STABILITY AND INSTABILITY PROPERTIES OF ROTATING BOSE-EINSTEIN CONDENSATES

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Abstract. We consider the mean-field dynamics of Bose-Einstein condensates in rotating harmonic traps and establish several stability and instability properties for the corresponding solution. We particularly emphasize the difference between the situation in which the trap is symmetric with respect to the rotation axis and the one where this is not the case.

1. Introduction

In this note, we consider the dynamics of (harmonically) trapped *Bose-Einstein condensates* (BEC), subject to an external rotating force. Because of their ability to display quantum effects at the macroscopic scale, BEC have become an important subject of research, both experimentally and theoretically. In particular, the expression of quantum vortices in rapidly rotating BEC has been an ongoing topic of interest over the last few decades, see, e.g., [1, 4, 7, 10, 12, 13, 25] and the references therein. It is well-known that in the mean-field regime, BEC can be accurately described by the celebrated *Gross-Pitaevskii equation* (GP) for ψ, the macroscopic wave function of the condensate, see [22, 26, 27]. In dimensionless units, the GP equation with general nonlinearity reads

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(x)\psi + a|\psi|^{2\sigma}\psi - (\Omega\cdot L)\psi\,, \quad \psi_{|t=0} = \psi_0(x)\,. \tag{1.1}$$

Here, $a \in \mathbb{R}$, $\sigma > 0$ and $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ with $d = 2$, or 3, respectively. The former situation thereby corresponds to the case of an *effective* two-dimensional BEC, obtained via strong confining forces, see, e.g., [23] for more details. The external potential $V(x) \in \mathbb{R}$ is assumed to be *harmonic*, i.e.,

$$V(x) = \frac{1}{2}\sum_{j=1}^{d}\omega_j^2 x_j^2, \tag{1.2}$$

where the parameters $\omega_j \in \mathbb{R}\setminus\{0\}$ represent the respective trapping frequencies in each spatial direction. As we shall see, the smallest trapping frequency denoted by

$$0 < \omega \equiv \min_{j=1,\dots,d}\{\omega_j\},$$

will play a particular role in our analysis.

We further assume that the BEC is subject to a rotating force along a given *rotation axis* $\Omega \in \mathbb{R}^3$ and denote by

$$L = -ix \wedge \nabla,$$

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the quantum mechanical angular momentum operator. Note that in dimension $d = 2$, we always have

$$\Omega \cdot L = -i|\Omega|\big(x_1\partial_{x_2} - x_2\partial_{x_1}\big), \tag{1.3}$$

corresponding to the case where $\Omega = (0, 0, |\Omega|) \in \mathbb{R}^3$.

The nonlinearity in (1.1) describes the mean-field self-interaction of the condensate particles. The physically most relevant case is given by a cubic nonlinearity, i.e. $\sigma = 1$, but for the sake of generality we shall in the following allow for more general $\sigma > 0$. We shall also allow for both *attractive* $a < 0$ and *repulsive* $a > 0$ interactions, satisfying Assumption 1 below. Vortices are generally believed to be unstable in the former case (see, e.g., [7, 9, 25]), while they are known to form stable lattice configurations in the latter [1, 10, 13].

In this work, we shall not be interested in the dynamical features of individual vortices, but rather study bulk properties of the condensate, as described by (1.1). To this end, we recall that the natural energy space associated to (1.1) is given by

$$\Sigma = \{u \in H^1(\mathbb{R}^d) \,:\, |x|u \in L^2(\mathbb{R}^d)\},$$

equipped with the norm

$$\|u\|_\Sigma^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \||x|u\|_{L^2}^2\,.$$

We also impose the following sub-criticality condition on the nonlinearity:

Assumption 1. *One of the following holds:*
- $a > 0$ *(defocusing) and* $0 < \sigma < \frac{2}{(d-2)_+}$*, or*
- $a < 0$ *(focusing) and* $0 < \sigma < \frac{2}{d}$.

Under these hypotheses, the existence of a unique global in-time solution $\psi \in C(\mathbb{R}_t; \Sigma)$ to (1.1) has been proved in [2]. In particular, the restriction $\sigma < \frac{2}{d}$ in the focusing case ($a < 0$) ensures that *no finite-time blow-up* can occur. In addition, the global solution $\psi(t, \cdot) \in \Sigma$ is known to conserve the *total mass*, i.e.

$$N(\psi(t, \cdot)) = \int_{\mathbb{R}^d} |\psi(t, x)|^2\, dx = N(\psi_0), \quad \forall\, t \in \mathbb{R}, \tag{1.4}$$

as well as

$$E_\Omega(\psi(t, \cdot)) = E_\Omega(\psi_0), \quad \forall\, t \in \mathbb{R}, \tag{1.5}$$

where E_Ω denotes the associated *Gross-Pitaevskii energy*:

$$E_\Omega(\psi) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla\psi|^2 + V(x)|\psi|^2 + \frac{a}{\sigma + 1}|\psi|^{2\sigma+2} - \overline{\psi}(\Omega \cdot L)\psi\, dx. \tag{1.6}$$

Note that the last term within E_Ω is sign indefinite.

In the following, we shall focus on various stability and/or instability properties of solutions ψ to (1.1): Our first task will be to study the orbital stability of nonlinear *ground states* associated to (1.1). These are solutions to (1.1) given by

$$\psi(t, x) = e^{-i\mu t}\varphi(x), \quad \mu \in \mathbb{R},$$

where φ is obtained as a constrained minimizer of the energy functional $E_\Omega(\varphi)$. In [20, 26, 27], the onset of vortex nucleation is linked to a symmetry breaking phenomenon for minimizers of $E_\Omega(\varphi)$, which is proved to happen for $|\Omega|$ above a certain critical speed $\Omega_{\text{crit}} > 0$, even in the case of radially symmetric traps V with $\omega_1 = \omega_2 = \omega_3$ (see Section 3 for more details). In our first main result below, we shall prove that under Assumption 1 and for $|\Omega| < \omega$, the set of all energy minimizers is indeed orbitally stable under the time-evolution of (1.1). In turn, this will allow us to conclude several new results of orbital stability for a class of rotating solutions to nonlinear Schrödinger equations *without* the angular momentum term $\propto \Omega$.

