Remarks on the Tree Property and its Generalizations

by

Carl Thomas Dean
B.S., Duquesne University, 2016
M.S., University of Illinois at Chicago, 2019

THESIS

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Defense Committee:
Dima Sinapova, Chair and Advisor
Maxwell Levine, University of Freiburg
Fillipo Calderoni
James Freitag
David E. Marker

This thesis is dedicated to my parents, whose support is unwavering.

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SUMMARY

In this thesis we contribute to the outstanding problem of determining the extent to which the tree property and super tree property can hold simultaneously in the set theoretic universe. The main results proven in order of appearance are the following:

Theorem. *Let $\tau < \kappa$ with τ regular and κ supercompact. Let $\mathbb{M}$ be the forcing to make $\kappa = \tau^{++}$ with ITP holding at κ. In $V[\mathbb{M}]$ adding Cohen subsets of τ does not destroy ITP at τ^{++}.*

Theorem. *Let $(\kappa_n : n < \omega)$ and $(\lambda_n : n < \omega)$ be increasing sequences of supercompact cardinals with $\sup_n \kappa_n < \lambda_0$, and $\kappa_0 = \kappa$. There is a forcing extension in which κ is singular strong limit and the tree property holds simultaneously at κ^{+n} for each natural number $n \geq 1$.*

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CHAPTER 1

INTRODUCTION

Set theory has taken many forms since its humble beginnings when Cantor proved $|\mathbb{R}| > |\mathbb{N}|$. An important area of research in set theory is the study of large cardinals, which are infinite cardinals whose existence prove that ZFC is consistent. By Gödel's Incompleteness Theorem we know that such cardinals cannot be proven to exist using only the ZFC axioms - provided that ZFC is consistent - and so it is natural to wonder why anyone would be interested in studying these cardinals at all. One response is that asserting the existence of certain large cardinals provide a *set-theoretic* description of axioms we know must exist from Gödel. Even further, these large cardinal properties are often generalizations of properties about ω and can be understood without having formal training in logic.

This thesis concerns itself primarily with the tree property and one of its generalizations. The tree property is a compactness property that takes inspiration from König's Lemma that every finitely branching tree with infinitely many nodes has a branch. The natural extension of this result – asking whether there is a branch through every tree of height ω_1 where each level has countably many nodes – was proven false by Aronszajn and was described in (Kurepa, 1935). Aronszajn's original construction involved a tree whose nodes were certain sequences of rationals, and where a branch through this tree, if it existed, was an increasing sequence of rationals of length ω_1. Counterexamples to this aforementioned statement have become known as ω_1-Aronszajn trees.

In general, an infinite cardinal κ has the tree property if there is a branch through every tree of height κ whose levels have size less than κ. Using this terminology we know ω has the tree property due to König's Lemma and $\aleph_1$ does not have the tree property due to Aronszajn. What about for cardinals greater than $\aleph_1$? Interestingly, results of Mitchell and Silver (Mitchell, 1972) show that it is consistent that $\aleph_2$ has the tree property if and only if it is consistent that there is a weakly compact cardinal. Given a weakly compact cardinal, Mitchell's result generalizes to the consistency of the tree property at τ^{++} for a regular τ. A general discussion about the Mitchell forcing may be found later in Section 1.8.

Given that the consistency of large cardinals are required to show that $\aleph_2$ has the tree property, it is natural to wonder whether or not large cardinals are required to show that any $\kappa > \aleph_1$ has the tree property. In the case where κ is singular it is straightforward to argue that the tree property always fails at κ, and so a more precise question is whether or not large cardinals are required to show that any regular $\kappa > \aleph_1$ has the tree property. In the case where κ is inaccessible, κ is weakly compact exactly when the tree property holds at κ.

Going further, one can ask if it is consistent for the tree property to hold *simultaneously* at every regular cardinal $\kappa > \aleph_1$. This question was originally due to Magidor in the 1970s. A first attempt would be to iterate Mitchell's original forcing, but this fails due to interference of the Cohen parts of these posets (see Section 1.10). Further, by results of Abraham, this would require large cardinal hypotheses stronger than a weakly compact. Specifically, it is known that $0^{\#}$ has consistency strength greater than the existence of a weakly compact, and Abraham showed that if the tree property holds simultaneously at $\aleph_2$ and $\aleph_3$ then $0^{\#}$ exists (Abraham, 1983).

The results in this thesis work towards this overarching goal of getting the tree property to hold simultaneously everywhere. More precisely, we are interested in using forcing and large cardinal hypotheses to construct models of ZFC in which the tree property holds simultaneously at consecutive cardinals. The first results of this kind came from the paper of Abraham just mentioned, where he constructed modulo large cardinals a forcing in which the tree property holds simultaneously at both $\aleph_2$ and $\aleph_3$. Expanding on the work of Abraham, Cummings and Foreman constructed modulo large cardinals a forcing in which the tree property holds simultaneously at $\aleph_n$ for natural numbers $n \geq 2$ (Cummings and Foreman, 1998).

One of the obstacles when trying to get the tree property everywhere occurs when trying to get the tree property at the double successor of a singular κ. More specifically, Specker proved that if $\kappa^{<\kappa} = \kappa$ then the tree property fails at κ^+ (Specker, 1949). So, for the tree property to hold at the double successor of a singular κ, SCH must fail at κ. This presents a host of difficulties, but was eventually shown to be possible in (Sinapova, 2016). Sinapova and Unger later were able to add collapses to show that it was consistent to have the tree property simultaneously at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ (Sinapova and Unger, 2018).

Instead of working with the tree property, a related area of research involves asking similar questions about a well-known generalization of the tree property, the super tree property (denoted ITP). This was originally studied by Magidor who showed that the super tree property is a combinatorial charaterization of supercompactness in the same way that the tree property is a combinatorial characterization of weak compactness (Magidor, 1974). More specifically, if κ is inaccessible then κ is supercompact exactly when the super tree property holds at κ.

Similar to the tree property, Magidor additionally proved that, if we start with a supercompact, the super tree property holds at $\aleph_2$ after forcing with Mitchell poset. Recently, Unger (Unger, 2014) and Fontenella (Fontanella, 2013) independently showed that the super tree property holds simultaneously at $\aleph_n$ for each $n \geq 2$ after forcing with the Cummings-Foreman forcing in (Cummings and Foreman, 1998). This past year it was shown that it is consistent to have a singular strong limit κ with SCH failing at κ and ITP holding at κ^+ (Cummings et al., 2020a). Even further, it was shown that κ could be made to be $\aleph_{\omega^2}$, although it is unknown at the time of this thesis whether or not κ can be made to be $\aleph_\omega$. These results are a necessary building block to argue later that it is consistent to get ITP at κ^+ and κ^{++} with κ singular strong limit.

The rest of Chapter 1 provides a brief summary of key facts about forcing, the relevant forcing posets necessary to understand later chapters, as well as a background on trees and the so-called branch lemmas that are used when arguing that the (super) tree property holds after forcing. We note that Section 1.7 is original work and provides an abstract framework for understanding the results in Chapter 4.

1.1 Notation

- $\mathcal{P}(X)$ denotes the power set of X
- $\mathcal{P}_\kappa(\lambda)$ denotes the set of all subsets $X \subseteq \lambda$ where $|X| < \kappa$
- ORD denotes the class of ordinals

- $V[\mathbb{P}]$ denotes the generic extension $V[G]$ where G is some $\mathbb{P}$-generic filter over V
- $V[\mathbb{P}] \subseteq V[\mathbb{Q}]$ means for each $\mathbb{Q}$-generic H there is a $\mathbb{P}$-generic G such that $V[G] \subseteq V[H]$.
- $V^{\mathbb{P}}$ denotes the set of all $\mathbb{P}$-names in V.
- $p \Vdash \dot{x} \in V$ means that for each $q \leq p$ there is an $r \leq q$ and y such that $r \Vdash \dot{x} = \check{y}$.
- Given a poset $(\mathbb{P}, \leq)$ and $A \subseteq \mathbb{P}$, we write $A \downarrow$ to denote the downwards closure of A and $A \uparrow$ to denote the upwards closure of A.
- If s is a sequence, then $|s|$ is the length of the sequence.
- If λ is a cardinal, then $H_\lambda = \{x \colon |tc(x)| < \lambda\}$.

1.2 Forcing

We also recall some elementary properties of forcing posets:

Definition 1.1. Let κ be an infinite cardinal and $\mathbb{P}$ be a forcing poset.

- $\mathbb{P}$ is κ-c.c. if $|A| < \kappa$ for each antichain $A \subseteq \mathbb{P}$.
- $\mathbb{P}$ is κ-Knaster if, whenever $A \subseteq \mathbb{P}$ and $|A| = \kappa$, then there is a $B \subseteq A$ with $|B| = \kappa$ where elements of B are pairwise compatible.
- $\mathbb{P}$ is κ-closed if every decreasing $(p_i \colon i < \theta)$ with $\theta < \kappa$ has a lower bound.
- $\mathbb{P}$ is (canonically) κ-directed closed if every directed set $D \subseteq \mathbb{P}$ with $|D| < \kappa$ has a (greatest) lower bound. Recall that D is directed if every two elements in D has a common extension in D.
- $\mathbb{P}$ is κ-distributive if, whenever $f \in V[G]$ is a function from $\lambda \leq \kappa$ into ORD, then $f \in V$.

