DIMENSION REDUCTION FOR ROTATING BOSE-EINSTEIN CONDENSATES WITH ANISOTROPIC CONFINEMENT

FLORIAN MÉHATS AND CHRISTOF SPARBER

Abstract. We consider the three-dimensional time-dependent Gross-Pitaevskii equation arising in the description of rotating Bose-Einstein condensates and study the corresponding scaling limit of strongly anisotropic confinement potentials. The resulting effective equations in one or two spatial dimensions, respectively, are rigorously obtained as special cases of an averaged three dimensional limit model. In the particular case where the rotation axis is not parallel to the strongly confining direction the resulting limiting model(s) include a negative, and thus, purely repulsive quadratic potential, which is not present in the original equation and which can be seen as an effective centrifugal force counteracting the confinement.

1. Introduction and main result

We are interested in the dimension reduction problem arising in the description of *rotating Bose-Einstein condensates* with strongly *anisotropic confinement* potential. In physics experiments such potentials are used to obtain effective one-dimensional (called cigar-shaped) or two-dimensional (called pancake-shaped) condensates which, among other features, exhibit different stability and instability properties than the usual three dimensional case (for a general introduction to the physics of Bose-Einstein condensates, see, e.g., [18, 19]). The present work aims to give a rigorous justification to the use of these approximate lower-dimensional models. In comparison with earlier studies in the mathematics literature, see [3, 4, 5, 6, 7], the main novelty in our work is the presence of an additional angular momentum rotation term, whose strong interaction with the confinement will, in general, result in a nontrivial effect within the limiting model obtained.

The starting point of our investigation is the three-dimensional *Gross-Pitaevskii equation*, describing the Bose-Einstein condensate in a mean-field approximation, cf. [16, 17, 19]. Rescaled into dimensionless form (see, e.g., [7]) and in a rotating reference frame (which is customary used throughout the literature), this *nonlinear Schrödinger equation* (NLS) reads

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + \left(\frac{|x|^2}{2} + \frac{z^2}{2\varepsilon^4}\right)\psi + i\Omega\cdot(\mathbf{x}\wedge\nabla)\,\psi + \beta^\varepsilon|\psi|^{2\sigma}\psi, \tag{1.1}$$

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with an initial data $\psi(t=0,\mathbf{x}) = \varepsilon^{-1/2}\psi_0(x,z/\varepsilon)$ and where $\Omega \equiv (\Omega_1,\Omega_2,\Omega_z)$ is a given vector of $\mathbb{R}^3$, describing the *rotation axis*. Here, the exponent of the nonlinearity is assumed to be $1 \leq \sigma < 2$. In particular, this means that the nonlinearity is $H^1(\mathbb{R}^3)$-subcritical, cf. [10]. Note that this includes the cubic case $\sigma = 1$, which is the physically most relevant situation. The space variable $\mathbf{x} \in \mathbb{R}^3$ splits into $\mathbf{x} = (x,z) \in \mathbb{R}^2 \times \mathbb{R}$, with $x \equiv (x_1,x_2)$. Finally, we assume that $\varepsilon \in (0,1]$ is a *small parameter* describing the anisotropy within the confining potential. In the following we shall be interested in the limit $\varepsilon \to 0$ for solutions to (1.1). Note that we thereby assume that the initial wave function is already confined at the scale epsilon in the z-direction (an assumption which is consistent with the asymptotic limiting regime considered) and such that its total *mass* $\|\psi(t=0,\cdot)\|_{L^2}^2 = 1$, uniformly in ε.

Let us rescale the variables. We set

$$x' = x, \quad z' = \frac{z}{\varepsilon}, \quad \psi^\varepsilon(t,x',z') = \varepsilon^{1/2}\psi\left(t,x',\varepsilon z'\right)$$

and assume that $\beta^\varepsilon = \lambda\varepsilon^\sigma$, where $\lambda \in \mathbb{R}$ is fixed. We are thus in a weak interaction regime similar to [6, 7]. Under this rescaling the NLS becomes (dropping the primes in the variables for simplicity)

$$\begin{aligned} i\partial_t\psi^\varepsilon = & \frac{1}{\varepsilon^2}\mathcal{H}_z\psi^\varepsilon - \frac{i}{\varepsilon}\left(\Omega_2x_1 - \Omega_1x_2\right)\partial_z\psi^\varepsilon + \mathcal{H}_x\psi^\varepsilon - \Omega_zL_z\psi^\varepsilon \\ & - i\varepsilon z\left(\Omega_1\partial_{x_2} - \Omega_2\partial_{x_1}\right)\psi^\varepsilon + \lambda|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \end{aligned} \tag{1.2}$$

with $\psi^\varepsilon(t=0,x,z) = \psi_0(x,z)$. Here, and in the following, we denote

$$\mathcal{H}_z = -\frac{1}{2}\partial_z^2 + \frac{z^2}{2}, \qquad \mathcal{H}_x = -\frac{1}{2}\Delta_x + \frac{|x|^2}{2}, \quad L_z = ix_2\partial_{x_1} - ix_1\partial_{x_2}\,,$$

where L_z is the angular momentum operator associated to a rotation around the negative z-axis.

In order to get rid of the singular rotation term proportional to ε^{-1} in (1.2), we shall invoke the following (unitary) change of unknown. Setting

$$\psi^\varepsilon(t,x,z) = e^{i\varepsilon z(\Omega_1x_2 - \Omega_2x_1)}u^\varepsilon(t,x,z), \tag{1.3}$$

we obtain

$$\begin{aligned} i\partial_tu^\varepsilon = & \frac{1}{\varepsilon^2}\mathcal{H}_zu^\varepsilon + \mathcal{H}_xu^\varepsilon - \frac{1}{2}\left(\Omega_2x_1 - \Omega_1x_2\right)^2u^\varepsilon - \Omega_zL_zu^\varepsilon + \lambda|u^\varepsilon|^{2\sigma}u^\varepsilon \\ + & \frac{3\varepsilon^2}{2}\left(\Omega_1^2 + \Omega_2^2\right)z^2u^\varepsilon - \varepsilon\Omega_z\left(\Omega_1x_1 + \Omega_2x_2\right)zu^\varepsilon + 2i\varepsilon z(\Omega_2\partial_{x_1} - \Omega_1\partial_{x_2})u^\varepsilon, \end{aligned} \tag{1.4}$$

subject to initial data

$$u^\varepsilon(t=0) = u_0^\varepsilon = e^{-i\varepsilon z(\Omega_1x_2 - \Omega_2x_1)}\psi_0. \tag{1.5}$$

Note that in (1.4), the only singular term left is $\varepsilon^{-2}\mathcal{H}_z$. We consequently expect that by filtering the associated rapid oscillations the function $e^{it\mathcal{H}_z/\varepsilon^2}u^\varepsilon(t)$ will converge to some finite limit $\phi(t)$, as $\varepsilon \to 0$.

A suitable functional framework for the analysis of our problem is the scale of Sobolev spaces adapted to $\mathcal{H}_z$ and $\mathcal{H}_x$. For any real number $s \geq 0$, we denote

$$\Sigma^s := \left\{u \in H^s(\mathbb{R}^3)\ :\ |\mathbf{x}|^su \in L^2(\mathbb{R}^3)\right\}.$$

According to [14, 6], this Hilbert space can be equipped with the following equivalent norms:

$$\|u\|_{\Sigma^s}^2 := \|u\|_{H^s}^2 + \||\mathbf{x}|^s u\|_{L^2}^2 \simeq \|\mathcal{H}_z^{s/2} u\|_{L^2}^2 + \|\mathcal{H}_x^{s/2} u\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{1.6}$$

It is also useful to recall that, for all $0 \le \ell \le s$, we have

$$\left\| |\mathbf{x}|^{s-\ell} (-\Delta)^{\ell/2} u \right\|_{L^2} \lesssim \|u\|_{\Sigma^s} \quad \text{and} \quad \left\| (-\Delta)^{\ell/2} |\mathbf{x}|^{s-\ell} u \right\|_{L^2} \lesssim \|u\|_{\Sigma^s}. \tag{1.7}$$

For $s > 3/2$, Σ^s is an algebra. Moreover, the self-adjoint operators $\mathcal{H}_z$ and $\mathcal{H}_x$ generate the groups of isometries $\theta \mapsto e^{i\theta\mathcal{H}_z}$ and $\theta \mapsto e^{i\theta\mathcal{H}_x}$ on any Σ^s, $s \ge 0$.

To derive the limit model as $\varepsilon \to 0$, we need to introduce the following nonlinear function:

$$\begin{aligned} F(\theta, u) &= e^{i\theta\mathcal{H}_z} \left(\left| e^{-i\theta\mathcal{H}_z} u \right|^{2\sigma} e^{-i\theta\mathcal{H}_z} u \right) \\ &= e^{i\theta(\mathcal{H}_z - 1/2)} \left(\left| e^{-i\theta(\mathcal{H}_z - 1/2)} u \right|^{2\sigma} e^{-i\theta(\mathcal{H}_z - 1/2)} u \right), \end{aligned} \tag{1.8}$$

and study the behavior of $F\left(t/\varepsilon^2, u\right)$, as $\varepsilon \to 0$. For $s > 3/2$, Σ^s is an algebra and it is readily seen that $F \in C(\mathbb{R} \times \Sigma^s, \Sigma^s)$. Moreover, since the spectrum of the quantum harmonic oscillator $\mathcal{H}_z$ is $\{n + 1/2,\ n \in \mathbb{N}\}$, the operator $e^{i\theta(\mathcal{H}_z - 1/2)}$ is 2π periodic with respect to θ, so F is also 2π periodic with respect to θ. Denoting the average of this function by

$$\begin{aligned} F_{\mathrm{av}}(u) &:= \lim_{T \to \infty} \frac{1}{T} \int_0^T F(\theta, u)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta\mathcal{H}_z} \left(\left| e^{-i\theta\mathcal{H}_z} u \right|^{2\sigma} e^{-i\theta\mathcal{H}_z} u \right) d\theta, \end{aligned} \tag{1.9}$$

the *limit model* as $\varepsilon \to 0$ reads

$$i\partial_t \phi = -\frac{1}{2}\Delta_x \phi + \frac{1}{2}\left(|x|^2 - (\Omega_2 x_1 - \Omega_1 x_2)^2 \right) \phi - \Omega_z L_z \phi + \lambda F_{\mathrm{av}}(\phi) \tag{1.10}$$

with the initial data $\phi(t=0) = \psi_0$. The Gross-Pitaevskii type energy associated to this equation is

$$\begin{aligned} E(\phi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x \phi|^2 dx dz &+ \frac{1}{2} \int_{\mathbb{R}^3} \left(|x|^2 - (\Omega_2 x_1 - \Omega_1 x_2)^2 \right) |\phi|^2 dx dz \\ &- \Omega_z \langle L_z \phi, \phi \rangle_{L^2} + \frac{\lambda}{2\pi(\sigma+1)} \int_{\mathbb{R}^3} \int_0^{2\pi} \left| e^{-i\theta\mathcal{H}_z} \phi \right|^{2\sigma+2} d\theta\, dx\, dz, \end{aligned}$$

where here and in the following,

$$\langle u, v \rangle_{L^2} = \mathrm{Re} \int_{\mathbb{R}^3} u \bar{v}\, dx dz.$$

Note that (1.10) is still a model in *three* spatial dimensions. Except for the nonlinear averaging operator $F_{\mathrm{av}}(\phi)$, however, the variable z only enters as a parameter and hence the linear part of the dynamics with respect to z is, in fact, trivial. This allows to derive from (1.10) an *effective two-dimensional* limiting model, provided the initial data is *polarized* on a single mode of $\mathcal{H}_z$, see Corollary 1.2 below.