The question of whether the condition $|\Omega| < \omega$ is only needed for the existence of ground states, or also has a nontrivial effect in the solution of the time-dependent equation (1.1), then leads us to our second line of investigation. A theorem based on the Ehrenfest equations associated to (1.1), shows that in the case of *non-istotropic potentials* V, a resonance-type phenomenon can occur for $|\Omega| \geqslant \omega$. This leads to solutions ψ whose Σ-norm is growing (forward or backward) in time with a rate that can even be exponential, depending on the choice of Ω and ω_j. Physically, this can be interpreted as a manifestation of non-trapped solutions of (1.1) whose mass is pushed towards spatial infinity.

The paper is organized as follows: In Section 2 below we shall prove the existence of nonlinear ground states. Their orbital stability (and several further consequences) is proved in Section 3. Finally, we turn to the analysis of possible resonances in Section 4.

2. Existence of ground states

In this section we shall prove the existence of time-periodic solutions $\psi(t,x) = e^{-i\mu t}\varphi(x)$ to (1.1), which satisfy the following nonlinear elliptic equation

$$\mu\varphi = \Big(-\frac{1}{2}\Delta + V(x) - (\Omega\cdot L)\Big)\varphi + a|\varphi|^{2\sigma}\varphi. \tag{2.1}$$

Note that if φ solves this equation, then so does $\varphi e^{i\theta}$ with $\theta \in \mathbb{R}$, i.e., we have symmetry under gauge transformations.

For any given total mass $N > 0$, a particular class of solutions $\varphi \in \Sigma$ to (2.1), called ground states, is obtained by considering the following constrained minimization problem:

$$e(N,\Omega) := \inf\{E_\Omega(\varphi) : \varphi \in \Sigma,\ N(\varphi) = N\}, \tag{2.2}$$

where the infimum can be replaced by a minimum whenever the energy functional (1.6) is bounded from below. In this case $e(N,\Omega) > -\infty$ denotes the ground state energy. Note that $E_\Omega(\varphi)$ is well-defined for any $\varphi \in \Sigma$, since Assumption 1 and Sobolev's imbedding imply $\Sigma \hookrightarrow L^{2\sigma+2}$ provided $\sigma < \frac{2}{(d-2)_+}$. Moreover, for any $\gamma > 0$ we have

$$|\langle\psi, (\Omega\cdot L)\psi| \leqslant \|(\Omega\wedge x)\psi\|_{L^2}\|\nabla\psi\|_{L^2} \leqslant \frac{1}{2\gamma}|\Omega|^2\|x\psi\|_{L^2}^2 + \frac{\gamma}{2}\|\nabla\psi\|_{L^2}^2, \tag{2.3}$$

which in itself follows by rewriting $\Omega\cdot L = (\Omega\wedge x)\cdot\nabla$ and employing Young's inequality.

The existence and orbital stability of ground state solutions will be proved by the same method as in [8, 11]. To this end, we shall first show that the energy functional (1.6) is coercive, provided the angular velocity $|\Omega|$ is less than the smallest trapping frequency:

Proposition 2.1. *Let $|\Omega| < \omega$ and Assumption 1 hold. Then for any $\varphi \in \Sigma$ with $\|\varphi\|_{L^2}^2 = N$, there is a $\delta > 0$ such that*

$$E_\Omega(\varphi) \geqslant \delta\|\varphi\|_\Sigma^2 - C_N \geqslant 0, \tag{2.4}$$

Moreover, $\varphi \mapsto E_\Omega(\varphi)$ is weakly lower semicontinuous in Σ, i.e. for $\{\varphi_k\}_{k=1}^\infty \subset \Sigma$ such that $\varphi_k \rightharpoonup \varphi \in \Sigma$, we have

$$E_\Omega(\varphi) \leqslant \liminf_{k\to\infty} E_\Omega(\varphi_k).$$

Proof. The coercivity follows from (2.3) and the fact that $V(x) \geqslant \frac{1}{2}\omega^2|x|^2$ where $\omega > 0$ is defined above. Thus one finds, for $0 < \gamma < 1$:

$$E_\Omega(\varphi) \geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\|x\varphi\|_{L^2}^2 + \frac{a}{\sigma+1}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{2.5}$$

In the case $a > 0$, we directly obtain

$$E_\Omega(\varphi) \geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\|x\varphi\|_{L^2}^2 \geqslant \delta\|\varphi\|_\Sigma^2 - \frac{N}{2},$$

where we choose $\gamma \in (0,1)$ such that $\frac{|\Omega|^2}{\omega^2} < \gamma < 1$, and we set

$$\delta = \min\left\{\frac{1-\gamma}{2}, \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\right\} > 0.$$

In the case $a < 0$, we first note from the Gagliardo-Nirenberg inequality that

$$\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant C_{\sigma,d}\|\nabla u\|_{L^2}^{d\sigma}\|u\|_{L^2}^{2+\sigma(d-2)}, \tag{2.6}$$

with the optimal constant $C_{\sigma,d} > 0$ obtained in [28], i.e.,

$$C_{\sigma,d} = \frac{\sigma+1}{\|Q\|_{L^2}^{2\sigma}},$$

where Q satisfies

$$\frac{d\sigma}{2}\Delta Q - \Big(1 + \frac{\sigma(d-2)}{2}\Big)Q + Q^{2\sigma+1} = 0.$$

