On Restricted Tangent Bundles Of Grassmannian, And Betti Numbers Of The Moduli Of Stable Sheaves On $\mathbb{P}^2$

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THESIS

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To my parents

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SUMMARY

This thesis is based on work done on two different problems. The first problem is regarding restricted tangent bundles of the Grassmannian to rational curves. Let $n \geq 4$, $2 \leq r \leq n-2$ and $e \geq 1$. We show that the intersection of the locus of degree e morphisms from $\mathbb{P}^1$ to $G(r,n)$ with the restricted universal sub-bundles having a given splitting type and the locus of degree e morphisms with the restricted universal quotient-bundle having a given splitting type is non-empty and generically transverse. As a consequence, we get that the locus of degree e morphisms from $\mathbb{P}^1$ to $G(r,n)$ with the restricted tangent bundle having a given splitting type need not always be irreducible.

The second problem is regarding the Betti numbers of the moduli space of sheaves on the projective plane. Let $r \geq 2$ be an integer, and let a be an integer coprime to r. We show that if $c_2 \geq n + \lfloor \frac{r-1}{2r}a^2 + \frac{1}{2}(r^2+1) \rfloor$, then the $2n$th Betti number of the moduli space $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(r, \mathcal{O}_{\mathbb{P}^2}(a), c_2)$ stabilizes.

CHAPTER 1

INTRODUCTION

In this thesis, we study the following two items:

- the locus of restricted tangent bundle of the Grassmannian to rational curves with a given splitting type, and
- the Betti numbers of the moduli space of stable sheaves on $\mathbb{P}^2$ with a fixed Chern character.

We show that

Theorem (Corollary 3.3.7)**.** *When $2 \leq r \leq (n-2)$, the locus of morphisms f from $\mathbb{P}^1$ to $G(r,n)$ of degree $e \geq 1$ with the restricted tangent bundle having a specified splitting may not always be irreducible.*

Moreover, we also show that

Theorem (Theorem 4.4.1)**.** *Assume that the rank $r \geq 2$ and the first Chern class a are coprime. If $c_2 \geq N + \lfloor \frac{r-1}{2r}a^2 + \frac{1}{2}(r^2+1) \rfloor$, then the $2N$th Betti numbers of the moduli space of stable sheaves on $\mathbb{P}^2$ of rank r, first Chern class a, and second Chern class c_2 stabilizes.*

1.1 Rational curves and the Grassmannian

Rational curves play a central role in the study of algebraic geometry of projective varieties. Let X be a non-singular projective variety over an algebraically closed field $\mathbb{K}$

of characteristic zero, and let $C \subset X$ be a rational curve. We can study vector bundles on X by studying their restrictions to rational curves. This approach is often useful due to Grothendieck's theorem which tells us that given any vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank $r \geq 1$, there exists a unique collection of integers $a_1 \leq \cdots \leq a_r$ such that $\mathcal{E}$ is isomorphic to the direct sum of the line bundles $\mathcal{O}_{\mathbb{P}^1}(a_i)$, for $1 \leq i \leq r$. We call this collection of integers $a_1, \cdots, a_r$, the *splitting type* of $\mathcal{E}$.

The two bundles which are especially important to study are $T_X|_C$ and $N_{C/X}$ because they help us in understanding the deformations of C in X and in understanding the geometry of the tangent space of smooth rational curves on X. These vector bundles have been studied by Eisenbud and Van de Ven (1), (2), and by Ghione and Sacchiero (3), (4), (5) who characterized the possible splitting types of the normal bundle of rational curves in $\mathbb{P}^3$ and showed that the locus of rational curves in $\mathbb{P}^3$ with whose normal bundles have a specified splitting type is irreducible of the expected dimension. Ran (6) determined the splitting type of a generic genus-0 curve with one or two components in $\mathbb{P}^n$, as well as the way the bundle deforms locally with a general deformation of the curve. More recently, Coskun and Riedl (7), (8) showed that the locus of nondegenerate rational normal curves in $\mathbb{P}^n$ of fixed degree having a specified splitting type of the normal bundle can be reducible when $n \geq 5$.

In a similar vein, Verdier (9) and Ramella (10) showed that the locus of nondegenerate rational curves in $\mathbb{P}^n$ with a given splitting type of the restricted tangent bundle is irreducible of expected codimension. Strømme (11) examined a nice compactification of this locus as a certain Quot scheme and computed the Chow ring of this compactification.

In this paper, we study the locus of degree e morphism from $\mathbb{P}^1$ to the Grassmannian variety with a specified splitting type of the restricted tangent bundle.

Let $G(r,n)$ denote the Grassmannian variety of r-dimensional subspaces of the n-dimensional vector space $\mathbb{K}^{\oplus n}$. The Grassmannian variety has two special vector bundles, the universal sub-bundle $\mathcal{S}$ of rank r and the universal quotient bundle $\mathcal{Q}$ of rank $n-r$. Given a r-dimensional subspace Λ of $\mathbb{K}^{\oplus n}$, let $p_\Lambda \in G(r,n)$ be the point corresponding to this subspace. Then, we have

$$\mathcal{S}|_{p_\Lambda} = \Lambda \qquad \text{and} \qquad \mathcal{Q}|_{p_\Lambda} = \mathbb{K}^{\oplus n}/\Lambda$$

Moreover, these vector bundles fit together in an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{G(r,n)}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Additionally, the tangent bundle to the Grassmannian variety $G(r,n)$, denoted $T_{G(r,n)}$, is isomorphic to $\mathcal{S}^* \otimes \mathcal{Q}$. We denote by $Mor_e(\mathbb{P}^1, G(r,n))$ the scheme parameterizing degree e morphisms from $\mathbb{P}^1$ to $G(r,n)$. We know (see Lemma 2.1.2) that this scheme is a non-singular quasi-projective variety of dimension $r(n-r)+ne$. We denote by $M(b_\bullet)$ the locus of morphisms f in $Mor_e(\mathbb{P}^1_{\mathbb{K}}, G(r,n))$ with $f^*(\mathcal{Q})$ having splitting type $0 \le b_1 \le \cdots \le b_{n-r}$, and by $M'(a_\bullet)$ be the locus of morphism f with $f^*(\mathcal{S}^*)$ having splitting type $a_1 \ge \cdots \ge a_r \ge 0$. We first show that

Proposition [Proposition 2.1.6]. *The loci $M(b_\bullet)$ and $M'(a_\bullet)$ are smooth of the expected codimension.*

This follows as a consequence of a Corollary due to Le Potier (12)[Corollary 15.4.3]. We then show that

Theorem [Theorem 3.2.9]**.** *Let $n \geq 4$ and $2 \leq r \leq n-2$. The intersection of the loci $M(b_\bullet)$ and $M'(a_\bullet)$ is nonempty and generically transverse.*

Note that the locus of degree e morphisms f from $\mathbb{P}^1$ to $G(r,n)$ with $f^*(T_{G(r,n)})$ having a specified splitting type is stratified by the intersection loci $M(b_\bullet) \cap M'(a_\bullet)$ coming from possible splitting types $\{a_\bullet\}$ and $\{b_\bullet\}$ of $f^*(\mathcal{S}^*)$ and $f^*(\mathcal{Q})$ respectively. We know that $\overline{M(b_\bullet)}$ is the union of $M(b'_\bullet)$ where $0 \leq b'_1 \leq \cdots \leq b'_{n-r}$, $b'_1 + \cdots + b'_{n-r} = e$ and $b'_j + \cdots + b'_{n-r} \geq b_j + \cdots + b_{n-r}$ for all $1 \leq j \leq n-r$. Similarly, $\overline{M'(a_\bullet)}$ is the union of $M'(a'_\bullet)$ where $a'_1 \geq \cdots \geq a'_r \geq 0$, $a'_1 + \cdots + a'_r = e$ and $a'_1 + \cdots + a'_i \geq a_1 + \cdots + a_i$ for all $1 \leq i \leq r$. Thus, there exists intersection loci $M(b_\bullet) \cap M'(a'_\bullet)$ which are closed in the locus of all morphisms with restricted tangent bundle having a specified splitting type. Consequently,

Corollary [Proposition 3.3.4, Corollary 3.3.7]**.** *The locus of morphisms f with $f^*(T_{G(r,n)})$ having a given splitting type can be reducible in general, and it has at least one irreducible component coming from a closed intersection loci $M(b_\bullet) \cap M'(a_\bullet)$.*

For example, (as a consequence of Corollary 3.3.7 and Lemma 3.3.8) the locus of morphisms in $Mor_e(\mathbb{P}^1, G(2,4))$ with restricted tangent bundle having splitting type c_1, c_2, c_3, c_4 with $c_1 \leq c_2 < c_3 \leq c_4$ and $c_1 + c_2 + c_3 + c_4 = 4e$ has at least two irreducible components.

This is in sharp contrast with the results of Verdier (9) and Ramella (10) who have shown that the locus of morphisms f in $Mor_e(\mathbb{P}^1, \mathbb{P}^n)$ with the restricted twisted tangent bundle $f^*(T_{\mathbb{P}^n}(-1))$ having splitting type $a_1, \cdots, a_n$ with $a_1 \geq \cdots \geq a_n \geq 0$ and $a_1 + \cdots + a_n = e$ is a nonempty, smooth, irreducible subvariety.

1.2 Betti numbers of the moduli space of stable sheaves on $\mathbb{P}^2$

Let X be a smooth projective surface over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let H be an ample divisor on X. Given a torsion-free, coherent sheaf $\mathcal{F}$ on X, we define its H-slope $\mu_H(\mathcal{F})$ and discriminant $\Delta(\mathcal{F})$ as follows:

$$\mu_H(\mathcal{F}) = \frac{ch_1(\mathcal{F}) \cdot H}{ch_0(\mathcal{F})H^2} \qquad \text{and} \qquad \Delta(\mathcal{F}) = \frac{ch_1(\mathcal{F})^2 - 2ch_0(\mathcal{F})ch_2(\mathcal{F})}{2ch_0(\mathcal{F})^2}$$

We denote the Chern character of the torsion-free coherent sheaf $\mathcal{F}$ by $\gamma = (r, c, \Delta(\mathcal{F}))$, where r is the rank and c is the first Chern class. We say that $\mathcal{F}$ is H-slope *(semi)stable* if for all subsheaves $\mathcal{E}$ of smaller rank, we have $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})$. We denote by $M_{X,H}(\gamma)$, the moduli-space parameterizing H-slope semistable sheaves with Chern character γ. These spaces were constructed by Gieseker (13) and Maruyama (14), and play a central role in many areas of mathematics including algebraic geometry, topology, representation theory, etc. For example, they are used to study linear systems on curves and in the Donaldson theory of 4-manifolds.

Given a Chern character γ, assume that all H-slope semistable sheaves with Chern character γ are H-slope stable, and that such stable sheaves do exist. Then the moduli-space $M_{X,H}(\gamma)$ is a smooth projective variety of dimension $1 - \chi(\gamma, \gamma)$, where χ denotes the Euler characteristic. A crucial step to understand the geometry of such moduli spaces is by scrutinizing the cohomology groups associated with them. Consequently, determining the Betti numbers of these spaces are of utmost importance. The general philosophy of Donaldson, Gieseker and Li is that the geometry of the moduli space $M_{X,H}(\gamma)$ behaves better as Δ tends to infinity. O'Grady (15) showed that $M_{X,H}(\gamma)$ is irreducible and generically smooth if Δ is sufficiently large. Li (16) showed the stabilization of the first and the second Betti numbers of $M_{X,H}(\gamma)$ when the rank is two. In this paper, we look at the special case when $X = \mathbb{P}^2$ and $H = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$. We show that

Theorem (Theorem 4.4.1). *Assume that the rank $r \geq 2$ and the first Chern class aH are coprime. If $c_2 \geq N + \lfloor \frac{r-1}{2r}a^2 + \frac{1}{2}(r^2+1) \rfloor$, then the $2N$th Betti numbers of the moduli space $M_{\mathbb{P}^2,H}(r, aH, c_2)$ stabilizes.*

The following theorem due to Coskun and Woolf (17) tells us how to compute these stable Betti numbers by describing their generating function.

Theorem ((17), Corollary 7.7). *Let X be a rational surface, and H be a polarization on X such that $K_X \cdot H < 0$. Assume that all semistable sheaves of rank r and first Chern class c are stable. Then the Poincaré polynomial of $M_{X,H}(r, c, \Delta)$ stabilizes as $\Delta \to \infty$, and the generating function for the stable Betti numbers is given by*

$$(1 - t^2) \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i})^{\chi_{top}(X)}}$$

Consequently, we can determine the Betti numbers for a large collection of such moduli spaces $M_{X,H}(\gamma)$. We list of the first few stable Betti numbers when $X = \mathbb{P}^2$ and $H = \mathcal{O}_{\mathbb{P}^2}(1)$ in Table I.

i	0	2	4	6	8	10	12
$b_{stab,\,i}$	1	2	6	13	29	57	113

TABLE I: Table showing the first few stable Betti numbers for $\mathbb{P}^2$

Given a collection of smooth projective varieties X_d for $d \geq 0$, with Poincaré polynomials $P_d(t) = \sum_{i=0}^{s_d} a_{i,d} t^d$ respectively, we say that $\{P_d\}$ stabilizes (see Definition 2.2.1) if for all $i \geq 0$, there exists an integer $d_0(i)$ depending on i such that for all integers $d \geq d_0(i)$, we have $a_{i,d} = a_{i,d+1}$. We know (see Corollary 2.2.4) that the Poincaré polynomials $\{P_d\}$ stabilizes iff for each $i \geq 0$, the coefficient of t^i in the series $(1-q)\sum_{d=0}^{\infty} P_d(t)q^d$ is a polynomial in q. Additionally, the generating function of the stable coefficients is given by taking the limit of this series as $q \to 1$. Hence, to understand the stability of the Betti numbers or equivalently, the Poincare polynomials, it is enough to study the series $(1-q)\sum_{d=0}^{\infty} P_d(t)q^d$.

The stability of the Betti numbers and the Poincaré polynomials have been studied extensively by several mathematicians. For instance, Macdonald,'62 (18) showed stabilization of the Poincaré polynomials for the family of symmetric products of a smooth projective surface X, and determined their sum

$$\zeta_X(q,t) := \sum_{d=0}^{\infty} P_{X^{(d)}(t)} q^d = \frac{(1+qt)^{b_1(X)}(1+qt^3)^{b_3(X)}}{(1-q)(1-qt^2)^{b_2(X)}(1-qt^4)}$$

Similarly, Göttsche,'90 (19) studied stabilization of the Poincaré polynomials for the family $\{X^{[n]}\}$ comprising Hilbert scheme of n points on a smooth projective surface X, and showed that

$$F_X(q,t) = \sum_{n=0}^{\infty} P_{X^{[n]}}(t) q^n = \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t)$$

When the rank is one, the moduli space $M_{X,H}(1,c,\Delta)$ is isomorphic to $Pic^c(X) \times X^{[\Delta]}$ The Künneth formula yields

$$G_X(q,t) = \sum_{\Delta=0}^{\infty} P_{M_{X,H}(1,c,\Delta)}(t) q^{\Delta} = (1+t)^{b_1(X)} \prod_{m=1}^{\infty} \zeta_X(t^{2m-2}q^m, t)$$

Therefore, the Betti numbers of $M_{X,H}(1,c,\Delta)$ stabilizes as Δ tends to infinity. In the special case when $X = \mathbb{P}^2$ and $H = \mathcal{O}_{\mathbb{P}^2}(1)$, the stabilization of the Betti numbers of $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(1,c,\Delta)$ was shown by Ellingsrud and Strømme, '87 (20). Furthermore, they computed explicit formulas to describe the Betti numbers. We list the first few Betti numbers in Table II.

Stabilization of the Betti numbers of $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(2,-1,c_2)$ was worked out by Yoshioka, '94 (21). We list the first few Betti numbers in Table III .

Similarly, stabilization of the Betti numbers of $M_{\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)}(3,-1,c_2)$ was shown by Manschot, '11 (22), and furthermore, analyzed the rank 4 case building on the work of Mozgovoy (23). We list the first few Betti numbers in Table IV.

Upon scrutinizing entries of Table III and Table IV, we deduce that in the rank 2 case, if $c_2 \geq N+1$ then b_{2N} stabilizes, and in the rank 3 case, if $c_2 \geq N+2$ then

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}
1	1	1									
2	1	2	3								
3	1	2	5	6							
4	1	2	6	10	13						
5	1	2	6	12	21	24					
6	1	2	6	13	26	39	47				
7	1	2	6	13	28	49	74	83			
8	1	2	6	13	29	54	94	131	150		
9	1	2	6	13	29	56	105	167	232	257	
10	1	2	6	13	29	57	110	189	298	395	440

TABLE II: Ellingsrud and Strømme's table for rank 1

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}	b_{28}
1	1														
2	1	2	3												
3	1	2	6	9	12										
4	1	2	6	13	24	35	41								
5	1	2	6	13	29	51	85	113	129						
6	1	2	6	13	29	57	106	175	262	337	370				
7	1	2	6	13	29	57	113	200	342	527	746	922	1002		
8	1	2	6	13	29	57	113	208	372	625	995	1464	1978	2390	2556

TABLE III: Yoshioka's table for rank 2 and $c_1 = -1$

c_2	b_0	b_2	b_4	b_6	b_8	b_{10}	b_{12}	b_{14}	b_{16}	b_{18}	b_{20}	b_{22}	b_{24}	b_{26}
2	1	1												
3	1	2	5	8	10									
4	1	2	6	12	24	38	54	59						
5	1	2	6	13	28	52	94	149	217	273	298			
6	1	2	6	13	29	56	108	189	322	505	744	992	1200	1275

TABLE IV: Manschot's table for rank 3 and $c_1 = -1$

b_{2N} stabilizes. We expect these kind of inequalities to hold in general. If we apply our Theorem (see Theorem 4.4.1), we get that in the rank 2 case, if $c_2 \geq N + 2$ then b_{2N} stabilizes, and in the rank 3 case, if $c_2 \geq N + 5$ then b_{2N} stabilizes. We loose a little bit because our inqualities work for any rank and any first Chern class. However, we can get the actual inequality if we fix the rank to be 2 and the first Chern class $c_1 = -1$ (see Proposition 4.4.2).

Stabilization of the Betti numbers of the moduli space have been studied for other surfaces as well. For example, Yoshioka (24), (25) and Göttsche (26) computed the Betti and Hodge numbers of $M_{X,H}(\gamma)$ when X is a ruled surface and the rank is two. Yoshioka (24), (27) observed the stabilization of the Betti numbers for rank two bundles on ruled surfaces. Göttsche (28) extended his results to rank two bundles on rational surfaces with polarizations which are K_X-negative. The stabilization of the Betti numbers is known for smooth moduli space of sheaves on $K3$ surfaces. By works of Mukai (29), Huybrechts (30), and Yoshioka (31), smooth moduli spaces of sheaves on a $K3$ surface X are deformations of the Hilbert scheme of points on X of the same dimension. In particular, they are diffeomorphic to the Hilbert scheme of points, and hence, their Betti numbers stabilizes. Yoshioka (32) obtained similar results for moduli spaces of sheaves on abelian surfaces. A smooth moduli space of sheaves $M_{X,H}(\gamma)$ on an abelian surface X is deformation equivalent to the product of the dual abelian surface of X and a Hilbert scheme of points on X. Consequently, the Betti numbers stabilizes.

CHAPTER 2

PRELIMINARIES

In this chapter, we will set-up notations and go over preliminary results. We split this chapter into two sections. Section 2.1 deals with rational curves and Grassmannain varieties while Section 2.2 deals with Betti numbers of the moduli space of sheaves on the projective plane.

2.1 Rational curves and the Grassmannian

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^1$ of rank r and degree e. Grothendieck's theorem tells us that there are uniquely determined integers $a_1, \cdots, a_r$ with $a_1 \leq \cdots \leq a_r$ and $a_1 + \cdots + a_r = e$ such that $\mathcal{E}$ is isomorphic to $\oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$. We call this collection of integers the *splitting type* of $\mathcal{E}$. We say that $\mathcal{E}$ is *balanced* if $a_j - a_i \leq 1$ for all $1 \leq i, j \leq r$.

Let $n \geq 2$ and $1 \leq r \leq n-1$. We denote by $G(r,n)$ the Grassmannian variety of r-dimensional subspaces of the n-dimensional vector space $\mathbb{K}^{\oplus n}$. We can think of $G(r,n)$ as a subvariety of $\mathbb{P}(\bigwedge^r \mathbb{K}^{\oplus n}) = \mathbb{P}^{\binom{n}{r}-1}$ via the Plücker embedding, which given r linearly independent vectors $v_1, \cdots, v_r$, it sends the subspace spanned by the v_i's to the point $[v_1 \wedge \cdots \wedge v_r]$. We see that the Grassmannian variety $G(r,n)$ is a smooth projective variety of dimension $r(n-r)$. For example, when $r=2$ and $n=4$, let $x_{1,2}$, $x_{1,3}$, $x_{1,4}$, $x_{2,3}$, $x_{2,4}$, $x_{3,4}$ denote the co-ordinates of $\mathbb{P}(\bigwedge^2 \mathbb{K}^{\oplus 4})$, then $G(2,4)$ has dimension 4 and the image of $G(2,4)$ under the Plücker embedding is given by the zero locus of the homogeneous polynomial $x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}$.

The Grassmannian variety has two special vector bundles, the universal sub-bundle $\mathcal{S}$ of rank r and the universal quotient bundle $\mathcal{Q}$ of rank $n-r$. Given a r-dimensional subspace Λ of $\mathbb{K}^{\oplus n}$, let $p_\Lambda \in G(r,n)$ be the point corresponding to this subspace. Then, we have

$$\mathcal{S}|_{p_\Lambda} = \Lambda \qquad \text{and} \qquad \mathcal{Q}|_{p_\Lambda} = \mathbb{K}^{\oplus n}/\Lambda$$

Moreover, these vector bundles fit together in an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{G(r,n)}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Moreover, the tangent bundle to the Grassmannian variety $G(r,n)$, denoted $T_{G(r,n)}$, is isomorphic to $\mathcal{S}^* \otimes \mathcal{Q}$.

Given integers $e \geq 1$ and $n \geq 1$, we can look at the locus of degree e morphisms from $\mathbb{P}^1$ to $\mathbb{P}^n$. A degree e morphisms f from $\mathbb{P}^1$ to $\mathbb{P}^n$ is uniquely determined upto scalars by a collection of $n+1$ homogeneous polynomials on $\mathbb{P}^1$ of degree e, namely the functions $x_i \circ f$ for $0 \leq i \leq n$, where x_i's are the co-ordinate functions of $\mathbb{P}^n$. Thus, this locus of degree e morphisms can be identified with a open subvariety of the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(e))^{\oplus n+1})$. Hence, this locus is smooth of dimension $n + (n+1)e$.

Similarly, in a more general setting, given $e \geq 1$ and $2 \leq r \leq n-2$, we can look at the locus of degree e morphisms from $\mathbb{P}^1$ to $G(r,n)$. We denote by M the scheme $Mor_e(\mathbb{P}^1, G(r,n))$ parameterizing such morphisms. Similar to our previous case, it is natural to expect that M is a smooth quasi-projective variety of dimension $r(n-r)+ne$. Our next goal is to show that this is indeed the case.

We glean the following Lemma 2.1.1 from the universal property of Grassmannian

Lemma 2.1.1. *A degree e morphism $\mathbb{P}^1 \longrightarrow G(r,n)$ corresponds uniquely to a vector bundle E of rank r and degree e together with a surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E$.*

Proof. Given a morphism $\varphi : \mathbb{P}^1 \longrightarrow G(r,n)$, we take $E = \varphi^*(\mathcal{S}^*)$, where $\mathcal{S}$ is the universal sub-bundle, and we clearly have a surjection $v_\varphi : \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow \varphi^*(\mathcal{S}^*)$.

Conversely, given a surjection $v : \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E$ where E is a vector bundle of rank r and degree e, let $s_1, \cdots, s_n$ form a basis for image of $H^0(\mathcal{O}_{\mathbb{P}^1}^{\oplus n})$ in $H^0(E)$, we have a morphism $\varphi_v : \mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{n}{r}}$ with co-ordinates given by $s_{i_1} \wedge \cdots \wedge s_{i_r}$ for $1 \leq i_1 < \cdots < i_r \leq n$, and we see that the image lies in $G(r,n)$ because the co-ordinates satisfy Plücker relations, and the resulting map has degree e because E has degree e. $\square$

Subsequently, using Lemma 2.1.1, we can think of a morphism from $\mathbb{P}^1$ to $G(r,n)$ as an element of the quot scheme $Quot^{r,e}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{K}}$, which parameterizes quotient sheaves of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ of rank r and degree e. Strømme (11, Theorem 2.1) showed that this quot scheme is an irreducible, rational, nonsingular, projective variety of dimension $r(n-r)+ne$. In particular, we can think of M as a subscheme of $Quot^{r,e}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{K}}$.

Lemma 2.1.2. *M is an open subscheme of the quot scheme $Quot^{r,e}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{K}}$. Therefore, M is a smooth quasi-projective variety of dimension $r(n-r)+ne$.*

Proof. Note that any coherent sheaf E on $\mathbb{P}^1$ has a unique decomposition $E = E' \oplus T$, where E' is locally free and T is torsion. Given any $1 \leq i \leq e$, let X_i be the image of the map

$$Quot^{r,e-i}_{\mathcal{O}^{\oplus n}_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{K}} \times \mathbb{P}^1 \times \cdots_{(i \text{ times })} \cdots \times \mathbb{P}^1 \longrightarrow Quot^{r,e}_{\mathcal{O}^{\oplus n}_{\mathbb{P}^1}/\mathbb{P}^1/\mathbb{K}}$$

which sends $(E', x_1, \cdots, x_i)$ to $E' \oplus T$ where T is the structure sheaf of the closed subscheme of $\mathbb{P}^1$ defined by $\{x_1, \cdots, x_i\}$. We see that X_i is closed and irreducible because it is the image of a proper irreducible variety. We have

$$dim(X_i) \leq r(n-r) + n(e-i) + i < r(n-r) + ne$$

Since every coherent sheaf E of rank r and degree e which is not locally free lies in some X_i, we conclude that M is the complement of the union of the X_i's for $1 \leq i \leq e$. $\square$

Let S be a smooth variety and X be a smooth projective variety. Let $\mathcal{E}$ be a coherent S-flat sheaf on $S \times X$. For every $s \in S$, let $\mathfrak{m}_s \subset \mathcal{O}_{S,s}$ be the ideal sheaf of the point s, and let $\mathcal{E}_s$ be the induced sheaf on X. We have an exact sequence

$$0 \longrightarrow (T_s S)^* = \mathfrak{m}_s/\mathfrak{m}_s^2 \longrightarrow \mathcal{O}_{S,s}/\mathfrak{m}_s^2 \longrightarrow \mathcal{O}_{\{s\}} \longrightarrow 0$$

Tensoring with $\mathcal{E}$, we get an exact sequence

$$0 \longrightarrow T_s S^* \otimes \mathcal{E}_s \longrightarrow \mathcal{E}/\mathfrak{m}_s^2 \mathcal{E} \longrightarrow \mathcal{E}_s \longrightarrow 0$$

This exact sequence gives rise to an element $\omega \in Ext^1(\mathcal{E}_s, T_sS^* \otimes \mathcal{E}_s)$, a posteriori, a linear map

$$\omega : T_sS \longrightarrow Ext^1(\mathcal{E}_s, \mathcal{E}_s)$$

We call this linear map ω, the *Kodaira - Spencer infinitesimal deformation map* at the point $s \in S$.