One should also note that in (1.10) there is a non-trivial effect due the presence of the rotation. Indeed, in the case where the rotation axis is not parallel to the z-axis, i.e., if $\Omega_1, \Omega_2 \neq 0$, a repulsive quadratic potential is present in the limiting model (see also the discussion at the beginning of Section 2.1 below). The reason for this effect becomes apparent from the scaling of equation (1.2), which includes a rotation term of order $O(\varepsilon^{-1})$. The latter becomes large in the limit of strong confinement $\varepsilon \to 0$, resulting in an effective centrifugal force counteracting the original trap. In the physics literature, it seems that it is almost always assumed that the rotation axis is equal to the z-axis, and hence, this effect is almost never considered. We finally remark that, at least formally, a second order averaging procedure (similar to [5, 11]) can be used to derive (1.10) from (1.2) directly. In order to make this procedure rigorous, though, uniform (in ε) energy estimates are needed which seem to be rather difficult to obtain on the level of (1.2) (given its singular scaling). Thus, instead of working with (1.2) directly, we use the change of variables (1.3) which yields the same effect as the second order averaging and also allows us to use the better behaved model (1.4).

Our main result is the following theorem.

Theorem 1.1. *Let $1 \leq \sigma < 2$ and $\psi_0 \in \Sigma^2$. Then the following holds.*
(i) *The limit model* (1.10) *admits a unique maximal solution $\phi \in C([0, T_{\max}), \Sigma^2) \cap C^1([0, T_{\max}), L^2)$, with $T_{\max} \in (0, +\infty]$, such that for all $t \in [0, T_{\max})$:*

$$\|\phi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad E(\phi(t)) = E(\psi_0), \quad \langle \mathcal{H}_z \phi(t), \phi(t)\rangle_{L^2} = \langle \mathcal{H}_z \psi_0, \psi_0\rangle_{L^2}.$$

Moreover, we have the blow-up alternative:

$$\text{if} \quad T_{\max} < +\infty \quad \text{then} \quad \lim_{t \to T_{\max}} \|\nabla_x \phi(t)\|_{L^2} = +\infty.$$

(ii) *For all $T \in (0, T_{\max})$, there exists $\varepsilon_T > 0$, $C_T > 0$ such that, for all $\varepsilon \in (0, \varepsilon_T]$,* (1.2) *admits a unique solution $\psi^\varepsilon \in C([0,T], \Sigma^2) \cap C^1([0,T], L^2)$, which is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_T]$ in $L^\infty((0,T), \Sigma^2)$ and satisfies the error bound*

$$\max_{t \in [0,T]} \left\| \psi^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2} \phi(t) \right\|_{L^2} \leq C_T\, \varepsilon.$$

As an immediate corollary we have:

Corollary 1.2. *Denote by $(\chi_n, \lambda_n)_{n\in\mathbb{N}}$ the n-th eigenfunction/eigenvalue-pair of the one-dimensional harmonic oscillator $\mathcal{H}_z$. Assume that $\psi_0 \in \Sigma^2$ is such that*

$$\psi_0(x, z) = \varphi_0(x)\chi_n(z).$$

Then for all $T \in (0, T_{\max})$ we have

$$\max_{t \in [0,T]} \left\| \psi^\varepsilon(t) - e^{-it\lambda_n/\varepsilon^2} \varphi(t)\chi_n \right\|_{L^2} \leq C_T\, \varepsilon,$$

where $\varphi(t, x)$ solves the effective two-dimensional model

$$i\partial_t \varphi = -\frac{1}{2}\Delta_x \varphi + \frac{1}{2}\left(|x|^2 - (\Omega_2 x_1 - \Omega_1 x_2)^2\right)\varphi - \Omega_z L_z \varphi + \kappa_n |\varphi|^{2\sigma}\varphi \tag{1.11}$$

with $\varphi(t = 0, x) = \varphi_0(x_1, x_2)$ and

$$\kappa_n := \lambda \int_{\mathbb{R}} |\chi_n(z)|^{2\sigma+2} dz,$$

the effective nonlinear coupling constant in the n-th energy band.

This result follows from Theorem 1.1 (ii) and the fact that (1.10) preserves the initial polarization, i.e., admits solutions of the form $\phi(t,x,z) = \varphi(t,x)\chi_n(z)$. To see this, recall that the eigenfunctions $\{\chi_m\}_{L^2}$ form an orthonormal basis of $L^2(\mathbb{R})$. Using this we can write

$$e^{-it\mathcal{H}_z} f(z) = \sum_{m\in\mathbb{N}} e^{-it\lambda_m} \chi_m(z) \langle \chi_m, f\rangle_{L^2}$$

and hence (1.9) implies

$$F_{\rm av}(\varphi\chi_n) = \frac{1}{2\pi} |\varphi|^{2\sigma}\varphi \sum_{m\in\mathbb{N}} \int_0^{2\pi} e^{i\theta(\lambda_m-\lambda_n)} d\theta\, \chi_m \langle \chi_m, |\chi_n|^{2\sigma}\chi_n\rangle_{L^2}.$$

However, $\lambda_m - \lambda_n \in \mathbb{N}$ and thus, this integral is identically zero unless $m = n$, for which it is equal to 2π. Thus, the whole sum collapses to one term only and we consequently obtain that in the case of polarized solutions, (1.10) reduces to (1.11). The latter is an effective two-dimensional model describing the degrees of freedom in the unconstrained direction.

The paper is organized as follows: In Section 2 we shall, as a first step, establish well-posedness of the Cauchy problem corresponding to both the three dimensional NLS (1.2) and the averaged limiting model (1.10). Once this is done, rigorous error estimates between the exact and the approximate solution will be established in Section 3. In there, we shall also indicate how to obtain an improved error estimate, provided ψ^ε satisfies sufficiently strong regularity assumptions. Finally, in Section 4 we shall show how to adapt our results to the situation with strong confinement in two spatial dimensions, and derive the associated limiting model.

2. Analysis of the Cauchy problems

In this section we shall prove local and global well-posedness results for equation (1.2), i.e., the original NLS in $d = 3$ dimensions, and for the formal limiting model (1.10). The analysis of the former is relatively standard and follows along the lines of [2]. We shall therefore only sketch the main ideas and rather focus on the Cauchy problem corresponding to (1.10).

2.1. The Cauchy problem corresponding to the averaged NLS model.

In this subsection we shall analyze the Cauchy problem

$$i\partial_t\phi = -\frac{1}{2}\Delta_x\phi + \frac{1}{2}\left(|x|^2 - (\Omega_2 x_1 - \Omega_1 x_2)^2\right)\phi - \Omega_z L_z\phi + \lambda F_{\rm av}(\phi) \tag{2.1}$$

with some general initial data $\phi(t=0) = \phi_0 \in \Sigma^1$. Recall that ϕ depends on the space variables $x \in \mathbb{R}^2$ and $z \in \mathbb{R}$, but in this problem dispersive effects *only* occur in the x direction (due to the lack of a second order derivative in z). The basic existence proof therefore requires several changes from the standard approach.

To this end, let us first derive Strichartz estimates adapted to the situation at hand. We recall that, in dimension *two*, a pair (q,r) is said to be admissible if $2 \le r < \infty$ and

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{r}.$$

We denote by $U(t)=e^{itH}$ the strongly continuous group of unitary operators generated by the Hamiltonian

$$H=-\frac{1}{2}\Delta_x+\frac{1}{2}\left(|x|^2-(\Omega_2x_1-\Omega_1x_2)^2\right)-\Omega_zL_z. \tag{2.2}$$

This operator can be seen as a special case of the Weyl-quantization of a (real-valued) second order polynomial $H(x,\xi)$. It is thus essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)\subset\Sigma^1\equiv\{f\in H^1(\mathbb{R}^2):|x|f\in L^2(\mathbb{R}^2)\}$, cf. [15]. In the case without rotation $\Omega_z=0$, H is of the form of an anisotropic harmonic oscillator with potential

$$V(x)=\frac{1-\Omega_2^2}{2}\,x_1^2+\frac{1-\Omega_1^2}{2}\,x_2^2-\Omega_1\Omega_2x_1x_2. \tag{2.3}$$

Clearly, this potential becomes repulsive if $\Omega_1,\Omega_2>1$. Physically speaking this results in a loss of confinement, and thus, the destruction of the condensate. On the other hand, by means of Young's inequality, one easily sees that a sufficient condition for confinement, i.e., $V(x)\to+\infty$ as $|x|\to\infty$, is

$$\Omega_1^2+\Omega_2^2<1.$$

In this case H has only pure point spectrum with no finite accumulation points. Similarly, in the case with rotation $\Omega_z\neq0$ the operator H remains confining provided Ω_z is sufficiently small (with respect to Ω_1,Ω_2). This can be seen by rewriting (2.2) in the form of a magnetic Schrödinger operator

$$H=\frac{1}{2}\big(\nabla+A(x)\big)^2+V(x)-\frac{1}{2}\Omega_z^2|x^\perp|^2$$

where V is as before and $A(x)=\Omega_zx^\perp$ with $x^\perp=(x_2,-x_1)$. In this form, the effect of the rotation term L_z has been split into Coriolis and centrifugal forces. The latter is seen to act as a repulsive quadratic potential, counteracting the confinement. Depending on the size of $\Omega_1,\Omega_2,\Omega_z$ we thus might have de-confinement due to the combined effects of the rotation and the strong confinement. This also has an influence on the question of global existence of solutions to the NLS, see the Remark 2.4 below.

Lemma 2.1 (Vectorial Strichartz estimates)**.** *There exists $\delta>0$ such that the following properties hold true.*
(i) For any admissible pair (q,r), there exists C such that, for all $\phi\in L^2(\mathbb{R}^3)$,

$$\|U(t)\phi\|_{L_t^qL_x^rL_z^2}\leq C\|\phi\|_{L^2}\,, \tag{2.4}$$

where $L_t^qL_x^rL_z^2$ stands for $L_t^q((-\delta,\delta),L_x^r(\mathbb{R}^2,L_z^2(\mathbb{R})))$.
(ii) For any admissible pairs (q,r) and (γ,ρ), there exists C such that, for all $f=f(t,x,z)$,

$$\left\|\int_{(-\delta,\delta)\cap\{s\leq t\}}U(t-s)f(s)ds\right\|_{L_t^qL_x^rL_z^2}\leq C\|f\|_{L_t^{\gamma'}L_x^{\rho'}L_z^2}\,. \tag{2.5}$$

Proof. This lemma relies on the usual Strichartz estimates for the group $U(t)$ acting on functions depending only on x and on Minkowski's inequality. The existence of Strichartz estimates for $U(t)$ thereby follows from the results in [15] which can be directly applied to the Hamiltonian given in (2.2). The fact that H in general will have eigenvalues, implies that dispersive effects will only be present for small $|t|<\delta$, preventing the existence of global-in-time Strichartz estimates, cf. [8] for a more detailed discussion on this issue.

Let us introduce a Hilbert basis $(e_p)_{p\in\mathbb{N}}$ of $L^2_z(\mathbb{R})$. Decomposing the function $\phi \in L^2(\mathbb{R}^3)$ as $\phi(x,z) = \sum_p \phi_p(x)e_p(z)$, one obtains:

$$\begin{aligned}
\|U(t)\phi\|^2_{L^q_tL^r_xL^2_z} &= \left\|\left\|\|U(t)\phi\|_{L^2_z}\right\|_{L^r_x}\right\|^2_{L^q_t} = \left\|\left\|\|U(t)\phi\|^2_{L^2_z}\right\|_{L^{r/2}_x}\right\|_{L^{q/2}_t} \\
&= \left\|\left\|\sum |U(t)\phi_p|^2\right\|_{L^{r/2}_x}\right\|_{L^{q/2}_t} \\
&\le \left\|\sum \|U(t)\phi_p\|^2_{L^r_x}\right\|_{L^{q/2}_t} \\
&\le \sum \left\|\|U(t)\phi_p\|_{L^r_x}\right\|^2_{L^q_t} \\
&\le C\sum \|\phi_p\|^2_{L^2_x} = C\|\phi\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Here, we have used twice the Minkowski's inequality [13] in the third line (notice that $q/2 \ge 1$ and $r/2 \ge 1$) and the usual Strichartz estimate for each $U(t)\phi_p$ in the fourth line. This proves (2.4).