Then applying (2.6) to (2.5) and employing Young's inequality with

$$(p,q) = \Big(\frac{2}{d\sigma}, \frac{1}{1-d\sigma/2}\Big)$$

yields the following lower bound for any $\varepsilon > 0$:

$$\begin{aligned}
E_\Omega(\varphi) &\geqslant \frac{1-\gamma}{2}\|\nabla\varphi\|_{L^2}^2 - \frac{|a|}{\|Q\|_{L^2}^{2\sigma}}\|\nabla\varphi\|_{L^2}^{d\sigma}\|\varphi\|_{L^2}^{2+\sigma(d-2)} + \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\|x\varphi\|_{L^2}^2 \\
&\geqslant \Big(\frac{1-\gamma}{2} - \frac{d\sigma|a|\varepsilon^p}{2\|Q\|_{L^2}^{2\sigma}}\Big)\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\|x\varphi\|_{L^2}^2 \\
&\quad - \frac{|a|(1-\frac{d\sigma}{2})}{\|Q\|_{L^2}^{2\sigma}\varepsilon^q}\|\varphi\|_{L^2}^{\frac{2+\sigma(d-2)}{1-\frac{d\sigma}{2}}}.
\end{aligned}$$

Now recall $p = \frac{2}{d\sigma} > 1$ and choose $\gamma \in (0,1)$, as above, such that $\frac{|\Omega|^2}{\omega^2} < \gamma < 1$, and then $\varepsilon > 0$ such that

$$\frac{d\sigma|a|\varepsilon^p}{2\|Q\|_{L^2}^{2\sigma}} = \frac{1-\gamma}{4}.$$

After recalling that $\|\varphi\|_{L^2}^2 = N$, we find

$$\begin{aligned}
E_\Omega(\varphi) &= \frac{1-\gamma}{4}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\|x\varphi\|_{L^2}^2 - \frac{|a|(1-\frac{d\sigma}{2})}{\|Q\|_{L^2}^{2\sigma}\varepsilon^q}N^{\frac{2+\sigma(d-2)}{2-d\sigma}} \\
&\geqslant \tilde{\delta}\|\varphi\|_\Sigma^2 - C(a,d,\sigma,N,\|Q\|_{L^2}^{2\sigma})
\end{aligned}$$

where

$$\tilde{\delta} = \min\left\{\frac{1-\gamma}{4}, \frac{1}{2}\left(\omega^2 - \frac{|\Omega|^2}{\gamma}\right)\right\}.$$

Moreover, since the Σ-norm is weakly lower semicontinuous, the estimate (2.4) directly implies the same holds for E_Ω, since its quadratic part together with a multiple of the L^2-norm forms a norm on Σ equivalent to the usual one. □

To proceed further, we recall the following compactness result (see, e.g., [18, 29]).

Lemma 2.2. *For $2 \leqslant q < \frac{2d}{(d-2)_+}$, the embedding $\Sigma \hookrightarrow L^q$ is compact.*

Using this we can prove existence of a (constrained) minimizer.

Proposition 2.3. *Let $|\Omega| < \omega$ and Assumption 1 hold. Then for a given $N > 0$, there exists a $\varphi_\infty \in \Sigma$ such that $\|\varphi_\infty\|_{L^2}^2 = N$ and*

$$E_\Omega(\varphi_\infty) = \min_{\varphi \in \Sigma} E_\Omega(\varphi) = e(\Omega, N).$$

In addition, φ_∞ is a weak solution to (2.1) with $\mu \in \mathbb{R}$ a Lagrange multiplier associated to the mass constraint.

Proof. Choose a minimizing sequence $\{\varphi_k\}_{k=1}^\infty \subset \Sigma$ such that $\|\varphi_k\|_{L^2}^2 = N$. First we show $\{\varphi_k\}_{k=1}^\infty$ is a bounded sequence in Σ. From Proposition 2.1 we know that $0 < E_\Omega(\varphi_k) < \infty$ and the coercivity implies that any minimizing sequence $\{\varphi_k\}_{k=1}^\infty$ is a bounded sequence in Σ. By Banach-Alaoglu, there exists a weakly convergent subsequence $\{\varphi_{k_j}\}_{j=1}^\infty \subset \{\varphi_k\}_{k=1}^\infty$ such that

$$\varphi_{k_j} \rightharpoonup \varphi_\infty \quad \text{as } j \to \infty,$$

for some $\varphi_\infty \in \Sigma$. The compact embedding of Lemma 2.2 implies that $\varphi_{k_j} \to \varphi_\infty$ strongly (and hence in norm) in L^2 and in $L^{2\sigma+2}$, provided $\sigma < \frac{2}{(d-2)_+}$. In particular

$$\|\varphi_\infty\|_{L^2}^2 = \lim_{j\to\infty} \|\varphi_{k_j}\|_{L^2}^2 = N. \tag{2.7}$$

By the lower semicontinuity of the functional E_Ω we have

$$E_N := \inf_{\varphi\in\Sigma, \|\varphi\|_2^2 = N} E_\Omega(\varphi) \leqslant E_\Omega(\varphi_\infty) \leqslant \lim_{j\to\infty} \inf E_\Omega(\varphi_{k_j}) = E_N.$$

Furthermore, since $e(N,\Omega) \equiv E_\Omega(\varphi_\infty) = \lim_{j\to\infty} E_\Omega(\varphi_{k_j})$, we see that $\|\varphi_{k_j}\|_\Sigma \to \|\varphi_\infty\|_\Sigma$, as $j \to \infty$. Together with the weak convergence of the minimizing sequence this implies strong convergence to some $\varphi_\infty \in \Sigma$.