Definition 2.1.3. We say that the sheaf $\mathcal{E}$ defines a *complete family* parameterized by S if the Kodaira - Spencer infinitesimal deformation map is surjective at every point $s \in S$

In our case, we have a canonical map

$$\varPhi : M \times \mathbb{P}^1 \longrightarrow G(r, n)$$

which sends a pair (f, x) to $f(x)$. Let $\mathcal{S}$ denote the universal bundle over $G(r, n)$. We can look at the pullback vector bundle $\varPhi^*(\mathcal{S}^*)$ which is clearly M-flat and coherent. We have

Lemma 2.1.4. *The family of vector bundles parametrized by M via $\varPhi^*(\mathcal{S}^*) \longrightarrow M \times \mathbb{P}^1$ is a complete family.*

Proof. Let $E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$ and $K = \mathcal{O}_{\mathbb{P}^1}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-b_{n-r})$, where $deg(E) = -deg(K) = e$, and consider the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E \longrightarrow 0$$

We first observe that if f is the morphism corresponding to $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E$, then $\Phi^*(\mathcal{S}^*)|_f = E$. We look at the following commutative diagram

$$\begin{array}{ccc} T_f(M) & \longrightarrow & Ext^1(\Phi^*(\mathcal{S}^*)|_f, \Phi^*(\mathcal{S}^*)|_f) \\ \downarrow & & \downarrow \\ Hom(K, E) & \longrightarrow & Ext^1(E, E) \end{array}$$

where the vertical maps are isomorphisms, the top horizontal map is the Kodaira-Spencer map, and the bottom horizontal map is obtained by applying $Hom(\bullet, E)$ to the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E \longrightarrow 0$$

Since the next term in the long exact sequence is $Ext^1(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, E) = H^1(E)^{\oplus n} = 0$, the bottom horizontal map is surjective. Hence, the Kodaira-Spencer map is surjective, and so the family is complete. $\square$

Let X be a smooth projective curve, and let $\mathcal{E}$ be a coherent torsion free sheaf on X of degree d and rank r. We define the *slope* of $\mathcal{E}$ to be

$$\mu(\mathcal{E}) = \frac{d}{r}$$

We say that $\mathcal{E}$ is *stable (respectively semistable)* if for all nonzero proper subsheaves $\mathcal{F}$ of smaller rank, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$). Given any coherent torsion-free sheaf $\mathcal{E}$ on X, there exists a unique filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable for all $1 \leq i \leq l$ and moreover, we have $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ for all $1 \leq i \leq l-1$. This filtration is called the *Harder-Narasimhan filtration* of $\mathcal{E}$.

We will now use the following corollary due to Le Potier to conclude that the locus of quotient vector bundles in M of given splitting type has expected codimension.

Proposition 2.1.5 ((12), Cor 15.4.3)**.** *Let X be a smooth projective curve of genus g. Let E_s be a complete family of vector bundles of rank r and degree d parametrized by a smooth variety S. For integers $l, r_i > 0$ and d_i, set*

$$\mu_i = \frac{d_i}{r_i}$$

The points $s \in S$ such that the Harder-Narasimhan filtration (if it exists) has length l and such that the Harder-Narasimhan grading $gr_i(E_s)$ of E_s has rank r_i and degree d_i, for $i = 1, \cdots, l$, form a locally closed smooth subvariety of codimension

$$\sum_{i<j} r_i r_j(\mu_i - \mu_j + g - 1)$$

Observe that when $g = 0$, we have $E_s = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ for some integers $a_1, \cdots, a_r$, and so,

$$\sum_{i<j} r_i r_j (\mu_i - \mu_j - 1) = ext^1(E_s, E_s) = \sum_{i,j} \max\{a_i - a_j - 1, 0\} \tag{2.1}$$

Now we fix two collection of non-negative integers $a_1 \geq \cdots \geq a_r \geq 0$ and $0 \leq b_1 \leq \cdots \leq b_{n-r}$ such that $a_1 + \cdots + a_r = b_1 + \cdots + b_{n-r} = e > 0$. Let $M(b_\bullet)$ be the locus of morphisms in M with the restricted universal quotient bundle being isomorphic to $\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_{n-r})$, and let $M'(a_\bullet)$ be the locus of morphisms in M with the restricted universal sub-bundle being isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_r)$. Our goal is to show that the intersection locus $M(b_\bullet) \cap M'(a_\bullet)$ is nonempty and generically transverse, a posteriori, has an irreducible component of expected codimension

$$\sum_{1 \leq i,j \leq r} \max\{a_i - a_j - 1, 0\} + \sum_{1 \leq i,j \leq n-r} \max\{b_i - b_j - 1, 0\}$$

We see that

Proposition 2.1.6. *The locus $M(b_\bullet)$ is smooth of codimension*

$$\sum_{i,j} \max\{b_i - b_j - 1, 0\}$$

Similarly, $M'(a_\bullet)$ is smooth of codimension

$$\sum_{i,j} \max\{a_i - a_j - 1, 0\}$$

Proof. The first part of the Lemma follows from Lemma 2.1.4, Proposition 2.1.5, and equation Equation 2.1.

To conclude the second part, we note that the canonical map

$$Quot^{r}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{K}} \longrightarrow Quot^{n-r}_{\mathcal{O}_{\mathbb{P}^1}^{\oplus n}/\mathbb{P}^1/\mathbb{K}}$$

which sends $[\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E]$ to $[\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow \mathcal{K}^*]$, where $\mathcal{K}$ is the kernel of the map $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \longrightarrow E$, induces an isomorphism between $Mor_e(\mathbb{P}^1, G(r,n))$ and $Mor_e(\mathbb{P}^1, G(n-r,n))$. Hence, the second part of the Lemma follows from the first part. □

Therefore, we need to show that the intersection of $M(b_\bullet)$ and $M'(a_\bullet)$ is nonempty, and we need to find a point in $M(b_\bullet) \cap M'(a_\bullet)$ where the intersection is transverse. We show these in section 3.1 and 3.2.

Definition 2.1.7. Given a collection of non-negative integers $a_1, \cdots, a_l$, we define its *polygonal line* to be

$$\mathfrak{P}(a_1, \cdots, a_l) = (a'_1, a'_1 + a'_2, \cdots, a'_1 + \cdots + a'_l)$$

where $a'_1, \cdots, a'_l$ is a rearrangement of the a_i's such that $a'_1 \geq \cdots \geq a'_l$. Additionally, given another such collection $b_1, \cdots, b_l$ with rearrangement $b'_1 \geq \cdots \geq b'_l$, we define inequality

$$\mathfrak{P}(b_\bullet) \geq \mathfrak{P}(a_\bullet) \quad \text{if} \quad \sum_{j=1}^{i} b'_j \geq \sum_{j=1}^{i} a'_j, \quad \text{for all } 1 \leq i \leq l$$

Note that if $\mathcal{E}$ is a vector bundle of rank r on $\mathbb{P}^1$ with splitting type $a_1 \geq \cdots \geq a_r \geq 0$, then $\mathfrak{P}(a_1, \cdots, a_r)$ is the tuple consisting of the degrees of the subbundles appearing in the Harder-Narasimhan filtration of $\mathcal{E}$.

It follows as a consequence of Proposition 1.2 due to Ramella (10)

Proposition 2.1.8. *Given two collection of non-negative integers* $0 \leq b_1 \leq \cdots \leq b_{n-r}$ *and* $0 \leq b'_1 \leq \cdots \leq b'_{n-r}$ *with* $b_1 + \cdots + b_{n-r} = b'_1 + \cdots + b'_{n-r} = e$. *We have*

$$\overline{M(b'_\bullet)} \supset M(b_\bullet) \quad \textit{iff} \quad \mathfrak{P}(b'_\bullet) \leq \mathfrak{P}(b_\bullet)$$

Similar result holds for $M'(a_\bullet)$.

Since M is stratified by $M(b_\bullet)$ for all possible $0 \leq b_1 \leq \cdots \leq b_{n-r}$ with $b_1+\cdots+b_{n-r} = e$, and by $M'(a_\bullet)$ for all possible $a_1 \geq \cdots \geq a_r \geq 0$ with $a_1 + \cdots + a_r = e$, Proposition 2.1.8 yields the following Corollary.

Corollary 2.1.9. *The closure of the locus* $M(b_\bullet)$ *in* $Mor_e(\mathbb{P}^1, G(r,n))$ *is*

$$\overline{M(b_\bullet)} = \bigcup_{\substack{0\leq b'_1\leq\cdots\leq b'_{n-r} \\ b_1+\cdots+b_{n-r}=e \\ \mathfrak{P}(b'_\bullet)\geq\mathfrak{P}(b_\bullet)}} M(b'_\bullet)$$

Similarly, we have

$$\overline{M'(a_\bullet)} = \bigcup_{\substack{a'_1\geq\cdots\geq a'_r\geq 0 \\ a'_1+\cdots+a'_r=e \\ \mathfrak{P}(a'_\bullet)\geq\mathfrak{P}(a_\bullet)}} M'(a'_\bullet)$$

2.2 Betti numbers of the moduli space of sheaves on $\mathbb{P}^2$

Let X be a smooth projective surface over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let H be an ample divisor on X. Throughout this thesis, we are going to assume that all sheaves are coherent and torsion free. Given a sheaf $\mathcal{F}$, we define the *H-slope* of $\mathcal{F}$ as

$$\mu_H(\mathcal{F}) = \frac{ch_1(\mathcal{F}) \cdot H}{ch_0(\mathcal{F}) \cdot H^2}$$

Additionally, we define the *Chern character* of $\mathcal{F}$ as $\gamma = (r, c, \Delta)$ where r is the rank, c is the first Chern class, and Δ is the discriminant defined as

$$\Delta(\mathcal{F}) = \frac{ch_1(\mathcal{F})^2 - 2ch_0(\mathcal{F})ch_2(\mathcal{F})}{2ch_0(\mathcal{F})^2} \tag{2.2}$$

We define a sheaf $\mathcal{F}$ to be *μ_H-semistable* if for every nonzero proper subsheaf $\mathcal{E}$, we have $\mu_H(\mathcal{E}) \leq \mu_H(\mathcal{F})$. Likewise, we define a sheaf $\mathcal{F}$ to be *μ_H-stable* if the inequality is strict.

Given any sheaf $\mathcal{F}$, there exists a unique filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$$

such that the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are μ_H-semistable for all $1 \leq i \leq l$, and moreover, we have $\mu_H(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu_H(\mathcal{F}_{i+1}/\mathcal{F}_i)$ for all $1 \leq i \leq l-1$. We call this filtration the *Harder-Narasimhan filtration* of $\mathcal{F}$ (see (33)[Section 1.3]).

Furthermore, given any μ_H-semistable sheaf $\mathcal{F}$, there exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}$$

such that the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are stable and have H-slope $\mu_H(\mathcal{F})$ for all $1 \leq i \leq l$. We call such a filtration, a *Jordan-Holder filtration* of $\mathcal{F}$ (see (33)[Section 1.5]). Up to isomorphism, the direct sum of the subquotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $1 \leq i \leq l$ does not depend on the Jordan-Holder filtration. We say two μ_H-semistable sheaves are *S-equivalent* if the direct sum of subquotients appearing in their corresponding Jordan Hölder filtrations are isomorphic.

Given a Chern character $\gamma = (r, c, \Delta)$, we denote by $M_{X,H}(\gamma)$ the moduli space of S-equivalence classes of μ_H-semistable sheaves with Chern character γ. These spaces were constructed by Gieseker (13) and Maruyama (14). When X is smooth projective surface and H is ample divisor with $K_X \cdot H < 0$, the moduli space $M_{X,H}(\gamma)$ is smooth at every stable sheaf $\mathcal{F}$ because $ext^2(\mathcal{F}, \mathcal{F}) = hom(\mathcal{F}, \mathcal{F} \otimes K_X) = 0$. Consequently, assuming $M_{X,H}(\gamma)$ is nonempty, if all μ_H-semistable sheaves with Chern character γ are μ_H-stable, then $M_{X,H}(\gamma)$ is a smooth projective variety of dimension $ext^1(\gamma, \gamma) = 1 - \chi(\gamma, \gamma)$. We denote by $\mathcal{M}_{X,H}(\gamma)$ the moduli stack of μ_H-semistable sheaves with Chern character γ.

Given a rank r and first Chern class c, assume that $M_{X,H}(r, c, \Delta)$ is nonempty smooth projective variety for all $\Delta \geq 0$, for example when $r \cdot H^2$ and $c \cdot H$ are coprime. Let

$\gamma = (r, c, \Delta)$ be the Chern character. To understand the Betti numbers of $M_{X,H}(\gamma)$, we look at the Poincaré polynomial

$$P_{M_{X,H}(\gamma)}(t) = \sum_{i=0}^{2(1-\chi(\gamma,\gamma))} b_i(M_{X,H}(\gamma))t^i$$

Intuitively, stabilization of the Betti numbers, or equivalently the Poincaré polynomials mean that for each $i \geq 0$, the Betti number $b_i(M_{X,H}(\gamma))$ becomes the same for sufficiently large Δ.

In general, consider a collection of polynomials $P_d(t) = \sum_{i=0}^{s_d} a_{i,d}t^i$ indexed by integers $d \geq N$, for some integer N. We look at the corresponding collection of shifted polynomials $\tilde{P}_d(t) = \sum_{j=-s_d}^{0} b_{j,d}t^j$, where $b_{j,d} = a_{j+s_d,d}$.

Definition 2.2.1. We say that the collection of polynomials $\{P_d(t)\}_{d\geq N}$ *stabilize* if for each j there exists an integer $d_0(j)$ such that for all $d \geq d_0(j)$ we have $b_{j,d} = b_{j,d+1}$. In this case, we define the *stable limit* to be $\tilde{P}_\infty(t) = \sum_{j=-\infty}^{0} \beta_j t^j$, where $\beta_j = b_{j,d}$ for any $d \geq d_0(j)$.

In our case, we fix r and c and look at the collection of polynomials $P_{M_{X,H}(r,c,\Delta)}$ for $\Delta \geq 0$. If this collection of polynomials stabilize, we say that the Betti numbers of $M_{X,H}(r, c, \Delta)$ *stabilize.*

Consider the generating function

$$\tilde{F}(q, t) = \sum_{d=N}^{\infty} \tilde{P}_d(t)q^d \tag{2.3}$$

We have

Proposition 2.2.2 ((17), Proposition 3.1)**.** *The polynomials $P_d(t)$ stabilize iff the coefficient of t^i in $(1-q)\tilde{F}(q,t)$ is a Laurent polynomial in q. Moreover, if the polynomials stabilize, the stable limit is obtained by evaluating $(1-q)\tilde{F}(q,t)$ at $q=1$.*

The proof of Proposition 2.2.2 due to Coskun and Woolf (17) essentially follows from the following Lemma.

Lemma 2.2.3. *For any $j \geq 0$, the coefficient of $t^{-j}q^d$ in $(1-q)\tilde{F}(q,t)$ is zero for $d \geq d_0(j)$ iff $b_{j,d} = b_{j,d+1}$ for all $d \geq d_0(j)-1$.*

Proof. Let us define $b_{j,d} = 0$ for $j < -s_d$. It follows from equation Equation 2.3 that

$$\tilde{F}(q,t) = \sum_{d \geq N, j \leq 0} b_{j,d} t^j q^d$$

whence,

$$(1-q)\tilde{F}(q,t) = \sum_{d \geq N, j \leq 0} (b_{j,d} - b_{j,d-1}) t^j q^d$$

□

Additionally, let

$$F(q,t) = \sum_{d=N}^{\infty} P_d(t) q^d$$

and assume that the polynomials $P_d(t)$ satisfy Poincaré duality i.e. $t^{s_d} P_d(t^{-1}) = P_d(t)$ for $d \gg 0$, then we have

Corollary 2.2.4 ((17), Corollary 3.2). *The polynomials $P_d(t)$ stabilize iff the coefficient of t^i in $(1-q)F(q,t)$ is a Laurent polynomial in q, and in this case, we get the generating function for the stable coefficients by evaluating $(1-q)F(q,t)$ at $q=1$.*

Let $K_0(var_{\mathbb{K}})$ denote the Grothendieck ring of varieties over the field $\mathbb{K}$, which we think of as a quotient of the free abelian group of varieties of finite type over $\mathbb{K}$ by the scissor relations

$$[X] = [Y] + [Z]$$

where X is a disjoint union of locally closed subvarieties Y and Z. Multiplication in $K_0(var_{\mathbb{K}})$ is defined as

$$[X] \cdot [Y] = [X \times Y]$$

As a consequence of Hironaka's resolution of singularities (34), the Grothendieck ring $K_0(var_{\mathbb{K}})$ is generated by the classes of smooth projective varieties. The Poincaré polynomials for smooth varieties induces (35) the *virtual Poincaré polynomial* map

$$P(t) : K_0(var_{\mathbb{K}}) \longrightarrow \mathbb{Z}[t]$$

Let $\mathbb{L}$ denote the class $[\mathbb{A}^1]$ in $K_0(var_{\mathbb{K}})$. Consider the ring $R = K_0(var_{\mathbb{K}})[\mathbb{L}^{-1}]$. We have a $\mathbb{Z}$-graded filtration $\mathfrak{F}$ on R, where for any given variety Y, we have

$$[Y]\mathbb{L}^a \in \mathfrak{F}^i \quad \text{iff} \quad dim(Y) + a \leq -i$$

We define the ring A^- to be the inverse limit

$$A^- := \varprojlim_{i\geq 0} R/(\mathfrak{F}^i \otimes_{\mathfrak{F}^0} R) \tag{2.4}$$

Since $\mathbb{L}$ and $\mathbb{L}^i - 1$ for $i > 0$ are invertible in A^-, we have a well-defined map from $R[\{(\mathbb{L}^i - 1)\,|\,i > 0\}]$ to A^-. Our notion of dimension extends from $K_0(var_{\mathbb{K}})$ to A^-. Similarly, the virtual Poincaré polynomial extends to R and A^- where it takes values in $\mathbb{Z}[t, t^{-1}]$ and $\mathbb{Z}((t^{-1}))$ respectively.

Definition 2.2.5. We say that a sequence of elements a_i in A^- for $i \geq 0$ *stabilize* to a iff the sequence $a_i\mathbb{L}^{-dim(a_i)}$ converges to a.

Given any smooth projective variety Y of dimension d, it follows from Poincaré duality that

$$P_{[Y]}(t) = t^{2d}P_{[Y]}(t^{-1}) = P_{[Y]\mathbb{L}^{-d}}(t^{-1}) \tag{2.5}$$

Therefore, we have

Lemma 2.2.6. *Given a collection of smooth projective varieties $[X_i]$ of dimension d_i, if they stabilize in A^- then their respective Poincaré polynomials also stabilize.*

Moreover, we know

Proposition 2.2.7 ((17), Proposition 3.6)**.** *A sequence of elements $a_i \in A^-$ for $i \geq 0$ converges to a iff the generating function $(1-q)\sum_{i\geq 0} a_i q^i$ is convergent at $q = 1$, and in this case, evaluating the generating function $(1-q)\sum_{i\geq 0} a_i q^i$ at $q = 1$ yields a.*

In particular, we see that

Remark 2.2.8. If for all $N \geq 0$, there exists $\Delta_0(N) > 0$ such that the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\sum_{i\geq 0}[X_i]\mathbb{L}^{-d_i}q^i$ is zero, then for all $N \geq 0$ the coefficient of $\mathbb{L}^{-N}$ in $(1-q)\sum_{i\geq 0}[X_i]\mathbb{L}^{-d_i}q^i$ is a Laurent polynomial of q of degree at most $\Delta_0(N)$. As a result, it follows from Proposition 2.2.7 that the generating function $(1-q)\sum_i[X_i]\mathbb{L}^{-d_i}q^i$ is convergent at $q = 1$, whence Lemma 2.2.6 yields the Poincaré polynomials of $[X_i]$ also stabilize. Consequently, it follows from equation Equation 2.5, Lemma 2.2.3, and definition 2.2.1 that the $2N$th Betti number of X_Δ stabilize when $\Delta \geq \Delta_0(N) - 1$.

Given a smooth projective surface X, we have the following equality of generating functions due to Göttsche

$$\sum_{\Delta=1}^{\infty}[X^{[\Delta]}]\mathbb{L}^{-2\Delta}q^{\Delta} = \prod_{m=1}^{\infty}\left(\sum_{n=0}^{\infty}[X^{(n)}]\mathbb{L}^{(-m-1)n}q^{mn}\right)$$

Vakil and Wood (36)[Conjecture 1.25] conjecture that the sequence $[X^{(\Delta)}]\mathbb{L}^{-2\Delta}$ converges in A^-. Using above equality, this conjecture implies that the sequence $[X^{[\Delta]}]\mathbb{L}^{-2\Delta}$ also converges. This conjecture is known in the case when X is a rational surface. Coskun and Woolf (17) showed that when X is a rational surface and H is an ample line bundle, and $K_X \cdot H < 0$, the sequence $[\mathcal{M}_{X,H}(r,c,\Delta)]\mathbb{L}^{-r^2(2\Delta-\chi(\mathcal{O}_X))}$ converges to the same limit in A^-. They studied the generating function

$$G_{X,H,r,c}(q) = \sum_{\Delta=0}^{\infty}[\mathcal{M}_{X,H}(r,c,\Delta)]\mathbb{L}^{-r^2(2\Delta-\chi(\mathcal{O}_X))}q^{r\Delta}$$

Using Proposition 2.2.7, they showed convergence of the generating function $(1-q)G_{X,H,r,c}(q)$ at $q = 1$ and evaluated it.

In the special case when $X = \mathbb{P}^2$ and $H = \mathbb{O}_{\mathbb{P}^2}(1)$, we define generating function

$$G_{r,c}(q) = \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{P}^2,H}(r,c,\Delta)]\mathbb{L}^{r^2(1-2\Delta)} q^{r\Delta} \tag{2.6}$$

To study convergence of this generating function, we look at the blow-up of $\mathbb{P}^2$ at a point and study convergence of a similar generating function on the blow-up.

Given any integer $e \geq 0$, we denote by $\mathbb{F}_e$ the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. The Picard group of $\mathbb{F}_e$ is the abelian group generated by E which is the class of a section of the canonical map $\pi : \mathbb{F}_e \longrightarrow \mathbb{P}^1$ and F which is the class of fibers of π, satisfying the relations

$$E^2 = -e, \quad E \cdot F = 1, \quad F^2 = 0$$

The canonical class of $\mathbb{F}_e$ is $K_{\mathbb{F}_e} = -2E - (e+2)F$. Since $-K_{\mathbb{F}_e}$ is effective, $K_{\mathbb{F}_e} \cdot H < 0$ for every ample divisor H. The nef cone of $\mathbb{F}_e$ is spanned by F and $E + eF$. In the special case when $e = 1$, we think of $\mathbb{F}_1$ as the blow-up of $\mathbb{P}^2$ at a point p. We denote by E the exceptional divisor and by F the fiber class.

We are going to look at the moduli stacks $\mathcal{M}_{\mathbb{P}^2,H}(r,c,\Delta)$ and $\mathcal{M}_{\mathbb{F}_1,E+F}(r,\tilde{c},\tilde{\Delta})$ where $\gamma=(r,c,\Delta)$ is Chern character on $\mathbb{P}^2$ and $\tilde{\gamma}=(r,\tilde{c},\tilde{\Delta})$ is Chern character on $\mathbb{F}_1$. We define generating functions

$$G_{r,c}(q)=\sum_{\Delta\geq 0}[\mathcal{M}_{\mathbb{P}^2,H}(r,c,\Delta)]\mathbb{L}^{r^2(1-2\Delta)}q^{r\Delta}$$

and (2.7)

$$\tilde{G}_{r,\tilde{c}}(q)=\sum_{\tilde{\Delta}\geq 0}[\mathcal{M}_{\mathbb{F}_1,E+F}(r,\tilde{c},\tilde{\Delta})]\mathbb{L}^{r^2(1-2\tilde{\Delta})}q^{r\tilde{\Delta}}$$

Coskun and Woolf have shown that

Theorem 2.2.9 ((17), Theorem 5.4, Corollary 5.5)**.** *The generating function* $(1-q)G_{r,c}(q)$ *converges at* $q=1$ *to* $\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-i})^3}$. *Similarly, the generating function* $(1-q)G_{r,\tilde{c}}(q)$ *converges at* $q=1$ *to* $\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-i})^4}$.

Our goal is to determine lower bounds for the stabilization of the Betti numbers for the moduli space $M_{\mathbb{P}^2,H}(r,c,\Delta)$ in the special case when r and $c\cdot H$ are coprime. The way we do this is by relating the stabilization of the Betti numbers with the convergence of the generating function $(1-q)G_{r,c}(q)$ at $q=1$. A key ingredient in this method is to relate the classes of the moduli stack and the moduli space in A, which was shown by Coskun and Woolf, where A is the quotient of A^{-} by relations $[P]=[X][PGL_n]$ whenever $P\longrightarrow X$ is an étale PGL_n-torsor.

Proposition 2.2.10 ((17), Proposition 7.3)**.** *The moduli stack and moduli space of μ_H-stable sheaves on X, denoted $\mathcal{M}^s_{X,H}(\gamma)$ and $M^s_{X,H}(\gamma)$ respectively, are related in A as follows:*

$$[M^s_{X,H}(\gamma)] = (\mathbb{L} - 1)[\mathcal{M}^s_{X,H}(\gamma)] \tag{2.8}$$

By our assumption, r and $c \cdot H$ are coprime, a posteriori, all μ_H-semistable sheaves are μ_H-stable. As a consequence, we can use Proposition 2.2.10 to relate the moduli stack and the moduli space.

CHAPTER 3

RESTRICTED TANGENT BUNDLE OF GRASSMANNIAN TO RATIONAL CURVES

In this chapter, we study the locus of restricted tangent bundle of the Grassmannian to rational curves with a given splitting type. More precisely, we show that this locus is stratified by intersection loci $M(b_\bullet) \cap M'(a_\bullet)$ which are nonempty and generically transverse.

3.1 The intersection locus is nonempty

In this section we show that the intersection of the locus of degree e morphisms from $\mathbb{P}^1$ to $G(r,n)$ with the restricted universal sub-bundle having given splitting type and the locus of degree e morphisms with restricted universal quotient bundle having given splitting type is non-empty. In particular, we want to show that given two sequences of non-negative integers $a_1 \geq \cdots \geq a_r \geq 0$ and $0 \leq b_1 \leq \cdots \leq b_{n-r}$ such that $a_1+\cdots+a_r = b_1+\cdots+b_{n-r} = e > 0$, there exits an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-b_{n-r}) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \xrightarrow{v} \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r) \longrightarrow 0$$

By dualizing the sequence if necessary, we may assume without loss of generality that $(n-r) \leq r$.

Before doing the general case, we would like to do the case $r = n-r = 2$. We have $a_1 \geq a_2$, $b_1 \leq b_2$ and $a_1 + a_2 = b_1 + b_2 = e$.

Proposition 3.1.1. *There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \xrightarrow{u} \mathcal{O}^{\oplus 4} \xrightarrow{v} \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \longrightarrow 0$$

Proof. Note that we must have $a_1 \geq b_1$, otherwise

$$b_1 + b_2 \geq 2b_1 > 2a_1 \geq a_1 + a_2$$

which is a contradiction. We define

$$v = \begin{pmatrix} x^{a_1} & y^{a_1} & 0 & x^{a_1-b_1}y^{b_1} \\ 0 & x^{a_2} & y^{a_2} & 0 \end{pmatrix} \qquad u = \begin{pmatrix} -y^{b_1} & 0 \\ 0 & x^{a_1-b_1}y^{a_2} \\ 0 & -x^{b_2} \\ x^{b_1} & -y^{b_2} \end{pmatrix}$$

where x and y denote the co-ordinate functions of $\mathbb{P}^1$. The minor corresponding to the first two columns of v is $x^{a_1+a_2}$ and the minor corresponding to the second and third column of v is $y^{a_1+a_2}$. Since these two monomials do not vanish simultaneously on $\mathbb{P}^1$, we conclude that v is surjective.

Similarly, by looking at the minor corresponding to first and fourth row, and the minor corresponding to third and fourth row, we conclude that u is injective.

Finally, one can check that $v \circ u = 0$. □

Now we discuss the general case when $(n - r) \leq r$. We define

Definition 3.1.2.

$$A(j) = \begin{cases} 0, & \text{if } j \leq 0 \\ a_1 + \cdots + a_j, & \text{if } 1 \leq j \leq r \\ a_1 + \cdots + a_r, & \text{if } j \geq r \end{cases} \quad B(i) = \begin{cases} 0, & \text{if } i \leq 0 \\ b_1 + \cdots + b_i, & \text{if } 1 \leq i \leq n-r \\ b_1 + \cdots + b_{n-r}, & \text{if } i \geq n-r \end{cases}$$

To describe the matrices, we need to use the following lemma.