Let us prove (2.5): For $f(t,x,z) = \sum_p f_p(t,x)e_p(z)$, denoting

$$g_j(t,x) = \int_{(-\delta,\delta)\cap\{s\le t\}} U(t-s)f_j(s)ds,$$

we estimate similarly (we have again $q/2 > 1$ and $r/2 \ge 1$)

$$\begin{aligned}
\left\|\int_{(-\delta,\delta)\cap\{s\le t\}} U(t-s)f(s)ds\right\|^2_{L^q_tL^r_xL^2_z} &= \left\|\left\|\left\|\sum g_pe_p\right\|^2_{L^2_z}\right\|_{L^{r/2}_x}\right\|_{L^{q/2}_t} \\
&= \left\|\left\|\sum |g_p|^2\right\|_{L^{r/2}_x}\right\|_{L^{q/2}_t} \\
&\le \left\|\sum \|g_p\|^2_{L^r_x}\right\|_{L^{q/2}_t} \le \sum \left\|\|g_p\|_{L^r_x}\right\|^2_{L^q_t} \\
&\le C\sum \|f_p\|^2_{L^{\gamma'}_tL^{\rho'}_x} = C\sum \left\|\left\||f_p|^2\right\|_{L^{\rho'/2}}\right\|_{L^{\gamma'/2}_t} \\
\le C\left\|\sum \left\||f_p|^2\right\|_{L^{\rho'/2}}\right\|_{L^{\gamma'/2}_t} &\le C\left\|\left\|\sum |f_p|^2\right\|_{L^{\rho'/2}}\right\|_{L^{\gamma'/2}_t} = C\|f\|^2_{L^{\gamma'}_tL^{\rho'}_xL^2_z}.
\end{aligned}$$

Here, we have used, in the fourth line, the Strichartz inequality for each g_j and, in the fifty line, the reverse Minkowski's inequality [13] (note that we have necessarily $\gamma'/2 < 1$ and $\rho'/2 < 1$). The proof of Lemma 2.1 is complete. □

Remark 2.2. *We note that the existence of anistropic Strichartz estimates has also been established in other works, cf.* [20, 1]. *In particular,* [1] *showed that in the case of partial harmonic confinement one can bypass the spectral decomposition used above.*

Proposition 2.3. *Let $\phi_0 \in \Sigma^1$. Then there exists $T_{\max} \in (0,+\infty]$ such that* (2.1) *admits a unique maximal solution $\phi \in C([0,T_{\max}),\Sigma^1)$, in the sense that*

$$\textit{if}\quad T_{\max} < +\infty \quad \textit{then} \quad \lim_{t\to T_{\max}} \|\nabla_x\psi^\varepsilon(t)\|_{L^2} = +\infty.$$

Moreover, the following conservation laws hold

$$\|\phi(t)\|_{L^2} = \|\phi_0\|_{L^2}, \quad E(\phi(t)) = E(\phi_0), \quad \|\mathcal{H}_z^{1/2}\phi(t)\|_{L^2} = \|\mathcal{H}_z^{1/2}\phi_0\|_{L^2}.$$

Furthermore, if $\phi_0 \in \Sigma^2$*, then* $\phi \in C([0, T_{\max}), \Sigma^2) \cap C^1([0, T_{\max}), L^2)$.

Note that for $\phi(t) \in \Sigma^2$ we have the strong form of the conservation law, i.e.,

$$\langle \mathcal{H}_z\phi(t), \phi(t)\rangle_{L^2} = \langle \mathcal{H}_z\phi_0, \phi_0\rangle_{L^2}.$$

Proof. Step 1: uniqueness in Σ^1. Let $u \in L^\infty((0,T), \Sigma^1)$, $\widetilde{u} \in L^\infty((0,T), \Sigma^1)$ be two weak solutions of (2.1), where, without loss of generality, we can assume that $0 < T \le \delta$. From (2.1), it follows that

$$(u - \widetilde{u})(t) = i\lambda \int_0^t U(t-s)\left(F_{\rm av}(u) - F_{\rm av}(\widetilde{u})\right) ds$$

and then, by (2.5), for any admissible pairs (q,r) and (γ, ρ),

$$\|u - \widetilde{u}\|_{L^q_t L^r_x L^2_z} \le C \left\|F_{\rm av}(u) - F_{\rm av}(\widetilde{u})\right\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}.$$

Let us choose $r > 2$ and $\rho > 2$ such that $\frac{1}{r} + \frac{1}{\rho} = 1 - \frac{\sigma}{2}$, where we recall that $1 \le \sigma < 2$. Then, denoting $v = e^{i\theta\mathcal{H}_z}u$ and $\widetilde{v} = e^{i\theta\mathcal{H}_z}\widetilde{u}$, one gets by Hölder,

$$\begin{aligned}
\left\|F_{\rm av}(u) - F_{\rm av}(\widetilde{u})\right\|_{L^{\rho'}_x L^2_z} &\le \frac{1}{2\pi}\int_0^{2\pi} \left\||v|^{2\sigma}v - |\widetilde{v}|^{2\sigma}\widetilde{v}\right\|_{L^{\rho'}_x L^2_z} d\theta \\
&\le C\int_0^{2\pi} \left\|(|v|^{2\sigma} + |\widetilde{v}|^{2\sigma})|v - \widetilde{v}|\right\|_{L^{\rho'}_x L^2_z} d\theta \\
&\le C\int_0^{2\pi} (\|v\|^{2\sigma}_{L^4_x L^\infty_z} + \|\widetilde{v}\|^{2\sigma}_{L^4_x L^\infty_z})\|v - \widetilde{v}\|_{L^r_x L^2_z} d\theta \\
&\le C\int_0^{2\pi} (\|v\|^{2\sigma}_{\Sigma^1} + \|\widetilde{v}\|^{2\sigma}_{\Sigma^1})\|v - \widetilde{v}\|_{L^r_x L^2_z} d\theta \\
&= C\big(\|u\|^{2\sigma}_{\Sigma^1} + \|\widetilde{u}\|^{2\sigma}_{\Sigma^1}\big)\|u - \widetilde{u}\|_{L^r_x L^2_z}.
\end{aligned}$$

Here, we have used the unitarity of $e^{i\theta\mathcal{H}_z}$ in L^2_z and Σ^1, and the embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^4_x(\mathbb{R}^2, L^\infty_z(\mathbb{R}))$ and $H^1(\mathbb{R}^3) \hookrightarrow L^r_x(\mathbb{R}^2, L^2_z(\mathbb{R}))$ (see the Appendix of [7]). Since u and $\widetilde{u}$ belong to $L^\infty((0,T), \Sigma^1)$, this yields

$$\|u - \widetilde{u}\|_{L^q_t L^r_x L^2_z} \le C\|u - \widetilde{u}\|_{L^{\gamma'}_t L^r_x L^2_z}.$$

Since $\gamma' < 2 < q$, this inequality is enough to conclude that $u = \widetilde{u}$, see Lemma 4.2.2 in [10].

Step 2: local existence. Let us adapt to (2.1) the proof of well-posedness of NLS with a quadratic potential of [8], using Kato's strategy (see for instance [10]) and the vectorial Strichartz estimates given in Lemma 2.1. The main technical difficulty here is the fact that we are working with an NLS in three spatial dimensions, but we can only utilize the dispersive properties of the two-dimensional Schrödinger group $U(t) = e^{itH}$. In order to remedy this, an important ingredient will be the anisotropic Sobolev imbeddings proved in [7].

With this in mind, local in-time existence for solutions in $\phi(t) \in \Sigma^1$ can be proved by means of a fixed point theorem applied to Duhamel's representation of (1.10), i.e.,

$$\phi(t) = U(t)\phi_0 - i\lambda \int_0^t U(t-s)F_{\rm av}(\phi)(s)ds =: \Phi(\phi)(t),$$

where, as before, $U(t) = e^{itH}$ and H is given by (2.2). We want to show that for $\phi_0 \in \Sigma^1$ and sufficiently small $T > 0$, Φ is a contraction mapping in the complete metric space

$$\begin{aligned} X_{T,M} = \big\{\psi \in C([0,T];\Sigma^1) \ : \ \psi, \mathbf{x}\psi, \nabla\psi \in L^q_t([0,T]; L^r_x(\mathbb{R}^2; L^2_z(\mathbb{R}))), \\ \|\psi\|_{L^\infty_t(\Sigma^1)} + \|\psi\|_{L^q_t L^r_x L^2_z} + \|\mathbf{x}\psi\|_{L^q_t L^r_x L^2_z} + \|\nabla\psi\|_{L^q_t L^r_x L^2_z} \leq M\big\}, \end{aligned}$$

equipped with the distance

$$\mathrm{d}(u,v) = \|u-v\|_{L^\infty_t L^2_x L^2_z} + \|u-v\|_{L^q_t L^r_x L^2_z}.$$

The real numbers r, q, ρ, γ are taken as in Step 1 above and T, M are to be chosen later. To prove that $X_{T,M}$ is stable by Φ, one first checks that the commutator

$$[\partial_z, H] = [z, H] = 0,$$

whereas

$$[x, H] = \nabla_x - i\Omega_z x^\perp,$$

with $x^\perp = (x_1, -x_2)$. Similarly, we find

$$[\nabla_x, H] = \nabla_x V + i\Omega_z \nabla_x^\perp$$

where V is as in (2.3), and hence $\nabla_x V$ is in fact linear in x. We consequently obtain that the combination of ϕ, $\mathbf{x}\phi$ and $\nabla\phi$ form a closed coupled system of equations. Therefore, by applying the operators $\mathbf{x}$ and ∇ to (2.1) and by using Lemma 2.1, we obtain for all $\phi \in X_{T,M}$,

$$\begin{aligned} &\|\Phi(\phi)\|_{L^\infty_t(\Sigma^1)} + \|\Phi(\phi)\|_{L^q_t L^r_x L^2_z} + \|\mathbf{x}\Phi(\phi)\|_{L^q_t L^r_x L^2_z} + \|\nabla\Phi(\phi)\|_{L^q_t L^r_x L^2_z} \\ &\quad\lesssim \|\phi_0\|_{L^2} + \|\mathbf{x}\phi_0\|_{L^2} + \|\nabla\phi_0\|_{L^2} + \|x\phi\|_{L^1_t L^2_x L^2_z} + \|\nabla_x\phi\|_{L^1_t L^2_x L^2_z} \\ &\qquad + \|F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z} + \|\mathbf{x}F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z} + \|\nabla F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}, \\ &\quad\lesssim \|\phi_0\|_{\Sigma^1} + TM + \|F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z} + \|\mathbf{x}F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z} + \|\nabla F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z} \end{aligned}$$

where $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 0$. Denoting $v = e^{i\theta\mathcal{H}_z}\phi$, one gets

$$\begin{aligned} \|\nabla_x F_{\rm av}(\phi)\|_{L^{\rho'}_x L^2_z} &\leq \frac{1}{2\pi}\int_0^{2\pi} \big\||v|^{2\sigma}\nabla_x v\big\|_{L^{\rho'}_x L^2_z} d\theta \\ &\lesssim \int_0^{2\pi} \|v\|^{2\sigma}_{L^4_x L^\infty_z}\|\nabla_x v\|_{L^r_x L^2_z} d\theta \\ &\lesssim \int_0^{2\pi} \|v\|^{2\sigma}_{\Sigma^1}\|\nabla_x v\|_{L^r_x L^2_z} d\theta = 2\pi\|\phi\|^{2\sigma}_{\Sigma^1}\|\nabla_x\phi\|_{L^r_x L^2_z} \end{aligned}$$

where we have used the fact that $e^{i\theta\mathcal{H}_z}$ is unitary in L^2_z and Σ^1, together with a Hölder estimate and the embedding $H^1(\mathbb{R}^3)\hookrightarrow L^4_x(\mathbb{R}^2,L^\infty_z(\mathbb{R}))$. Hence,

$$\|\nabla_x F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}\lesssim M^{2\sigma}\|\nabla_x\phi\|_{L^{\gamma'}_t L^r_x L^2_z}\leq T^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma}\|\nabla_x\phi\|_{L^q_t L^r_x L^2_z}$$
$$\leq T^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma+1}.$$