It is then straightforward to compute the first variation $\langle \frac{\delta E_\Omega}{\delta \varphi}, \chi \rangle = 0$ to see that a minimizer $\varphi_\infty \in \Sigma$ indeed solves (2.1) in the weak sense, i.e.

$$\mu \int_{\mathbb{R}^d} \overline{\varphi}_\infty \chi \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \nabla \overline{\varphi}_\infty \cdot \nabla \chi + V(x)\overline{\varphi}_\infty \chi - \overline{\varphi}_\infty (\Omega \cdot L)\chi + a|\varphi_\infty|^{2\sigma} \overline{\varphi}_\infty \chi \, dx,$$

for all $\chi \in \Sigma$. □

Remark 2.4. It is straightforward to generalize all of the results in this section to GP equations with general confinement potentials $V(x) \to +\infty$, as $|x| \to \infty$, provided an appropriate energy space Σ is chosen.

3. Orbital stability

The set of all ground states with a given mass N will be denoted by

$$\mathcal{G}_\Omega = \Big\{ \varphi \in \Sigma : E_\Omega(\varphi) = e(N, \Omega) \text{ and } N(\varphi) = N \Big\} \neq \emptyset. \tag{3.1}$$

Recall that, by gauge symmetry, $\varphi \in \mathcal{G}_\Omega$ if and only if $e^{i\theta}\varphi \in \mathcal{G}_\Omega$, for some $\theta \in \mathbb{R}$. In the case without rotation, i.e., $\Omega \equiv 0$, and for radially symmetric potentials V with $\omega_1 = \omega_2 = \omega_3$, one can show that the energy minimizer is indeed radially symmetric and positive on all of $\mathbb{R}^d$, see [18, 19] and the references therein. In other words, in this case

$$\mathcal{G}_0 = \{ u e^{i\theta}, \ u \equiv u(|x|) > 0, \, \theta \in \mathbb{R} \}. \tag{3.2}$$

Moreover, since the action of $\Omega \cdot L$ vanishes on radially symmetric functions, any radially symmetric $\varphi \in \mathcal{G}_\Omega$ is also in $\mathcal{G}_0$, and hence of the form above. However, the symmetry breaking results in [26, 27] imply that for $|\Omega| \neq 0$, a minimizer $\varphi_\infty \in \mathcal{G}_\Omega$ is in general *not radially symmetric*. More precisely, it is proved in there that for

$|\Omega| > \Omega_{\text{crit}} > 0$ no eigenfunction of the angular momentum operator L can be a minimizer (and a radial function u is an eigenfunction with zero eigenvalue), even if the GP functional is invariant under rotations around the Ω-axis. This implies that φ_∞ in the case *with* rotation cannot be unique (up to gauge transforms), since by rotating a minimizer one obtains another minimizer. In this context, an estimate for the critical rotation speed Ω_{crit} in $d = 2$ can be found in [20]. In summary, these results show that $\mathcal{G}_\Omega$, in general, will be a more complicated set than $\mathcal{G}_0$. Moreover, $\mathcal{G}_\Omega$ should also be distinguished from the set of rotationally symmetric vortex solutions studied in, e.g., [17].

Our first main result is as follows:

Theorem 3.1 (Orbital stability of ground states). *Let $|\Omega| < \omega$ and Assumption 1 hold. Then the set of ground states $\mathcal{G}_\Omega \neq \emptyset$ is orbitally stable in Σ. That is, for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that if $\psi_0 \in \Sigma$ satisfies*

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi_0 - \varphi\|_\Sigma < \delta,$$

then the solution $\psi \in C(\mathbb{R}_t, \Sigma)$ to (1.1) with $\psi(0, x) = \psi_0 \in \Sigma$ satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\varphi \in \mathcal{G}_\Omega} \|\psi(t, \cdot) - \varphi\|_\Sigma < \varepsilon.$$

This theorem generalizes earlier results on the orbital stability of standing waves in nonlinear Schrödinger equations with (unbounded) potential (see, e.g., [8, 14, 15, 18, 29, 30] and the references therein) to the case with harmonic potential and additional rotation. Note that for $\Omega = 0$, the simple structure of $\mathcal{G}_0$, given in (3.2), allows one to rephrase the infimum over $\mathcal{G}_0$ as an infimum over $\theta \in \mathbb{R}$. Also note Theorem 3.1 holds for defocusing and focusing nonlinearities satisfying Assumption 1 (see also Remark 3.2 below). In this context, we also mention the papers [14, 16], in which the authors study various instability properties of standing wave solutions to focusing nonlinear Schrödinger equations with potentials.

Proof. By way of contradiction, assume that the set of ground states $\mathcal{G}_\Omega \neq \emptyset$ is unstable. Then there exist $\varepsilon_0 > 0$, $\varphi_0 \in \mathcal{G}_\Omega$, a sequence of initial data $\{\psi_0^k\}_{k \in \mathbb{N}} \subset \Sigma$ satisfying

$$\|\psi_0^k - \varphi_0\|_\Sigma \to 0 \quad \text{as} \quad k \to \infty,$$

and a sequence of times $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$, such that

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi^k(t_k, \cdot) - \varphi\|_\Sigma > \varepsilon_0.$$

Here $\psi^k(t, x) \in C(\mathbb{R}, \Sigma)$ is the unique global solution to (1.1) with initial data ψ_0^k. For simplicity set $u_k(x) := \psi^k(t_k, x)$. From mass conservation (1.4) we have, as $k \to \infty$:

$$\|u_k\|_{L^2}^2 \equiv \|\psi^k(t_k, \cdot)\|_{L^2}^2 = \|\psi_0^k\|_{L^2}^2 \xrightarrow{k \to \infty} \|\varphi_0\|_{L^2}^2 = N.$$

Moreover, by energy conservation (1.5) it also follows that

$$E_\Omega(u_k) \equiv E_\Omega(\psi^k(t_k, \cdot)) = E_\Omega(\psi_0^k) \xrightarrow{k \to \infty} E_\Omega(\varphi_0) = e(N, \Omega).$$

