given a similar situation. Without the ability to anticipate the result of a scheme, a student may determine relationships through the actions of solving the problem. To summarize, a scheme has a situation perceived by the learner, activity associated with situation, and the result of the activity (von Glasersfeld, 1995). Given a consistent protocol, the student's activity related to the problem-solving for the situation is used to make inferences about her reasoning processes. Think aloud protocol (TAP) interviews allow for observing the results of students' activity in a consistent manner. A description of the purpose and procedure for TAP follows.

Think Aloud Protocol (TAP) Interviews

Think aloud protocol interviews provide a means to observe student reasoning. Think aloud interviewing is a procedure for gathering data and interpreting the analyzed results. This method is used to measure specific constructs, often to validate psychological or educational tests. A cognitive model is identified, then a means for creating, collecting and recording responses with consistency towards producing data is developed. This research identified a cognitive model for multiplicative concepts. The interview tasks need to have a problematic nature that is likely to elicit participants to process information in ways that will allow observations to make inferences related to the cognitive model(s) taking into account social-emotional factors and sample size needed for inferential statistics as appropriate (Leighton, 2017).

Inferences regarding how students' anticipated actions, such as combinations of gestures, written work, and words, might be connected to cognitive models, are based on two assumptions: that students' thinking reflects their current understanding of the task, and the task is problematic for the student. Researchers have noted the importance of a problematic aspect of the task in order to reveal student thinking (Kouba, 1989; Lamon, 1996; Steffe & Olive, 2010). When the arrays are larger than students' knowledge of math facts, determining how to find the quantities problematizes the situation such that the student will need to develop a plan or structure a solution strategy. To fully respond, (1) a student needs to determine what is being asked, (2) reference what is already known given his/her understanding of the description of the activity or task, (3) recognize the quantities and the relationships with regard to the question at hand, (4) toggle between what is being asked and prior understanding to create a response, and occasionally (5) reflect on the appropriateness of the response. When the learner is perplexed about how to approach a problem in a situation, she may revert back to a previous level of understanding in order to make sense of the situation. Pirie and Kieren (1994) describe this as *folding back*. Students' choices at each of these steps provide opportunities to observe how students are thinking about the array given the potential models a student may choose to impose on the rectangular grid.

Verbalizations for making inferences about student thinking include observations of a student reading or recounting well-known facts but also include asking students to describe what they are thinking, which may require a re-coding of data. The introduction to the TAP includes making sure the interviewee is clear on what it means to think-aloud; the interviewer may need to demonstrate thinking aloud. There is a fine line for the interviewer, as the prompts to help the interviewee share her thinking are important for retrieving information to form inferences, but the prompts should not be invasive to the students' thinking such that they change what the student thinking about in order to respond to the interviewer's prompts. Consequently, appropriate questioning is essential to make students reasoning evident in the analysis.

Observing Thinking/Reasoning

Students do communicate with more than just words. How students move and use their hands to communicate can become sources of communication to tap for learning about student thinking. With the increased opportunities to use video, capturing students' motions is easier.

Role of gesture for explaining reasoning. Gesture is an underutilized tool for gathering information about what a student knows and can do. Speech and written work are important but sometimes incomplete resources for evidence of a student's reasoning and for recognizing aspects of student thinking that often accompany students' speech. Although communication with both gestures and speech is widely accepted, assessing knowledge conveyed through gesture is not as prevalent. "Gestures tend to be meaningful movements produced along with speech." (Goldin-Meadow, S., Cook, S.W. & Mitchell, Z.A., 2009, p. 267). Goldin-Meadow and others have studied how hand movements are part of communication and also part of learning, initially part of learning a language, later helping to learn concepts through the description of ideas with gestures. Also, gestures allow deaf individuals to interconnect and provide a means for speaking persons to communicate without using speech (Goldin-Meadow and Alibali, 2013). Gestures and speech flow out of cognitive processes (Hostetter & Alibali, 2008).

The gestures are used to help explain – the gestures complement verbal explaining and the verbal explaining complements the gestures to communicate the ideas. Communication with gestures and speech is tied to thinking as a communication source. Novack and Goldin-Meadow identify ways gestures can support teaching-learning:

> *How* does gesture promote learning? There are likely many mechanisms through which gesture has its effects. For example, gesture can link abstract concepts in the immediate environment (Alibali et al., 2014), or gesture can reduce cognitive load (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001; Hu, Ginns, & Bobis, 2015; Ping & Goldin-Meadow, 2010), and gesture can enhance spoken communication. (Hostetter, 2011). (Novack, M., & Goldin-Meadow, S., 2015 p. 409).

How a student is enacting an explanation in an interiorized way may be communicated (or rehearsed) using gestures alongside speech (Church & Goldin–Meadow, 1986). For example, in a study by Williams, et al., 2012, students' reasoning about the triangle inequality theorem was able to construct justifications for the theorem with gestures and a narrative versus algebraic proof. . Specific gestures, speech, and written work can make student's reasoning processes more visible. Sometimes students may explain an idea in gestures that they did not share through their speech. In this way, "attending to students' gestures in tandem with language can contribute significantly to understanding students' mathematical reasoning and proving practices, particularly when attending to dynamic gestures depicting relationships that may be difficult to communicate verbally" (Williams-Pierce, Pier, Walkington, Boncoddo, Clinton, Alibali, & Nathan, 2017, p. 257).

When gestures, speech, and written work match, there are multiple measures mutually supportive of the rating choice that can help confirm a rating. In some studies, student gestures related to their oral explanation with reference to the diagram may be conflicting. For example, a student might reference the quantity of squares in a four-by-five array saying four and five but makes an oval motion over the four-by-five array. The gesture indicates thinking about all the squares, not just the ones along the edges, but the words infer thinking about adding side lengths. Whether the multiple measures match or not, information for instructional decision-making is provided. Pursuing to see what might be the potential for instructional decision-making when there are mismatches in students' gestures and speech, Church and Goldin-Meadow performed an additional study involving students' Piagetian understanding, to conclude that students whose gestures and speech do not match are more likely to benefit from additional instruction than students for whom there is not a mismatch (Church & Goldin-Meadow, 1986). In a later study, Goldin-Meadow and Alibali (2013) maintain the same conclusion: "After language has been mastered, gesture continues to predict learning in both children and adults. Learners who convey information in gesture that differs from the information in speech on a particular task are likely to learn when given instruction in that task" (p. 276).

In recent years the role gesture plays in communicating thinking has been studied as embodied cognition. Two principles that guide the development of the embodied cognition perspective described by Nathan (2008) are (1) cognition has a contextual, situated basis and (2) that cognitive work is supported by the environment or activity. This work explores how students use spatial representations to help support their thinking and reasoning processes. DeSutter and Steiff (2017) describe learning environment design to promote learning through spatial approaches when problem-solving as well as other thinking. They distinguish between embodiment as a mental action and embodied actions as the physical movements, gestures, etc. that are observable. They "operationalize embodied actions as the purposeful body positions and movements that an individual engages in during a learning activity" (p. 8). In this work, unitizing is an interiorized or mental action that can be inferred by students' embodied actions, speech, and written work.

Gestures, speech and written work agreement can indicate similar understanding and may confirm the student's cognitive model, in this case, unitizing levels. Student gestures related to their oral explanation and reference to the diagram may help. As more is known about embodied cognition (Nathan, 2012), how gestures in combination with speech may support students communication of proof (Williams-Pierce, et al, 2017) and how incongruent stories from speech and gestures may indicate instructional next steps (Goldin-Meadow & Alibali, 2013), there will be more support for using gesture to support teaching and learning.

Role of diagram within the assessment prompt. When a diagram is present, there are opportunities to gain information from a visual/pictorial channel. The abstract and formal aspects of multiplication calculations are supposed to be minimized by providing a diagram. However, a student needs to choose to use the diagram provided, and the student's background knowledge about the representation will impact its' use.

"Representational fluency is the ability to work within and translate among representations" (Bieda & Nathan, 2009). Bieda and Nathan found that when students got *stuck* in their problem-solving with one representation, the student might choose another representation to try a new path. In a Cartesian graph context, they found patterns in the way students reasoned using the graphic representation which benefitted from the idea of *grounded* representations. Grounded representations provide sense-making about an abstract, unclear idea/concept, because the representation comes to have a meaning that illustrates that idea/concept that not only gives clarity but also supports further understanding (both accurate and inaccurate) about the relationships of the idea/concept that would otherwise be opaque (Bieda & Nathan, 2009). Students with background knowledge about a specific representation, say an array representing a multiplication equation, are expected to interact with it as needed. Using an array diagram as the context for the problem to solve means there will be support for some students who see it as a grounded representation, where the columns can represent a group amount and the number of columns represent the number of groups or vice versa. Others may not need the support for multiplication yet it may still be a tool to extend understanding. In addition, some students may interact with the array diagram, but not as a grounded representation. Students who learned multiplication facts before understanding multiplication as *n* groups with *m* in a group may interact with the diagram to find numbers to make a math fact, but do not know the fact will be represented by the number of squares in the array.

Role of speech. Especially at this age band, students have more mature oral language than written language, so oral explanations of procedures or reasoning can provide critical information about student thinking including evidence for how they can unitize a quantity. When students are sharing thinking, their general language ability, especially for students with limited English proficiency, is a constant concern. This poses the need to elicit student thinking that is not swayed by the question.

Summary

This study seeks to develop inference-making techniques about students' coordination of their mathematical knowledge of units within array multiplication settings. Recognizing the recursive nature of interiorizing increasingly more sophisticated units, understanding unit coordination and unitizing processes may support inferences for number sense development. Unitizing and unit coordination are not typically part of the current classroom teacher's toolbox. However, identifying the foundational construct for operating with quantities, coordinating units, or nesting number when students are learning multiplication has the potential to provide important insights for instructional next steps. Using TAP with special attention to gestures and the array as a grounded representation should support unit coordination inference-making for use as formative assessment data. The study is designed to complement other research on students' development and coordination of mathematical knowledge of unit with emphasis on practical applications in learning environments. In particular the study is focused on (1) revealing unit coordination schemes in students' array multiplication solution processes,

(2) developing a scoring guide using common classroom materials to meaningfully distinguish between unit coordination levels across array multiplication problem types, and (3) exploring the predictive validity of unit coordination scores from students’ array multiplication solution processes relative to future multiplication performances such as 1-digit and 2-digit multiplication.

4. DESCRIPTION OF PARTICIPANTS AND TASKS

This chapter provides the context for obtaining the study data. The participants, materials and procedures as well as the activities are presented here.

Study Context

The data for this study are derived from an NSF funded research project entitled: *Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach*. The NSF study was focused on evaluating various curriculum embedded assessment activities within the *Everyday Mathematics* and *Math Trailblazers* curricula, with an explicit focus on issues related to aspects of the validity of the assessments. All data are derived from the performance of consenting students in classrooms of teachers who agreed to participate in the NSF study under University of Illinois at Chicago (UIC) approved Institutional Review Board (IRB) and district approved IRB protocols (Appendix D).

Participants

Participants were fourth grade students in four classrooms drawn from three schools located in suburban and city schools in a major Midwest metropolitan area. Data were collected during the 2011-12 and 2012-13 school years at which time all four classrooms were using the *Math Trailblazers* Grade 4 Field Test materials. A total of seventy-eight students completed the student interview protocol with adequate accompanying video. The distribution of students across the four classrooms and three schools is shown below in Table 2.

Table 2 Distribution of Student Participants across Schools and Classrooms

School	Classroom	# Participants
S28	424076	16
S28	424058	27
S32	524117	12
S33	524120	23

School 28 is in an urban school district in the Midwest. About one-fifth of the students are described as Asian American, one-tenth African American, two-fifths Hispanic American and one-fourth Caucasian. About 90% of the students are low income. For 2012 reporting on the mathematics portion of the state test, 81% of the school's students met or exceeded state standards in mathematics. Requirements for meeting and exceeding state standards changed in this state in 2013 and in that year 53% students met or exceeded state standards in mathematics (see Table 3).

Table 3
Percent Students in 2012 and 2013 Who Meet or Exceed State Standards on State Mathematics Achievement Test

	2012 Results			2013 Results		
School	School Data	District Data	State Data	School Data	District Data	State Data
S28	81%	74%	82%	53%	49%	59%
S32	89%	84%	82%	50%	58%	59%
S33	84%	84%	82%	52%	58%	59%

Note. Requirements for meeting and exceeding state standards changed in 2013

School 32 is in a suburban school district in the Midwest. About 50% (47% in 2012; 50% in 2013) of the students are Caucasian, about 41% in 2012 and 45% in 2013 are Hispanic American, 4% are African American and 2% are Asian American. About half of the students are designated as low income. For 2012 state mathematics test reporting, 89% of students met or exceeded state standards. In 2013, 50% met or exceeded state standards (see Table 3).

School 33 is in a suburban school district in the Midwest. About 51% of the students were Caucasian, 44% in 2012 and 45% in 2013 percent were Hispanic American, about 1.5% African American and 1.5% Asian American. About half (53%) were designated as low income. About a fifth (20-21%) had English Language Learner needs. For the 2012 state mathematics test reporting, 84% of the students met or exceeded the state standards; in 2013, 52% met or exceeded state standards with the changed requirements (see Table 3).

Materials and Procedures

After students in each of the four classrooms completed the lessons in *Math Trailblazers* Grade 4 Unit 3, "Products and Factors" with their class, each consenting student was individually interviewed using think-aloud interview procedures in an area other than the classroom setting where the original group instruction occurred. The interview protocol (Kaduk, C., 2012) designed to validate items used to assess multiplication understanding, is a Think Aloud Protocol (TAP).   Designed to validate curricular items, when the curriculum indicated that a student could use tools such as calculators, multiplication tables or square-inch tiles to solve problems, these tools were made available to the students during the TAP.

During the interview each student was asked to talk aloud as he/she solved a set of multiplication related questions, explaining his/her thinking as part of the process. The interviewer presented a sample of thinking aloud as a model. Interviewers were trained to use the TAP methods by an experienced researcher and the author, with new interviewers trained by existing interviewers. The interview protocol author (this researcher) was not able to conduct interviews due to other responsibilities. Although the interviewers received answers to questions interviewers poses, feedback on their approach to the interviews was limited which led to some variation in how interviewers probed for student explanations about student strategy choices, resulting in variation in how follow-up questions elicited data. Interviewers were dutiful about following the explicit aspects of the protocol, including the essential element of asking students to share their thinking without leading students to think the interviewer wanted to hear a particular method. Students' gestures and written work were not affected by interviewer's questioning practices.

A videographer controlled the video recorder for the initial interviews, but when more interviews were conducted per week, the videographer was not available for all interviews. Instead, a video camera was strategically placed to continuously record the students' actions and their verbalizations before starting the interview in order to maximize the view of the students' hands for gestures and written work. Occasionally the child might move, the view was incomplete for the range of movement, or objects such as hair would fall between the camera lens blocking the view of the child's actions, limiting some gesture data collection. Written work was collected in addition to the camera view of student work. In a few cases, the sound was not turned on, background sounds made it hard to identify what the student said or the camera view made a section of the interview lack necessary information for scoring although the rest of the interview data could be scored. However, with the large number of cases, a significant amount of useful data was collected for each item.

The interview materials include questions for a variety of mathematics tasks to which each student was asked to respond. These include the focal task and supplementary tasks specific to particular aspects of the focal task. As part of the NSF research study team, this researcher created the supplementary tasks for the NSF study by modifying other parts of the curriculum. Most supplementary tasks were from the same chapter as the break-apart multiplication (BAM) task but the Rectangle Task was modeled after an instructional item for area, with the intent of identifying the extent to which students needed to see or touch the array diagram to determine the multiplication. The complete interview script is provided in Appendix A. Although initially designed to support validation of the multiplication items in the fourth grade MTB field test Chapter 3 for the NSF study, as the study ended, this researcher realized that the video data from the study would also support the development of ways to identify students' dynamic understanding of units within a given assessment. The tasks selected from the NSF study to support understanding unitizing and their rationale are discussed below.

A timeline of the tasks presented during the interview is presented in Figure 4. After establishing expectations for the student to talk aloud while thinking, the focal task was presented. This included the "Count Accurately" task, which uses a focal task diagram. Students complete the focal task prior to the auxiliary tasks to minimize the potential impact of student thinking about auxiliary tasks affecting the results on the focal task. The auxiliary tasks are ordered by task complexity, starting with knowledge of math facts, listed as "Assess Fact Retrieval" on the chart. The last task, "Evidence of Model Use", provided an opportunity to calculate a two-digit by one-digit multiplication using any method, thus indicating students' selection of models and methods. This task included two items, but if the child was unduly tired after completing the first item, the second was not included in the interview. Each task will now be explained in more detail.

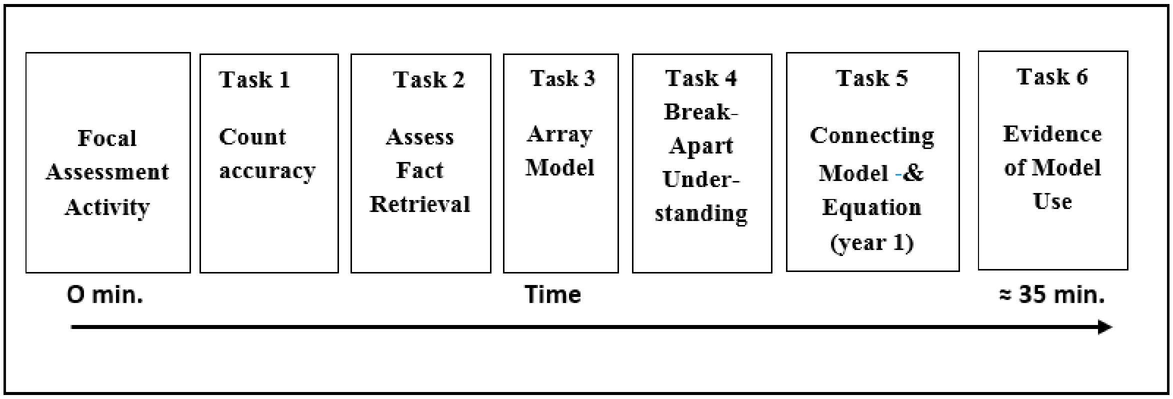


Figure 4. Timeline of Interview Task

The break-apart multiplication item selected from the Embedded Assessments fourth grade Multiplication Case Study interview is the basis for investigating unitizing in activities promoting multiplication. Classroom observations indicated that students received direct instruction for the break-apart method using a rectangular array diagram, such as the example shown earlier in Figures 1 and 2. The present work assumes that students are motivated to use the most efficient method with which they have confidence and show thinking that is typical for them because of the novelty of a one-on-one student protocol interview, and the interest shown by the trained interviewer. Students' schemes for finding the number of squares in this same rectangle are expected to vary, implying different cognitive models underlying additive or multiplicative strategies. Tasks associated with the aforementioned interview are now described in the order presented to each student (see Figure 4).

Task 1: Focal Activity of the Multiplication Case Study

The Focal Activity of the Multiplication Case Study interview developed by the author includes 3 items from the Math Trailblazers "Products and Factors" Unit Test, questions 1, 8 and 10 (see Figure 5). For question 1, students are to determine the number of tiles in a row, given that there are 16 tiles and 4 rows. Successful completion requires close reading, as students must identify 16 as the *total* amount of tiles, not the number of tiles in a row. This item was selected to observe how students interact with an array and understanding of vocabulary. Focal Activity Question 8 requires students to create a box to hold candy given certain constraints of number of candy pieces in a layer, with more than two layers. It requires close reading and making a diagram that is likely to include an array. It was selected to see if and how students might use array information in a problem setting, and to identify potential variation of patterns in the solution strategies. Question 10 was selected to indicate how students might use the break-apart model, part of a continuum of methods for instruction on calculating products, where the rectangle provides a visual representation to help students think about the multiplication concept. The present unitizing research study uses question 10 (Q10) as the focal item.

Task 1

You may use calculators, multiplication tables, or square-inch tiles to solve the following problems.

1. Tom made a rectangle with 16 tiles. If there were 4 rows, how many tiles were in each row? Sketch a picture of this rectangle.

8. Design a box for the TIMS Candy Company that will hold 36 pieces of candy and that has more than two layers.

Tell how many layers are in your box. Also, tell how many pieces of candy are in each layer. Each layer must hold the same number of pieces.

10. Find the number of squares in the rectangle below using the break-apart method.

- Break the rectangle into parts to make it easier to multiply. Write number sentences to show the number of squares in each part.
- Write a number sentence to show how you found the total number of squares in the large rectangle.

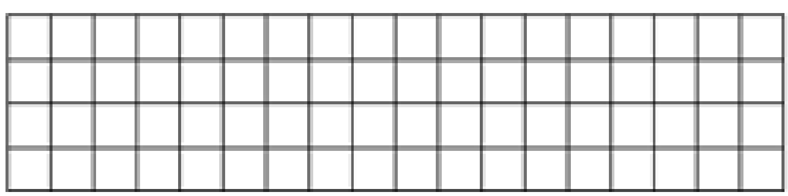


Figure 5. Multiplication Focal Activity, with Emphasis on Item 10. .Adapted from "Unit 3 Test" by TIMS Project, 2010, *MTB4 Unit Resource Guide Unit 3 Field Test Edition.* Chicago, IL: Kendall Hunt and University of Illinois at Chicago, pp.208 - 210. Copyright 2010 by TIMS Project.

Students learned the *break-apart* method in Unit 3 lessons 5 and 10, after using arrays at the beginning of Unit 3. According to the Math Trailblazers Fourth Grade Teacher's Guide, to find the number of squares in the rectangle using the break-apart method, the method begins with identifying the side lengths of the rectangle with the assumption the student identifies the side lengths as the factors for the multiplication. The rectangle is to be split into smaller rectangles with side lengths that will make calculation easier for that student. A multiplication sentence to match each smaller rectangle is to be recorded on or near the smaller rectangles. Then a number sentence showing the sum of the products for the smaller rectangles is to be recorded. A sample number sentence to match Item 10 might be: $4 \times 18 = 40 + 32 = 72$ (with the number phrase 4×10 on one smaller rectangle and 4×8 on the other smaller rectangle). Classroom instruction encourages students to make choices for breaking apart the rectangle that use their multiplication facts background knowledge. This provides differentiation, as there are multiple ways to break apart the rectangle that account for varying background knowledge of facts. This activity provides a visual model for future work with the distributive property.

Since the task has several opportunities for students to match a multiplication equation to a rectangle, there are more examples for the researcher to observe how students unitize a quantity, as well as analyze the choices students make relative to theoretical expectations. According to the MTB fourth grade Unit 3 Guide (2010), the break-apart method combines the amount calculated for each sub-rectangle part to produce the product for the total number of squares in the rectangle. Consequently, students use both addition and multiplication to find the number of squares in the diagram. As students work with both operations, the opportunity to observe actions for inference-making regarding students' conceptual thinking about quantities increases. Students can present the quantity in terms of m rows of y squares in a column plus n

rows of *y* squares in a column to find the total number of squares, count the singleton squares or determine the quantity using other strategies. When the number is determined by counting squares individually, the expected mental model is singleton units. Talking about a row or column as a composite number presumes the mental model is a composite unit, where the student can think about the number as *n* groups, but also see the quantity as singleton squares. Varied solution strategies for finding the number of squares link to different ways of thinking about units, which makes the array item type especially useful for observing unitizing as an underlying aspect of number sense at the point where students are transitioning from additive strategies to multiplicative ones.

Students do two break-apart multiplication problems, but the second instance is slightly different in the second year of data collection due to changes in the protocol interview. In year two, students are asked to show two different ways to break-apart the Q10 rectangle. In year one data collection, showing a second way to break- apart the rectangle occurs in Task 5B. Although showing the second way to break-apart the rectangle occurs at different locations in the order of tasks in different years, these are considered the same task for the present study.

The *Count Accurately* task is connected to a diagram in the Focal Task Item 10. The student is asked to count the number of squares in the top row of the rectangle used in Item 10 (Figure 5). The interviewer asks the student to count the top row of the focal activity rectangle to determine that the student can count, to see if the student will skip count, and to determine if a given student is counting the squares or the line segments. This task provides evidence whether the student has one-to-one correspondence of number to the quantity of squares, necessary for connecting the number of squares in the rectangle to the proposed multiplication. Students who count the lines are predisposed to interpret the task as counting lines instead of finding the area (square units) of the rectangle, the basis of the comparison. The task is designed to assure that the student is counting the squares and not the lines (Battista, 1998).

Task 2: Assessment of Multiplication Fact Retrieval

Next in the interview sequence, (see Figure 4) students are asked to give answers to single digit multiplication problems. The order is mixed up in the interview, but the three categories are (1) facts with twos or fives, (2) fours facts, and (3) other facts. During the task, the interviewer shows a card (see Figure 6) saying the fact question out loud to the student. The interviewer explains to each student that the task is to say the answer to the multiplication as quickly as he/she can, giving each the option of saying “pass” if the product is unknown. Students are not asked to share their strategies. As soon as an answer is given and the interviewer records it, the next card is shown as listed on the Appendix A sheet for the Task 2 activity.

Figure 6. Multiplication Facts Cards used in Task 2 from the MTB4 Multiplication Case Study Student Interviews. From “Task 2” by C Kaduk, 2012. Student Interview: Math Trailblazers Grade 4, Unit 3.10 Unit 3 Test in *Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach.*

Students' automaticity for recalling multiplication facts can provide a measure of familiarity with ranges of numbers. Throughout the interview there are opportunities for students without automatic facts knowledge to use what they know to find answers. This auxiliary task provides a baseline for multiplication fact automaticity.

At this point in the curriculum, students are expected to know *2's, 10's* and *5's* facts. Predicting the difficulty of the focal task for a given child is in part based on their ability to recall math facts, as there may be a problem-size effect. When tasks use 5s or 10's or when the task includes smaller numbers as factors, there is a better chance of known products, thus easier for students (Campbell & Graham, 1985; LeFevre et al., 1996; Miller, Perlmutter, & Keating, 1984). Consequently, variability in the level of fact difficulty for a given student may impact the need for unitizing actions. An aspect of this research is to identify ways to uncover student's thinking about number when memorized math facts mask less robust understanding of units for measuring quantity.

Task 3 Rectangular Array

In Task 3 (see Figure 7) students are asked (1) to find how many square units there are in a given rectangle where some of the lines are missing in the display, and then (2) to write a multiplication sentence to match the given rectangle on the card. The diagram is then rotated 90 degrees and students are again asked to write a multiplication sentence to match the rectangle. The drawings include some lines to show squares, but also have a section where there are no lines. Students are told the squares are in the rectangle, but that some are covered. This activity is patterned after classroom activities where rectangles are drawn completely but this diagram does not show all the lines to delineate the squares (TIMS Project, 2011) in order to recognize if ability (Steffe, 2013). Through observation, the student may demonstrate (1) perceptions regarding the array – viewing it as a set of objects, whether unitary objects or composite ones, and (2) operationally, if the rectangle is perceived as an array of *x* rows of *y* objects, or something else.

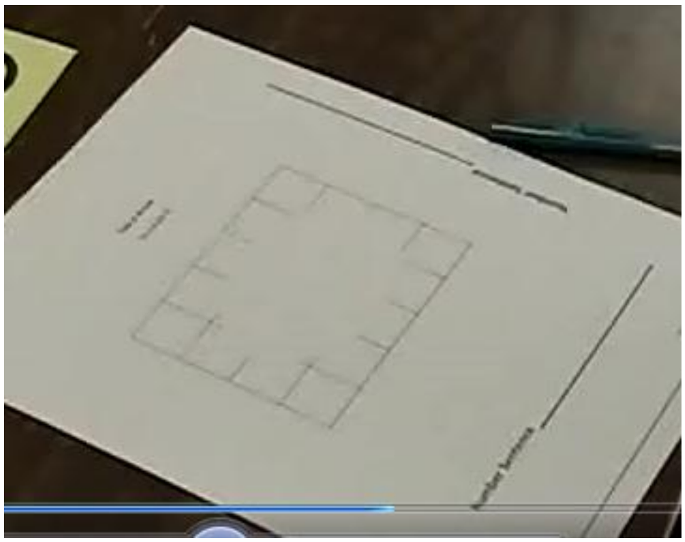


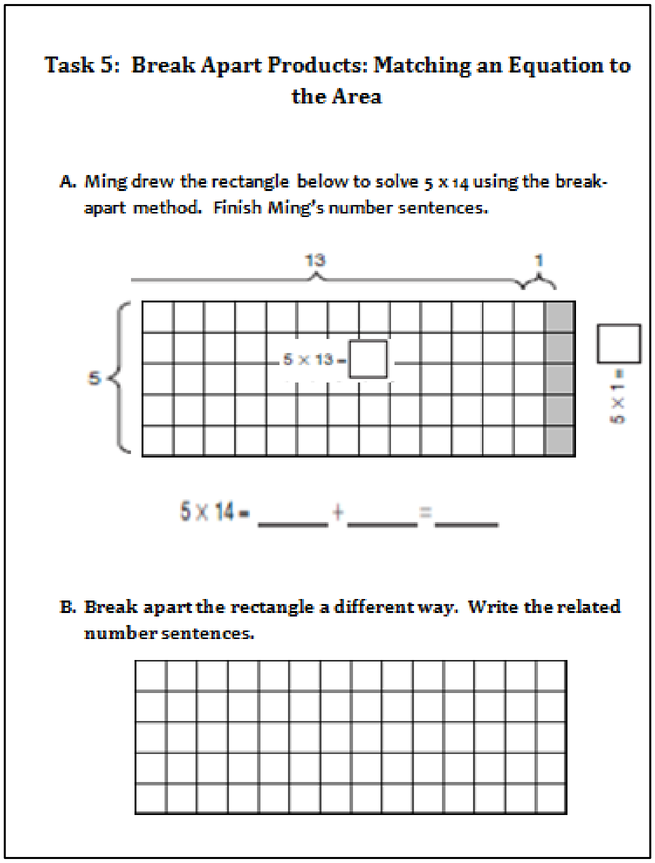
Figure 7. Array Diagram Example from Task 3 of the MTB4 Multiplication Case Study Student Interviews.From “Task 3” by C Kaduk, 2012. Student Interview: Math Trailblazers Grade 4, Unit 3.10 Unit 3 Test in *Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach.*

Task 3 is designed to provide more evidence of students’ solution models for rectangular array problems. Recognizing students are not required to use a given strategy, this task provides data regarding student awareness of the array model and finding the quantity of squares in the rectangle. Students were presented with three different rectangle display sizes (2x6, 4x5, 6x9) to account for variation in strategy selection when the size increases or when there is familiarity with a given side length factor.   The researcher hypothesized students who did not know the corresponding fact for the array would be more likely to nest smaller rectangles in the problem rectangle using the BAM method to find the number of squares. This can potentially provide information about students' unitizing schemes relative to the size of the quantity because students have choice for solution strategies.

Information from the math fact fluency task may help identify if Task 3 has different meaning to students who already know certain multiplication facts and those who do not. Task 3 may reveal student thinking about part-part-whole number sense as related to unitizing ability and calculation strategies if the rectangle is large enough to be problematic. If students know the math fact related to the rectangle size, it was predicted that the item will be much less likely to provide information about student thinking because there is no problematic aspect to finding a solution.

Tasks 4 and 5

Tasks 4 and part of Task 5 will not be used in the present study. Task 4 is not included because the data do not provide useful information for the present study regarding unitizing schemes. Task 5A was excluded due to lack of interview data on this item for two classrooms.