Similarly, we obtain

$$\|xF_{\rm av}(u)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}\lesssim T^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma+1}.$$

To estimate $zF_{\rm av}(u)$ and $\partial_z F_{\rm av}(u)$, we use (1.6) several times:

$$\begin{aligned}
&\|zF_{\rm av}(\phi)\|_{L^{\rho'}_x L^2_z}+\|\partial_z F_{\rm av}(\phi)\|_{L^{\rho'}_x L^2_z}\\
&\lesssim\left\|\mathcal{H}_z^{1/2}\int_0^{2\pi}e^{i\theta\mathcal{H}_z}\left(|v|^{2\sigma}v\right)d\theta\right\|_{L^{\rho'}_x L^2_z}=\left\|\int_0^{2\pi}e^{i\theta\mathcal{H}_z}\mathcal{H}_z^{1/2}\left(|v|^{2\sigma}v\right)d\theta\right\|_{L^{\rho'}_x L^2_z}\\
&\lesssim\int_0^{2\pi}\left\|\mathcal{H}_z^{1/2}\left(|v|^{2\sigma}v\right)\right\|_{L^{\rho'}_x L^2_z}d\theta\lesssim\int_0^{2\pi}\left\||v|^{2\sigma}\left(|zv|+|\partial_z v|\right)\right\|_{L^{\rho'}_x L^2_z}d\theta\\
&\lesssim\int_0^{2\pi}\|v\|^{2\sigma}_{L^4_x L^\infty_z}\left(\|zv\|_{L^r_x L^2_z}+\|\partial_z v\|_{L^r_x L^2_z}\right)d\theta\\
&\lesssim\int_0^{2\pi}\|v\|^{2\sigma}_{\Sigma^1}\|\mathcal{H}_z^{1/2}v\|_{L^r_x L^2_z}d\theta=2\pi\|\phi\|^{2\sigma}_{\Sigma^1}\|\mathcal{H}_z^{1/2}\phi\|_{L^r_x L^2_z}\\
&\lesssim\|\phi\|^{2\sigma}_{\Sigma^1}\left(\|z\phi\|_{L^r_x L^2_z}+\|\partial_z\phi\|_{L^r_x L^2_z}\right),
\end{aligned}$$

which yields, again

$$\|zF_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}+\|\partial_z F_{\rm av}(\phi)\|_{L^{\gamma'}_t L^{\rho'}_x L^2_z}\lesssim T^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma+1}.$$

Finally, we have proved that

$$\begin{aligned}
&\|\Phi(\phi)\|_{L^\infty_t(\Sigma^1)}+\|\Phi(\phi)\|_{L^q_t L^r_x L^2_z}+\|\mathbf{x}\Phi(\phi)\|_{L^q_t L^r_x L^2_z}+\|\nabla\Phi(\phi)\|_{L^q_t L^r_x L^2_z}\\
&\leq C\|\phi_0\|_{\Sigma^1}+CTM+CT^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma+1}.
\end{aligned}$$

We now set

$$M=2C\|\phi_0\|_{\Sigma^1}$$

and choose T small enough so that

$$CTM+CT^{\frac{q-\gamma'}{q\gamma'}}M^{2\sigma+1}\leq\frac{M}{2}.$$

It follows that $\Phi(\phi)\in X_{T,M}$. The contraction property can then be proved by following the same lines as the proof of uniqueness in Step 1: it can be proved that, for T small enough, we have

$$\mathrm{d}(\Phi(\phi),\Phi(\widetilde{\phi}))\leq\frac{1}{2}\mathrm{d}(\phi,\widetilde{\phi})$$

for all $\phi,\widetilde{\phi}\in X_{T,M}$. Hence, by Banach's fixed point theorem, Φ has a unique fixed point, which is a mild solution of (2.1).

Step 3: blow-up alternative. From the uniqueness result and from the fact that the existence time in Step 1 only depends on $\|\phi_0\|_{\Sigma^1}$, one can define the maximal solution $\phi \in C([0, T_{\max}), \Sigma^1)$ and obtain a first blow-up alternative in terms of the whole Σ^1 norm:

$$\text{if} \quad T_{\max} < +\infty \quad \text{then} \quad \lim_{t \to T_{\max}} \|\phi(t)\|_{\Sigma^1} = +\infty.$$

Then, we compute from (2.1)

$$\frac{d}{dt}\|x\phi(t)\|_{L^2}^2 = 2\,\mathrm{Im}\int_{\mathbb{R}^3} x \cdot \nabla_x \phi(t,x)\,\overline{\phi}(t,x)\,dxdz \le \|x\phi(t)\|_{L^2}^2 + \|\nabla_x\phi(t)\|_{L^2}^2.$$

Hence, a bound on $\|\nabla_x \phi\|_{L^2}$ yields a bound on $\|x\phi(t)\|_{L^2}$ by the Gronwall lemma. Since the L^2 norm of ϕ is conserved, it is clear that $\lim_{t\to T_{\max}} \|\phi(t)\|_{\Sigma^1} = +\infty$ implies that $\lim_{t\to T_{\max}} \|\nabla\phi(t)\|_{L^2} = +\infty$. We have proved the blow-up alternative as it is stated in the Proposition.

Step 4: conservation laws. In order to prove the conservation laws stated above, it is enough to consider the case of local-in-time solutions $\phi(t)$ which are sufficiently smooth and decaying. By following a standard regularization procedure (as given in, e.g., [10]), this can then be extended to general $\phi(t) \in \Sigma^1$. Conservation of mass then follows from the fact that H is self-adjoint and hence

$$\frac{d}{dt}\|\phi(t)\|_{L^2}^2 = \mathrm{Re}\left(\frac{2}{i}\langle H\phi + \lambda F_{\mathrm{av}}(\phi), \phi\rangle_{L^2}\right) = 2\lambda\,\mathrm{Im}\langle F_{\mathrm{av}}(\phi), \phi\rangle_{L^2}.$$

However,

$$\begin{aligned}
\mathrm{Im}\langle F_{\mathrm{av}}(\phi), \phi\rangle_{L^2} &= \frac{1}{2\pi}\,\mathrm{Im}\left\langle \int_0^{2\pi} e^{i\theta\mathcal{H}_z}\left(\left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma} e^{-i\theta\mathcal{H}_z}\phi\right) d\theta, \phi\right\rangle_{L^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}\left\langle \left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma} e^{-i\theta\mathcal{H}_z}\phi, e^{-i\theta\mathcal{H}_z}\phi\right\rangle_{L^2} d\theta = 0,
\end{aligned}$$

which implies $\|\phi(t)\|_{L^2}^2 = \|\phi_0\|_{L^2}^2$.

A similar argument can be used to prove that $\|\mathcal{H}_z^{1/2}\phi(t)\|_{L^2} = \|\mathcal{H}_z^{1/2}\phi_0\|_{L^2}$, having in mind that both $\mathcal{H}_z$ and H are self-adjoint, and that the following commutation relations hold: $[\mathcal{H}_z^{1/2}, H] = 0$ as well as $[\mathcal{H}_z^{1/2}, e^{i\theta\mathcal{H}_z}] = 0$, see [6].

Finally, in order to prove the conservation of the energy it is useful to first note that $E(\phi)$ is formally equal to

$$E(\phi) = \langle H\phi, \phi\rangle_{L^2} + \lambda\int_{\mathbb{R}^3} N_{\mathrm{av}}(\phi)dx\,dz,$$

where

$$N_{\mathrm{av}}(\phi) := \frac{1}{2\pi(\sigma+1)}\int_0^{2\pi}\left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma+2} d\theta.$$

We then compute

$$\begin{aligned}
\frac{d}{dt}\langle H\phi, \phi\rangle_{L^2} &= 2\,\mathrm{Im}\langle F_{\mathrm{av}}(\phi), H\phi\rangle_{L^2} = 2\,\mathrm{Im}\langle F_{\mathrm{av}}(\phi), i\partial_t\phi - \lambda F_{\mathrm{av}}(\phi)\rangle_{L^2} \\
&= -2\lambda\,\mathrm{Re}\langle F_{\mathrm{av}}(\phi), \partial_t\phi\rangle_{L^2}
\end{aligned}$$

From here, it follows that $E(\phi)$ is conserved provided that

$$2\,\mathrm{Re}\langle F_{\mathrm{av}}(\phi), \partial_t\phi\rangle_{L^2} = \frac{d}{dt}\int_{\mathbb{R}^3} N_{\mathrm{av}}(\phi)dx\,dz.$$

This, however, can be seen by a direct computation, using the definition of $F_{\rm av}$ and $N_{\rm av}$, i.e.,

$$\begin{aligned}\mathrm{Re}\langle F_{\rm av}(\phi), \partial_t \phi\rangle_{L^2} &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left\langle \left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma} e^{-i\theta\mathcal{H}_z}\phi, \partial_t\left(e^{-i\theta\mathcal{H}_z}\phi\right)\right\rangle_{L^2} d\theta\\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_{\mathbb{R}^3}\left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma}\partial_t\left|e^{-i\theta\mathcal{H}_z}\phi\right|^2 d\theta\, dx\, dz\\ &= \frac{1}{4\pi(\sigma+1)}\frac{d}{dt}\int_0^{2\pi}\int_{\mathbb{R}^3}\left|e^{-i\theta\mathcal{H}_z}\phi\right|^{2\sigma+2} d\theta\, dx\, dz.\end{aligned}$$

Step 5: Σ^2 regularity. Assume that $\phi_0 \in \Sigma^2$. Since Σ^2 is an algebra and $e^{i\theta\mathcal{H}_z}$ is unitary on Σ^2, it is easy to see that $F_{\rm av}$ is locally Lipschitz continuous on Σ^2 provided $\sigma \geq 1$. Hence, by a standard fixed-point technique, we can show the existence of a unique maximal solution $\phi \in C([0,T_1),\Sigma^2)\cap C^1([0,T_1),L^2)$, with $0 < T_1 \leq T_{\max}$. Let us prove that $T_1 = T_{\max}$ by contradiction. To this end, we assume that $T_1 < T_{\max}$ and consequently deduce that

$$\lim_{t\to T_1}\|\phi(t)\|_{\Sigma^2} = +\infty \quad \text{and} \quad \sup_{t\in[0,T_1]}\|\phi(t)\|_{\Sigma^1} < +\infty.$$

Therefore, it suffices to find $\tau < T_1$ such that $\|\phi(t)\|_{\Sigma^2}$ is bounded on the interval (τ, T_1) to have the desired contradiction. Let $\tau \geq 0$, to be fixed later, be such that $0 < T_1 - \tau < \delta$ (δ is defined in Lemma 2.1). By differentiating (2.1) with respect to time, we obtain that

$$\partial_t\phi(t) = U(t)\partial_t\phi(\tau) - i\lambda\int_\tau^t U(t-s)\partial_t F_{\rm av}(\phi)(s)ds.$$