- $\mathbb{P}$ is $<\kappa$-distributive if $\mathbb{P}$ is λ-distributive for each $\lambda < \kappa$.

If we have a forcing notion $\mathbb{P}$ which has some chain condition and another $\mathbb{Q}$ which is closed, it is useful to understand the closure and chain condition of these forcings after iterating $\mathbb{P}$ followed by $\mathbb{Q}$ or vice versa. This is the content of Easton's Lemma:

Lemma 1.2 (Easton's Lemma)**.** *Let κ be regular. If $\mathbb{P}$ is κ-cc and $\mathbb{Q}$ is κ-closed, then*

1. $\Vdash_{\mathbb{Q}} \mathbb{P}$ *is κ-cc.*
2. $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is $<\kappa$-distributive.*
3. *If G is $\mathbb{P}$-generic over V and H is $\mathbb{Q}$-generic over V then G and H are mutually generic (i.e G is $\mathbb{P}$-generic over $V[H]$ and vice versa).*

Proof. We do each in turn:

1. If otherwise, fix a $\mathbb{Q}$-name $\dot{f}$ for a function from $\check{\kappa}$ into $\check{\mathbb{P}}$ whose range is an antichain. Then, in V, use the closure of $\mathbb{Q}$ to construct a decreasing sequence $(q_i \colon i < \kappa)$ and a set $\{p_i \colon i < \kappa\} \subseteq \mathbb{P}$ where $q_i \Vdash \dot{f}(i) = p_i$. Since the range of $\dot{f}$ is forced to be an antichain, it follows that $\{p_i \colon i < \kappa\}$ is an antichain of size κ, a contradiction.
2. Assuming (3) is true: Let G be $\mathbb{P}$-generic over V and H be $\mathbb{Q}$-generic over $V[G]$. Further, let $f \in V[G][H]$ be a function from some $\tau < \kappa$ into ORD. By (3), we have that $V[G][H] = V[H][G]$. By (1), $\mathbb{P}$ is still κ-cc in $V[H]$ and so we may find a $\mathbb{P}$-name $\dot{f} \in V[H]$ for f of size $< \kappa$. Since $\mathbb{Q}$ is κ-closed in V, we have $\dot{f} \in V$ and so $f = \dot{f}[G] \in V[G]$.

3. It is enough to show that G is $\mathbb{P}$-generic over $V[H]$. To see this observe that if $A \subseteq \mathbb{P}$ is a maximal antichain in $V[H]$, then it has size $< \kappa$ by (1). But then $A \in V$ because $\mathbb{Q}$ is closed, and so $G \cap A \neq \emptyset$ because A is still a maximal antichain in V.

$\square$

Lemma 1.3 (Easton's Lemma Variant)**.** *Let κ be regular. If $\mathbb{P}$ is κ-closed and $\mathbb{Q}$ is $< \kappa$-distributive, then*

1. $\Vdash_{\mathbb{Q}} \mathbb{P}$ *is κ-closed.*
2. $\Vdash_{\mathbb{P}} \mathbb{Q}$ *is $< \kappa$-distributive.*

Proof. Similar to the proof of Easton's Lemma:

1. If H is $\mathbb{Q}$-generic over V and $(p_i \colon i < \tau) \in V[H]$ is a sequence with $\tau < \kappa$, then $(p_i \colon i < \tau) \in V$ because $\mathbb{Q}$ is τ-distributive and so has a lower bound because $\mathbb{P}$ is κ-closed in V.
2. If G is $\mathbb{P}$-generic over V, H is $\mathbb{Q}$-generic over $V[G]$, and $f \in V[G][H]$ is a function from some $\tau < \kappa$ into ORD. By the product lemma we have that $V[G][H] = V[H][G]$ and by (1) we have that $\mathbb{P}$ is still κ-closed in $V[H]$. This implies that $\mathbb{P}$ is $< \kappa$-distributive in $V[H]$ and so $f \in V[H]$. Since $\mathbb{Q}$ is $< \kappa$-distributive in V it follows that $f \in V$ as desired.

$\square$

Lemma 1.4 (Mixing Lemma)**.** *Assume $\mathbb{P}$ is a poset and $A \subseteq \mathbb{P}$ is an antichain. Then for any function $f \colon A \to V^{\mathbb{P}}$ there is a $\mathbb{P}$-name σ such that $a \Vdash f(a) = \sigma$ for each $a \in A$.*

Proof. Define a $\mathbb{P}$-name σ as follows:

$$\sigma = \{(\tau, p)\colon \exists a \in A,\ p \leq a \text{ and } p \Vdash \tau \in f(a)\}.$$

Technically σ might not be a set, but this can be remedied by choosing only the (τ, p) where $\tau \in \mathrm{dom}(f(a))$. To see that σ works, fix $a \in A$ and a $\mathbb{P}$-generic object G such that $a \in G$. We want to show that $f(a)[G] = \sigma[G]$. If $x \in f(a)[G]$, then we can find a $\mathbb{P}$-name τ such that $x = \tau[G]$. Since $V[G] \models \tau[G] \in f(a)[G]$, it follows that there is a $p \leq a$ such that $p \in G$ and $p \Vdash \tau \in f(a)$. But then $(\tau, p) \in \sigma$ by definition and so $x = \tau[G] \in \sigma[G]$. Conversely, if $x \in \sigma[G]$ then by definition we can find $(\tau, p) \in \sigma$ such that $p \in G$, $x = \tau[G]$, $p \leq b$ for some $b \in A$, and $p \Vdash \tau \in f(b)$. Since $a \in G$ we know that p and a are compatible. Since A is an antichain it must be that $a = b$. Therefore $p \Vdash \tau \in f(a)$ and so $\tau[G] \in f(a)[G]$. □

Lemma 1.5 (Maximality of the forcing language)**.** *If $\mathbb{P}$ is a poset and $p \in \mathbb{P}$ such that $p \Vdash \exists x \varphi(x)$, then there is a $\mathbb{P}$-name σ such that $p \Vdash_{\mathbb{P}} \varphi(\sigma)$.*

Proof. Fix $p \in \mathbb{P}$ with $p \Vdash \exists x \varphi(x)$, and define D to be the following set:

$$D = \{q \in \mathbb{P}\colon \text{there is a } \mathbb{P}\text{-name } \tau \text{ such that } q \Vdash \varphi(\tau)\}.$$

Given our assumption on p, we have that D is dense below p. Let $A \subseteq D$ be a maximal antichain in D and for each $a \in A$ let τ_a witness that $a \in D$. By the mixing lemma, there is a $\mathbb{P}$-name σ such that $a \Vdash \tau_a = \sigma$ for each $a \in A$. To show $p \Vdash \varphi(\sigma)$ we claim that $\{q \in \mathbb{P}\colon q \Vdash \varphi(\sigma)\}$ is dense below p. To see this, fix $q \leq p$ and find $d \in D$ such that $d \leq q$. Since A is a maximal antichain in D, there is an $a \in A$ and $p' \in \mathbb{P}$ such that $p' \leq a$ and $p' \leq d \leq q$. By definition of σ and τ_a, we have that $a \Vdash \varphi(\sigma)$. Therefore $p' \Vdash \varphi(\sigma)$ as desired. $\square$

Lemma 1.6. *Assume σ is a $\mathbb{P}$-name for a subset of an ordinal μ, $(p_1, p_2) \in \mathbb{P} \times \mathbb{P}$, and $(p_1, p_2) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\Gamma_L] = \sigma[\Gamma_R]$, where Γ_L/Γ_R is the canonical name for the generic of the left/right coordinate. Then $p_i \Vdash \sigma \in V$ for $i \in \{1, 2\}$.*

Proof. We just show that $p_1 \Vdash_{\mathbb{P}} \sigma \in V$. To do this we prove the following:

Claim 1.7. *For each $\alpha < \mu$, either $p_1 \Vdash_{\mathbb{P}} \alpha \in \sigma$ or $p_1 \Vdash_{\mathbb{P}} \alpha \notin \sigma$.*

Proof of Claim. Otherwise there would be some $\alpha < \mu$ as well as conditions $q_1, q_2 \leq p_1$ such that $q_1 \Vdash \alpha \in \sigma$ and $q_2 \Vdash \alpha \notin \sigma$. Without loss of generality find $q \leq p_2$ such that $q \Vdash \alpha \in \sigma$. If G is $\mathbb{P}$-generic over V containing q_2 and H is $\mathbb{P}$-generic over $V[G]$ containing q, the contradiction follows after observing that $V[G][H] \models \sigma[G] \neq \sigma[H]$.

$\square$

To finish the proof of the lemma, define $S = \{\alpha < \mu \colon p \Vdash_{\mathbb{P}} \alpha \in \sigma\}$. It is straightforward to check that $p_1 \Vdash_{\mathbb{P}} \sigma = \check{S}$. This implies $p_1 \Vdash_{\mathbb{P}} \sigma \in V$ as desired. $\square$

1.3 Trees

Let us recall the following notions related to the tree property:

Definition 1.8. Let κ be an infinite cardinal.

- $(T, <)$ is a tree if, for each $x \in T$, $pred(x) = \{y \in T \colon y < x\}$ is well-ordered.