Lemma 3.1.3. *Let $a_1 \geq \cdots \geq a_r \geq 0$ and $0 \leq b_1 \leq \cdots \leq b_{n-r}$ be two sequence of non-negative integers with $(n-r) \leq r$ and $A(r) = B(n-r)$. Then for all $0 \leq l \leq (n-r)$, we have $A(2r-n+l) \geq B(l)$.*

Proof. Let $s(l) = A(2r-n+l) - B(l)$ for any $0 \leq l \leq (n-r)$. Clearly, $s(0) \geq 0$. Let $1 \leq l_0 < n-r$ be the least integer such that $s(l_0 - 1) \geq 0$ and $s(l_0) < 0$.

Since $s(l_0) = s(l_0-1) + a_{2r-n+l_0} - b_{l_0}$, we must have $a_{2r-n+l_0} - b_{l_0} < 0$. This in turn implies that

$$a_r \leq \cdots \leq a_{2r-n+l_0+1} \leq a_{2r-n+l_0} < b_{l_0} \leq b_{l_0+1} \leq \cdots \leq b_{n-r}$$

which gives

$$s(n-r) = s(l_0) + (a_{2r-n+l_0+1} - b_{l_0+1}) + \cdots + (a_r - b_{n-r}) \leq s(l_0) < 0$$

But we know $s(n-r) = 0$, thus we have a contradiction. $\square$

The description of the matrices depend on how the $A(j)$'s and $B(i)$'s are ordered. For example, let $r = n - r = 5$ and let's assume the following order

$$B(1) < B(2) < A(1) < A(2) < B(3) < A(3) < B(4) < A(4) < B(5) = A(5)$$

For ease of notation, let us define $s_{j,i} = A(j) - B(i)$ for any given integers i, j. Let x and y denote the co-ordinate functions of $\mathbb{P}^1$. Then the first matrix v is given as follows :

$$\begin{pmatrix} x^{a_1} & y^{a_1} & 0 & 0 & 0 & 0 & x^{s_{1,1}}y^{-s_{0,1}} & x^{s_{1,2}}y^{-s_{0,2}} & 0 & 0 \\ 0 & x^{a_2} & y^{a_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^{a_3} & y^{a_3} & 0 & 0 & 0 & 0 & x^{s_{3,3}}y^{-s_{2,3}} & 0 \\ 0 & 0 & 0 & x^{a_4} & y^{a_4} & 0 & 0 & 0 & 0 & x^{s_{4,4}}y^{-s_{3,4}} \\ 0 & 0 & 0 & 0 & x^{a_5} & y^{a_5} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

The second matrix u is given as follows :

$$\begin{pmatrix} -y^{b_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x^{s_{1,2}}y^{-s_{1,3}} & 0 & 0 \\ 0 & 0 & -x^{s_{2,2}}y^{-s_{2,3}} & 0 & 0 \\ 0 & 0 & 0 & x^{s_{3,3}}y^{-s_{3,4}} & 0 \\ 0 & 0 & 0 & 0 & x^{s_{4,4}}y^{-s_{4,5}} \\ 0 & 0 & 0 & 0 & -x^{b_5} \\ x^{b_1} & -y^{b_2} & 0 & 0 & 0 \\ 0 & x^{b_2} & -y^{b_3} & 0 & 0 \\ 0 & 0 & x^{b_3} & -y^{b_4} & 0 \\ 0 & 0 & 0 & -x^{b_4} & -y^{b_5} \end{pmatrix} \tag{3.2}$$

It is easy to see that the v is surjective, u is injective, and $v \circ u = 0$.

We now proceed to define the matrices v and u in general. We define two increasing sequences of non-negative integers $\{i_l\}_{l\geq 0}$ and $\{j_l\}_{l\geq 0}$ recursively in the following manner:

We define $i_0 = 0$, and j_0 to be the largest non-negative integer such that $j_0 \leq r$ and $A(j_0) \leq B(1)$. For each $l \geq 1$, we define i_l to be the largest non-negative integer such that $i_l \leq n - r$ and $B(i_l) \leq A(j_{l-1} + 1)$ and j_l to be the largest non-negative integer such that $j_l \leq r$ and $A(j_l) \leq B(i_l + 1)$. It follows that for $l \gg 0$, we have $j_l = r$ and $i_l = n - r$. We define α to be the least positive integer such that $j_{\alpha+1} = r$. It follows from Lemma 3.1.3 that in general, there are two possible orderings:

if $a_1 > b_1$, we see that $i_0 = j_0 = 0$ and we have:

$B(1) \leq \cdots \leq B(i_1) \leq A(1) \leq \cdots \leq A(j_1) \leq$

$B(i_1 + 1) \leq \cdots \leq B(i_2) \leq A(j_1 + 1) \leq \cdots \leq A(j_2) \leq \cdots \leq$

$B(i_\alpha + 1) \leq \cdots \leq B(n - r - 1) \leq A(j_\alpha + 1) \leq \cdots \leq A(r - 1) \leq A(r) = B(n - r)$

if $a_1 \leq b_1$, we have:

$A(1) \leq \cdots \leq A(j_0) \leq B(1) \leq \cdots \leq B(i_1) \leq A(j_0 + 1) \leq \cdots \leq A(j_1) \leq$

$B(i_1 + 1) \leq \cdots \leq B(i_2) \leq A(j_1 + 1) \leq \cdots \leq A(j_2) \leq \cdots \leq$

$B(i_\alpha + 1) \leq \cdots \leq B(n - r - 1) \leq A(j_\alpha + 1) \leq \cdots \leq A(r - 1) \leq A(r) = B(n - r)$

We define the first matrix $v_{r \times n}$ as follows: we have a $r \times (r + 1)$ block matrix and a $r \times (n - r - 1)$ block matrix comprising the matrix $v_{r \times n}$. The $r \times (r + 1)$ block matrix has diagonal and super-diagonal entries defined as follows:

$$v_{i,i} = x^{a_i}, \text{ for } i = 1, \cdots, r; \qquad v_{i,i+1} = y^{a_i}, \text{ for } i = 1, \cdots, r;$$

All the remaining entries of this block are zero. The $r \times (n - r - 1)$ block has non-zero entries only in rows $j_0 + 1, j_1 + 1, \cdots, j_\alpha + 1$, and all other rows have all zero entries. For

$0 \leq l \leq \alpha - 1$, the row $j_l + 1$ have non-zero entries in columns $r+2+i_l$ upto $r+1+i_{l+1}$ and zero entries for all other columns. The non-zero entries are:

$$v_{j_l+1,r+2+i_l} = x^{A(j_l+1)-B(i_l+1)}y^{B(i_l+1)-A(j_l)}, \cdots, v_{j_l+1,r+1+i_{l+1}} = x^{A(j_l+1)-B(i_{l+1})}y^{B(i_{l+1})-A(j_l)}$$

The row $j_\alpha + 1$ has non-zero entries in columns $r+2+i_\alpha$ upto n, and zero entries in all other columns. The non-zero entries are:

$$v_{j_\alpha+1,r+2+i_\alpha} = x^{A(j_\alpha+1)-B(i_\alpha+1)}y^{B(i_\alpha+1)-A(j_\alpha)}, \cdots, v_{j_\alpha+1,n} = x^{A(j_\alpha+1)-B(n-r-1)}y^{B(n-r-1)-A(j_\alpha)}$$

We now proceed to define the second matrix $u_{n\times(n-r)}$. The matrix u comprises of three blocks, a $(j_0+1)\times(n-r)$ block $u1$ consisting of the first j_0+1 rows of u, a $(r-j_0-1)\times(n-r)$ block $u2$ consisting of rows j_0+2 upto r of u, and a $(n-r)\times(n-r)$ block $u3$ consisting of rows $r+1$ upto n of u.

The matrix $u1$ has non-zero entries in the first column and zero entries in all remaining columns. The non-zero entries are:

$$u_{1,1} = -y^{b_1}, u_{2,1} = (-1)^2x^{A(1)}y^{B(1)-A(1)}, \cdots, u_{j_0+1,1} = (-1)^{j_0+1}x^{A(j_0)}y^{B(1)-A(j_0)}$$

The matrix $u2$ has non-zero entries in columns $i_1+1, i_2+1, \cdots, i_\alpha+1$ and $(n-r)$, and zero entries in all other columns. For any $1 \leq l \leq \alpha$, the column i_l+1 has non-zero entries in rows $j_{l-1}+2, \cdots, j_l+1$ and zero entries in all other rows. The non-zero entries are:

$$u_{j_{l-1}+2,i_l+1} = (-1)^{j_{l-1}+2-(j_{l-1}+2)} x^{A(j_{l-1}+1)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+1)}$$
$$u_{j_{l-1}+3,i_l+1} = (-1)^{j_{l-1}+3-(j_{l-1}+2)} x^{A(j_{l-1}+2)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+2)}$$
$$\vdots$$
$$u_{j_l+1,i_l+1} = (-1)^{j_l+1-(j_{l-1}+2)} x^{A(j_l)-B(i_l)} y^{B(i_l+1)-A(j_l)}$$

The $(n-r)$th column has non-zero entries in rows $j_\alpha+2$ upto r, and has zero entries in all other rows. The non-zero entries are:

$$u_{j_\alpha+2,n-r} = (-1)^{j_\alpha+2-(j_\alpha+2)} x^{A(j_\alpha+1)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+1)}, \cdots$$
$$\cdots u_{r,n-r} = (-1)^{r-(j_\alpha+2)} x^{A(r-1)-B(n-r-1)} y^{B(n-r)-A(r-1)}$$

The non-zero entries of matrix $u3$ are along the diagonal, the sub-diagonal, and in the $(n-r)$th column. The diagonal entries are:

$$u_{r+i,i} = \begin{cases} 0, & \text{if } i=1 \\ -y^{b_i}, & \text{if } 2 \le i \le (n-r) \end{cases}$$

The sub-diagonal entries are:

$$u_{r+1+i,i} = (-1)^{\beta_i} x^{b_i}, \text{ for } i = 1, \cdots, n-r-1$$

where β_i denotes the number of $A(j)$'s lying strictly in between $B(i)$ and $B(i-1)$. We also have $u_{r+1,n-r} = (-1)^{\beta_{n-r}} x^{b_{n-r}}$. All other entries are zero.

Proposition 3.1.4. *The matrix v is surjective, u is injective, and $v \circ u = 0$. In particular, we have an exact sequence*

$$0 \longrightarrow \oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(-b_j) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \xrightarrow{v} \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow 0$$

Proof. It follows from the definition of v that every entry in the ith row of v is either zero or a monomial of degree a_i in x and y, where x and y are the co-ordinate functions of $\mathbb{P}^1$. Hence, v defines a morphism from $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ to $\oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Similarly, it follows from definition of u that every entry in the jth column of u is either zero or a monomial of degree b_j in x and y, a posteriori, defining a morphism from $\oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(-b_j)$ to $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$.

To show v is surjective, we look at two $r \times r$ minors of v, the first one consisting of the first r columns and the second one consisting of columns $2, \cdots, (r+1)$

$$(v_{p,q})_{1 \leq p,q \leq r} \text{ and } (v_{p,q})_{1 \leq p \leq r, 2 \leq q \leq r+1}$$

The determinant of first one is $x^{a_1+\cdots+a_r}$ and second one is $y^{a_1+\cdots+a_r}$, which do not vanish simultaneously at any point of $\mathbb{P}^1$.

Similarly, to show u is injective, we look at two $(n-r)\times(n-r)$ minors, the first one consisting of rows $(r+1),\cdots,n$ and the second one consisting of row 1 and rows $r+2,\cdots,n$

$$(u_{p,q})_{r+1\leq p\leq n, 1\leq q\leq n-r} \text{ and } (u_{p,q})_{p=1, r+2\leq p\leq n, 1\leq q\leq n-r}$$

The determinant of first one is $(-1)^{\beta_1+\cdots+\beta_{n-r}+n-r+1}x^{b_1+\cdots+b_{n-r}}$ and for the second one is $(-1)^{n-r}y^{b_1+\cdots+b_{n-r}}$, which do not vanish simultaneously at any point of $\mathbb{P}^1$.

Before we begin proof of third part, we would like to explicitly write down the $\beta_i's$ used in the description of the matrix u. Recall that β_i is the number of $A(j)'s$ lying strictly in between $B(i)$ and $B(i-1)$. Thus, when we are in first case where $a_1 > b_1$, we have

$$\beta_i = \begin{cases} j_l - j_{l-1}, & \text{if } i = i_l + 1,\ 1 \leq l \leq \alpha \\ r - j_\alpha - 1, & \text{if } i = n-r \\ 0, & \text{otherwise} \end{cases}$$

and when we are in second case where $a_1 \leq b_1$, we have

$$\beta_i = \begin{cases} j_0, & \text{if } i = 1 \\ j_l - j_{l-1}, & \text{if } i = i_l + 1,\ 1 \leq l \leq \alpha \\ r - j_\alpha - 1, & \text{if } i = n-r \\ 0, & \text{otherwise} \end{cases}$$

Let v_p denote the pth row of v, and u_q denote the qth column of u. Our goal is to show that for any $1 \le q \le (n-r)$, we have $v_p \cdot u_q = 0$ for every $1 \le p \le r$, and hence we can conclude that $v \cdot u = 0$.

We first analyze the case when $a_1 > b_1$. Now u_1 has nonzero entry in the first and $(r+2)$th row, and v_1 is the only row in v with nonzero entries in the respective columns. We see

$$v_1 \cdot u_1 = x^{a_1} \cdot (-y^{b_1}) + x^{A(1)-B(1)} y^{B(1)} \cdot x^{b_1} = -x^{a_1} y^{b_1} + x^{a_1} y^{b_1} = 0$$

Thus, for any $1 \le p \le r$ we have $v_p \cdot u_1 = 0$.

For $2 \le q \le n-r-1$ and $q \ne i_1+1, \cdots, i_\alpha+1$, u_q has nonzero entry in $(r+q)$th and $(r+1+q)$th row. By construction, $u_{r+q,q} = -y^{b_q}$ and $u_{r+q+1,q} = x^{b_q}$. Let $A(j_l) \le B(q) \le A(j_l+1)$ for some $0 \le l \le \alpha$ as per our chosen ordering, then the $(r+q)$th and $(r+1+q)$th columns of v have nonzero entry only in row j_l+1, and the entries are $v_{j_l+1,r+q} = x^{A(j_l+1)-B(q-1)} y^{B(q-1)-A(j_l)}$ and $v_{j_l+1,r+1+q} = x^{A(j_l+1)-B(q)} y^{B(q)-A(j_l)}$. Thus,

$$\begin{aligned} v_{j_l+1} \cdot u_q &= x^{A(j_l+1)-B(q-1)} y^{B(q-1)-A(j_l)} \cdot (-y^{b_q}) + x^{A(j_l+1)-B(q)} y^{B(q)-A(j_l)} \cdot x^{b_q} \\ &= -x^{A(j_l+1)-B(q-1)} y^{B(q)-A(j_l)} + x^{A(j_l+1)-B(q-1)} y^{B(q)-A(j_l)} = 0 \end{aligned}$$

Hence, for any $1 \le p \le r$ and $2 \le q \le n-r-1$, $q \ne i_1+1, \cdots, i_\alpha+1$, we have $v_p \cdot u_q = 0$.

Suppose $q = i_l + 1$ for some $1 \leq l \leq \alpha$, by construction u_{i_l+1} has nonzero entries in rows $j_{l-1}+2, \cdots, j_l+1$, $r+1+i_l$ and $r+1+(i_l+1)$. By our chosen ordering, we have

$$B(i_l) \leq A(j_{l-1}+1) \leq \cdots \leq A(j_l) \leq B(i_l+1) \leq A(j_l+1)$$

Clearly the rows $j_{l-1}+1, \cdots, j_l+1$ of v are the only ones in which there is a nonzero entry in the columns corresponding to the aforementioned rows of u. We have

$$\begin{aligned} v_{j_{l-1}+1} \cdot u_{i_l+1} &= y^{a_{j_{l-1}+1}} \cdot x^{A(j_{l-1}+1)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+1)} \\ &\quad + x^{A(j_{l-1}+1)-B(i_l)} y^{B(i_l)-A(j_{l-1})} \cdot (-y^{b_l}) \\ &= x^{A(j_{l-1}+1)-B(i_l)} y^{B(i_l+1)-A(j_{l-1})} - x^{A(j_{l-1}+1)-B(i_l)} y^{B(i_l+1)-A(j_{l-1})} = 0 \end{aligned}$$

$$\begin{aligned} v_{j_l+1} \cdot u_{i_l+1} &= x^{a_{j_l+1}} \cdot (-1)^{j_l+1-(j_{l-1}+2)} x^{A(j_l)-B(i_l)} y^{B(i_l+1)-A(j_l)} \\ &\quad + x^{A(j_l+1)-B(i_l+1)} y^{B(i_l+1)-A(j_l)} \cdot (-1)^{j_l-j_{l-1}} x^{b_l} \\ &= (-1)^{j_l-j_{l-1}-1} x^{A(j_l+1)-B(i_l)} y^{B(i_l+1)-A(j_l)} \\ &\quad + (-1)^{j_l-j_{l-1}} x^{A(j_l+1)-B(i_l)} y^{B(i_l+1)-A(j_l)} = 0 \end{aligned}$$

For $c = 2, 3, \cdots, j_l - j_{l-1}$, we have

$$
\begin{aligned}
v_{j_{l-1}+c} \cdot u_{i_l+1} &= x^{a_{j_{l-1}+c}} \cdot (-1)^{j_{l-1}+c-(j_{l-1}+2)} x^{A(j_{l-1}+c-1)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+c-1)} \\
&\qquad + y^{a_{j_{l-1}+c}} \cdot (-1)^{j_{l-1}+c+1-(j_{l-1}+2)} x^{A(j_{l-1}+c)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+c)} \\
&= (-1)^{c-2} x^{A(j_{l-1}+c)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+c-1)} \\
&\qquad + (-1)^{c-1} x^{A(j_{l-1}+c)-B(i_l)} y^{B(i_l+1)-A(j_{l-1}+c-1)} = 0
\end{aligned}
$$

Hence, for any $1 \leq p \leq r$ and $q = i_l + 1$ for $1 \leq l \leq \alpha$, we have $v_p \cdot u_q = 0$.

By construction, u_{n-r} has a non-zero entry in rows $j_\alpha + 2, \cdots, r, r+1$ and n. The rows v_p of v such that there is a non-zero entry in any of the columns corresponding to non-zero rows of u_{n-r} are $p = j_\alpha + 1, \cdots, r$. We have

$$
\begin{aligned}
v_{j_\alpha+1} \cdot u_{n-r} &= y^{a_{j_\alpha+1}} \cdot (-1)^{j_\alpha+2-(j_\alpha+2)} x^{A(j_\alpha+1)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+1)} \\
&\qquad + x^{A(j_\alpha+1)-B(n-r-1)} y^{B(n-r-1)-A(j_\alpha)} \cdot (-y^{b_{n-r}}) \\
&= x^{A(j_\alpha+1)-B(n-r-1)} y^{B(n-r)-A(j_\alpha)} - x^{A(j_\alpha+1)-B(n-r-1)} y^{B(n-r)-A(j_\alpha)} = 0
\end{aligned}
$$

Similarly,

$$
v_r \cdot u_{n-r} = x^{a_r} \cdot (-1)^{r-(j_\alpha+2)} x^{A(r-1)-B(n-r-1)} y^{B(n-r)-A(r-1)} + y^{a_r} \cdot (-1)^{\beta_{n-r}} x^{b_{n-r}}
$$

Recall that $\beta_{n-r} = r - j_\alpha - 1$ and $A(r) = B(n-r)$. Thus, we have

$$v_r \cdot u_{n-r} = (-1)^{r-j_\alpha-2} x^{A(r)-B(n-r-1)} y^{B(n-r)-A(r-1)} + (-1)^{r-j_\alpha-1} x^{b_{n-r}} y^{a_r}$$

$$= (-1)^{r-j_\alpha-2} x^{B(n-r)-B(n-r-1)} y^{A(r)-A(r-1)} + (-1)^{r-j_\alpha-1} x^{b_{n-r}} y^{a_r}$$

$$= (-1)^{r-j_\alpha-2} x^{b_{n-r}} y^{a_r} + (-1)^{r-j_\alpha-1} x^{b_{n-r}} y^{a_r} = 0$$

For any $2 \leq c \leq r - j_\alpha - 1$, we have

$$v_{j_\alpha+c} \cdot u_{n-r} = x^{a_{j_\alpha+c}} \cdot (-1)^{j_\alpha+c-(j_\alpha+2)} x^{A(j_\alpha+c-1)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+c-1)}$$

$$+ y^{a_{j_\alpha+c}} \cdot (-1)^{j_\alpha+c+1-(j_\alpha+2)} x^{A(j_\alpha+c)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+c)}$$

$$= (-1)^{c-2} x^{A(j_\alpha+c)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+c-1)}$$

$$+ (-1)^{c-1} x^{A(j_\alpha+c)-B(n-r-1)} y^{B(n-r)-A(j_\alpha+c-1)} = 0$$

Thus, we have $v_p \cdot u_{n-r} = 0$ for any $1 \leq p \leq r$.

We now analyze the case $a_1 \leq b_1$. Observe that for $i_1 + 1 \leq q \leq (n-r)$, the proof of the fact that $v_p \cdot u_q = 0$ for any $1 \leq p \leq r$ is exactly same as above. We only need to work out the cases $1 \leq q \leq i_1$.

By construction, the column u_1 has non-zero entries in rows $1, 2, \cdots, j_0+1$ and $r+2$. The only rows of v which has non-zero entry in corresponding columns are $1 \leq p \leq j_0+1$. We have

$$v_1 \cdot u_1 = x^{a_1} \cdot (-y^{b_1}) + y^{a_1} \cdot (-1)^2 x^{A(1)} y^{B(1)-A(1)} = -x^{a_1} y^{b_1} + x^{a_1} y^{b_1} = 0$$

For $2 \leq c \leq j_0$, we have

$$v_c \cdot u_1 = x^{a_c} \cdot (-1)^c x^{A(c-1)} y^{B(1)-A(c-1)} + y^{a_c} \cdot (-1)^{c+1} x^{A(c)} y^{B(1)-A(c)}$$
$$= (-1)^c x^{A(c)} y^{B(1)-A(c-1)} + (-1)^{c+1} x^{A(c)} y^{B(1)-A(c-1)} = 0$$

and lastly

$$v_{j_0+1} \cdot u_1 = x^{a_{j_0+1}} \cdot (-1)^{j_0+1} x^{A(j_0)} y^{B(1)-A(j_0)} + x^{A(j_0+1)-B(1)} y^{B(1)-A(j_0)} \cdot (-1)^{\beta_1} x^{b_1}$$
$$= (-1)^{j_0+1} x^{A(j_0+1)} y^{B(1)-A(j_0)} + (-1)^{j_0} x^{A(j_0+1)} y^{B(1)-A(j_0)} = 0$$

Thus, $v_p \cdot u_1 = 0$ for all $1 \leq p \leq r$.

For $2 \leq q \leq i_1$, observe that u_q has non-zero entry in row $r+1+q-1$ and $r+1+q$. Clearly, j_0+1 is the only row in v with non-zero entry in the corresponding columns. We have

$$v_{j_0+1} \cdot u_q = x^{A(j_0+1)-B(q-1)} y^{B(q-1)-A(j_0)} \cdot (-y^{b_q}) + x^{A(j_0+1)-B(q)} y^{B(q)-A(j_0)} \cdot x^{b_q}$$
$$= -x^{A(j_0+1)-B(q-1)} y^{B(q)-A(j_0)} + x^{A(j_0+1)-B(q-1)} y^{B(q)-A(j_0)} = 0$$

Thus, $v_p \cdot u_q = 0$ for all $1 \leq p \leq r$ and $2 \leq q \leq i_1$.

In conclusion, we have $v \circ u = 0$. $\square$

Recall from section 2.1 that $M(b_\bullet)$ is the locus of morphisms in $Mor_e(\mathbb{P}^1, G(r,n))$ with the restricted universal quotient bundle being isomorphic to $\mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_{n-r})$, and $M'(a_\bullet)$ is the locus of morphisms in $Mor_e(\mathbb{P}^1, G(r,n))$ with the restricted universal sub-bundle being isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_r)$. We see that

Corollary 3.1.5. *The intersection of the loci $M(b_\bullet)$ and $M'(a_\bullet)$ is nonempty.*

Proof. It follows from Proposition 3.1.4 that we have an exact sequence

$$0 \longrightarrow \oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(-b_j) \xrightarrow{u} \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \xrightarrow{v} \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow 0 \tag{3.3}$$

The surjection v in equation Equation 3.3 corresponds uniquely to an element of $Mor_e(\mathbb{P}^1, G(r,n))$, say φ_v. Moreover, it follows from our identification of v and φ_v in Lemma 2.1.1 and from equation Equation 3.3 that $\varphi_v^*(\mathcal{S})$ is isomorphic to $\oplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(-a_i)$ and $\varphi_v^*(\mathcal{Q})$ is isomorphic to $\oplus_{j=1}^{n-r}\mathcal{O}_{\mathbb{P}^1}(b_j)$, where $\mathcal{S}$ is the universal sub-bundle and $\mathcal{Q}$ is the universal quotient bundle of $G(r,n)$. Hence, the intersection of $M(b_\bullet)$ and $M'(a_\bullet)$ is non-empty. □

3.2 The intersection locus is generically transverse

In this section, we are going to show that there is a point in $M(b_\bullet) \cap M'(a_\bullet)$ where the intersection is transverse. As a consequence, we see that $M(b_\bullet)$ and $M'(a_\bullet)$ intersect generically transversely.

More precisely, we want to show that there exists an exact sequence

$$0 \longrightarrow \oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(-b_j) \xrightarrow{u} \mathcal{O}^{\oplus n} \xrightarrow{v} \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow 0 \tag{3.4}$$

where $a_1 \geq \cdots \geq a_r \geq 0$, $0 \leq b_1 \leq \cdots \leq b_{n-r}$, $(n-r) \leq r$, and $a_1 + \cdots + a_r = b_1 + \cdots + b_{n-r} = e$, such that $M(b_\bullet)$ and $M'(a_\bullet)$ intersect transversely at the morphism φ_v corresponding to the surjection v (see Lemma 2.1.1).

For ease of notation, let $E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ and $K = \mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_{n-r})$. Applying $Hom(K, \bullet)$ and $Hom(\bullet, E)$ to equation Equation 3.4, we obtain two long exact sequences

$$0 \longrightarrow Hom(K,K) \longrightarrow Hom(K, \mathcal{O}^{\oplus n}) \longrightarrow Hom(K,E) \longrightarrow Ext^1(K,K) \longrightarrow 0$$

and

$$0 \longrightarrow Hom(E,E) \longrightarrow Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K,E) \longrightarrow Ext^1(E,E) \longrightarrow 0 \tag{3.5}$$

We observe that

Remark 3.2.1. To show that $M(b_\bullet)$ and $M'(a_\bullet)$ intersect transversely at φ_v, it is enough to show that the kernels of the maps $Hom(K,E) \longrightarrow Ext^1(K,K)$ and $Hom(K,E) \longrightarrow Ext^1(E,E)$ intersect transversely.

Let W_1 be the kernel of the map $Hom(K,E) \longrightarrow Ext^1(K,K)$ and W_2 be the kernel of the map $Hom(K,E) \longrightarrow Ext^1(E,E)$. Using elementary linear algebra, we deduce the following Lemma.

Lemma 3.2.2. *The subspaces W_1 and W_2 of $Hom(K,E)$ intersect transversely iff they span $Hom(K,E)$.*

Proof. Note that W_1 and W_2 intersect transversely if and only if

$$codim(W_1 \subset Hom(K, E)) = codim((W_1 \cap W_2) \subset W_2)$$

Furthermore, it is a known fact that for any two subspaces W_1 and W_2, we have

$$codim((W_1 \cap W_2) \subset W_2) = codim(W_1 \subset (W_1 + W_2))$$

Our assertion follows from these two equations. □

We infer from the exact sequences in equation Equation 3.5 that W_1 is the image of the map

$Hom(K, \mathcal{O}^{\oplus n}) \longrightarrow Hom(K, E)$, and W_2 is the image of the map $Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K, E)$.