Task 5: Break Apart Products: Matching an Equation to the Area

A. Ming drew the rectangle below to solve 5 x 14 using the break-apart method. Finish Ming's number sentences.

5 × 14 = ____ + ____ = ____

B. Break apart the rectangle a different way. Write the related number sentences.

Figure 8. Break Apart Rectangle in Task 5A & Task5B used in Year 1 of the MTB4 Multiplication Case Study Student Interviews. From "Task 5" by C Kaduk, 2012. Student Interview: Math Trailblazers Grade 4, Unit 3.10 Unit 3 Test in *Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach.*

Task 5 (see Figure 8), is an activity taken from a textbook introduction to the break-apart method, and it provides another opportunity for demonstrating the break-apart method. Task 5 was presented to students in the first year of Multiplication Case Study interviews, but this task was discontinued in the second year due to lack of useful variation in the students' Task 5 Part A performance and overlap with the information provided relative to other tasks. The Task 5 Part B (see Figure 8) asks students to show a second way to break- apart the given rectangle. Task 5 Part B (Task 5B) from the first year of data collection is used in the present study. In the second year of data collection, Task 1B replaced Task 5B. Consequently, the data for Task 5B in year one of data collection and the data for Task 1B in year two of the data collection are used as the same task for analysis purposes. Even though they appear at different points in the interview, each is preceded by an opportunity to complete a problem using the break-apart strategy.

Task 6 Evidence of Strategy Use in Two-digit by one-digit Multiplication

In Task 6 students are asked to solve two-digit by one-digit multiplication problems where the item does not specify the strategy choice (see Figure 9). This is similar to Unit 3, Lesson 10 instruction and multiplication problems in later MTB grade 4 units. This activity is last in the sequence (see Figure 4) because it does not use the rectangular array to scaffold calculation. It is used to help connect student performance on multiplication calculations and student use of the array diagram. Observing students' actions and thinking while they calculate in Task 6 (see Figure 9) is designed to identify potential models that students generate when multiplying with multi-digit numbers – whether break-apart strategies, related strategies or other methods. For the present study, each student's actions are coded for the strategy and for accuracy.

Task 6: Break-Apart Products with Larger Numbers

Solve the following problems using rectangles, expanded form or another way. Tell how you are solving the problem as you are doing it.

A. 6 x 21

B. 34 x 7

Figure 9. Products with Larger Numbers, Task 6. From "Task 6" by C. Kaduk, 2012. Student Interview: Math Trailblazers Grade 4, Unit 3.10 Unit 3 Test in *Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach.*

Error Check: Counting Squares Not Lines

Recognizing that one common array work concern is whether the student is counting squares or line segments. A task to determine if students were counting squares or something else was included in the TAP interview. Counting lines that form the squares instead of the square itself is a common error (Battista, 1998). To avoid construct irrelevant variance, each child was asked to count a row of squares in a rectangular array for the Counting Squares task. Students for whom there are data (n = 74) did count squares, with no data for three of the 77 students. Another three did not count *correctly* across the entire row, but did count squares and not line segments. This information was used to help interpret student's work on array items. None of the video data were eliminated due to counting something other than squares.

5. FRAMEWORK AND PROCESS FOR DEVELOPING A UNITIZING SCORING GUIDE

Recognizing that classroom decision-making at the third, fourth, and fifth grade levels would benefit from an awareness of students' use of multiplicative reasoning versus additive reasoning, a scoring guide using the array model to complement existing classroom activities was developed to meet this need. The array model is common to this grade level band and is used in a variety of curricula. This section outlines the development and refinement of the scoring guide (SG). Based on prior unitizing and unit coordination research for use in problematic situations for students, the SG design is in keeping with the guidelines for identifying schemes in conceptual analysis described in Chapter 3. The SG underwent several revisions during this study, based on pilot data results and expert review.

Considerations from Research

The initial conceptual framework is structured around the way students organize number into composite units along a continuum. This unitizing continuum includes coordinating levels of units with the use of a diagram or story to help keep track of the quantity as well as using *interiorized* models. Predicted second-order models, that is, models of what adults anticipate the student is thinking specific to the array rectangle item type were conceived. First, a developmental framework of schemes for Q10 was established by making predictions for break-apart multiplication actions from previous teaching experiments and other related studies that are referenced in the *Unit Coordination Progression* section of Chapter 2 in this document (Hackenberg & Tillema, 2009; Norton & Wilkins, 2012; Sophian, 2008; Steffe, 2014; Steffe & Olive, 2010; Tzur et. al., 2013). A master table to organize the evidence from these various schemes identified the specific researcher whose work prompted looking for a given behavior.

The chart lists the activity/situation, specific actions, matching unitizing level operations, and what are called *Steffe descriptors*, a progression of unitizing skill based on Steffe's constructs (Olive, 2001, Steffe, 2010) and MC levels (Hackenberg & Tillema, 2009). By organizing the actions related to a give unit coordination level in the literature, comparable actions, verbal explanations or written work for array multiplication were placed on the scoring guide. Figure 10 is a chart sample. By analyzing the actions described in schemes from prior teaching experiment research with computerized sticks or story problem activities, student actions that are likely to demonstrate similar unitizing actions in the array diagram use were determined.

Authors	Activity/ Situation	Scheme Name or Level	Unit Level Operations	Action Level	Goal	Steffe descriptors
Thompson et al 2013	Magnitude Meanings for "quantity"	Measure magnitude	"person equates a quantity's size and its measure, where 'measure' means 'number of units.'	"a quantity taken as a measured whole and a conception of the quantity segmented into parts, each part taken nominally to be the same size and collectively taken to constitute the whole" p 3		units of units
Amy Hackenberg, 2010	Reversible Multiplicative Relationship (RMR) problems for example, *The Peppermint Stick Problem*	interiorized 2 levels of units	students know ahead of time that the insertion can be made. "thus their multiplying scheme is anticipatory: Upon recognition of a situation of the scheme, the activity of the scheme is available to them in re-presentation; however, they may still need to carry out the units-coordinating activity to determine the results of the scheme." p391; CREATE THE 3-LEVEL IN ACTIVITY	"two sixes is twelve, and two twelves is twenty-four. Is created in 2 levels of activity. For problem of combining 48 muffins in a second batch, could arrive at 72 muffins, but likely the number of rows of 6 muffins each in the 72 muffins would be new-(operating with the 24 muffins and 48 muffins, students not retain these quantities as 3-levels-of -units structures. p.391		Units of units, or, units of units of unit in activity
Hackenberg & Tillema 2009	whole number multiplicative concepts	first multiplicative concept	inserting a composite unit into the units of another composite unit.	units into each of 3 units" p3; "Vacation problem. Susan went on vacation for 3 weeks. How many days was she on vacation. ... counting on by ones from the first 7 days, monitoring the number of times she or he has counted to seven (sometimes with the aid of physical markers like fingers or marks on paper), " p3; Sara partitioned a "cake" into 15 parts; then Sara partitioning 1/15th into 2 parts, but could not find answer	"student has constructed an operation of recursion that she or he can use to repeatedly insert the same number of units into a composite unit."	units of units in activity
Hackenberg 2010	Reversible Multiplicative Relationship (RMR) problems for example, *The Peppermint Stick Problem*	simplest whole number multiplying scheme		insert a composite unit, six muffins, into each unit of another composite unit, four rows, producing four sixes....in activity means the student needs to enact the coordinating in situations of the scheme; and activity and results of units coordinating are not available to the student in assimilation." p 391		units of units in activity
		Multiplicative	"recognizing a given number of composite units, each consisting of the same number of 1s." p 90 Child anticipates total (24) is a composite unit (CU) composed of another CU (4 bags) each of		"goal is to figure out the total of 1s in this compilation of compaosit units, and, and the activity is simultaneous, coordinated	units of units; Units o units of units in activity; Explicitly Nested Number Sequence (Steffe

Figure 10 Sample from the spreadsheet to organize pertinent schemes research.

After predicting schemes based on prior research, the codes for gestures, speech and written work for observing the schemes in array multiplication were developed. In order to make the tool easier to use, the unitizing scoring guide disaggregates working with only the mental model, or *interiorization,* from the unitizing made possible with the support of a diagram or other activity. In developing the actions to match unitizing indicators, the students' actions related to unitizing were grouped related to increased amount of flexible unit use. Generally, these schemes were designed by combining scheme-based findings with other research findings on students' mathematical thinking, leading to this progression: (1) counting singletons (unitary counting to reach a total), labeled *Pre-MC*, (2) using units of units *during the activity* of determining the quantity, considered *MC1*, (3) using units of units (counting by a group of units, where child realizes the group of units can also be that number of ones-being able to insert a quantity into another quantity, a grouping of groups) labelled *MC2 emergent* and (4) *MC2 elaborated,* where students can keep track of two levels of groups and within the process of solution-finding, can keep track of a third level of units while holding two levels of units in their head. A code for when it was not possible to identify the unitizing level was also developed.

A preliminary scoring guide (SG) for the break apart rectangular array problem (Table 4) was designed to connect the student's behaviors to inferences about the student's *units* understanding, described as MC levels. For example, drawing individual squares to make an array is connected to unitary counting (*Pre-MC*). Verbally counting with an emphasis on the last number, 1,2,3, **4**, 5, 6, 7, **8**, 9, 10, 11, **12**… may be counted as part of the evidence for *units of units in activity* (*MC1*) thinking. Table 4 presents the initial SG. The author revised the scoring guide several times over the course of the study: after the pilot study and conversation with a dissertation committee member, and after scoring all the data. Key changes included changing the name for each level from *Units* labels to MC labels as well as increasing scoring evidence. The SG assumes students have had some prior instruction in BAM.

Considerations Based on Examinee Characteristics

The SG presented here is designed to indicate shifts in the way students organize number into composite units along a continuum from observing the student's solution strategy in the array problems. It is important to note that students in the study were exposed to the *break-apart* multiplication process through classroom instruction, and student thinking and resulting solutions are reviewed in light of instruction as described in the curriculum. Within learning the break-apart multiplication process, students are exposed to instruction encouraging them to make a shift in their thinking – instead of breaking apart a number like 14 into tens and ones, the lesson is about breaking apart a rectangle with 14 across a row into 10 columns and 4 columns. However, each student may not interpret this information in the same way. The SG connects the students' solution actions to inferences about students' thinking to determine a given level of unit coordination.

Table 4
First Version of Scoring Guide: Mental Model Predicted for Observable Action in Solving Rectangular Array Problems

Model of Mathematical Thinking and Learning	Observable Actions in a Predicted Scheme (Actions to result in finding number of squares in a rectangle)
Level 1(Pre-MC): UNITARY COUNTING Use of unitary counting to find a quantity in units The rectangle is seen as a set of unitary objects.	Keep track one at a time Touch squares individually to count by ones. For example, student might put a pencil dot in each square or represent the quantity with tally marks and count the marks.
Level 2 (MC1): UNITARY COUNTING Units of units in activity	Keep track of quantity as more than singletons. Touch row or column to keep track of multiples, verbally count with an emphasis on the last

Model of Mathematical Thinking	Observable Actions
Counting on is used. Counting units of singleton units or keeping track of multiples, but not keeping track of singleton objects at the same time	number (1,2,**3**, … 4,5,**6**…7,8,**9,** etc.) with a hand sweeping gesture or drawing a line to go over the squares being counted on the diagram. Counting groups of square units, candy or other pieces where the rectangle is seen as a set of composite ones, but the student interacts with the diagram in order to arrive at grouping.
Level 3 (MC2 emergent): UNITS OF UNITS Use of Units of units without activity	Keep track in multiples without activity. Verbal count in multiples or with verbal/gesture indication of matching row/column to the array area without having to touch each area before calculating. For example, extends finger for each count – 4, 8, 12, 16, or, touches a row of 9, touch the next row and say 18; Thinking in composite units where each is also considered a distinct quantity, such as 5 of 4 rows [5 of 4-units].
Level 4 MC2 elaborated): UNITS OF UNITS: Units of units of units in activity	Working with multiple groups with multiples in a group while using the diagram to keep track. Through facial expressions, pointing, verbalizations or other gestures show concurrently keeping track of rectangles within the larger rectangles and the larger rectangle at the same time; being able to keep track of the amount in smaller arrays and be able to add the smaller array amounts while decomposed by using the diagram or other activity. For example, saying “three rows of 4 and six rows of 4 makes nine rows of 4, or 36 square units” or similar ideas with appropriate gestures.
Level 5 (MC3): UNITS OF UNITS OF UNITS *Grouping groups of numbers* *Nested units of units*	Keep track of multiples of multiples Through facial expressions, verbalizations or gesture, show ease of completion for task showing knowledge of concurrently keeping track of rectangles within the larger rectangles and the larger rectangle at the same time; being able to keep track of the amount in smaller arrays and be able to add the smaller array amounts while decomposed. For example, 3 rows of 4 and 6 rows of 4 makes 9 rows of 4 or 36 square units. Student can go back and forth to describe relationships between the three levels of units in

	the diagram without redoing information-gathering.
Other	Gesture, facial expression (squished eyebrows, shake head back and forth) or verbalization to indicate lack of strategy to find the number of squares after receiving additional prompts regarding the directions may indicate lack of understanding. Other actions may not fit other categories will be stored here for further review.

Initial Application of Unitizing Scoring Guide

The first version of the scoring guide was used to identify schemes using Focal Activity Q10 *break apart multiplication* video data, from a random sample of 30 of the 78 interviews. In this array multiplication problem type called break-apart multiplication (BAM), the scheme *situation* is the mental image for finding the number of squares in the diagram from typical classroom learning.

Students' break-apart multiplication problem-solving video data was transcribed for pertinent gestures, utterances, and speech for use with the unitizing scoring guide using Inqscribe software. Since gestures can enlighten instructors regarding the nature of their students' mathematical thinking (Alibali & Nathan, 2012), specific gestures are described in the guide for inferring unitizing levels as well as specific verbal phrases and written work, including diagrams. Student's written work from the interview as well as the video data provided evidence for the unitizing scheme choices. Within the data analysis, additional gestures for making inferences were identified.

To detect students' dynamic understanding within the BAM problem situation, the researcher reviewed the video looking for evidence of given actions on the SG in order to make a wholistic rating. A unitizing rating was based on which of the SG's model had the most evidence. Collectively reviewing verbal, written, or movements by which a student kept track of the number of array squares was one source of information for determining a rating. Listening and watching the student explain how and why they determined the number of squares for their answer was the other main source of information. Various segments of the interview included video did not provide the opportunity to view student actions, so not every item has scorable responses from all students. Some students did not initially share their reasoning for solution choices, and the interviewers did not always encourage students to explain their reasoning, which occasionally diminished the verbal evidence opportunities. When there was not enough evidence to make inferences about a scheme, the data was not included in the item analysis. More information about the available data for a specific item or group of items is included in the discussion for that item(s).

Video data from four classrooms of students completing the fourth grade *Math Trailblazers* Unit 3 test Q10 were available for 78 students. Thirty of the 78 available video clips were randomly selected and reviewed as an initial review to improve the scoring guide, in keeping with the top-down and bottom-up design. The researcher selected every 3rd interview in a continuous loop of the list of available interviews to randomly select the pilot sample. Table 5 indicates the number of students per classroom.

Table 5
Students in Pilot Study by Classroom

Classroom #	N Students Pre(all)	N Students (pilot)
058	28	13
076	15	5
117	12	5
120	23	7
All classes	78	30

In the sample of 30 students, the majority of students used one of two methods to break apart the rectangle with 4 rows of 18, either (a) split the rectangle in half, calculate *4* x *9* and double it (30%), or (b) privilege *10,* splitting the array into *4* x*10* and *4* x *8* (53%). Two students (6.7%) used multiple groups of *4* x *4* or *4* x *5* arrays, one student used *4* x *7* and *4* x *11* to calculate, and one confused student used *70* x *4* and *2* x *4*. Twenty-four of the 30 students (80%) provided accurate answers. -

In the pilot sample of 30 students completing the BAM task Q10, three students' behavior indicated level 1 (Pre-MC). Ten students at level 2 (MC1) were able to keep track of both how many groups of a composite unit as well as keep track of the number of singletons at the same time with the support of interacting with the rectangle. Eleven students demonstrated, level 3 (MC2 emergent) and four students demonstrated level 4 (MC2 elaborated) coordinating three levels of units with the help of interacting with the rectangular array or other supportive activity. Video and student work for two students were inconclusive. Of the six students with an inaccurate answer, two presented Unitary Counting (Pre-MC) scheme use (Level 1), two gave Level 3 (MC2 emergent) evidence and two students' actions were inconclusive (Table 6).

Table 6
Unitizing Pilot Scoring Results

Unitizing Level	Name of Unitizing Level	Number of students
1 (Pre-MC)	*UNITARY COUNTING* *Use of unitary counting to find a quantity in units*	3
2 (MC1)	*UNITARY COUNTING* *Units of units in activity*	10
3 (MC2 emergent)	*UNITS OF UNITS* *Use of units of units without activity*	11
4 (MC2 elaborated)	*UNITS OF UNITS:* *Units of units of units in activity*	4
0	*Other*	2
All		30

Scoring Guide Refinement

From this initial scoring work, it became clear there was not enough evidence to support the identification of the category called using *units of units of units (Level 5, MC3)*. Scoring for this category might be possible with Q10 if questions beyond the initial prompt were used that could evoke the activity needed to infer the student's thinking to demonstrate this scheme use. Consequently, the Level 5, *MC3* category was removed from the scoring guide. Recommendations for future development include changes to the protocol interview that will support making inferences for *MC3*, Level 5.

In keeping with the combined top-down and bottom-up approach, adjustments were made to the scoring guide based on scoring these 30 interviews. Videotape and written work data annotations for pertinent actions help document why certain actions or writing/drawing match a given rating for an inferred scheme. This helped document connections between written, spoken or gestural actions that have potential for use as evidence of a particular unit coordinating level, thus providing support for further development of a rubric tool.

The environment at the time of the interview, including the location (background sounds) and the interviewer's style influenced scoring opportunities. In some situations, the camera view was blocked or insufficient and in others, the background sounds could make it difficult to hear the child being interviewed. There were differences in students' linguistic abilities with regard to sharing their thinking and in how the interviewer sought to learn about the students' thinking. To account for the inability to gather complete information, a rating that shows the student can coordinate units "for at least this level" was added as well.   In the scoring, adding ".01" to the rating score shows "at least" that score to indicate that more evidence of interiorized levels of units might have surfaced if the data had been more complete. Starting with the remaining 48 Focal Task Q10 interviews, scoring the break apart Focal Task Item T5-10b from year 2 data collection and the corresponding break apart item from Task 5B from year 1 data collection began using the revised scoring guide.

The scoring guide underwent another revision after review of unusual cases with a knowledgeable committee member. About this time, scoring videos using the scoring guide led to the realization that the scoring guide vocabulary could be improved. Given the continued use of the term *unit coordination* in the related literature, it made more sense to talk about how students will coordinate units in multiplicative concepts stages instead of using the *units of units* vocabulary. After consulting with a knowledgeable committee member regarding this change and conferring with advisors, the scoring guide language was standardized to include *unit coordination* labels instead of references to thinking in *units* or *units of units.*

Decisions about the starting point for the unitizing levels were expanded while scoring of Item T5-10b because the T5-10b video data presented schemes indicating limited units understanding compared to the existing scoring guide. As a result, some modifications to the guide were made. *Unitary Counting*, with a rating score of .5 was added to account for activity where the array is seen as a set of unitary objects, where the student can only count starting with one and counting all. The break-apart rectangle activity scoring at this point in time included 5 levels: Unitary Counting, Additive Coordination of Composite Units, Coordinating Two Levels of Units in Activity, Interiorizing Two Levels of Units, and Coordinating Three Levels of Units in Activity. With the need to account for items where only conflicting or inconclusive evidence was available at the time of scoring, a level 0 was added.

Following the above revisions, video data for the rest of the break-apart multiplication items were scored with similar procedures for recording gesture, utterance, and speech. Table 7 presents the predicted schemes and actions for rectangular array problems. Student work and some transcript data for levels are provided in Appendix B.

Table 7
Revised Unitizing Scoring Guide - after Pilot Study & Discussion

MC level	Model of Mathematical Thinking and Learning	Observable Actions in a Predicted Scheme (Actions to result in finding number of squares in a rectangle)
Pre-MC (count all)	*UNITARY COUNTING Use of unitary counting to find a quantity in units* (Level .5) The rectangle is seen as a set of unitary objects.	Keep track one at a time Counting to keep track of the number of squares. Touch squares individually to count by ones. For example, student might put a pencil dot in each square or represent the quantity with tally marks and count the marks.
Pre-MC (count on)	*ADDITIVE COORDINATION Use of addition to find a quantity in units* (Level1) The rectangle is seen as a set of unitary objects in each part, and the parts are added together.	Keep track of just top row - “11” and “7” so that the student can think about all at the same time – the “11” and “7” are part of the “18” , but not recognize each square in the top row is part of a column. Count by ones for each part of break-apart and add amounts together using the rectangular array diagram.
MC1	*COORDINATING 2 LEVELS OF UNITS IN ACTIVITY* (Level 2) In the activity of using rectangle to keep track of the quantity both as singleton units and as composite units inserted into another composite unit at the same time	Keep track of quantity as more than singletons in activity. For instance, touch row or column to keep track of multiples, verbally count with an emphasis on the last number (1,2,**3**... 4,5,**6**…7,8, **9,** etc.) with a hand sweeping gesture or drawing a line to go over the squares being counted on the diagram. Counting groups where the rectangle is seen as a set of composite ones, but the student interacts with the diagram in order to arrive at grouping. Interacting with the rectangle diagram or in other activity can keep track of the singleton amount, and the number of groups at the same time.
Emergent MC2	*INTERIORIZING 2 LEVELS OF UNITS* (Level 3) The array is seen as composed of units inserted into other units prior to interacting with the diagram or story problem	Student sees an array as composed of multiple units prior to activity. Verbal count in multiples or with verbal/gesture indication of matching row/column to the array area without having to touch each area unit. For example, extends a finger for each count – 4, 8, 12, 16, or, touches a row of 9, touches the next row and says 18; Thinking in composite units where each is also considered a distinct quantity, such as 5 groups of 4 rows [5 of 4-units]. Making a sweeping motion over a row of squares as well as a student explaining “eight and eight is 16, then 16 plus 16 is 6 +6

		is 12 and 10 + 10 is 20, so then 12 + 20 is 32, so its 32" or "twelve is ten plus two more."
Elaborated MC2	*COORDINATING THREE LEVELS OF UNITS IN ACTIVITY* (Level 4) Working with multiple groups with multiple units in each group while using the actions with the diagram or aspects of the problem to consider three levels of units at one time	Working with multiple groups with multiple units in each group while using the diagram to keep track of their activity. Through facial expressions, pointing, verbalizations or other gestures show concurrently keeping track of rectangles within the larger rectangles and the larger rectangle at the same time. For example, 10 rows of 4 is 40 and 8 x 4 is 32, so 18 x 4 = 72 where students reason with 10 rows of 4 units and 8 rows of 4 units, understanding that both are contained in, and constitute, the entire rectangle. Because they establish this unit structure in activity, they still need to act on their diagram, hands, etc. to know that the 72 is also 18 rows of 4.
	UNKNOWN (Level 0)	There is conflicting or inconclusive evidence at the time of scoring.

6. RESULTS OF APPLICATION OF SCORING GUIDE TO BAM PROBLEMS

This section builds on the details about students' expected actions described in the Scoring Guide. Scoring as well as sample cases are used to illustrate how students' solution choices regarding strategy choices, array diagram use, and number sentence completion are used to make inferences about students' unitizing levels. Using the scoring guide with BAM items, details about gestures surfaced that could increase the opportunities to make scoring inferences. Observations during the scoring regarding pointing to side lengths and motions over the diagram when describing solution strategies became useful SG additions. Missing data and issues involving unexpected variations in scores between the Q10 and T5-10b data are described, including how BAM data were scored using the refined scoring guide. Memos preserved observations, scoring concerns, and questions about student actions. Scoring as well as sample cases illustrate how students' solution choices regarding strategy choices, array diagram use, and number sentence completion are used to make inferences about students' unitizing levels.

Scoring Break Apart Multiplication Items, Q10 and T5-10b

To produce a unitizing score, student actions from the protocol interview video were compared to the expected actions described at the various levels on the Scoring Guide. While rating unitizing levels using the scoring guide, the researcher looked for additional gestures and combinations of actions that were consistent at a given score. These additional gestures and combinations of actions are described in this section if selected to provide more support for a rating. Triangulating words, gestures and written work helped confirm a rating. Discrepancies between verbal or written work and gestures may indicate a student's instructional needs (Goldin-Meadow, 1997). Although the data for this study is videotape of students from a prior study that does not allow for follow-up with students, for in-classroom use of the SG, students with gesture and verbal/written work mismatches would be candidates for intense instruction specifically related to the mismatches.

Additional Gestures to Support MC Levels Involve Interaction with Rectangle Side Lengths

Students' visible hand movements or eye contact with the rectangle are compared to the expected actions to make inferences regarding students thinking. The type of movements as well as the quantity of interaction regarding the number of squares on a side can support inference-making. Instructional materials describe the array diagram as a conceptual representation for multiplication and the corresponding equation.

Counting the squares in a row or a column multiple times indicates thinking in singleton units or a need for active work with the array diagram in order to find the number of squares. Counting squares in a row or column once to determine the side length number of squares and referencing this amount thereafter shows minimal use of the array diagram to keep track of singleton units. Sometimes students find the number of squares in a side length and *remember* that amount for additional sections of the rectangle that have the same number of squares for a width or length. For example, if the *4* x *18* rectangle is broken into two parts, *4* x *10* and *4* x *8*, if the student recognizes the count of 4 is the same for both rectangles and does not need to recount, there is less interaction with the array with the corresponding inference of more interiorized unit coordination or spatial knowledge. This may indicate that the student is aware the height of the column does not change across the rectangle as a spatial attribute of the rectangle. The researcher infers that the student views the rectangle as one entity split into parts, where each part will have the same side length. Thus, the student recognizes the multiplicand is four for both parts.

Gestures may indicate if the student is thinking about all the squares or just one row. Student interactions with the rectangle that describe both the factors and the resulting product imply thinking more comprehensively about finding the rectangle's number of squares. Students who move the pointer back and forth over a row or a column may be focused on how they found the numbers to use in the equation from verbal responses, but students whose gestures only involve the row or column of squares may only be thinking about the single row and not a quantity in each row times the number of squares in a given row. As a representation of multiplication, the latter example may indicate additive thinking versus multiplicative. Gesture references to the array diagram can help communicate the process by which the quantity of squares is determined, thus providing another source of potential evidence for inferring how the student is using units to determine the quantity within the solution process.

Gestures while Counting Squares Help Infer MC Levels

Although counting the number of squares in a side row or column is typical, the interaction with the array varies. Students' gestures when counting may indicate whether the student is thinking about individual squares or groups of squares. While the researcher scored interview data, anecdotal notes and other records of student actions led to noticing patterns in the way students counted the number of squares on a rectangle's side. This encouraged the researcher to more specifically note a student's finger motions when counting the squares on a side. From these interview data tallies, students who only slightly lift the finger used to point between squares were more likely to treat the squares in the row or column being counted as a composite unit. From the researcher's general observations, students who coordinate two levels of units in an interiorized way (MC2) were more likely to move their finger or writing instrument close to the surface of the paper to count the number of squares. Students who need the diagram to keep track seem more likely to make bigger hops or jumps when going between squares to count them. When the counting acts include hopping motions or higher jumps, the student may be focused on identifying the square as a unit while counting the number of squares. When the counting acts have a smoother movement across the row or column, it may indicate the student is thinking about the quantity in the row, seeing the row or column as the unit versus the square as the unit.

To find the size of an array, almost all students counted the number of squares on the rectangle's sides. Students point at a square, say the appropriate number name in the sequence, move the pointer to the next square and repeat the sequence until there were no more squares to count. The interesting observation is not that the student is counting squares, but in the way the student interacts with the square to know the total number of squares on a side. In going from one square to the next, a finger could go up from one square and down to touch the next, or basically slide between the squares. This difference in height between transferring a finger from one square to the next is identified as a student's finger jump height between squares. The range in the distance from the paper to the highest point above the paper when the student is moving from square to square to count squares is the student's finger jump height. This aspect of moving between squares reveals information – generally, the more the student demonstrates coordinating units, the less of a pronounced jump between the squares is observed. Jumps between squares were more likely to occur when students were less able to coordinate units most of the time. When students exhibited the behaviors for the Pre-MC level, such as marking the square with a dot or shading each square as well as touching it, the student was more likely to make more pronounced jumps during counting. For students exhibiting the behaviors for MC2, indicating more unit coordination, low jumps or a smooth move across the squares were more likely to occur. The Counting Squares Jump Height chart (Table 8) describes the movements with the inferred relationship to unitizing levels based on the researcher's observations during scoring.

Table 8
Counting Squares Jump Height

Finger movement between squares	Description	Example	Typical Level
Marking	Touches square and marks the square in some way	Touches square and marks the square with a dot; lifts up pen to go to next square and mark with a dot	1
Hop – Jump	Pointer is lifted up from one square and dropped on to the adjacent square, possibly in a pronounced way	Student lifts the pointer (finger, pencil, etc.) between squares which is often accompanied by a verbal count	2 -3
Glide	Student moves pointer over the squares	Student moves pointer over the squares with little to no vertical motion	3-4

Typical Gestures Connecting the Number Sentence and the Array Diagram

The BAM includes writing a number sentence to indicate break-apart choices. The curriculum suggests a format (see Figure 8) which includes the dimensions of the large rectangle. Thus, including the dimensions of the large rectangle in the equation can be considered an indicator of understanding. For example, Q10 may be represented as *4* x *18 = 40 + 32 = 72*, where *4* x 10 would be recorded in the appropriate rectangle section and *4* x 8 would be recorded on the other rectangle section. The number sentence for the given rectangle may be evidence of early distributive property understanding, but that was not explored in this research.

Students' number sentence explanations often included various gestures to support their verbal explanation. Students who provided an explanation frequently talked about the equation, then referenced previous solution actions by pointing to sections of the equation or pointing to the rectangle as part of the explanation. The scorer can use these gestures to infer how a student is thinking about the factors (from the student's interaction with squares on a side) or the product (oval motion over the rectangle) as a quantity. For instance, a student might be talking about the product, but move the pointer in a linear movement not an oval motion to indicate the group of squares formed by the insertion of a unit for every square in that line. Instead, the linear movement may indicate more of a focus on only the squares in the line that represents the factor. These gesture observations were added to the final version of the SG (Table 26). Next, a sample case for each level is shared to illuminate the differences between levels.

Example of Pre-Multiplicative Concept / Unitizing Level 1 – Excerpts from Stephanie's Case

A student's reasoning ability is inferred from their verbal and non-verbal actions while problem-solving within array problems. Students who counted one item at a time to determine a quantity were rated Pre-MC /level 1. Some students counted one at a time but then stopped counting and seemed to make a guess about the quantity. About a third of the Pre-MC/level 1 students physically marked the squares one at a time to keep track, and all students touched the squares, most with an explicit pointing touch. To illustrate the actions of a student at Pre-MC/level 1, selected excerpts from the protocol interview with Stephanie (Student 1970) follow.

In explaining what she is to do for the BAM task, Stephanie noted that she will need to "break this rectangle apart into 2 or more pieces" while her left hand performs a karate chop movement. Then when she said, "umm, make it easier to multiply," she placed her hands with palms up and fingers out, as if ready to use them to count. This is a significant action as it can be interpreted as an indication that when she is doing *multiplying*, she is going to be using her fingers to keep track. In the actual solution process, she created a smaller rectangle by shading in each square individually by columns to create a *4* x *9* array. Then she used a hopping motion such that the pen landed on the next square. This indicates her perceptual need to keep track of the count. Even though she had just shaded in multiple rows, she needed to touch and count the squares in a row to determine the amount for writing the *9* as a factor in the multiplication.