Denoting $\psi = e^{-i\theta\mathcal{H}_z}\phi$, we compute now

$$\partial_t F_{\rm av}(\phi) = \frac{\sigma+1}{2\pi}\int_0^{2\pi} e^{i\theta\mathcal{H}_z}\left(|\psi|^{2\sigma}\partial_t\psi\right)d\theta + \frac{\sigma}{2\pi}\int_0^{2\pi} e^{i\theta\mathcal{H}_z}\left(|\psi|^{2\sigma-2}\psi^2\partial_t\overline{\psi}\right)d\theta$$

and, using the same admissible pairs (q,r) and (γ,ρ) as in Step 1, we estimate similarly

$$\|\partial_t F_{\rm av}(\phi)\|_{L_x^{\rho'}L_z^2} \leq C\|\phi\|_{\Sigma^1}^{2\sigma}\|\partial_t\phi\|_{L_x^r L_z^2}.$$

Using the Strichartz estimates (2.4) and (2.5), this yields

$$\|\partial_t\phi\|_{L_t^q L_x^r L_z^2} \leq C\|\partial_t\phi(\tau)\|_{L^2} + C\|\partial_t\phi\|_{L_t^{\gamma'}L_x^r L_z^2}$$

where the time integral is computed on the interval $I = (T_1-\tau, T_1)$. Since $\gamma' < q$, if $T_1 - \tau$ is small enough, this implies

$$\|\partial_t\phi\|_{L_t^q L_x^r L_z^2} \leq C\|\partial_t\phi(\tau)\|_{L^2}$$

and, applying again the Strichartz estimates, we get

$$\|\partial_t\phi(t)\|_{L_t^\infty L_{x,z}^2} \leq C\|\partial_t\phi(\tau)\|_{L^2} \leq C\|\phi(\tau)\|_{\Sigma^2}.$$

For the last inequality, we used (2.1) at $t = \tau$ and the following estimate deduced from a Sobolev inequality and from interpolation inequality

$$\|F_{\rm av}(u)\|_{L^2} \leq C\int_0^{2\pi}\|e^{-i\theta\mathcal{H}_z}u\|_{L^{4\sigma+2}}^{2\sigma+1}d\theta \leq C\|u\|_{H^s}^{2\sigma+1} \leq C\|u\|_{H^2}^{\alpha}\|u\|_{H^1}^{2\sigma+1-\alpha}, \quad (2.6)$$

with $s = \frac{3\sigma}{2\sigma+1}$ and $\alpha = \max(\sigma-1, 0) < 1$.

Now, we remark that, by integrations by parts, a direct calculation gives
$$\|L_z\phi\|^2 = -\langle x_1^2\phi, \partial_{x_2}^2\phi\rangle - \langle x_2^2\phi, \partial_{x_1}^2\phi\rangle + 2\,\mathrm{Re}\langle x_1x_2\phi, \partial_{x_1}\partial_{x_2}\phi\rangle - 2\|\phi\|_{L^2}^2$$
so, using directly (2.1), we estimate
$$\|\Delta_x\phi\|_{L^2} \le 2\|\partial_t\phi\|_{L^2} + C\||x|^2\phi\|_{L^2} + C\|L_z\phi\|_{L^2} + \|F_{\mathrm{av}}(\phi)\|_{L^2}$$
$$\le 2\|\partial_t\phi\|_{L^2} + C\||x|^2\phi\|_{L^2} + C\|\Delta_x\phi\|_{L^2}^{1/2}\||x|^2\phi\|_{L^2}^{1/2} + C\|\phi\|_{L^2} + C\|\phi\|_{H^2}^{\alpha}\|\phi\|_{H^1}^{2\sigma+1-\alpha}$$
and then, using the bounds of $\|\phi\|_{\Sigma^1}$ and $\|\partial_t\phi\|_{L^2}$, we deduce that, for all $t \in I$
$$\|\Delta_x\phi(t)\|_{L^2} \le C + C\|\phi(\tau)\|_{\Sigma^2} + C\||x|^2\phi(t)\|_{L^2}. \tag{2.7}$$
Next we can proceed similarly as above to estimate $\||x|^2\phi(t)\|_{L^2}$. We have
$$|x|^2\phi(t) = U(t)(|x|^2\phi(\tau)) - i\int_\tau^t U(t-s)\left(2\phi + 2x\cdot\nabla_x\phi + \lambda|x|^2F_{\mathrm{av}}(\phi)\right)(s)ds.$$
Hence, using that
$$\||x|^2F_{\mathrm{av}}(\phi)\|_{L_x^{\rho'}L_z^2} \le C\|\phi\|_{\Sigma^1}^{2\sigma}\||x|^2\phi\|_{L_x^rL_z^2}$$
and that, by (2.7) and (1.7),
$$\|\Delta_x\phi + x\cdot\nabla_x\phi\|_{L^2} \le C + C\|\phi(\tau)\|_{\Sigma^2} + C\||x|^2\phi\|_{L_x^2L_z^2},$$
we get by Strichartz estimates
$$\||x|^2\phi\|_{L_t^\infty L_x^2L_z^2} + \||x|^2\phi\|_{L_t^qL_x^rL_z^2} \le C\Big(1 + \|\phi(\tau)\|_{\Sigma^2} + \||x|^2\phi\|_{L_t^1L_x^2L_z^2} + \||x|^2\phi\|_{L_t^{\gamma'}L_x^rL_z^2}\Big).$$
From $\gamma' < q$, it is easy to conclude that, for $T_1 - \tau$ small enough,
$$\||x|^2\phi\|_{L_t^\infty L_x^2L_z^2} + \||x|^2\phi\|_{L_t^qL_x^rL_z^2} \le C + C\|\phi(\tau)\|_{\Sigma^2} \tag{2.8}$$
and, with (2.7), that
$$\|\Delta_x\phi\|_{L_t^\infty L_{x,z}^2} \le C + C\|\phi(\tau)\|_{\Sigma^2}. \tag{2.9}$$
Finally, consider the equation satisfied by $\mathcal{H}_z\phi$. We have
$$\mathcal{H}_z\phi(t) = U(t)(\mathcal{H}_z\phi(\tau)) - i\lambda\int_\tau^t U(t-s)\mathcal{H}_zF_{\mathrm{av}}(\phi)(s)ds$$
and, denoting again $\psi = e^{-i\theta\mathcal{H}_z}\phi$, one computes
$$\|\mathcal{H}_zF_{\mathrm{av}}(\phi)\|_{L_z^2} \le C\int_0^{2\pi}\left\||\psi|^{2\sigma}|\partial_z^2\psi| + |\psi|^{2\sigma-1}|\partial_z\psi|^2 + |z|^2|\psi|^{2\sigma+1}\right\|_{L_z^2}d\theta$$
$$\le C\int_0^{2\pi}\left(\|\psi\|_{L_z^\infty}^{2\sigma}\|\partial_z^2\psi\|_{L_z^2} + \|\psi\|_{L_z^\infty}^{2\sigma-1}\|\partial_z\psi\|_{L_z^4}^2 + \|\psi\|_{L_z^\infty}^{2\sigma}\||z|^2\psi\|_{L_z^2}\right)d\theta$$
$$\le C\int_0^{2\pi}\|\psi\|_{L_z^\infty}^{2\sigma}\|\mathcal{H}_z\psi\|_{L_z^2}d\theta,$$
where we used the following Gagliardo-Nirenberg inequality in dimension 1:
$$\|\partial_zu\|_{L^4} \le C\|u\|_{H^2}^{1/2}\|u\|_{L^\infty}^{1/2} \le C\|\mathcal{H}_zu\|_{L^2}^{1/2}\|u\|_{L^\infty}^{1/2}.$$
Hence, as above, we get
$$\|\mathcal{H}_zF_{\mathrm{av}}(\phi)\|_{L_x^{\rho'}L_z^2} \le C\int_0^{2\pi}\|\psi\|_{\Sigma^1}^{2\sigma}\|\mathcal{H}_z\psi\|_{L_x^rL_z^2}d\theta = C\|\phi\|_{\Sigma^1}^{2\sigma}\|\mathcal{H}_z\phi\|_{L_x^rL_z^2},$$

which enables to conclude again with Strichartz inequalities that, for $T_1 - \tau$ small enough,

$$\|\mathcal{H}_z\phi\|_{L_t^\infty L_x^2 L_z^2} + \|\mathcal{H}_z\phi\|_{L_t^q L_x^r L_z^2} \leq C + C\|\phi(\tau)\|_{\Sigma^2}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), we deduce that $\|\phi(t)\|_{\Sigma^2}$ is uniformly bounded on the interval $I = (\tau, T_1)$: the proof is complete. □

Remark 2.4. *There are certainly situations for which $T_{\max} = +\infty$. In view of the discussion at the beginning of Section 2.1, this will be true, in particular, if $\Omega_1^2 + \Omega_2^2 < 1$ and the nonlinearity is defocusing $\lambda > 0$, since in this case, the results given in* [2] *apply. However, if the effective centrifugal force is too big, the resulting repulsive quadratic potential requires particular techniques, see* [9] *which would need to be combined with the effect of the rotation term.*

2.2. **The Cauchy problem for the NLS equation in 3D.** Before we can proceed to the proof of convergence for solutions ψ^ε of (1.2) as $\varepsilon \to 0$, we need, as a final preparatory step, a suitable existence result for the three-dimensional Cauchy problem (1.2)

Proposition 2.5. *Let $\psi_0 \in \Sigma^1$. Then, for any fixed $\varepsilon > 0$, there exists $T_1^\varepsilon \in (0, +\infty]$ such that* (1.2) *admits a unique maximal solution $\psi^\varepsilon \in C([0, T_1^\varepsilon), \Sigma^1)$, i.e.,*

$$\text{if} \quad T_1^\varepsilon < +\infty \quad \text{then} \quad \lim_{t \to T_1^\varepsilon} \|\nabla\psi^\varepsilon(t)\|_{L^2} = +\infty.$$

Furthermore, if $\psi_0 \in \Sigma^s$, $s > 1$, then $\psi^\varepsilon \in C([0, T_1^\varepsilon), \Sigma^s) \cap C^1([0, T_1^\varepsilon), L^2)$.

Proof. The proof follows along the same lines as the one for Proposition 2.3 above, i.e., through a fixed point argument for Duhamel's formula in a suitable metric space. Indeed, it is even easier, since (1.2) is a standard three-dimensional NLS equation with quadratic potential and rotation term. For such equations, the local and global existence theory in Σ^1, based on Strichartz estimates and the use of energy-methods, has been studied in detail in [2] (see also [8]). Additional smoothness for $\psi_0 \in \Sigma^s$ with $s > 1$, then follows by the same arguments as given in Step 5 in the proof of Proposition 2.3. □

One might be concerned that, as $\varepsilon \to 0$, the existence time $T_1^\varepsilon \to 0$, but our convergence proof below will show that this is indeed not the case. We finally, note that (1.2) admits the usual conservation laws for the mass and the total energy. The latter, however, is in general indefinite, due to the appearance of the angular momentum operator. Since, we shall not use any of these conservation laws in the following, we omit a more detailed discussion of these issues and refer the reader to [2].

3. Convergence proof and error estimates

This section is devoted to the proof of our main Theorem 1.1. Item (i) of this theorem is a consequence of Proposition 2.3. We prove Item (ii) in Subsection 3.2 after we have obtained some uniform estimates.

3.1. **Uniform estimates.** Let $0 < T < T_{\max}$. By Proposition 2.3, we know that the solution ϕ of (1.10) belongs to $C([0,T],\Sigma^2)\cap C^1([0,T],L^2)$. Using the Sobolev embedding $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ and the unitarity of $e^{-it\mathcal{H}_z/\varepsilon^2}$ in Σ^2, we have

$$\|e^{-it\mathcal{H}_z/\varepsilon^2}\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leq C\|e^{-it\mathcal{H}_z/\varepsilon^2}\phi\|_{L^\infty((0,T),\Sigma^2)} = C\|\phi\|_{L^\infty((0,T),\Sigma^2)} < +\infty,$$

so the following quantity is finite;

$$M := \sup_{\varepsilon>0}\|e^{-it\mathcal{H}_z/\varepsilon^2}\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)}. \tag{3.1}$$

Notice that, in particular, we have $\|\psi_0\|_{L^\infty} = \|\phi(t=0)\|_{L^\infty}\leq M$.