- For $x \in T$, the height of x, denoted $ht(x)$, is the order-type of $pred(x)$. The height of T is $ht(T) = \sup_{x \in T} \{ht(x) + 1\}$.

- For $\alpha < ht(T)$, the α^{th}-level of T is defined as $Lev_\alpha(T) = \{x \in T \colon ht(x) = \alpha\}$.

- A κ-tree T is a tree of height κ where each level of T has size strictly less than κ.

- A branch b through T is a totally ordered subset of T such that $b \cap Lev_\alpha(T)$ is non-empty for each $\alpha < ht(T)$.

- The tree property holds at κ, denoted TP_κ, if every κ-tree T has a branch. The witness to the failure of TP_κ is called a κ-Aronszajn tree.

This definitions are motivated from (Weiss, 2010) and are found in (Cummings et al., 2020a) and (Hachtman and Sinapova, 2019):

Definition 1.9. Let κ and μ be cardinals with κ regular.

- A sequence $D = (d_x \colon x \in \mathcal{P}_\kappa(\mu))$ is a $\mathcal{P}_\kappa(\mu)$-list if for each $x \in \mathcal{P}_\kappa(\mu)$, $d_x \subseteq x$.

- A subset $b \subseteq \mu$ is a cofinal branch of a $\mathcal{P}_\kappa(\mu)$-list D if for all $x \in \mathcal{P}_\kappa(\mu)$ there is a $y \supseteq x$ such that $d_y \cap x = b \cap x$.

- A subset $b \subseteq \mu$ is an ineffable branch of a $\mathcal{P}_\kappa(\mu)$-list D if there is a stationary set $S \subseteq \mathcal{P}_\kappa(\mu)$ such that $d_x = b \cap x$ for each $x \in S$.

- A $\mathcal{P}_\kappa(\mu)$-list D is thin if for each $x \in \mathcal{P}_\kappa(\mu)$, $|\{d_y \cap x \colon x \subseteq y \text{ and } y \in \mathcal{P}_\kappa(\mu)\}| < \kappa$.

- $\mathrm{STP}(\kappa, \mu)$ holds if every thin $\mathcal{P}_\kappa(\mu)$-list has a cofinal branch.

- The strong tree property holds at κ if $\mathrm{STP}(\kappa, \mu)$ holds for each regular $\mu \geq \kappa$.

- ITP(κ, μ) holds if every thin $\mathcal{P}_\kappa(\mu)$-list has an ineffable branch.
- ITP holds at κ if $ITP(\kappa, \mu)$ holds for each regular $\mu \geq \kappa$.

The following definitions are critical when arguing that ITP holds after forcing with $\mathbb{P}$.

Definition 1.10. Let $\kappa > \omega$ be regular and $W \models ZFC$.

- A $\mathbb{P}$-name $\dot{b}$ for a subset of an ordinal μ is κ-approximated in W if $\Vdash_{\mathbb{P}} \dot{b} \cap x \in W$ for each $x \in (\mathcal{P}_\kappa(\mu))^W$.
- A $\mathbb{P}$-name $\dot{b}$ for a subset of an ordinal μ is thinly κ-approximated in W if $\Vdash_{\mathbb{P}} \dot{b} \cap x \in W$ and $|\{y \in W \colon (\exists p \in \mathbb{P})\ p \Vdash_{\mathbb{P}} y = \dot{b} \cap x\}| < \kappa$ for each $x \in (\mathcal{P}_\kappa(\mu))^W$.
- A poset $\mathbb{P}$ has the (thin) κ-approximation property in W if $\Vdash_{\mathbb{P}} \dot{b} \in W$ for each $\dot{b}$ which is (thinly) κ-approximated in W.

1.4 Branch Lemmas

In tree property arguments we frequently use that a new branch through a tree T obtained by forcing must split at unboundedly many levels of the tree. This is crucially used when showing that certain types of forcings cannot add branches.

Lemma 1.11 (Branch Splitting Lemma)**.** *Suppose $\mathbb{P}$ is a forcing and $\dot{b}$ a name for a branch through some tree T. Further assume that $\Vdash_{\mathbb{P}} \dot{b} \notin V$. Then for each $\alpha < ht(T)$ and $p \in \mathbb{P}$, there are $p_1, p_2 \leq p$ and $\beta \geq \alpha$ such that p_1 and p_2 decide different values of $\dot{b} \cap Lev_\beta(T)$.*

Proof. Assume the lemma is false. Fix $\alpha < ht(T)$ and $p \in \mathbb{P}$ witnessing this. The contradiction follows after showing that $p \Vdash \dot{b} \in V$. Towards that end, define

$$d = \{x \in T \colon \exists q \leq p,\ q \Vdash \check{x} \in \dot{b}\}.$$

We claim that $p \Vdash \dot{b} = \check{d}$. Towards that end, fix a $\mathbb{P}$-generic object G containing p. If $\sigma[G] \in \dot{b}[G]$, then we can find an $x \in T$ and $q \leq p$ in G such that $x = \sigma[G]$ and $q \Vdash \check{x} \in \dot{b}$. But this implies that $\sigma[G] = x \in d$ by definition. Going in the other direction, let $x \in d$ and fix $q \leq p$ such that $q \Vdash \check{x} \in \dot{b}$. Let $\beta = \max\{\alpha, ht(x)\}$ and find $p_1 \leq q$ and $x_1 \in Lev_\beta(T)$ such that $p_1 \Vdash \check{x}_1 \in \dot{b}$. Since $\dot{b}[G]$ is a branch through T there is a $x_2 \in \dot{b}[G] \cap Lev_\beta(T)$ and $p_2 \leq p$ such that $p_2 \in G$ and $p_2 \Vdash \check{x}_2 \in \dot{b}$. Since the lemma fails at α and p, it must be that $x_1 = x_2$. This implies that $p_2 \Vdash \check{x} \in \dot{b}$ and therefore that $x \in \dot{b}[G]$ since $p_2 \in G$. Therefore, $p \Vdash \dot{b} = \check{d} \in V$, a contradiction. ☐

The previous lemma is classical and deals with the splitting that can occur with branches through trees. For ITP arguments we need a newer type of splitting lemma that deals with sufficiently approximated subsets of ordinals. The following may be found in (Unger, 2014).

Lemma 1.12 (ITP Splitting Lemma)**.** *Suppose $\mathbb{P}$ is a forcing and $\dot{b}$ is a $\mathbb{P}$-name for a subset of some ordinal μ that is κ-approximated in V. Further assume $\Vdash_{\mathbb{P}} \dot{b} \notin V$. Then for all $x \in \mathcal{P}_\kappa(\mu)$ and $p \in \mathbb{P}$, there are $p_1, p_2 \leq p$ and $y \supseteq x$ with $y \in \mathcal{P}_\kappa(\mu)$ such that p_1 and p_2 decide different values of $\dot{b} \cap y$.*

Proof. Assume the lemma is false. Fix $x \in \mathcal{P}_\kappa(\mu)$ and $p \in \mathbb{P}$ witnessing this. The proof is similar to the previous lemma. Namely, define

$$d = \{\alpha < \mu \colon \exists q \leq p,\ q \Vdash \alpha \in \dot{b}\}.$$

We again check that $p \Vdash \dot{b} = \check{d}$. If G is a generic object and $p \in G$, we just check the trickier direction that $d \subseteq \dot{b}[G]$. Towards that end, let $\alpha \in d$ and fix $q \leq p$ such that $q \Vdash \alpha \in \dot{b}$. Let $y = x \cup \{\alpha\} \in \mathcal{P}_\kappa(\mu)$. Since $\dot{b}$ is κ-approximated in V we know that $\Vdash \dot{b} \cap y \in V$. This implies that we may find $p_1 \leq q$, $p_2 \leq p$, and $v_1, v_2 \in \mathcal{P}_\kappa(\mu)$ such that

1. $p_1 \Vdash \dot{b} \cap y = v_1$,
2. $p_2 \Vdash \dot{b} \cap y = v_2$, and
3. $p_2 \in G$.

Note that (1) implies that $\alpha \in v_1$. Since we assumed the lemma is false at x and p we have that $v_1 = v_2$ and therefore that $\alpha \in v_2 \subseteq \dot{b}[G]$ as desired. $\square$

Assume that we have a tree T in our ground model. Of great importance in tree property arguments is knowing which properties about a poset $\mathbb{P}$ allow you to conclude that you do not add a branch through T after forcing with $\mathbb{P}$. The first noteworthy branch lemma is that Knaster forcings do not add branches through κ-trees. Normally the result assumes that the κ-tree is branchless in the ground model (as in (Cummings and Foreman, 1998) and (Unger, 2013)), and Unger says that it “seems like it should be able to be eliminated” (Unger, 2013).

We note that it can be eliminated and we present the proof below. The following result when $\kappa = \omega_1$ is proven in (Baumgartner, 1983).