She also recounted across the column to find the digit *4* and used a calculator to determine what four times nine equals. With the use of the calculator, Stephanie determined that each sub-rectangle would be 36 squares. When it came to combining the subtotals for the squares, though, instead of doubling the number of singletons on one of the two equal parts, she used her fingers to add nine and nine because her focus is on doubling the nine, the side length, showing a lack of coordinating two levels of units. The nine which is the number that is a factor in the multiplication is the complete number of squares in her scheme, not just the squares in the top row. She repeats this action with the four, the other side length's number of squares. With lines drawn from each of the sub-rectangle equations (*9* x *4 = 36*), she created a new phrase, *8x18,* and added the equal sign. After she used her fingers for more calculation, she wrote *168* next to the equal sign to make *8* x *18 = 168.* This indicates a lack of connection between the number of singletons generated by the calculator for the smaller *4* x *9* arrays and the total number for the squares in the entire rectangle. Probably aware that doubling is involved, she doubled the number of squares in a side length, numbers that she actually determined herself by counting perceptual items. She does not seem to recognize the *36* as a meaningful part of finding the number of squares, as it is not referenced in her explanation or used in her calculation. Although she knew how to use the calculator to generate the number *36*, there is no evidence that she sees *36* representing the number of squares on a sub-rectangle.

Stephanie, similar to other students at Pre-MC/level 1, counted the squares by ones to obtain the amount. In her written response and matching verbal explanation, she focuses on adding, not making groups of groups: “I added four times four. I added four and four and I got eight, and then nine plus nine and I got eighteen, and then I came up with, a hundred and (pause) a hundred and sixteeeey-eight”. At this point, her hands are clasped together above the desk height, but between the desk and her body and she took a big breath. There is the feeling that she is waiting for something to happen, indicating a lack of confidence in her final response. I think she recognized her answer for the number of squares is not reasonable, but she didn’t know what to do about it.

Although Stephanie’s actions suggest a Pre-MC/level 1 for unitizing, her gestures are mismatched in spots, indicating that focused instruction might impact her learning. In describing the BAM problem, Stephanie moves her right hand with pointer finger extended in a circular motion by and over the rectangle, while saying, “Write a sentence for all of it together”. This shows that she is planning to find the total of squares for more than a row. However, her later actions indicate her reasoning was more linearly based. Since she seemed so apprehensive about her total number, perhaps her spatial sense of quantity helped her realize her answer was not reasonable, or perhaps she was not confident about her method. In addition, in the second BAM problem, when she shaded in the squares for one side, she shaded squares in groups of four –

first going down each column, but for the last four columns, instead of going down to count, she made groups of four going across. Although her speech and equation-writing continued to be focused on using side lengths for actual calculating as she used the calculator to find the products of the sub-rectangular parts, her gestures indicate grouping - both with her circular motion gesture and the way she shaded squares in groups of four. Stephanie's circular motion is an indicator of thinking about an array of squares versus a line of squares counted by ones, with limited additional information due to calculator use. Her gesture indicates a different understanding than her spoken response focused on counting. Situations with such a mismatch indicate the gestures indicate an emerging understanding (Goldin-Meadow, 1986). If this were a classroom setting, more opportunities for Stephanie to do a double count of the number of groups while skip-counting (four in this situation) and to explain her grouping choices as well as other actions might help her move forward in her emergent understanding.

Example of Multiplicative Concept 1 / Unitizing Level 2 –Excerpts from Nick's Case

Students like Stephanie counted by ones to find a quantity, but students in the second unitizing level are able to use the array diagram or another representation to help them keep track of a quantity in groups of singletons as well as the singleton unit count. Examples from Nick's work are featured to illuminate MC1/Level 2. When he described how he will find the number of squares, Nick (Student 2096) says, "You can do break it and multiply. I'll just do like four (He shades in the entire leftmost column with top to bottom motion over all four squares.) times… (He started to point the pen to shade one square and then another square, but stops to move the pen back and forth, shading the bottom row of squares.) ...I do four …times… eighteen". Before saying "eighteen" Nick pointed his pen at each square from left to right with a jump between each square as if counting, but there is no audible response until he says "eighteen".

Unlike Stephanie, Nick shaded in the entire row or column in a broader movement of the pen, not square by square. This is non-verbal evidence of thinking of the quantity of the row or column as a unit, not just a collection of singletons. When Nick went to the row to count the 18 squares, he made jumps between the squares, showing his active work with individual squares. In shading all the squares, he shows the entity of a unit of 18, but in counting the 18 squares, the jumps indicate the importance to him of keeping track of each square. With more unit coordination expertise, a student might glide over the squares to see how many are in the shaded unit, internalizing the 18-unit, but for Nick, each square is an entity to think about as well as the 18-unit that is represented on the paper.

Even though Nick told the interviewer that he needs to break the rectangle, he proceeded to use the entire rectangle to find the number of squares. Later, as he recognized there should be a second rectangle, and he added four times one, the left-most shaded column, as part of the equation. The focus for an MC1/level 2 student finding the number of squares in an array is coordinating two levels of units within the activity of *solving.*   In using the array diagram for solving the problem, the students might use it to keep track of insertions, or to keep track of the quantity of the unit being inserted into the other unit, such as the amount of a column being inserted into each *square* of the top row. Students at MC1 might break the large array diagram into two parts, and only find the number of squares for one, or might find the amount of squares for each of the smaller parts, but then, the student may be at a loss when thinking about combining the sub-rectangles to find the grand total of squares. The child might work with parts in a way that segments the problem enough to find the answer through the use of the diagram and other means.

Nick decided to use a standard paper and pencil calculation method to find the number of squares for *18 x 4*, and he tackled the task of finding eight times four using his fingers as counters. Nick gave a verbal clue indicating he is doing an emphasized count by fours when he says, “I do four times eight,” following this in a whisper voice with, “seventeen, eighteen, nineteen, *twenty*.” He made a fist and then extended one finger at a time until four fingers were extended. He continued in a pattern of opening and closing fingers, extending 4 fingers/thumb each time. Nick’s other hand was under the table, so we don’t know how or if he was using it to keep track of the counts (8:32-9:31). Eventually, he said, “thirty-two?” Using the activity of his fingers he was able to keep track of groups of four for a particular number of groups, in this case, eight groups of four.

Nick used the activity of moving his fingers to find eight groups of four, but he did not use the array diagram to find the total number of squares, using standard calculation incorrectly instead. To find the number of squares in the rectangle, he combined the amount for the left-most column (four times one) and the entire array that includes the leftmost column. This is an indicator that he is trying to make sense of the directions, but is not fully understanding how the array acts as a representation for multiplication. He is accounting for the shaded column twice. Other students did not count the column that was shaded in the total count. The mismatch between the actual number of squares and the number sentence representing the squares due to the child’s interpretation of a shaded column could be related to spatial awareness or to unitizing skill/ability or to something else.

Example of Multiplicative Concept 2 Emergent /Level 3 –Excerpts from Leah’s Case

While students in the MC1/level 2 are able to insert a group into another group with their actions in some way, MC2 emergent students are able to coordinate two levels of units in an interiorized way. Leah’s work can provide an example. In the following excerpts from Leah’s BAM solution, we can see Leah (Student 1952) does not need to touch the array or her fingers the way Stephanie or Nick does, and she is more articulate about grouping in her explanation. She starts using a typical strategy, splitting the rectangle into two equal parts. Forty-six percent of the students for whom there are data in the study chose to split the rectangle in half in a BAM problem. To find the place for the break apart bar, Leah’s fingers jumped from square to square for the first count of the number of top row squares, but after determining there are 18 squares, she worked to determine what half would be in her head: “Eighteen times that equals. Eighteen plus what (pause) Ohh (pause) Nine” (6:00). In counting out nine squares from the left to find where to place the line, her jump from square to square is closer to the paper, and she only counted the first nine. She drew the line between the eighth and ninth square, but at a later inspection, she recounted in order to redraw the vertical line correctly. Both times, after counting the first squares in the row, the squares that are left are already established as a unit of *nine*. In an atypical fashion, she divides the squares widthwise as well, first saying “Then since its four on a side, break that into two” and then without counting goes directly to the first column between the 2nd and 3rd squares and traces the line horizontally across the rectangle. So now the array is split into four parts (see Figure 11).

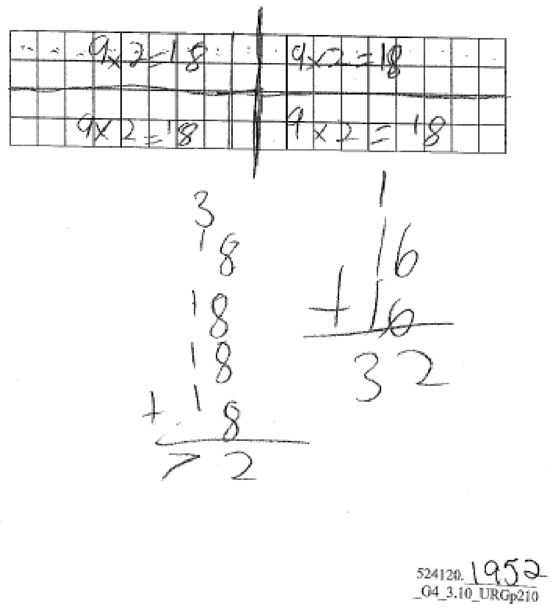


Figure 11 Leah's Q10 work

From her actions the researcher inferred that Leah knew in advance that each of the four parts is the same. She chose to break apart the rectangle into smaller rectangles that represent an established multiplication fact for her, and she completed the steps with a confident rhythm. However, evidence that this is not a rote strategy for her is needed.

More evidence for being able to keep track of two levels of units in her head included that she calculated the value of four eights for her need to find the sum of 18 +18 + 18 +18. She used the doubling strategy in her head, noting two eights is 16, and two 16s would be the same as six plus six plus twenty. There is support from her response to the interviewer to infer that her teacher did not stress using the curriculum number sentence format she used. When the interviewer asked Leah to show her the number sentence, Leah points to the vertical list of the four 18s and the answer on the line below it confidently indicating that this was her number sentence. There was not a look of thinking to remember a number sentence format learned in class.

Leah's work also represents the students (29%) who broke the rectangle into more than two parts to create subsections that are easy to calculate at least once. For the first array BAM solution, she splits the rectangle in half both by the length and the width to create four *2* x *9* rectangles, with the corresponding multiplication fact that most students in the class know, *2* x *9 = 18.* For her second array BAM solution she chose to create an array that represents a multiplication fact *(4* x *4)* practiced in this curricular unit. What is surprising about solution is that she recognized that if she makes rectangles that have four as the dimension for the longer side, she will not be able to have the same size shapes for all of the sub-rectangles because four is not a factor of 18. She decides to use four as a side length anyway, with the last rectangle not four by four, but two by four. By recognizing that she will not be able to make all the sub-rectangles the same, and being able to execute appropriate rectangles, she demonstrated making all the rectangles the same is not a rote procedure. The researcher infers that Leah selected sub-rectangles that match multiplication facts that are well established for her, supporting the inference that she is aware of the relationship between the number of squares on an array side and factors in a multiplication sentence, thus supporting the inference that she is coordinating two levels of units.

It is possible that Leah could coordinate three levels of units, but there is not enough evidence in her interview to indicate this. For instance, if her teacher had addressed writing number sentences that included the dimensions of the entire rectangle and she produced a number sentence that included the whole rectangle WRD in the format reminiscent of the distributive property as described in MC2 elaborated/level 4, or if she could appropriately respond to questions about how combining the number of rows in one section to rows in another section to determine the total number of rows as well as the number of singletons, there would be some evidence for MC2 elaborated/level 4 or beyond.

Example of Multiplicative Concept 2 Elaborated/ Unitizing Level 4 –Excerpts from Michelle's Case

MC2 elaborated students are able to think about the quantity as groups of groups, as well as singletons and with the use of a diagram, fingers, or other activity that supports keeping track of a quantity. They are able to reason about an additional layer of units, such as the quantity of rows of a given column length when the rows are being added. There are several significant aspects of Michelle's solutions (Student 1286) that help elucidate the actions to indicate a MC2 elaborated/level 4 – the efficiency in Michelle's solution process, how she focused on the diagram, and the details in her elaborate number sentences and explanations.

Michelle's motions for counting the number of squares are mostly similar to MC2 emergent/level 3 students - counting quickly with the pen moving smoothly across the paper. She uses an additional approach, counting by fours to find the place to draw the break apart bar by moving the pen across four squares while saying, "Four, eight, nine." MC2 Emergent behavior does not always include counting multiples across a row, but MC2 behaviors include skip-counting to determine amounts. Michelle shows she has established four and is counting in groups of four to get to the desired number of singletons. Her pen point moved smoothly over the paper, indicating a focus on determining the quantity of the selected section of row, not focusing on the singleton aspects of the amount, similar to expected actions for an MC2 emergent student.

To break apart the rectangle she used a common method, splitting the array in half. About 46% of the students in this study used this method. Michelle knows that the multiplication fact for each half is $4 \times 9 = 36$, but unlike others who may know that multiplication fact, she quickly connected these to the whole rectangle's number of squares: "4 times 9 plus 4 times 9 and then 4 times 18 equals 36 plus 36." Connecting the entire rectangle to the subparts using the activity of the situation is characteristic for this level. Michelle recorded her actions in a number sentence that included the factors for the entire rectangle as well as the factors for the sub-rectangles, the number of singletons for the sub-rectangles, and the number of singletons for the entire rectangle. This is characteristic for students at level four if they have received instruction on this format. Although this format is presented in a previous lesson (fifth), only some students in this study use this level of detail in their work, and they are all in classroom 076. Using this equation format is considered MC2 elaborated evidence because not all students in the 076 class can reproduce this format without a model.

Since memorizing the more detailed equation format for its use on BAM problems is still a possibility, it is important to find corroborating evidence in the explanation or other means to use this as supporting evidence. Initial interactions with the diagram at this level are to gather information regarding the number of squares on each dimension, and this is completed without hopping type movement. Michelle's actions are typical of elaborated MC2 students. As she gazed at the second array to determine the sub-arrays, she is not touching the paper but glides the pen over the second row of the array. This might be interpreted as thinking about columns and not merely one row of squares. Finding the number of squares is completed as directed by the interaction in addition to this interiorized thinking would be expected at this level if questions related to MC3 were asked, like a question asking the student to explain how the diagram supports the distributive property.

Looking at Michelle's actions, support for MC2 elaborated comes from the way she explained her equations, and how she referred to the array diagram within her explanation. Since there is no explanation for her number sentence in the first BAM item, we look to the second instance. In explaining the number sentence for the 5 x 14 array, she pointed at side lengths and then at the middle of the sub-rectangle, "so here is five times seven," doing this for both sub-parts of the rectangle. This indicates she is using the side lengths to determine the quantity for that portion. Next, in describing how she found *5* x *14*, she referred to the two *35s* in the next line of the equations (see Figure 12) to identify the total. She says:

> and then 5 times 14 equals … so what's 5 times 7 equals? [00:26:19.12] … (Points the marker in the center of the array) so I put 35 [00:26:22.07] and over here is the same, is 35. (Points the marker in the center of the array) [00:26:23.14]   So 5 times 14 equals, so I put the answer that equals 35 plus 35 so 70.

Here we see that her indexical gestures do not focus merely on writing the numbers in the equation, but try to connect the arrays to specific aspects of the equation to tell her thinking. This indicates her thinking involves the entire rectangle as a unit, as well as the units of rows and columns and she identifies there are 70 singleton units. She shows interiorizing 2 levels of units and with the help of the diagram she is coordinating 3 levels of units, the description for elaborated MC2.

The preceding descriptive cases help to illuminate major scoring nuances for BAM items and the next section compares the overall scoring results for the two instances of BAM followed by how issues of missing data for determining the BAM item type score were resolved.

Figure 12 Example of equation format on Q10 for Student 1286

BAM Item Performance and Resolving Issues Associated with Missing Data

Of the 78 student cases with adequate video, there are 13 cases with only one suitable instance of the break-apart multiplication item to score for unit coordination, and one case where neither instance was suitable for scoring (Table 9). This provided 77 cases with at least one instance of break-apart multiplication (BAM), and 64 cases with two instances of BAM. Of the 64 cases with two scored items, the students were observed using the same MC level in 83% (n=53) of the cases.   Resolving missing data issues to maximize the use of all cases is discussed first followed by reconciling cases where both BAM scores were at different MC levels.

Reviewing the data, the majority of the instances of two data points for unitizing scores on the BAM items were in agreement.   Missing evidence for scoring BAM items are evenly distributed between Q10 and T5-10b items (Table 10), suggesting that the item design or placement in the protocol is not likely to be a source of scoring difficulty. Table 9 indicates the number of students with missing data by classroom.

Table 9
Students with Insufficient Evidence to Score Q10 or Item T5-10b

Class	Q10 Insufficient evidence	T5-10b insufficient evidence
058	1636	
058	1638	
058		1642
058		1648
058	1653	
058		1654
058	1809	1809
076	1826	
076		1277
117		2086
120	1951	
120		1954
120	1967	
120		1972

Table 10
Number of Students by Item and Classroom with Insufficient Evidence to score Q10 or T5-10b

Class	**Q10**	**T5-10b**
058	4	4
076	1	1
117	0	1
120	2	2

Three factors impacted the ability to use students' solution data to rate a student's unit coordination on BAM items were viewed as related to data collection: (1) ability to see students' actions on the video, (2) consistent opportunity for productive struggle and (3) students' explanations of their reasoning. Essential conditions for determining unit coordination skill include: (1) adequate sound quality and opportunity to view students' actions including gestures, (2) having a situation where the student has a goal of finding a solution to a problematic situation versus using tools to find an answer to the problem, and (3) the student's thinking is evident for both the process steps and reasoning regarding why they used this process. Unitizing scores were not produced without these conditions. Some interviews lacked these conditions, which led to insufficient evidence for scoring some cases. More explanation on these factors is found in Chapter 8.

Using this information, and following consultation with advisors, the decision was made to use the single available BAM score to maintain the power of the study. To produce a uniform BAM score for each student, the unitizing ratings from Q10 and T5-10b were integrated into a single score, called the *combined* unitizing score/unit coordination level (cBAM). As noted earlier, the majority of scores for the remaining 64 students are in agreement, but not all. Determining a score to represent MC when the two BAM scores vary is discussed next.

For the 64 students with 2 scores, score agreement fell into two bins: (a) the observed MC was the same (83%) or (b) the scores were not in the same (17%). Scores that agreed held the same score. If the rating was unitizing Pre-MC and MC1, students are considered to be thinking in singleton units, although for MC1, the student's actions led to an inference that the student demonstrated the coordination of two levels of units within the activity (MC1). Since the student demonstrated coordinating two levels of units for one of the two instances, MC1 is used.

When the two unitizing scores were not in the same multiplicative concept level, the activities were reviewed. For five cases (1646, 1962, 1973, 1959 and 1649) the MC score decreased for the second BAT item. In reviewing the video of both items, these students were perplexed when asked to show BAM a second way. This suggests that students may have practiced the procedure one way enough to produce it one time but not enough understanding to produce BAM a second time.

In some instances, the video revealed the student interpreted the item to produce a situation outside of an on-grade level problem to solve. The following provides an example. There were two instances of a .5 score. Both cases are part of the four cases where the student's initial rectangle break apart choice required the use of math facts out of the expected grade level multiplication fact range making the task more sophisticated. For example, in making the split *5* and *13*, the resulting multiplications would be *5* x *4* and *5* x *13*. The difficulty of finding the quantity of an array in that problem setting may have caused the student to revert to less efficient strategies in an attempt to find a solution. Pirie and Kieren (1994) describe students reverting to less efficient strategies when the problem difficulty increases as *folding back*. If solving a break apart multiplication appeared difficult for the child due to a larger number as a factor, *folding back* aspects of problem-solving were considered. The score indicating the student's best performance was selected for the cBAM score. In one case, the student miscounted, making the problem easier than grade-level expectations, prompting the use of the score for the more on-grade level instance. For the two cases with a .5 rating, the combined BAM score became 1 for one student and 2 for the other student. In addition, there were three other cases with scores in different MC levels and in these cases, the score with greater evidence to support it was used. Appendix C contains the specific rationale for each score. This discussion highlights the importance of using a problem setting within an expected range of complexity to increase the reliability of resulting scores. With the 64 cases with two scores resolved, the researcher determined the MC levels for the BAM problem type (Table 11).

Table 11
Number of students in cBAM Activity Demonstrating *At Least* a Given Multiplicative Concept Level

Multiplicative Concept (MC)	Number	Percent
Pre-Multiplicative	7	9%
1st MC	16	21%
2nd MC	54	70%
All students	77	100%

Comparing cBAM Unitizing Results by Classroom

Unitizing levels are presented in Table 12 by classroom, with the number of students and the percent of students at each MC level. From general inspection of Table 12, the percent of students at a given level generally follows the same pattern as the aggregated scores. The researcher compared the values of MC1 and MC2 for each classroom to the expected values. The relation between these variables was not significant, X2 (3, N= 77), p=.29 thus indicating no significant differences in a given classroom and the aggregate data for MC1 or MC2. However, due to the small number of students at the *pre-multiplicative concept level*, this group was not included in the chi-square analysis. The next section compares MC scores from using the SG with the BAM item type and Rectangular Array item type.

Table 12
Frequency (Percentage) of Students in BAM Activity Demonstrating at Least a Given Multiplicative Concept Level in a Classroom

Multiplicative Concept	Classroom				
(MC)	058	076	117	120	All
Pre-Multiplicative	2 (7%)	0 (0%)	1 (8%)	4 (17%)	7 (9%)
1st MC	4 (15%)	2 (13%)	4 (33%)	6 (26%)	16 (21%)
2nd MC	21 (78%)	13 (87%)	7 (58%)	13 (56%)	54 (70%)
All students	27(100%)	15 (100%)	12 (99%)	23 (99%)	77 (100%)

7. APPLYING UNITIZING ANALYSIS TO PERFORMANCE ON OTHER TASKS

Although the Scoring Guide was designed specifically to distinguish MC levels using the break-apart multiplication (BAM) items, the Scoring Guide descriptors can be observed in other array diagram problems. Two related problem types are available in the video data: (1) finding the number of squares in a rectangular array where part of the array grid is covered, and (2) a story problem where an array is the expected diagram. In order to identify the suitability of the Scoring Guide for broader embedded assessment use, data for these additional array multiplication problem types were scored using the scoring guide. The following sections describe the differences and similarities for scoring various items followed by the data analysis.

Analysis of Task 3: Nuances of the Rectangular Array Task (RA) Compared to BAM Task

When the researcher scored the with the Rectangular Array problems (RA) with the Scoring Guide designed with BAM problems, the incomplete grid aspect of the array diagram emerged as an important factor in student's solution processes. In Task 3 Rectangular Arrays, (see Figure 7) students are asked (1) to find how many square units there are in a given rectangular array where part of the array grid is covered, and then asked (2) to write a multiplication sentence to match the given rectangle. Like BAM items, students find the number of squares in a rectangular array. Unlike BAM items: (1) the Rectangular Array grid has a smaller number of squares with array side lengths less than nine but larger individual squares, (2) the RA directions do not specify the use of break-apart multiplication and significantly, (3) the RA diagrams have an incomplete array grid. This means students cannot *see* each square.

In the RA items, students are not specifically asked to break the array into parts as in BAM items. However, it seemed reasonable to assume that some students would break-apart the rectangular arrays to find the number of squares for the larger rectangle. In this way, the RA activity provided an opportunity for students to transfer what they had learned about the break apart process to a new setting. Surprisingly, there were only two instances where a student chose to use the BAM method, providing a way to show MC elaborated. Student thinking is reported at Pre-MC, MC1 or MC2.

The incomplete grid in the rectangular array diagram is significant because the entire square shape is not in view for every square. The lack of definition for all squares is an added challenge for students who need to see a square in order to count it. Some students chose to complete the missing grid by drawing in the lines, thus making all the squares visible and others left the grid alone. Each student's choice regarding the grid, accuracy and unitizing scores was recorded.

Piaget's theory of intellectual development (Copeland, 1974) provides a context for thinking about student responses with regard to the missing grid array display. Being able to interiorize an entire square without its complete picture is more developmentally advanced thinking than with a visible square. Piaget describes developmental periods of growth with a progression. Operational thinking begins with thinking based on more concrete experiences or observations (concrete operational), leading to more formal ways of reasoning with abstract symbols or ideas (formal operations stage). With an incomplete array grid to represent multiplication, students who operate with concrete objects may need to see the square in order to count it. This type of student may use different strategies to solve the Rectangular Array items where the squares are not all visible than the strategies used with the visible squares that provide more visual support (BAM items).

In Piaget's intellectual development theory, initial mental structures are specifically tied to the environment and direct actions, but about the time children enter pre-school, children's mental structures can *represent* the environment. Called the pre-operational stage in Piaget's Intellectual Development theory, children ages two to seven are considered to be the typical range for this stage (Copeland, 1974). Copeland adds, "However, this is only a rough guide. For some mathematical concepts, children do not leave the preoperational stage until nine or ten years of age." (Copeland, 1974, p. 26). Most students in this study are ages nine or ten, so there is not an expectation that many students would be influenced by an incomplete grid, but the Rectangular Array item type was included to see if the lack of a full grid might influence student responses in ways that might produce inferences about units understanding or more globally add to research regarding representations used in classroom assessment.

Rectangular Array Problems, Task 3: Scheme and Unitizing Results

Students' interaction with the missing grid and the three sizes of rectangular arrays is considered with regard to the MC levels identified with the SG. How students interpreted the task and how they found the number of squares is reported, followed by the score consistency.

Video is available for 79 cases for RA items. Data from three students could not be scored for any of the three RA problems. Across all the problems, about 10% of the data were not suitable for scoring for similar reasons to the BAM missing data: students' actions were not visible or the student did not provide enough talk aloud evidence to support a score, providing 64 cases with data for all three items. In order to develop a score to represent efforts on Rectangular Array items for comparison purposes, a combined score for the Rectangular Array problems (cRA) for the 64 students is determined by the median of the three scores. For 62 of the 64 students, at least two of the three scores are the same. Table 13 presents the summary of the students MC ratings on the cRA. Scores are reported in terms of MC levels. The results are reported by individual item to look for patterns in the MC scores by the rectangle size.

Table 13
Unitizing Score Frequencies for Rectangle Items for Students with Response for All Rectangle Items (n=64)

Multiplicative Concept (MC)	Rectangle A 2x6	Rectangle B 4x5	Rectangle C 6x9	Median cRA Score*	Drew Lines in at Least One Rectangle
Pre-MC	19 (30%)	16 (25%)	13 (21%)	13 (20%)	7 (54%)
1st MC	23 (37%)	21 (33%)	19 (30%)	23 (36%)	11 (48%)
2nd MC	22 (34%)	27 (42%)	32 (48%)	28 (44%)	3 (11%)
Total	64	64	64	64 (100%)	21 (33%)

* Median combined Rectangular Array (cRA) score is determined by finding a median MC from items A, B, and C per student as an MC rating for the combined RA scores

Solution strategies by size. Students' solution strategies on the three different array sizes did not match the researcher's prediction that the smaller array would be more familiar, and thus more likely to be counted with more advanced strategies. Actually, more students exhibited pre-MC behaviors on the 2x 6 rectangle and fewer on the *6* x *9* rectangle. In some Rectangular Array interviews, there was a significant switch in the strategy to find the number of squares in the *2* x *6* and *4* x *5* arrays compared to the *6* x *9* array. Some students counted the squares for the *2* x *6* rectangle, but when confronted with the *6* x *9* rectangle, the same students counted rectangle side lengths and determined the product by using these side lengths as factors with a multiplication table or calculator to find the number of squares as the product.

The *2* x *6* RA was the first instance of an array missing part of the grid which may have affected students' actions and explanations. A missing grid is not a typical diagram for learning multiplication. Without the grid in place, more students may have touched each square to establish the unit. Since touching squares is a behavior used for a pre-MC rating, the need to establish the square as a unit by touching the partial square may have caused more students to touch squares and hence produce more pre-MC ratings. When combining the results of the three RA instances and excluding missing data, 25% of the students exhibited behaviors of the pre-MC. There is the possibility that some students touched squares to determine the number even though other strategies were available to them. This is discussed in more detail later. MC levels based on scoring the RA problems are presented in percentage bar graphs (Figure 13).

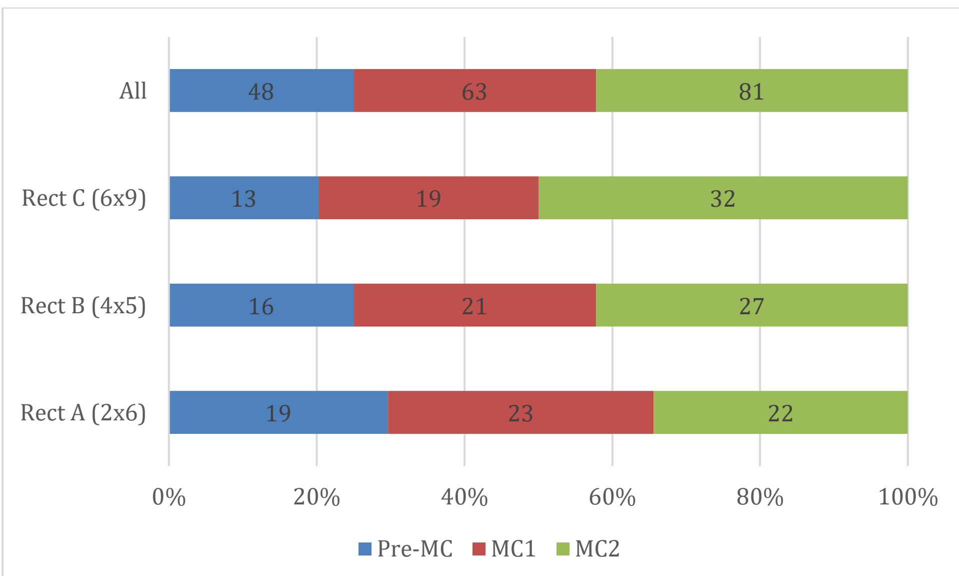


Figure 13 Comparing the MC ratings for the Rectangular Array problems for students who completed all (n=64)

Six students were rated Pre-MC on their first encounter with the missing grid array -- the *2x6* array -- but the rest of their ratings yielded an MC score. Student 1961 is one of these students. In solving for the *2x6* array, she said, "12 …since the line was cut off here and here, I tried to imagine where the line was, and I noticed the lines here are missing…" (15:46). This shows she was actively engaged in figuring out the representation for the array from the diagram with missing lines. However, she did not seem to need to do this for the rest of the RA problems. Somehow, after interacting with the *2* x *6* diagram, she developed a strategy to account for the visible, incomplete, and non-visible squares in the other two problems.

Student 1951 was rated pre-MC for the *2* x *6* rectangle because the explanation focused on counting by ones, yet this student used skip-counting for the *4* x *5* rectangle and used multiplication language for the *4* x *5* and *6* x *9* arrays. When asked to write a number sentence for *2* x *6* array, he writes 6 + 6 = 12. When the rectangle is turned, he indicates the amount is the same and says he counted: "the same...I counted." But for the other rectangles he describes multiplication sentences. He touches each square in the left-hand column with finger of different hand, says "20." He explains, "The first one was 5 so I multiplied 5 times…" He moves pen down the left column. "4." Writes *5* x *4= 20.* This leads to pondering if his "I counted" on the *2* x *6* array referred to counting six and doubling it or the student may have counted each square. Given the ease for counting the *2* x *6* array, the child may have counted anticipating it would take the same amount of effort whether counting or doubling a number. Overall, the absence of part of the grid did seem to affect some students at the onset, but not impact solution processes after the first one for a handful of students.

Drawing in missing lines. Variation in how students demonstrated MC in RA solution processes included whether or not students drew missing lines. The lines form the squares to count or manipulate for grouping. The missing grid may be a challenge for students who have not made the transition to the concrete operation level from the pre-operational level because missing or incomplete squares, eliminates the visual support for thinking about a quantity with groups within groups. Using the combined RA (cRA) unitizing score, 54% of the students at Pre-MC drew lines in at least one rectangle, and 48% of the students atMC1 drew in lines (Figure 13). In contrast, 11% of the students rated MC2 for that item drew in the lines. Students who draw in lines to make the grid may need the diagram to determine the number of squares which suggests they are less likely to have the second multiplicative concept.