By Proposition 2.5, (1.2) admits a unique maximal solution $\psi^\varepsilon\in C([0,T_1^\varepsilon),\Sigma^2)\cap C^1([0,T_1^\varepsilon),L^2)$. We recall that the function u^ε defined by

$$u^\varepsilon = e^{-i\varepsilon z(\Omega_1x_2-\Omega_2x_1)}\psi^\varepsilon$$

satisfies (1.4). By (1.7), it is clear that, for all $t\in[0,T_1^\varepsilon)$,

$$(1-C_1\varepsilon)\|\psi^\varepsilon(t)\|_{\Sigma^2}\leq\|u^\varepsilon(t)\|_{\Sigma^2}\leq(1+C_2\varepsilon)\|\psi^\varepsilon(t)\|_{\Sigma^2}.$$

Moreover, since

$$e^{-i\varepsilon z(\Omega_1x_2-\Omega_2x_1)}-1 = -i\varepsilon z(\Omega_1x_2-\Omega_2x_1)\int_0^1 e^{-i\varepsilon z(\Omega_1x_2-\Omega_2x_1)s}\,ds,$$

we have also that

$$\|u^\varepsilon(t)-\psi^\varepsilon(t)\|_{L^2}\leq C\varepsilon\|u^\varepsilon(t)\|_{\Sigma^2}. \tag{3.2}$$

This will allow us to infer the desired approximation result for ψ^ε, once we have a sufficiently good estimate on the difference between $e^{it\mathcal{H}_z/\varepsilon^2}u^\varepsilon$ and the limit ϕ.

From a Gagliardo-Nirenberg inequality, we get

$$\|u_0^\varepsilon-\psi_0\|_{L^\infty}\leq C\|u_0^\varepsilon-\psi_0\|_{L^2}^{1/4}\|u_0^\varepsilon-\psi_0\|_{H^2}^{3/4}\leq C_1\varepsilon^{1/4},$$

Hence, for $\varepsilon<\varepsilon_1:=(M/2C_1)^4$, we have

$$\|u^\varepsilon(0)\|_{L^\infty}\leq\|u_0^\varepsilon-\psi_0\|_{L^\infty}+\|\psi_0\|_{L^\infty}<3M/2$$

and we can define

$$T^\varepsilon = \sup\left\{t\in[0,T_1^\varepsilon)\,:\,\text{for all } s\in[0,t],\ \|u^\varepsilon(s)\|_{L^\infty}\leq 2M\right\}. \tag{3.3}$$

Lemma 3.1. *There exists a constant C_M such that, for $0<\varepsilon<\varepsilon_1$ and for all $t\in[0,T^\varepsilon]$, we have*

$$\|u^\varepsilon(t)\|_{\Sigma^2}\leq C_M.$$

Proof. We first recall from (1.6), that

$$\|u\|_{\Sigma^2}^2\simeq\|\mathcal{H}_zu\|_{L^2}^2+\|\mathcal{H}_xu\|_{L^2}^2+\|u\|_{L^2}^2.$$

We will now derive suitable a priori estimates for these three parts of the Σ^2-norm. To this end, we first multiply (1.4) by $\overline{u^\varepsilon}$, integrate over $\mathbb{R}^3$, and take the real part of the resulting expression. This yields

$$\frac{d}{dt}\|u^\varepsilon\|_{L^2}^2=0,$$

i.e., the conservation of mass. For the other two parts of the Σ^2-norm, we first compute the commutation relations

$$[\mathcal{H}_z,\mathcal{H}_x]=[\mathcal{H}_z,L_z]=[\mathcal{H}_x,L_z]=[\mathcal{H}_z,\Omega_1x_1\pm\Omega_2x_2]=0,$$

(where in the third expression we have used that $\mathcal{H}_x$ is rotationally symmetric), as well as

$$[\mathcal{H}_z, z] = -\partial_z, \text{ and } [\mathcal{H}_z, z^2] = -(1 + 2z\partial_z).$$

Keeping these relations in mind, we can thus apply $\mathcal{H}_z$ to (1.4), commute, and, after multiplying by $\mathcal{H}_z\overline{u}$, integrate over $\mathbb{R}^3$. Taking the real part of the resulting expression, we obtain

$$\begin{aligned}\frac{d}{dt}\|\mathcal{H}_z u^\varepsilon\|_{L^2}^2 &= -3\varepsilon^2(\Omega_1^2+\Omega_2^2)\operatorname{Im}\langle \mathcal{H}_z u^\varepsilon, u^\varepsilon + 2z\partial_z u^\varepsilon\rangle_{L^2} \\ &\quad + 2\varepsilon \operatorname{Im}\langle \mathcal{H}_z u^\varepsilon, \Omega_z(\Omega_1 x_1 + \Omega_2 x_2)\partial_z u^\varepsilon - 2(\Omega_2\partial^2_{x_1,z}u^\varepsilon - \Omega_1\partial^2_{x_2,z}u^\varepsilon)\rangle_{L^2} \\ &\quad + 2\lambda \operatorname{Im}\langle \mathcal{H}_z u^\varepsilon, \mathcal{H}_z(|u^\varepsilon|^{2\sigma}u^\varepsilon)\rangle_{L^2}.\end{aligned}$$

After several integrations by parts and using Cauchy-Schwarz, this yields the following estimate

$$\begin{aligned}\frac{d}{dt}\|\mathcal{H}_z u^\varepsilon\|_{L^2}^2 \leq \varepsilon^2 C_1\left(\|\mathcal{H}_z u^\varepsilon\|_{L^2}^2 + \|u^\varepsilon\|_{L^2}^2\right) + \varepsilon C_2\left(\|\mathcal{H}_z u^\varepsilon\|_{L^2}^2 + \|\mathcal{H}_x u^\varepsilon\|_{L^2}^2\right) \\ + C_3|\lambda|\left(\|\mathcal{H}_z u^\varepsilon\|_{L^2}^2 + \|\mathcal{H}_z(|u^\varepsilon|^{2\sigma}u^\varepsilon)\|_{L^2}^2\right),\end{aligned}$$

where C_1, C_2, C_3 are some constants depending only on Ω_1, Ω_2, and Ω_z, but not on ε. Similarly, a computation shows

$$\begin{aligned}\frac{d}{dt}\|\mathcal{H}_x u^\varepsilon\|_{L^2}^2 =& \Omega_2^2 \operatorname{Im}\langle \mathcal{H}_x u^\varepsilon, u^\varepsilon + 2x_1\partial_{x_1}u^\varepsilon\rangle_{L^2} + \Omega_1^2 \operatorname{Im}\langle \mathcal{H}_x u^\varepsilon, u^\varepsilon + 2x_2\partial_{x_2}u^\varepsilon\rangle_{L^2} \\ &- 2\Omega_1\Omega_2 \operatorname{Im}\langle \mathcal{H}_x u^\varepsilon, x_2\partial_{x_1}u^\varepsilon + x_1\partial_{x_2}u^\varepsilon\rangle_{L^2} \\ &+ 2\varepsilon\Omega_z \operatorname{Im}\langle \mathcal{H}_x u^\varepsilon, z(\Omega_1\partial_{x_1}u^\varepsilon + \Omega_2\partial_{x_2}u^\varepsilon - (\Omega_1 x_2 - \Omega_2 x_1)u^\varepsilon)\rangle_{L^2} \\ &+ 2\lambda \operatorname{Im}\langle \mathcal{H}_x u^\varepsilon, \mathcal{H}_x(|u^\varepsilon|^{2\sigma}u^\varepsilon)\rangle_{L^2},\end{aligned}$$

and using again Cauchy-Schwarz yields the analogous estimate for $\|\mathcal{H}_x u^\varepsilon\|_{L^2}$. Combining the three estimates obtained above, allows us to write

$$\frac{d}{dt}\|u^\varepsilon\|_{\Sigma^2}^2 \leq K_1\|u^\varepsilon\|_{\Sigma^2}^2 + |\lambda|K_2\||u^\varepsilon|^{2\sigma}u^\varepsilon\|_{\Sigma^2}^2 \tag{3.4}$$

where $K_{1,2} = K_{1,2}(\varepsilon, \Omega_1, \Omega_2, \Omega_z) > 0$ are both bounded as $\varepsilon \to 0$. Now, we use the fact that, by Sobolev's imbedding,

$$\||u^\varepsilon|^{2\sigma}u^\varepsilon\|_{\Sigma^2} \leq C\|u^\varepsilon\|_{L^\infty}^{2\sigma}\|u^\varepsilon\|_{\Sigma^2} \leq CM^{2\sigma}\|u^\varepsilon\|_{\Sigma^2}$$

equation (3.4) implies

$$\frac{d}{dt}\|u^\varepsilon\|_{\Sigma^2}^2 \leq K_1\|u^\varepsilon\|_{\Sigma^2}^2 + |\lambda|K_3M^{4\sigma}\|u^\varepsilon\|_{\Sigma^2}^2.$$

Gronwall's lemma consequently implies that $\|u^\varepsilon\|_{\Sigma^2}$ stays bounded for all $t \in [0, T_\varepsilon]$. ∎

We note that an important consequence of this lemma is that

$$\text{if} \quad T^\varepsilon < +\infty \quad \text{then } T^\varepsilon < T_1^\varepsilon \quad \text{and} \quad \|u^\varepsilon(T^\varepsilon)\|_{L^\infty} = 2M. \tag{3.5}$$

3.2. **Proof of the error estimate.** In this section, we prove Item (ii) of Theorem 1.1. Consider the function $v^\varepsilon = e^{it\mathcal{H}_z/\varepsilon^2}u^\varepsilon$. This function satisfies the equation

$$i\partial_t v^\varepsilon = -\frac{1}{2}\Delta_x v^\varepsilon + \frac{1}{2}\left(|x|^2 - (\Omega_2 x_1 - \Omega_1 x_2)^2\right)v^\varepsilon - \Omega_z L_z v^\varepsilon + \lambda F\left(\frac{t}{\varepsilon^2}, v^\varepsilon\right) + \varepsilon r_1^\varepsilon + \varepsilon^2 r_2^\varepsilon$$

where F is defined in (1.8),

$$v^\varepsilon(t=0) = u_0^\varepsilon = e^{-i\varepsilon z(\Omega_2 x_1 - \Omega_1 x_2)}\psi_0,$$

and where we also denote

$$r_1^\varepsilon = Je^{it\mathcal{H}_z/\varepsilon^2} z e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon, \quad J := \left(2(\Omega_2\partial_{x_1} - \Omega_1\partial_{x_2}) - \Omega_z\left(\Omega_1 x_1 + \Omega_2 x_2\right)\right),$$
$$r_2^\varepsilon = \frac{3}{2}\left(\Omega_1^2 + \Omega_2^2\right)e^{it\mathcal{H}_z/\varepsilon^2} z^2 e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon.$$

We have $v^\varepsilon \in C([0, T_1^\varepsilon], \Sigma^2) \cap C^1([0, T_1^\varepsilon], L^2)$ and, by Lemma 3.1,

$$\max_{t\in[0,T^\varepsilon]} \|v^\varepsilon(t)\|_{\Sigma^2} = \max_{t\in[0,T^\varepsilon]} \|u^\varepsilon(t)\|_{\Sigma^2} \le C_M.$$