Proposition 1.13. *Let κ be a regular cardinal. If $\mathbb{P}$ is κ-Knaster, then forcing with $\mathbb{P}$ does not add a new branch through a tree T of height κ.*

Proof. Assume otherwise and fix a name $\dot{b}$ for a branch such that $\Vdash \dot{b} \notin V$. For notation let $\mathbb{F}$ be all $t \in T$ forced into $\dot{b}$ by some element $p \in \mathbb{P}$. For each $\alpha < \kappa$, find a $u_\alpha \in \mathbb{F}$ and $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash u_\alpha \in Lev_\alpha(T) \cap \dot{b}$. Without loss of generality we may assume that each p_α is distinct, and so Knasterness of $\mathbb{P}$ implies that there is some unbounded $B \subseteq \kappa$ such that for $\alpha < \beta$ in B, p_α and p_β are compatible. Then the set $\{u_\alpha \colon \alpha \in B\}$ induces a branch d in V. (This is where the usual argument ends, as this would contradict that T is branchless in the ground model.)

Observe that since $\Vdash \dot{b} \notin V$, for any element $t \in d$ there must be incompatible $t_0, t_1 \in \mathbb{F}$ such that $t \leq t_0, t_1$. Further, since d is a branch through T, either $t_0 \notin d$ or $t_1 \notin d$ (or both). Define $\mathbb{F}' \subseteq \mathbb{F}$ as the set of all $t \in \mathbb{F}$ such that $t \notin d$ and $s \in d$ for each $s <_T t$. By definition, distinct elements of $\mathbb{F}'$ are incompatible. Further, since d is unbounded we have that $|\mathbb{F}'| = \kappa$. If we enumerate $\mathbb{F}'$ as $\{t_\alpha \colon \alpha < \kappa\}$ and let q_α witness that $t_\alpha \in \mathbb{F}$, then we have that $\{q_\alpha \colon \alpha < \kappa\}$ is an antichain of size κ, a contradiction. $\square$

The next result is due to Unger and appears as Lemma 2.4 in (Unger, 2014). It is, in fact, a generalization of Proposition 1.13 because if $\mathbb{P}$ is κ-Knaster then $\mathbb{P} \times \mathbb{P}$ is κ-cc. Further, any branch through a tree of height κ is κ-approximated.

Lemma 1.14. *Let κ be a regular cardinal. If $\mathbb{P} \times \mathbb{P}$ is κ-cc, then $\mathbb{P}$ has the κ-approximation property in V.*

Proof. Assume that $\mathbb{P}$ does not have the κ-approximation property in V. Let $\dot{b}$ be a name for a κ-approximated subset of μ where, without loss of generality, $\Vdash \dot{b} \notin V$. Construct sequences $((p_i, q_i) \colon i < \kappa)$ in $\mathbb{P} \times \mathbb{P}$ and $(x_i, y_i \colon i < \kappa)$ of elements of $\mathcal{P}_\kappa(\mu)$ such that, for each $i < \kappa$,

1. p_i and q_i decide different values of $\dot{b} \cap x_i$,
2. p_i and q_i both force that $\dot{b} \cap \bigcup_{j<i} x_j = y_i$, and
3. $(x_i \colon i < \kappa)$ and $(y_i \colon i < \kappa)$ are both $\subseteq$-increasing.

To do this, assuming we have defined (p_i, q_i) and x_i for $i < \theta$, fix some $r \in \mathbb{P}$ and y_θ such that $r \Vdash \dot{b} \cap \bigcup_{i<\theta} x_i = y_\theta$. Next, we finish the construction by appealing to Lemma 1.12 to find $x_\theta \supseteq y_\theta$ and $(p_\theta, q_\theta) \leq (r, r)$ such that p_θ and q_θ decide different values of $\dot{b} \cap x_\theta$.

But, then we have that $\{(p_i, q_i) \colon i < \kappa\}$ is an antichain of size κ. To see this, assume otherwise that (p_i, q_i) and (p_j, q_j) are compatible with $i < j$. Assume that $p \leq p_i, p_j$ and $q \leq q_i, q_j$. By construction of (p_i, q_i) there are $v_1 \neq v_2$ such that $p \Vdash$ "$\dot{b} \cap x_i = v_1$" and $q \Vdash$ "$\dot{b} \cap x_i = v_2$". However, by construction of (p_j, q_j) we have the following:

1. $p \Vdash v_1 = \dot{b} \cap x_i = \dot{b} \cap \left(\bigcup_{k<j} x_k\right) \cap x_i = y_j \cap x_i$, and
2. $q \Vdash v_2 = \dot{b} \cap x_i = \dot{b} \cap \left(\bigcup_{k<j} x_k\right) \cap x_i = y_j \cap x_i$.

However, this is a contradiction because it follows that $v_1 = v_2$. $\square$

The other classical branch lemma involves various situations when $\mathbb{P}$ is sufficiently closed, as opposed to having the right amount of chain condition. A proof may be found in (Abraham, 1983).

Lemma 1.15 (Silver's Branch Lemma)**.** *Let κ be a regular cardinal. Assume that T is a κ-tree and $\mathbb{P}$ is τ^+-closed for some cardinal $\tau < \kappa$ such that $2^\tau \geq \kappa$. Then forcing with $\mathbb{P}$ does not add new branches through T.*

The following generalization appears as Proposition 2.1.12 in (Weiss, 2010). In the same way that Lemma 1.12 is a generalization of the classical splitting lemma for branches through a tree, the following is a newer generalization of Silver's Branch Lemma showing that closed forcings - in the correct context - cannot add branches through lists.

Lemma 1.16 (Generalized Silver's Branch Lemma)**.** *Let κ be a regular cardinal. Assume that $\mathbb{P}$ is τ^+-closed for some cardinal $\tau < \kappa$ such that $2^\tau \geq \kappa$. Then $\mathbb{P}$ has the thin κ-approximation property in V.*

Proof. Assume that $\dot{b}$ is a name for a subset of μ and further that $\dot{b}$ is thinly κ-approximated in V. Towards a contradiction assume that $\Vdash \dot{b} \notin V$. Let τ be minimal such that $2^\tau \geq \kappa$. The idea is similar to the proof of Silver's Branch Lemma, where in this scenario we use Lemma 1.12 to decide 2^τ distinct values of some $\dot{b} \cap x$. This would contradict that $\dot{b}$ is thinly approximated.

Towards that end, we define sequences $(p_\sigma, v_\sigma \colon \sigma \in 2^{<\tau})$ and $(x_\alpha \colon \alpha < \tau)$ by induction on the length of σ with the following properties:

1. For each $\sigma \in 2^{<\tau}$, $p_\sigma \in \mathbb{P}$ and $v_\sigma, x_{|\sigma|} \in \mathcal{P}_\kappa(\mu)$

2. For each $\sigma \in 2^{<\tau}$ with $|\sigma| = \alpha$, $p_\sigma \Vdash v_\sigma = \dot{b} \cap x_\alpha$

3. For each $\sigma \in 2^{<\tau}$, $v_{\sigma^\frown 0} \neq v_{\sigma^\frown 1}$

4. If $\sigma_0 \subset \sigma_1$ then $p_{\sigma_1} \leq p_{\sigma_0}$, $v_{\sigma_0} \subset v_{\sigma_1}$, and $x_{|\sigma_0|} \subset x_{|\sigma_1|}$.

Base Case: Set $p_\varnothing = 1_{\mathbb{P}}$ and $x_0 = v_\varnothing = \varnothing$.

Successor Case: Assume $\alpha < \tau$ is an ordinal and that we have defined p_σ and v_σ when $|\sigma| \leq \alpha$ and x_β for $\beta \leq \alpha$. For each $\sigma \in 2^\alpha$, use Lemma 1.12 to find $y_\sigma \supset x_\alpha$, $q_{\sigma^\frown 0}, q_{\sigma^\frown 1} \leq p_\sigma$, and distinct $w_{\sigma^\frown 0}, w_{\sigma^\frown 1}$ such that $q_{\sigma^\frown i} \Vdash w_{\sigma^\frown i} = \dot{b} \cap y_\sigma$ for $i < 2$. Define $x_{\alpha+1} = \bigcup_{\sigma \in 2^\alpha} y_\sigma$. Since τ is minimal such that $2^\tau \geq \kappa$, we have that $2^\alpha < \kappa$ and therefore $x_{\alpha+1} \in \mathcal{P}_\kappa(\mu)$ because κ is regular. By construction observe that $x_\alpha \subset x_{\alpha+1}$. To finish this case, for each $i < 2$ and $\sigma \in 2^\alpha$, choose $p_{\sigma^\frown i} \leq q_{\sigma^\frown i}$ and $v_{\sigma^\frown i}$ such that $p_{\sigma^\frown i} \Vdash v_{\sigma^\frown i} = \dot{b} \cap x_{\alpha+1}$. This may be done because $\dot{b}$ is κ-approximated in V.

Limit Case: Assume that $\theta < \tau$ is a limit ordinal and that we have defined p_σ and v_σ when $|\sigma| < \theta$ and x_β for $\beta < \theta$. For each $\sigma \in 2^\theta$, note that $\{p_{\sigma \restriction \alpha} \colon \alpha < \theta\}$ is a decreasing sequence. Since $\mathbb{P}$ is τ^+-closed, we may find a lower bound q_σ for $\{p_{\sigma\restriction\alpha} \colon \alpha < \theta\}$. Further, set $x_\theta = \bigcup_{\beta<\theta} x_\beta$. Then, choose $p_\sigma \leq q_\sigma$ and v_θ such that $p_\sigma \Vdash v_\sigma = \dot{b} \cap x_\theta$. This completes the construction.