Consequently, the continuity in time implies that u_k is a minimizing sequence in Σ. By the proof of Proposition 2.3, there exists a subsequence such that $u_{k_j} \to \varphi_\infty \in \Sigma$ strongly, as $j \to \infty$. Thus

$$\inf_{\varphi \in \mathcal{G}_\Omega} \|\psi^{k_j}(t_{k_j}, \cdot) - \varphi\|_\Sigma \leqslant \|u_{k_j} - \varphi_\infty\|_\Sigma \xrightarrow{j \to \infty} 0,$$

which contradicts our assumption. $\square$

Remark 3.2. It is possible to generalize this result to the case of an attractive ($a<0$) mass-critical nonlinearity $\sigma=\frac{2}{d}$, under the assumption that $N<\|Q\|_{L^2}^2$, see, e.g., [30, 31] for analogous results in the case without rotation. We shall not go into further details here, but note that the associated question of a blow-up profile as $N\to\|Q\|_{L^2}^2$ in the case with rotation has recently been studied in [21].

Theorem 3.1 has the following interesting consequence: Recall that $\Omega\cdot L$ is the generator of rotations around the Ω-axis, in the sense that

$$e^{t\Omega\cdot L}u(x)=u\left(e^{t\Theta}x\right),\quad \forall\, u\in L^2(\mathbb{R}^d),$$

where Θ is the skew symmetric matrix given by

$$\Theta=\begin{pmatrix}0 & |\Omega|\\ -|\Omega| & 0\end{pmatrix}\text{ for } d=2,\text{ and }\Theta=\begin{pmatrix}0 & \Omega_3 & -\Omega_2\\ -\Omega_3 & 0 & \Omega_1\\ \Omega_2 & -\Omega_1 & 0\end{pmatrix}\text{ for } d=3.$$

Clearly, this is a unitary operator on both $L^2(\mathbb{R}^d)$ and Σ. It was shown in [2] that if $\psi(t,x)$ solves (1.1), i.e., the GP equation with rotation, then

$$\Psi(t,x):=\left(e^{t\Omega\cdot L}\psi(t,\cdot)\right)(x), \tag{3.3}$$

solves the following nonlinear Schrödinger equation with time-dependent potential:

$$i\partial_t\Psi=-\frac{1}{2}\Delta\Psi+W_\Omega(t,x)\Psi+a|\Psi|^{2\sigma}\Psi,\quad \Psi_{|t=0}=\psi_0(x)\,. \tag{3.4}$$

Here, the new potential W_Ω is given by

$$W_\Omega(t,x):=e^{t\Omega\cdot L}V(x)\equiv V\left(e^{t\Theta}x\right).$$

The global existence result for (1.1) then directly translates to the existence of a unique global solution $\Psi\in C(\mathbb{R}_t;\Sigma)$ to (3.4) (see also [6] for related results). Moreover, we have that (3.4) conserves the total mass, i.e., $N(\Psi(t,\cdot))=N(\psi_0)$ for all $t\in\mathbb{R}$. The associated energy, however, is *no longer conserved* unless $V(x)$ is rotationally or at least axisymmetric w.r.t. Ω, cf. [2] for more details.

Corollary 3.3. *Under the same assumptions as in Theorem 3.1 it holds: For all $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that if $\psi_0\in\Sigma$ satisfies*

$$\inf_{\varphi\in\mathcal{G}_\Omega}\|\psi_0-\varphi\|_\Sigma<\delta,$$

then the solution $\Psi\in C(\mathbb{R}_t,\Sigma)$ to (3.4) with $\psi(0,x)=\psi_0\in\Sigma$ satisfies

$$\sup_{t\in\mathbb{R}}\inf_{\varphi\in\mathcal{G}_\Omega}\|\Psi(t,\cdot)-e^{t\Omega\cdot L}\varphi(\cdot)\|_\Sigma<\varepsilon.$$

In other words, we have orbital stability of the set $e^{t\Omega\cdot L}\mathcal{G}_\Omega$ under the dynamics of (3.4). To the best of our knowledge, this is the only orbital stability result for nonlinear Schrödinger equations with a time-dependent potential available to date.

In the particular situation where V is *rotationally symmetric*, i.e., $V(x)=\frac{1}{2}\omega^2|x|^2$, one finds

$$W_\Omega(t,x)=V(x),\quad \text{for any }\Omega\in\mathbb{R}^d,$$

yielding the usual Gross-Pitaevskii equation for (harmonically) trapped Bose gases

$$i\partial_t\Psi=-\frac{1}{2}\Delta\Psi+\frac{1}{2}\omega^2|x|^2+a|\Psi|^{2\sigma}\Psi,\quad \Psi_{|t=0}=\psi_0(x)\,, \tag{3.5}$$

In contrast to (3.4), this equation does conserve the associated Gross-Pitaveskii energy, $E_0(\Psi(t,\cdot))=E_0(\psi_0)$, for all $t\in\mathbb{R}$. The orbital stability result proved above then has the following consequence:

Corollary 3.4. *Let Assumption 1 hold and V be rotationally symmetric. Then*

$$\mathcal{O} = \cup_{(\Omega\in\mathbb{R}^d;|\Omega|<\omega)} \left(e^{t\Omega\cdot L}\mathcal{G}_\Omega\right),$$

is an orbitally stable set of solutions to (3.5).

The usual orbital stability result for ground states associated to (3.5) applies to $\mathcal{G}_0$, see, e.g., [8]. Note that if, for some Ω, all minimizers $\varphi \in \mathcal{G}_\Omega$ are rotationally symmetric, then $e^{t\Omega\cdot L}\mathcal{G}_\Omega = \mathcal{G}_\Omega = \mathcal{G}_0$. However, the results of [20, 26, 27] show that, in general, $\varphi \in \mathcal{G}_\Omega$ is *not* rotationally symmetric, in which case $e^{t\Omega\cdot L}\mathcal{G}_\Omega$, does *not* contain stationary solutions to (3.5) given by $\Psi(t,x) = \Phi(x)e^{i\mu t}$. Again, to the best of our knowledge, this is the only orbital stability result for (3.5) based on non-stationary solutions.