Additional possible explanations for pointing to squares on the smallest rectangle include choosing to point to squares even though more advanced strategies are known. Since the *2* x *6* was the introduction to the missing grid diagram, and it was a smaller grid, perhaps some students chose to count the small number of squares because counting a small number does not take long, and the diagram was unusual. So even though they knew more advanced strategies, they may have counted because it would not take long and provide certitude. Two students completed each of the grids, but their use of grouping strategies indicated MC2. These students, along with others who completed one or two of the grids, did not appear to use the grid to determine the quantity. Also, some students may have interpreted the directions such that completing the grid was a prerequisite to solving the RA problem. For example, one student started to draw lines for the *6* x *9* rectangle after drawing them for the two smaller ones, and then asked, “Can I just write a multiplication sentence?” (Student 1278). Stating “Can I just” infers that drawing in the lines was not necessary for a solution for this student, but past experience in school math may have encouraged the students to infer the need to complete the squares as part of the school problem-solving expectation. Completing the grid was not considered for scoring here because students may have completed the grid to solve the task, but there are alternate reasons as well. In future versions of the SG with protocol wording that discourages drawing in lines without purpose, adding lines to see complete squares will be considered as an indicator for Pre-MC or MC1.

MC consistency across three rectangle sizes. Of the 64 students who answered all three Rectangular Array items, sixty percent maintained the same score across the three items. For 37% of the students, two of the three scores were the same but the third score did not match. Twelve students' scores matched for items B and C, five students' scores matched for items A and B, with six students' scores matching for items A and C. However, for three percent (two students), all the scores were different. Although further talk-aloud information from these students might be useful to identify ways to make scoring more reliable, this opportunity was not available.

The 4x4 Array, Question 1 (Q1) Analysis

Question 1 (Q1) requires finding the number of tiles in a row of an array given the quantity of squares and the number of rows (see Figure 5). The Unitizing Scoring Guide developed with the BAM items was applied to performance on Q1 using the video and written work data given the problem's array context. This item is unique in that it is the only word problem in the set of tasks and it is the only problem where students find a factor instead of finding a product, thus requiring students to work in reverse. As with the Rectangular Array problems, the scoring opportunities are less robust in part because the task as stated does not encourage actions that require coordinating three levels of units in the activity (MC Elaborated). There are 77 cases with video for Q1 and there were two cases with insufficient evidence to produce a score, so 75 cases were scored. Over 90% of the students demonstrated MC with about one-third at MC1 and just over half of the students rated at MC2 level (Table 14).

Table 14
Question 1 (Q1) Percent of Students at Each MC

Q1 Multiplicative Concept	Frequency	Percent
Pre-MC	6	8%
MC 1	26	34%
MC 2	43	55.5%
No score	2	2.5%
Total	77	100%

Interpreting students' activity on Q1. Question 1 (Q1) was the first problem in the students' protocol interview, and the only item in a story problem format included in this study. The problem is included because it has specific directions to include an array diagram. In order to solve Q1, students needed to comprehend the following scenario: "Tom made a rectangle with 16 tiles. If there were 4 rows, how many tiles are in each row? Sketch a picture of this rectangle."

Interpreting the problem was not uniform. Ten of the 75 students (13%) interpreted both numbers in the story to be factors, making the number 16 a factor (the number of squares on the length of a side) instead of the total quantity of squares within the rectangle. Although more (four) of these ten students are from the classroom with more of the students who are learning English as a second language, the rest are evenly distributed across the other three classrooms (two per classroom).

Regarding the instructions to draw an array, all but two students followed this direction. Knowing that pre-MC and MC1 students benefit from activity such as interacting with the diagram, whether the drawing was completed first or the answer was shared first was tabulated. Of the students who drew arrays, slightly more students drew the array before answering the question (56%) than those who answered the question first (46%) (Table 15). Of the students rated Pre-MC or MC1, over four times as many drew the rectangle first (26) and six drew the rectangle after answering. One could infer that students may draw to help support their thinking. This suggests the prompt to make a drawing may help more students reason through the problem by encouraging the use of representation.

Table 15
Drawing Array First or Telling Answer First (Frequency)

Unitizing Level	MC Level	Draw Rectangle First (frequency)	Draw Rectangle Last (frequency)
1	Pre-MC	4	2
2	MC1	22	4
3	MC2	14	25
4	MC2	1	1
Total		41	32

Unitizing Scoring: Item Type Difficulty

Students' MC scores for each type of problem (Figure 14) are compared to explore issues related to consistency. The bar graph of MC levels for each of the problems (Figure 14 displays percentages for the MC levels. Discussion about the variation between cBAM and other items follows to guide item use and SG modifications.

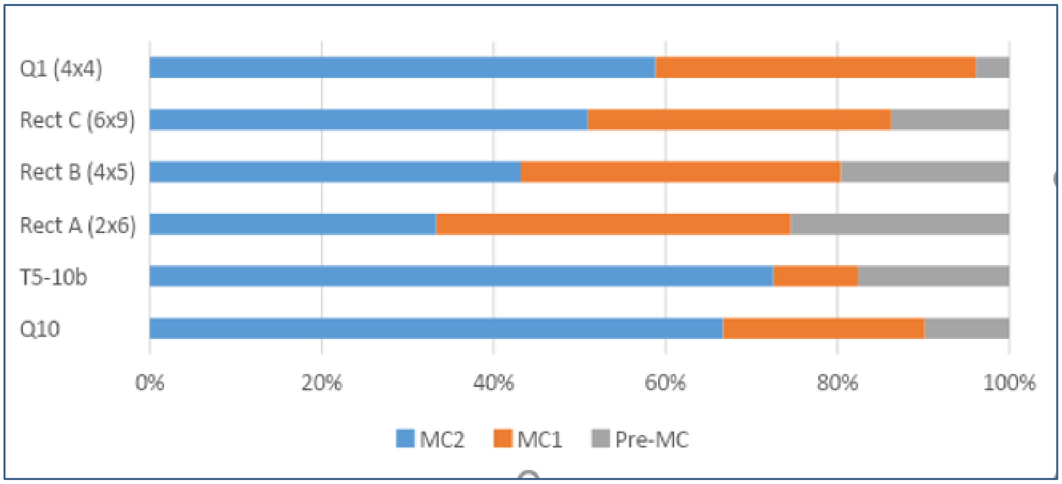


Figure 14 Percent of students with MC scores for all items: MC levels by item (n=51)

Comparing MC Classification on the BAM and RA Item Types – Role of Representation

The distribution of students' responses indicating a given Multiplicative Concept for Break Apart Multiplication (cBAM) and combined Rectangle Array (cRA) problems in shown in Table 16. Grey boxes highlight those students who gave the same response for both.

Table 16
Correspondence of Students' MC Ratings Across cRA and cBAM Problem Types

cRA / cBAM	Pre MC	MC 1	MC 2	Total
Pre MC	2	4	0	6
MC 1	5	4	1	10
MC 2	6	15	26	47
Totals	13	23	27	63

The goal of finding the number of squares in an array is the same for both activities. However, each activity is different due to the affordances of the diagram (complete squares vs.

covered squares) and the presence or absence of instruction with that item type prior to the interview. The distribution of MC scores for the two problem types is different - For example, the number of students with a Pre-MC rating is at least twice as high for the combined Rectangular Array (cRA) score as it is for the cBAM score (Table 16).

Looking at specifics to help explain differences, five students worked at Pre-MC for Rectangular Array item type (cRA) but were rated at MC1 on the BAM item type (cBAM). In addition, six students worked at Pre-MC on the cRA but rated at the MC2 level on cBAM. This means that 11 of the 13 students (85%) who did not demonstrate coordinating units on the Rectangular Array problems demonstrated coordinating units on the cBAM problems.

These students were observed interacting with the RA diagrams differently than with the BAM problems. More students used schemes with a counting or additive basis (Pre-MC) or actively engaged with the diagram, making marks or using fingers to keep track of how many same size groups were needed in order to find the number of squares (MC1) in Rectangular Array items than for the BAM items. The missing grid on the Rectangular Array is a distinctive feature for students who use the diagram to coordinate units, leading to the inference that students who had access to the diagram of BAM items and were familiar with the BAM process were able to produce evidence for more advanced MC because the diagram provided more support and students were more familiar with the process. It is also possible that the rater made assumptions about student’s actions regarding the BAM problems without confirming evidence from explanations.

Four students who demonstrated behaviors for Pre-MC on the BAM problem were able to show more unit coordination behaviors (MC1) on the RA items. Since the RA items have smaller rectangles, and students could draw in the necessary lines, one hypothesis is that the smaller rectangle size did not produce an overload for working memory, allowing these students to use the smaller array whereas the larger array may have been overwhelming.

Focusing on the array as a representation for multiplication, the BAM problem type uses larger rectangles (*4* x *18* and *5* x *14*) with complete grid diagrams and connects to recent instruction. The Rectangular Array problems use smaller rectangles (*2* x *6*, *4* x *5*, and *6* x *9*) with incomplete grids for the diagram which are not part of previous instruction or practice. This analysis suggests that the role of the diagram in the BAM problems was undervalued for the students who need the diagram to solve problems. Students receiving support from the BAM diagram may have displayed actions or shared phrases that provided evidence for more ability to coordinate units than these same students could display in the missing grid diagram of the Rectangular Array problems because they were more familiar with the procedure and recognized the use of tools could support finding their answer.

As noted earlier, students who may have to skip count squares for the *4* x *5* array chose to determine rectangle side lengths and multiply with tools to find the number of squares in the *6* x *9* RA. The BAM rectangles, even larger than the RA, had an even smaller percentage of the students at the Pre-MC level. One hypothesis is that when faced with counting a large number of squares, students look for more efficient strategies and use what they have learned about the relationship between the number of squares on the side of an array and factors to determine the number of squares in the array as the product. It is possible that students no longer thought counting would be the most efficient strategy for the larger rectangle, so the student switched strategies to a procedure learned in class. Counting side lengths to use multiplication facts are actions that may give evidence for higher levels of unit coordination given additional support for the rating during the interview. Gestures during counting side lengths can provide corroborating evidence for inferring MC levels.

In addition, the different affordances of the diagram seemed to matter. Although the goal of finding the number of squares in an array is the same, there is variation by item types. A pronounced difference in the item types is in the corresponding array diagram. The complete grid of BAM problems versus the incomplete RA grid may have influenced students' scores. Since the MC1 level is based on coordinating two levels of units in activity, the visible diagram and the familiarity of the setting may have influenced how students could reason about the quantities. The Rectangular Array problems did not have a complete array diagram, so this may have diminished some students' ability to reason about the quantity because the entire quantity was not visible.

Comparing MC Classification on BAM and Q1 Items –The Role of Interpretation

The overall distribution of MC scores for the Q1 task is closer to the distribution of cBAM scores (Figure 14) than for the RA item type. Sixty-two percent of the students' actions produced the same MC for both the BAM and Q1 activities compared to 49% when comparing the MC for the cBAM and cRA. Looking more closely, the consistency between the Q1 and cBAM ratings improves as the unit coordination level increases. Two of these seven students (29%) with a Pre-MC rating for cBAM also have a Pre-MC rating for Q1. Eight of the 15 (53%) students with an MC1 rating for cBAM have the same MC1 rating for Q1. Thirty-six of the 52 students (69%) with an MC2 rating for cBAM also have an MC2 rating for Q1 (Table 17). The BAM items include an array for a larger quantity while the array for Q1 is a smaller quantity.

Making a sketch is a required part of the Q1 story problem, whereas the array diagram is presented in BAM.

Table 17
Correspondences of Students' MC Ratings Across Q1 Story Problems and cBAM Problems

4x4-Q1	Pre-MC	MC1	MC2	Total
cBAM				
Pre-MC	2	4	1	7
MC1	1	8	6	15
MC2	2	14	36	52
Totals	**5**	**26**	**43**	**74**

An individual's scheme use was not consistent on these two items. The question arises of how the student's interpretation of the problem influences the students' thinking. Over half of the 26 students rated MC1 (14 students) were rated at MC1 for Q1 but were rated at the MC2 level for the cBAM score. The researcher inferred student were coordinating units in their head (MC2) for the cBAM problem, but using the activity of the problem and diagram to coordinate units in the story problem. The story problem is slightly different in that the student is asked to find the missing factor instead of the product. One hypothesis is determining the strategy use from the story may have affected the need to draw but the scoring guide did not distinguish what might be necessary to solve the problem versus the actions used to produce the answer or explain the problem. The requirement to draw an array may have encouraged students to do more with the drawing, which would result in more observable actions for MC1, important considerations for the SG revisions.

Comparing the scores from the cBAM and Q1 takes into account three key item type differences, (1) the absence or presence of an array diagram as part of the problem, (2) the size of the factors, and (3) variation in the linguistic aspects of the problem – interpreting the task or communicating one's thinking. As discussed earlier, 10 students, in misinterpreting the problem, created a break-apart multiplication situation using the *total* number of squares in the problem description as a factor instead of the product. These students then broke apart a *16* x *4* rectangle instead of finding the side length for a rectangle with 16 squares. One interpretation is that the students did not know how to reverse the operation of multiplication to find the factor when given the product. Students, such as English as a second language or with other language comprehension concerns may find interpreting the problem more difficult which may affect solution strategies.

The smaller factor and product size in Q1 may have impacted the use of unit coordination in the problem solution. Four students (5%) presented Pre-MC on the cBAM but MC1 on Q1 (chart 6). The story problem, with smaller factors where the student made their own diagram (Q1) was a setting where four students were able to use the diagram or story to group numbers when they were not able to demonstrate this on the BAM problems. Six students (8%) were observed using grouping without the use of the diagram for Q1 but needed to use the cBAM diagram for grouping to solve the cBAM problems. Perhaps these students unitized more on the story problem because the smaller quantity is easier to interiorize, and students at this point in the curriculum are studying 4's facts.

Other students needed to use the array or other Q1 activity to coordinate two levels of units but appeared to be able to coordinate two levels of units prior to activity on cBAM problems. Fourteen (19%) students were rated at MC2 on cBAM but MC1on Q1. These 14 students did not interact with the BAM diagram except to find the number of squares on a side and split the rectangle as prescribed in the BAM instructions. Given the calculator and multiplication chart as tools, these students may have been able to use their understanding of BAM activity to present actions that infer using MC2. These same students used tiles or drew squares to determine there would be 4 rows of four using 16 tiles. These students could not partition the 16 tiles in an interiorized way, such as counting back by fours, repeatedly subtracting four or using fact family information for $4 \times 4 = 16$ to determine there are four rows of four in the array. For these students, working from the product to find the factor needed more visual support. Of these students, five of the fourteen drew lines in the rectangular array problems, but nine did not. This indicates the use of tools and a familiar procedure may point to coordinating two levels of units but in a less familiar setting, the activity of problem solution is needed to coordinate units.

From the theoretical base for scoring, we know that students as the Pre-MC and MC1 make more use of the diagram and other activity within the action of solving the problem than do students who use interiorized strategies for multiplication. Comparing the MC scores on Q1 and cBAM, the analysis suggests that the size of the factors/product may impact students' grouping options and choices, matching general expectations for learning multiplication in CCSSM (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) or in the OGAP problem structures (Ongoing Assessment Project, January, 2017).

MC Variation

Score variation across items is expected when students have choices for problem solution. The curricular instruction for doing BAM created a structure with some student choice. For example, choices in how to break apart the rectangle in BAM can make the next steps for finding the number of squares harder or easier. Also, students can choose strategies that are not the most efficient or ones they think they are supposed to use based on past experiences or the way directions are stated. Students' past experiences influence background knowledge for problem solution, too.

Confusing situations. As students move towards more sophisticated reasoning, a student may use less advanced reasoning to find a solution when confused within that situation. Students may fold back to easier models (Pirie & Kieren, 1994). Seeking the solution to an overly hard problem for a student-led to random answers to end the process or reverting back to a simpler unit structure. As students selected how to break apart the larger quantity array in the BAM items to make finding the number of squares more manageable, some students picked a number for the split that created an easy-to-find quantity paired with a difficult one. In seeking to solve the more difficult part, some students returned to less efficient strategies, indicating less ability to coordinate quantities. It is likely that if the student chose to create a more typical split, the solution strategy would have been scored differently. This points to the variation in scoring potential. How students choose to solve the problem influences what methods they use which may affect MC inferences.

Student decisions. Also, students choose strategies without attempting to show what the observer will think is the most advanced strategy, they are working to solve the problem- it is the observer making second-order models. As students find solutions to the task at hand, they use “conceptual tools that are designed for specific purposes and situations.” (Lesh & Carmona, 2003, p72). However, the student’s interpretation of what is appropriate in-the-moment may not be the student’s most efficient strategy, as observed in the smallest Rectangular Array item. Students who drew lines because they thought they should, whether from interpreting the prompt or past experiences made personal choices that were not based on showing the most efficienct strategy alone.

A starting point. This researcher is in agreement with Cognitively Guided Instruction (CGI) authors’ view of CGI analysis: “We do not propose that the CGI analysis completely accounts for all children’s mathematical thinking and problem-solving, but it is a place for teachers to start.” (Carpenter, Fennema, & Franke, 1996, p. 13). Recognizing MC from students’ actions provides a starting point for instructional decisions, but variation between items is expected because elements of specific problems can impact students’ solution paths. For example, the multiplication table and calculator tool use allowed students to find a calculation based on a procedure they learned – whether they really understand how the procedure works or not.

Although this work determined a rating based on a single item, it is more appropriate from a reliability as well as a practical view to use multiple events to form a conclusion about a general unitizing level. That said, MC scoring guide background knowledge supports gathering information from a single event for *in the moment* data. The researcher or educator interprets the student’s actions at that time and place – looking at what child can do in the given situation, with the constraints of that situation, to identify a dynamic understanding and then help the student based on that understanding.

In the next section, all available items are used to develop a combined MC score for each student to predict what students might do on two-digit by one-digit problems where there are no special directions or a special context such as using the BAM method.

Establishing MC Levels for Students Based on All Available Items

In order to develop an overall unitizing rating from the available data, all available MC scores from the six items in the protocol interview were used to find the median MC score for each student, producing an MC level for 77 students (Table 18). This combined MC (cMC) was determined using the available Q10, T5-10b, RA-A, RA-B, RA-C, and Q1 items for each student, although the amount of evidence for each student varied. This researcher determined the cMC from the available data instead of using only cases with scores for all items because there was a disproportionate number of students with Pre-MC and MC1 ratings who had incomplete score sets. In order to provide a better representation for comparing how students demonstrated MC on array problems with strategy choices for two-digit by one-digit multiplication problems, all cases are included. There were 10 scores where the median was halfway between MC levels, three between Pre-MC and MC1 and seven between MC1 and MC2. In keeping with past decisions, these scores were raised to indicate the more advanced level of reasoning, since that reasoning was demonstrated in some of the items. The median MC score becomes the student's MC level for making comparisons with students' actions regarding performance on the two-digit by one-digit multiplication problems.

Table 18 displays the percentages of students categorized at each MC level. There is a greater percentage of students in the Pre-MC level (12%) when using all the items than for the cBAM (9%), and a smaller percentage in the MC2 level (58%) versus the percentage in MC2 using just the cBAM (70%) data. As discussed earlier, the role of the incomplete grid did increase the number of Pre-MC and MC1 scores. The strategy use and accuracy for two-digit by one-digit multiplication work is reviewed in light of the median MC score based on the available student array multiplication scores in the next section to see if the MC rating has predictive value.

Table 18
Percentages of Students at Various MC Levels Using All Available Data

Unitizing Score	MC Level	Number of Ss	Percent
1	Pre MC	9	12%
2	MC1	23	30%
3-4	MC2	45	58%
Total		77	100%

Beyond Determining the MC Score: Using it as a Predictor for Students' Actions on Two-Digit by One-Digit Multiplication

In Task 6 students solved the two-digit by one-digit multiplication problems with a personally selected method.   Hypotheses regarding two-digit by one-digit calculation performance based on the array multiplication MC include the following:

- Students who are rated as pre-MC or MC1 will be more likely to choose a method that involves a figure or drawing that can help support the calculation.
- Students who are rated as pre-MC are not expected to think about quantities in terms of some number of groups with *y* quantity in a group.
- Students in MC2 are more likely to use a strategy that involves grouping of groups.
- Students in MC2 are more likely to produce an accurate answer for a multiplication problem.

To test these hypotheses, students' unitizing scores from the array multiplication problems are compared to general calculation strategy choices and accuracy for the two-digit by one-digit multiplication problems. For each student, performance on the 2-digit by 1-digit multiplication is matched to the MC level using the student's MC median score from the array multiplication problems. There is no unit coordination scoring for the 2-digit by 1-digit multiplication (Task 6) solutions because the interview protocol was not designed to elicit this type of evidence.

Task 6 problem solutions were coded for general calculation strategy choice based on strategies described by the curriculum, such as array use, expanded form, standard form, etc., and calculation choices and accuracy. Sixty-two percent of the students calculated *6* x *21* correctly. Students who demonstrated Pre-MC in array multiplication are less likely to get a problem like *6 x 21* correct according to this data and students who demonstrated MC1 have about an equally likely chance of getting a problem like *6* x *21* correct according to the data in Table 19. However, students who can coordinate two levels of units in their head MC2 got this problem correct about four times more than students who were incorrect. Students' solution methods for *6* x *21* as well as solution accuracy are presented in Table 20.

Table 19
Calculation Accuracy for *6x21* Problem by the Multiplicative Concept Levels Determined from the cBAM

Multiplicative Concept Levels:	Inaccurate Response	Accurate Response	Total
Pre-MC	10	3	13
MC1	10	12	22
MC2	8	33	41
No MC level			1
Total	28	48	77

Table 20
Frequency for Multiplication Strategies by Accuracy for 6 x 21

Solution Method	Inaccurate Frequency	Accurate Frequency	All Frequency
Standard Column Multiplication	2	2	4
Standard method with something else	1	2	3
Expanded Form Multiplication	2	24	26
Break Apart Multiplication	10	15	25
Repeated Addition of Larger #	2	3	5
Repeated Addition of Smaller #	1	1	2
Draw pictures (Tiles or Base 10)	11	0	11
Mental Math	0	1	1
Total	**29**	**48**	**77**

Close to two-thirds of the students used one of two strategies - either break apart multiplication solution strategy (31%) or the expanded multiplication solution strategy (34%). Up to this point, there was little instruction about standard multiplication calculation methods, but classes 117 and 120 had more exposure to the expanded form for multiplication than other classes. *Expanded* multiplication is a blend of break apart multiplication and standard multiplication features (see Figure 15). The BAM multiplication strategy is typically taught prior to expanded form.

$$\begin{array}{lcr} 23 & = & 20 + 3 \\ \underline{\times\ 6} & & \underline{\times\ 6} \\ & & 120 + 18 = 138 \end{array}$$

Figure 15 Example of Expanded Form for multiplication

Students can solve the two-digit by one-digit multiplication problems in Task 6 with any method. Students who exhibit MC1 behaviors on the array multiplication problems are expected to prefer to use a method that involves a figure or a drawing to support thinking about multiplication and they are expected to focus on counting, as there was no evidence for any type of grouping in their earlier work. Students who exhibit MC2 behaviors on the array multiplication items are expected to use a strategy that involves grouping groups. As students choose their solution strategies for Task 6 problems, *6* x *21* and *34* x *7* the solution strategy can provide evidence to support or refute these preferences.

Given the identified solution choices listed in Table 20, the students' array multiplication MC level is aligned to the strategies each student selected to find answers on the *6* x *21* problem. Table 21 shows the solution method selected by students at a given MC level:

Table 21

Frequency of MC Level by Strategy

Solution Method	Pre-MC	MC1	MC2	Total
Standard Column Multiplication	0	2	3	5
Expanded Form Multiplication	3	6	18	27
Break Apart Multiplication	3	4	18	25
Repeated Addition of Larger #	0	2	3	5
Repeated Addition of Smaller #	0	1	1	2
Draw pictures (Tiles or Base 10)	3	8	1	12
Mental Math	0	1	0	1
Total	9	24	44	77

Students with a Pre-MC score on array items used a variety of different strategies such as drawing pictures, BAM and expanded form. About a third of the students drew pictures, another third attempted BAM and another third used expanded form. About two-thirds of the Pre-MC students did make a diagram or draw pictures, but the only students who used the expanded notation answered correctly.

Students demonstrating the MC1 on array items were more likely to use the *expanded form* or to draw pictures to find the answer to *6* x *21* but every type of strategy was used by at least one student exhibiting MC1 behaviors on array problems. These students are expected to understand the idea of grouping groups through the activity of solving the problem. In the *6* x *21* problem MC1 students most frequently drew pictures or used the expanded form, with students who used the expanded form most likely to get the correct answer. There were mixed accuracy results for students who used repeated adding or BAM to solve the problems. Students who drew pictures did not get correct answers. This suggests that students who used the most recent strategy learned in the classroom and/or used tools were more likely to get accurate answers for the two-digit by one-digit problems than students using drawings to solve a two-digit by one-digit number multiplication. Also, some students may learn how to do a solution procedure and be successful using the procedure, even though the underlying conceptual understanding regarding nesting units within units is not fully established. Students with correct answers may look successful and do well on tests but may need more opportunities to gain conceptual understanding/number sense for when it is needed to learn a more advanced concept. Over three-fourths of the students demonstrating MC2 used *expanded form* or BAM strategies.

Table 22 shows the choice of strategies for correct and incorrect answers. Overall, students who used the expanded form were likely to answer correctly (92%), students who used the BAM method answered correctly about 60% of the time, and students who were drawing pictures to solve these problems only produced correct answers once (8%). Next students' accuracy is shared for the various MC levels and strategy choices.

Table 22

Performance by MC Level (from all available items) and Solution Method on the 6 x 21 Problem

Solution Method	Pre-Multiplicative Concept (Pre-MC)		Multiplicative Concept 1 (MC1)		Multiplicative Concept 2 (MC2)		Total
	Inc.	Correct	Inc.	Correct	Inc.	Correct	
Standard Form Multiplication	0	0	1	1	1	2	5
Expanded Form Multiplication	0	3	0	6	1	17	27
Break Apart Multiplication	3	0	4	0	3	15	25
Repeated Addition of Larger #	0	0	1	1	1	2	5
Repeated Addition of Smaller #	0	0	1	0	0	1	2
Draw pictures (Tiles / Base 10)	3	0	7	1	1	0	12
Mental Math	0	0	0	1	0	0	1
Total	6	3	11	12	7	37	77

The first problem, *6* x *21*, does not require regrouping for a correct solution, so a correct solution can be achieved without understanding aspects of place value if the digits are aligned. Students demonstrating the Pre-MC level used varied strategies, but only students using the expanded form strategy had correct answers. Students demonstrating the MC1 in the array problems who used expanded form had accurate responses whereas students who drew pictures did not have accurate answers. This suggests that students who did not demonstrate the ability to coordinate two levels of units in cBAM were more successful in doing the *6* x *21* calculation. Given that no regrouping was required to solve *6* x *21* when using a defined procedure such as expanded form, where the digits that represent the quantity are used in a set procedure which can be learned as a set of steps, students may determine accurate answers even though their understanding of place value units may be weaker.

Due to protocol constraints, not all students completed the *34* x *7* multiplication problem. The protocol indicated that if the child was getting tired due to the burden of the first multiplication problem, this last problem would not be used. Sixty of the 77 students did do this problem. Sixteen students only completed the first multiplication, *6* x *21*. One student attempted but did not complete either problem. The strategy the student used is evident, so the data for the student's strategy is used, but there is no accuracy score for this student.

For the 60 students who completed both problems, 78% used the same solution strategy for both: 16 of the 17 students who used BAM used the BAM strategy both times, 21 of the 25 students who used expanded form used this strategy both times and 10 of the 13 who used other methods used the same strategy both times, for example, repeated addition or drawing pictures. This suggests that students view both problems as similar, or that many students viewed their strategy as successful and worthy of repeated use. Table 23 is a summary of the student accuracy and strategy use when considering both two-digit by one-digit problems.

Table 23
Multiplication Accuracy by Problem and Strategy

	6 x 21		*34 x 7*		*6 x 21*	*34 x 7*
Strategy	Correct	Incorrect	Correct	Incorrect	All	
Break Apart Multiplication	15	14	7	11	29	18
Expanded Multiplication	26	2	17	9	28	26
Standard Form Multiplication	1	1	3	2	2	5
Other	5	11	4	7	16	11
No data	1				1	0
Did not complete					1	17
TOTAL	48	28	31	29	**77**	**77**

Success rates were expected to be low for these two problems given little exposure to 2-digit by 1-digit problems. Of the 60 students who did both problems, 28 got both right (47%) and 16 answered one of the two correctly (27%). Of these 16 students, 13 answered only 6 x 21 correctly and three answered only 34 x 7 correctly. For the 16 students who only completed one problem, slightly less than half (44%) were able to produce an accurate answer. Twenty-five of the 76 students (33%) did not get any of the attempted problems correct. Table 24 shows the problem accuracy. The accuracy on both items is viewed using the lens of MC levels (Table 25).

Table 24

Accuracy Scores for Number of Completed Multiplication Problems

N correct	0	1	2	Total
Both	16	16	28	60
Only 6 x 21	9	7	.	16
Neither				1
Total	25	23	28	77

Table 25

Accuracy Scores by Multiplicative Concept Level for Completed Multiplication Problems

	2 problems			1 problem		No problems	Total
# correct	0	1	2	0	1	Null	
Pre-MC	6	2	2	3	0	0	13
MC1	6	2	10	3	1	1	23
MC2	4	12	16	3	6	0	41
Total	25	16	28	9	7	1	77

Students’ accuracy on the two-digit by one-digit multiplication problems increased with an increase in MC levels. Looking only at the students who completed both problems, half of the six students at the Pre-MC level responded incorrectly, but about half answered at least one problem correctly. Similarly, of the students (14) who demonstrated MC1, half answered both questions incorrectly, and half answered at least one problem correctly. Of the 40 MC2 students who completed both, 15% answered both questions incorrectly and 85% of these 40 students answering at least one question correctly. The data suggest that students who demonstrate greater ability to coordinate units are more likely to be accurate, but all who are accurate are not using the same MC thinking.

In this section, students' unit coordination scores from array problems are compared to the accuracy and the strategy type a given student selects for up to two two-digit by one-digit multiplication problems to answer the third research question.   The results of this study have only a few instances of future performances available per student demonstrating a given MC level.

Students who demonstrated higher MC on the array multiplication problems were more likely to get a correct answer on the two-digit by one-digit problems completed in any fashion (Table 25). Considering strategy use, students demonstrating MC2 or MC1 were most likely to use Expanded Form or BAM. All students using Expanded Form were accurate except one student (demonstrating MC2). The variation in accuracy with BAM use is interesting in that none of the four students demonstrating MC1 who chose to use the BAM strategy were accurate but 15 of the 18 students demonstrating MC2 who chose BAM strategy were accurate. Although students can get accurate answers using procedural calculation approaches to find answers to a multiplication calculation such as Expanded Form, the ability to solve problems where there are relationships between the numbers or quantities, as demonstrated in the BAM strategy, does seem to benefit from coordinating units in an interiorized way. This suggests that it may be useful to specifically identify unit coordination ability/skill using formative assessments as calculation accuracy is dependent on multiple factors. Calculation success may mask the need to develop more unit coordination skill needed to understand situations where number relationships are critical, such as fractions or algebraic equations. The next section is focused on the first research question, "Given what is known about unit coordination, how might a continuum of students' unit coordination be revealed through students' array multiplication solution processes?" and the last section is focused on the second research question: To what extent might the identified unit coordination schemes provide a model for use in teachers' formative assessment processes?