To complete the proof, set $x = \bigcup_{\alpha<\tau} x_\alpha$ and, for each $f \in 2^\tau$, use the closure of $\mathbb{P}$ to find a lower bound $q_f \in \mathbb{P}$ for the decreasing sequence $\{p_{f\restriction\alpha} \colon \alpha < \tau\}$. Like above, choose $p_f \leq q_f$ and v_f such that $p_f \Vdash v_f = \dot{b} \cap x$. For distinct sequences $f, g \in 2^\tau$, we claim that $v_f \neq v_g$. To see this, let $i < \tau$ be least such that $f(i) \neq g(i)$ and set $\sigma = f \restriction i = g \restriction i$. Without loss of generality assume that $f(i) = 0$ and $g(i) = 1$. Then we have simultaneously that

1. $p_f \Vdash v_f \cap x_{i+1} = \dot{b} \cap x \cap x_{i+1} = \dot{b} \cap x_{i+1} = v_{\sigma^\frown 0}$,

2. $p_g \Vdash v_g \cap x_{i+1} = \dot{b} \cap x \cap x_{i+1} = \dot{b} \cap x_{i+1} = v_{\sigma \frown 1}$.

This implies that $v_f \cap x_{i+1} = v_{\sigma \frown 0}$ and $v_g \cap x_{i+1} = v_{\sigma \frown 1}$. By construction we have that $v_{\sigma \frown 0} \neq v_{\sigma \frown 1}$ and therefore $v_f \neq v_g$. However, this implies that we have decided at least 2^τ distinct values of $\dot{b} \cap x$. This is a contradiction because $2^\tau \geq \kappa$ and we assumed that $\dot{b}$ is thinly κ-approximated. $\square$

We also have the following key branch lemma. Originally appearing in (Unger, 2012), it is a variant of Silver's branch lemma after forcing with a poset that has sufficient chain condition. It will be generalized in a later section to be used in ITP arguments and so the proof is omitted until then.

Lemma 1.17. *Let κ be a regular cardinal. Assume that $\mathbb{P}$ is τ^+-cc, $\mathbb{Q}$ is τ^+-closed for some $\tau < \kappa$ such that $2^\tau \geq \kappa$, and T is a κ-tree in $V[\mathbb{P}]$. Then in $V[\mathbb{P}]$ forcing with $\mathbb{Q}$ does not add new branches through T.*

If $\dot{b}$ is approximated in V, there is no reason that $\dot{b}$ must remain approximated after forcing with some poset. Therefore it it important to keep track of situations in which this does occur. This form originally appeared in (Unger, 2014).

Lemma 1.18. *Assume $\dot{b}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a subset of μ, which is (thinly) κ-approximated in V. Further assume that $\mathbb{P}$ has κ-cc and G is $\mathbb{P}$-generic over V. Then, $\dot{b}$ is (thinly) κ-approximated in $V[G]$. (Note that in $V[G]$ we are thinking about $\dot{b}$ as a $\mathbb{Q}$-name.)*

Proof. First assume that $\dot{b}$ is κ-approximated in V. So we have that $\Vdash^V_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{b} \cap x \in V$ for each $x \in \mathcal{P}_\kappa(\mu)^V$. We need to show that $\Vdash^{V[G]}_{\mathbb{Q}} \dot{b} \cap x \in V[G]$ for each $x \in \mathcal{P}_\kappa(\mu)^{V[G]}$. Since $\mathbb{P}$ is κ-cc, we may find a $y \supseteq x$ such that $y \in \mathcal{P}_\kappa(\mu)^V$. This implies that $\Vdash_{\mathbb{Q}}^{V[G]} \dot{b} \cap x = (\dot{b} \cap y) \cap x$. By hypothesis we have that $\Vdash_{\mathbb{Q}}^{V[G]} \dot{b} \cap y \in V$. Since $x \in V[G]$, we have that $\Vdash_{\mathbb{Q}}^{V[G]} \dot{b} \cap y \in V[G]$ as desired.

Now, assume additionally that $\dot{b}$ was thinly κ-approximated in V. We have to show $D_x = \{v \in V[G] \colon \exists q \in \mathbb{Q}, q \Vdash_{\mathbb{Q}}^{V[G]} v = \dot{b} \cap x\}$ has size less than κ in $V[G]$. To do this, let y be from the previous paragraph and define $C_y = \{w \in V \colon \exists (p, \dot{q}),\ (p, \dot{q}) \Vdash_{\mathbb{P} * \dot{\mathbb{Q}}}^{V} w = \dot{b} \cap y\}$. Notice that the argument from above implies that each $v \in D_x$ has the form $w \cap x$ for some $w \in C_y$. This implies that in $V[G]$ the map $w \mapsto w \cap x$ is a surjection of some subset of C_y onto D_x. Since $\dot{b}$ is thinly approximated we have that $|C_y| < \kappa$ and so the result follows. $\square$

1.5 Large Cardinals

In this thesis we concern ourselves with a few types of large cardinals, whose properties are summarized below. Since we are interested in getting the (super) tree property at successive cardinals, large cardinal hypotheses involving supercompact cardinals will be used most frequently.

Definition 1.19. A cardinal κ is weakly compact if and only if for each function $F \colon [\kappa]^2 \to 2$ there is a $H \subseteq \kappa$ such that $F \restriction [H]^2$ is a constant.

Although this definition of weakly compact is in terms of partitions, the name "weakly compact" comes from an equivalent formulation asserting that certain infinitary languages satisfy the Weak Compactness Theorem.

Definition 1.20. A cardinal κ is measurable if and only if there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ where $M \subseteq V$ is a transitive class such that ${}^{\kappa}M \subseteq M$.

Lemma 1.21. *If κ is measurable, then the tree property holds at κ.*

Proof. Fix a κ-tree $T \in V$ and an elementary embedding $j\colon V \to M$ witnessing that κ is measurable. Elementarity implies that $j(T)$ has height $j(\kappa)$. Since $j(\kappa) > \kappa$, we can fix $u \in Lev_{\kappa}(j(T))$. Since T is a κ-tree, it follows that the subtree $j(T) \restriction \kappa$ is isomorphic to T, and so $\{x \in T\colon j(x) <_{j(T)} u\}$ is a branch through T. $\square$

Although the previous lemma can also be proven when κ is weakly compact, the proof given above is illustrative and appears in more involved contexts later in this thesis. The following definition may be found in (Kanamori, 2003).

Definition 1.22. A cardinal κ is strongly compact if for each $\mu \geq \kappa$ there is an elementary embedding $j\colon V \to M$ with $\operatorname{crit}(j) = \kappa$ where $M \subseteq V$ is a transitive class such that for every $X \subseteq M$ with $|X| \leq \mu$ there is a $Y \supseteq X$ with $Y \in M$ and $|Y|^M < j(\kappa)$. Such an embedding j is called a μ-strong compactness embedding.

Lemma 1.23. *If κ is strongly compact, then the strong tree property holds at κ.*

Proof. Let $D = (d_x\colon x \in \mathcal{P}_{\kappa}(\mu))$ be a thin $\mathcal{P}_{\kappa}(\mu)$-list where $\mu \geq \kappa$ is regular, and let $j\colon V \to M$ be a μ-strong compactness embedding. By elementarity, we have $j(D) = \left(e_x\colon x \in \mathcal{P}_{j(\kappa)}(j(\mu))^M\right)$ is a thin $j\left(\mathcal{P}_{\kappa}(\mu)\right)$-list in M. Note that $j``\mu \subseteq M$ and has size $\mu$, and so there is a $Y \supseteq j``\mu$ with $Y \in M$ and $|Y|^M < j(\kappa)$. If we set $Z = Y \cap j(\mu)$ it follows that $Z \in j(\mathcal{P}_{\kappa}(\mu))$. Next, define $b = \{\alpha < \mu\colon j(\alpha) \in e_Z\}$. We claim that b is a cofinal branch for D. In other words, for each $x \in \mathcal{P}_\kappa(\mu)$ we must show there is a $y \supseteq x$ such that $b \cap x = d_y \cap x$. If $Lev(x) = \{d_y \cap x \colon y \supseteq x\}$, then since D is a thin list we have that $|Lev(x)| < \kappa$ and so elementarity implies $j(Lev(x)) = j``Lev(x) = \{e_{j(y)} \cap j(x) \colon y \supseteq x\}$. Now, observe that $e_Z \cap j(x) \in j(Lev(x))$ by definition of $j(Lev(x))$ and so there is a $y \supseteq x$ such that $e_Z \cap j(x) = e_{j(y)} \cap j(x)$. This implies that $b \cap x = d_y \cap x$ as desired. $\square$

Definition 1.24. A cardinal κ is supercompact if for each $\mu \geq \kappa$ there is an elementary embedding $j \colon V \to M$ with $\mathrm{crit}(j) = \kappa$ where $M \subseteq V$ is a transitive class such that ${}^{\mu}M \subseteq M$. Such an embedding j is called a μ-supercompactness embedding.