4. A resonance-type phenomenon in non-isotropic potentials

All the preceding results are obtained under the condition $|\Omega| < \omega$, which is necessary for the existence of nonlinear ground states. However, one may wonder (in particular in view of Corollary 3.4) if there are any qualitative changes to the time-dependent solution of (1.1) for $|\Omega| \geqslant \omega$. At least in the case of *non-isotropic potentials* $V(x)$, we will see below that this is indeed the case.

To this end, we denote for $\psi(t,\cdot) \in \Sigma$, the quantum mechanical mean position and momentum by

$$X(t) := \int_{\mathbb{R}^d} x|\psi(t,x)|^2\,dx, \quad P(t) := -i\int_{\mathbb{R}^d} \overline{\psi}(t,x)\nabla\psi(t,x)\,dx.$$

Lemma 4.1. *Let $\psi \in C(\mathbb{R}_t;\Sigma)$ be a solution to* (1.1)*, then, for all $t \in \mathbb{R}$:*

$$\begin{aligned} X(t) &= X(0) + \int_0^t P(s) - \Omega\wedge X(s)\,ds \\ P(t) &= P(0) - \int_0^t \nabla V(X(s)) + \Omega\wedge P(s)\,ds. \end{aligned} \tag{4.1}$$

This system can be regarded as a generalization of the results in [24, Section 6], obtained for $\Omega = 0$. Note that the nonlinearity does not enter in (4.1).

Proof. We shall assume that ψ is sufficiently smooth (and decaying) such that all of our computations below are rigorous. A classical density argument, combined with the continuous dependence of ψ on its initial data, then allows us to extend the result to solutions $\psi \in C(\mathbb{R}_t;\Sigma)$.

We start by calculating the time derivative of X:

$$\begin{aligned} \dot{X} &= 2\mathrm{Re}\langle\partial_t\psi, x\psi\rangle = 2\mathrm{Re}\langle i(\tfrac{1}{2}\Delta\psi - V(x)\psi - a|\psi|^{2\sigma}\psi + (\Omega\cdot L)\psi), x\psi\rangle \\ &= \mathrm{Re}\langle i\Delta\psi, x\psi\rangle + 2\mathrm{Re}\langle i(\Omega\cdot L)\psi, x\psi\rangle + 2\mathrm{Im}\,\underbrace{\langle V(x)\psi + a|\psi|^{2\sigma}\psi, x\psi\rangle}_{\in\mathbb{R}} \\ &\equiv J_1 + J_2. \end{aligned}$$

An integration by parts then implies

$$J_1 = \mathrm{Re}\langle -i\nabla\psi, \nabla(x\psi)\rangle = \mathrm{Im}\langle\nabla\psi, x\nabla\psi\rangle + \mathrm{Re}\langle -i\nabla\psi, \psi\nabla x\rangle = P.$$

The term J_2 can be rewritten using $(\Omega\cdot L) = -i(\Omega\wedge x)\cdot\nabla$ and integration by parts

$$\begin{aligned} J_2 &= 2\mathrm{Re}\langle(\Omega\wedge x)\cdot\nabla\psi, x\psi\rangle = 2\mathrm{Re}\sum_{\ell,j=1}^d \langle\partial_{x_j}\psi, (\Omega\wedge x)_j x_\ell\psi\rangle e_\ell \\ &= -2\sum_{j=1}^d \langle\psi, (\Omega\wedge x)_j\psi\rangle e_j - 2\mathrm{Re}\langle x\psi, (\Omega\wedge x)\cdot\nabla\psi\rangle = -2\langle\psi, (\Omega\wedge x)\psi\rangle - J_2, \end{aligned}$$

which implies that

$$J_2 = -\langle \psi, (\Omega \wedge x)\psi \rangle = -\Omega \wedge X.$$

In summary this yields the following equation of motion for X:

$$\dot{X} = P - \Omega \wedge X, \tag{4.2}$$

which is the time-differentiated version of the first equation in (4.1).

Next, we calculate the time-derivative of P as:

$$\begin{aligned}\dot{P} &= 2\mathrm{Re}\langle i\partial_t\psi, \nabla\psi\rangle = 2\mathrm{Re}\langle V(x)\psi + a|\psi|^{2\sigma}\psi - \tfrac{1}{2}\Delta\psi - (\Omega\cdot L)\psi, \nabla\psi\rangle \\ &\equiv I_1 + I_2 + I_3 + I_4.\end{aligned}$$

For the first term, a straightforward integration by parts yields

$$I_1 = -2\mathrm{Re}\langle \nabla\big(V\psi\big), \psi\rangle = -2\int_{\mathbb{R}^d} \nabla V(x)|\psi(t,x)|^2\, dx - I_1,$$

which implies

$$I_1 = -\int_{\mathbb{R}^d} \nabla V(x)|\psi(t,x)|^2\, dx = -\nabla V(X),$$

since $\nabla V(x) = \sum_{j=1}^d \omega_j^2 x_j$. Furthermore, I_2 vanishes, since

$$I_2 = \frac{a}{\sigma+1}\int_{\mathbb{R}^d} \nabla\big(|\psi|^{2(\sigma+1)}\big)\, dx = 0,$$

and one also finds $I_3 = -\mathrm{Re}\langle\Delta\psi, \nabla\psi\rangle = 0$. Finally, we compute, using standard vector identities

$$I_4 = -2\mathrm{Re}\langle(\Omega\cdot L)\psi, \nabla\psi\rangle = -2\Omega\wedge P - I_4,$$

which implies that

$$\dot{P} = -\nabla V(X) - \Omega \wedge P, \tag{4.3}$$

i.e., the differential version of the second line in (4.1). □

Given that (4.1) constitutes a closed system for X and P, one can study its solution independently of (1.1). As a first step, we have the following global existence result.