Consider the map

$$\Psi : Hom(K, \mathcal{O}^{\oplus n}) \times Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K, E)$$

given by $\Psi(\varphi, \psi) = \psi \circ u + v \circ \varphi$. Clearly, W_1 and W_2 span $Hom(K, E)$ iff Ψ is surjective.

Consider the bilinear map of vector spaces

$$\Phi : Hom(K, \mathcal{O}^{\oplus n}) \times Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K, E)$$

given by $\Phi(\varphi, \psi) = \psi \circ \varphi$. We see that Φ is a bilinear smooth map, so we can look at $D\Phi_{(u,v)}$. Identifying the tangent spaces with the original vector space, we get a map

$$D\Phi_{(u,v)} : Hom(K, \mathcal{O}^{\oplus n}) \times Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K, E)$$

given by $D\Phi_{(u,v)}(\varphi, \psi) = \psi \circ u + v \circ \varphi$. Therefore, we have $D\Phi_{(u,v)} = \Psi$ which yields

Lemma 3.2.3. *The subspaces W_1 and W_2 intersect transversely iff $D\Phi_{(u,v)}$ is surjective.*

We want to show that there exists a pair (u, v) with u injective, v surjective, $v \circ u = 0$, and $D\Phi_{(u,v)}$ is surjective. Before we proceed to show this, we make a couple of observations.

Proposition 3.2.4. *The map Φ is surjective.*

Proof. Let $P = (P_{i,j})_{r\times(n-r)}$ be an element of $Hom(K, E)$. We need to find elements $A \in Hom(K, \mathcal{O}^{\oplus n})$ and $B \in Hom(\mathcal{O}^{\oplus n}, E)$ such that $P = A \circ B$. Clearly, $P_{i,j}$ is a homogeneous element of degree $a_i + b_j$ and hence, there exists homogeneous polynomials $R_{i,j}$ of degree b_j and $Q_{i,j}$ of degree a_i such that

$$P_{i,j} = x^{a_i} \cdot R_{i,j} + Q_{i,j} \cdot y^{b_j}$$

Consider the matrix $A = (A_{i,j})_{r\times n}$ and $B = (B_{i,j})_{n\times(n-r)}$ defined as follows :

$$A_{i,j} = \begin{cases} x^{a_i}, \text{ if } i = j \\ Q_{i,j-r}, \text{ if } r+1 \leq j \leq n \\ 0, \text{ otherwise} \end{cases} \qquad B_{i,j} = \begin{cases} R_{i,j}, & \text{if } 1 \leq i \leq r \\ y^{b_j}, & \text{if } i = r+j \\ 0, & \text{otherwise} \end{cases}$$

Let A_i denote the ith row of A and B_j denote the jth column of B. It follows from construction that $A_i \cdot B_j = x^{a_i} R_{i,j} + Q_{i,j} y^{b_j} = P_{i,j}$. Hence, $\Psi(A,B) = A \circ B = P$. $\square$

Proposition 3.2.5. *When K or E is balanced, then $D\Phi_{(u,v)}$ is surjective.*

Proof. Let $K = \mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_{n-r})$ is balanced. Then, we have

$$Ext^1(K,K) = H^1(\mathbb{P}^1, K^* \otimes K) = H^0(\mathbb{P}^1, K^* \otimes K \otimes \mathcal{O}(-2))^* \quad \text{by Serre's duality}$$

Clearly,

$$K^* \otimes K \otimes O(-2) = \oplus_{i,j} \mathcal{O}(b_i - b_j - 2)$$

Since K is balanced, $b_i - b_j - 2 < 0$ for all $1 \leq i,j \leq n-r$. Hence, $Ext^1(K,K) = 0$. It follows from exact sequence stated earlier (see equation Equation 3.5) that the map

$$Hom(K, \mathcal{O}^{\oplus n}) \longrightarrow Hom(K, E)$$

is surjective, and hence the map

$$D\Phi_{(u,v)} : Hom(K, \mathcal{O}^{\oplus n}) \times Hom(\mathcal{O}^{\oplus n}, E) \longrightarrow Hom(K, E)$$

is also surjective.

We argue similarly when E is balanced. □

We now proceed to show that there exists a pair (u, v) with $D\Phi_{(u,v)}$ is surjective. Before tackling the general case, we look at special case when $r = n - r = 2$.

Proposition 3.2.6. *When $n = 4$ and $r = 2$, then there exits a pair (u, v) with u injective, v surjective, $v \circ u = 0$, and $D\Phi_{(u,v)}$ is surjective.*

Proof. Recall that in Proposition 3.1.1, we constructed a pair (u, v) with v surjective, u injective, and $v \circ u = 0$. Let P be an element of $Hom(K, E)$. We can think of P as a 2×2 matrix $P = (P_{i,j})$ whose (i, j)th entry $P_{i,j}$ is a homogeneous polynomial of degree $a_i + b_j$.

We need to find a 4×2 matrix $R = (R_{i,j})$ and a 2×4 matrix $Q = (Q_{i,j})$, where $R_{i,j}$ has degree b_j and $Q_{i,j}$ has degree a_i, which satisfies the equation

$$P = v \circ R + Q \circ u$$

Comparing the entries of the matrices, we get the following equations

$$P_{1,1} = x^{a_1}R_{1,1} + y^{a_1}R_{2,1} + x^{a_1-b_1}y^{b_1}R_{4,1} - Q_{1,1}y^{b_1} + Q_{1,4}x^{b_1}$$

$$P_{1,2} = x^{a_1}R_{1,2} + y^{a_1}R_{2,2} + x^{a_1-b_1}y^{b_1}R_{4,2} + Q_{1,2}x^{a_1-b_1}y^{a_2} - Q_{1,3}x^{b_2} - Q_{1,4}y^{b_2}$$

$$P_{2,1} = x^{a_2}R_{2,1} + y^{a_2}R_{3,1} - Q_{2,1}y^{b_1} + Q_{2,4}x^{b_1}$$

$$P_{2,2} = x^{a_2}R_{2,2} + y^{a_2}R_{3,2} + Q_{2,2}x^{a_1-b_1}y^{a_2} - Q_{2,3}x^{b_2} - Q_{2,4}y^{b_2}$$

We solve these equations from bottom to top. First, set $R_{3,2}, Q_{2,2}, Q_{2,3}$ to be zero, and solve $R_{2,2}, Q_{2,4}$ for the equation $P_{2,2} = x^{a_2}R_{2,2} - Q_{2,4}y^{b_2}$. Then, set $R_{3,1} = 0$, and solve for $R_{2,1}, Q_{2,1}$ in the equation $P_{2,1} - Q_{2,4}x^{b_1} = x^{a_2}R_{2,1} - Q_{2,1}y^{b_1}$. Then, set $R_{4,2}, Q_{1,2}, Q_{1,3}$ to be zero, and solve for $R_{1,2}, Q_{1,4}$ in the equation $P_{2,1} - y^{a_1}R_{2,2} = x^{a_1}R_{1,2} - Q_{1,4}y^{b_2}$. Finally, set $R_{4,1} = 0$ and solve $R_{1,1}, Q_{1,1}$ in the equation $P_{1,1} - y^{a_1}R_{2,1} - Q_{1,4}x^{b_1} = x^{a_1}R_{1,1} - Q_{1,1}y^{b_1}$.

This shows that the map $D\Phi_{(u,v)}$ is surjective. □

We now proceed to the general case.

Proposition 3.2.7. *Given any $n \geq 4$ and $2 \leq r \leq n-2$ satisfying $(n-r) \leq r$, there exists a pair (u,v) with u injective, v surjective, $v \circ u = 0$, and $D\Phi_{(u,v)}$ is surjective.*

Proof. Recall that we constructed matrices v and u in the paragraphs preceding Proposition 3.1.4, and proved that v is surjective, u is injective, and $v \circ u = 0$. We just need to show that $D\Phi_{(u,v)}$ is surjective for this pair (u,v).

Let P be an element of $Hom(K, E)$. We can think of P as $(P_{i,j})$ which is a $r \times (n-r)$ matrix with $P_{i,j}$ being a homogeneous polynomial of degree $a_i + b_j$. We need to show that there exits elements $R \in Hom(K, \mathcal{O}^{\oplus n})$ and $Q \in Hom(\mathcal{O}^{\oplus n}, E)$ such that $P = v \circ R + Q \circ u$. We can think of R as $(R_{i,j})$ which is a $(n-r) \times n$ matrix with $R_{i,j}$ being homogeneous polynomial of degree b_j, and $Q = (Q_{i,j})$ a $r \times n$ matrix with $Q_{i,j}$ being homogeneous polynomial of degree a_i.

Observe that by comparing both sides of equation $P = v \circ R + Q \circ u$, get that for any i, j, we have

$$P_{i,j} = x^{a_i} R_{i,j} - Q_{\alpha_i, \beta_j} y^{b_j} + \text{ other terms}$$

We try to solve these equations in the following order

$$P_{r,n-r}, \cdots, P_{r,1}, P_{r-1,n-r}, \cdots, P_{r-1,1}, \cdots, P_{1,n-r}, \cdots, P_{1,1}$$

in the following manner :

Assume that all equations for $P_{i,j}$ where $i > i_o$, or $i = i_o$ and $j > j_o$ are solved. As mentioned earlier we have equation

$$P_{i_o,j_o} = x^{a_{i_o}} R_{i_o,j_o} - Q_{\alpha_{i_o}, \beta_{j_o}} y^{b_{j_o}} + \text{ other terms}$$

where the "other terms" has a bunch of $R_{\alpha,\beta}$'s and $Q_{\alpha',\beta'}$'s occurring in them, some of which are already determined in some previous equation, and some are not. If they are not determined, then set them to be 0. Then we solve for R_{i_0,j_0} and $Q_{\alpha_{i_0},\beta_{j_0}}$ in the equation

$$P_{i_0,j_0} - \text{ other terms } = x^{a_{i_0}} R_{i_0,j_0} - Q_{\alpha_{i_0},\beta_{j_0}} y^{b_{j_0}}$$

We claim that we can solve for all the equations $P_{r,n-r}, \cdots, P_{1,1}$ in aforementioned method. Suppose not, consider the first P_{i_0,j_0} for which a conflict occurs. Only possible conflict at this step is that R_{i_0,j_0} or $Q_{\alpha_{i_0},\beta_{j_0}}$ has been already determined at some previous step. But this is not possible, because by construction of the matrices u and v, we have that in each column of v in which x^{a_i} appears, all the entries below x^{a_i} in that column are 0; similarly, in each row of u in which $-y^{b_j}$ appears, all the entries to the right of $-y^{b_j}$ in that row are 0; and hence, R_{i_0,j_0} and $Q_{\alpha_{i_0},\beta_{j_0}}$ does not appear in any of the previous equations. □

As a corollary, we get

Corollary 3.2.8. *There exists an exact sequence*

$$0 \longrightarrow \oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(-b_j) \xrightarrow{\;u\;} \mathcal{O}^{\oplus n} \xrightarrow{\;v\;} \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \longrightarrow 0$$

such that the loci $M(b_\bullet)$ and $M'(a_\bullet)$ intersect transversely at the morphism φ_v corresponding to the surjection v.

In particular, the loci $M(b_\bullet)$ and $M'(a_\bullet)$ intersect generically transversely and has an irreducible component of codimension

$$\sum_{i,j} \max\{a_i - a_j - 1, 0\} + \sum_{i,j} \max\{b_i - b_j - 1, 0\}$$

Moreover, if either of the splitting type $\{a_\bullet\}$ or $\{b_\bullet\}$ is balanced, then the intersection is transverse.

Proof. The first assertion of the corollary follows from Remark 3.2.1, Lemma 3.2.2, Lemma 3.2.3, and Proposition 3.2.7.

The second assertion follows from the first one and Proposition 2.1.6.

The third assertion follows from Remark 3.2.1, Lemma 3.2.2, Lemma 3.2.3, and Proposition 3.2.5. □

In summary, it follows from Corollary 3.1.5 and 3.2.8 that

Theorem 3.2.9. *The intersection of the loci $M(b_\bullet)$ and $M'(a_\bullet)$ is nonempty and generically transverse. Furthermore, if either of the splitting types $\{a_\bullet\}$ or $\{b_\bullet\}$ is balanced, then the intersection is transverse.*

3.3 Analyzing the locus with restricted tangent bundle having fixed splitting type

In this section, we are going to show that the locus of morphisms in $Mor_e(\mathbb{P}^1, G(r,n))$ with the restricted tangent bundle having fixed splitting type need not always be irreducible. This is in sharp contrast with the results of Verdier (9) and Ramella (10), who have shown that given a collection of integers $a_1, \cdots, a_n$ with $a_1 \geq \cdots \geq a_n$ and $\sum_{i=1}^n a_i = e$, the locus of morphisms φ in $Mor_e(\mathbb{P}^1, \mathbb{P}^n)$ with the restricted twisted tangent bundle $\varphi^*(T_{\mathbb{P}^n}(-1))$ having splitting type $(a_1, \cdots, a_n)$ is empty if $a_n < 0$, else it is nonempty, smooth and connected of codimension

$$\sum_{i,j} \max\{a_i - a_j - 1, 0\}$$

Recall that given a morphism $\varphi : \mathbb{P}^1 \longrightarrow G(r,n)$, the restricted tangent bundle $\varphi^*(T_{G(r,n)})$ is isomorphic to $\varphi^*(\mathcal{S}^*) \otimes \varphi^*(\mathcal{Q})$, where $\mathcal{S}$ and $\mathcal{Q}$ are the universal sub-bundle and universal quotient bundle of $G(r,n)$. Now let us fix a splitting type $c_1, \cdots, c_{r(n-r)}$ for the restricted tangent bundle $\varphi^*(T_{G(r,n)})$. We define

Definition 3.3.1. A *filling* for the splitting type $\{c_l\}_{1 \leq l \leq r(n-r)}$ to be a $r \times (n-r)$ matrix A with entries $a_{i,j} = c_l$ for some l depending on i, j such that

- For all $1 \leq i \leq r-1$ and $1 \leq j \leq n-r-1$, we have $a_{i,j} \leq a_{i+1,j}$ and $a_{i,j} \leq a_{i,j+1}$.
- For all $1 \leq i \leq r-1$ we have $a_{i,n-r} \leq a_{i+1,n-r}$, and for all $1 \leq j \leq n-r-1$ we have $a_{r,j} \leq a_{r,j+1}$.
- For all $1 \leq i \leq r-1$ the difference $a_{i+1,j} - a_{i,j}$ is independent of j, and for all $1 \leq j \leq n-r-1$ the difference $a_{i,j+1} - a_{i,j}$ is independent of i.

Moreover, we define

Definition 3.3.2. A collection of integers $\alpha_1, \cdots, \alpha_\nu$ is *non-negative* if α_i are non-negative integers for all $1 \le i \le \nu$. A collection of integers $\alpha_1, \cdots, \alpha_\nu$ is *increasing* if $\alpha_1 \le \cdots \le \alpha_\nu$.

The exigency of these definitions is due to the following Lemma.

Lemma 3.3.3. *A filling for the splitting type $\{c_l\}_{1 \le l \le r(n-r)}$ uniquely determines the non-negative increasing splitting type of $\varphi^*(\mathcal{S}^*)$ and $\varphi^*(\mathcal{Q})$.*

Proof. Let $\varphi^*(\mathcal{S}^*)$ be isomorphic to $\oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$, and let $\varphi^*(\mathcal{Q})$ be isomorphic to $\oplus_{j=1}^{n-r} \mathcal{O}_{\mathbb{P}^1}(b_j)$. We can determine the a_i's and b_j's uniquely by the following equations

$$
\begin{aligned}
e &= \frac{1}{n} \sum_{i,j} a_{i,j} \\
a_i &= \frac{1}{n-r} \left(\sum_{j=1}^{n-r} a_{i,j} - e \right) && \text{for all } 1 \le i \le r \\
b_j &= \frac{1}{r} \left(\sum_{i=1}^{r} a_{i,j} - e \right) && \text{for all } 1 \le j \le n-r
\end{aligned}
$$

Conversely, given a splitting type $\{a_\bullet\}$ for $\varphi^*(\mathcal{S}^*)$ and $\{b_\bullet\}$ for $\varphi^*(\mathcal{Q})$ with $0 \le a_1 \le \cdots \le a_r$ and $0 \le b_1 \le \cdots \le b_{n-r}$, we define a filling whose (i,j)th entry is $a_i + b_j$. $\square$

Let $\{a_\bullet\}$ and $\{a'_\bullet\}$ be two non-negative increasing splitting types for $\varphi^*(\mathcal{S}^*)$, and let $\{b_\bullet\}$ and $\{b'_\bullet\}$ be two non-negative increasing splitting types for $\varphi^*(\mathcal{Q})$. If $\{a_\bullet\}$ is different from $\{a'_\bullet\}$ (i.e. the corresponding vector bundles are not isomorphic) or $\{b_\bullet\}$ is different from $\{b'_\bullet\}$, then the intersection of loci $M(b_\bullet) \cap M'(a_\bullet)$ and $M(b'_\bullet) \cap M'(a'_\bullet)$ must be empty because a morphism $\varphi : \mathbb{P}^1 \longrightarrow G(r,n)$ uniquely determines the splitting type for $\varphi^*(\mathcal{S}^*)$ and $\varphi^*(\mathcal{Q})$. Hence, it follows from Lemma 3.3.3 that

Proposition 3.3.4. *The locus of morphisms φ in $Mor_e(\mathbb{P}^1, G(r,n))$ with the restricted tangent bundle having the splitting type $\{c_l\}_{1\leq l\leq r(n-r)}$ is stratified by the loci $M(b_\bullet)\cap M'(a_\bullet)$ where $\{a_\bullet\}$ and $\{b_\bullet\}$ are non-negative increasing splitting types for $\varphi^*(\mathcal{S}^*)$ and $\varphi^*(\mathcal{Q})$ arising from the distinct fillings for $\{c_l\}_{1\leq l\leq r(n-r)}$.*

Recall that given a collection of non-negative increasing integers $\alpha_1, \cdots, \alpha_\nu$, we defined its *polygonal line* (see Definition 2.1.7) to be $\mathfrak{P}(\alpha_\bullet) = (\alpha_\nu, \alpha_\nu + \alpha_{\nu-1}, \cdots, \alpha_\nu + \cdots + \alpha_1)$.

Definition 3.3.5. We say a filling $\{a_{i,j}\}_{1\leq i\leq r,\, 1\leq j\leq n-r}$ of a splitting type $\{c_l\}_{1\leq l\leq r(n-r)}$ to be *minimal* if the following holds:

Let $a_1, \cdots, a_r$ be the non-negative increasing splitting type of $\varphi^*(\mathcal{S}^*)$ and $b_1, \cdots, b_{n-r}$ be the non-negative increasing splitting type of $\varphi^*(\mathcal{Q})$ uniquely determined by the filling (see Lemma 3.3.3). Then for every possible non-negative increasing collection of integers $a'_1, \cdots, a'_r$ and $b'_1, \cdots, b'_{n-r}$ with $a'_1 + \cdots + a'_r = b'_1 + \cdots + b'_{n-r} = e$ satisfying $\mathfrak{P}(a'_\bullet) \geq \mathfrak{P}(a_\bullet)$ and $\mathfrak{P}(b'_\bullet) \geq \mathfrak{P}(b_\bullet)$ with atleast one of the inequality being strict, the matrix $\{a'_i + b'_j\}_{1\leq i\leq r,\, 1\leq j\leq n-r}$ is not a filling for $\{c_l\}$.

It follows as a consequence of Corollary 2.1.9 that

Lemma 3.3.6. *Suppose $\{a_\bullet\}$ and $\{b_\bullet\}$ be the non-negative increasing splitting types for $\varphi^*(\mathcal{S})^*$ and $\varphi^*(\mathcal{Q})$ respectively arising from a minimal filling of a given splitting $\{c_l\}$, then the loci $M(b_\bullet)\cap M'(a_\bullet)$ is closed in the locus of all degree e morphisms with restricted tangent bundle having splitting type $\{c_l\}$.*

Hence, we see that

Corollary 3.3.7. *The number of irreducible components of the locus of degree e morphisms from $\mathbb{P}^1$ to $G(r, n)$ with the restricted tangent bundle having a given splitting type is bounded below by the number of distinct minimal fillings of the given splitting type. In particular, this locus need not always be irreducible.*

Proof. The proof follows from Proposition 3.3.4 and Lemma 3.3.6, Corollary 3.1.5 and 3.2.8. $\square$

For example, let $r = 2$, $n = 4$ and $e = 6$. The locus $Mor_6(\mathbb{P}^1, G(2, 4))$ of degree 6 morphisms from $\mathbb{P}^1$ to $G(2, 4)$ has dimension 28. Consider the splitting type $3, 5, 7, 9$ for the restricted tangent bundle. We have two possible fillings

$$\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 7 \\ 5 & 9 \end{pmatrix}$$

Corresponding to the first filling we have non-negative increasing splitting types $(a_1, a_2) = (1, 5)$ and $(b_1, b_2) = (2, 4)$, and to the second filling we have $(a_1, a_2) = (2, 4)$ and $(b_1, b_2) = (1, 5)$. Since both the fillings are minimal (see Lemma 3.3.8), the locus of morphisms in $Mor_6(\mathbb{P}^1, G(2, 4))$ is the disjoint union of the loci $M(2, 4) \cap M'(1, 5)$ and $M(1, 5) \cap M'(2, 4)$. Now the loci $M(1, 5)$ and $M'(1, 5)$ have codimension 3 in $Mor_6(\mathbb{P}^1, G(2, 4))$ which follows from Proposition 2.1.6. Similarly, the loci $M(2, 4)$ and $M'(2, 4)$ have codimension 1 in $Mor_6(\mathbb{P}^1, G(2, 4))$. It follows from Corollary 3.1.5 that the locus $M(2, 4) \cap M'(1, 5)$ is nonempty. Similarly, $M(1, 5) \cap M'(1, 5)$ is also nonempty, and since $M(2, 4)$ and $M(1, 5)$ are disjoint, the intersection $M(2, 4) \cap M'(1, 5)$

must be proper subset of $M'(1,5)$. Moreover, since $M(2,4)$ has codimension 1 in $Mor_6(\mathbb{P}^1, G(2,4))$, the intersection locus $M(2,4) \cap M'(1,5)$ must have codimension 1 in $M'(1,5)$, and hence, it must have codimension 4 in $Mor_6(\mathbb{P}^1, G(2,4))$. Similarly, the intersection locus $M(1,5) \cap M'(2,4)$ has codimension 4 in $Mor_6(\mathbb{P}^1, G(2,4))$. Hence, the locus of degree 6 morphisms from $\mathbb{P}^1$ to $G(2,4)$ with restricted tangent bundle having splitting type $3,5,7,9$ has codimension 4 and has atleast two irreducible components arising from the two distinct fillings.

We see that Proposition 3.3.4 exhorts us to determine the possible fillings of a splitting type as a key step towards understanding the locus of morphisms in $Mor_e(\mathbb{P}^1, G(r,n))$ with restricted tangent bundles having the given splitting type. To this end, we have the following Lemmas.

Lemma 3.3.8. *Let $r = 2$ and $n = 4$, and let $\{c_1, c_2, c_3, c_4\}$ be a splitting type of the restricted tangent bundle with $c_1 \le c_2 < c_3 \le c_4$. Then $\{c_1, c_2, c_3, c_4\}$ has two possible fillings*

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}$$

Moreover, both the fillings are minimal.

Similarly, we have

Lemma 3.3.9. *Let $r = 3$ and $n = 5$. A splitting type $\{c_1, \cdots, c_6\}$ of the restricted tangent bundle with $c_1 \leq \cdots \leq c_6$ has exactly one filling except when $\{c_1, \cdots, c_6\} = \{c_1, c_1 + \lambda, \cdots, c_1 + 5\lambda\}$ for some integer λ in which case there are two possible fillings*

$$\begin{pmatrix} c_1 & c_1 + \lambda \\ c_1 + 2\lambda & c_1 + 3\lambda \\ c_1 + 4\lambda & c_1 + 5\lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_1 + 3\lambda \\ c_1 + \lambda & c_1 + 4\lambda \\ c_1 + 2\lambda & c_1 + 5\lambda \end{pmatrix}$$

Additionally, all the fillings are minimal.

Proof of Lemma 3.3.8 and 3.3.9. We will briefly sketch the proof of Lemma 3.3.9. One can prove Lemma 3.3.8 in a similar fashion.

Let $r = 3$ and $n = 5$. Given a splitting type $\{c_1, \cdots, c_6\}$ of a restricted tangent bundle $\varphi^*(T_{G(r,n)})$, there is at least one filling (since $\varphi^*(T_{G(r,n)}) = \varphi^*(\mathcal{S}^*) \otimes \varphi^*(\mathcal{Q})$), say A, which is a 3×2 matrix. After subtracting the $(1, 1)$th entry from every other entry of A, we get a new matrix of form

$$\begin{pmatrix} 0 & \lambda \\ \rho_2 & \rho_2 + \lambda \\ \rho_3 & \rho_3 + \lambda \end{pmatrix}$$

for some non-negative integers λ, ρ_2, ρ_3 with $\rho_2 \leq \rho_3$. We now look at every possible permutations with the $(1, 1)$th entry being zero and the $(3, 2)$th entry being $\rho_2 + \lambda$, and force the conditions of definition 3.3.1 which gives us some equations which must be compatible. This gives us all the possibilities. A similar brute force method works for $r = 2$ and $n = 4$.

The proof of minimality of the fillings in the cases $r = 2$, $n = 4$ and $r = 3$, $n = 5$ are in a similar flavor. The key idea is to use the fact that the $(1, 1)$th and $(r, n-r)$th entries are the same for every possible filling. For instance, when $r = 3$, $n = 5$, let $\{c_1, \cdots, c_6\} = \{c_1, c_1 + \lambda, \cdots, c_1 + 5\lambda\}$ and let $\{a_\bullet\}$ and $\{b_\bullet\}$ be the corresponding induced splittings. For any $\{a'_\bullet\}$ with $\mathfrak{P}(a'_\bullet) \geq \mathfrak{P}(a_\bullet)$ and $\{b'_\bullet\}$ with $\mathfrak{P}(b'_\bullet) \geq \mathfrak{P}(b_\bullet)$, we must have $a'_3 + b'_2 = a_3 + b_2$ and since $a'_3 \geq a_3$ and $b'_2 \geq b_2$, we get $a'_3 = a_3$, $b'_2 = b_2$. This gives $b'_1 = b_1$ and $a'_2 + a'_1 = a_2 + a_1$. Since we must have $a'_1 + b'_1 = a_1 + b_1$, we get $a'_i = a_i$ for all $i = 1, 2, 3$ and $b'_j = b_j$ for all $j = 1, 2$. $\square$

Using a similar method as in proof of Lemma 3.3.9, we deduce that when $r = 4$ and $n = 6$, a splitting type of the restricted tangent bundle of the form $\{c_1, c_1, c_2, c_2, c_3, c_3, c_4, c_4\}$ with $0 \leq c_2 - c_1 = c_3 - c_2 = c_4 - c_3$ has three possible fillings

$$\begin{pmatrix} c_1 & c_1 \\ c_2 & c_2 \\ c_3 & c_3 \\ c_4 & c_4 \end{pmatrix}, \quad \begin{pmatrix} c_1 & c_2 \\ c_1 & c_2 \\ c_3 & c_4 \\ c_3 & c_4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 & c_3 \\ c_1 & c_3 \\ c_2 & c_4 \\ c_2 & c_4 \end{pmatrix}$$

However, in general, we found it impossible to determine all possible fillings using this brute force method.

Additionally, we observe from these special cases that the number of fillings seems to increase as we increase r, n and e. We don't know how the fillings of a given splitting type depend on r and n, but we can provide a very crude upper bound for the number of possible fillings.

