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FRENET-FRAME CONTINUITY AND EXISTENCE OF THE SERRET-FRENET EQUATIONS AT ZERO-CURVATURE POINTS

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ABSTRACT

The continuity of the Frenet frame and existence of the Serret-Frenet equations at zero-curvature points are established for a general three-dimensional curve with arbitrary parameterization. To this end, the Frenet-Euler angles, referred to for brevity as *Frenet angles*, are used. Definition of the *Frenet bank angle* is used to prove existence of the curve normal vector at the zero curvature points.

1. INTRODUCTION

The Frenet frame defines the geometry of a curve, described in its parametric form by the equation $\mathbf{r}(t)=\begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$, where t is the curve parameter [1 – 2]. The Frenet frame is defined by three orthogonal unit vectors that represent the curve tangent vector $\mathbf{t}$, normal vector $\mathbf{n}$, and the bi-normal vector $\mathbf{b}$. The transformation matrix that defines the orientation of the Frenet frame is written as $\mathbf{A}_f=\begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix}$. The unit vector tangent to the curve is defined by differentiating the curve equation with respect to the curve arc length s, that is, $\mathbf{t}=\partial\mathbf{r}/\partial s=\mathbf{r}_s$, where $a_s=\partial a/\partial s$. Norm of the curvature vector $\mathbf{r}_{ss}$, defined as $\mathbf{r}_{ss}=\partial\mathbf{t}/\partial s$, defines the curve curvature κ as $\kappa=|\mathbf{r}_{ss}|=1/R$, where $R=R(s)$ is the radius of curvature of the curve. The unit vector $\mathbf{n}$ normal to the curve is defined along the curvature vector $\mathbf{r}_{ss}$ according to $\mathbf{n}=\mathbf{r}_{ss}/|\mathbf{r}_{ss}|$. The bi-normal vector $\mathbf{b}$ is defined using the cross product $\mathbf{b}=\mathbf{t}\times\mathbf{n}$ [1 – 2].

At points on the curve with zero curvature, the procedure described above fails to define the normal vector $\mathbf{n}$ and the Frenet frame of the curve at these zero-curvature points. Consequently, at these points, the bi-normal vector $\mathbf{b}$ and its derivative $\mathbf{b}_s$ cannot be defined. At the points of curvature singularity, the Serret-Frenet equations $\mathbf{t}_s=\kappa\mathbf{n}$, $\mathbf{n}_s=-\kappa\mathbf{t}+\tau\mathbf{b}$, and $\mathbf{b}_s=-\tau\mathbf{n}$, where τ is the curve torsion, fail and all the derivatives of the unit vectors that define the Frenet frame cannot be determined. Furthermore, the curve torsion τ at the zero-curvature points cannot be determined using the norm of the vector $\mathbf{b}_s$.

This paper provides a proof that the Frenet frame and the Serret-Frenet equations are well defined at the zero-curvature points. The curvature vector $\mathbf{r}_{ss}$ should be viewed as a vector along a well-defined normal vector $\mathbf{n}$ and such a curvature vector has a norm defined by the curve curvature κ that varies continuously and can assume zero value without affecting the definition of the normal vector $\mathbf{n}$ or the continuity of the Frenet frame. Consequently, all derivatives that appear in the Serret-Frenet equations are well-defined at all points at which tangent vector $\mathbf{t}$ can be defined. The proof provided in this paper is established using the concept of the *Frenet angles*, use of these angles facilitate demonstrating that the Frenet frame and the Serret-Frenet equations are well-defined regardless of the value of the curve curvature. The analysis presented also demonstrates the continuity of the centrifugal force $m\dot{s}^2/R$ of a vehicle or a particle with mass m at the inflection points which have zero curvature [3 – 4].

2. TANGENT VECTOR AND FRENET ANGLES

The *tangent vector* is defined as $\mathbf{r}_t = \partial \mathbf{r}/\partial t = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T$, where $a_t = \partial a/\partial t$. The norm of this vector is $|\mathbf{r}_t| = \sqrt{x_t^2 + y_t^2 + z_t^2}$. Therefore, the unit tangent to the curve is defined as $\mathbf{t} = \partial \mathbf{r}/\partial s = \begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T / |\mathbf{r}_t|$. This unit tangent can always be written in terms of two angles as

$$\mathbf{t} = \mathbf{r}_s = \frac{1}{|\mathbf{r}_t|}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\theta \\ \sin\psi\cos\theta \\ \sin\theta \end{bmatrix} \tag{1}$$

where the angles ψ and θ are defined according to

$$\left.\begin{aligned} &\cos\psi = x_t \Big/ \sqrt{x_t^2 + y_t^2}, \quad \sin\psi = y_t \Big/ \sqrt{x_t^2 + y_t^2}, \\ &\cos\theta = \sqrt{x_t^2 + y_t^2} \Big/ |\mathbf{r}_t|, \quad \sin\theta = z_t / |\mathbf{r}_t|, \end{aligned}\right\} \tag{2}$$

The angles ψ and θ are referred to as the *Frenet horizontal curvature angle* and the *Frenet vertical-development angle* [3 – 4]. They can be used in a three-angle Euler sequence of rotations to define a Frenet frame that does not suffer from discontinuities and also define the derivatives that appear in the Serret-Frenet equations. The third angle ϕ, called the *Frenet bank angle*, is used to measure the deviation of the centrifugal force vector from the horizontal plane. The sequence of rotation used for these angles is $Z,-Y,-X$ with the negative signs used to give a physical interpretation used in the constructions of the roads and railroad tracks [3 – 4].