8. IDENTIFYING UNITIZING ABILITY - WHAT HAVE WE LEARNED?

A given student's unit coordination scores were more likely when at least one of the ratings (Pre-MC or MC1) indicated the student used the activity of solving the problem to develop the reasoning she used to determine a response. This suggests that when a student is using the activity of the problem to coordinate two levels of units or may not be coordinating units in their reasoning, the types of support in the diagram or prior knowledge related to the problem structure may affect student thinking. This section highlights lessons learned regarding (1) gathering scoring evidence about MC levels as well as articulating factors that impacted that process and (2) unit coordination and instructional implications for building mathematical competence, especially in the area of number sense. Key findings are summarized and SG use with arrays is connected to well-known curricular learning maps.

Gathering Evidence of Unitizing Ability/Skill

Taking a learning sciences perspective, the measurement approach and descriptors within the array multiplication SG are based on cognitive theory and research, where students' gestures and use of diagrams as representations for multiplication were given significant attention alongside verbal explanations. In the SG, identified gestures, explanations, and diagram use during typical early multiplication explanation of work are matched to MC to encourage classroom decision-making with data on how students think about quantities.

In order to make inferences about unitizing ability, problematic activities are needed to invoke reasoning about a quantity such that students' thinking can be revealed through speech, actions or written work. The SG was used to interpret gestures, speech, and written work and produce ratings that represented inferences about a student's grouping practices/unit coordination.

Gestures. Students' movements can be an indicator of unit coordination. As students interacted with the diagram, students' hand motions over squares in the array diagram were viewed as an indicator of unit coordination. Students whose pointer fingers make more pronounced jumps while counting squares were considered likely to be thinking about each square as an individual unit versus students whose finger glides across a row or column of squares. Smoother finger-pointing motions are seen as an indicator of thinking of the side length as a unit of squares.

Students' gestures when describing the product were incorporated into the SG. Making a circular motion over a group of squares to indicate the product of the side lengths, helped support related talk about the quantity as an amount in a column. Students who pointed at a side length, motioning back and forth across the side length were more likely to focus on the factors as the numbers needed to find the answer. These students' reasoning often matched the number of squares on a side to the number in the expression used to find the answer versus indicating each column as a group for each square in the referent row (or vice versa).

In addition to student gestures, marking or other drawing on the diagram or elsewhere provided evidence for inferences about MC. Students marked squares or added lines to complete a rectangular array of squares be able to identify where squares are located in the diagram. How the student interacted with the diagram helped determine the MC rating.

Verbal. Written or spoken explanations describing how to find the number of squares could include counting sequences, descriptions of grouping, emphasized counting or references to number sentences. At the beginning of the think-aloud protocol, students were encouraged to share their thinking, but in the process of doing the work, students may have been so focused on solving the problem or not mindful of explaining and consequently did not provide an adequate trail of their reasoning. To minimize this in the future, interviewers can use phrases to encourage students to share their thinking without creating bias. Successful phrases are more open-ended such as “Can you walk me through how you did that?” used with Student 1273, “How did you know to…” used with Student 1649, “What are you thinking?” used with Student 1955, or, “Explain what you did there-” used with Student 1974.

Calculations. How students calculated the number of squares could demonstrate MC reasoning. Writing numbers in counting order, writing multiples or comparing the whole rectangle dimension to the sub-rectangle amounts as arrays show increasing amounts of unit coordination. When students selected to use a tool such as a calculator or multiplication chart opportunities for obtaining MC evidence were substantially reduced or eliminated. Consequently, to obtain more evidence for MC, less use of calculation tools is suggested.

Triangulating evidence. The opportunity for triangulation of speech, gesture and written work strengthens the reliability of a unitizing score with consistency across the three modalities. In situations where a student’s actions and speech do not match, such as pointing in a circular motion for the number of squares and indicating grouping but little auditory explanation about grouping, the mismatch between students’ gestures and their speech may indicate partial understanding or the ability to make some sense of the relations but not the full concept yet. Research has indicated that in such situations, students may be particularly receptive to targeted follow-up instruction, with especially productive results (Church & Goldin-Meadow, 1986; Goldin-Meadow, 1997). Since the SG can help to identify when there is a mismatch between what a student says/writes and the meaning of their gestures, the SG may increase awareness of important timing for effective instruction.

The scoring guide (Table 26) was revised by distinguishing spoken, written and gestural cues for each level instead of putting all the expected actions for operating at a given MC together.

Table 26
Scoring Guide -Revised After Data Analysis

MC level	Grouping Model for finding a quantity	Auditory Observable Actions	Written Observable Actions	Gestural Observable Actions
Pre-MC (count all) UNITARY COUNTING (Rating = .5)	*UNITARY COUNTING* Each item is a unit The rectangle is seen as a set of **individual objects**. Keep track one at a time without other evidence of grouping	**Counting one at a time** to keep track of the number of squares.	For example, a student **might put a pencil dot** in each square or represent the quantity with tally marks and count the marks.	**Touch squares individually** to count by ones
Pre-MC (count on) (Rating = 1)	*ADDITIVE COORDINATION* Use of addition to find a quantity in units. Student does not coordinate a group of units. The rectangle is seen as a **set of unitary objects in each part, and the parts are added together.** May keep track of just top row - “11” and “7” so that the student can think about all at the same time –not recognize each square in the top row is part of a column of squares to include in count.	May do a verbal count by ones, record the amount and then do a verbal count for another amount, and may verbalize **adding the two groups together with a count-on strategy**	Count by ones for each sub-rectangle of break-apart and **add amounts together using the rectangular array diagram, determining one section, then counting on for the next section.**	**Touches the squares with more of a jump** between squares as if each square is a separate unit.
MC1	*COORDINATING 2 LEVELS OF UNITS IN ACTIVITY* (Rating = 2) Students can **keep track of one level of unit and use her actions to coordinate 2 levels of units** in the array	Verbal counting with an **emphasis on the last number** (For instance,1,2,**3**, ... 4,5,**6,** …7,8, **9,** etc.)	May add drawings in the activity of solving the problem; may mix up the operation symbol – put addition sign where multiplication sign belongs; may write	**Touches the squares on a row or column with a bit of a jump,** orientation for counting each row stays the

MC level	Grouping Model for finding a quantity	Auditory Observable Actions	Written Observable Actions	Gestural Observable Actions
	Uses the rectangle to keep track of the quantity Sees multiple squares say a row or a column both as a group and as singleton units in doing the problem.		entire number in the calculation but only use the tens digit to compute; may use doubling and halving strategy	same and pauses at the end of the row or column. May see a hand sweeping gesture or as if drawing a line over the squares being counted on the diagram May touch row or column to keep track of multiples (ie row or column quantity)
Emergent *MC2*	*INTERIORIZING 2 LEVELS OF UNITS* (Rating = 3) The student can keep track of 2 levels units **prior** to problem-solving activity. Thinking in composite units where each is also considered a distinct quantity, such as 5 groups of 4 rows [5 of 4-units].	**Verbal count in multiples** or with verbal/gesture indication of matching row/column to the array area without having to touch each area unit. For example, extends a finger for each verbal or whispered count - 4, 8, 12, 16, or, touches a row of 9, touches the next row and says 18; or, "eight and eight is 16, then 16 plus 16 is 6 +6 is 12 and 10 + 10 is 20, so then 12 + 20 is 32, so its 32"	May write out: "eight and eight is 16, plus 8 is 24 plus 8 is 32 Writing often includes an equation(s) of some sort. Writes total number of squares on each sub-rectangle and knows to add these.	When counting squares on a side length, touches the squares on a row or column with a bit of a jump. For example, extends a finger for each verbal or whispered count - 4, 8, 12, 16, or, **touches a row of 9, touches the next row and says 18;** Making a sweeping motion over a row of squares as well as a student **explaining how putting groups of quantities together**

MC level	Grouping Model for finding a quantity	Auditory Observable Actions	Written Observable Actions	Gestural Observable Actions
Elaborated *MC2*	*COORDINATING THREE LEVELS OF UNITS IN ACTIVITY* (Rating = 4) Student can coordinate two levels of units **prior** to activity and uses his action to coordinate 3 levels of units within the activity of the problem solving. While actively working, can **keep track of rectangles within the larger rectangles and the larger rectangle at the same time**,	For example, "10 rows of 4 is 40 and 8 x 4 is 32, so 18 x 4 = 72" **students' reason with 10 rows of 4 units and 8 rows of 4 units while interacting with diagram hands, etc**. to know that the 72 is also 18 rows of 4 in addition to the rest.	May record numbers to keep track of sub-totals they are finding in their head. Work out calculations – cannot determine rating with calculations without writing may have memorized algorithm; writes equations that include the entire rectangle as equal to the parts or equal to the number of singletons; Labels the sub-rectangles with quantity-often with equation.	Students reason (perhaps out loud) with 10 rows of 4 units and 8 rows of 4 units while interacting with diagram- pointing with hands, etc. to know that the 72 is also 18 rows of 4 in addition to the rest.
Emergent *MC3*	*COORDINATING THREE LEVELS OF UNITS* In their mind, use the relations of the rows and columns for two different groups and connect them	Students say, **"3 rows of 5 is 15 and 11 rows of 5 is 55 while also realizing that the 3 rows of 5 and the 11 rows of 5 make 14 rows of 5 which will be 70.** May find amount for 4 groups of 5 and then realize 12 groups of 5 is the same as 3 of the 4 groups of 5	Writing might include 3 x 5 = 15, 11 x 5 = 55, 14 x 5 = 15 + 55 = 70 and then students can indicate 5 groups of 14 is 70 or 11 groups of 5 and 3 groups of 5 is 70.	Students count squares in a diagram with finger moving along as a slide along the row or column
	UNKNOWN (Rating = 0)			

Influential Factors for Gathering Data about Unitizing Ability/Skill

Factors that influence the reliability of a score, as well as factors to consider if comparing items are reviewed in this section. When SG use was expanded to other array multiplication problems besides BAM, the multiple item types and the resulting score variation helped identify elements that impact unitizing scores in students' array multiplication solution processes. Elements include the problem situation, the type of diagram or other supports for activity, and communication, both the examinee understanding the problem to be solved and the interviewer's ability to elicit students' responses for making inferences Since the MC score is an inference about a student's reasoning in a given situation, recognizing how these elements may vary in the different problem situations can help make sense of the resulting scores across various activities.

The role of the situation in making MC inferences. Recognizing that unitizing is observed through a conceptual analysis of students' problem-solution work, the SG is designed for use in a situation that is problematic for the student. When students' work was too easy or too hard, score determination was tricky or inconclusive. The problematic situation provides the activity for the scheme to be identified from students' actions. If the teacher is using this as an embedded assessment, the interviewer/rater is more likely to know the student and be able to identify if the selected task is appropriately problematic. When the student is actively engaged in problem-solving the rater has the opportunity to observe actions and hear explanations about the actions' purpose that are more likely to be evidence of students' reasoning for why they selected a given action as well as explaining the actions. In the video data, there is evidence of less available data when thinking processes were not activated as much as needed in situations where the problem was too simple or a student's thinking processes shut down if the problem was too hard. The data demonstrated that student solution of an overly easy problem led to the use of typically less efficient strategies. As in the case of the Rectangular Array *2* x *6*, an overly easy problem may be solved with enough ease that more sophisticated methods are not used. It does not take much more time to count each square for the *2* x *6* than to count six squares for a side length and double it.

Familiarity with the task can make a difference as well. Students who are doing break-apart as a rote procedure to follow are less likely to demonstrate thinking in ways that will reveal unit coordination because it will be harder for students to remember why this procedure is appropriate. As a classroom assessment, the person gathering data is likely to be aware of the amount of prior instruction and can take that into account when reviewing results.

In some situations where the instructional steps were memorized students could look more capable by following a rote procedure leading to an inflated rating on that problem. From observing interviews, some students can produce an answer using a strategy successfully (say split rectangle in half, or split into rows of 10 for one part, and the rest for the other part). The protocol interview includes doing the BAM task with a different break apart choice, and when students were asked to break apart the rectangle in a second way, some students could not perform the BAM another way – they only knew one way using the rote procedure. These students were able to complete the first BAM without much extra diagram interaction the first time, but they were not able to complete a second BAM another way without significant interaction or other activity. Consequently, asking the student to complete a BAM a second time using a different way may be an important step for gathering evidence about how the student is reasoning about the quantity.

The use of tools such as the calculator and multiplication table hindered the observer's ability to identify an MC level. The initial test from which the items were extracted may have allowed tools because the item was designed to determine if students understood the steps for the BAM procedure. However, in this research study, the use of tools instead of student's invented solution methods eliminated opportunities to observe student thinking. When the assessment tool is designed to make judgments about student thinking and cognition, the calculator or multiplication table permitted use should be evaluated carefully.

In summary, embedded assessment in classrooms can provide opportunities for gathering information about unitizing or other aspects of mathematical thinking. Classroom assessments are often administered by people who know the students and can gauge what is problematic and at an appropriate range of challenge. The importance of a problematic situation for gathering data about unitizing skill, or other aspects of mathematical thinking should not be underestimated. The amount of challenge needs to be within an appropriate range. Consequently, the prompt for thinking needs to elicit thinking from students even though there may be a wide range of student abilities, and scoring needs to take into account the relative challenge of the situation. This means that in gathering data for unitizing scores, the interviewer will need to pay attention to the way the student is engaged in the problem solution. If the student solves the problem in a rote manner, it will be especially important to ask the student why these actions are appropriate.

Determining a score is not possible without evidence for the students' thinking. People administering classroom assessments can dismiss evidence from routine, non-problematic activity. The evidence cannot be obtained if the student is not actively processing the problematic situation. Training in the use of the SG is designed to help people observing students in classrooms to learn how to connect students' written, spoken and gestural actions in problematic situations to MC levels. This is in hopes that scoring guide use will increase the opportunities for making inferences about unitizing – and in particular, unit coordination.

The role of the diagram as a representation for multiplication. At the level of early multiplication, activity with an array, use of fingers or other items has been shown to support thinking about grouping. Pre-MC and MC1 students are expected to interact with a diagram or use other activities to solve problems, where the array diagram is viewed as a representation for multiplication. The use of the diagram makes the problem accessible for some, may be limited support for some and not needed by yet others. Consequently, the potential affordances of the diagram in the problem need to be considered when comparing scores. Comparing scores from the three problem types, (1) BAM, (2) Rectangular Arrays, and an (3) array-oriented story problem, the Rectangular Array items with part of the grid covered posed the greatest difficulty. If a student did not add the lines, the diagram would not be as supportive of coordinating two levels of units using the diagram or counting squares. Some students did not include squares they could not see from the count. This suggests that the incomplete grid may have been problematic for students who are depending on the diagram to help them look at a quantity as a group of groups, such as x rows with y in a row.

The role of communication. The story problem item solution is designed to produce a small grid, but not all students interpreted the problem as designed. Some students' interpretation did not address that one number represented the total number of squares (product) within the problem wording versus a factor. This created non-standard responses for the story problem. Although students' actions can be indicators for unitizing, for story problems the situation in which the actions occur will be less uniform due to multiple interpretations of the problem. This suggests that students' solution activity may not match the expected actions on the Unitizing Scoring Guide. Variation in opportunities to present understanding may arise from problem interpretation. The story problem (Q1) misinterpretation was consistent, leading to the potential to use this problem along with an inference for the misinterpretation.

Eliciting reasoning. In the protocol interview, the interviewer continually monitors the student and encourages the student to provide evidence regarding the purpose of an action, i.e. *why* do a particular move as well as encouraging the student to *describe* the move, without changing the direction or flow of the student's thinking. The interviewer's questioning techniques matter – a student's problem-solving processes need to be revealed but not altered as students transform quantities to solve problems. In the video data, some students did not share their thinking without a prompt, eliminating the verbal source of evidence. Prompts are important, but over prompting can alter thinking. Students were less likely to share their reasoning without a reminder to do so. Not all students readily shared their reasoning, in part due to knowledge of the English language.

To increase opportunities to learn about students' reasoning about a quantity, interviewer training, the use of additional questions, and expanding opportunities to observe students need to be further explored. Training in how to support students sharing thinking for the interviews in this study was focused on encouraging students to share their thoughts. For future elicitation of student thinking, training to appropriately encourage students to share *why* an action was chosen as well as describing their steps could lead to more scorable information. Although the initial interview questions did not support finding evidence for coordinating three levels of units, asking students to explain the quantity in terms of the entire quantity as amounts in a row or column as well as in singletons, or, posing the question of finding a new quantity of squares if *x* rows or columns were added and what the total might be, are questions that could support identification for coordinating three levels of units. Another variation for a classroom setting is teacher use of the Unitizing Scoring Guide while students prompt each other to share their thinking about a problem solution as the teacher observes the group.

***Learning environment*.** Protocol interviewers were trained to support students' active participation at their own pace. The students were told that their work would not affect grades, and they should do their best to explain their thinking for a positive, active engagement environment. This is important for data collection: students who are afraid of consequences for sharing their thinking may state what they think the adult wants to hear and not what they are thinking. On a more basic level, students who are interviewed in a noisy space are less likely to be heard. Although the impact of the learning environment was not explicitly part of this study, it is important to note that efforts were made to provide students with the most optimum space for learning, both physically, intellectually, and emotionally, for the interview, but the options were sometimes limited.

In summary, there are aspects of data collection that must be considered: (1) providing a problematic situation for the child, (2) having the student work where the observer will have a full view of gestures and the enactment of written work and the opportunity for audible speech, (3) encouraging complete explanations without leading the student's thoughts, (4) considering diagram and problem interpretation, and (5) providing a positive learning environment.

Unitizing, Multiplicative Reasoning, and Mathematics Instruction

Multiplicative reasoning. Making inferences about whether student reasoning in a situation is additive or multiplicative is supported by describing student thinking with MC levels.

Mathematically, the concept of unit includes (1) recognizing flexibility in what is considered a unit (how much or how many is considered one) and (2) establishing that units have equivalencies such that the way a unit measure can be compared to other units (Sophian, 2008; Lamon, 1996). Students who do not demonstrate that a quantity can be viewed as both singletons and as a unit composed of singletons are not likely to understand *times as many* without the activity of figuring out the answer to support the idea (MC1). Experiences and cognitive development both impact interiorization of *grouping groups* within a quantity. Opportunities to purposefully work with materials to construct units thinking versus memorizing math facts may help interiorize unit relationships to understand the *times as many* concept or understand relationships between amounts based on *times as many* information.

Array diagram interaction. Although there is an expectation for fourth graders to understand the concrete operational level, in the Rectangular Array problems, where part of the array is *covered*, one-fourth of the students' diagram interaction indicated thinking in singletons, and about a third of the interviewed students used interaction with the diagram to make groups of singleton units, representing more pre-operational behaviors. This shows that about half of the students in the study were affected by an incomplete grid when attempting to determine the number of squares in the array. The absence of the visual display for some of the singleton units produced a different scheme than either completely displaying the array in the BAM problems or requiring the student to completely create the array diagram as in the Q1 story problem.

Distributive property. BAM, as a physical representation for the distributive property of multiplication over addition, can make it easier to understand the distributive property providing a reference for future learning. A typical definition for the distributive property is:

> The most common distributive property is the distribution of multiplication over addition. It says that when a number is multiplied by the sum of two other numbers, the first number can be handed out or distributed to both of those two numbers and multiplied by each of them separately. Here's the distributive property in symbols: $a * (b + c) = a * b + a * c$. (National Council of Teachers of Mathematics, 2019)

Reversing the action, the distributive property is used to combine like terms: $ba + ca = (b + c)a$. Student work on BAM problems can provide some insight regarding learning about the distributive property, but understanding the distributive property concept involves operating at MC3. An explanation follows.

The BAM process for finding the number of squares in the array begins with splitting the array into two parts. Students who cannot use at least MC1 in the BAM situation are likely to only consider one row of the array to determine the quantity, usually the top or bottom row. MC1 students may be able to split the rectangle into two parts and find the number of squares in each part, but for some, it is too much grouping to combine the number of squares from each of the parts into a total number of squares, so their answer becomes the number of squares in one part, or the numbers at hand are used in an invented procedure that fits with the students' present thinking. Others operating at MC1 make great efforts to record what they know on the paper and using the activity of keeping track of the quantity are able to follow the procedure. So, students who coordinate two groups of units in the action of solving the problem (MC1) can complete a representation for the distributive property, but it takes knowing a procedure and only thinking about one part at a time to produce the number of squares, without the conceptual aspect of the

DP. Students can interact with the diagram in the activity of finding the number of squares with varying amounts of interiorized work. It takes at least MC1 thinking to appropriately interact with the BAM diagram.

MC2 elaborated students can record information on the diagram to keep track of the sub-quantities as well as the number of squares in the entire rectangle, three levels of units, by determining the number of squares with flexible groupings and more interiorized work. In addition to interiorizing two levels of units, they use the diagram to keep track of three levels of units. MC3 students are able to describe the number of squares in the rectangle as singletons, as a number of row/columns for each part and as a number of rows/columns for the entire rectangle going between the three using the diagram to explain, not calculate. Students using MC3 students can think about the three levels of units and then reverse their thinking to the starting point in an interiorized fashion. So, three levels of unit coordination are accomplished in an interiorized fashion. This is the expectation for students who are using the distributive property to solve equations for an unknown. Students who can think about hierarchically related units in a BAM situation, i.e. 3 groups of 2 units within 7 other units and can rearrange the grouping to 3 groups of 7 units within 2 other units and recalculate amounts in an interiorized way are demonstrating they are ready to conceptualize the distributive property and are more likely to be successful making the transition to using the distributive property to solve problems including a missing factor (side length).

Consequently, it makes sense to focus on multiplication without an expectation for mastery of distributive property (DP) in fourth grade. DP is not expected in fourth grade (TIMS Project, 2011), yet making students fluent in the BAM process provides a representation for when DP is a learning expectation.

Number sense. With regard to number sense, when there is an ability think about a quantity in a variety of groupings of chunks (unit) with a corresponding number of chunks (unit), then more flexible thinking is possible in problem-solving and elsewhere. For this age band, providing students with experiences that encourage deeper understanding of multiplicative thinking as shown with the BAM connection to distributing multiplication over addition, may mean these students will have a better chance of reaching middle school with essential units understanding for recognizing and using multiplicative relationships.

An advantage of using formative assessment to recognize scheme use is matching instruction to meet student needs. Students who demonstrate Pre-MC or MC1 behaviors or whose scheme choices vary across representation demonstrate different needs. These students need to learn to group a quantity and also think of it as singletons versus memorizing fact family numbers. Students without success on visually incomplete diagrams of the singleton units might benefit from focus on more activities with finding the number of squares with tiles and a physical cover, or finding area of rectangular shapes (CCSSM 4.MD.3) with materials or shading grid paper by rows or columns with another paper as a cover and with appropriate questioning to highlight groupings versus a focus on memorizing multiplication facts. In addition to making the connection between area as the product of the number of squares in a side length, students can practice skip-counting with a physical scaffold as a springboard to skip-counting, like skip-counting while covering rows of an array, that also builds a mental image of the grouping skip-counting represents until the concept is established and the support is removed.

As teachers gain awareness of potential variation in students' MC levels from embedded assessment of unitizing ability, the researcher expects that this awareness will impact how instructors structure classroom math talk. In addition to opportunities to meaningfully work with objects or drawings to develop unitizing, encouraging classroom mental math talk can develop unitizing. The change to explicitly highlighting units understanding in number talks can become part of best practices as another way to strengthen understanding of grouping a quantity in multiple ways, thus building number sense.

Teachers with a background understanding of unitizing and MC are likely to be more effective in coaching such discussions because the questions they pose are more likely to match likely structures for students' reasoning. With more understanding of multiplicative thinking from third, fourth and fifth grade activity, hopefully fewer students will hit a wall when meeting learning standards for fraction use and algebra in middle school or high school. With more research focus on how students use unit coordination ideas in fraction understanding (Steffe and Olive 2010; Hackenberg, Norton & Wright, 2016), there is greater awareness of how students' unit coordination impacts fraction understanding and operating with fractions. Teachers predicting which students are likely to have difficulty with fractions and a sense of what to do about it based on unit coordination assessment during multiplication and division learning provides a way to guide differentiating instruction to support learning. In addition, teaching based on evidence of students' unit coordination levels can address equity issues such as power and positionality or access and achievement (Tillema & Hackenberg, 2017).

Comparing Fourth Graders' Unit Coordination Levels to CCSSM and OGAP Progressions

This work adds to the resources about student achievement and cognitive levels and relates unitizing to a general progression of multiplication strategies such as demonstrated by OGAP. There are several grade bands where array multiplication is targeted, providing opportunities to use the Scoring Guide.

Unit coordination and CCSSM. The results of this study are compared to expectations on national standards. Table 27, a copy of Table 18, indicates the percentage of students judged to be at each of the MC levels using the median of available data. The unit coordination abilities for the four classrooms of fourth-grade students in the study are compared to the expectations identified in the Common Core State Standards for Mathematics (CCSSM).

Table 27
Percent of 4^{th} Grade Study Participants with Video Data at Each of the MC Levels Determined Using All Available Data from Six Items

Unitizing Score	MC Level	Number of Ss	Percent
1	Pre MC	9	12%
2	MC1	23	30%
3-4	MC2	45	58%
All	All	77	100%

The CCSSM Expectations for Multiplicative Concept (MC) state that students at the end of 3^{rd} grade should have a conceptual understanding of multiplication -- CCSSM expectation 3.OA (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In the present study, student work was collected in the early winter of the fourth-grade academic year after the unit of study in which the focal items were first presented. Consequently, students in the study should be demonstrating conceptual understanding of multiplication according to these standards.

Conceptual understanding of multiplication in unitizing terms means students need to coordinate at least two levels of units. Using the percentages of students at the various levels in this study, over half of the students are rated at unitizing levels 3 – 4, or MC2, indicating the ability to coordinate at least two levels of units in an *interiorized* way. Consequently, this sample shows about 58% of the second-trimester fourth grade students in this study demonstrated the ability to make groups of groups in an interiorized way, using numbers in the range of the curriculum expectations for array problems. Thirty percent of the students rated at the MC1 level should be able to coordinate two levels of units within the activity of solving the problem. About 12% of the students demonstrated pre-MC, where there theevidence infers that the students do not have a conceptual understanding of multiplication.

Pre-MC students need support to achieve the CCSSM third grade operations and algebra expectations. Although some fourth grade students may not achieve 3.OA because they missed initial instruction or they have trouble attending, the Pre-MC rating indicates that the student needs more experiences with quantity to construct grouping ideas, to the extent that working memory and long-term memory with allow. Reviewing math facts is not likely to produce the same results as focused learning activities including physically putting equal groups of objects together and taking them apart. Strategies to encourage grouping practices to determine a multiplication fact are established (Van de Walle, Karp & Bay-Williams, 2019) but instruction that highlights the unitizing ideas may make the instruction more effective – a topic for further research.

Relating unit coordination identified by the SG to OGAP. The Scoring Guide, as a formative assessment tool for recognizing the MC level students use in a given array situation/problem, provides inferences about a student's mathematical thinking. It is designed to support teacher decision-making for subsequent instruction based on a progression. The Ongoing Assessment Project (OGAP) is also designed to help teachers choose instruction based on moving their students along a pathway of growth with progressions described in the OGAP Multiplicative Framework (Ongoing Assessment Project, Ongoing Assessment Project, January 2017). The OGAP Multiplicative Framework describes an expected progression of strategy uses and error types. OGAP specializes in professional development to help guide teachers' use of formative assessment in decision-making regarding instructional practices. OGAP Progressions help teachers identify students' readiness for transitioning from additive to multiplicative thinking.

By connecting the unitizing levels identified in the array problems to the protocol interview one-digit by two-digit problems strategies students selected, the strategy choices used on the one-digit by two-digit multiplication problems were linked to that student's median MC on array items. Next, the strategy data are organized so that the strategies used by students with a given median MC are aligned with the strategies in the OGAP progression. For a snapshot comparison of the two, the percent of students at a given MC level using strategies at a given level of the OGAP Progression of Multiplication Strategies was determined (Table 28). On the OGAP Progression, the least complex strategies are early additive strategies like counting by ones, modeling with objects or drawing. Theoretically, one would expect the students who were reasoning at the pre-MC levels to use the least complex strategies. 33% of the nine students in this category do draw pictures, but the rest use more advanced strategies on the OGAP progression. Students showing MC1 understanding are expected to use additive strategies due to the lack of supportive diagram or storyline to support coordinating units, and 46% of these 23 students chose strategies that do not use multiplicative strategies. Students showing MC2 understanding are expected to use transitional strategies such as BAM or multiplicative strategies on the OGAP progression. 89% of the 44 students are using multiplicative strategies on the OGAP progression.

In this study, strategy use in the one-digit by two-digit problems by students using Pre-MC thinking in earlier problems is varied, without a typical strategy choice. Strategy use in the one-digit by two-digit problems by students using MC2 thinking in earlier problems is likely to be a BAM or expanded form strategy. This suggests that students who are not reasoning about quantities as groups of groups (MC2) are more likely to have random strategy choices but the students who coordinate two levels of units in their head (MC2), are more likely to choose what is considered a multiplicative strategy on the OGAP multiplication progression.

Table 28
Strategy Use by MC Level on the OGAP Progression

	OGAP Progression of Multiplication Strategies				
MC Levels	Early Additive (modeling with objects, drawing and/or counting by ones)	Additive (repeated addition with or without a model)	Early Transitional (skip counting, equal groups in circles or an array, single digit area model considering both dimensions)	Transitional (BAM or open area model, considering both dimensions)	Multiplicative (partial products, expanded form, doubling and halving, standard form)
Pre MC (9 Ss)	Draw Pictures (33%)			BAM (33%)	Expanded (33%)
MC1 (23 Ss)	Draw Pictures (33%)	Repeated Addition (13%)		BAM (17%)	Expanded (25%) Standard (8%) Mental Math (4%)
MC2 (44 Ss)	Pictures (2%)	Repeated Addition (9%)		BAM (41%)	Expanded (41%) Standard (7%);

9. CONCLUSIONS AND IMPLICATIONS: RECOGNIZING MC USING COMMON CLASSROOM MATERIALS WITH A SCORING GUIDE

Previous studies have identified MC in teaching experiments comparing rod or tower lengths. This study suggests that classroom curricular items based on arrays can also provide a setting for making judgments about a student's MC. For the two BAM items, MC ratings agreed for 83% of the 64 students where both items could be scored. This suggests that continued refinement of the scoring guide and validation of the guide with specific items for school use is appropriate. Study data for fourth-grade students' interpretation of an array diagram suggests that the diagram can be a tool to distinguish additive thinking from beginning levels of multiplicative thinking. This supports using array multiplication in classroom activities as embedded assessments for making inferences about MC and related unitizing reasoning. Study evidence also indicated that the array diagram can represent the distributive property but expectations for all fourth-graders should not include comprehensive understanding of the distributive property. This section describes the implications of this study for classroom use and future research directions.

Understanding unitizing has far-reaching consequences beyond whole number multiplication, including number sense, rational number, algebra in that the relationship between number and unit to represent quantity in unitizing is also present in comprehending rational number and equations (Hackenburg, Norton & Wright, 2016; Thompson, P.W., Carlson, M., Byerley, C., & Hatfield, N., 2013). Inferences about multiplicative thinking for eight to eleven-year-olds are important because there are opportunities to encourage the development of multiplicative thinking before middle school and high school, when the absence of multiplicative thinking ability creates a roadblock for students' rational number and algebraic conceptual development. Providing classroom tools to identify multiplicative thinking when teachers have time to help the child construct this understanding is critical. Educator use of the SG use with array problems gives evidence to support instructional decision-making more appropriate for the 21st century to provide student understanding inferences that go beyond fact or algorithm memorization.