Lemma 3.3.10. *The total number of distinct fillings of a splitting type $\{c_l\}_{1\leq l\leq r(n-r)}$ of the restricted tangent bundle is bounded above by $\binom{r(n-r)-2}{n-r-1}$.*

Proof. It follows from definition 3.3.1 that every filling must have the same $(1,1)$th and $(r, n-r)$th entry. Furthermore, we see that every filling is uniquely determined by the entries $(1,2)$, $\cdots$, $(1, n-r)$. Hence, a clumsy upper bound for the total number of fillings is the number of choices for these entries, which is $\binom{r(n-r)-2}{n-r-1}$. □

On a more positive note, we see that

Lemma 3.3.11. *If the splitting type of the restricted tangent bundle $\varphi^*(T_{G(r,n)})$ is balanced, then the splitting type of the restricted universal sub-bundle $\varphi^*(\mathcal{S})$ and the splitting type of the restricted universal quotient bundle $\varphi^*(\mathcal{Q})$ must be balanced.*

Proof. Let us choose a filling for the splitting type of the restricted tangent bundle, and let $a_1, \cdots, a_r$ and $b_1, \cdots, b_{n-r}$ be non-negative increasing splitting types of $\varphi^*(\mathcal{S}^*)$ and $\varphi^*(\mathcal{Q})$ respectively. Since the splitting type of the restricted tangent bundle is balanced, we must have $(a_r + b_{n-r}) - (a_1 + b_1) \leq 1$, which yields $a_r - a_1 \leq 1$ and $b_{n-r} - b_1 \leq 1$. Hence, the splitting types of $\varphi^*(\mathcal{S})$ and $\varphi^*(\mathcal{Q})$ must be balanced. □

In conclusion, the locus of morphisms in $Mor_e(\mathbb{P}^1, G(r,n))$ need not always be irreducible. For example, when $r = 2$ and $n = 4$, and let c_1, c_2, c_3, c_4 be non-negative increasing splitting type of the restricted universal tangent bundle, with $c_2 < c_3$. It follows from Lemma 3.3.8 that this locus has at least two irreducible components.

CHAPTER 4

BETTI NUMBERS OF MODULI SPACE OF SHEAVES ON $\mathbb{P}^2$

In this chapter, we determine bounds for stabilization of the Betti numbers of the moduli space of stable sheaves on $\mathbb{P}^2$ when the rank is atleast two and it is coprime to the first Chern class.

4.1 Estimating the Generating Functions when the rank is one

In this section, our goal is to analyze the generating functions $G_{1,c}(q)$ and $\tilde{G}_{1,\tilde{c}}(q)$. More precisely, we are going to show that when $\Delta > 2N$ the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in the generating functions $(1-q)G_{1,c}(q)$ and $(1-q)\tilde{G}_{1,\tilde{c}}(q)$ is zero. As a consequence, we are going to show that the $2N$th Betti number of $M_{\mathbb{P}^2,H}(1,c,c_2)$ stabilize when $c_2 \geq 2N$.

Recall that given a smooth projective surface X with an ample divisor H on X, the moduli space $M_{X,H}(1,c,c_2)$ is isomorphic to $Pic^c(X) \times X^{[c_2]}$, where $Pic^c(X)$ is the abelian variety of line bundles on X with first Chern class c, and $X^{[n]}$ is the Hilbert scheme of n points on X. The Betti numbers of $X^{[n]}$ were computed by Göttsche (19). Using the Künneth formula, Coskun and Woolf (17)[Proposition 3.3] showed that the Betti numbers of $M_{X,H}(1,c,c_2)$ stabilize as c_2 tends to infinity. In the special case when $X = \mathbb{P}^2$, the moduli space $M_{\mathbb{P}^2,H}(1,c,c_2)$ is isomorphic to $\mathbb{P}^{2[c_2]}$. Ellingsrud and Stromme (20)[Theorem 1.1, Corollary 1.3] computed the Betti numbers of $\mathbb{P}^{2[c_2]}$ and showed that the $2N$th Betti number stabilize when $c_2 \geq 2N$. In this section, our goal is to re-derive this result in a flavor similar to the higher rank case.

We infer from equation Equation 2.7 that

$$G_{1,c}(q) = \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{P}^2,H}(r,c,\Delta)]\mathbb{L}^{(1-2\Delta)}q^{\Delta}$$

and

$$\tilde{G}_{1,\tilde{c}}(q) = \sum_{\tilde{\Delta} \geq 0} [\mathcal{M}_{\mathbb{F}_1,E+F}(r,\tilde{c},\tilde{\Delta})]\mathbb{L}^{(1-2\tilde{\Delta})}q^{\tilde{\Delta}}$$

We have

Proposition 4.1.1. *For $\Delta > 2N$, the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{1,c}(q)$ is zero. Same for $(1-q)\tilde{G}_{r,\tilde{c}}(q)$.*

Proof. We have the following equality of generating functions due to Göttsche (37)[Example 4.9.1]

$$\sum_{\Delta=0}^{\infty} [(\mathbb{P}^2)^{[\Delta]}]q^{\Delta} = \prod_{m=1}^{\infty} \frac{1}{(1-\mathbb{L}^{m-1}q^m)(1-\mathbb{L}^m q^m)(1-\mathbb{L}^{m+1}q^m)}$$

Replacing q with $\mathbb{L}^{-2}q$ in above equation, we get

$$\sum_{\Delta=0}^{\infty} [(\mathbb{P}^2)^{[\Delta]}]\mathbb{L}^{-2\Delta}q^{\Delta} = \prod_{m=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-(m-1)}q^m)(1-\mathbb{L}^{-m}q^m)(1-\mathbb{L}^{-(m+1)}q^m)}$$

Note that we have

$$[\mathcal{M}_{\mathbb{P}^2,H}(1,c,\Delta)] = (\mathbb{L}-1)^{-1}[(\mathbb{P}^2)^{[\Delta]}]$$

Thus, we get

$$\begin{aligned}(1-q)G_{1,c}(q) &= \frac{(1-q)\mathbb{L}}{(\mathbb{L}-1)}\sum_{\Delta=0}^{\infty}[(\mathbb{P}^2)^{[\Delta]}]\mathbb{L}^{-2\Delta}q^{\Delta} \\ &= \frac{(1-q)}{(1-\mathbb{L}^{-1})}\prod_{m=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-(m-1)}q^m)(1-\mathbb{L}^{-m}q^m)(1-\mathbb{L}^{-(m+1)}q^m)} \\ &= \prod_{m_1=2}^{\infty}\frac{1}{(1-\mathbb{L}^{-(m_1-1)}q^{m_1})}\prod_{m_2=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-2m_2}q^{m_2})}\prod_{m_3=0}^{\infty}\frac{1}{(1-\mathbb{L}^{-(m_3+1)}q^{m_3})} \\ &= \prod_{m_1=2}^{\infty}\left(\sum_{\alpha_1=0}^{\infty}\mathbb{L}^{-(m_1-1)\alpha_1}q^{m_1\alpha_1}\right)\times\prod_{m_2=1}^{\infty}\left(\sum_{\alpha_2=0}^{\infty}\mathbb{L}^{-m_2\alpha_2}q^{m_2\alpha_2}\right)\times \\ &\qquad\prod_{m_3=0}^{\infty}\left(\sum_{\alpha_3=0}^{\infty}\mathbb{L}^{-(m_3+1)\alpha_3}q^{m_3\alpha_3}\right)\end{aligned}$$

Each non-zero term contributing to the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{1,c}(q)$ arises from a pair of equations

$$\begin{aligned}\Delta &= \sum_{j=1}^{\delta_1}m_1^{(j)}\alpha_1^{(j)}+\sum_{j=1}^{\delta_2}m_2^{(j)}\alpha_2^{(j)}+\sum_{j=1}^{\delta_3}m_3^{(j)}\alpha_3^{(j)} \\ -N &= \sum_{j=1}^{\delta_1}-(m_1^{(j)}-1)\alpha_1^{(j)}+\sum_{j=1}^{\delta_2}-m_2^{(j)}\alpha_2^{(j)}+\sum_{j=1}^{\delta_3}-(m_3^{(j)}+1)\alpha_3^{(j)}\end{aligned}$$

where $\alpha_1^{(j)},\alpha_2^{(j)},\alpha_3^{(j)}\geq 0$ for all $j\geq 1$, and $m_1^{(j)}\geq 2$, $m_2^{(j)}\geq 1$, $m_3^{(j)}\geq 0$ for all $j\geq 1$. Therefore, we see that

$$\Delta-N=\sum_{j=1}^{\delta_1}\alpha_1^{(j)}-\sum_{j=1}^{\delta_3}\alpha_3^{(j)}\leq\sum_{j=1}^{\delta_1}(m_1^{(j)}-1)\alpha_1^{(j)}\leq N$$

Hence, for $\Delta>2N$ the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{1,c}(q)$ must be zero.

In a similar fashion as above, we use the following equality of generating functions due to Göttsche (37)[Example 4.9.3]

$$\sum_{\tilde{\Delta}=0}^{\infty}[\mathbb{F}_1^{[\tilde{\Delta}]}]q^{\tilde{\Delta}} = \prod_{m=1}^{\infty}\frac{1}{(1-\mathbb{L}^{m-1}q^m)(1-\mathbb{L}^m q^m)^2(1-\mathbb{L}^{m+1}q^m)}$$

Replacing q with $\mathbb{L}^{-2}q$ and using the fact $[\mathcal{M}_{\mathbb{F}_1,E+F}(1,\tilde{c},\tilde{\Delta})] = (\mathbb{L}-1)^{-1}[\mathbb{F}_1^{[\tilde{\Delta}]}]$, we obtain the following equation

$$(1-q)\tilde{G}_{1,\tilde{c}}(q) = \prod_{m_1=2}^{\infty}\left(\sum_{\alpha_1=0}^{\infty}\mathbb{L}^{-(m_1-1)\alpha_1}q^{m_1\alpha_1}\right)\times\prod_{m_2=1}^{\infty}\left(\sum_{\alpha_2=0}^{\infty}\mathbb{L}^{-m_2\alpha_2}q^{m_2\alpha_2}\right)^2\times$$
$$\prod_{m_3=0}^{\infty}\left(\sum_{\alpha_3=0}^{\infty}\mathbb{L}^{-(m_3+1)\alpha_3}q^{m_3\alpha_3}\right)$$

Each non-zero term contributing to the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{1,\tilde{c}}(q)$ arises from a pair of equations

$$\Delta = \sum_{j=1}^{\delta_1} m_1^{(j)}\alpha_1^{(j)} + \sum_{j=1}^{\delta_{2,1}} m_2^{(j,1)}\alpha_2^{(j,1)} + \sum_{j=1}^{\delta_{2,2}} m_2^{(j,2)}\alpha_2^{(j,2)} + \sum_{j=1}^{\delta_3} m_3^{(j)}\alpha_3^{(j)}$$
$$-N = \sum_{j=1}^{\delta_1} -(m_1^{(j)}-1)\alpha_1^{(j)} + \sum_{j=1}^{\delta_{2,1}} -m_2^{(j,1)}\alpha_2^{(j,1)} + \sum_{j=1}^{\delta_{2,2}} -m_2^{(j,2)}\alpha_2^{(j,2)} + \sum_{j=1}^{\delta_3} -(m_3^{(j)}+1)\alpha_3^{(j)}$$

where $\alpha_1^{(j)}, \alpha_2^{(j,1)}, \alpha_2^{(j,2)}, \alpha_3^{(j)} \geq 0$ for all $j \geq 1$, and $m_1^{(j)} \geq 2$, $m_2^{(j,1)}, m_2^{(j,2)} \geq 1$, $m_3^{(j)} \geq 0$ for all $j \geq 1$. Therefore, we see that

$$\Delta - N = \sum_{j=1}^{\delta_1}\alpha_1^{(j)} - \sum_{j=1}^{\delta_3}\alpha_3^{(j)} \leq \sum_{j=1}^{\delta_1}(m_1^{(j)}-1)\alpha_1^{(j)} \leq N$$

Hence, for $\Delta > 2N$ the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\tilde{G}_{1,\tilde{c}}(q)$ must be zero. $\square$

As a consequence of above Proposition 4.1.1, we have the following:

Proposition 4.1.2. *When $c_2 \geq 2N$, the $2N$-th Betti number of $M_{\mathbb{P}^2,H}(1,c,c_2)$ stabilize.*

Proof. Note that all μ_H-semistable sheaves of rank one on $\mathbb{P}^2$ are μ_H-stable, because the rank is coprime to the first Chern class. As a consequence, we can use Proposition 2.2.10 due to Coskun and Woolf and the fact that $c_2 = r\Delta + \frac{r-1}{2r}c_1^2$ to get the following equality of generating functions

$$(1-q)\sum_{c_2\geq 0}[M_{\mathbb{P}^2,H}(\gamma)]\mathbb{L}^{-ext^1(\gamma,\gamma)}q^{c_2} = (1-\mathbb{L}^{-1})(1-q)G_{1,c}(q)$$

where γ denotes the Chern character (r,c,Δ).

Each term contributing to the coefficient of $\mathbb{L}^{-N}q^{d}$ in $(1-\mathbb{L}^{-1})(1-q)G_{1,c}(q)$ comes from a pair of equations

$$d = \Delta - N = \varepsilon - N'$$

where $\varepsilon \in \{-1,0\}$ accounts for the contribution of the coefficient coming from $(1-\mathbb{L}^{-1})$, and (Δ, N') accounts for the contribution coming from the terms in coefficient of $\mathbb{L}^{-N'}q^{\Delta}$ in $(1-q)G_{1,c}(q)$. It follows from Proposition 4.1.1 that for the coefficient of $\mathbb{L}^{-N'}q^{\Delta}$ to be nonzero, we must have $\Delta \leq 2N'$. Consequently, we must have $d \leq 2N$. Hence, for $d > 2N$, the coefficient of $\mathbb{L}^{-N}q^{d}$ in $(1-\mathbb{L}^{-1})(1-q)G_{1,c}(q)$ must be zero. Therefore, using Remark 2.2.8, we conclude that the $2N$th Betti number of $M_{\mathbb{P}^2,H}(1,c,c_2)$ stabilize for $c_2 \geq 2N$. $\square$

4.2 Estimating the generating function $\tilde{G}_{r,\tilde{c}}(q)$ when rank is at least two

In this section, our goal is to show that there is a constant C_0 depending only on r and $\tilde{c}$ such that when $\Delta > N + C_0$, the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\tilde{G}_{r,\tilde{c}}(q)$ is zero. We are going to show this in a couple of steps. First, we are going to use Mozgovoy's theorem (23)[Theorem 1.1] and estimate a generating function in A^- expressed in terms of the classes of the moduli stack $\mathcal{M}_{\mathbb{F}_1,F}(\gamma)$. Then, we are going to use Joyce's theorem (38)[Theorem 6.21] to relate the classes of the moduli stacks $\mathcal{M}_{\mathbb{F}_1,E+F}(\gamma)$ and $\mathcal{M}_{\mathbb{F}_1,F}(\gamma)$ in A^-. Lastly, we are going to use key ideas of Coskun and Woolf (17) and Manschot (22), (39) to derive our estimate (see Proposition 4.2.5).

Throughout this section, we are going to assume that r is at least two. We recall two theorems due to Mozgovoy (23) and Joyce (38) respectively.

Let $\mathcal{M}_{\mathbb{F}_1,F}(\gamma)$ denote the moduli stack of torsion free μ_F semistable sheaves on $\mathbb{F}_1$ with Chern character $\gamma = (r, c, \Delta)$. We define generating function

$$H_{r,c}(q) = \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)]q^{r\Delta} \tag{4.1}$$

Let $Z_{\mathbb{P}^1}(q) = \frac{1}{(1-q)(1-\mathbb{L}q)}$ be the motivic Zeta function for $\mathbb{P}^1$. Then, we have

Theorem 4.2.1 ((23)[Theorem 1.1)**.** *If $r \nmid c \cdot F$, then $\mathcal{M}_{\mathbb{F}_1,F}(\gamma)$ is empty, and hence $H_{r,c}(q) = 0$. Otherwise, we have*

$$H_{r,c}(q) = \frac{1}{(\mathbb{L}-1)} \prod_{i=1}^{r-1} Z_{\mathbb{P}^1}(\mathbb{L}^i) \prod_{k=1}^{\infty} \prod_{i=-r}^{r-1} Z_{\mathbb{P}^1}(\mathbb{L}^{rk+i} q^k)$$

Before proceeding to Joyce's theorem, in a similar vein as in Proposition 4.1.1, we would like to show that for $\Delta \gg N$, the coefficient of $\mathbb{L}^{-N} q^{\Delta}$ in the generating function $(1-q)\sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)] \mathbb{L}^{r^2(1-2\Delta)} q^{r\Delta}$ vanishes.

Proposition 4.2.2. *If $\Delta > N$, the coefficient of $\mathbb{L}^{-N} q^{\Delta}$ in the generating function*

$$(1-q) \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)] \mathbb{L}^{r^2(1-2\Delta)} q^{r\Delta}$$

is zero.

Proof. Clearly we can assume that $r \mid c \cdot F$, because otherwise by Mozgovoy's theorem (Theorem 4.2.1) we have $[\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)] = 0$. Observe that

$$(1-q) \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)] \mathbb{L}^{r^2(1-2\Delta)} q^{r\Delta} = (1-q) \mathbb{L}^{r^2} H_{r,c}(\mathbb{L}^{-2r} q) \tag{4.2}$$

Moreover, we have the following equations

$$\frac{1}{(\mathbb{L}-1)}\prod_{i=1}^{r-1}\frac{1}{(1-\mathbb{L}^{i})(1-\mathbb{L}^{i+1})}=\frac{L^{-r^2}}{(1-\mathbb{L}^{-r})}\prod_{i=1}^{r-1}\frac{1}{(1-\mathbb{L}^{-i})^2}$$

$$\prod_{k=1}^{\infty}\prod_{i=-r}^{r-1}\frac{1}{(1-\mathbb{L}^{-rk+i}q^{k})(1-\mathbb{L}^{-rk+i+1}q^{k})}=\prod_{k_1=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-(rk_1+r)}q^{k_1})}\times$$
$$\prod_{k_2=1}^{\infty}\prod_{i=-r+1}^{r-1}\frac{1}{(1-\mathbb{L}^{-(rk_2-i)}q^{k_2})^2}\times$$
$$\prod_{k_3=1}^{\infty}\frac{1}{(1-\mathbb{L}^{-(rk_3-r)}q^{k_3})}$$

Therefore, we have

$$(1-q)\mathbb{L}^{r^2}H_{r,c}(\mathbb{L}^{-2r}q)=\left(\sum_{\alpha_1=0}^{\infty}\mathbb{L}^{-r\alpha_1}\right)\prod_{i=1}^{r-1}\left(\sum_{\alpha_2=0}^{\infty}\mathbb{L}^{-i\alpha_2}\right)^2\prod_{k_1=1}^{\infty}\left(\sum_{\alpha_3=0}^{\infty}\mathbb{L}^{-(rk_1+r)\alpha_3}q^{k_1\alpha_3}\right)\times$$
$$\prod_{k_2=1}^{\infty}\prod_{j=-r+1}^{r-1}\left(\sum_{\alpha_4=0}^{\infty}\mathbb{L}^{-(rk_2-j)\alpha_4}q^{k_2\alpha_4}\right)^2\prod_{k_3=2}^{\infty}\left(\sum_{\alpha_5=0}^{\infty}\mathbb{L}^{-(rk_3-r)\alpha_5}q^{k_3\alpha_5}\right)$$

Each non-zero term contributing to the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\mathbb{L}^{r^2}H_{r,c}(\mathbb{L}^{-2r}q)$ corresponds to a pair of equations

$$\Delta=\sum_{j_1=1}^{\delta_1}k_1^{(j_1)}\alpha_3^{(j_1)}+\sum_{j_2=1}^{\delta_2}\sum_{j=-r+1}^{r-1}k_2^{(j_2,j)}(\alpha_4^{(j_2,j,1)}+\alpha_4^{(j_2,j,2)})+\sum_{j_3=1}^{\delta_3}k_3^{(j_3)}\alpha_5^{(j_3)}$$
$$-N=-r\alpha_1+\sum_{i=1}^{r-1}-i(\alpha_2^{(i,1)}+\alpha_2^{(i,2)})+\sum_{j_1=1}^{\delta_1}-(rk_1^{(j_1)}+r)\alpha_3^{(j_1)}+$$
$$\sum_{j_2=1}^{\delta_2}\sum_{j=-r+1}^{r-1}-(rk_2^{(j_2,j)}-j)(\alpha_4^{(j_2,j,1)}+\alpha_4^{(j_2,j,2)})+\sum_{j_3=1}^{\delta_3}-(rk_3^{(j_3)}-r)\alpha_5^{(j_3)}$$

where all the α's are non-negative integers and all the δ's and k's are positive integers except $k_3^{(j_3)}$ which is at least 2, for all $1 \leq j_3 \leq \delta_3$. We see that

$$r\Delta - N \leq \sum_{j_2=1}^{\delta_2} \sum_{j=-r+1}^{r-1} j(\alpha_4^{(j_2,j,1)} + \alpha_4^{(j_2,j,2)}) + \sum_{j_3=1}^{\delta_3} r\alpha_5^{(j_3)}$$

Since $j \leq r-1$ and $k_3^{(j_3)} \geq 2$, we see that $(rk_2^{(j_2,j)} - j) \geq 1$ and $(rk_3^{(j_3)} - r) \geq r$. Hence, we have

$$\begin{aligned}\sum_{j_2=1}^{\delta_2} \sum_{j=-r+1}^{r-1} j(\alpha_4^{(j_2,j,1)} + \alpha_4^{(j_2,j,2)}) + \sum_{j_3=1}^{\delta_3} r\alpha_5^{(j_3)} &\leq (r-1) \sum_{j_2=1}^{\delta_2} \sum_{j=-r+1}^{r-1} (rk_2^{(j_2,j)} - j)(\alpha_4^{(j_2,j,1)} + \alpha_4^{(j_2,j,2)}) \\ &+ \sum_{j_3=1}^{\delta_3} (rk_3^{(j_3)} - r)\alpha_5^{(j_3)} \leq (r-1)N\end{aligned}$$

Hence for $\Delta > N$, the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\sum_{\Delta\geq 0}[\mathcal{M}_{\mathbb{F}_1,F}(r,c,\Delta)]\mathbb{L}^{r^2(1-2\Delta)}q^{r\Delta}$ is zero. $\square$

We now proceed to state Joyce's theorem. Let X be a surface. Given a Chern character γ, the ample cone of X admits a chamber decomposition where for all ample divisors H in a given chamber the moduli stacks $\mathcal{M}_{X,H}(\gamma)$ are isomorphic. When the ample divisor H crosses a wall, certain sheaves in $\mathcal{M}_{X,H}(\gamma)$ become destabilized and other unstable sheaves may become semistable. Joyce gives an inductive formula for computing the change in the classes $[\mathcal{M}_{X,H}(\gamma)]$ in term of the possible Harder-Narasimhan filtration for unstable sheaves.

Let H_1 and H_2 be two ample line bundles on X. Let $\mathcal{M}_{X,H_1}(\gamma)$ (respectively $\mathcal{M}_{X,H_2}(\gamma)$) denote the moduli stack of torsion free μ_{H_1} (respectively μ_{H_2}) semistable sheaves on X with Chern character $\gamma = (r, c, \Delta)$. Let $\gamma_1, \cdots, \gamma_l$ be Chern characters such that $\sum_{i=1}^{l} \gamma_i = \gamma$. Assume that $l \geq 2$, and consider the following conditions for all $1 \leq i \leq l-1$

$$\begin{aligned} &\text{A) } \mu_{H_1}(\gamma_i) > \mu_{H_1}(\gamma_{i+1}) \text{ and } \mu_{H_2}(\sum_{j=1}^{i} \gamma_j) \leq \mu_{H_2}(\sum_{j=i+1}^{l} \gamma_j) \\ &\text{B) } \mu_{H_1}(\gamma_i) \leq \mu_{H_1}(\gamma_{i+1}) \text{ and } \mu_{H_2}(\sum_{j=1}^{i} \gamma_j) > \mu_{H_2}(\sum_{j=i+1}^{l} \gamma_j) \end{aligned} \tag{4.3}$$

Let u be the number of times that Case B occurs. We define

$$S^{\mu}(\gamma_1, \cdots, \gamma_l; H_1, H_2) = \begin{cases} 1, & \text{if } l = 1 \\ (-1)^u, & \text{if } l \geq 2, \text{ and Case A or B occurs for all } 1 \leq i \leq l-1 \\ 0, & \text{otherwise} \end{cases} \tag{4.4}$$

Theorem 4.2.3 ((38),Theorem 6.21). *If H_1 and H_2 are ample line-bundles on X satisfying $K_X \cdot H_1 < 0$ and $K_X \cdot H_2 < 0$, then we have the following equation*

$$[\mathcal{M}_{X,H_2}(\gamma)] = \sum_{\sum_{i=1}^{l} \gamma_i = \gamma} S^{\mu}(\gamma_1, \cdots, \gamma_l; H_1, H_2) \mathbb{L}^{-\sum_{1 \leq i < j \leq l} \chi(\gamma_j, \gamma_i)} \prod_{i=1}^{l} [\mathcal{M}_{X,H_1}(\gamma_i)]$$

In our case, we would like to take $X = \mathbb{F}_1$, $H_1 = F$ and $H_2 = E + F$. Clearly, since $K_{\mathbb{F}_1} = -2E - 3F$, we have $K_{\mathbb{F}_1} \cdot H_1 < 0$ and $K_{\mathbb{F}_1} \cdot H_2 < 0$. However, H_1 is not ample and so we cannot use Joyce's theorem (Theorem 4.2.3) as stated. Luckily the following observation due to Coskun and Woolf (17)[Corollary 4.4] saves the day.

Remark 4.2.4. Joyce's theorem (Theorem 4.2.3) holds if H_1 and H_2 are nef, as long as the sum on the right side of equation is convergent.

Moreover, Coskun and Woolf shows (17)[Corollary 5.3] that we can use Joyce's equation in our case. Hence, we have

$$\sum_{\Delta \geq 0} \mathcal{M}_{\mathbb{F}_1, E+F}(\gamma) q^{r\Delta} = \sum_{\Delta \geq 0} \sum_{\sum_{i=1}^{l} \gamma_i = \gamma} S^{\mu}(\gamma_1, \cdots, \gamma_l; F, E+F) \, \mathbb{L}^{-\sum_{1 \leq i < j \leq l} \chi(\gamma_j, \gamma_i)} \times \left(\prod_{i=1}^{l} [\mathcal{M}_{\mathbb{F}_1, F}(\gamma_i)] \right) q^{r\Delta} \tag{4.5}$$

Let $\gamma_i = (r_i, c_i, \Delta_i)$ for all $1 \leq i \leq l$. Further, we define $\mu_i = \frac{c_i}{r_i}$ for all $1 \leq i \leq l$. We would like to manipulate equation Equation 4.5 so that the left hand side term of equation Equation 4.5 becomes $\tilde{G}_{r,c}(q)$ and get rid of Δ from the right hand side term of equation Equation 4.5.