3. CURVATURE VECTOR

For a general three-dimensional curve, one has the differential relationship $ds = |\mathbf{r}_t| dt$, or alternatively $\partial s/\partial x = |\mathbf{r}_t| = \sqrt{x_t^2 + y_t^2 + z_t^2}$. It follows that $\partial\left(1/|\mathbf{r}_t|\right)/\partial x = -\left(x_t x_{tt} + y_t y_{tt} + z_t z_{tt}\right)/|\mathbf{r}_t|^3$. The *curvature vector* is $\mathbf{r}_{ss} = \partial^2 \mathbf{r}/\partial s^2 = \left(\partial\left(\begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T / |\mathbf{r}_t|\right)/\partial t\right)\left(\partial t/\partial s\right)$, which can be written using the equation $\partial t/\partial s = 1/|\mathbf{r}_t|$ as $\mathbf{r}_{ss} = \left(1/|\mathbf{r}_t|\right)\left(\partial\left(\begin{bmatrix} x_t & y_t & z_t \end{bmatrix}^T / |\mathbf{r}_t|\right)/\partial t\right)$. This equation yields

$$\begin{aligned} \mathbf{r}_{ss} &= \frac{1}{|\mathbf{r}_t|^4}\left(-\left(x_t x_{tt} + y_t y_{tt} + z_t z_{tt}\right)\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + |\mathbf{r}_t|^2 \begin{bmatrix} x_{tt} \\ y_{tt} \\ z_{tt} \end{bmatrix}\right) \\ &= \frac{1}{|\mathbf{r}_t|^4}\left(-\left(\mathbf{r}_t^T \mathbf{r}_{tt}\right)\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + \left(\mathbf{r}_t^T \mathbf{r}_t\right)\begin{bmatrix} x_{tt} \\ y_{tt} \\ z_{tt} \end{bmatrix}\right) = \frac{1}{|\mathbf{r}_t|^2}\begin{bmatrix} x_{tt} - \alpha_c x_t \\ y_{tt} - \alpha_c y_t \\ z_{tt} - \alpha_c z_t \end{bmatrix} \end{aligned} \tag{3}$$

where $\alpha_c = \left(x_t x_{tt} + y_t y_{tt} + z_t z_{tt}\right)/|\mathbf{r}_t|^2 = \mathbf{r}_t^T \mathbf{r}_{tt}/|\mathbf{r}_t|^2$. The *curvature* κ is magnitude of the curvature vector defined as

$$\begin{aligned} \kappa = \kappa(t) &= \sqrt{\left(x_{tt} - \alpha_c x_t\right)^2 + \left(y_{tt} - \alpha_c y_t\right)^2 + \left(z_{tt} - \alpha_c z_t\right)^2} \Big/ |\mathbf{r}_t|^2 \\ &= \sqrt{\left(x_{tt}^2 + y_{tt}^2 + z_{tt}^2\right) - \left(\alpha_c\right)^2 |\mathbf{r}_t|^2} \Big/ |\mathbf{r}_t|^2 \end{aligned} \tag{4}$$

At zero-curvature points, $x_{tt} = y_{tt} = z_{tt} = 0$. Two orthogonal unit vectors $\mathbf{n}_h$ and $\mathbf{n}_v$, which are orthogonal to the tangent vector $\mathbf{t}$, can be defined as

$$\left.\begin{aligned}\mathbf{n}_h &= \left(1\Big/\sqrt{x_t^2+y_t^2}\right)\begin{bmatrix}-y_t & x_t & 0\end{bmatrix}^T,\\ \mathbf{n}_v &= \left(1\Big/|\mathbf{r}_t|\sqrt{x_t^2+y_t^2}\right)\begin{bmatrix}-x_t z_t & -y_t z_t & x_t^2+y_t^2\end{bmatrix}^T\end{aligned}\right\} \tag{5}$$