As demonstrated in this research and elsewhere, tools such as a multiplication table or calculator or memorizing a procedure without conceptual understanding can help a student find the answer to a calculation. Although a student can consistently get correct answers with tools or procedural fluency, it is important to also determine whether conceptual understanding exists as well to ensure the reasoning needed for future learning/use is developed.

Since students miscalculate for a variety of reasons, the instructional follow-up should be different to increase the likelihood of student achievement. Follow-up for a student who has trouble coordinating units is different than follow-up for a student who cannot recall due to inattentiveness during directions. People can use SG indicators to infer whether a student is using multiplicative reasoning, is able to calculate with an algorithm, has predominantly additive thinking, or something else.

Unitizing Scoring Guide (SG) Implications for Classroom Use

Revising the scoring guide with specific item types in school settings and training teachers to use this scoring guide are the topics for future research. Validation of item types for school use and the suggestions for developing professional development based on the results of this study are discussed in this section.

The scoring guide can introduce teachers to unitizing and unit coordination as underlying cognitive aspects of number sense, encourage observing gestures as part of formative assessments and remind elementary grades teachers about the broader implications of multiplicative thinking. Using the SG as a basis for a classroom instrument, specific observable measures for scoring unitizing levels can be listed, perhaps on an electronic form, where the results can be tabulated to determine a potential score. The scoring guide uses gestures, written work, and speech to triangulate observations. This work can help encourage classroom use of corroborating evidence from written work, gesture, and speech as an important technique for classroom formative assessment in general. Using a unitizing formative assessment tool can promote more awareness of the role of unit coordination in teaching multiplication which can lead to significant changes to instruction to accommodate constructing *units* understanding, especially MC. While scoring students, teachers can match the work students do on a daily basis to the MC ratings to help build understanding of this continuum. Scoring may help teachers develop better ways to support students at the different levels of understanding because there will be more specific information available for a given student about what it means to operate at a given level.

Professional development. In addition to developing a more efficient instrument, modules describing unitizing ideas, and specific teacher instruction for scoring student activity will be part of the next steps. The training will include explanations about unitizing as is relates to multiplicative thinking to help recognize meaningful actions for scoring, and then specific training related to the specific items and scoring guide would be developed.

Professional development on unitizing and how to use the SG can build on this work by using some video cases created from study data and pertinent research articles used to develop scoring and background. Developing indicators and observing student solution work may transpose into a classroom observation tool and a more formal assessment instrument for classroom use.

Professional developers or education coursework designers might consider including unitizing as a topic. In addition to developing a more efficient instrument, training modules describing unitizing ideas, practice interviewing, and specific instruction for scoring student activity will be part of the next steps for this work. Training scorers will include several parts. Identifying the connections between unitizing and multiplicative thinking, observing and analyzing gestures related to unitizing, and recognizing how to be sensitive to bias when asking students to share their thinking. Training will include noting when gestures show advanced thinking compared to an explanation or written work in that the mismatch may indicate the student will benefit from instruction specific to the mismatch. Training related to the specific items and scoring guide would be developed using the findings described in Chapter 8. Sample cases based on project data can be a springboard for discussion about how students think about units. Training components can be illustrated with examples from the data.

The exemplary cases for each SG level presented in Chapter 6 are available for professional development. In a sample presentation to teachers in a suburban elementary school in the Midwest, theoretical aspects of unitizing were described, followed by explanations for the levels using video from these sample cases highlighting how to observe students' specific gestures, words, and written work for evidence of students' dynamic understanding of units while engaged in rectangular array activities.

For example, the presentation described specific gestures to help identify how a student is grouping a quantity. The interaction with the array diagram motioning to the side lengths followed by an oval motion over the total number of squares to indicate that the amount in a column produces the total quantity was viewed and analyzed. Also, the way a student points to a square while counting the number of squares was shown to indicate whether the student viewed the squares as individual units or a group. With support from professional development on this topic, educators may recognize the potential meaning of students' gestures that combined with other evidence may allow more information to be collected *in the moment* as well as in a formal assessment setting.

Models for instructional practices to build math inquiry as follow-up instruction were discussed using specific questions and tasks for different ways of operating on a quantity given the level. Level-specific questions are based on constructing student knowledge by asking variations of "How do you know this is always going to work?" and "How can we be sure this is true?" Teacher feedback from the sample presentation session indicated likely use of the SG indicators to develop instructional next steps. Teachers indicated that recognizing that gestures, especially combined with written work production and explanations could help identify student thinking was an important takeaway (E. Liszka, L. Ralph, K. Stanford, K. Gborigi, personal communication, April 2019).

Diagram use in classrooms. Using the SG with a variety of different array diagrams has highlighted that the diagram can play a role in supporting or distracting student thinking, especially for students at Pre-MC and MC1. When a student's score indicated the ability to coordinate two levels of units in their head (MC2) it was more likely the student's score would be stable across the various items. Information from the diagram can support determining the number of squares, in keeping with the perceptual needs of students at the Pre-MC and MC1 levels. This suggests that when a given student cannot interiorize two levels of units, the diagrams in the item types will impact comparing results of items. Since diagrams are an aspect of curriculum and assessment design, this work suggests some general guidelines about diagram use in curriculum and assessment, such as encouraging students to write or draw on the diagram to make sense of it beyond extracting information about the problem from it Since diagrams can support students' ability to group and ungroup a quantity, including explicit instruction on how to use diagrams such as the array diagram in curricular materials may be useful.

Curriculum. Curriculum developers might consider including more information about unitizing to help build awareness of cognitive aspects of conceptual development. With encouragement within a curriculum guide, teachers may pursue more information through professional development for collecting both *in the moment* as well as more formal data on how students are thinking about quantities.

Curriculum developers might consider more explicit emphasis on using diagrams. From general observations, students who would benefit from interacting with the array diagram did not always do so to find the number of squares. Students working at MC1 did not always use the diagram to help them keep track, but instead guessed, used fingers, or made other marks. Instruction and activity directions that encourage effective use of diagrams within curriculum materials may help more students' reason through the problem by encouraging the use of a diagram or other representation to solve problems.

Future instrument. An instrument to measure unitizing based on the findings of this study could be developed to help teachers learn about unitizing/unit coordination as well as help them assess their students multiplicative reasoning abilities. Improvements are needed to make the instrument appropriate for general classroom use. Three areas for revision are highlighted in

Chapter 8, the role of the situation, the diagram, and communication. To start, the situation for observations needs to be engaging for students who are in the act of solving what is a problem for them. The Rectangular Array A problem was generally too easy, and might be more appropriate as an example of the problem type instead of providing data. Active engagement in the activity is important for valid scores. For a classroom assessment, teachers can identify if their students are actively engaged when they are gathering data and consider this when making inferences. Since BAM problems are part of the instrument, students need to have exposure to the BAM method of multiplication taught in many curricula. However, too much recent practice introduces the concern that a strategy is memorized.

BAM items could provide opportunities to demonstrate MC3 in addition to Pre-MC, MC1 and MC2 given the addition of an interview question to elicit MC3 thinking. Students who can thing about subunits in the smaller rectangles as well as the sum of the subparts in the large rectangle and the number of singletons would demonstrates MC3. How the student chooses to group the quantities can demonstrate MC3 also realizing that five rows of 14 in an array can be arranged as five rows of two groups of seven and rearrange the amounts to make 10 groups of seven would also demonstrate MC3.

In the study data, some students could complete the BAM task easily the first time, but not a second way. Asking for another way may minimize the number of students who seem successful because they have memorized a procedure, but are not using the corresponding schemes.   To minimize false positives, the same BAM diagram will be used twice, where the second time, students are asked to use the break-apart method another way in order to minimize the number of students who seem successful because they have memorized a procedure. For the next version, BAM scoring will focus on students’ efforts to produce two ways to break apart the same rectangle. More emphasis on obtaining an indication for why they made their choices may also minimize false positives.

Second, study results match the theoretical expectation that some students use the array diagram as a perceptual tool or to interact with the diagram to coordinate two levels of units. In a future instrument for measuring unitizing at the level of multiplicative reasoning, items like the Rectangular Array problems with partially covered squares with more item-specific scoring could be piloted as a screener to distinguish students who are at least MC2.

Asking students to solve a problem that includes drawing an array is the third item type. In Q1, the story problem, students drew arrays. Students' responses provided another type of evidence – the range of the drawing included individual squares or rows and columns providing visual support for a rating. The diagram is created providing opportunities to see how a student combines, partitions, or iterates a quantity. In addition, the item asked students to reverse their thinking to find a factor, not the product. Although some students may have missed this item from reading incorrectly, the ability to see situation when the question focuses on the inverse operation should be considered. Generally, problem interpretation consistency is a concern prompting additional review of word problem interpretation. Students could be asked to draw an array asking for the value of the side length without using the story, or include a procedure to ensure that the student interpreted the story as intended is needed.

In summary, items for a classroom use instrument could include the partially covered array with more item-specific scoring is included to help determine if a student demonstrates MC2. The BAM item is included but scoring will be based on both presenting a BAM solution and a second, different BAM solution using scoring developed to minimize false positives. A third item type that requires drawing an array similar to Q1 would be considered given more item-specific scoring for the drawing. The next iteration of an assessment would look for evidence across a set of items like the ones described here, marking whether or not an indicator is present in the student's actions with a wholistic score determined by the evidence across all items.

In this study, there were few items of the same type. To strengthen reliability, more items of a given type are needed for a more formal diagnostic tool. Two pairs of BAM problems would provide better evidence while keeping the test a manageable length. Using two of the Rectangular Array problems (B and C), removing the smallest rectangle will make room for more BAM items and eliminate an item that was not *problematic* for most students. The Q1 word problem with drawing the array would need to be paired with another similar problem or eliminated.

Implementation ideas. In addition to person-to-person interviews to gather data on student thinking, technology can provide other ways to obtain evidence for inferences about student thinking. To offset the need to observe many students at one time, a technology where students can make short videos can be employed.

Using interactive whiteboard apps, such as ExplainEverything © or Flipgrid ©, is an alternative to using a video recording tool. These whiteboard apps for i-Pads or other electronic devices allow students to record themselves explaining something. Students' speech as well as their writing is recorded, which may minimize time issues related to gathering students thinking explanations. Data for multiple students could potentially be gathered at the same time. However, modeling will be needed for students to recognize the need to share why they are making the choices they make and practice sharing explanations that include reasoning for actions.

The interviewers did a great job of not leading students to provide answers they thought the interviewer wanted, but the inconsistency of the follow-up questions to learn what students are thinking when students do not explain why they chose a given solution path posed concerns for gathering evidence about students' thinking. Interviewer training should be addressed. When the interviewer is focused on how a student knows she is correct or student reasoning for a given action the interviewer can ask follow-up questions to help expose student thinking which can be compared to gestures and work. Students also have to understand the expectations for sharing their thinking. Using recordings is possible only after students have an idea of what is being asked of them and the expectations for explaining their reasoning. When students understand the expectations, and possibly with some prompts to share thinking, teachers can have multiple students making short recordings that can be viewed at a later time for data analysis. With this option, a teacher might collect video from all students, but not look at each video with the same intensity. Using the SG, evidence for the student's understanding at that moment in time can be data for instructional decisions.

An instrument for classroom use should take consider the need to routinely gather information about a student's progress over time. Variations on the same item type are needed, with similarly sized arrays and protocols for multiple implementations. Knowing that formative assessment occurs over time, varying activities or items with the same properties as the original item set may be used as indicators.

Gathering data through informal review of students' classroom actions and work using the unitizing indicators for array multiplication problem solutions is another approach. A student's actions may be observed by knowledgeable teachers informally reviewing students while walking around the classroom stopping to observe and ask questions to elicit thinking. Classroom use of this scoring guide may go beyond a teacher-to-student interview to teacher observations of students working with other students. The teacher can foster student-to-student conversations likely to expose reasoning in speech, gesture or written work as well as have one-on-one conversations with a student to gather the evidence to make a rating. A version of the present SG can support this approach to the dynamic assessment of unitizing, where teachers can direct student work in the moment and plan activities to help students construct stronger multiplicative reasoning through unit coordination.

Students' gesture use while interacting with other students or the gesture use by an individual student explaining the process to the teacher while doing general monitoring activities can support inferences about thinking. A student's finger movements on diagrams can be noted while the student is working independently or in a group. As a classroom tool, the day to day activity observed by the teacher can provide a backdrop for viewing a student's problem-solving actions in light of current instruction.

Scoring students' performances on array items may help identify students who need more support and those who need more challenge. Activities for productive struggle matching the unit coordination evidence can be selected, with the potential for productive struggle and greater engagement. For example, Ron Tzur's Please Go and Bring for Me described in *Goldrick Elementary's Big Leap in Math Achievements* (Colorado University Denver School of Education & Human Development, 2017), where students play a game that involves gathering a quantity of linkercubes in particular configurations supports MC1 learning. An important part of these activities is asking for explanations of why the student knows her thinking is correct. The instructor can scaffold the discussion as appropriate for the student's unit coordination level with prompts for student-to-student work if the teacher is not participating. Riddles which require thinking about layers of units or distributive property diagrams with missing side lengths may provide more challenge.

Reliability and Validity for a Future Tool

In the present study, this researcher suggests SG use can identify MC within array multiplication situations with general descriptive statistics and qualitative evidence. Item identification for a more formal instrument and additional statistical support are suggested.

Although unitizing ratings using the SG were initially developed for BAM items, additional item were scored with the SG. A version producing a rating based on multiple items should be considered. To the extent possible, the environment for the task enactment will be refined including eliciting verbal responses with more explanation for why and more consistent number phrase or equation writing, as well as taking measures to increase the visibility of students' hands and writing to ensure adequate collection of evidence.

Data collected with principled use of tools designed to elicit multiplicative reasoning will confirm evidence by triangulating gestures, speech and written work within a problem solution episode. For example, if the student's speech will indicate a certain level, if the gestures and written work generally fall within the same MC level there would be confirming evidence. Considering response processes, students would need exposure to the BAM model and understand the need and have skill at explaining their thinking for what they did and why they did it. Interviewers would need to know how to elicit thinking without encouraging a particular line of thought. Statistical analysis will support validation and reliability. By considering gesture, verbal reasoning and written work as separate scores to be entered for each of the items, a Partial Credit model or Rating Scale model could be employed to evaluate the extent of agreement between the wholistic rating and the expected verbal explanations, gestures, and written work by analyzing item fit and person fit statistics. Principal component analysis performed with this data could examine unidimensionality.

A protocol and rubric based on this study's findings could be developed. New raters, some with less familiarity as well as those with knowledge of unit coordination, would receive training based on this study, implement the protocol and score using the rubric. These interviews would be video-recorded to review the fidelity of protocol implementation and improve the training. Using the revised scoring guide, the interviewer as well as another rater would rate the student, pairing experienced with experienced and novice with novice, Cohen's kappa would be used as a measure of inter-rater reliability, with Kendall's tau or Spearman's rho for measuring agreement. Reviewing changes in the various iterations of the scoring guide document may uncover patterns of learning that will be useful for professional development.

Limitations

This study does present ways to make inferences about students' multiplicative reasoning with scheme use for MC levels with opportunities to triangulate spoken, gesture and written evidence from video from the student's actions while actively seeking to answer a problem, using the video data for students' solution processes. The sample used in this study is not representative of a particular population or geographic area, and is not a random sample.

While the video data provided a wonderful opportunity to analyze over 75 students' work, the opportunity to have follow-up sessions with the same students or to ask the student more about their thinking was not possible. Consequently, there was no recourse for incomplete evidence on video data, and no way to follow-up with claims by gathering more evidence. The protocol for the data collection allowed students to use tools, including a calculator, a multiplication chart or tiles. When students chose the calculator or used the multiplication chart, less opportunity to observe how they determined the number of squares meant the score was based on less evidence. Due to the limited evidence in some situations where a student made few verbal comments but produced a correct answer, the claims were based on less evidence than desired. This study lays a foundation for a future, validated instrument(s).

Future Work

Building awareness for unitizing ability/skill, encouraging the development of educator expertise to identify unitizing and creating the related instructional plans across the range of number and operations in elementary school mathematics requires professional development. Training modules about unitizing as related to additive and multiplicative thinking for teachers and pre-service teachers could include video cases from this study to provide a springboard for active discussion about what unitizing is and what it looks like. In addition, interviewer training to learn how to rate a student's MC level based on their actions in a given activity for valid results could include video cases from the study to support scoring guide training.

This research supports making inferences about students' thinking from gesture information, adding to the body of knowledge for observing gestures to make instructional decisions. The above-mentioned training may encourage more active use of gesture within classroom assessment. Explicit training for gesture use can support actively noticing and encouraging gestures in teaching-learning environments.

Open response items provide student choice, where student choices pose new research questions. In the BAM items, choosing a more typical split, say *tens plus the rest* meant the resulting sub-rectangles were not likely to be more difficult than expected. What might the student's decision to break-apart the rectangle into parts that include a known array and a difficult array, such as splitting *18* x *4* into *5* x *4* and 13 x 4 mean? Choices for the split might indicate number knowledge or might be related to organization and planning skills. Student choices add to the variation in scoring potential because the quantity may affect their solution choices and related evidence indicating MC.

From general observation, students using Pre-MC or MC1 schemes did not always use the diagram to solve problems even though the use of the diagram such as counting squares or grouping squares with lines or circles could have supported their work. This prompts the question: if given specific instruction to use the diagram to determine the number of squares, would students choose to do so, and would it help them recognize patterns in determining the quantity? Determining what type of instruction or teaching methods might support student use of diagram might help students use a diagram more effectively or use a representation to solve problems more effectively. This is a topic for further research.

The *2* x 6 Rectangular Array is based on a known fact for students at this point in the curriculum, making it less problematic for many students. Consequently, the researcher predicted that more students might demonstrate an advanced unitizing level relative to the BAM problems, but actually, more students were touching and counting each square, typical for Pre-

MC. The question arises: Did some students see counting squares as the most efficient strategy in the smallest array problem or were they *in the mood* to count? In the section comparing BAM and Rectangular Arrays, the likelihood of students touching the *2* x *6* array due to the missing grid is considered. Another interpretation is that the effort to count the 12 squares was not much more than figuring out 6 + 6 for these students, so that a student might consider counting the 12 squares an efficient strategy. Little is known from this study how the student determined what is the most efficient strategy or why the student might choose to use what she perceives is most efficient versus something else. Patterns in the solution strategies can be explored using both the size of the array and MC levels as variables. Factors to consider in further exploration of students' strategy choices include student perceptions of their skills, the problem situation and classroom expectations.

In addition, the researcher noted a shift in how some students interacted with the array based on the size of the array: students who were focused on counting squares for the *2* x *6* or *4* x *5* rectangle switched their focus to counting the number of squares on the side lengths and seeking to find the product of the two numbers instead of counting the squares for the *6* x *9* rectangle. Student reasoning for these actions is not abundant, so inferences for why students might do this is left to future research.

The potential connection to early algebra may be explored through a more thorough analysis of how students record their calculation choices in an early algebra format. The activities used in this study included directions to include number sentences to match the use of the diagram to find the number of squares. Casual observations of the format for writing number sentences indicate that students who demonstrated higher MC wrote equations and students who did not demonstrate interiorized MC were more likely to write expressions. Further exploration of the nuances of writing or talking about the number sentences/expressions related to the array may correlate with unitizing level indicators. As students use a rectangular array as a spatial tool to help them determine the number of squares as well as recording their results as an equation may indicate an early algebra component to the BAM exercise that was not explored, but may help connect understanding about interactions between spatial and early algebraic thinking, especially in relation to the distributive property. The role of instruction as a factor in how students write number sentences versus phrases and the effects of how teachers instruct students to record number sentences/phrases in this setting may be useful to include as factors in a future study.

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APPENDICES

APPENDIX A

Student Interview: Math Trailblazers Grade 4, Unit 3.10 Unit 3 Test

Expectation 1 Feedback Box: "Use arrays to solve multiplication and division problems (Q# 1.8, 10)."

DR-K12 Embedded Assessment Project (2011-2012)

Student Protocol: MTB Grade 4

Location: **Unit 3, Lesson 10 "Unit 3 Test"**

Getting Ready:

- Secure A Location:
  - *Request a quiet space (e.g., library, unused classroom/office) to insure recording quality*
- Prepare Equipment:
  - *Video camera(s): ONE for each researcher conducting an interview plus tripods.*
  - *External microphone, extension cords, markers, student protocol folder.*
  - *Class student roster showing all students that consented to be interviewed, index cards with student ID numbers, manipulatives for the activity.*
- Positioning of the Video Camera:
  - At a 45° angle, facing the student: showing the student's face, hands and work.
  - If possible, please conduct one interview in each room (or position the students so that the tape recorders do not pick up the other voices)
  - Task time limit — 45 minutes max

Build rapport while walking to the interview:

- Make the student feel comfortable:
  - "Hey, how are you?"
  - "Thanks for coming out."
  - "We will get you back to class soon. You won't miss out on instruction."

INTRODUCE YOURSELF:

"Hi, my name is [_____________] and I'm here to do some math activities with you."

➔ OPENING QUESTIONS:

"What's your name?"

APPENDIX A (continued)

<child responds>

"How old are you?"

<child responds>

"What grade are you in?"

<child responds>

"Who is your teacher?"

<child responds>

"What are you working on in math class right now?"

<child responds>

➔ <u>EXPLAIN PURPOSE OF INTERVIEW:</u>

"I am going to ask you some questions about math because I want to learn how you think about the different kinds of math you do in class and for homework. Is that okay with you?"

<child responds>

(Note: If the child says that it is not alright to conduct the interview, try to find out why it is not ok. Regardless of the child's rationale for declining to participate, or if you notice that the child expresses discomfort, then stop the interview and escort the child back to the classroom and thank the child. Make a note of the incident, and then disclose the incident to the PIs by the end of the day.)

Great. I'm going to show you an activity from UNIT 3 in your math book. As you read the directions and solve the problems, I want you to read everything out loud. Tell me what you are doing and thinking as you solve the problems

<u>SEE THINK ALOUD ACTIVITY ON NEXT PAGE</u>

You can use the scratch paper when you need it.

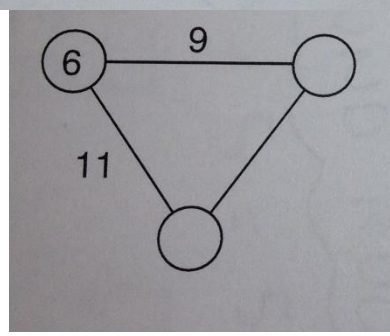
I won't be able to help you solve the problems. If you make a mistake, it's ok to change your answer. What I really want to know is how you think and what you look at when you solve the problems. No one is going to give you a grade or even show it to your teacher. So just remember to talk and think out loud as you solve the problems. Okay?"

APPENDIX A (continued)

Student Questions

E Triangle Sums 3

Notice that numbers in the circles are missing in each of the triangle problems below. Find the missing numbers.



APPENDIX A (continued)

Task 1: Focal Activity, Using Arrays Break-Apart Multiplication

"UNIT 3 TEST #1, 8, and 10"

Materials

"UNIT 3 Test" activity (L3.10, URG, p.210, #1, 8 and 10; see Appendix A)
Markers
Multiplication Tables
Calculators
Square inch Tiles
Blank sheets of paper
Have square inch tiles, calculator, multiplication chart and plain paper in sight of the child, but do not give to the child unless the child asks for them.

PROCEDURE

PRESENT THE TASK

"Can you please read the directions aloud?"

*****If student can't read, ask if he/she needs help reading*****

<child reads the directions>

"What is the problem asking you to do?"

<child responds>

WORKING OUT THE PROBLEMS: USE DARKER COLOR MARKER

- OBSERVE/TAKE NOTES:
  - Take notes so you can later ask, "How did you know…?"
- Remind the child to vocalize and ask the child for justifications:
  - DO NOT interrupt the child's thinking process; wait until they have completed an action.
- Ask the child to tell you how he solved the problem after each item.
  - "How did you figure this out?"
  - How did you know ___?
- If the student's verbalization is unclear, restate/echo:
  - — "So you're ______?"
- If the student is not on task**:**

APPENDIX A (continued)

- **First:** "Tell me how you are thinking about this problem."
- **Second**: re-read
- **Third**: move to the next problem.

Problem #10

- **Use interviewer record sheet**
- if the student does not write number sentences, remind students of the directions.
    - "Now that you have parts of the rectangle, be sure to label them with the number sentences."
    - "Please write a number sentence to show how you found a total number of squares."

After Problem #10 – *If there is no evidence how the student found the length of the array, ask the student to count the top row of the array*

- Cover up all except the top row of squares in # 10:
- "Please count the squares in this row so I can see how you count."

Let's look at some different problems.

APPENDIX A (continued)

Interviewer Sheet Task 1

 # 10

Horizontal line distance __________ (18),

Note if count by 1's ____, 2's______, or another way ______________.

APPENDIX A (continued)

Task 1

You may use calculators, multiplication tables, or square-inch tiles to solve the following problems.

1. Tom made a rectangle with 16 tiles. If there were 4 rows, how many tiles were in each row? Sketch a picture of this rectangle.

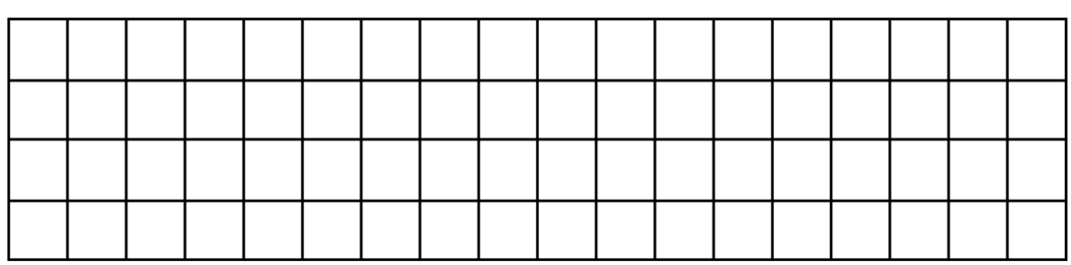
8. Design a box for the TIMS Candy Company that will hold 36 pieces of candy and that has more than two layers.

Tell how many layers are in your box. Also, tell how many pieces of candy are in each layer. Each layer must hold the same number of pieces.

10. Find the number of squares in the rectangle below using the break-apart method.

- Break the rectangle into parts to make it easier to multiply. Write number sentences to show the number of squares in each part.

- Write a number sentence to show how you found the total number of squares in the large rectangle.



APPENDIX A (continued)

Task 2

Knowledge of multiplication facts

Materials

Flash cards
Interviewer record sheet

PROCEDURE

" I am going to show you a card with a multiplication fact and then you tell me the answer. I will write down your answers on this sheet.

- record the answers and take notes if the student is talking out loud

Do these problems as best you can. If you cannot figure one out, it's okay to say "Pass" and I will show you the next problem. Ready?

- **Do not encourage thinking aloud, but if it happens, listen and go on.**

PRESENT THE FLASHCARDS:

- Go through cards

"Let's look at some different problems."

APPENDIX A (continued)

Interviewer Sheet Task 2

Task 2: Multiplication Facts

Record student's response to each.

A. 4 x 10 =	B. 2 x 9 =	C. 3 x 4 =
D. 4 x 5 =	E. 6 x 4 =	F. 4 x 4 =
G. 10 x 4 =	H. 5 x 8 =	J. 4 x 8 =
K. 6 x 8 =	L. 4 x 3 =	M. 9 x 2 =
N. 5 x 4 =	P. 4 x 6 =	Q. 9 x 4 =
R. 2 x 18 =	S. 4 x 2 =	

APPENDIX A (continued)

Task 3

Arrays as a representation to show multiplication

Materials

6 inch x 2 inch rectangle
4 inch x 5 inch rectangle
6 inch x 9 inch rectangle
Marker

Procedure

A. Show 6 by 2 array in front of the child, point and say
    a. "Part of this rectangular array is covered up. Can you tell me how many square units would be in the complete rectangle?"

<Child responds>

Watch for the way the child responds- quickly says answer, counts—by 2's, by two rows of 6 or counts by ones, other…

B. Then say,
    a. "Please write a multiplication sentence for this rectangle here."

C. Turn the rectangular array so that it faces the child as 2 by 6
    a. "How many square units are in this rectangle?"

<child responds>

<Watch to see if the child recognizes the area will be the same or if the child does a procedure to find the area.>

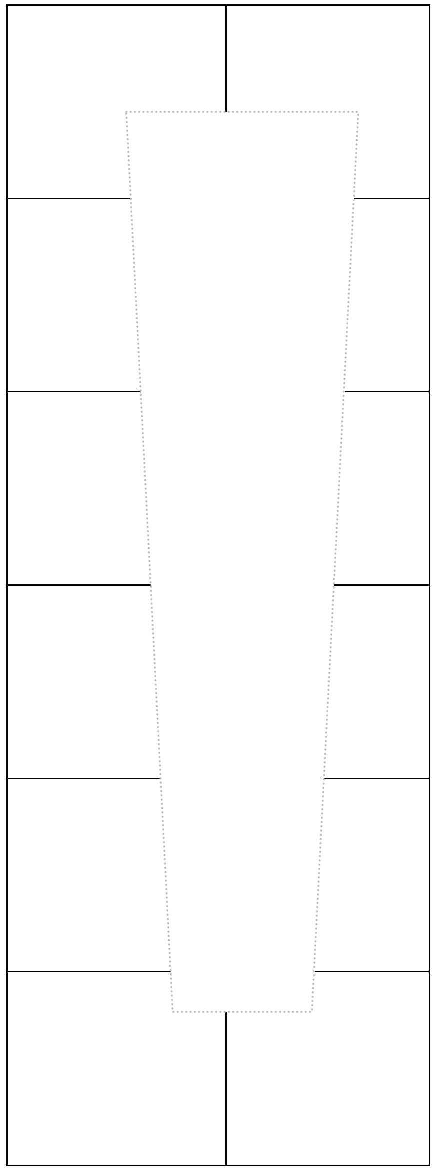
D. DO THE SAME WITH THE 4 BY 5 AND 6 BY 9 ARRAYS

Let's look at some different problems.