Lemma 1.25. *If κ is supercompact, then ITP holds at κ.*

Proof. Let $D = (d_x \colon x \in \mathcal{P}_\kappa(\mu))$ be a thin $\mathcal{P}_\kappa(\mu)$-list where $\mu \geq \kappa$ is regular, and let $j \colon V \to M$ be a μ-supercompactness embedding. By elementarity, we have $j(D) = \left(e_x \colon x \in \mathcal{P}_{j(\kappa)}(j(\mu))^M\right)$ is a thin $j\left(\mathcal{P}_\kappa(\mu)\right)$-list in M. Since M is closed under μ-sequences, we have that $j``\mu \in \mathcal{P}_{j(\kappa)}(j(\mu))^M$. We claim that $e_{j``\mu}$ induces an ineffable branch through D. In particular, define $b = \{\alpha < \mu \colon j(\alpha) \in e_{j``\mu}\}$. We need to show that $S = \{x \in \mathcal{P}_\kappa(\mu) \colon d_x = b \cap x\}$ is stationary. Recall that measure one sets are stationary and so it is enough to show that $S$ is measure one. To do this it is enough to check that $j``\mu \in j(S)$ as then S is in the measure generated by the seed $j"\mu$. By definition, $j``\mu \in j(S)$ is equivalent to $e_{j``\mu} = j(b) \cap j``\mu$. This last statement follows from the definition of b and so the result follows. $\square$

Observe that the previous lemmas all had the form of showing that some version of the tree property would hold if κ was a certain large cardinal. These results may all be reversed, which is summarized in the following:

Lemma 1.26. *Assume that κ is inaccessible. Then the following hold:*

1. *(Folklore) κ is weakly compact if and only if the tree property holds at κ.*

2. *(Jech) κ is strongly compact if and only if the strong tree property holds at κ.*

3. *(Magidor) κ is supercompact if and only if ITP holds at κ.*

Originally appearing in (Laver, 1978), the following result is crucial for lifting arguments found in later sections by allowing us to correctly guess what happens at the κ^{th}-stage of particular forcing iterations:

Lemma 1.27. *Assume that κ is supercompact. Then there is a function $F\colon \kappa \to V_\kappa$ such that if for each μ and each $x \in H_{\mu^+}$ there is a μ-supercompactness embedding j such that $j(F)(\kappa) = x$. The function F is called a Laver function for κ.*

Definition 1.28. A supercompact cardinal κ is indestructibly supercompact if for any κ-directed closed forcing $\mathbb{P}$, κ remains supercompact in $V[\mathbb{P}]$.

Importantly, being supercompact is equiconsistent with being indestructibly supercompact. More specifically, using Laver functions we can force to make any supercompact cardinal indestructible. This result also appeared in (Laver, 1978):

Lemma 1.29. *Assume κ is supercompact. For each $\theta < \kappa$ there is a κ-cc and θ-directed closed forcing $\mathbb{P}_\theta$ such that κ is indestructibly supercompact in $V[\mathbb{P}_\theta]$.*

1.6 Projections

We summarize some well-known information on projections as they are crucial for analyzing the various forcings later in this paper.

Definition 1.30. We say that $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a projection[1] if the following hold:

1. $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$
2. for any $p, q \in \mathbb{P}$, if $p \leq q$ then $\pi(p) \leq \pi(q)$,
3. for any $q \in \mathbb{Q}$, $p \in \mathbb{P}$, if $q \leq \pi(p)$, then there is an $r \leq p$ such that $\pi(r) \leq q$.

Lemma 1.31. *If G is $\mathbb{P}$-generic over V, then the upwards closure $H = (\pi``G)\uparrow$ is $\mathbb{Q}$-generic over V. In particular, $V[\mathbb{Q}] \subseteq V[\mathbb{P}]$.*

Proof. It is not hard to show that H is a filter, so it is enough to show that H is generic. Fix a dense set $D \subseteq \mathbb{Q}$ and define $D' = \{p \in \mathbb{P}\colon \exists q \in D\ \pi(p) \leq q\ \}$. Since π is a projection, D' is readily seen to be dense and so we may fix $p \in D' \cap G$. So there is some $q \in D$ such that $\pi(p) \leq q$. But then $q \in D \cap H$ by definition of H. This shows that H is $\mathbb{Q}$-generic over V. For the second part of the lemma, observe that H is definable from $(\mathbb{Q}, \leq)$, G, and π. So $H \in V[G]$ and so $V[H] \subseteq V[G]$ because the generic extension is minimal. $\square$

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[1] Often a slightly weaker notion of projection is used; namely, that there is a projection from $\mathbb{P}$ to $\mathbb{Q}$ if, given any $\mathbb{P}$-generic G, we can define a $\mathbb{Q}$-generic H such that $V[H] \subseteq V[G]$. This implies that there is a projection (in our sense) from $\mathbb{P}$ to $\mathrm{RO}(\mathbb{Q})$, where $\mathrm{RO}(\mathbb{Q})$ is the complete boolean algebra that $\mathbb{Q}$ densely embeds into.

Definition 1.32. Let $\pi\colon \mathbb{P} \to \mathbb{Q}$ be a projection and H be $\mathbb{Q}$-generic over V. In $V[H]$, define $\mathbb{P}/H$ to be the poset whose underlying set is $\pi^{-1}[H]$ and whose ordering is induced from $\mathbb{P}$. Note that $\mathbb{P}/H$ is non-empty because $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}} \in H$.

Lemma 1.33. *If G is $\mathbb{P}/H$-generic over $V[H]$, then it is also $\mathbb{P}$-generic over V. Further, $H = (\pi``G)\uparrow$.*

Proof. To show that G is a $\mathbb{P}$-generic filter over V, we have to first check that it is a filter when viewed as a subset of $\mathbb{P}$. Since G is already a $\mathbb{P}/H$-filter, the only thing that could now fail is that G is no longer upwards closed with respect to $\mathbb{P}$. However, given $g \leq p$ with $g \in G$ and $p \in \mathbb{P}$, we know that $\pi(g) \leq \pi(p)$ since π is a projection. Since $g \in G$ we have that $\pi(g) \in H$ and therefore $\pi(p) \in H$ as well. This implies that $p \in \mathbb{P}/H$. Since G is $\mathbb{P}/H$-upwards closed, we have that $p \in G$ as desired.

To show genericity, fix a dense $D \subseteq \mathbb{P}$ in V and define $D' = D \cap (\mathbb{P}/H)$. Since G is $\mathbb{P}/H$-generic over $V[H]$, it is enough to show that D' is a dense subset of $\mathbb{P}/H$. Towards that end, fix $p \in \mathbb{P}/H$, and consider the set $D'' = \pi[D \cap (p\downarrow)]$.

Claim 1.34. *$D'' \subseteq \mathbb{Q}$ is dense below $\pi(p)$.*

Proof of Claim. Given $q \leq \pi(p)$ we may find a $p' \leq p$ such that $\pi(p') \leq q$. Since D is dense, let $d \in D$ with $\pi(d) \leq \pi(p') \leq q$. Then $\pi(d) \in D''$ as desired. $\square$

Since $\pi(p) \in H$, it follows by genericity of H that $H \cap \pi[D \cap (p\downarrow)]$ is nonempty. So, we may find some $\pi(q) \in H$ with $q \leq p$ and $q \in D$. But, then by definition we have that $q \in D'$. This shows that D' is dense as desired.

Next we show the second part of the lemma that $H = (\pi``G)\uparrow$. First, the harder direction: fix $h \in H$. Consider the set $D = \{p \in \mathbb{P} \colon \pi(p) \leq h \text{ or } \pi(p) \perp h\}$. Using the definition of projection, it is readily seen that D is a dense subset of $\mathbb{P}$. It follows that there is some $q \in G \cap D$. Since $G \subseteq \mathbb{P}/H$, we have $\pi(q) \in H$. Since $h \in H$, we cannot have $\pi(q) \perp h$. So, $\pi(q) \leq h$, or $h \in (\pi``G)\uparrow$. Conversely, fix $h \in \mathbb{Q}$ and $q \in G$ such that $\pi(q) \leq h$. Since $G \subseteq \mathbb{P}/H$, we have that $\pi(q) \in H$. By upwards closure of H, we have that $h \in H$. □

Corollary 1.35. *Let $\pi \colon \mathbb{P} \to \mathbb{Q}$ be a projection. Given a $\mathbb{Q}$-generic filter H over V, there is a $\mathbb{P}$-generic filter G over V such that $V[H] \subseteq V[G]$. Note that this is similar to the statement $V[\mathbb{Q}] \subseteq V[\mathbb{P}]$, but here we start with a $\mathbb{Q}$-generic object instead of a $\mathbb{P}$-generic object.*

Proof. Given a $\mathbb{Q}$-generic filter H over V, fix any $\mathbb{P}/H$-generic G over $V[H]$. By the previous lemma, G is $\mathbb{P}$-generic over V. Further, H is definable from $(\mathbb{Q}, \leq)$, G, and π. This implies that $H \in V[G]$ and further that $V[H] \subseteq V[G]$. □

Lemma 1.36. *If $\mathbb{P}$-generic filter G over V and $H = (\pi``G)\uparrow$ is the $\mathbb{Q}$-generic filter induced by π, then G is a $\mathbb{P}/H$-generic filter over $V[H]$. Note that this is the converse to Lemma 1.33.*

Proof. First observe that $G \subseteq \mathbb{P}/H$. To show genericity, it is enough to show that G meets every open dense subset of $\mathbb{P}/H$. Towards that end, let $D \subseteq \mathbb{P}/H$ be open dense and in $V[H]$. By the definition of H, we may fix some $g \in G$ and $\mathbb{Q}$-name $\dot{D}$ such that $\dot{D}[H] = D$ and $\pi(g) \Vdash_{\mathbb{Q}}$ "$\dot{D}$ is open dense in $\mathbb{P}/\Gamma$", where Γ is the canonical $\mathbb{Q}$-name for the generic filter.