It is easy to see that

$$-\sum_{1 \leq i < j \leq l} \chi(\gamma_j, \gamma_i) = -\frac{1}{2} \left(\sum_{i<j} \chi(\gamma_j, \gamma_i) + \chi(\gamma_i, \gamma_j) \right) - \frac{1}{2} \left(\sum_{i<j} \chi(\gamma_j, \gamma_i) - \chi(\gamma_i, \gamma_j) \right)$$

We now list down some equations expressing the various Euler characteristics

- $\chi(\gamma_j, \gamma_i) - \chi(\gamma_i, \gamma_j) = r_i r_j (\mu_j - \mu_i) \cdot K_{\mathbb{F}_1}$
- $\chi(\gamma, \gamma) = r^2(1 - 2\Delta)$, and $\chi(\gamma_i, \gamma_i) = r_i^2(1 - 2\Delta_i)$ for all $1 \leq i \leq l$.
- $\sum_{i<j} \chi(\gamma_j, \gamma_i) + \chi(\gamma_i, \gamma_j) = \chi(\gamma, \gamma) - \sum_{i=1}^{l} \chi(\gamma_i, \gamma_i)$

Using the above equations we get

$$-\sum_{i<j}\chi(\gamma_j,\gamma_i) = -\frac{1}{2}r^2(1-2\Delta)+\frac{1}{2}\sum_{i=1}^{l}r_i^2(1-2\Delta_i)-\frac{1}{2}\sum_{i<j}r_ir_j(\mu_j-\mu_i)\cdot K_{\mathbb{F}_1} \quad (4.6)$$

We now replace q by $\mathbb{L}^{-2r}q$ in both sides of equation Equation 4.5, multiply both sides of Equation 4.5 by $\mathbb{L}^{r^2}$, and use equation Equation 4.6. We get

$$\begin{aligned}\sum_{\Delta\geq 0}[\mathcal{M}_{\mathbb{F}_1,E+F}(\gamma)]\mathbb{L}^{r^2(1-2\Delta)}q^{r\Delta} = \sum_{\Delta\geq 0}\sum_{\sum_{i=1}^{l}\gamma_i=\gamma} & S^{\mu}(\gamma_1,\cdots,\gamma_l;F,E+F)\times \\ & \mathbb{L}^{\frac{1}{2}r^2(1-2\Delta)+\frac{1}{2}\sum_{i=1}^{l}r_i^2(1-2\Delta_i)}\,\mathbb{L}^{-\frac{1}{2}\sum_{i<j}r_ir_j(\mu_j-\mu_i)\cdot K_{\mathbb{F}_1}}\times \\ & \left(\prod_{i=1}^{l}[\mathcal{M}_{\mathbb{F}_1,F}(\gamma_i)]\right)q^{r\Delta}\end{aligned} \quad (4.7)$$

Note that we are yet to get rid of Δ from right hand side term in equation Equation 4.7. To do that, we need to use Yoshioka's relation for discriminants (25)[Equation 2.1]

$$r\Delta = \sum_{i=1}^{l}r_i\Delta_i - \sum_{i=2}^{l}\frac{1}{2r_i\left(\sum_{j=1}^{i}r_j\right)\left(\sum_{j=1}^{i-1}r_j\right)}\left(\sum_{j=1}^{i-1}r_ic_j-r_jc_i\right)^2 \quad (4.8)$$

It follows from Yoshioka's relation that the difference $r\Delta-\sum_{i=1}^{l}r_i\Delta_i$ depends only on (r,c) and (r_i,c_i) for $1\leq i\leq l$. So we rewrite the first exponent of $\mathbb{L}$ in equation Equation 4.7

$$\frac{1}{2}r^2(1-2\Delta)+\frac{1}{2}\sum_{i=1}^{l}r_i^2(1-2\Delta_i) = \frac{1}{2}(r^2+\sum_{i=1}^{l}r_i^2)-r(r\Delta-\sum_{i=1}^{l}r_i\Delta_i)-\sum_{i=1}^{l}r_i(r+r_i)\Delta_i \quad (4.9)$$

Using equation Equation 4.9 back in equation Equation 4.7 yields

$$\tilde{G}_{r,c}(q) = \sum_{\Delta \geq 0} \sum_{\sum_{i=1}^{l} \gamma_i = \gamma} S^{\mu}(\gamma_1, \cdots, \gamma_l; F, E+F) \mathbb{L}^{\frac{1}{2}\left(r^2 + \sum_{i=1}^{l} r_i^2\right)} \mathbb{L}^{-\frac{1}{2}\sum_{i<j} r_i r_j (\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}} \times \\ (\mathbb{L}^{-r} q)^{r\Delta - \sum_{i=1}^{l} r_i \Delta_i} \left(\prod_{i=1}^{l} [\mathcal{M}_{\mathbb{F}_1,F}(\gamma_i)] \left(\mathbb{L}^{-(r+r_i)} q\right)^{r_i \Delta_i} \right) \tag{4.10}$$

Observe that all the terms except the last one involving products on right hand side of equality in equation Equation 4.10 depends only on (r, c) and (r_i, c_i) for $1 \leq i \leq l$, and the last term depends only on the Δ_i's for $1 \leq i \leq l$. Therefore, we have

$$\tilde{G}_{r,c}(q) = \sum_{\sum_{i=1}^{l} \gamma_i = \gamma} S^{\mu}(\gamma_1, \cdots, \gamma_l; F, E+F) \mathbb{L}^{\frac{1}{2}\left(r^2 + \sum_{i=1}^{l} r_i^2\right)} \mathbb{L}^{-\frac{1}{2}\sum_{i<j} r_i r_j (\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}} \times \\ (\mathbb{L}^{-r} q)^{r\Delta - \sum_{i=1}^{l} r_i \Delta_i} \sum_{\Delta_1, \cdots, \Delta_l} \left(\prod_{i=1}^{l} [\mathcal{M}_{\mathbb{F}_1,F}(\gamma_i)] \left(\mathbb{L}^{-(r+r_i)} q\right)^{r_i \Delta_i} \right) \tag{4.11}$$

Recall that we previously defined in equation Equation 4.1 the generating function

$$H_{r,c}(q) = \sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{F}_1,F}(r, c, \Delta)] q^{r\Delta}$$

The second summation term in equation Equation 4.11 can be expressed in terms of $H_{r,c}(q)$ as follows

$$\sum_{\Delta_1, \cdots, \Delta_l} \left(\prod_{i=1}^{l} [\mathcal{M}_{\mathbb{F}_1,F}(\gamma_i)] \left(\mathbb{L}^{-(r+r_i)} q\right)^{r_i \Delta_i} \right) = \prod_{i=1}^{l} H_{r_i, c_i}(\mathbb{L}^{-(r+r_i)} q) \tag{4.12}$$

Therefore, we have

$$\tilde{G}_{r,c}(q) = \sum_{\sum_{i=1}^{l}(r_i,c_i)=(r,c)} S^{\mu}(\gamma_1,\cdots,\gamma_l;F,E+F)\mathbb{L}^{\frac{1}{2}\left(r^2-\sum_{i=1}^{l} r_i^2\right)}\mathbb{L}^{-\frac{1}{2}\sum_{i<j} r_i r_j(\mu_j-\mu_i)\cdot K_{\mathbb{F}_1}} \times \\ (\mathbb{L}^{-r}q)^{r\Delta-\sum_{i=1}^{l} r_i\Delta_i}\prod_{i=1}^{l}\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q) \tag{4.13}$$

It follows from the definition of $S^{\mu}(\gamma_1,\cdots,\gamma_l;F,E+F)$ in equation Equation 4.4 and from Mozgovoy's theorem (Theorem 4.2.1) that all the terms on right hand side of equality of equation Equation 4.13 depends only on (r,c) and (r_i,c_i) for $1\le i\le l$. Our next goal is to analyze the exponents of each of these terms further and show that for $\Delta \gg N$ the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\tilde{G}_{r,c}(q)$ vanishes.

Proposition 4.2.5. *There is a constant C_0 depending only on r and c such that if $\Delta > N + C_0$, then coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\tilde{G}_{r,c}(q)$ is zero. Moreover, we can take C_0 to be $\frac{1}{2}(r^2+1)$.*

Proof. Our approach is to look at each summand of $(1-q)\tilde{G}_{r,c}(q)$ corresponding to a equation

$$(r,c) = \sum_{i=1}^{l}(r_i,c_i)$$

and find a lower bound for Δ corresponding to the term

$$(1-q)S^{\mu}(\gamma_1,\cdots,\gamma_l;F,E+F)\mathbb{L}^{\frac{1}{2}\left(r^2-\sum_{i=1}^{l} r_i^2\right)}\mathbb{L}^{-\frac{1}{2}\sum_{i<j} r_i r_j(\mu_j-\mu_i)\cdot K_{\mathbb{F}_1}} \times \\ (\mathbb{L}^{-r}q)^{r\Delta-\sum_{i=1}^{l} r_i\Delta_i}\prod_{i=1}^{l}\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q) \tag{4.14}$$

If $l = 1$, then equation Equation 4.14 becomes

$$(1-q)S^{\mu}(\gamma; F, E+F)\mathbb{L}^{r^2}H_{r,c}(\mathbb{L}^{-2r}q) \tag{4.15}$$

It follows from Proposition 4.2.2 and equation Equation 4.2 that for $\Delta > N$, the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)\mathbb{L}^{r^2}H_{r,c}(\mathbb{L}^{-2r}q)$ is zero.

Assume $l \geq 2$. We would like to estimate a lower bound for Δ_i' such that the coefficient of $\mathbb{L}^{-N_i'}q^{\Delta_i'}$ in $\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q)$ is zero, and then use that to figure out a lower bound for Δ in equation Equation 4.14. It follows from Mozgovoy's theorem (Theorem 4.2.1) that

$$\begin{aligned}\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q) &= \mathbb{L}^{r_i^2}\frac{1}{(\mathbb{L}-1)}\prod_{j=1}^{r_i-1}Z_{\mathbb{P}^1}(\mathbb{L}^j)\prod_{k=1}^{\infty}\prod_{j=-r_i}^{r_i-1}Z_{\mathbb{P}^1}(\mathbb{L}^{-(rk-j)}q^k)\\ &= \frac{1}{(1-\mathbb{L}^{-r_i})}\left(\prod_{j=1}^{r_i-1}\frac{1}{(1-\mathbb{L}^{-j})^2}\right)\prod_{k=1}^{\infty}\left\{\frac{1}{(1-\mathbb{L}^{-(rk+r_i)}q^k)}\times\right.\\ &\qquad\left.\left(\prod_{j=-r_i+1}^{r_i-1}\frac{1}{(1-\mathbb{L}^{-(rk-j)}q^k)^2}\right)\frac{1}{(1-\mathbb{L}^{-(rk-r_i)}q^k)}\right\}\end{aligned}$$

Thus, we get

$$\begin{aligned}\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q) &= \left(\sum_{\alpha_1=0}^{\infty}\mathbb{L}^{-r_i\alpha_1}\right)\left(\prod_{j_1=1}^{r_i-1}\left(\sum_{\alpha_2=0}^{\infty}\mathbb{L}^{-j_1\alpha_2}\right)^2\right)\times\\ &\prod_{k=1}^{\infty}\left\{\left(\sum_{\alpha_3=0}^{\infty}\mathbb{L}^{-(rk+r_i)\alpha_3}q^{k\alpha_3}\right)\left(\prod_{j_2=-r_i+1}^{r_i-1}\left(\sum_{\alpha_4=0}^{\infty}\mathbb{L}^{-(rk-j_2)\alpha_4}q^{k\alpha_4}\right)^2\right)\right.\\ &\qquad\left.\left(\sum_{\alpha_5=0}^{\infty}\mathbb{L}^{-(rk-r_i)\alpha_5}q^{k\alpha_5}\right)\right\}\end{aligned}$$

Each nonzero term contributing to the coefficient of $\mathbb{L}^{-N_i'}q^{\Delta_i'}$ in $\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q)$ arises from a pair of equations

$$\Delta_i' = \sum_{j=1}^{\delta}\left\{k^{(j)}\alpha_3^{(j)} + \sum_{j_2=-r_i+1}^{r_i-1} k^{(j)}(\alpha_4^{(j,j_2,1)} + \alpha_4^{(j,j_2,2)}) + k^{(j)}\alpha_5^{(j)}\right\}$$

$$-N_i' = -r_i\alpha_1 + \sum_{j_1=1}^{r_i-1} -j_1(\alpha_2^{(j_1,1)} + \alpha_2^{(j_1,2)}) + \sum_{j=1}^{\delta}\Bigg\{ -(rk^{(j)}+r_i)\alpha_3^{(j)} + \left(\sum_{j_2=-r_i+1}^{r_i-1} -(rk^{(j)}-j_2)(\alpha_4^{(j,j_2,1)}+\alpha_4^{(j,j_2,2)})\right) - (rk^{(j)}-r_i)\alpha_5^{(j)}\Bigg\}$$

where all the α's are non-negative integers, δ and the k's are positive integers. Hence, we get

$$r\Delta_i' - N_i' \leq \sum_{j=1}^{\delta}\left(\sum_{j_2=-r_i+1}^{r_i-1} j_2(\alpha_4^{(j,j_2,1)}+\alpha_4^{(j,j_2,2)})\right) + r_i\alpha_5^{(j)}$$

Since $j_2 \leq r_i - 1$ and $k^{(j)} \geq 1$, we see that $j_2 \leq r_i(rk^{(j)} - j_2)$. Moreover, because $l \geq 2$ we have $r_i \leq (r-1)$, and so $r_i \leq r_i(rk^{(j)} - r_i)$. These two inequalities yield

$$\sum_{j=1}^{\delta}\left(\sum_{j_2=-r_i+1}^{r_i-1} j_2(\alpha_4^{(j,j_2,1)}+\alpha_4^{(j,j_2,2)})\right) + r_i\alpha_5^{(j)} \leq r_iN_i'$$

In summary, we get $r\Delta_i' - N_i' \leq r_iN_i' \leq (r-1)N_i'$, a posteriori, $\Delta_i' \leq N_i'$.

Going back to equation Equation 4.14, we see that each non-zero term contributing to the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in equation Equation 4.14 arises from a pair of equations

$$\begin{aligned}
&\Delta' = \varepsilon + \left(r\Delta - \sum_{i=1}^{l} r_i\Delta_i\right) + \sum_{i=1}^{l}\Delta_i' \\
&-N' = \frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\left(\sum_{i<j} r_i r_j(\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}\right) - r\left(r\Delta - \sum_{i=1}^{l} r_i\Delta_i\right) + \sum_{i=1}^{l} -N_i'
\end{aligned}$$

where $\varepsilon \in \{0, 1\}$ which accounts for contribution to the coefficient coming from $(1-q)$, and (Δ_i', N_i') accounts for the contribution of terms to the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ coming from terms of coefficient of $\mathbb{L}^{-N_i'}q^{\Delta_i'}$ appearing in $\mathbb{L}^{r_i^2}H_{r_i,c_i}(\mathbb{L}^{-(r+r_i)}q)$. Since $\Delta_i' \leq N_i'$ for all $1 \leq i \leq l$ and $\varepsilon \leq 1$, we see that

$$\Delta' \leq N'+1+\frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\left\{\left(\sum_{i<j} r_i r_j(\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}\right) + 2(r-1)\left(r\Delta - \sum_{i=1}^{l} r_i\Delta_i\right)\right\} \tag{4.16}$$

Clearly, to bound Δ', we need to bound the last term in above equation Equation 4.16. We are going to show later (in Lemma 4.2.6) that

$$2(r-1)\left(r\Delta - \sum_{i=1}^{l} r_i\Delta_i\right) + \left(\sum_{i<j} r_i r_j(\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}\right)$$

is bounded below by a constant κ which depends only on (r, c) and r_i for all $1 \leq i \leq l$, except when $l = 2$ and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$. Thus, we have

$$\Delta' \leq N' + 1 + \frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\kappa$$

We would like to scrutinize the special case when $l = 2$ and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$. Note that it follows from Mozgovoy's theorem (Theorem 4.2.1) that $H_{r,c}$ only depends on whether or not $r \mid c \cdot F$. Let $r = r_1 + r_2$, $c = aE + bF$, $c_1 = r_1 a_1 E + b_1 F$ and $c_2 = r_2 a_2 E + b_2 F$. We will denote H_{r_i, c_i} by H_{r_i} for $i = 1, 2$ because we are assuming that $r_i \mid c_i \cdot F$ for $i = 1, 2$. It follows from equation Equation 4.4 that for $S^{\mu}(\gamma_1, \gamma_2; F, E+F)$ to be nonzero, we must have $\mu_{E+F}(\gamma_1) \leq \mu_{E+F}(\gamma_2)$, or equivalently, we have $b_2 \geq \frac{br_2}{r}$. Furthermore, we see that

$$-\frac{1}{2} r_1 r_2 (\mu_2 - \mu_1) \cdot K_{\mathbb{F}_1} = r_1 r_2 (a_2 - a_1) + r b_2 - r_2 b$$

and

$$r\Delta - r_1 \Delta_1 - r_2 \Delta_2 = \frac{r_1 r_2}{2r}(a_2 - a_1)^2 - (a_2 - a_1) b_2 + b \frac{r_2(a_2 - a_1)}{r}$$

Using these equations together with the fact that $a_2 - a_1 = -1$, we see that equation Equation 4.14 transforms to

$$(1-q)\mathbb{L}^{\frac{1}{2}(r^2 - r_1^2 - r_2^2)} \mathbb{L}^{-r_1 r_2} q^{\frac{r_1 r_2}{2r} - \frac{br_2}{r}} q^{b_2} \prod_{i=1}^{2} \mathbb{L}^{r_i^2} H_{r_i}(\mathbb{L}^{-(r+r_i)} q)$$

whenever $b_2 \geq \frac{br_2}{r}$ and is zero otherwise. Adding all these terms for $b_2 \geq \frac{br_2}{r}$ yields

$$\mathbb{L}^{\frac{1}{2}(r^2 - r_1^2 - r_1^2) - r_1 r_2} q^{\frac{r_1 r_2}{2r} - \frac{br_2}{r}} q^{\lceil \frac{br_2}{r} \rceil} \prod_{i=1}^{2} \mathbb{L}^{r_i^2} H_{r_i}(\mathbb{L}^{-(r+r_i)} q) \tag{4.17}$$

Each nonzero term appearing in the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in equation Equation 4.17 arises from a pair of equations

$$\Delta' = \frac{r_1 r_2}{2r} - \frac{br_2}{r} + \left\lceil \frac{br_2}{r} \right\rceil + \Delta_1' + \Delta_2'$$
$$-N' = \frac{1}{2}(r^2 - r_1^2 - r_2^2) - r_1 r_2 - N_1' - N_2' = -N_1' - N_2'$$

where (Δ_i', N_i') accounts for contribution coming from terms of coefficient of $\mathbb{L}^{-N_i'}q^{\Delta_i'}$ in $\mathbb{L}^{r_i^2}\ H_{r_i}(\mathbb{L}^{-(r+r_i)}q)$. We have shown before that we must have $\Delta_i' \leq N_i'$ for $i = 1, 2$. Hence, we must have

$$\Delta' \leq N' + \frac{r_1 r_2}{2r} + \left(\left\lceil \frac{br_2}{r} \right\rceil - \frac{br_2}{r}\right)$$

In conclusion, we have

$$\Delta' \leq N' + C_0$$

where C_0 is the supremum of 0, the terms $1 + \frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\kappa$ corresponding to $l \geq 2$ and $r_1 + \cdots + r_l = r$, and the terms $\frac{r_1 r_2}{2r} + \left(\left\lceil \frac{br_2}{r} \right\rceil - \frac{br_2}{r}\right)$ corresponding to $l = 2$, $r_1 + r_2 = r$, and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$.

It follows from equation Equation 4.26 that κ is bounded below by $-(r-1)$. Clearly, $\left(r^2 - \sum_{i=1}^{l} r_i^2\right)$ is bounded above by $r^2 - r$. Hence, we see that

$$1 + \frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\kappa \ \leq \ \frac{1}{2}\,(r^2 + 1)$$

Clearly $\left(\left\lceil \frac{br_2}{r} \right\rceil - \frac{br_2}{r}\right) \leq 1$ and $\frac{r_1(r-r_1)}{2r}$ is bounded above by $\frac{r}{8}$, whence the terms corresponding to $r = r_1 + r_2$ and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$ are bounded above by $\frac{r}{8} + 1$.

In summary, we can take C_0 to be $\frac{1}{2}(r^2+1)$. Hence, for $\Delta' > N' + \frac{1}{2}(r^2+1)$, the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in $(1-q)\tilde{G}_{r,c}(q)$ is zero. $\square$

Lemma 4.2.6. *The following expression*

$$2(r-1)\left(r\Delta - \sum_{i=1}^{l} r_i\Delta_i\right) + \left(\sum_{i<j} r_i r_j(\mu_j - \mu_i)\cdot K_{\mathbb{F}_1}\right) \tag{4.18}$$

is bounded below by some constant κ which depends only on (r,c) and r_i for $1 \leq i \leq l$, except when $l = 2$ and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$.

Proof. We can assume that $r_i \mid c_i \cdot F$ for each $1 \leq i \leq l$, otherwise the entire summand (equation Equation 4.14) vanishes due to Mozgovoy's theorem (Theorem 4.2.1). Let $c = aE + bF$ and for each $1 \leq i \leq l$, let $c_i = r_i a_i E + b_i F$. Note that every term in the generating function $\tilde{G}_{r,c}(q)$ is invariant under the action of tensoring by line bundles, whence, we can assume that $0 \leq a, b \leq (r-1)$. Furthermore, we define $s_i = \sum_{j=i}^{l} b_j$ for all $1 \leq i \leq l$.

Following Manschot (39)[Proof of Proposition 4.1] we see that

$$r\Delta - \sum_{i=1}^{l} r_i\Delta_i = \sum_{i=2}^{l} \frac{r_i}{2\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)}\left(\sum_{j=1}^{i-1} r_j(a_i - a_j)\right)^2 - \sum_{i=2}^{l}(a_i - a_{i-1})s_i$$
$$+\ b\sum_{i=2}^{l} \frac{\sum_{j=1}^{i-1} r_i r_j(a_i - a_j)}{\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)}$$

Similarly, following Manschot (39)[Proof of Proposition 4.1] we see that

$$\sum_{i<j} r_i r_j(\mu_j - \mu_i)\cdot K_{\mathbb{F}_1} = \sum_{i<j} r_i r_j(a_i - a_j) - 2\sum_{i=2}^{l}(r_i + r_{i-1})s_i + 2(r - r_1)b$$

Using these two equations we get

$$\begin{aligned}
&2(r-1)\left(r\Delta-\sum_{i=1}^{l} r_i\Delta_i\right)+\left(\sum_{i<j} r_ir_j(\mu_j-\mu_i)\cdot K_{\mathbb{F}_1}\right)= \\
&\left\{2(r-1)\sum_{i=2}^{l}\frac{r_i}{2\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)}\left(\sum_{j=1}^{i-1} r_j(a_i-a_j)\right)^2+\sum_{i<j} r_ir_j(a_i-a_j)\right\} \\
&\qquad\qquad +\left\{2(r-1)b\sum_{i=2}^{l}\frac{\sum_{j=1}^{i-1} r_ir_j(a_i-a_j)}{\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)}\right. + \\
&\left.-2(r-1)\sum_{i=2}^{l}(a_i-a_{i-1})s_i-2\sum_{i=2}^{l}(r_i+r_{i-1})s_i+2(r-r_1)b\right\}
\end{aligned} \tag{4.19}$$

We would like to show that both the first and second summand of right hand side of equation Equation 4.19 are bounded below. Let us call the first summand S_1 and the second summand S_2.

We now proceed to scrutinize S_1 to determine its lower bound. We are going to use the following identity of Manschot (39)[Proof of Proposition 4.1]

$$\sum_{i=2}^{l}\frac{r_i}{2\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)}\left(\sum_{j=1}^{i-1} r_j(a_i-a_j)\right)^2=\frac{1}{2r}\left(\sum_{i=1}^{l} r_i(r-r_i)a_i^2-2\sum_{1\le i<j\le l} r_ir_ja_ia_j\right) \tag{4.20}$$

Since $a=\sum_{i=1}^{l} r_ia_i$, it follows from equation Equation 4.20 that

$$S_1=(r-1)\sum_{i=1}^{l} r_ia_i^2-\frac{r-1}{r}a^2+\sum_{i=1}^{l} a_ir_i\left(\sum_{j=i+1}^{l} r_j-\sum_{j=1}^{i-1} r_j\right)$$

Consider the smooth polynomial function

$$f(x_1, \cdots, x_l) = \sum_{i=1}^{l} r_i x_i^2 - \frac{1}{r} a^2 + \sum_{i=1}^{l} x_i \frac{r_i}{r-1} \left(\sum_{j=i+1}^{l} r_j - \sum_{j=1}^{i-1} r_j \right)$$

Clearly, the Hessian of f, given by $\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right)$ is positive definite. We define

$$g(x_1, \cdots, x_l) = \sum_{i=1}^{l} r_i x_i - a$$

Our goal is to minimize f along the locus of $g = 0$ for integer values of the x_i's. Using the Lagrange's multiplier method, we see that f assumes minima at

$$a_i = \frac{a}{r} - \frac{1}{2(r-1)} \left(\sum_{j=i+1}^{l} r_j - \sum_{j=1}^{i-1} r_j \right), \qquad \text{for } i = 1, \cdots, l$$

Clearly $\left| \sum_{j=i+1}^{l} r_j - \sum_{j=1}^{i-1} r_j \right| \leq (r-1)$, and hence we get $\frac{a}{r} - \frac{1}{2} \leq a_i \leq \frac{a}{r} + \frac{1}{2}$ for all $1 \leq i \leq l$. Thus, to find a lower bound for S_1 we need to find the minimum value of f when $x_i \in \{-1, 0, 1, 2\}$ for all $1 \leq i \leq l$. We have the following partition

$$\{1, \cdots, l\} = \{i_\alpha\}_{1 \leq \alpha \leq p} \cup \left\{j_\beta\right\}_{1 \leq \beta \leq q} \cup \{k_\gamma\}_{1 \leq \gamma \leq s} \cup \{m_\delta\}_{1 \leq \delta \leq t}$$

where $x_{i_\alpha} = -1$, $x_{j_\beta} = 1$, $x_{k_\gamma} = 2$, and $x_{m_\delta} = 0$. We see that

$$\begin{aligned} r(r-1)f = (12r-9)\left(\sum_{i_\alpha > k_\gamma} r_{i_\alpha} r_{k_\gamma}\right) + (6r-4)\left(\sum_{i_\alpha > j_\beta} r_{i_\alpha} r_{j_\beta} + \sum_{k_\gamma < m_\delta} r_{k_\gamma} r_{m_\delta}\right) \\ + (2r-1)\left(\sum_{i_\alpha > m_\delta} r_{i_\alpha} r_{m_\delta} + \sum_{j_\beta > k_\gamma} r_{j_\beta} r_{k_\gamma} + \sum_{j_\beta < m_\delta} r_{j_\beta} r_{m_\delta}\right) \\ + (6r-9)\left(\sum_{i_\alpha < k_\gamma} r_{i_\alpha} r_{k_\gamma}\right) + (2r-4)\left(\sum_{i_\alpha < j_\beta} r_{i_\alpha} r_{j_\beta} + \sum_{k_\gamma > m_\delta} r_{k_\gamma} r_{m_\delta}\right) \\ + (-1)\left(\sum_{i_\alpha < m_\delta} r_{i_\alpha} r_{m_\delta} + \sum_{j_\beta < k_\gamma} r_{j_\beta} r_{k_\gamma} + \sum_{j_\beta > m_\delta} r_{j_\beta} r_{m_\delta}\right) \end{aligned} \tag{4.21}$$

Note that since $r \geq 2$ all the summands in equation Equation 4.21 except the last one have non-negative coefficient. By further examining the summands with non-negative coefficient, we see that together they must be bounded below by $(2r-4)$ because all the inequalities in the summations cannot be simultaneously compatible. Moreover, the negative summand is bounded below by $-(r^2 - r)$. Hence, S_1 is bounded below by $-r + 3 - \frac{4}{r}$.