Hence, by (1.7), we get for $t \in [0, T^\varepsilon]$

$$\begin{aligned}\|r_1^\varepsilon\|_{L^2} &\le C\|\mathcal{H}_x^{1/2} e^{it\mathcal{H}_z/\varepsilon^2} z e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{L^2} \\ &\le C\|e^{it\mathcal{H}_z/\varepsilon^2} z e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{\Sigma^1} = C\|z e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{\Sigma^1} \\ &\le C\|e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{\Sigma^2} = C\|v^\varepsilon\|_{\Sigma^2} \le C_M, \end{aligned} \tag{3.6}$$

$$\begin{aligned}\|r_2^\varepsilon\|_{L^2} &= C\|e^{it\mathcal{H}_z/\varepsilon^2} z^2 e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{L^2} = C\|z^2 e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{L^2} \\ &\le C\|e^{-it\mathcal{H}_z/\varepsilon^2} v^\varepsilon\|_{\Sigma^2} = C\|v^\varepsilon\|_{\Sigma^2} \le C_M. \end{aligned} \tag{3.7}$$

We are now ready to estimate the difference $w^\varepsilon(t) = v^\varepsilon(t) - \phi(t)$ for $0 \le t \le \min(T, T^\varepsilon)$. This function satisfies

$$\begin{aligned} w^\varepsilon(t) &= U(t)(u_0^\varepsilon - \psi_0) + \lambda\int_0^t U(t-s)\left(F\left(\frac{s}{\varepsilon^2}, v^\varepsilon(s)\right) - F\left(\frac{s}{\varepsilon^2}, \phi(s)\right)\right)ds \\ &\quad + \lambda\int_0^t U(t-s)\left(F\left(\frac{s}{\varepsilon^2}, \phi(s)\right) - F_{\mathrm{av}}(\phi(s))\right)ds \\ &\quad + \varepsilon\int_0^t U(t-s)(r_1^\varepsilon(s) + \varepsilon r_2^\varepsilon(s))ds \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

Since $U(t)$ is unitary on L^2, (3.2) yields

$$\|A_1\|_{L^2} = \|u_0^\varepsilon - \psi_0\|_{L^2} \le C\varepsilon.$$

Moreover, for $0 \le t \le \min(T, T^\varepsilon)$, (3.1) and (3.3) imply that

$$\begin{aligned}\|A_2\|_{L^2} &\le C\int_0^t \left\| |u^\varepsilon(s)|^{2\sigma}u^\varepsilon(s) - |e^{-is\mathcal{H}_z/\varepsilon^2}\phi(s)|^{2\sigma} e^{-is\mathcal{H}_z/\varepsilon^2}\phi(s)\right\|_{L^2} ds \\ &\le C\int_0^t \left(\|u^\varepsilon(s)\|_{L^\infty}^{2\sigma} + \left\|e^{-is\mathcal{H}_z/\varepsilon^2}\phi(s)\right\|_{L^\infty}^{2\sigma}\right)\|u^\varepsilon(s) - e^{-is\mathcal{H}_z/\varepsilon^2}\phi(s)\|_{L^2} ds \\ &\le CM^{2\sigma}\int_0^t \|w(s)\|_{L^2} ds \end{aligned}$$

and (3.6) and (3.7) give

$$\|A_4\|_{L^2} \leq C\varepsilon.$$

Let us estimate A_3. To this aim, we introduce the following function, defined on $\mathbb{R} \times \Sigma^2$,

$$\mathcal{F}(\theta, u) = \int_0^\theta (F(s,u) - F_{\rm av}(u))ds.$$

Since $F(\cdot, u)$ is 2π-periodic and $F_{\rm av}$ is its average, $\mathcal{F}(\theta, u)$ is periodic with respect to θ. Hence, it is readily seen that this function satisfies the following properties:

$$\begin{aligned} &\text{if} \quad \|u\|_{\Sigma^2} \leq R \quad \text{then} \quad \sup_{\theta\in\mathbb{R}} \|\mathcal{F}(\theta,u)\|_{\Sigma^2} \leq CR^{2\sigma+1}, \\ &\text{if} \quad \|u\|_{\Sigma^2} + \|v\|_{L^2} \leq R \quad \text{then} \quad \sup_{\theta\in\mathbb{R}} \|D_u\mathcal{F}(\theta,u)\cdot v\|_{L^2} \leq CR^{2\sigma+1}. \end{aligned}$$

Recall that $U(t) = e^{itH}$, where H is the Hamiltonian defined by (2.2). Hence

$$\begin{aligned} &U(t-s)\left(F\left(\frac{s}{\varepsilon^2}, \phi(s)\right) - F_{\rm av}(\phi(s))\right) \\ &= \varepsilon^2 \frac{d}{ds}\left(U(t-s)\mathcal{F}\left(\frac{s}{\varepsilon^2}, \phi(s)\right)\right) + i\varepsilon^2 U(t-s) H\mathcal{F}\left(\frac{s}{\varepsilon^2}, \phi(s)\right) \\ &\quad - \varepsilon^2 U(t-s) D_u\mathcal{F}\left(\frac{s}{\varepsilon^2}, \phi(s)\right)\cdot \partial_t\phi(s), \end{aligned}$$

and then

$$\begin{aligned} \|A_3\|_{L^2} &\leq \varepsilon^2|\lambda| \left\|\mathcal{F}\left(\frac{t}{\varepsilon^2}, \phi(t)\right)\right\|_{L^2} + \varepsilon^2|\lambda| \int_0^t \left\|H\mathcal{F}\left(\frac{s}{\varepsilon^2}, \phi(s)\right)\right\|_{L^2} ds \\ &\quad + \varepsilon^2|\lambda| \int_0^t \left\|D_u\mathcal{F}\left(\frac{s}{\varepsilon^2}, \phi(s)\right)\cdot\partial_t\phi(s)\right\|_{L^2} ds \\ &\leq C\varepsilon^2, \end{aligned}$$

where we used that $\phi \in L^\infty([0,T], \Sigma^2)$ and $\partial_t\phi \in L^\infty([0,T], L^2)$. In summary, we have proved that, for all $t \in [0, \min(T, T_\varepsilon)]$,

$$\|w^\varepsilon(t)\|_{L^2} \leq C\varepsilon + C\int_0^t \|w^\varepsilon(s)\|_{L^2} ds.$$

Thus, Gronwall's lemma yields

$$\|u^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{L^2} = \|v^\varepsilon(t) - \phi(t)\|_{L^2} = \|w^\varepsilon(t)\|_{L^2} \leq C\varepsilon. \tag{3.8}$$

In particular, we deduce from (3.1), from a Gagliardo-Nirenberg inequality, from (3.8) and from Lemma 3.1 that

$$\begin{aligned} \|u^\varepsilon(t)\|_{L^\infty} &\leq M + \|u^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{L^\infty} \\ &\leq M + \|u^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{L^2}^{1/4} \|u^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{H^2}^{3/4} \\ &\leq M + C\varepsilon^{1/4}\left(\|u^\varepsilon(t)\|_{\Sigma^2} + \|\phi(t)\|_{\Sigma^2}\right)^{3/4} \\ &\leq M + C\varepsilon^{1/4}. \end{aligned}$$

Hence, for $\varepsilon < \varepsilon_T := (M/2C)^4$, we have

$$\forall t \leq \min(T, T^\varepsilon), \quad \|u^\varepsilon(t)\|_{L^\infty} < 3M/2. \tag{3.9}$$

It is clear then that $T_\varepsilon \geq T$. Indeed, otherwise this would imply that $T^\varepsilon < +\infty$ thus, by (3.5), that $\|u^\varepsilon(T_\varepsilon)\| = 2M$, which contradicts (3.9). Consequently, (3.8) is valid on $[0,T]$ and, together with (3.2) and Lemma 3.1, this yields

$$\|\psi^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{L^2} \leq \|\psi^\varepsilon(t) - u^\varepsilon(t)\|_{L^2} + \|u^\varepsilon(t) - e^{-it\mathcal{H}_z/\varepsilon^2}\phi(t)\|_{L^2} \leq C\varepsilon$$

for all $t \in [0,T]$. We have proved Item (ii) of Theorem 1.1. □

We note that under sufficiently high regularity assumptions on $\psi^\varepsilon(t)$, a slightly stronger approximation result can be proved. In this case, one can show that, indeed, $\|A_4\|_{L^2} \leq C\varepsilon^2$, and not only $\mathcal{O}(\varepsilon)$ as shown above. To see this, we expand

$$R_1^\varepsilon := \varepsilon \int_0^t U(t-s) r_1^\varepsilon(s)\, ds,$$

using the eigenfunctions of $\mathcal{H}_z$. Writing $v^\varepsilon(t,x,z) = \sum_{m\in\mathbb{N}} v_m^\varepsilon(t,x)\chi_m(z)$ we obtain

$$R_1^\varepsilon = J \int_0^t U(t-s) \sum_{m\neq n\in\mathbb{N}} \langle z\chi_m, \chi_n\rangle_{L^2}\, e^{is(\lambda_m-\lambda_n)/\varepsilon^2} v_m^\varepsilon(s)\chi_m\, ds,$$

where J is as above. Here, we have also used that $z|\chi_m|^2$ is odd and thus only indices $m \neq n$ appear in the double sum above and R_1^ε is consequently seen to be highly oscillatory. After an integration by parts in time (which requires the improved regularity of v^ε), one obtains that $R_1^\varepsilon = \mathcal{O}(\varepsilon^2)$. Using this one can show that, for $0 < T < T_{\max}$ and $0 < \varepsilon \leq \varepsilon_T$, the following *improved* error estimate holds

$$\max_{t\in[0,T]} \left\| \psi^\varepsilon(t) - e^{i\varepsilon z(\Omega_1 x_2 - \Omega_2 x_1)} e^{-it\mathcal{H}_z/\varepsilon^2} \phi^\varepsilon(t) \right\|_{L^2} \leq C_T\, \varepsilon^2,$$

where ϕ^ε is the solution of (1.10) with initial data

$$\phi^\varepsilon(t=0) = e^{-i\varepsilon z(\Omega_1 x_2 - \Omega_2 x_1)}\psi_0. \tag{3.10}$$

Note, however, that if we apply this ε-correction to the Cauchy data, the solution ϕ^ε does not remain polarized any more, in which case the reduced model is still posed in three spatial dimensions.

4. The case of strong two-dimensional confinement

In this section, we briefly discuss how to obtain the limiting model in the case of strong *two-dimensional* confinement within the original (three-dimensional) Gross-Pitaevskii equation. To this end, we start with the analog of (1.1), given by

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + \left(\frac{|x|^2}{2\varepsilon^4} + \frac{|z|^2}{2}\right)\psi + i\Omega\cdot(\mathbf{x}\wedge\nabla)\,\psi + \beta^\varepsilon|\psi|^{2\sigma}\psi, \tag{4.1}$$

subject to $\psi(t=0,\mathbf{x}) = \varepsilon^{-1}\psi_0(x/\varepsilon, z)$, where as before $\mathbf{x} = (x,z) \in \mathbb{R}^3$ with $x = (x_1,x_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$. Note that in comparison to (1.1) the roles of x and z have been *reversed*. Thus, in (4.1), the x variables are now the ones which represent the strongly confined directions, and we consequently aim to derive an effective model depending on the z-variable only. To this end, we rescale

$$x' = \frac{x}{\varepsilon}, \quad z' = z, \quad \psi^\varepsilon(t,x',z') = \varepsilon\psi\left(t, \varepsilon x', z'\right)$$

and assume that $\beta^\varepsilon = \lambda \varepsilon^{2\sigma}$, i.e., an even weaker interaction regime as before. The rescaled NLS equation then becomes

$$\begin{aligned} i\partial_t \psi^\varepsilon = &\frac{1}{\varepsilon^2}\mathcal{H}_x \psi^\varepsilon + \mathcal{H}_z \psi^\varepsilon - \frac{i}{\varepsilon} z\left(\Omega_1 \partial_{x_2}\psi^\varepsilon - \Omega_2 \partial_{x_1}\psi^\varepsilon\right) - \Omega_z L_z \psi^\varepsilon \\ &- i\varepsilon\left(\Omega_2 x_1 - \Omega_1 x_2\right)\partial_z \psi^\varepsilon + \lambda |\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \end{aligned} \tag{4.2}$$

with $\psi^\varepsilon(t=0,x,z) = \psi_0(x,z)$. In order to get rid of the singular perturbation we invoke the *same* change of variables (up to a sign) as in the case of a strong one-directional confinement, i.e.,

$$\psi^\varepsilon(t,x,z) = e^{i\varepsilon z(\Omega_2 x_1 - \Omega_1 x_2)} u^\varepsilon(t,x,z).$$