APPENDIX A (continued)

Task 3: Arrays

Rectangle A



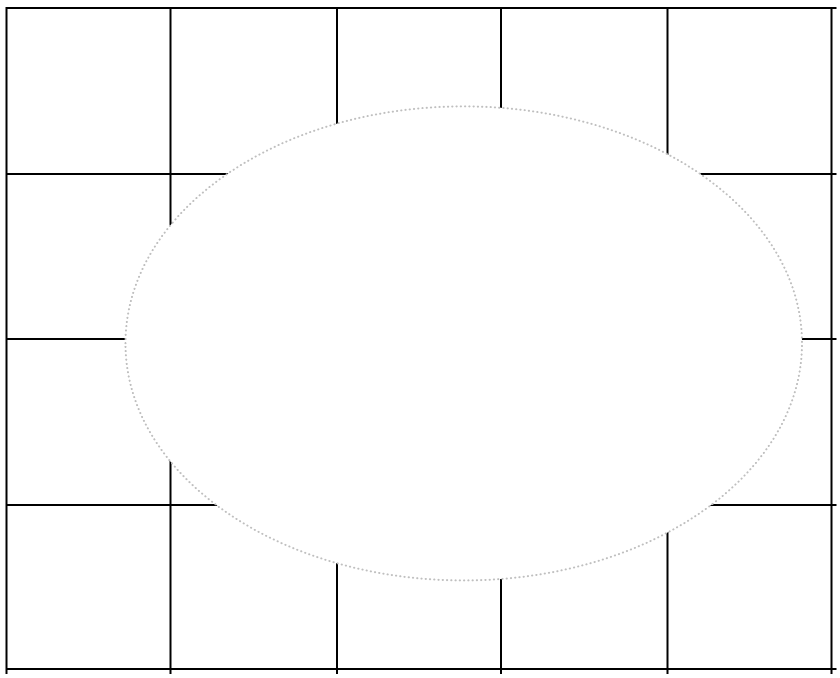
Number Sentence

Number Sentence ______________________________

APPENDIX A (continued)

Task 3: Arrays

Rectangle B



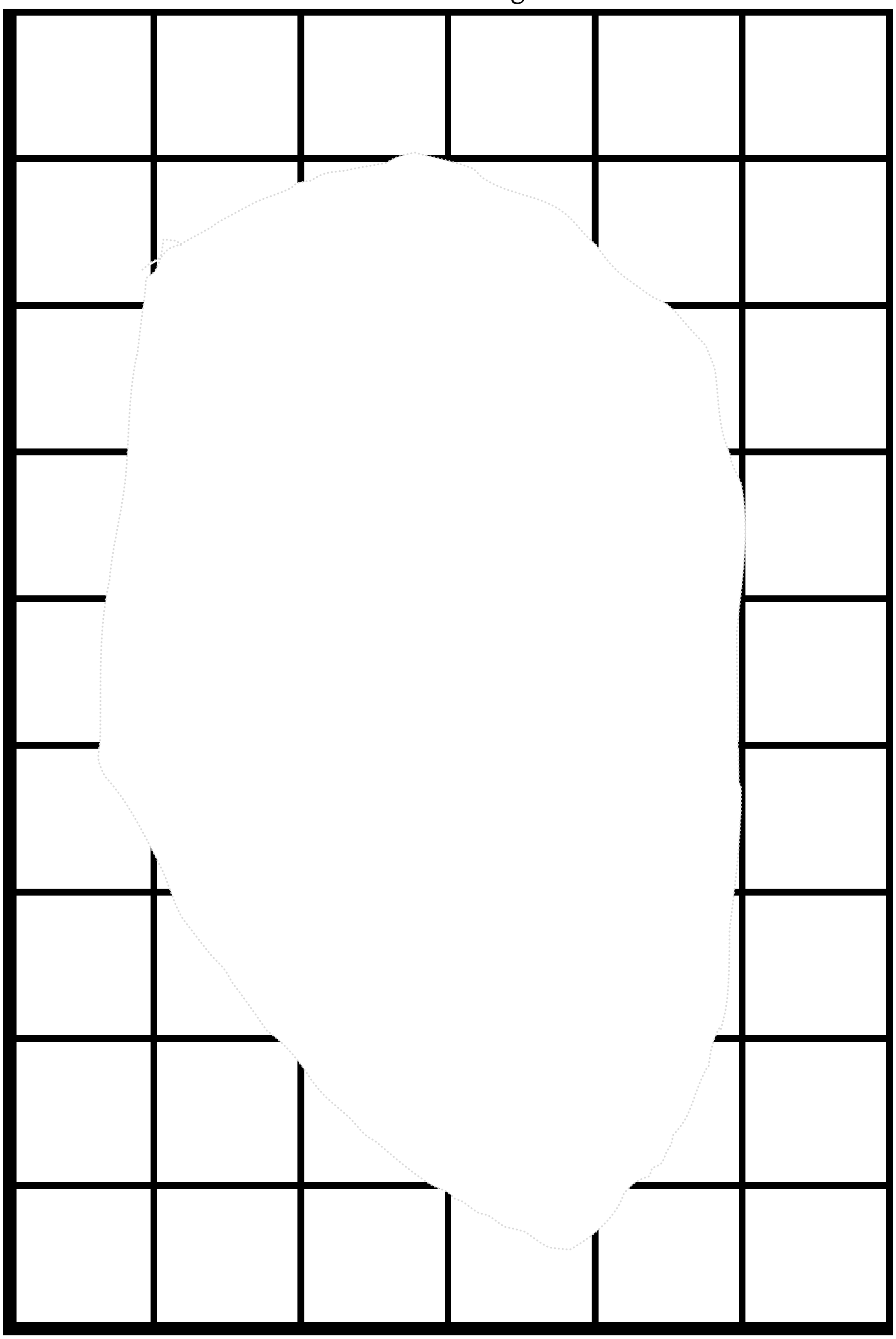
Number Sentence

Number Sentence ____________________

APPENDIX A (continued)

Task 3: Arrays

Rectangle C



Number Sentence

Number Sentence ______________________________

APPENDIX A (continued)

Task 4

Breaking a rectangle into smaller rectangles, multiply and recombine

Materials

Break Apart Matching Cards:
Number phrase cards
Rectangle array cards

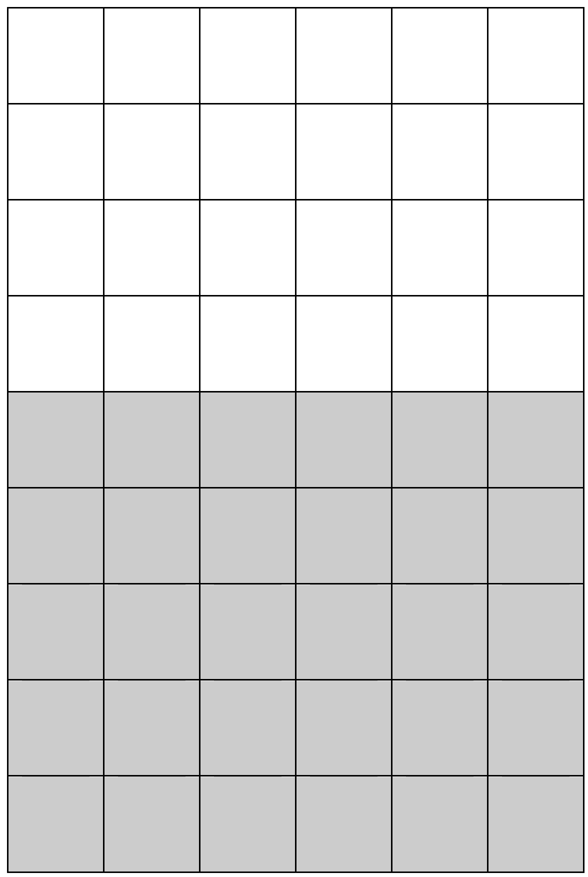
Procedure:

- Put the rectangle card on table
    - "Here are some examples of rectangles broken into parts".
- Present number phrase cards, one at a time:
    - "Are there any rectangles that match this card?"
- RECORD STUDENT RESPONSE
- Ask students to share their thinking.
    - "Why does this card match this rectangle?"
- ***Put the rectangle array card back on the table after each time it is matched***
- REPEAT WITH THE OTHER NUMBER PHRASES

Let's look at some different problems.

APPENDIX A (continued)

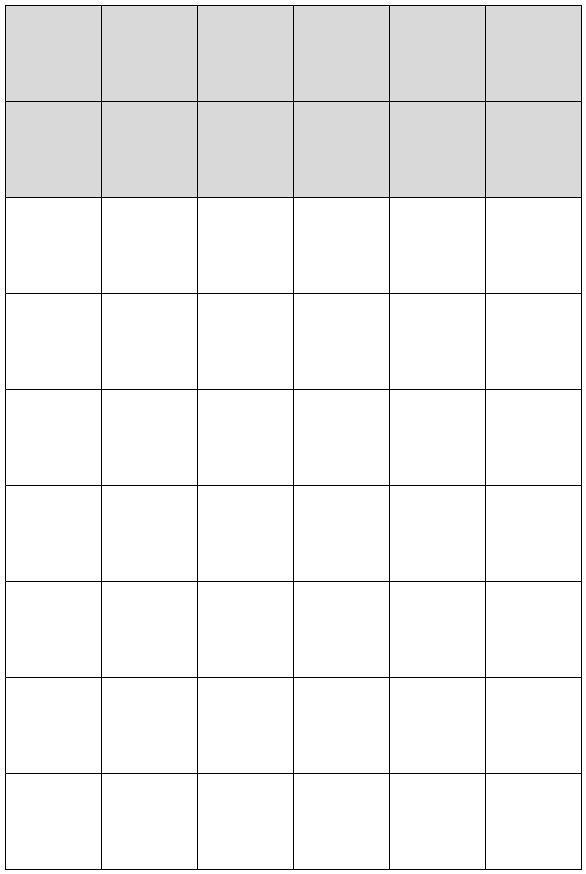
Task 4: Rectangle array card



Rectangle A

APPENDIX A (continued)

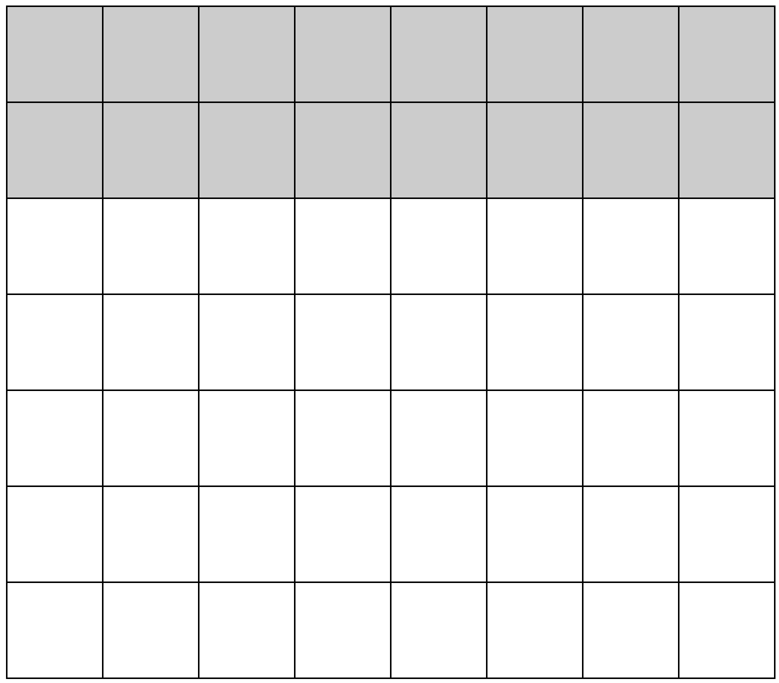
Task 4: Rectangle array card



Rectangle B

APPENDIX A (continued)

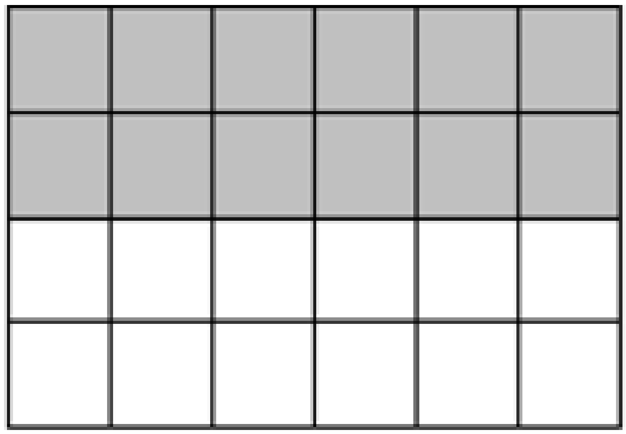
Task 4: Rectangle array card



Rectangle C

APPENDIX A (continued)

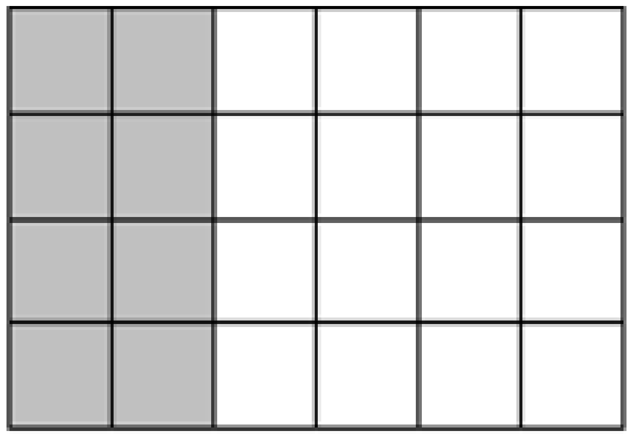
Task 4: Rectangle array card



Rectangle D

APPENDIX A (continued)

Task 4: Rectangle array card



Rectangle E

APPENDIX A (continued)

6 x 7 + 6 x 2
Rectangle with a side equal to 7.
4 x 2 + 4 x 4
Rectangle with a factor of 8.
Rectangle with a side equal to 6.
2 x 8 + 5 x 8
Rectangle that shows 6 x 9 = 54
5 x 6 + 4 x 6
6 x 2 + 6 x 2
Rectangle that shows 4 x 6 = 24.
Rectangle with a factor of 4.

APPENDIX A (continued)

Task 4 INTERVIEWER RECORD SHEET Student #_______________________

1. **6 x 7 + 6 x 2** ____________
2. **Rectangle with a side equal to 7** ____________
3. **4 x 2 + 4 x 4** ____________
4. **Rectangle with a factor of 8** ____________
5. **Rectangle with a side equal to 6** ____________
6. **2 x 8 + 5 x 8** ____________
7. **Rectangle that shows 6 x 9 = 54** ____________
8. **5 x 6 + 4 x 6** ____________
9. **6 x 2 + 6 x 2** ____________
10. **Rectangle that shows 4 x 6 = 24** ____________
11. **Rectangle with a factor of 4** ____________

APPENDIX A (continued)

Task 5

Break-Apart Multiplication – Match Equation to Area

Materials

Marker – at least 2 colors

Procedure

PROCEDURE

PRESENT THE TASK

"Can you please read the directions aloud?"

*****If student can't read, ask if he/she needs help reading*****

<child reads the directions>

"What is the problem asking you to do?"

<child responds>

WORKING OUT THE PROBLEMS: USE DARKER COLOR MARKER

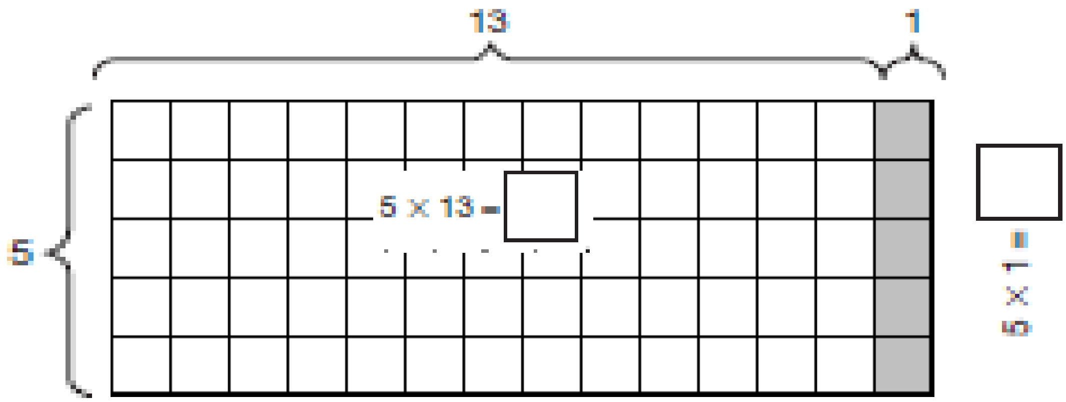
- OBSERVE/TAKE NOTES:
  - Take notes so you can later ask, "How did you know…?"
- Remind the child to vocalize and ask the child for justifications:
  - DO NOT interrupt the child's thinking process; wait until they have completed an action.
- Ask the child to tell you how he solved the problem after each item.
  - "How did you find this number sentence?"
  - Is there anything else you can tell me about this rectangle? How did you know this is correct?
  - Is there anything else you can tell me about this rectangle?
- If the student's verbalization is unclear, restate/echo:
  - — "So you're ______?"
- If the student is not on task**:**
  - **First:** "Tell me how you are thinking about this problem."
  - **Second**: re-read
  - **Third**: move to the next problem.
- **Have student read each item out loud**

  FOR PART C, If the child does **not** break apart by 10's, say "That's good.   Can you also show me that with 10's and 1's?"

APPENDIX A (continued)

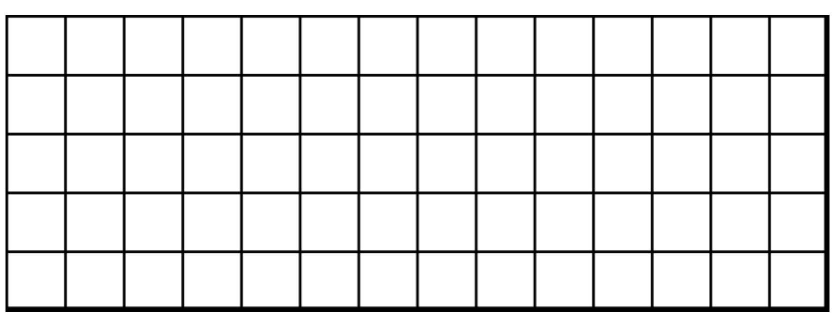
Task 5: Break Apart Products: Matching an Equation to the Area

A. Ming drew the rectangle below to solve 5 x 14 using the break-apart method. Finish Ming's number sentences.



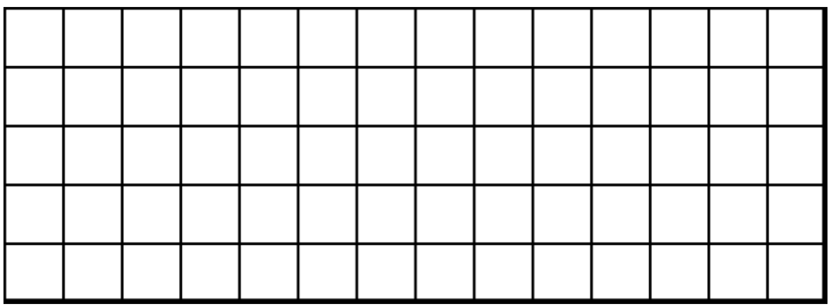
5 x 14 = ______ + ______ = ______

B. Break apart the rectangle a different way. Write the related number sentences.



APPENDIX A (continued)

**C. Break apart the rectangle again in a DIFFERENT way.
Break apart the rectangle into tens and ones (if you did not do this in part B). Write the related number sentences.**



APPENDIX A (continued)

Task 6
Multiplication with larger numbers

Materials

Note card that says 6 x 21
Markers
Grid paper
Scratch paper
Grid paper, larger paper and several markers should be in line of sight of the child.

Procedure

PRESENT THE TASK

"Can you please read the directions aloud?"

*****If student can't read, ask if he/she needs help reading*****

<child reads the directions>

"What is the problem asking you to do?"

<child responds>

WORKING OUT THE PROBLEMS: USE DARKER COLOR MARKER

- OBSERVE/TAKE NOTES:
  - Take notes so you can later ask, "How did you know…?"
  - "Tell me how you are thinking about this problem."
- Remind the child to vocalize and ask the child for justifications:
  - DO NOT interrupt the child's thinking process; wait until they have completed an action.
- Ask the child to tell you how he solved the problem after each item.
  - "How did you figure this out?"
  - How did you know that equals 6 x 21?
- If the student's verbalization is unclear, restate/echo:
  - — "So you're ______?"

- After the child finished 6 x 21, cover up the work and say:
  - "Now I am going to give you a challenge problem. Imagine I did not give you any paper or a pencil. Can you explain to me how you could solve this problem in your head without a paper and pencil?"
    (if the child does not know "in your head" you can say using "mental math")

APPENDIX A (continued)

- If the child was able to solve 6 x 21 in mentally, then show them 34 x 7 and say:
    - "Now I am going to give you another challenge problem. Can you try doing this one in your head".
      Have the child solve 34 x 7 mentally, then ask the child to check their work with paper-and-pencil:
    - "Can you check your work with paper-and-pencil?"

THE END

Okay, that's it. Thanks! .

Do you have any questions?

APPENDIX A (continued)

Task 6: Break-Apart Products with Larger Numbers

Solve the following problems using rectangles, expanded form or another way. Tell how you are solving the problem as you are doing it.

A. 6 x 21

B. 34 x 7

APPENDIX B

Interview Data Excerpts to Demonstrate Unitizing Schemes

Five unitizing schemes are used to infer how students interpret a quantity of squares in fourth grade multiplication array activities using the available video data. Unitizing is the interiorized action that modifies units understanding in a recursive fashion for developing and coordinating students' mathematical knowledge of unit. As the levels increase, (1) a student can organize a quantity with increasing amount of nesting of *unit* i.e. grouping, and (2) the student demonstrates more unitizing ability and consequently more flexibility in using number, which is considered part of number sense. The interview dialogue is between the interviewer (I) and the child (C).

Scoring Examples

Unitizing Level .5

Model	*Observable Actions in a Predicted Scheme for the Model*
UNITARY COUNTING	Keep track one at a time
Use of unitary counting to find a quantity in units (Rating = .5) The rectangle is seen as a set of unitary objects.	Counting to keep track of the number of squares. Touch squares individually to count by ones. For example, student might put a pencil dot in each square or represent the quantity with tally marks and count the marks.

Example: Student 1638 in Class 058

Transcript (T5-10b):

I: so what is this problem asking you to do?

C: that thing, broken another, broke another broke another uhhh rectks rectangle. We asked to, we asked to ... break it apart.

I: okay [00:29:31.21]

C: Writes numbers on the top row of squares using right hand. In the upper left-hand corner, writes *1*, and then puts *2* in the next square to the right, and so on, until the eighth square from the right has an *8.* In the next square to the right, she writes a *1*; then the next square to the right she writes *2* and so on until the far right square has *6* written on it. Then she draws a vertical line between the eighth and ninth columns.)

APPENDIX B (continued)

C: 'kay. 1, 2, 3, 4, 5 (points to each square starting on the top square in the far-left column while saying a number each time. While writing the same, says) five times eight equal. [00:29:53.15] one, two, three, four, five, (puts a dot in the top row square in the 9th column, which is labelled *1* and continues to say the number name while putting a dot in the square.) five times six equal (writes this while she is saying it, then pauses) [00:29:58.28]

C: Shifts eye gaze to the left side of rectangle - says) eight (holds out left hand all fingers out, then puts fingers in a fist and counts on) six, seven, eight, nine, ten (each time extends another finger/thumb on left hand, as she says a new number. Returns hand to fist and says the next number when extending a finger) eleven, twelve, thirteen, thirteen (then writes 13 on the rectangle next to the equal sign.) umm. (Moves pen hand to the equal sign in *5x6 =*)[00:30:07.26] Left hand is in a fist, extends it; is in a fist, extends it and closes it, says) 6, sev uh (as brings a finger/thumb out, and then another out) [00:30:11.20] Leaves pinky extended and says) 6 (then says another number in this sequence as extends another finger:) 7, 8, 9, 10. (then writes 10 next to the equal sign in "5 x 6 =") I got 10 because, umm, I got 13 here, (points to the 13) because I, I, I skipped I counted [00:30:23.04] I skip-count, then, THen I skip-count by this (emphatically points to 5x6 = 10 with right hand pointer finger and generally moving fingers back and forth over that side of rectangle. [00:30:26.23]

C: if I could do one, two, three, four, five, six (is moving pen tip along the top row of the right side of rectangle going to the right. Then moves pen tip to the 2nd row , left most square and continues to count while moving pen tip to the next square to the right, after saying seven in the left most square of the 2nd row.) seven, eight, nine, ten, eleven, twelve (continues to the third row doing the same process) thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, (moves to row below) nineteen, twenty, twenty-one, twenty-two, twenty- (moves pen away) [00:30:37.17] So I was, I was four times six (moves pen up and down the columns) I mean five times six (again moves pen to go up and down columns on the right side of the rectangle.) will equal ten. (moves pen to the left side and points to the *5x8*) and five times eight equals 13.

I: Could you write a number sentence for your answer at the bottom? [00:30:49.05]

C: ehayy. (She moves pen to below the rectangle. and writes in the air. Puts pen down and writes what she is saying) Five times 18 = 13 plus 10

Explanation: The subparts of the rectangle have a number sentence the child identifies as a multiplication where the factors are based on number of squares on the side lengths, but the child used counting on from one of the *factors* by the amount of the other *factor* to find the quantity. The child uses the multiplication equation format, but only the smaller side length matches the number in the equation for the entire rectangle. When explaining how she got the number of squares for the right side of the rectangle with the equation 5 x 6 = 10, she started counting the

APPENDIX B (continued)

squares and stopped at 22, even though she had not finished touching all the squares. She did not attempt to explain why her answer, (*ten),* did not match the count of squares.

Unitizing Level 1

Model	*Observable Actions in a Predicted Scheme for the Model*
ADDITIVE COORDINATION Use of addition to find a quantity in units (Rating = 1) The rectangle is seen as a set of unitary objects in each part, and the parts are added together.	Keep track of just top row - "11" and "7" so that the student can think about all at the same time – the "11" and "7" are part of the "18" , but not recognize each square in the top row is part of a column. Count by ones for each part of break-apart and add amounts together using the rectangular array diagram.

Example 1: Student 1970 in Class 120 A key feature for level 1 is that the student shades each square individually while counting. Student 1970 is using the diagram to help her visualize the squares as the objects being counted which will be an amount. She combines the quantities she created by breaking the large rectangle apart into two same size arrays, but instead of adding the two quantities of squares, she doubles the number of squares on each side. She was not able to take one quantity and add it to the other quantity; rather, she invented another procedure that involved a multiplication algorithm. She does not give a reason for choosing these steps versus adding.

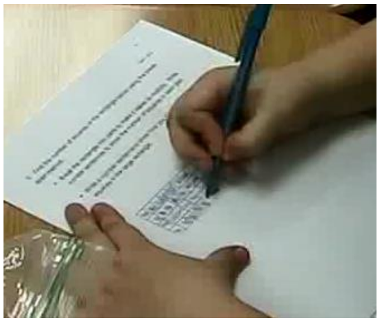


Figure 16 Student 1970 colors in each square individually

Student 1970 Transcript (Q10):

Gazes on the multiplication chart in upper left of desk. [00:11:26.07]

APPENDIX B (continued)

Puts pen in the middle of each square that is shaded on the top row. Writes "9 x 4 =" under the shaded squares, then gazes at the multiplication chart for several seconds and writes "36" next to the equal sign. [00:12:18.19]

C: (She touches each of the squares in the unshaded column closest to the shaded squares with a *hopping motion* and writes "4 x" underneath the white squares. Then she moves the pen cap along the top row of unshaded squares in a similar fashion, and makes a "9 = 36" next to the "x" in the spot below the white squares.

Then continues to solve the problem. When she has finished, [00:14:22.20] the interviewer asks,

I: So tell me what you did there?

C: Well, I umm, I did what they asked me, and shaded in (points and moves hand over the left side of the rectangle that is shaded in.) Well I colored in (moves left hand back away from the rectangle and points with right hand, moving hand back and forth.) one side of the umm.. rectangle. (Hands are now at edge of table with fingers lined up and perpendicular to table.) and then (points to the unshaded part of the rectangle) and then I did nine times four equals thirty-six, (points with left hand to the shaded part of the rectangle) cause I knew that. ... and then four times nine equals thirty-six also, (hands are now clasped between her body and the desk) so umm I added four (quickly points to *4* in the equation to the left) times four (quickly points to the equation on the right); I added four (right hand extended pointer finger to the right equation) and four (left hand moves out slightly and pointer finger points to equation on the left) and I got 8 (points to the 8 as the first number in the equation below the two diagonal lines) and then (hands are clasped together above height of desk, between desk and body) nine plus nine and I got eighteen, and then I came up with, (takes a breath) a hundred and ... a hundred and sixtee eight.

I: Okay, all right. [00:15:02.26]

Example 2: Student 1824 in Class 076 A key feature for level 1 is that the student shades each square individually while counting. Student 1824 counts by ones touching each square in the 4 x 8 section of the BAM, but gets “28” – the video does not give enough information to be certain, but it seems the column next to the vertical line was not included in the count. Instead of adding the two quantities of squares, he takes the two numbers and creates a multiplication phrase. He wrote “40 x 28” to find the quantity of the two parts giving an answer of “80.”   It is inferred that he multiplied four times two to get eight in the tens place and zero times eight to get zero in the ones place. It seems the child cannot take the result of counting the squares on the left as the starting point for adding the two quantities together.

APPENDIX B (continued)

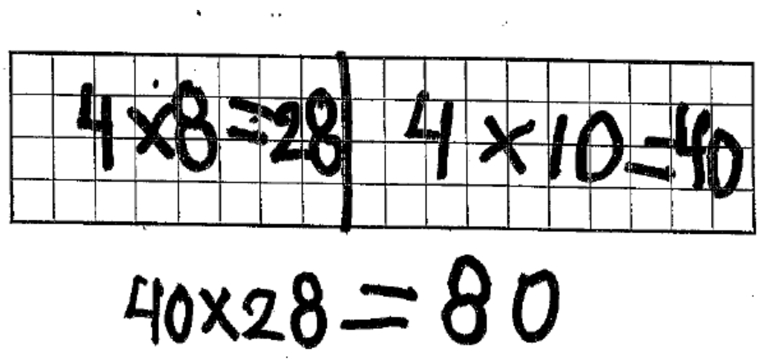


Figure 17 Level 1 Example Student 1824 (424076_1824_Q10)

Example 3: Student 2092 in Classroom 117 Student 2092 accounts for the top row only to find the total number of squares. "At the end, the interviewer asks him 'so the 9 represents the number of squares in the whole rectangle?' (12:26) Student response is that "9" is only one part. There is also "9" for the other part. Even though Student 2092 read the directions fluently, and highlighted shading when explaining the directions in his own words, it is not clear how child is thinking about anything below the top row, except for the shading of all squares on the left 2 minutes earlier. He only seems to be able to think about the quantity going across a row, versus rows and columns.

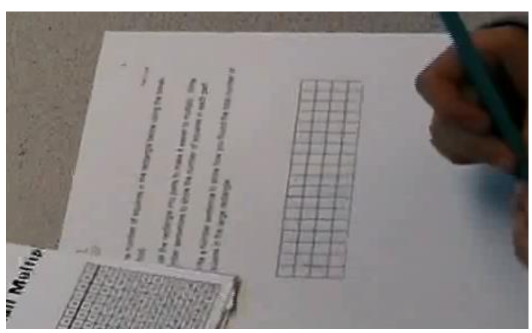


Figure 18 Level 1 Example Student 2092 in Classroom 117 doing Q10

Student 2092 Transcript Q10

Can you read the directions for me? [00:07:22.09]

C: (reads fluently with expression - deleted "to" and left of an "s")[00:07:46.27]

I: Okay, so tell me what you need to do.

APPENDIX B (continued)

C: I need to ... I'm going to count (touches the top row leftmost square and then moves pen along the top row) these squares (moves to the left and to the right with pen along top rows) and then would equals (moves pen in the middle) well, I'm going to cut it in half (moves pen from above top of rectangle to below rectangle about in the center of the rectangle if going from left to right) and if that part, fourteen, (moves pen along top row to the left and then to the right, stopping in the middle) and that part, thirteen, (moves pen to the far right and then to the left so pen is back in middle of top row) I'm going to shade that part in (moves pen as if shading in the left side of rectangle).

I: okay [00:08:04.21] sounds good.

C: Starting at the leftmost square in the top row, student points at square, jumps to next one across the top row. Pauses with right hand gripping pen over right side of directions, and face looking forward, not at paper, moves body out of side of camera. [00:08:52.02] and reaches for multiplication table. The next minute includes counting and recounting across a row, and shading the left side of the rectangle.

[00:10:01.22] He puts pen cap in left hand on pen in right hand and then takes off pen cap, moving pen to right side of rectangle, and then goes to far left top square. He points to each square in the top row that is shaded, and moves pen to lower part of shaded rectangle then moves head to look closely at rectangle and pen.[00:10:12.10] Moves head back, moves pen to above the blank space between the directions and rectangle. [00:10:19.22] Moves head closer to paper, as if looking at rectangle. Moves pen towards rectangle [00:10:43.25] Writes "9 + 9 = 18 under the rectangle. Below this, writes "I did 9 + 9 = 18" and moves pen hand to point pen at directions. [00:11:19.12] Moves pen from equation to the rectangle and back to pointing at the equation under the rectangle. [00:11:24.26] Writes "be" but gets pencil and crosses it out. He picks up pen to continue writing on the second line: "and I" then goes to form a 3rd line "got 9 from counting the squares" Puts down pen and moves paper. [00:12:08.13]

I: So explain to me what you your work here.