Consider the set $D' = \{p \in \mathbb{P} \colon (\exists q \in \mathbb{Q})(\pi(p) \leq q \text{ and } q \Vdash_{\mathbb{Q}} p \in \dot{D})\}$. Intuitively, $p \in D'$ means that p is forced by a weaker condition to be in D. Note that it is enough to show that D'

is dense below g, because then by genericity of G we may fix $p \in D' \cap G$, and $q \in \mathbb{Q}$ witnessing this. Since $\pi(p) \leq q$, we have that $q \in H$. By choice of q, we have that $p \in \dot{D}[H] \cap G = D \cap G$ as desired.

Towards showing that D' is dense below g, fix $r \leq g$. This implies that $\pi(r) \Vdash_{\mathbb{Q}}$ "$\dot{D}$ is open dense in $\mathbb{P}/\Gamma$." Observe also that $\pi(r) \Vdash_{\mathbb{Q}} r \in \mathbb{P}/\Gamma$. Fix some $\mathbb{Q}$-generic H_0 containing $\pi(r)$. Since $\dot{D}[H_0]$ is an open dense subset of $\mathbb{P}/H_0$, we may find some $d \leq r$ such that $V[H_0] \models d \in \dot{D}[H_0] \subseteq \mathbb{P}/H_0$. Observe that $\pi(d) \in H_0$, so we may find some $q \leq \pi(d) \leq \pi(r)$ such that $q \Vdash_{\mathbb{Q}} d \in \dot{D}$. By the definition of projection, we may find a $p \leq d$ such that $\pi(p) \leq q$. Since $q \Vdash_{\mathbb{Q}}$ "$\dot{D}$ is open dense and $d \in \dot{D}$", we have finally that $q \Vdash_{\mathbb{Q}} p \in \dot{D}$. Since $p \leq d \leq r$ this completes the proof. □

Corollary 1.37. *Given a $\mathbb{P}$-generic filter G over V, there is a $\mathbb{Q}$-generic H over V such that $V[G] = V[H][G]$. In other words, forcing with $\mathbb{P}$ is equivalent to first forcing with $\mathbb{Q}$ and then forcing with $\mathbb{P}/H$.*

Proof. Fix a $\mathbb{P}$-generic G over V and set $H = (\pi``G) \uparrow$. By the previous lemma we have that G is $\mathbb{P}/H$-generic over $V[H]$. It follows that the expression $V[H][G]$ makes sense and $V[G] \subseteq V[H][G]$ because $V \subseteq V[H]$. Conversely, since H is definable from G we have that $V[H] \subseteq V[G]$. Since $V[H][G]$ is the minimal model of ZFC containing $V[H]$ and G, it follows that $V[H][G] \subseteq V[G]$. □

Projections can also give us a sufficient condition on when we may lift an elementary embedding. For this we first state a result by Silver.

Lemma 1.38 (Silver's Lifting Criterion)**.** *Let $j\colon V \to M$ be an elementary embedding. Suppose that G is $\mathbb{P}$-generic over V and G^* is $j(\mathbb{P})$-generic over N. If $j"G \subseteq G^*$, then we may lift j to an elementary embedding $j\colon V[G] \to M[G^*]$, where $j(G) = G^*$. Furthermore, if $G^* \in V[G][H]$ for some generic extension of $V[G]$, then our lifted embedding is definable in $V[G][H]$.*

Lemma 1.39. *Assume that $j\colon V \to M$ is an elementary embedding, $\pi\colon j(\mathbb{P}) \to \mathbb{P}$ is a projection, and $q \leq_{j(\mathbb{P})} j(\pi(q))$ for any $q \in j(\mathbb{P})$. Then, for any $\mathbb{P}$-generic G over V we may find a $j(\mathbb{P})$-generic H over M letting us lift our embedding to $j\colon V[G] \to M[H]$.*

Proof. Fix some $\mathbb{P}$-generic G over V. By Silver's Lifting Criterion it is enough to find $j(\mathbb{P})$-generic H over M such that $j"G \subseteq H$. By Lemma 1.33 any $j(\mathbb{P})/G$-generic filter H over V is actually $j(\mathbb{P})$-generic over V (over M) and $G = (\pi"H)\uparrow$. To show $j``G \subseteq H$, fix $p \in G$ and find $q \in H$ such that $\pi(q) \leq_{\mathbb{P}} p$. By elementarity we have that $q \leq_{j(\mathbb{P})} j(\pi(q)) \leq_{j(\mathbb{P})} j(p)$. Since H is a filter we have $j"G \subseteq H$ as desired. $\square$

1.7 <u>p-Term Forcing</u>

In this section we generalize term forcing to include an additional parameter. The power of this analysis is that we create a family of two-step *products* that approximates the generic for a two-step *iteration*. A survey of the classical term forcing may be found in (Cummings, 2009).

Definition 1.40. For a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and $p \in \mathbb{P}$, define the p-term forcing $\mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ to be all $\mathbb{P}$-names for elements of $\dot{\mathbb{Q}}$ with the ordering $\dot{b} \leq_p \dot{a}$ if and only if $p \Vdash_{\mathbb{P}} \dot{b} \leq \dot{a}$.

We will also slightly abuse notation by writing $(q, \dot{b}) \leq_p (r, \dot{a})$ to abbreviate the product ordering on $\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$.

Lemma 1.41. *If* $p \Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ *is (canonically)* κ*-(directed) closed", then* $\mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ *is (canonically)* κ*-(directed) closed.*

Proof. For simplicity we do the case when $p \Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-closed." Let $\tau < \kappa$ and $(\dot{q}_i : i < \tau)$ be a sequences of names such that $p \Vdash_{\mathbb{P}}$ "$(\dot{q}_i : i < \tau)$ is a decreasing sequence." Since $p \Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-closed," it follows that there is a name $\dot{q}$ such that for all $i < \tau$ we have $p \Vdash_{\mathbb{P}} \dot{q} \leq \dot{q}_i$. But then $\dot{q}$ is a $\leq_p$-lower bound for the sequence $(\dot{q}_i : i < \tau)$ as desired. □

Definition 1.42. For $p \in \mathbb{P}$, we define the downwards closures of p as follows:

- $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \downarrow_p$ is the subset of $\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ of all conditions below $(p, 1)$.
- $(\mathbb{P} * \dot{\mathbb{Q}}) \downarrow_p$ is the subset of $\mathbb{P} * \dot{\mathbb{Q}}$ of all conditions below $(p, 1)$.

We emphasize that the difference between these two bullet points is the *ordering* not the underlying set.

Lemma 1.43. *For* $p \in \mathbb{P}$*, the identity map is a projection from* $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \downarrow_p$ *to* $(\mathbb{P} * \dot{\mathbb{Q}}) \downarrow_p$.

Proof. First note that the maximum element of both $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \downarrow_p$ and $(\mathbb{P} * \dot{\mathbb{Q}}) \downarrow_p$ is $(p, \dot{1})$. To check that it is ordering preserving, assume that $(a', \dot{b}') \leq_p (a, \dot{b})$ and both conditions are in $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \downarrow_p$. By definition we have that $p \Vdash \dot{b}' \leq \dot{b}$. Since $a' \leq p$, it follows that $a' \Vdash \dot{b}' \leq \dot{b}$ and so $(a', \dot{b}') \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (a, \dot{b})$ as desired. It is therefore enough to check the third condition of a projection. Assume that $(a', \dot{b}') \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (a, \dot{b})$ where $(a, \dot{b})$ is in the downwards closure of p. Our goal is to find $(a'', \dot{b}'')$ such that $(a'', \dot{b}'') \leq_p (a, \dot{b})$ and $(a'', \dot{b}'') \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (a', \dot{b}')$. Unwinding the definitions, this is equivalent to a condition $(a'', \dot{b}'')$ such that

1. $a'' \leq_{\mathbb{P}} a' \leq_{\mathbb{P}} a$,

2. $p \Vdash \dot{b}'' \leq \dot{b}$, and

3. $a'' \Vdash \dot{b}'' \leq \dot{b}'$.