Our next goal is to determine a lower bound for S_2. We are going to use the following identities of Manschot (39)[Proof of Proposition 4.1]

$$\sum_{i=2}^{l} \frac{r_i}{\left(\sum_{j=1}^{i} r_j\right)\left(\sum_{j=1}^{i-1} r_j\right)} \left(\sum_{j=1}^{i-1} r_j(a_i - a_j)\right) = \frac{1}{r}\left(\sum_{i=2}^{l} (a_i - a_{i-1})\left(\sum_{j=i}^{l} r_j\right)\right) \tag{4.22}$$

and

$$\sum_{i=2}^{l} (r_i + r_{i-1})\left(\sum_{j=i}^{l} r_j\right) = (r - r_1)r \tag{4.23}$$

The identities in equations Equation 4.22 and Equation 4.23 yields

$$S_2 = 2\sum_{i=2}^{l}((r-1)(a_i-a_{i-1})+(r_i+r_{i-1}))\left(\frac{b}{r}\left(\sum_{j=i}^{l} r_j\right)-s_i\right)$$

Following Coskun and Woolf (17)[Proof of Theorem 5.4], we interpret the definition of $S(\{\gamma_\bullet\}; F,\ E+F)$ (equation Equation 4.3) in our current situation, we obtain for all $2 \leq i \leq l$

$$\begin{aligned} &\text{A) } (a_i - a_{i-1}) < 0 \text{ and } s_i \geq \frac{b}{r}\left(\sum_{j=i}^{l} r_j\right) \\ &\text{B) } (a_i - a_{i-1}) \geq 0 \text{ and } s_i < \frac{b}{r}\left(\sum_{j=i}^{l} r_j\right) \end{aligned} \tag{4.24}$$

In Case A, we see that $(r-1)(a_i-a_{i-1})+r_i+r_{i-1} \leq 0$ except when $l=2$ and $a_2-a_1=-1$, which is not possible by our assumption. Hence, the term

$$((r-1)(a_i-a_{i-1})+(r_i+r_{i-1}))\left(\frac{b}{r}\left(\sum_{j=i}^{l} r_j\right)-s_i\right) \tag{4.25}$$

is non-negative.

Similarly, in Case B, we see that $(r-1)(a_i-a_{i-1})+r_i+r_{i-1} \geq (r_i+r_{i-1})$, hence the term in equation Equation 4.25 is non-negative. Additionally, by using the fact that s_i are integers, it follows from equation Equation 4.24 that we have a slightly better bound of equation Equation 4.25

$$|(r-1)(a_i-a_{i-1})+r_i+r_{i-1}|\left(1-sgn\left(a_i-a_{i-1}+\frac{1}{2}\right)\left(1-2\left\{-\frac{b}{r}\sum_{j=i}^{l} r_j\right\}\right)\right)$$

where sgn is the sign function and $\{\bullet\}$ is the fractional part of any real number.

In conclusion, we can take κ to be

$$
\begin{aligned}
&-r+3-\frac{4}{r}\\
&+\sum_{i=2}^{l}|(r-1)(a_i-a_{i-1})+r_i+r_{i-1}|\left(1-sgn\left(a_i-a_{i-1}+\frac{1}{2}\right)\left(1-2\left\{-\frac{b}{r}\sum_{j=i}^{l}r_j\right\}\right)\right)
\end{aligned}
\tag{4.26}
$$

which is our lower bound for equation Equation 4.18. □

Now that we have shown that for $\tilde{\Delta} \gg \tilde{N}$, the coefficient of $\mathbb{L}^{-\tilde{N}}q^{\tilde{\Delta}}$ in $(1-q)\tilde{G}_{r,\tilde{c}}(q)$ vanishes (see Proposition 4.2.5), our goal is to relate $G_{r,c}(q)$ with $\tilde{G}_{r,\tilde{c}}$ using the blow-up formula, and conclude a similar result for $G_{r,c}(q)$.

4.3 Estimating the generating function $G_{r,c}(q)$ when rank is at least two

In this section, our goal is to show that there is a constant C depending only on r and c such that when $\Delta > N+C$, the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{r,c}(q)$ is zero. To show this, we are going to look at the blow-up $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ and use the blow-up formula due to Mozgovoy (23)[Proposition 7.3] to relate the generating functions $G_{r,c}(q)$ and $\tilde{G}_{r,\tilde{c}}(q)$ (see equation Equation 4.30) in A^-. We are going to scrutinize the terms appearing in this relation, and use Proposition 4.2.5 to derive our inequality (see Theorem 4.3.7).

Recall from section 2.2 that we have a blow-up $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ at point $p \in \mathbb{P}^2$. Let $\gamma = (r, c, \Delta)$ be a Chern character on $\mathbb{P}^2$. Let m be the multiplicity of c at the point

p. Let $\tilde{\gamma} = (r, c - mE, \tilde{\Delta})$ be a Chern character on $\mathbb{F}_1$. The blow-up formula due to Mozgovoy (23)[Proposition 7.3] is the following equation

$$\sum_{ch_2} [\mathcal{M}_{\mathbb{F}_1, E+F}(r, c - mE, ch_2)] q^{-ch_2} = F_m(q) \sum_{ch_2} [\mathcal{M}_{\mathbb{P}^2, H}(r, c, ch_2)] q^{-ch_2} \tag{4.27}$$

where

$$F_m(q) = \left(\prod_{k=1}^{\infty} \frac{1}{(1 - \mathbb{L}^{rk} q^k)^r} \right) \left(\sum_{\substack{\sum_{i=1}^{r} a_i = 0, \\ a_i \in \mathbb{Z} + \frac{m}{r}}} \mathbb{L}^{\sum_{i<j} \binom{a_j - a_i}{2}} q^{-\sum_{i<j} a_i a_j} \right) \tag{4.28}$$

Note that on $\mathbb{P}^2$, we have $-ch_2(\gamma) = r\Delta - \frac{c^2}{2r}$, while on $\mathbb{F}_1$, we have $-ch_2(\tilde{\gamma}) = r\tilde{\Delta} - \frac{c^2}{2r} + \frac{m^2}{2r}$. Hence, we can rewrite the blow-up equation (equation Equation 4.27)

$$\sum_{\Delta \geq 0} [\mathcal{M}_{\mathbb{P}^2, H}(r, c, \Delta)] q^{r\Delta} = \frac{q^{\frac{m^2}{2r}}}{F_m(q)} \sum_{\tilde{\Delta} \geq 0} [\mathcal{M}_{\mathbb{F}_1, E+F}(r, c - mE, \tilde{\Delta})] q^{r\tilde{\Delta}} \tag{4.29}$$

Replacing q by $\mathbb{L}^{-2r} q$ and multiplying both sides by $\mathbb{L}^{r^2}$ in equation Equation 4.29 yields

$$G_{r,c}(q) = \frac{(\mathbb{L}^{-2r} q)^{\frac{m^2}{2r}}}{F_m(\mathbb{L}^{-2r} q)} \tilde{G}_{r, c-mE}(q) \tag{4.30}$$

It follows from equation Equation 4.30 that in order to achieve our goal, we need to analyze $F_m(\mathbb{L}^{-2r} q)$ and find an estimate for Δ in this expression.

By examining the definition of F_m in equation Equation 4.28, we conclude that it depends only on the remainder of m modulo r, which we shall denote by $\bar{m}$, which we will think of as an integer between 0 and $r - 1$.

We see that

$$F_{\bar{m}}(\mathbb{L}^{-2r}q) = \prod_{k=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-rk}q^k)^r} \sum_{\substack{\sum_{i=1}^{r} a_i = 0, \\ a_i \in \mathbb{Z} + \frac{\bar{m}}{r}}} \mathbb{L}^{\sum_{i<j}\binom{a_j - a_i}{2} + 2r\sum_{i<j} a_i a_j} q^{-\sum_{i<j} a_i a_j} \tag{4.31}$$

Since $\sum_{i=1}^{r} a_i = 0$, we see that

$$-\sum_{1\le i<j\le r} a_i a_j = \frac{1}{2}\sum_{i=1}^{r} a_i^2$$

and

$$\sum_{1\le i<j\le r} \binom{a_j - a_i}{2} + 2r \sum_{1\le i<j\le r} a_i a_j = -\frac{r}{2}\left(\sum_{i=1}^{r} a_i^2\right) - \left(\sum_{i=1}^{r} i a_i\right)$$

We now use the following substitutions

$$\begin{aligned} a_i &= b_i + \frac{\bar{m}}{r}, \text{ where } b_i \in \mathbb{Z}, \text{ for } 1 \le i \le r-1, \\ a_r &= -\sum_{i=1}^{r-1}\left(b_i + \frac{\bar{m}}{r}\right) \end{aligned}$$

These substitutions yield the following equations

$$\begin{aligned} -\frac{r}{2}\left(\sum_{i=1}^{r} a_i^2\right) - \left(\sum_{i=1}^{r} i a_i\right) &= -r\left(-\frac{\bar{m}^2}{2r} + \frac{\bar{m}^2}{2} + \sum_{i=1}^{r-1} b_i^2 + \bar{m}\sum_{i=1}^{r-1} b_i + \sum_{1\le i<j\le(r-1)} b_i b_j\right) \\ &\quad + \left(\frac{(r-1)\bar{m}}{2} + \sum_{i=1}^{r-1}(r-i)b_i\right) \\ \frac{1}{2}\sum_{i=1}^{r} a_i^2 &= \left(\frac{-\bar{m}^2}{2r} + \frac{\bar{m}^2}{2} + \sum_{i=1}^{r-1} b_i^2 + \bar{m}\sum_{i=1}^{r-1} b_i + \sum_{1\le i<j\le(r-1)} b_i b_j\right) \end{aligned} \tag{4.32}$$

Employing the above equations Equation 4.32 leads to the following expression for $F_{\bar{m}}(\mathbb{L}^{-2r}q)$

$$F_{\bar{m}}(\mathbb{L}^{-2r}q) = \left(\prod_{k=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-rk}q^k)^r}\right) (\mathbb{L}^{-r}q)^{-\frac{(r+1)\bar{m}^2}{2r}} \mathbb{L}^{\frac{(r-1)\bar{m}}{2}} \times$$
$$\sum_{b_1,\cdots,b_{r-1}\in\mathbb{Z}} \mathbb{L}^{\sum_{i=1}^{r-1}(r-i)b_i} (\mathbb{L}^{-r}q)^{\bar{m}^2+\sum_{i=1}^{r-1} b_i^2+\bar{m}\sum_{i=1}^{r-1} b_i+\sum_{i<j} b_i b_j}$$

For sake of convenience, we define

$$\Lambda_d^{(\bar{m})} = \sum_{\substack{b_1,\cdots,b_{r-1}\in\mathbb{Z},\\ \bar{m}^2+\sum_{i=1}^{r-1} b_i^2+\bar{m}\sum_{i=1}^{r-1} b_i+\sum_{i<j} b_i b_j=d}} \mathbb{L}^{\sum_{j=1}^{r-1}(r-j)b_j} \tag{4.33}$$

Thus, we can think of the last summation term of $F_{\bar{m}}(\mathbb{L}^{-2r}q)$ as a power series

$$F_{\bar{m}}(\mathbb{L}^{-2r}q) = \left(\prod_{k=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-rk}q^k)^r}\right) (\mathbb{L}^{-r}q)^{-\frac{(r+1)\bar{m}^2}{2r}} \mathbb{L}^{\frac{(r-1)\bar{m}}{2}} \left(\sum_{d=0}^{\infty} \Lambda_d^{(\bar{m})}(\mathbb{L}) (\mathbb{L}^{-r}q)^d\right) \tag{4.34}$$

Remark 4.3.1. Recall that any power series of the form $f(x) = 1 + a_1 x + a_2 x^2 + \cdots$ is invertible, and its inverse is given by $1 + b_1 x + b_2 x^2 + \cdots$, where for any positive integer n, we have

$$b_n = \sum_{\substack{n_1+\cdots+n_l=n\\ n_i\in\mathbb{Z}_{>0}}} (-1)^l a_{n_1}\cdots a_{n_l}$$

To analyze $G_{r,c}(q)$, we need to invert $F_{\bar{m}}(\mathbb{L}^{-2r}q)$ (equation Equation 4.30), and a posteriori, we need to invert the power series $\sum_{d=0}^{\infty} \Lambda_d^{(\bar{m})}(\mathbb{L})(\mathbb{L}^{-r}q)^d$. To do this, we need to figure out the least non-negative integer d such that $\Lambda_d^{(\bar{m})}(\mathbb{L})$ is nonzero.

Lemma 4.3.2. *The smallest non-negative integer d for which $\Lambda_d^{(\bar{m})}$ is nonzero, is $\frac{\bar{m}^2+\bar{m}}{2}$. Additionally,*

$$\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L}) = \mathbb{L}^{-r\bar{m}} \sum_{\nu=\frac{\bar{m}^2+\bar{m}}{2}}^{r\bar{m}-\frac{\bar{m}^2-\bar{m}}{2}} \rho_\nu \mathbb{L}^\nu$$

where ρ_ν is the cardinality of the set $\{(j_1,\cdots,j_{\bar{m}}) \mid 1 \leq j_1 < \cdots < j_{\bar{m}} \leq r,\ j_1+\cdots+j_{\bar{m}} = \nu\}$, when ν is a positive integer, and $\rho_0 = 1$.

Proof. Note that

$$\bar{m}^2 + \sum_{i=1}^{r-1} b_i^2 + \bar{m}\sum_{i=1}^{r-1} b_i + \sum_{i<j} b_i b_j = \frac{1}{2}\left(\bar{m}^2 + \sum_{i=1}^{r-1} b_i^2 + \left(\bar{m} + \sum_{i=1}^{r-1} b_i\right)^2\right)$$

Consequently, we need to figure out the smallest value of $\bar{m}^2+\sum_{i=1}^{r-1} b_i^2+\left(\bar{m} + \sum_{i=1}^{r-1} b_i\right)^2$, where $b_i \in \mathbb{Z}$ for all $1 \leq i \leq r-1$.

If $\bar{m} = 0$, we see that the equation $\sum_{i=1}^{r-1} b_i^2 + \left(\sum_{i=1}^{r-1} b_i\right)^2 = 0$ has only one solution, the trivial one. Thus, $\Lambda_0^{(0)}(\mathbb{L}) = 1$.

Assume $1 \leq \bar{m} \leq r-1$. It follows from Lemma 4.3.3 (below), that the smallest value assumed by the expression $\sum_{i=1}^{r-1} b_i^2 + \left(\bar{m} + \sum_{i=1}^{r-1} b_i\right)^2$ occurs at $b_1 = \cdots = b_{r-1} = -\frac{\bar{m}}{r}$. As a result, we need to evaluate the expression when $b_i \in \{-1, 0\}$ for all $1 \leq i \leq r-1$, to figure out the minimum value of the expression for integer values. Suppose k of the b_i's are (-1) and the remaining are zero, the expression becomes $k + (\bar{m}-k)^2$. Clearly, the minimum value of $k+(\bar{m}-k)^2$ for integer values of k is $\bar{m}$, which occurs when $k = \bar{m}-1, \bar{m}$.

In summary, when $1 \leq \bar{m} \leq r-1$, the smallest value of the expression

$$\frac{1}{2}\left(\bar{m}^2 + \sum_{i=1}^{r-1} b_i^2 + \left(\bar{m} + \sum_{i=1}^{r-1} b_i\right)^2\right)$$

for integer values of b_i is $\frac{\bar{m}^2+\bar{m}}{2}$, which occurs when $\bar{m}-1$ or $\bar{m}$ of the b_i's are (-1) and the remaining are zero. Hence, we have

$$\Lambda_d^{(\bar{m})}(\mathbb{L}) = \sum_{1\leq j_1<\cdots<j_{\bar{m}-1}\leq r-1} \mathbb{L}^{j_1+\cdots+j_{\bar{m}-1}-(\bar{m}-1)r} + \sum_{1\leq j_1<\cdots<j_{\bar{m}}\leq r-1} \mathbb{L}^{j_1+\cdots+j_{\bar{m}}-r\bar{m}}$$

Factoring out $\mathbb{L}^{-r\bar{m}}$ leads to

$$\Lambda_d^{(\bar{m})}(\mathbb{L}) = \mathbb{L}^{-r\bar{m}} \sum_{1\leq j_1<\cdots<j_{\bar{m}}\leq r} \mathbb{L}^{j_1+\cdots+j_{\bar{m}}}$$

□

Before proceeding further, we need to tie the loose ends of Lemma 4.3.2 by analyzing the real valued polynomial function $y_1^2 + \cdots + y_n^2 + (A + y_1 + \cdots + y_n)^2$.

Lemma 4.3.3. *Consider the smooth real valued function*

$$f(y_1, \cdots, y_n) = y_1^2 + \cdots + y_n^2 + (A + y_1 + \cdots + y_n)^2$$

where A is any real number. The Hessian of f is positive definite. Furthermore, the function f has a global minima at $y_1 = \cdots = y_n = -\frac{A}{n+1}$, and the minimum value for f is $\frac{A^2}{n+1}$.

Proof. Clearly, we see that for $1 \leq k \leq n$

$$\frac{\partial f}{\partial y_k} = 2y_k + 2\left(A + y_1 + \cdots + y_n\right)$$

Subsequently, we see that for $1 \leq l \leq n$

$$\frac{\partial^2 f}{\partial y_l \partial y_k} = \begin{cases} 2, & \text{if } k \neq l \\ \\ 4, & \text{if } k = l \end{cases}$$

Let H be the $n \times n$ matrix with $H_{l,k} = \frac{\partial^2 f}{\partial y_l \partial y_k}$, then we see that

$$\left(y_1 \;\; \cdots \;\; y_n\right) \cdot H \cdot \left(y_1 \;\; \cdots \;\; y_n\right)^T = 2\left(\sum_{i=1}^{n} y_i^2\right) + 2\left(\sum_{i=1}^{n} y_i\right)^2$$

Thus, H is positive definite. As a consequence, f has a global minimum when $\frac{\partial f}{\partial y_k} = 0$ for all $1 \leq k \leq n$. This system of linear equations has a unique solution $y_1 = \cdots = y_n = -\frac{A}{n+1}$. It follows that the minimum value for f is $\frac{A^2}{n+1}$. □

Returning back to our track, we still need to analyze $F_{\bar{m}}(\mathbb{L}^{-2r}q)$. Using equation Equation 4.34 and Lemma 4.3.2, we see that

$$F_{\bar{m}}(\mathbb{L}^{-2r}q) = \left(\prod_{k=1}^{\infty} \frac{1}{(1-\mathbb{L}^{-rk}q^k)^r}\right) \left(\mathbb{L}^{-r}q\right)^{-\frac{(r+1)\bar{m}^2}{2r}} \mathbb{L}^{\frac{(r-1)\bar{m}}{2}}$$
$$\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L}) \left(\mathbb{L}^{-r}q\right)^{\frac{\bar{m}^2+\bar{m}}{2}} \sum_{d=0}^{\infty} \tilde{\Lambda}^{(\bar{m})}_d(\mathbb{L}) \left(\mathbb{L}^{-r}q\right)^d$$

where $\tilde{\Lambda}^{(\bar{m})}_d(\mathbb{L}) = \left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-1} \cdot \Lambda^{(\bar{m})}_{d+\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})$.

Finally, using remark 4.3.1, we can invert $F_{\bar{m}}(\mathbb{L}^{-2r}q)$.

$$(F_{\bar{m}}(\mathbb{L}^{-2r}q))^{-1} = \left(\prod_{k=1}^{\infty}(1-\mathbb{L}^{-rk}q^k)^r\right)(\mathbb{L}^{-r}q)^{-\frac{r\bar{m}-\bar{m}^2}{2r}}\mathbb{L}^{-\frac{(r-1)\bar{m}}{2}}\left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-1}$$
$$\left(1+\sum_{d=1}^{\infty}\left(\sum_{\substack{d_1,\cdots,d_l\in\mathbb{Z}_{>0}\\ d_1+\cdots+d_l=d}}(-1)^l\prod_{i=1}^{l}\tilde{\Lambda}^{(\bar{m})}_{d_i}\right)(\mathbb{L}^{-r}q)^d\right) \tag{4.35}$$

Before tackling $G_{r,c}(q)$, we would like to analyze $F_{\bar{m}}(\mathbb{L}^{-2r}q)^{-1}$ and produce bounds for Δ such that the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ vanishes.

Lemma 4.3.4. *If $\Delta > N - \frac{(r-\bar{m})\bar{m}}{2r}$, then the coefficient of $\mathbb{L}^{-N}q^{\Delta}$ in $F_{\bar{m}}(\mathbb{L}^{-2r}q)^{-1}$ is zero.*

Proof. We are going to produce an expression for $\left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-1}$, and use it alongwith the expression for $F_{\bar{m}}(\mathbb{L}^{-2r}q)^{-1}$ (see equation Equation 4.35) to determine the bound for Δ.

Using Lemma 4.3.2 and factoring $\mathbb{L}^{r\bar{m}-\frac{\bar{m}^2-\bar{m}}{2}}$, we get

$$\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L}) = \mathbb{L}^{-\frac{\bar{m}^2-\bar{m}}{2}}\sum_{\nu=0}^{-(r\bar{m}-\bar{m}^2)}\rho_{\nu+r\bar{m}-\frac{\bar{m}^2-\bar{m}}{2}}\mathbb{L}^{\nu}$$

In a similar fashion as in remark 4.3.1, it follows that

$$\left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-1} = \mathbb{L}^{\frac{\bar{m}^2-\bar{m}}{2}}\left(1+\sum_{\nu=-1}^{-\infty}\left(\sum_{\substack{\nu_1,\cdots,\nu_l\in\mathbb{Z}_{<0}\\ \nu_1+\cdots+\nu_l=\nu}}(-1)^l\prod_{i=1}^{l}\rho_{\nu_i+r\bar{m}-\frac{\bar{m}^2-\bar{m}}{2}}\right)\mathbb{L}^{\nu}\right)$$

It follows from equation Equation 4.35 that

$$\left(F_{\bar{m}}(\mathbb{L}^{-2r}q)\right)^{-1} = \prod_{k=1}^{\infty}\left(\sum_{\alpha=0}^{\infty}(-1)^{\alpha}\binom{r}{\alpha}\mathbb{L}^{-rk\alpha}q^{k\alpha}\right) \times \left(\mathbb{L}^{-r}q\right)^{-\frac{r\bar{m}-\bar{m}^2}{2r}}\mathbb{L}^{-\frac{(r-1)\bar{m}}{2}} \times \left(\left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-1}\right.$$
$$\left. + \sum_{d=-1}^{-\infty}\left(\sum_{\substack{d_1,\cdots,d_l\in\mathbb{Z}_{<0}\\ d_1+\cdots+d_l=d}}(-1)^{l}\left(\Lambda^{(\bar{m})}_{\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)^{-(l+1)}\prod_{i=1}^{l}\Lambda^{(\bar{m})}_{d_i+\frac{\bar{m}^2+\bar{m}}{2}}(\mathbb{L})\right)\left(\mathbb{L}^{-r}q\right)^{d}\right)$$

Each nonzero term appearing in the co-efficient of $\mathbb{L}^{-N}q^{\Delta}$ in $F_{\bar{m}}(\mathbb{L}^{-2r}q)^{-1}$ arises from a pair of equations

$$\Delta = \left(\sum_{j=1}^{\delta}k^{(j)}\alpha^{(j)}\right) - \left(\frac{r\bar{m}-\bar{m}^2}{2r}\right) + d$$
$$-N = \left(\sum_{j=1}^{\delta}-rk^{(j)}\alpha^{(j)}\right) + r\left(\frac{r\bar{m}-\bar{m}^2}{2r}\right) - \frac{(r-1)\bar{m}}{2} +$$
$$\left(\left(\frac{\bar{m}^2-\bar{m}}{2}(l+1) + \sum_{i=1}^{l+1}\nu_i\right) + \sum_{i=1}^{l}\sum_{j=1}^{r-1}(r-j)b_j^{(i)}\right) - rd$$

where the α's, the ν's, and l are non-negative integers; the k's are positive integers; and the $b_j^{(i)}$'s are integers satisfying

$$\bar{m}^2 + \sum_{j=1}^{r-1}\left(b_j^{(i)}\right)^2 + \left(\bar{m} + \sum_{j=1}^{r-1}b_j^{(i)}\right)^2 = 2d_i + \bar{m}^2 + \bar{m}, \qquad \text{for } 1 \le i \le l$$

Subsequently, we will show (in Lemma 4.3.5) that $\left(\sum_{j-1}^{r-1}(r-j)b_j^{(i)}\right) + \frac{\bar{m}^2-\bar{m}}{2} \le (r-1)d_i$, for all $1 \le i \le l$. Consequently, we have

$$\left(\sum_{i=1}^{l}\sum_{j=1}^{r-1}(r-j)b_j^{(i)}\right) + \frac{\bar{m}^2-\bar{m}}{2}l \le (r-1)d$$

Therefore, we see that

$$N+(r-1)\frac{r\bar{m}-\bar{m}^2}{2r}-\frac{(r-1)\bar{m}}{2}+\frac{\bar{m}^2-\bar{m}}{2}\geq\left(\sum_{j-1}^{\delta}k^{(j)}\alpha^{(j)}\right)-\frac{r\bar{m}-\bar{m}^2}{2r}+d=\Delta$$

and hence,

$$N-\frac{(r-\bar{m})\bar{m}}{2r}\geq\Delta$$

□

Before we continue, we need to wrap up the proof of Lemma 4.3.4 by proving the following:

Lemma 4.3.5. *Let d be a non-negative integer, and $\bar{m}$ be a non-negative integer less than r. Suppose $b_1,\cdots,b_{r-1}$ are integers satisfying*

$$\bar{m}^2+\sum_{j=1}^{r-1}b_j^2+\left(\bar{m}+\sum_{j=1}^{r-1}b_j\right)^2=2d+\bar{m}^2+\bar{m} \tag{4.36}$$

Then, we have $\sum_{j-1}^{r-1}(r-j)b_j\leq(r-1)d$.

Furthermore, if $r\geq 3$ and $2\leq\bar{m}\leq(r-1)$, then we have

$$\left(\sum_{j=1}^{r-1}(r-j)b_j\right)+\frac{\bar{m}^2-\bar{m}}{2}\leq(r-1)d$$

Proof. Before we begin the proof of Lemma 4.3.5, note that

Remark 4.3.6. Let $r_1, \cdots, r_n$ be positive integers satisfying $r_1 > \cdots > r_n$, and let $b_1, \cdots, b_n$ be integers satisfying $b_1 \geq \cdots \geq b_n$. Let σ be any permutation of $\{1, \cdots, n\}$. Then, we have

$$r_1 b_{\sigma(1)} + \cdots + r_n b_{\sigma(n)} \leq r_1 b_1 + \cdots + r_n b_n$$

Thus, if $b'_1, \cdots, b'_{r-1}$ be a rearrangement of $b_1, \cdots, b_{r-1}$ satisfying $b'_1 \geq \cdots \geq b'_{r-1}$, then we see that

$$\sum_{j=1}^{r-1} (r-j) b_j \leq \sum_{j=1}^{r-1} (r-j) b'_j$$

Moreover, let n_1, n_2, n_3 be non-negative integers such that

- $b'_{j_1} \geq \cdots \geq b'_{j_{n_1}} \geq 2$,
- $b'_{j_{n_1+1}} = \cdots = b'_{j_{n_1+n_2}} = 1$,
- $-1 \geq b'_{j_{n_1+n_2+1}} \geq \cdots \geq b'_{j_{n_1+n_2+n_3}}$, and
- $b'_j = 0$ for all $j \neq j_l$, $1 \leq l \leq n_1 + n_2 + n_3$.