The projection of the curvature vector $\mathbf{r}_{ss}$ along these two vectors leads to $\mathbf{r}_{ss} = \alpha_h \mathbf{n}_h + \alpha_v \mathbf{n}_v = \kappa \mathbf{n}$, which can be written as

$$\mathbf{n} = (\alpha_h/\kappa)\mathbf{n}_h + (\alpha_h/\kappa)\mathbf{n}_v = (\cos\phi)\mathbf{n}_h - (\sin\phi)\mathbf{n}_v \tag{6}$$

where $\kappa = \sqrt{\alpha_h^2 + \alpha_v^2}$, $\mathbf{n} = (\alpha_h/\kappa)\mathbf{n}_h + (\alpha_v/\kappa)\mathbf{n}_v$, $\tan\phi = -\alpha_v/\alpha_h$, and

$$\left.\begin{aligned}\alpha_h &= \left(y_{tt}x_t - x_{tt}y_t\right)\Big/|\mathbf{r}_t|^2\sqrt{x_t^2+y_t^2},\\ \alpha_v &= \left(z_{tt}\left(x_t^2+y_t^2\right) - z_t\left(x_t x_{tt} + y_t y_{tt}\right)\right)\Big/\left(|\mathbf{r}_t|^3\sqrt{x_t^2+y_t^2}\right),\end{aligned}\right\} \tag{7}$$

The angle ϕ, called *Frenet bank angle*, defines the super-elevation of the curve *osculating plane* (OP), which contains the normal vector and the centrifugal force. For a motion-trajectory curve with zero Frenet bank angle, the centrifugal force lies in a plane parallel to the horizontal plane. Since the vectors $\mathbf{n}_h$ and $\mathbf{n}_v$ are functions of the first derivatives only, their existence is ensured. Therefore, existence of the angle ϕ at zero-curvature points ensures existence of the normal vector and the Serret-Frenet equations despite $\kappa = \alpha_h = \alpha_v = 0$ at these points.

4. EXISTENCE OF ϕ AND n

The Frenet bank angle, as previously defined, can be computed using the ratio α_v/α_h as

$$\tan\phi = -\frac{\alpha_v}{\alpha_h} = -\frac{\left(z_{tt}\left(x_t^2+y_t^2\right) - z_t\left(x_t x_{tt} + y_t y_{tt}\right)\right)}{|\mathbf{r}_t|\left(y_{tt}x_t - x_{tt}y_t\right)} \tag{8}$$

At the points of curvature singularity, $\alpha_h = \alpha_v = 0$, and the Frenet bank angle is not defined. To define this ratio, L'Hopital rule is used with the curvature singularity conditions $\kappa = x_{tt} = y_{tt} = z_{tt} = 0$. Using this procedure, one has

$$\lim_{\kappa \to 0} \frac{\partial \alpha_v}{\partial \alpha_h} = \frac{z_{ttt}\left(x_t^2 + y_t^2\right) - z_t\left(x_t x_{ttt} + y_t y_{ttt}\right)}{\left|\mathbf{r}_t\right|\left(y_{ttt} x_t - x_{ttt} y_t\right)} \tag{9}$$

This limit defines the Frenet bank angle at the zero-curvature points.

To demonstrate existence of the normal and Serret-Frenet equations, the tangent vector. expressed in terms of Frenet angles is differentiated with respect to the arc length s. This leads to

$$\mathbf{r}_{ss} = \psi_s \cos\theta \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} + \theta_s \begin{bmatrix} -\cos\psi \sin\theta \\ -\sin\psi \sin\theta \\ \cos\theta \end{bmatrix} \tag{10}$$

Comparing this equation with previous development, one has

$$\alpha_h = \psi_s \cos\theta, \quad \alpha_v = \theta_s \tag{11}$$

Furthermore, the normal vector is well defined as

$$\mathbf{n} = \begin{bmatrix} -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi \\ \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi \\ -\cos\theta\sin\phi \end{bmatrix} = \cos\phi \begin{bmatrix} \sin\psi \\ -\cos\psi \\ 0 \end{bmatrix} - \sin\phi \begin{bmatrix} -\cos\psi\sin\theta \\ -\sin\psi\sin\theta \\ \cos\theta \end{bmatrix} \tag{12}$$

Therefore, the Frenet frame is defined by the equation

$$\begin{aligned} \mathbf{A}_f &= \begin{bmatrix} \mathbf{t} & \mathbf{n} & \mathbf{b} \end{bmatrix} \\ &= \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & -\sin\psi\sin\phi - \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\psi\sin\phi - \sin\psi\sin\theta\cos\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \end{aligned} \tag{13}$$

This equation defines the Frenet frame everywhere including at points with curvature singularities. According to this description, the Frenet frame vectors $\mathbf{t}, \mathbf{n}$, and $\mathbf{b}$ are all differentiable and therefore, the Serret-Frenet equations exist at the points of zero curvature.

5. CONCLUSIONS

Continuity of the Frenet frame and existence of the curve normal and Serret-Frenet equations at zero-curvature points are established using the Frenet angles. The analysis of this paper shows that the unit vectors that define the axes of the Frenet frame are all differentiable for smooth curves regardless of the value of the curvature.

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Compliance with Ethical Standards:

The author declares that they have no conflict of interest

Data Availability Statement

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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