After a somewhat lengthy computation, this yields the following analog of (1.4):

$$\begin{aligned} i\partial_t u^\varepsilon = &\frac{1}{\varepsilon^2}\mathcal{H}_x u^\varepsilon + \mathcal{H}_z u^\varepsilon - \frac{1}{2}(\Omega_1^2 + \Omega_2^2) z^2 u^\varepsilon - \Omega_z L_z u^\varepsilon + \lambda |u^\varepsilon|^{2\sigma} u^\varepsilon \\ &+ \varepsilon \Omega_z\left(\Omega_1 x_1 + \Omega_2 x_2\right) z u^\varepsilon + 2i\varepsilon(\Omega_1 x_2 - \Omega_2 x_1)\partial_z u^\varepsilon + \frac{3}{2}\varepsilon^2(\Omega_2 x_1 - \Omega_1 x_2)^2 u^\varepsilon. \end{aligned} \tag{4.3}$$

In order to average out the fast oscillations stemming from $\mathcal{H}_x$, we introduce

$$G(\theta,u) = e^{i\theta\mathcal{H}_x}\left(\left|e^{-i\theta\mathcal{H}_x}u\right|^{2\sigma} e^{-i\theta\mathcal{H}_x}u\right),$$

which satisfies $G \in C(\mathbb{R}\times\Sigma^s;\Sigma^s)$. Moreover, G is easily seen to be a 2π-periodic function in θ, since the spectrum of the two-dimensional harmonic oscillator $\mathcal{H}_x$ is given by $\{\lambda_n = n+1\,,\, n\in\mathbb{N}_0\}$. Note however, that the eigenspace corresponding to λ_n is $(n+1)$-fold degenerate. We consequently denote the associated averaged nonlinearity by

$$G_{\rm av}(u) := \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta\mathcal{H}_x}\left(\left|e^{-i\theta\mathcal{H}_x}u\right|^{2\sigma} e^{-i\theta\mathcal{H}_x}u\right)d\theta,$$

and find, that, as $\varepsilon\to 0$, the new *limiting model* becomes

$$i\partial_t\phi = -\frac{1}{2}\,\partial_z^2\phi + \frac{1}{2}\left(1-\left(\Omega_1^2+\Omega_2^2\right)\right)z^2\phi - \Omega_z L_z\phi + \lambda G_{\rm av}(\phi). \tag{4.4}$$

Again, we note the appearance of an additional negative (repulsive) quadratic potential, provided $\Omega_1,\Omega_2 \neq 0$.

The limiting model (4.4) has the drawback to still be an equation in three dimensions. But, having in mind that $[\mathcal{H}_x, L_z] = 0$, there exists a *joint* orthonormal basis of eigenfunctions $\{\chi_\alpha\}_{\alpha\in\mathbb{N}^2}$ where $\alpha = (\alpha_1,\alpha_2)$, cf. [12], such that

$$L_z\chi_\alpha = \mu_\alpha\chi_\alpha, \quad \mathcal{H}_x\chi_\alpha = \lambda_n\chi_\alpha, \quad \mu_\alpha,\lambda_n\in\mathbb{R},\, n=\alpha_1+\alpha_2.$$

The simplest situation is then obtained for initial data ϕ_0 concentrated in the eigenspace corresponding to the ground state energy $\lambda_0 \equiv 1$. This eigenvalue is known to be non-degenerate, i.e., $\phi_0(x_1,x_2,z) = \varphi_0(z)\chi_0(x_1,x_2)$. In addition, $\chi_0 \equiv \chi_{0,0}$ is known to be radially symmetric which implies $\mu_0 = 0$. By the same arguments as earlier (see the remarks below Corollary 1.2), we consequently obtain that (4.4) admits polarized solutions of the form

$$\phi(t,x,z) = e^{-it/\varepsilon^2}\varphi(t,z)\chi_0(x_1,x_2),$$

where φ solves the *one-dimensional* NLS equation

$$i\partial_t\varphi = -\frac{1}{2}\,\partial_z^2\varphi + \frac{1}{2}\left(1-\left(\Omega_1^2+\Omega_2^2\right)\right)z^2\varphi + \kappa_0|\varphi|^{2\sigma}\varphi, \tag{4.5}$$

where $\kappa_0 = \lambda \|\chi_0\|_{L^{2\sigma+2}}^{2\sigma+2}$.

Remark 4.1. *Concerning the local and global well-posedness of* (4.5)*, we remark that in it falls within the class of models studied in* [9]*. In particular we have global in-time existence of solutions* $\varphi(t) \in \Sigma^2$ *for* $\lambda > 0$.

Our main result in this section is then as follows:

Theorem 4.2. *Let* $1 \leq \sigma < 2$ *and* $\psi_0 \in \Sigma^2$*. Then the following holds:*
(i) *The limit model* (4.4) *admits a unique maximal solution* $\phi \in C([0, T_{\max}), \Sigma^2) \cap C^1([0, T_{\max}), L^2)$*, with* $T_{\max} \in (0, +\infty]$*, such that for all* $t \in [0, T_{\max})$*:*

$$\|\phi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad E(\phi(t)) = E(\psi_0), \quad \langle \mathcal{H}_x \phi(t), \phi(t)\rangle_{L^2} = \langle \mathcal{H}_x \psi_0, \psi_0\rangle_{L^2}.$$

(ii) *For all* $T \in (0, T_{\max})$*, there exists* $\varepsilon_T > 0$*,* $C_T > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_T]$*,* (4.2) *admits a unique solution* $\psi^\varepsilon \in C([0, T], \Sigma^2) \cap C^1([0, T], L^2)$*, which is uniformly bounded with respect to* $\varepsilon \in (0, \varepsilon_T]$ *in* $L^\infty((0, T), \Sigma^2)$ *and satisfies the error bound*

$$\max_{t \in [0,T]} \left\| \psi^\varepsilon(t) - e^{-it\mathcal{H}_x/\varepsilon^2} \phi(t) \right\|_{L^2} \leq C_T\, \varepsilon.$$

(iii) *If moreover the initial data is such that* $\psi_0(x, z) = \varphi_0(z)\chi_0(x)$*, then, for all* $T \in (0, T_{\max})$*, we have*

$$\max_{t \in [0,T]} \left\| \psi^\varepsilon(t) - e^{-it/\varepsilon^2} \varphi(t)\chi_0 \right\|_{L^2} \leq C_T\, \varepsilon,$$

where $\varphi(t, x)$ *solves* (4.5).

Proof. The approximation proof follows along the same lines as the one for Theorem 1.1, up to adjusting the notation (i.e., switching the roles of x and z). The main difference concerns the proof of well-posedness for the limiting equation (4.4). In contrast to the case of one-dimensional confinement, equation (4.4) only admits dispersive properties only in one direction, which might not be sufficient for the use of Strichartz estimates. However, since we are working in $\Sigma^2 \hookrightarrow L^\infty(\mathbb{R}^3_{x,z})$, local in-time well-posedness follows from standard arguments, see [10]. □

In comparison to initial data polarized along the ground state λ_0, the situation for initial data polarized along some higher energy eigenvalue λ_n, $n \geq 1$, is much more complicated, due to their $(n+1)$-fold degeneracy. The corresponding solutions are then of the form

$$\phi(t, x, z) = e^{-it\lambda_n/\varepsilon^2} \sum_{\alpha_1+\alpha_2=n} e^{-it\mu_\alpha} \varphi_\alpha(t, z)\chi_\alpha(x_1, x_2),$$

where the coefficients $\varphi_\alpha \equiv \varphi_{\alpha_1,\alpha_2}$ solve a system of $n+1$ coupled NLS. The latter mixes the φ_α through the nonlinearity and describes the dynamics within the n-th eigenspace. The precise form of the NLS system is rather complicated and hence, we shall leave its details to the reader, in particular, since one anyway might prefer the description of the dynamics via the effective model (4.4) instead.

References

[1] P. Antonelli, R. Carles, and J. D. Silva, *Scattering for nonlinear Schrödinger equation under partial harmonic confinement*, Commun. Math. Phys. **334** (2015), no. 1, 367–396.

[2] P. Antonelli, D. Marahrens, and C. Sparber, *On the Cauchy problem for nonlinear Schrödinger equations with rotation*, Discrete Contin. Dyn. Syst. **32** (2012), no. 3, pp. 703–715.

[3] W. Bao, N. Ben Abdallah, and Y. Cai, *Gross-Pitaevskii-Poisson equations for dipolar Bose-Einstein condensate with anisotropic confinement*, SIAM J. Math. Anal. **44** (2012), pp. 1713–1741.

[4] W. Bao, P. A. Markowich, C. Schmeiser, and R. M. Weishäupl, *On the Gross-Pitaevskii equation with strongly anisotropic confinement: formal asymptotics and numerical experiments*, Math. Models Meth. Appl. Sci. **15** (2005), pp. 767–782.

[5] N. Ben Abdallah, Y. Cai, F. Castella, and F. Méhats, *Second order averaging for the nonlinear Schrödinger equation with strongly anisotropic potential*, Kinet. Relat. Models **4** (2011), pp. 831–856.

[6] N. Ben Abdallah, F. Castella, and F. Méhats, *Time averaging for the strongly confined nonlinear Schrödinger equation, using almost-periodicity*, J. Differential Equ. **245** (2008), pp. 154–200.

[7] N. Ben Abdallah, F. Méhats, C. Schmeiser, and R. M. Weishäupl, *The nonlinear Schrödinger equation with a strongly anisotropic harmonic potential*, SIAM J. Math. Anal. **37** (2005), pp. 189–199.

[8] R. Carles, *Global existence results for nonlinear Schrödinger equations with quadratic potentials.* Discrete Contin. Dyn. Syst. **13** (2005), no. 2, pp. 385–398.

[9] R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications.* SIAM J. Math. Anal. **35** (2003), no. 4, 823–843.

[10] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics vol. 10, AMS Bookstore, 2003.

[11] F. Delebecque-Fendt and F. Méhats, *An effective mass theorem for the bidimensional electron gas in a strong magnetic field*, Comm. Math. Phys. **292** (2009), no. 3, 829–870.

[12] S. Flügge, *Practical Quantum Mechanics*, Springer Classics in Mathematics, Springer Verlag 1971.

[13] G. H. Hardy, J. E. Littlewood, and J. E., G. Pólya, *Inequalities.* Cambridge University Press, 1952.

[14] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques*, Société mathématique de France, 1984.

[15] H. Kitada, *On a construction of the fundamental solution for Schrödinger equations.* J. Fac. Sci. Univ. Tokyo Sec. IA, **27** (1980), pp. 193–226.

[16] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, Oberwolfach Seminars 34, Birkhäuser Verlag, Basel, 2005.

[17] E. Lieb and R. Seiringer, *Derivation of the Gross-Pitaevskii equation for rotating Bose gases*, Comm. Math. Phys. **264** (2006), pp. 505–537.

[18] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, Cambridge University Press, 2002.

[19] L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation*, Clarendon Press, Oxford, 2003.

[20] N. Tzvetkov and N. Visciglia, *Small data scattering for the nonlinear Schrödinger equation on product spaces*, Commun. Partial Differ. Equ. **37** (2012), no. 1, 125–135.

IRMAR, Université de Rennes 1 and INRIA, IPSO Project
E-mail address: florian.mehats@univ-rennes1.fr

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago
E-mail address: sparber@math.uic.edu