Lemma 4.2. *For any $(X_0, P_0) \in \mathbb{R}^{2d}$, the system (4.1) admits a unique global in-time solution $(X, P) \in C^\infty(\mathbb{R}_t; \mathbb{R}^{2d})$ with $(X(0), P(0)) = (X_0, P_0)$.*

Proof. Denote $\Xi = (X, P)^\top$, then (4.2), (4.3) are equal to

$$\dot{\Xi} = M_d\Xi, \quad \Xi(0) = \Xi_0, \tag{4.4}$$

where $\Xi_0 = (X_0, P_0)^\top$, and

$$M_2 = \begin{pmatrix} 0 & |\Omega| & 1 & 0 \\ -|\Omega| & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & |\Omega| \\ 0 & -\omega_2^2 & -|\Omega| & 0 \end{pmatrix} \quad \text{for } d = 2,$$

and

$$M_3 = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 & 1 & 0 & 0 \\ -\Omega_3 & 0 & \Omega_1 & 0 & 1 & 0 \\ \Omega_2 & -\Omega_1 & 0 & 0 & 0 & 1 \\ -\omega_1^2 & 0 & 0 & 0 & \Omega_3 & -\Omega_2 \\ 0 & -\omega_2^2 & 0 & -\Omega_3 & 0 & \Omega_1 \\ 0 & 0 & -\omega_3^2 & \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \quad \text{for } d = 3.$$

Equation (4.4) is a linear matrix-valued ordinary differential equation with constant coefficients. Thus, (4.4), and equivalently (4.1), admits a unique smooth solution given by:

$$\Xi(t) = e^{tM_d}\Xi_0, \quad \text{for all } t \in \mathbb{R}.$$

□

To simplify the following discussion, we shall assume that $\Omega \in \mathbb{R}^3$ is aligned with one of the coordinate axes, say, $\Omega = (0, 0, |\Omega|)^\top$. In this way, (1.3) automatically holds and thus the two-dimensional situation is included in what follows.

Proposition 4.3. *Let $\Omega = (0, 0, |\Omega|)^\top$. Assume that*

$$\omega_1 \neq \omega_2 \text{ and } \min\{\omega_1, \omega_2\} \leqslant |\Omega| \leqslant \max\{\omega_1, \omega_2\}. \tag{4.5}$$

Then for all $(X_0, P_0) \in \mathbb{R}^{2d} \setminus \mathcal{H}$, where $\mathcal{H} = \mathcal{H}(\omega_1, \dots, \omega_d, \Omega)$ is a linear subspace of $\mathbb{R}^{2d}$, it holds

$$\lim_{t\to+\infty} |X(t)| = \lim_{t\to+\infty} |P(t)| = +\infty, \quad \text{or} \quad \lim_{t\to-\infty} |X(t)| = \lim_{t\to-\infty} |P(t)| = +\infty.$$

Moreover, if both inequalities in (4.5) are strict, this growth is exponentially fast and $\dim \mathcal{H} = 2(d-1)$. If, however $|\Omega| \in \{\omega_1, \omega_2\}$, then the growth is only linear in time and $\dim \mathcal{H} = 2d - 1$.

Proof. Observe that for $\Omega = (0, 0, |\Omega|)^\top$, the matrix M_3 decomposes as a direct sum of M_2 and the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_3^2 & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of M_3 is

$$\det(\lambda - M_3) = \det(\lambda - M_2) \cdot \det(\lambda - A) = \det(\lambda - M_2) \cdot (\lambda^2 + \omega_3^2).$$

Note that $\lambda^2 + \omega_3^2$ has purely imaginary roots, leading to bounded oscillations in the solution of (4.1). Thus, for both $d = 2$ and $d = 3$ the characteristic polynomial of M_2 is the only possible source of growth in the solution. One finds that

$$\det(\lambda - M_2) = \lambda^4 + b\lambda^2 + c$$

with

$$b = 2|\Omega|^2 + \omega_1^2 + \omega_2^2 \quad \text{and} \quad c = \big(|\Omega|^2 - \omega_1^2\big)\big(|\Omega|^2 - \omega_2^2\big).$$

As a quadratic polynomial in λ^2, it has discriminant

$$D = \big(\omega_1^2 - \omega_2^2\big)^2 + 8|\Omega|^2\big(\omega_1^2 + \omega_2^2\big) > 0,$$

and thus $\lambda^2 \in \mathbb{R}$. This implies that a necessary condition for the fact that at least one of the two limits

$$\lim_{t\to\pm\infty} |\Xi(t)| = +\infty,$$

is that $\lambda^2 \geqslant 0$. This growth occurs on $\mathbb{R}^{2d} \setminus \mathcal{H}$, where $\mathcal{H}$ is the orthogonal complement of the eigenspace corresponding to the real eigenvalue(s) λ.

Computing the roots, we find that since $b > 0$, the root

$$\lambda^2 = \frac{-b - \sqrt{b^2 - 4c}}{2} < 0.$$

In addition, the other root satisfies

$$\lambda^2 = \frac{-b + \sqrt{b^2 - 4c}}{2} \geqslant 0, \text{ if and only if } c \leqslant 0.$$

The latter is equivalent to $\min\{\omega_1, \omega_2\} \leqslant |\Omega| \leqslant \max\{\omega_1, \omega_2\}$.