C: I did 9 plus 9 equals 18. (moves pen tip along the equation under the rectangle.)

I: mm hmm

C: and I got 9 counting those squares.

I: okay, so umm. so 9 plus 9 equals 18 (points to the equation) so the 9 represents the number of squares in the whole rectangle? [00:12:26.09]

C: no - for this side (points to the shaded side) I shaded that in because that part was 9 and then (moves hand to the right side) and that side's 9.

APPENDIX B (continued)

I: Okay. Alright. Sounds good. [00:12:34.11]

Unitizing Level 2

Model	*Observable Actions in a Predicted Scheme for the Model*
COORDINATING 2 LEVELS OF UNITS IN ACTIVITY (Rating = 2) In the activity of using rectangle to keep track of the quantity both as singleton units and as composite units inserted into another composite unit at the same time	Keep track of quantity as more than singletons in activity. For instance, touch row or column to keep track of multiples, verbally count with an emphasis on the last number (1,2,**3**, .. 4,5,**6**…7,8, **9,** etc.) with a hand sweeping gesture or drawing a line to go over the squares being counted on the diagram. Counting groups where the rectangle is seen as a set of composite ones, but the student interacts with the diagram in order to arrive at grouping. Interacting with the rectangle diagram or in other activity can keep track of the singleton amount, and the number of groups at the same time.

Example 1: Student 1975 in Classroom 120 In Level 2, the diagram, fingers or some other element in the activity of solving the problem situation provides the needed support to keep track of multiples while also aware of the amount of singletons in the quantity. Student 1975 indicated to solve Q10 she picked five as the number of squares for one side of a rectangle because she knows her *5s*, and she is using this fact to split the rectangle into as many 4x5s as is possible. (9:58). She made a motion over each group of five in a row in order to find *4 x 5*, which, given the timing, infers skip counting. With the rectangle to help keep track, she is able to show the multiplication sentence on the array of squares to help her see each section as composite units and a section of unitary ones. She found the amount for the entire rectangle by adding the number of squares in each section, unlike students in level 1. There are no gestures or phrases that indicate Student 1975 is thinking about *4x5* while also thinking about *20* for a given section. In determining the number of squares for *4x3*, she touched each square in a zigzag pattern while saying each number with equal softness. (9:41- 9:52)

Example 2: Student 1971 in Classroom 120 This student uses the diagram in Q10 to keep track of multiples. She splits the rectangle into two equal parts, finding the number of squares on the left by touching each of the nine squares in a row, but always starts a row count on the left for an emphasis on the last number. She wrote *9 x 4* above this section. To find the number of squares on the right, instead of counting across, she slides pen down a column (or part

APPENDIX B (continued)

of a column) 9 times. I infer she is counting by 4s each time her pen tip slides down a column. She writes *4 x 9* above the right section. The order of the factors is different just as the number iterated was different. She could not find the total number of squares for the rectangle when asked.   Since she used the diagram to help her keep track of a group of singletons within either the group of 4 rows or of 9 columns I infer she is using the activity of working with the diagram to enable her to insert one grouping into another, but cannot connect the two quantities she determined with emphasized counting or skip counting to each other.

Example 3: Student 2096 in Classroom 117   The student could not find a way to split the rectangle to solve Q10, but in using standard multiplication procedures, needed to find *32*. He

relied on fingers to represent a composite unit of four, extending the four fingers of his hand one at time, keeping track of 8 of such groups of 4 using his other hand (8:41 - 10:03). With the support of his finger activity he inserted a quantity of four into a group of eight to achieve the quantity of eight times four. Then he used his fingers to help add *32 +4* (10:25 - 10:32) in order to account for the *1* in *18 x 4*. Using his hands to keep track of the count and the number of counts, he was able to insert a unit amount into another unit, even though he could not figure out a way to do this with the array diagram.   He was not able to recognize the meaning of eighteen times four as ten times four plus eight times four without support.

Student 2096 Transcript for Q10

Writes *18 x 4 =* using standard column multiplication format. Moves fingers of pen hand one at a time starting with pinky. Moves left hand under the desk. [00:08:41.11] Makes a fist with pen hand, then extends fingers one at a time starting with first finger. Leaves finger extended when extends another. so umm [00:09:06.13] I do 4 times 8, (in whisper voice) 17, 18, 19, 20, (slight view of fingers on left hand shows them extended, as if using for counters. hands/fingers are moviing towards and away from the pen hand. Extends first finger [00:09:34.18] then closes pen hand fingers into a fist, but then extends both first and middle fingers. Continues in a pattern of opening and closing fingers.) 32?

I: okay [00:10:03.11]

APPENDIX B (continued)

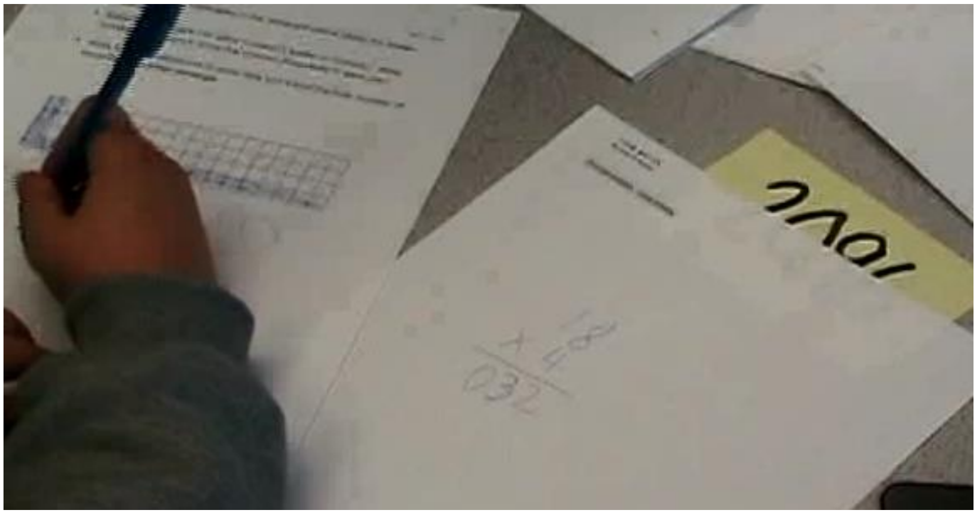


Figure 19 Level 2 Example student 2096 doing Q10

Example 4: Student 2080 in Classroom 117 In the problem Q10, Student 2080's method for determining eight times four indicates Level 2. She used an emphasized count of multiples of four keeping track of the number of multiples and the counting number at the same time, but needed the support of her hands and recording the number of singletons counted so far on paper to complete the task. She split the array into two parts, four rows of eight and 4 rows of ten. Unlike some students determined to be at Level 2, she used the diagram and her tens fact knowledge to recognize the number of squares would be 40 plus 32, but she could not connect this to the entire array as 18 groups of 4.

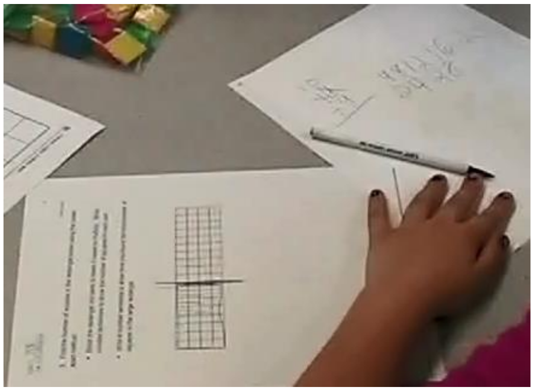


Figure 20 Level 2 example student 2080 in Class 117 doing Q10

Student 2080 Transcript for section of Q10

I: SO this is one (points to the left side of the line) and this is the other? (I: points to the right side of the line)

C: yeah

APPENDIX B (continued)

I: and how many did you get in each box?

C: umm... In this one it would be (moves hands and pen hand is next to where the 4, 8 was written earlier) four, eight, (moves pen tip back and forth), [00:12:15.12] 12, 16, (each time says a number from 17 to 20, hold out another finger) 17, 18, 19, 20. (writes 20) 24 (writes 24, and as says 25, 26, holds out one finger for each, then moves another finger and another finger,) 25, 26,...28 (writes 28). 29, 30, 31, 32 (each time says a number, holds out another finger on right hand, after saying 32, picks up pen and writes 32 on the scratch paper next to the 28.) One, two, three, four, five, six, (each time says a number points to a number in the list of multiples of 4 on the scratch paper with a hopping motion) seven, eight (picks up hand with pen and uses left hand to move hair to behind ear) [00:12:39.01]

I: okay [00:13:54.10] Then can you write a number sentence (points to the paper below the array) to show how you got the whole (moves finger around as in a circle around the array.) thing, like you were saying here?

C: Yeah, so, four times eight (writes *4 x 8* in a line, under the other calculation) four times ten (writes *4x10* under the *4x8*)

I: okay

C: and then ..umm. (draws a line under the 4 x 10) like that, and it would be plus (draws a plus sign next to the *4* of *4x10*)

I: ah okay

C: So it would be, 32 (points with pen to the top row) plus 40 (moves pen across the second row) and that would equal 72. (Points to the third line, "72" then picks up hands and moves them to lower part of the paper, past the sides of the paper, pen still in right hand.

I: perfect I've got your understanding now. [00:14:22.19]

APPENDIX B (continued)

Unitizing Level 3

Model	*Observable Actions in a Predicted Scheme for the Model*
INTERIORIZING 2 LEVELS OF UNITS (Rating = 3) The array is seen as composed of units inserted into other units prior to interacting with the diagram or story problem	Student sees an array as composed of multiple units prior to activity. Part-whole reasoning is developing. Verbal count in multiples or with verbal/gesture indication of matching row/column to the array area without having to touch each area unit. For example, extends a finger for each count – 4, 8, 12, 16, or, touches a row of 9, touches the next row and says 18; Thinking in composite units where each is also considered a distinct quantity, such as 5 groups of 4 rows [5 of 4-units]. Making a sweeping motion over a row of squares as well as a student explaining “eight and eight is 16, then 16 plus 16 is 6 +6 is 12 and 10 + 10 is 20, so then 12 + 20 is 32, so its 32” or “twelve is ten plus two more.”

Level 3 Example 1: Student 1952 in Classroom 120 For Level 3, a student can think about a quantity as an amount of singletons as well as a group times the number of groups and can go back and forth between the mental images. Student 1952 in Q10 breaks the rectangle into 4 equal parts with dimensions of two by nine. She recognizes the sum of four *18s* is the same as four 8s plus 4 tens, without needing to re-present. She uses knowledge of units within units to make the calculation within her reach. Splits the four 8s into 16 + 16, so recognizes the sum is the same if 16+16 or 8+8+8+8. She does not connect the number sentence for the parts and singleton quantity to the entire rectangle’s four times eighteen. Perhaps because to do so would involve connecting 18 fours to the grouping of four 8s plus 4 tens that she used to calculate 72 singletons.

Transcript for Student 1952 in Classroom 120 for Q10

C: [00:06:33.19] and then I could break it apart. (While writing what she is saying in the upper left hand 2x9 rectangle) Nine times two equals 18 (goes across to the right hand 2x9 rectangle) Nine times two equals ... 18 (goes to the lower left hand 2 x9 rectangle) Nine times two equals 18 (lifts hand up and goes to lower right hand rectangle writes while saying) Nine times two equals 18 [00:06:52.09]

C: And then, (moved pencil to below the rectangle and writes 18 each time the number is spoken with the 18's in a column) 18...plus 18, plus 18, plus 18 (Draws a line under the lowest 18 and lifts pencil.) [00:07:01.27] Eight (touching the 8 in the top "18"), sixteen, those two equal 16 (twirls pencil and it lands on paper eraser side touching paper. Left hand swipes the paper Changes position of pencil to lead tip down and writes what is saying) I'll do sixteen plus sixteen

(Touches the 6 and the other 6 and says) Six plus six is twelve, carry the one (writes 1 above the top 1 in "16", is 36. (writes 32 and lifts pencil up).[00:07:16.13] Put down the two (writes a two below the line under the 8 in 18) carry the three (writes a three above the 1 in the topmost "18") Three, four (pencil in air), five (pencil tip touches 1), six,(pencil tip touches 1) seven, (pencil tip touches 1, draws a 7 under the line under the 1s) and I got seventy-two. (looks up at interviewer.)

I: Okay [00:07:26.02] Umm … Did you write the number sentence to show (points to the directions that state this) how you found the number of squares in the rectangle?

Ch: no... I mean yes

I: Can you show me where?

Ch: Turns paper to face the interviewer and points to the line between the 18s and 72. [00:07:48.06]

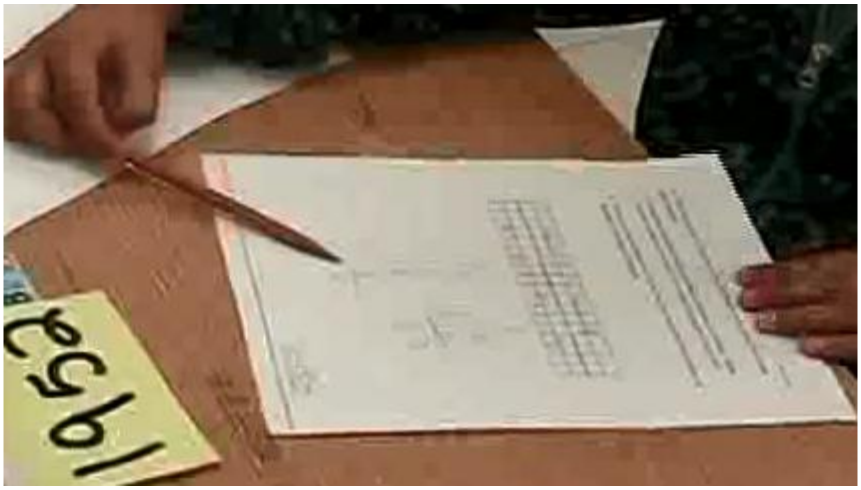


Figure 21 Level 3 example: student 1952 classroom 120 showing number sentence for Q10

Example 2: Student 2090 in Classroom 117 Student 2090 explains his special way to find a quantity - split the number to be multiplied into known factors and skip count. - May not know math facts well or know how to read well, but shows he can coordinate units by taking the composite unit of 8 and considering it to consist of 5 and 3 while at the same time being 8, and then use the 5 as part of a group of four 5s and use the 3 as part of a group of four 3s, showing he can use part-to-whole reasoning with regard to units. This is a characteristic for students who can coordinate two levels of units within their head. He combined the quantities of the smaller arrays to find the total number of squares, but in doing so did not keep track of the 18 rows of four aspect of the rectangle, focusing instead on the number of singleton squares.

Transcript for Student 2090 in Classroom 117 for Q10

APPENDIX B (continued)

C: And this is ... one, two, three (takes pen in right hand, touches each square in top row while saying next counting number) four, five, six, seven, eight. Four times eight (writes this on the unshaded squares while saying the words) equals (writes the equals sign next to the *8*) [00:11:43.28]

C: (makes a fist with left hand, pen still in right hand) Four, eight (says while putting out thumb, then scratches the wrist with the thumb and forefinger, as if getting a sticker off the wrist. Moves the pen hand to the paper with the pen point to the paper. (Draws *8 8 8 8.* Puts the pen point on the 8's farthest to the right.) Five. (Moves the pen tip to the next 8 to the left) Ten. (Moves the pen tip to the next *8* to the left) Fifteen. (Moves the pen tip to the first *8*) Twenty. (Writes *20* below the *8s* on the scratch paper. Puts the pen on the *8* on the left.) Three (Moves the pen to the *8* to the right.) Six. (Moves the pen tip to the *8* on the right.) Nine. (Moves the pen tip to the *8* on the right.) Twelve. (Writes a messy 12, where the *2* looks like a *7*. Draws a line under the *12* and under the line writes what looks like a *7* and then to the left of the *7*, writes a *3*) [00:12:09.08] I do it different than any other kid.

I: What is your strategy, then?

C: Ohh, I break up the number into smaller pieces without even breaking it.

Level 3 Example 3: Student 2083 in Classroom 117 In this example from Q10, the student skip counts 8s to find four times eight. This is an indicator of inserting a unit of 8 into a unit of 4, keeping track of the number of singletons.

Transcript for Student 2083 in Classroom 117 for Q10

C: Eight... times four. (moves pen to the shaded squares and writes *8 x 4 =*) Eight times four (pause) equals (transfers pen to the right hand and closes in fingers on left hand, moves left hand and puts out a thumb) Eight (moves left hand up and down, puts out index finger while leaving thumb extended) Sixteen... (moves another finger out) Twenty-four, (moves hand and puts out another finger) Thirty...two (puts out another finger, but then puts out pinky finger and quickly puts arm down on table.) Eight, (moves hand up and down and puts out thumb) sixteen, moves hand up and down and puts out first finger) thirty-two, (hand up and down and put out middle finger, keeping the rest extended) forty (puts out ring finger with other staying extended. Then makes a fist as he lowers hand and arm, but quickly moves fist up and extends the thumb [00:06:59.05] (In a whisper voice,) Eight (extends thumb) Sixteen (also extends first finger after moving hand up and down), moves hand sideways towards right, then back to left side of paper) Twenty-four, Thirty-two. (Raises hand to transfer pen to the left hand, and writes 32 while saying) Thirty-two. (Lifts hand with pen and moves it up and down about 2 inches.

APPENDIX B (continued)

The student also show he is thinking about putting all the square together. He makes parentheses to show the two parts to be combined and then inserts the appropriate factors inside. However, he does not represent or refer to the entire rectangle in terms of four row of 18 or vice-versa, which would be more typical of a student at level 4.

Unitizing Level 4

Model	*Observable Actions in a Predicted Scheme for the Model*
COORDINATING THREE LEVELS OF UNITS IN ACTIVITY (Rating = 4) Working with multiple groups with multiple units in each group while using the actions with the diagram or aspects of the problem to consider three levels of units at one time.	Working with multiple groups with multiple units in each group while using the diagram to keep track of their activity. Through facial expressions, pointing, verbalizations or other gestures show concurrently keeping track of rectangles within the larger rectangles and the larger rectangle at the same time. For example, 10 rows of 4 is 40 and 8 x 4 is 32, so 18 x 4 = 72 where students reason with 10 rows of 4 units and 8 rows of 4 units, understanding that both are contained in, and constitute, the entire rectangle. Because they establish this unit structure in activity, they still need to act on their diagram, hands, etc. to know that the 72 is also 18 rows of 4.

Example 1: Student 1960 in Classroom 120 There are two key pieces of evidence that show this student is thinking about the entire rectangle while also thinking of it as composed of the two smaller rectangles and the number of singleton pieces in it. First, the gestures used to describe the task included hand motions to show he is finding the number of squares for the entire rectangle by making circular motions over it, but then in explaining the process, also made smaller circular motion over the parts (7:10 – 7:23) indicating the smaller rectangular array squares fit into the larger one. He also wrote *4 x 18* as part of the equation at the onset. When he broke the rectangle apart and calculated the number of squares, he connected the two subparts to the whole in the equation writing and explained his answer in terms of finding the squares in a four by eighteen rectangle. Second, in finding the number of squares for the second rectangle, he did not recount the number of squares in a column. In finding the number of squares for nine times four, he used a standard hand mnemonic movement in a purposeful manner, focusing on placement of the finger to bend and the number of groups of nine. Without a prompt from the Interviewer he was able to find the number of squares in each part and add them together as shown in the transcript.

APPENDIX B (continued)

Transcript for Student 1960 in Classroom 120 for Q10:

C: [00:08:30.20] (lifts up pen and moves it to the *4 x 18* and writes = next to the *18*) I would do four times 18 equals 36 plus 36 equals [00:08:41.13] (pause moves pen to below *36s* and writes in column addition format while talking: *36 + 36=*) You do addition sentence ... six plus six equals twelve, carry the one, four times three equals seven. (He writes a *2* below the *6*, a *1* above the *3s* and the *7* under the *3s*. Moves pen to the equation and writes *72* while saying) the answer is 72.

Example 2: Student 1286 in Classroom 076 Student 1286 initially labels the large array sides in Q10, indicating her awareness of the entire rectangle, but also labels the smaller rectangles as well. It is how she talks about these rectangles that supports she is coordinating the eighteen groups of four is composed of nine groups of four along with another nine groups of four, and not providing a rote display of the directions. Also, she includes the parentheses to show the smaller rectangle, which many students who have been exposed to the practice do not do. She also counts in multiples across the top row, showing she could keep track of fours while keeping track of the target number nine. The *4* on the left side of the array is referenced for both of the small arrays as well as the large one, showing she can think about the large array as well as the two smaller arrays within it in interchangeable ways, giving an example of part-to-whole reasoning.

Transcript for Student 1286 in Classroom 076 in Q10

C: (while pen is close to the right side of each square in the top row, says the next number name as points to the next square:) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18. So I divide, mm, (moves pen across four squares in the top row) four, (moves pen over four more squares) eight, nine. [00:06:21.28] (Draws a line between the ninth and tenth columns. Writes *9* above the squares to the left of the vertical line and another *9* above the squares to the right of the vertical line. Writes a *4* to the left of the rectangle, then writes *4 x 9 = 36* within the left side of rectangle while saying) four times nine equals thirty-six and (writes *4 x 9 = 36* within the right side of the rectangle while saying) four times nine equals thirty-six. So I just [00:06:39.14] (moves pen hand to below the rectangle to write *4 x 18 = (4 x 9) + (4 x 9)* while saying) so I just put four times eight equals umm nine, wait, four times nine plus four times nine [00:06:53.17] and then (moves to the row below and writes *4 x 18 = 36 + 36* while quickly saying:) four times eighteen equals thirty-six plus thirty-six) [00:07:00.27] (Then moves to the next row under what has been written and writes *4 x 18 =* while saying:) four times eighteen equals, umm, two, umm, is seventy-two. (She writes *72* next to the equals sign. Lifts head up to look at interviewer while picking up pen.)

I: ahh. How did you decide to break it apart this way? [00:07:13.19]

APPENDIX B (continued)

C: because, there's like eighteen over here (moves pen from left to right across the top row of squares) so you want to put in half, so eighteen divided by two equals nine and so I count nine and broke it apart.

Example 3: Student 1286 in Classroom 076. Student 1713 show awareness of both the two smaller rectangles and the large rectangle at the same time. After labeling the large rectangle's sides and quickly creating and labeling two smaller rectangles within the larger one she describes splitting the large rectangle into parts using oval motions for the entire rectangle and ten by four rectangle and then moving her hand as if going row by row for the eight by four section as she talks about the multiplications for both. At the end she explains 18 x 4 in terms of the singletons from each, showing she can think about splitting eighteen groups of four into 10 groups of four with the eight groups of 4 levels of units with the help of the diagram.

Transcript for Student 1286 in Classroom 076 for Q10

I: so tell me how you figured that out- [00:06:45.00]

C: I did, umm (Switches from pencil point by the paper to pencil eraser next to the rectangle) I did this way because, by 10 x4 because it will make it easier to multiply. (moves pencil eraser left and right across all of the rectangle.)

I: mm hmm [00:06:51.24]

C: So I did ten times four equals forty (moves eraser tip to the left and right on the equation *10 x 4 = 40* on the left side of the rectangle) and then … I breaked it apart it (with pencil eraser retraces the vertical line in the middle of the rectangle. Moves pencil between left and right parts of rectangle.) So could, so like four (points to the *4* on the left side) and ten (points to the top of the left side of rectangle) could make it easier to multiply- goes up-down and then sideways with the pencil eraser.

I: mm hmm

C: and umm I did eight times four equals thirty-two so that I umm, you know there are eight columns (goes across the top with the pencil eraser) and four rows (points pencil at top just to right of rectangle, and then moves pencil eraser from top to bottom next to rectangle) and that equals thirty-two, because that might be the left over (moves pencil from left to right and right to left across the right side of rectangle that is marked *8x4*).

I: mm hmm

APPENDIX B (continued)

C: and then I did eighteen times four equals forty from the product (points to the right side labelled *4*, plus thirty-two, the product too (points to the right side where the equation shows *32*), and equals seventy-two. (Moves pencil from pointing at the equation up above the paper, to clasp

hands together, with the pencil between thumb of right hand and first finger and resting on the left hand) at the edge of the desk, in line with the rectangle paper.

I: okay, all right, let's move on to something else [00:07:26.21]

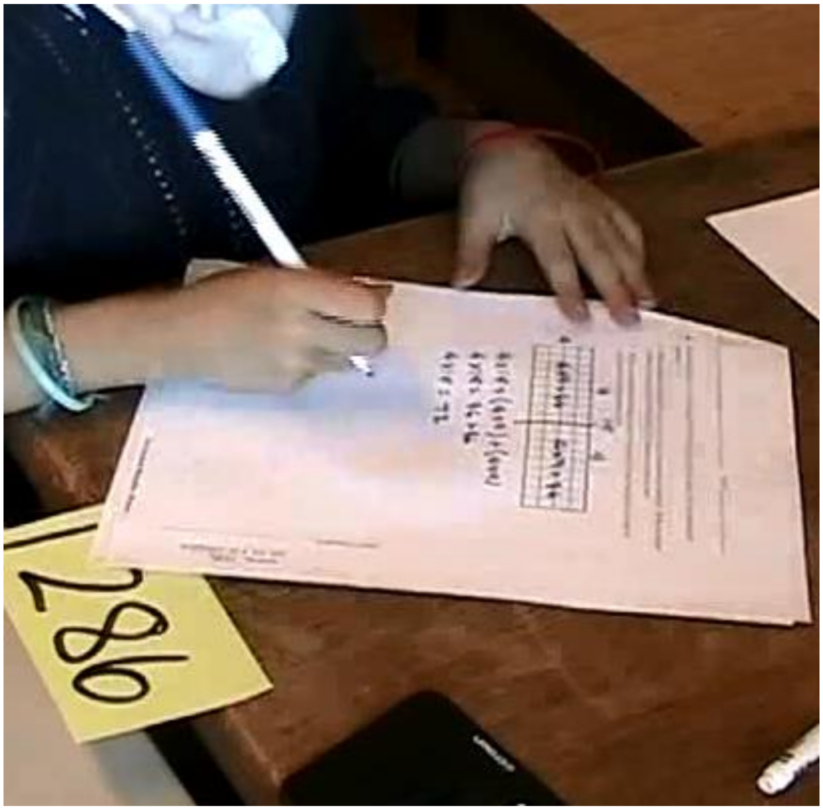


Figure 22 Level 4 example student 1286 doing Q10

Example 4: Student 1810 in Classroom 058 Students with Level 4 start with sense of the problem's rectangular array comprised of rows and/or columns that are in and of themselves a unit, so the description of a smaller array of squares is often referenced more frequently by the rows and columns as an array than by pointing. Student 1810 counts the 18 across the top row in multiples, then after she counts out the first 9 to find the halfway point, she does not count the rest, indicating she knows the two parts equal the total (6:29) and the multiplication sentences would be the same, and worked to make the "4x9" in the same factor order on both sides, as if thinking "4 rows of 9" as is in the instruction (6:58). She includes the dimensions of the large rectangle as equal to the products of the two smaller rectangles, (gestures make connections

APPENDIX B (continued)

between the equation and the square being counted 7:39-7:47). Although she calculates an incorrect total (I infer that in adding 36 + 36, she multiplied 3x3 instead of adding 3+3, thus getting “9” +1 to make 10 instead of “6” + 1 to make 7), she can think about splitting eighteen groups of four into nine groups of four and nine groups of four and use a process to find the number of singletons, (even though her calculation was incorrect) with the help of the diagram..

Transcript for Student 1810 in Classroom 058 for Q10

C: (whisper voice:) three (hand with pen is near the top row and is making broader jumps than 1 square at a time.   There's eighteen in all. (looks at interviewer, then writes "18" above the rectangle and lifts hand with pen away from paper. Looks at interviewer,) I'm going to break it apart so I could (pen hand is leaning on the rectangle now and eye focus is back on the paper.)[00:06:29.01] make it easier.

I: mm hmm.

C: picks up pen. So I'm going to break the 18 into halves. (moves pen up and down near middle of the rectangle.) So I'm doing 1, 2, 3, 4, 5, 6, 7, 8, 9, (as says the number, moves to the next square to the right with a small jump, starting with the upper left hand square. Draws a line between the 9th and 10th columns.) So I'm going 9...4 times 9 equals 36 (first wrote a 9, but changes it to a 4, followed by x9 = 36. Moves hand to the right side and writes 4 x 9 = 36 while saying:) 4 times 9 equals 36.[00:06:58.11] (moves pen hand off paper closer to body and shifts how she is sitting. Then returns pen paper) There I'm goin do a number sentence, as you see (moves pen from left to right across the rectangle) from there. I'm going to do 4 times 18 (writes *4x18 = 36 + 36 =*) equals 36 plus 36 equal... (adjusts seat; on the left side of paper, writes *36* with a *36* below it and a vertical line below that as in column addition. Flicks pen back and forth. Eyes flick, mouth moves, pen flicks; writes a *1* above the *3* and writes *102* below the lower line) equals a hundred two. (he looks at Interviewer)

I: A hundred two...... so how did you know to add 36 plus 36? [00:07:39.08]

C: because. ahh... I saw (moves pen in hand to make the pen cap edge touch the equation) .. cuz here (points with back of pen to the leftmost 36 in the equation; then makes pen point go around in a circle over the left side of rectangle and then points to the *4x9=36* on the left side), I did on my number sentence (moves pen top under the number sentence from left to right, picks up pen [00:07:47.01]

I: uh huh

C: and when we do break-apart (pen is pointing at top of rectangle pen goes back and forth, and then in the middle points to the line, then makes an oval over the left side and puts pen on the left

APPENDIX B (continued)

side equation again) we have to umm write what it equals (mimics the writing of the equation on the left side with the pen top above the left side) then plus (pen is pointing to the 36+36) and then (moves pen to the right side of rectangle and makes an oval over the top of the right side rectangle, then points to the equation under the rectangle) add that up (moves pen in hand so that the pen tip is by the paper near the equation.

APPENDIX C

The reasoning for the choice of cBAM score during the reconciliation of the three discrepant unit coordination levels where the Q10 and T5-10b scores were not the same.

1639 In one of the two instances, there was not enough information to produce a more advanced rating than indicated on other example

1962 The student constructed a situation that left the expected number range for the grade level expectations. An inference that the student did not see grouping clues that are in a more typical situation, so the student reverts to counting in the second instance.

1968 – The student used a perceptual strategy for 8x4 in Q10, but with known facts, the student's actions provided for using three levels of units in T5-10b.

APPENDIX D

Approval Notice
Final Report – Expedited Review

April 25, 2018

James Pellegrino, PhD
Psychology
Phone: (312) 996-2488 / Fax: (312) 413-7411

RE: Protocol # 2007-0485
"Evaluating the Cognitive, Psychometric, and Instructional Affordances of Curriculum-Embedded Assessments: A Comprehensive Validity-Based Approach"

Dear Dr. Pellegrino:

Members of Institutional Review Board (IRB) # 2 conducted a review of your final report of April 23, 2018 under expedited review procedures [45 CFR 46.110(b)(2)] on April 25, 2018 and acknowledged the completion of your research. We would like to thank you for your continued compliance with the UIC OPRS Human Subject Protection policies including the submission of a final report notification.

If you have any questions or need further help, please contact the OPRS office at (312) 996-1711 or me at (312) 996-0548. Please send any correspondence about this protocol to OPRS at 203 AOB, M/C 672.

Sincerely,

Brandi L. Drumgole, B.S.
Assistant Director
Office for the Protection of Research Subjects

cc: Michael E. Ragozzino, Psychology, M/C 285