Therefore, we have

$$\sum_{j=1}^{r-1} (r-j) b'_j \leq \sum_{l=1}^{n_1} (r-j_l) b'_{j_l} + \sum_{l=n_1+1}^{n_1+n_2} (r-j_l) \leq \frac{(r-1)}{2} \left(2 \left(\sum_{l=1}^{n_1} b'_{j_l} \right) + 2n_2 \right)$$

We observe that to complete our proof it is enough to show that

$$\left(\sum_{l=1}^{n_1} 2b'_{j_l} \right) + 2n_2 \leq 2d$$

Since $\left(b'_{j_l}\right)^2 \geq 2b'_{j_l}$ for $1 \leq l \leq n_1$ and $\left(b'_{j_l}\right)^2 = 1$ for $n_1 + 1 \leq l \leq n_1 + n_2$, it follows from equation Equation 4.36 that it is enough to show that

$$n_2 + \bar{m} \leq \sum_{l=n_1+n_2+1}^{n_1+n_2+n_3} \left(b'_{j_l}\right)^2 + \left(\left(\bar{m} + n_2 + \sum_{l=1}^{n_1} b'_{j_l}\right) + \sum_{l=n_1+n_2+1}^{n_1+n_2+n_3} b'_{j_l}\right)^2$$

If $n_2+\bar{m} \leq n_3$, then we are done because $\left(b'_{j_l}\right)^2 \geq 1$ for all $n_1+n_2+1 \leq l \leq n_1+n_2+n_3$. Otherwise, it follows from Lemma 4.3.3 that

$$\sum_{l=n_1+n_2+1}^{n_1+n_2+n_3} \left(b'_{j_l}\right)^2 + \left(\left(\bar{m} + n_2 + \sum_{l=1}^{n_1} b'_{j_l}\right) + \sum_{l=n_1+n_2+1}^{n_1+n_2+n_3} b'_{j_l}\right)^2 \geq \frac{1}{n_3+1}\left(\bar{m} + n_2 + \sum_{l=1}^{n_1} b'_{j_l}\right)^2$$

Since $b'_{j_l} \geq 2$ for $1 \leq l \leq n_1$ and $n_2 + \bar{m} \geq n_3 + 1$, we have

$$\frac{1}{n_3+1}\left(\bar{m} + n_2 + \sum_{l=1}^{n_1} b'_{j_l}\right)^2 \geq n_2 + \bar{m}$$

Now we are going to specialize to the case when $r \geq 3$ and $2 \leq \bar{m} \leq r-1$. Clearly, since $\bar{m} \geq 2$, we see that $\frac{\bar{m}^2-\bar{m}}{2} = 1 + \cdots + (\bar{m}-1)$. We define

$$b'_j = \begin{cases} b_j, & \text{if } 1 \leq j \leq (r-\bar{m}) \\ b_j + 1, & \text{if } (r-\bar{m}+1) \leq j \leq (r-1) \end{cases}$$

As a consequence, we see that

$$\left(\sum_{j=1}^{r-1} (r-j)b_j\right) + \frac{\bar{m}^2-\bar{m}}{2} = \sum_{j=1}^{r-1} (r-j)b'_j$$

Additionally, we can rewrite equation Equation 4.36 in terms of b_j''s as follows

$$\sum_{j=1}^{r-1}\left(b_j'\right)^2+\left(\sum_{j=1}^{r-1}b_j'\right)^2+2\left(\sum_{j=1}^{r-\bar{m}}b_j'\right)=2d$$

As a result, to prove our claim, it is enough to show that

$$\frac{(r-1)}{2}\left\{\sum_{j=1}^{r-1}\left(b_j'\right)^2+\left(\sum_{j=1}^{r-1}b_j'\right)^2+2\left(\sum_{j=1}^{r-\bar{m}}b_j'\right)\right\}-\left(\sum_{j=1}^{r-1}(r-j)b_j'\right)\geq 0$$

for integer values of b_j', for all $1\leq j\leq r-1$. Consider the smooth polynomial function

$$f(x_1,\cdots,x_{r-1})=\frac{(r-1)}{2}\left\{\sum_{j=1}^{r-1}x_j^2+\left(\sum_{j=1}^{r-1}x_j\right)^2+2\left(\sum_{j=1}^{r-\bar{m}}x_j\right)\right\}-\left(\sum_{j=1}^{r-1}(r-j)x_j\right)$$

We have

$$\frac{\partial f}{\partial x_k}=\begin{cases}\frac{(r-1)}{2}\left\{2x_k+2\left(\sum_{j=1}^{r-1}x_j\right)+2\right\}-(r-k), & \text{if } 1\leq k\leq (r-m)\\ \frac{(r-1)}{2}\left\{2x_k+2\left(\sum_{j=1}^{r-1}x_j\right)\right\}-(r-k), & \text{if } (r-\bar{m}+1)\leq k\leq (r-1)\end{cases}$$

and, the second partial derivatives are

$$\frac{\partial^2 f}{\partial x_l \partial x_k}=\begin{cases}2\frac{(r-1)}{2}, & \text{if } l\neq k\\ 4\frac{(r-1)}{2}, & \text{if } l=k\end{cases}$$

Since $r \geq 3$ and the Hessian matrix for f is $\frac{(r-1)}{2}$ times the Hessian matrix in Lemma 4.3.3, we conclude that our Hessian matrix is positive definite. Thus, f has a global minimum at the critical point

$$x_k = \begin{cases} -\frac{\bar{m}}{r} - \frac{1}{2} + \frac{(r-k)}{(r-1)}, & \text{if } 1 \leq k \leq (r - \bar{m}) \\ -\frac{\bar{m}}{r} + \frac{1}{2} + \frac{(r-k)}{(r-1)}, & \text{if } (r - \bar{m} + 1) \leq k \leq (r-1) \end{cases}$$

It follows from the bounds on k that in either case, we have $-\frac{1}{2} \leq x_k \leq \frac{1}{2}$. Hence, to show that f is non-negative for all integer values of x_j, for all $1 \leq j \leq (r-1)$, it is enough to show that f is non-negative for every element of the set $\{-1, 0, 1\}^{r-1}$. Let $(x_1, \cdots, x_{r-1})$ be an element of the set $\{-1, 0, 1\}^{r-1}$. Furthermore, assume that for $1 \leq j \leq (r - \bar{m})$, x of the x_j's are $(+1)$ and y of the x_j's are (-1). On a similar note, assume that for $(r - \bar{m} + 1) \leq j \leq (r-1)$, z of the x_j's are $(+1)$ and w of the x_j's are (-1). It follows from Remark 4.3.6 that

$$\begin{aligned} \sum_{j=1}^{r-1} (r-j)x_j \leq (r-1) + \cdots + (r-x) - \{\bar{m} + (\bar{m}+1) + \cdots + (\bar{m}+y-1)\} \\ + (\bar{m}-1) + \cdots + (\bar{m}-z) - \{1 + \cdots + w\} \\ = rx - \bar{m}y + \bar{m}z - \frac{x^2+x}{2} - \frac{y^2-y}{2} - \frac{z^2+z}{2} - \frac{w^2+w}{2} \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(x_1, \cdots, x_{r-1}) \geq \frac{(r-1)}{2}\{(x-y+z-w)^2 + 3x - y + z + w\} \\ - \left\{ rx - \bar{m}y + \bar{m}z - \frac{x^2+x}{2} - \frac{y^2-y}{2} - \frac{z^2+z}{2} - \frac{w^2+w}{2} \right\} \end{aligned} \tag{4.37}$$

For ease of notation, let's call the right hand side of inequality in equation Equation 4.37 as $g(x, y, z, w)$. Upon further scrutinizing, we deduce that

$$2g(x,y,z,w) = (r-1)(x-y+z-w)^2+(x^2+y^2+z^2+w^2)+(r-2)x+(2\bar{m}-r)y+(r-2\bar{m})z+rw$$

If $r = 2\bar{m}$, then $2g(x, y, z, w) \geq 0$ because x and w are non-negative integers. If $r > 2\bar{m}$, then we see that

$$2g(x, y, z, w) \geq (r - 2\bar{m})\{(x - y + z - w)^2 + (x - y + z - w)\} \geq 0$$

Similarly, if $r < 2\bar{m}$, then using the fact that $(r - 1) > (2\bar{m} - r)$, we get

$$2g(x, y, z, w) \geq (2\bar{m} - r)\{(-x + y - z + w)^2 + (-x + y - z + w)\} \geq 0$$

In conclusion, the function f is non-negative for all integer values of x_j, for all $1 \leq j \leq r - 1$. $\square$

We are finally ready to analyze $(1 - q)G_{r,c}(q)$.

Theorem 4.3.7. *If $\varDelta > N + \frac{(2-2r)\bar{m}^2 - r\bar{m}}{2r} + C_0$, where C_0 is the same constant as in Proposition 4.2.5, then the coefficient of $\mathbb{L}^{-N}q^{\varDelta}$ in $(1-q)G_{r,c}(q)$ is zero.*

Proof. Recall that if follows from the blow-up equation (equation Equation 4.30) that

$$(1-q)G_{r,c}(q) = \left(\mathbb{L}^{-2r}q\right)^{\frac{m^2}{2r}} \times \left(F_m(\mathbb{L}^{-2r}q)\right)^{-1} \times (1-q)\tilde{G}_{r,c-mE}(q)$$

Each nonzero term appearing in the co-efficient of $\mathbb{L}^{-N}q^{\varDelta}$ arises from a pair of equations

$$\varDelta = \frac{\bar{m}^2}{2r} + \varDelta_1 + \varDelta_2$$
$$-N = -\bar{m}^2 + (-N_1) + (-N_2)$$

where $(\varDelta_1, -N_1)$ accounts for the contribution of terms from the co-efficient of $\mathbb{L}^{-N_1}q^{\varDelta_1}$ in $(F_{\bar{m}}(\mathbb{L}^{-2r}q))^{-1}$, and $(\varDelta_2, -N_2)$ accounts for the contribution of terms from the co-efficient of $\mathbb{L}^{-N_2}q^{\varDelta_2}$ in $(1-q)\tilde{G}_{r,c-mE}(q)$.

It follows from Lemma 4.3.4 and Proposition 4.2.5 that

$$\varDelta_1 \leq N_1 - \frac{(r-\bar{m})\bar{m}}{2r}, \qquad \text{and} \quad \varDelta_2 \leq N_2 + C_0$$

These inequalities yield

$$\varDelta \leq N + \frac{(2-2r)\bar{m}^2 - r\bar{m}}{2r} + C_0$$

In conclusion, for $\Delta > N + \frac{(2-2r)\bar{m}^2 - r\bar{m}}{2r} + C_0$, the co-efficient of $\mathbb{L}^{-N}q^{\Delta}$ in $(1-q)G_{r,c}(q)$ is zero. □

4.4 Bounds for stabilization of Betti numbers

In this section, our goal is to determine lower bounds such that the Betti numbers of the moduli space stabilize. More precisely, we look at $\mathbb{P}^2$ equipped with the ample divisor $H = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$. We assume that r and a are coprime and consider the moduli space $M_{\mathbb{P}^2,H}(r, aH, c_2)$. Since r and a are coprime, all μ_H-semistable sheaves are μ_H-stable. Using Proposition 2.2.10 in conjunction with Theorem 4.3.7, we derive the lower bounds such that the Betti numbers of $M_{\mathbb{P}^2,H}(r, aH, c_2)$ stabilize. Lastly, we investigate some examples and show that we can improve this bound further.

Theorem 4.4.1. *Let r be at least two. Assume that r and a be coprime. There is a constant C depending only on r and a such that if $c_2 \geq N + C$, the $2N$th Betti number of the moduli space $M_{\mathbb{P}^2,H}(r, aH, c_2)$ stabilize. Moreover, we can take $C = \lfloor \frac{r-1}{2r}a^2 + \frac{1}{2}(r^2+1) \rfloor$.*

Proof. Let γ denote the Chern class (r, aH, c_2). By our assumption, r and a are co-prime, a posteriori, all μ_H-semistable sheaves are μ_H-stable. In this case, we know that $M_{\mathbb{P}^2,H}(\gamma)$ is a smooth projective variety of dimension $ext^1(\gamma, \gamma)$. We conclude using Remark 2.2.8 that to show that the $2N$th Betti number stabilize for $c_2 \geq N + C$, it is enough to show that the coefficient of $\mathbb{L}^{-N}q^d$ in the generating function

$$(1-q)\sum_{c_2 \geq 0} [M_{\mathbb{P}^2,H}(\gamma)]\mathbb{L}^{-ext^1(\gamma,\gamma)}q^{c_2}$$

is zero for $d > N + C$.

We note that $\chi(\gamma, \gamma) = 1 - ext^1(\gamma, \gamma)$ and $c_2 = r\Delta + \frac{r-1}{2r}c_1^2$. Proposition 2.2.10 yields the following equality in A

$$[M_{\mathbb{P}^2,H}(r, aH, c_2)] = (\mathbb{L} - 1)[\mathcal{M}_{\mathbb{P}^2,H}(r, aH, c_2)]$$

Thus, we have the following equality of generating functions

$$(1-q)\sum_{c_2 \geq 0}[M_{\mathbb{P}^2,H}(\gamma)]\mathbb{L}^{-ext^1(\gamma,\gamma)}q^{c_2} = q^{\frac{r-1}{2r}a^2}(1-\mathbb{L}^{-1})(1-q)G_{r,aH}(q)$$

Each term contributing to the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{r-1}{2r}a^2}(1-\mathbb{L}^{-1})(1-q)G_{r,aH}(q)$ arises from a pair of equations

$$d = \frac{r-1}{2r}a^2 + \Delta'$$
$$-N = \varepsilon - N'$$

where $\varepsilon \in \{-1, 0\}$ accounts for the contribution to the coefficient of $\mathbb{L}^{-N}q^d$ coming from $(1-\mathbb{L}^{-1})$, and (Δ', N') accounts for the contribution coming from the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in $(1-q)G_{r,aH}(q)$. It follows from Theorem 4.3.7 that for the coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ to be nonzero, we must have $\Delta' \leq N' + C_0$ (using $m = 0$). Moreover, it follows from Proposition 4.2.5 that we can take $C_0 = \frac{1}{2}(r^2+1)$. Consequently, for the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{r-1}{2r}a^2}(1-\mathbb{L}^{-1})(1-q)G_{r,aH}(q)$ to be nonzero, we must have

$$d \leq N + \left\lfloor \frac{r-1}{2r}a^2 + C_0 \right\rfloor$$

$\square$

For the remainder of this section, we look at some examples. Yoshioka (21)[Page 194] has computed the Betti numbers $b_{2N}(M_{\mathbb{P}^2,H}(2,-H,c_2))$, where $M_{\mathbb{P}^2,H}(2,-H,c_2)$ is the moduli space of μ_H-stable sheaves with Chern classes $(2,-H,c_2)$, which we will denote by γ. We observe from the table in (21)[Page 194] that the Betti numbers $b_{2N}(M_{\mathbb{P}^2,H}(\gamma))$ stabilize when $c_2 \geq N+1$. Since $r=2$ and $a=-1$, we get from Theorem 4.4.1 that the Betti numbers stabilize when $c_2 \geq N+2$. Therefore, we need to improve our lower bound.

Proposition 4.4.2. *If $c_2 \geq N+1$, the $2N$th Betti number of the moduli space $M_{\mathbb{P}^2,H}(2,-H,c_2)$ stabilize.*

Proof. Following the proof of Theorem 4.4.1, it is enough to show that when $d > N+1$,the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{1}{4}}(1-\mathbb{L}^{-1})(1-q)G_{2,-H}(q)$ is zero.

Each term contributing to the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{1}{4}}(1-\mathbb{L}^{-1})(1-q)G_{2,-H}(q)$ arises from a pair of equations

$$d = \frac{1}{4} + \Delta'$$
$$-N = \varepsilon - N'$$

where $\varepsilon \in \{-1, 0\}$ accounts for the contribution to the coefficient coming from $(1-\mathbb{L}^{-1})$, and (Δ', N'') accounts for the contribution coming from terms in coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in $(1-q)G_{2,-H}(q)$.

It follows from Theorem 4.3.7 that for the co-efficient of $\mathbb{L}^{-N'}q^{\Delta'}$ to be nonzero, we must have $\Delta' \leq N' + C_0$. Consequently, we must have

$$d - \frac{1}{4} = \Delta' \leq N' + C_0 = N + \varepsilon + C_0 \leq N + C_0 \tag{4.38}$$

As a result, for $d > N + \left\lfloor \frac{1}{4} + C_0 \right\rfloor$, the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{1}{4}}(1-\mathbb{L}^{-1})(1-q)G_{2,-H}(q)$ must be zero. Therefore, to complete the proof of our Claim, we need to figure out the value of C_0.

It follows from the proof of Proposition 4.2.5 that to compute C_0, we need to compute

$$\frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\kappa$$

where $l = 2$, $r = 2$, $r_1 = r_2 = 1$, and κ is a lower bound for

$$2\left(2\Delta - \Delta_1 - \Delta_2\right) + \left(c_2 - c_1\right) \cdot K_{\mathbb{F}_1}$$

except for the case $l = 2$ and $\left(c_2 - c_1\right) \cdot F = -1$.

Let $c_1 = a_1 E + b_1 F$ and $c_2 = a_2 E + b_2 F$. Since $c_1 + c_2 = -E - F$, we have $a_1 + a_2 = -1$ and $b_1 + b_2 = -1$. Moreover, we must have $a_2 - a_1 \neq -1$. Using Yoshioka's relation (equation Equation 4.8) yields

$$\left(2\Delta - \Delta_1 - \Delta_2\right) = -\frac{1}{4}\left(c_1 - c_2\right)^2 = \frac{1}{4}\left(2a_1 + 1\right)^2 - \frac{1}{2}\left(2a_1 + 1\right)\left(2b_1 + 1\right)$$

Since $K_{\mathbb{F}_1} = -2E - 3F$, we see that

$$\left(c_2 - c_1\right) \cdot K_{\mathbb{F}_1} = \left(2a_1 + 1\right) + 2\left(2b_1 + 1\right)$$

Therefore, we have

$$2(2\Delta - \Delta_1 - \Delta_2) + \left(c_2 - c_1\right) \cdot K_{\mathbb{F}_1} = 2a_1^2 + 2a_1 + 2b_1 - 4a_1 b_1 + \frac{5}{2}$$

Clearly $a_1^2 + a_1 \geq 0$ for all integer values of a_1. Thus, we need to find a lower bound for $2b_1(1 - 2a_1)$.

Recall that as per the definition of $S^{\mu}(\{1, c_1\}, \{1, c_2\}, F, E + F)$ (see equation Equation 4.3, Equation 4.4) we have two cases

A) $a_1 > -\frac{1}{2}$ and $b_1 \leq -\frac{1}{2}$

B) $a_1 \leq -\frac{1}{2}$ and $b_1 > -\frac{1}{2}$

Since a_1 and b_1 are integers, in Case A, we see that $a_1 \geq 0$ and $-b_1 \geq 1$. When $a_1 = 0$, we must have $a_2 = -1$, whence $a_2 - a_1 = -1$ which is not possible by our assumption. Hence, we must have $a_1 \geq 1$, which yields

$$2b_1(1 - 2a_1) = (2a_1 - 1)(-2b_1) \geq (2\,(1) - 1)\,(2(1)) = 2$$

Similarly, in Case B, we see that $-a_1 \geq 1$ and $b_1 \geq 0$, thereby yielding

$$2b_1(1 - 2a_1) \geq (2(0))\,(1 + 2(1)) = 0$$

In either case we see that $2b_1(1 - 2a_1) \geq 0$, and hence we can take $\kappa = \frac{5}{2}$.

Clearly, in our case $r = 2$ and $r_1 = r_2 = 1$, whence $\frac{1}{2}\,(r^2 - r_1^2 - r_2^2) = 1$. Following the proof of Proposition 4.2.5, we see that

$$C_0 = \max\left\{0, 1 + 1 - \frac{1}{2}\kappa, 1 - \frac{3}{4} + \left(\left\lceil\frac{-1}{2}\right\rceil - \frac{-1}{2}\right)\right\} = 2 - \frac{5}{4}$$

In summary, for the coefficient of $\mathbb{L}^{-N}q^d$ to be nonzero, we must have

$$d \leq N + \frac{1}{4} + 2 - \frac{5}{4} = N + 1$$

In conclusion, when $d > N+1$, the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{1}{4}}(1-\mathbb{L}^{-1})(1-q)G_{2,-H}(q)$ is zero. $\square$

Manschot (22)[Table 1], (39)[Table 1] computed the Betti numbers of the moduli space $M_{\mathbb{P}^2,H}(3,-H,c_2)$ and the virtual Betti numbers of the moduli space $M_{\mathbb{P}^2,H}(4,2H,c_2)$. We observe from the tables in these papers that the Betti numbers of $M_{\mathbb{P}^2,H}(3,-H,c_2)$ stabilize when $c_2 \geq N+2$ and the virtual Betti numbers of $M_{\mathbb{P}^2,H}(4,2H,c_2)$ stabilize when $c_2 \geq N+3$. In the first case, we have $r=3$ and $a=-1$, we get from Theorem 4.4.1 that the Betti numbers stabilize when $c_2 \geq N+5$.

As our second example, we scrutinize the Betti numbers of the moduli space $M_{\mathbb{P}^2,H}(4,H,c_2)$. In this case, Theorem 4.4.1 yields the stabilization of the Betti numbers when $c_2 \geq N+8$. We improve this bound in the following Proposition.

Proposition 4.4.3. *If $c_2 \geq N+5$, the $2N$-th Betti number of the moduli space $M_{\mathbb{P}^2,H}(4,H,c_2)$ stabilize.*

Proof. Following the proof of Theorem 4.4.1, it is enough to show that when $d > N+5$, the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{3}{8}}(1-\mathbb{L}^{-1})(1-q)G_{4,H}(q)$ is zero.

Each term contributing to the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{3}{8}}(1-\mathbb{L}^{-1})(1-q)G_{4,H}(q)$ arises from a pair of equations

$$d = \frac{3}{8} + \Delta'$$

$$-N = \varepsilon - N'$$

where $\varepsilon \in \{-1, 0\}$ accounts for the contribution to the coefficient coming from $(1 - \mathbb{L}^{-1})$, and (Δ', N') accounts for the contribution coming from the terms in coefficient of $\mathbb{L}^{-N'}q^{\Delta'}$ in $(1-q)G_{4,H}(q)$.

It follows from Theorem 4.3.7 that if the co-efficient of $\mathbb{L}^{-N'}q^{\Delta'}$ is non-zero, then we must have $\Delta' \leq N' + C_0$, whence, $d \leq N + \lfloor \frac{3}{8} + C_0 \rfloor$. Consequently, for $d > N + \lfloor \frac{3}{8} + C_0 \rfloor$, the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{3}{8}}(1-\mathbb{L}^{-1})(1-q)G_{4,H}(q)$ must be zero. Therefore, to complete our proof, we need to determine the value of C_0.

Adopting the notation used in proof of Proposition 4.2.5 and Lemma 4.2.6 in our situation, we get $r = 4$, $a = b = 1$. Recall that C_0 is the maximum of the terms $1 + \frac{1}{2}\left(r^2 - \sum_{i=1}^{l} r_i^2\right) - \frac{1}{2}\kappa$ except the case when $l = 2$ and $\mu_F(\gamma_2) - \mu_F(\gamma_1) = -1$ and the terms $\frac{r_1 r_2}{2r} + \left(\left\lceil \frac{br_2}{r} \right\rceil - \frac{br_2}{r}\right)$ for $r_1 + r_2 = r$, where $r = \sum_{i=1}^{l} r_i$, $a = \sum_{i=1}^{l} r_i a_i$, $s_i = \sum_{j=i}^{l} b_j$, $b = s_1$, and κ is lower bound for $S_1 + S_2$, where

$$S_1 = (r-1)\sum_{i=1}^{l} r_i a_i^2 - \frac{r-1}{r}a^2 + \sum_{i=1}^{l} a_i r_i \left(\sum_{j=i+1}^{l} r_j - \sum_{j=1}^{i-1} r_j\right)$$

and

$$S_1 = 2\sum_{i=2}^{l}((r-1)(a_i - a_{i-1}) + r_i + r_{i-1})\left(\frac{b}{r}\sum_{j=i}^{l} r_j - s_i\right)$$

When $l = 2$ and $(r_1, r_2) = (3, 1)$, we see that $S_1 \geq -\frac{3}{4}$ with equality occurring at $(a_1, a_2) = (0, 1)$. At the point $(0, 1)$ we get $S_2 \geq \frac{7}{2}$, and hence, $S_1 + S_2 \geq \frac{11}{4}$. Since there are no other points (a_1, a_2) satisfying $3a_1 + a_2 = 1$ at which $S_1 < \frac{11}{4}$, we can take $\kappa = \frac{11}{4}$, and we get $1 + \frac{1}{2}(r^2 - r_1^2 - r_2^2) - \frac{1}{2}\kappa = \frac{21}{8}$.

When $l = 2$ and $(r_1, r_2) = (1, 3)$, we see that $S_1 \geq \frac{21}{4}$ with equality occurring at $(a_1, a_2) = (1, 0)$, and $S_2 \geq 1$. Thus, we can take $\kappa = \frac{25}{4}$, and we get $1 + \frac{1}{2}(r^2 - r_1^2 - r_2^2) - \frac{1}{2}\kappa = \frac{7}{8}$.

When $l = 2$ and $(r_1, r_2) = (2, 2)$, there is no integer solution for $2a_1 + 2a_2 = 1$. Thus, we ignore this case.

When $l = 3$ and $(r_1, r_2, r_3) = (2, 1, 1)$, we see that $S_1 \geq -\frac{3}{4}$ with equality occurring at $(a_1, a_2, a_3) = (0, 0, 1)$. At this point we get $S_2 \geq 4$, whence $S_1 + S_2 \geq \frac{5}{2}$. The only other point (a_1, a_2, a_3) with $S_1 \leq \frac{5}{2}$ is $(0, 1, 0)$ at which $S_1 = \frac{5}{4}$ and $S_2 \geq \frac{9}{2}$, and thus $S_1 + S_2 \geq \frac{23}{4}$. Therefore, we can take $\kappa = \frac{5}{2}$, and we get $1 + \frac{1}{2}\left(r^2 - r_1^2 - r_2^2 - r_3^2\right) - \frac{1}{2}\kappa = \frac{19}{4}$.

When $l = 3$ and $(r_1, r_2, r_3) = (1, 2, 1)$, we see that $S_1 \geq -\frac{3}{4}$ with equality occurring at $(0, 0, 1)$. At this point, we see that $S_2 \geq 6$, whence $S_1 + S_2 \geq \frac{21}{4}$. At every other point (a_1, a_2, a_3) with $a_1 + 2a_2 + a_3 = 1$, we have $S_1 \geq \frac{21}{4}$. As a consequence, we can take $\kappa = \frac{21}{4}$, and we get $1 + \frac{1}{2}\left(r^2 - r_1^2 - r_2^2 - r_3^2\right) - \frac{1}{2}\kappa = \frac{27}{8}$.

When $l = 3$ and $(r_1, r_2, r_3) = (1, 1, 2)$, we see that $S_1 \geq \frac{5}{4}$ with equality occurring at $(a_1, a_2, a_3) = (-1, 0, 1)$. At this point, we see that $S_2 \geq 6$, and thus $S_1 + S_2 \geq \frac{29}{4}$. The other points (a_1, a_2, a_3) satisfying $a_1 + a_2 + 2a_3 = 1$ at which $S_1 \leq \frac{29}{4}$ are $(0, 1, 0)$, $(0, -1, 1)$, $(1, 0, 0)$. Analyzing S_1 and S_2 at these points, we see that $S_1 + S_2$ may attain the least possible value $\frac{25}{4}$. Thus, we take $\kappa = \frac{25}{4}$, and we see that $1 + \frac{1}{2}\left(r^2 - r_1^2 - r_2^2 - r_3^2\right) - \frac{1}{2}\kappa = \frac{23}{8}$.

When $l = 4$ and $(r_1, r_2, r_3, r_4) = (1, 1, 1, 1)$, we see that $S_1 \geq -\frac{3}{4}$ with equality occurring at $(a_1, a_2, a_3, a_4) = (0, 0, 0, 1)$. At this point, we see that $S_2 \geq 6$, and thus

$S_1+S_2 \geq \frac{21}{4}$. The other points (a_1, a_2, a_3, a_4) with $a_1+a_2+a_3+a_4 = 1$ at which $S_1 \leq \frac{21}{4}$ are $(-1,0,0,2)$, $(0,-1,1,1)$, $(-1,1,1,0)$, $(-1,1,0,1)$, $(-1,0,1,1)$, $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,1,0)$. However, we see that at each of these points we have $S_1 + S_1 \geq \frac{21}{4}$. Hence, we can take $\kappa = \frac{21}{4}$, and we get $1 + \frac{1}{2}\left(r^2 - r_1^2 - r_2^2 - r_3^2 - r_4^2\right) - \frac{1}{2}\kappa = \frac{35}{8}$.

Finally, since $b = 1$, $r = 4$, and $1 \leq r_2 \leq 3$, we see that $\frac{(r-r_2)r_2}{2r} + 1 - \frac{r_2}{r}$ attains maximum value of $\frac{9}{8}$ at $r_2 = 1$.

In conclusion, we can take $C_0 = \frac{19}{4}$, and we get that when $d > N + 5$ the coefficient of $\mathbb{L}^{-N}q^d$ in $q^{\frac{3}{8}}(1-\mathbb{L}^{-1})(1-q)G_{4,H}(q)$ is zero. $\square$

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