Now if $c < 0$ then $\lambda^2 > 0$. Hence, the system has a positive and a negative simple eigenvalue, implying exponential growth for $t \to \pm\infty$ and co-dimension of

$\mathcal{H}$ equal to 2. The fact that both X and P grow individually can be seen from computing the eigenvector $V = (v_1, v_2, v_3, v_4)^\top$ associated to λ. This can be done using the block structure of M_2 to derive a new eigenvalue equation for $(v_1, v_2)^\top$, given by

$$\begin{pmatrix} |\Omega|^2 - \omega_1^2 & -2\lambda|\Omega| \\ 2\lambda|\Omega| & |\Omega|^2 - \omega_2^2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

In addition, one finds that

$$\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} \lambda & |\Omega| \\ -|\Omega| & \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This yields the expression for V after which a straightforward but somewhat tedious analysis leads to the desired conclusion.

When $c = 0$ then $\lambda = 0$ is a double eigenvalue, in which case one needs to study the dimension $d_0 \in \mathbb{N}$ of the associated eigenspace. A straightforward computation shows that if $\omega_1 = \omega_2$ (the axisymmetric case), then $d_0 = 2$ is maximal and hence the solution does not grow in t. By contrast if $\omega_1 \neq \omega_2$, then $d_0 = 1$, and there exists a linearly independent solution $\propto t$, stemming from the eigenvector $V = (1, 0, 0, -|\Omega|)^\top$. □

Remark 4.4. In the case without rotation, i.e. $|\Omega| = 0$, one finds

$$\lambda^2 = -\frac{\omega_1^2 + \omega_2^2}{2} \pm \left|\frac{\omega_1^2 - \omega_2^2}{2}\right|,$$

which implies $\lambda = \pm i\omega_1, \pm i\omega_2$, and thus a purely oscillatory solution.

We are now in position to prove the second main result of this work.

Theorem 4.5 (Resonance in non-isotropic potentials)**.** *Let Assumption 1 hold and $\Omega = (0, 0, |\Omega|)^\top$. If condition (4.5) holds and if $\psi_0 \in \Sigma$ is such that the associated averages $(X_0, P_0) \notin \mathcal{H}$, then the solution $\psi \in C(\mathbb{R}_t; \Sigma)$ satisfies*

$$\lim_{t\to+\infty} \|\psi(t,\cdot)\|_\Sigma = +\infty, \ \text{or} \ \lim_{t\to-\infty} \|\psi(t,\cdot)\|_\Sigma = +\infty.$$

Proof. Recall that both (1.1) and (4.1) have unique solutions. Thus, if $\psi(t,\cdot)$ solves (1.1) with initial data $\psi_0 \in \Sigma$ and if $X_0 = \langle \psi_0, x\psi_0 \rangle$ and $P_0 = -i\langle \psi_0, \nabla\psi_0 \rangle$ are the initial data to (4.1), then

$$X(t) = \langle \psi(t,\cdot), x\psi(t,\cdot) \rangle, \quad P(t) = -i\langle \psi(t,\cdot), \nabla\psi(t,\cdot) \rangle, \quad \forall\, t \in \mathbb{R}.$$

By Cauchy-Schwarz

$$|X| \leqslant \|\psi\|_{L^2} \|x\psi\|_{L^2}, \quad |P| \leqslant \|\psi\|_{L^2} \|\nabla\psi\|_{L^2},$$

which together with the results of Proposition 4.3 and the mass conservation property (1.4) implies the assertion of the theorem. □

Remark 4.6. The fact that there are nontrivial $\psi_0 \in \Sigma$ for which the associated $(X_0, P_0) \notin \mathcal{H}$, can be easily seen by considering initial data of the form:

$$\psi_0(x) = e^{ip_0\cdot x} e^{-(x-x_0)^2/2}, \quad x_0, p_0 \in \mathbb{R}^d.$$

In this case, $X_0 = \pi^{d/2} x_0$ and $P_0 = \pi^{d/2} p_0$ and thus one obtains a growing Σ-norm of the solution ψ provided $(x_0, p_0) \notin \mathcal{H}$.

Indeed, the proof of Proposition 4.3 shows that if condition (4.5) holds, there are solutions to (1.1) for which

$$\|\nabla\psi(t,\cdot)\|_{L^2}, \|x\psi(t,\cdot)\|_{L^2} \to \infty,$$

if $t \to +\infty$, or $t \to -\infty$. In other words, these solutions develop frequencies which are larger than those controlled by the Σ-norm and, in addition, their mass is transferred to infinity, resulting in a weaker decay of ψ. This is in sharp contrast to the case $\omega_1 = \omega_2 = \omega_3$, where (1.1) is equivalent, up to the time-dependent change of variables (3.3), to the classical NLS with harmonic trapping (3.5). The latter conserves the energy $E_0(\Psi(t,\cdot)) = E_0(\psi_0)$, which in the defocusing case $a > 0$ directly yields the uniform bound

$$\|\Psi(t,\cdot)\|_\Sigma = \|\psi(t,\cdot)\|_\Sigma \leqslant E_0(\psi_0), \quad \forall\, t \in \mathbb{R}.$$

Remark 4.7. The growth of (higher order) Sobolev-norms of solutions to nonlinear Schrödinger equations with time-dependent, quadratic potentials was also studied in [6]. One can check that (3.4) (obtained from (1.1), via the change of variables) falls into the class of models for which exponentially growing upper bounds were established in [6]. Theorem 4.5 shows that, in general, such exponential growth indeed occurs, and that this is true even for linear Schrödinger equations. There exponential growth naturally occurs in the case of (even only partially) repulsive harmonic potentials. We finally mention that very recently a somewhat similar instability phenomenon for linear Schrödinger equations with quadratic time-dependent Hamiltonian has been established in [3].

It is very likely that additional (in-)stability phenomena appear for general $\Omega \in \mathbb{R}^3$, not necessarily aligned to one of the axis. However, the calculations of the roots of the associated degree 6 characteristic polynomial become extremely involved, see also [5]. Since our main goal was to establish an instability result for ψ we do not investigate the general case in full detail.

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