We define $a'' = a'$ and define $\dot{b}''$ to be the following name:

$$\dot{b}'' = \{(\sigma, r) \colon \text{Either (1) } r \leq a' \text{ and } r \Vdash \sigma \in \dot{b}' \text{ or (2) } r \perp a' \text{ and } r \Vdash \sigma \in \dot{b}\}.$$

Claim 1.44. $a'' \Vdash \dot{b}'' = \dot{b}'$

Proof of Claim. Assume that G is $\mathbb{P}$-generic and $a' = a'' \in G$. We must show that $\dot{b}''[G] = \dot{b}'[G]$. If $x \in \dot{b}'[G]$, then we can find (σ, r) such that $r \in G$, $\sigma[G] = x$, and $r \Vdash \sigma \in \dot{b}'$. But then $(\sigma, r) \in \dot{b}''$ and so $x \in \dot{b}''[G]$. Conversely, if $x \in \dot{b}''[G]$, then by definition we may find (σ, r) such that $r \in G$, $\sigma[G] = x$, and either (1) or (2) occur. Since elements of G are compatible, it must be that (1) occurs. Therefore $r \Vdash \sigma \in \dot{b}'$ and so $x \in \dot{b}'[G]$. $\square$

Claim 1.45. *If t incompatible with a'', then $t \Vdash \dot{b}'' = \dot{b}$.*

Proof of Claim. Assume that G is $\mathbb{P}$-generic and $t \in G$. We proceed as in the previous claim. If $x \in \dot{b}[G]$, then we can find (σ, r) such that $r \leq t$, $r \in G$, $\sigma[G] = x$, and $r \Vdash \sigma \in \dot{b}$. Since $r \leq t$, we must have that r is also incompatible with a' and therefore $(\sigma, r) \in \dot{b}''$. So $x \in \dot{b}''[G]$. Conversely, if $x \in \dot{b}''[G]$, then find (σ, r) such that $r \in G$, $\sigma[G] = x$, and either (1) or (2) occur. Note that (2) must occur as otherwise t and a' are compatible. It follows that $x \in \dot{b}[G]$. $\square$

Claim 1.46. *The set $\{q \colon q \Vdash \dot{b}'' \leq \dot{b}\}$ is dense.*

Proof of Claim. Fix $q \in \mathbb{P}$. If $q \perp a'$ then $q \Vdash \dot{b}'' = \dot{b}$ and the claim follows. If instead q and a' are compatible, then we may extend q to a condition r witnessing this. Since $r \leq a'$, we have that $r \Vdash \dot{b}'' = \dot{b}'$. Further, since $(a', \dot{b}') \leq_{\mathbb{A}} (a, \dot{b})$ we also have that $r \Vdash \dot{b}' \leq \dot{b}$. The conclusion follows. $\square$

The previous claim implies that $p \Vdash \dot{b}'' \leq \dot{b}$, as desired. $\square$

Lemma 1.47. *If $q \leq p$, then the identity map is a projection from $\mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ onto $\mathcal{A}_q(\mathbb{P}, \dot{\mathbb{Q}})$.*

Proof. The identity is order preserving because we assumed that $q \leq p$. It is enough to check the third condition of a projection. Assume that $\dot{b} \leq_q \dot{a}$. Our goal is to construct $\dot{c}$ such that $\dot{c} \leq_p \dot{a}$ and $\dot{c} \leq_q \dot{b}$. Towards that end, define

$$\dot{c} = \{(\sigma, r) \colon \text{Either (1) } r \leq q \text{ and } r \Vdash \sigma \in \dot{b} \text{ or (2) } r \perp q \text{ and } r \Vdash \sigma \in \dot{a}\}.$$

As in the previous lemma, we may argue that $q \Vdash \dot{c} = \dot{b}$ and $t \Vdash \dot{c} = \dot{a}$ when $t \perp q$. This implies that $p \Vdash \dot{c} \leq \dot{a}$, as desired. $\square$

Corollary 1.48. *If $q \leq p$, then the identity map is a projection from $\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ onto $\mathbb{P} \times \mathcal{A}_q(\mathbb{P}, \dot{\mathbb{Q}})$.*

Lemma 1.49. *Assume that $G = \mathcal{P} * \mathcal{Q}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic, $q \leq p$, and $q \in \mathcal{P}$. Then in $V[G]$ the identity map is a projection from $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$ onto $(\mathbb{P} \times \mathcal{A}_q(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$.*

Proof. We check the third condition of a projection. Assume that $(a', \dot{b}') \leq_q (a, \dot{b})$ where both are in G. We need $(a'', \dot{b}'') \in G$ such that $(a'', \dot{b}'') \leq_p (a, \dot{b})$ and $(a'', \dot{b}'') \leq_q (a', \dot{b}')$. First note that since $(a', \dot{b}') \in G$, it follows that there is some $(a'', \dot{c}) \in G$ extending $(a', \dot{b}')$ and $(q, \dot{1})$ with respect to the ordering on $\mathbb{P} * \dot{\mathbb{Q}}$. As in the proof of Lemma 1.47, we can find a name $\dot{b}''$ such that $q \Vdash \dot{b}'' = \dot{b}'$ and $1 \Vdash \dot{b}'' \leq \dot{b}$. Then $(a'', \dot{b}'') \leq_p (a, \dot{b})$ and $(a'', \dot{b}'') \leq_q (a', \dot{b}')$. Finally, since $a'' \leq q$, we have that $a'' \Vdash \dot{b}'' = \dot{b}'$. This implies that $(a'', \dot{c}) \leq_{\mathbb{P} * \dot{\mathbb{Q}}} (a'', \dot{b}'')$ and therefore $(a'', \dot{b}'') \in G$. □

The following is the main lemma of the section. It makes precise the claim that the generics for the p-term forcing products $\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})$ approximate the generic for $\mathbb{P} * \dot{\mathbb{Q}}$.

Lemma 1.50. *Assume the following:*

- *$G = \mathcal{P} * \mathcal{Q}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic,*
- *G^* is $(\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}))/G$-generic,*
- *For $p \in \mathcal{P}$, G_p is $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$-generic induced from G^*.*

Then $G = \bigcup_{p \in \mathcal{P}} G_p$.

Proof. By definition observe that $G_p \subseteq G$. Therefore it is enough to show $G \subseteq \bigcup_{p \in \mathcal{P}} G_p$. Towards that end, fix $a \in G$ and define $D = \{c \colon c \leq_p a \text{ for some } p \in \mathcal{P}\}$.

Claim 1.51. *D is dense in* $(\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$.

Proof of Claim. Fix $b \in (\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$. Since G is a filter, we may find $c \in G$ extending both a and b with respect to the ordering $\leq_{\mathbb{P} * \dot{\mathbb{Q}}}$. If the left coordinate of c is $p \in \mathcal{P}$, then we have that $c \leq_p a$ and $c \leq_p b$. Since $(\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$ projects onto $(\mathbb{P} \times \mathcal{A}_p(\mathbb{P}, \dot{\mathbb{Q}})) \cap G$, we may find $c' \in G$ such that $c' \leq_1 b$ and $c' \leq_p c$. But then $c' \in D$ and $c' \leq_1 b$ as desired. □

Since G^* is $(\mathbb{P} \times \mathcal{A}(\mathbb{P}, \dot{\mathbb{Q}}))/G$-generic over $V[G]$, we may find $b \in G^* \cap D$ where $p \in \mathcal{P}$ witnesses that $b \in D$. By the third hypothesis of the lemma, it follows that $b \in G_p$, which in turn implies that $a \in G_p$ because G_p is a filter.

$\square$

1.8 Mitchell Forcing

In this section we define a collection of Mitchell's original forcing $\mathbb{M}$ from (Mitchell, 1972) and present the argument that if κ is supercompact, then ITP holds at ω_2 after forcing with $\mathbb{M}$. Understanding the structure of this argument is critical for understanding the more involved (super) tree property arguments.

Definition 1.52. For $\tau < \mu < \kappa$ with τ and μ regular and κ inaccessible. Let conditions in $\mathbb{M}(\tau, \mu, \kappa)$ have the form (p, r) where

1. $p \in Add(\tau, \kappa)$,
2. r is a partial function on κ such that $|\mathrm{dom}(r)| < \mu$,
3. for each $\alpha \in \mathrm{dom}(r)$, we have $\Vdash_{Add(\tau,\alpha)} r(\alpha) \in \dot{Add}(\mu, 1)$.

Define the ordering on $\mathbb{M}(\tau, \mu, \kappa)$ by saying $(p, r) \leq (p', r')$ if

1. $p \leq p'$,
2. $\mathrm{dom}(r') \subseteq \mathrm{dom}(r)$ and for each $\alpha \in \mathrm{dom}(r')$, $p \restriction \alpha \Vdash r(\alpha) \leq r'(\alpha)$.

We abbreviate $\mathbb{M}(\tau, \tau^+, \kappa)$ by $\mathbb{M}(\tau, \kappa)$. Although the definition is convoluted at first glance, the motivation for this definition is concrete and gives us the following projections:

Lemma 1.53. $\mathbb{M}(\tau, \mu, \kappa)$ *projects onto* $Add(\tau, \kappa)$.

Proof. The map $(p, r) \mapsto p$ is a projection. □

Lemma 1.54. *For each* $\alpha < \kappa$, $\mathbb{M}(\tau, \mu, \kappa)$ *projects onto* $Add(\tau, \alpha) * Add(\dot{\mu}, 1)$.

Proof. The map $(p, r) \mapsto (p \upharpoonright \alpha, r(\alpha))$ is a projection. □

Lemma 1.55. *There is a* μ*-closed forcing* $\mathbb{Q}$ *such that* $Add(\tau, \kappa) \times \mathbb{Q}$ *projects onto* $\mathbb{M}(\tau, \mu, \kappa)$ *via the identity map.*

Proof. $\mathbb{Q}$ is defined to be the forcing whose underlying set is $(0, r) \in \mathbb{M}(\tau, \mu, \kappa)$ with the ordering induced from $\mathbb{M}(\tau, \mu, \kappa)$. Notice that we can think about $\mathbb{Q}$ as a product of 1-term forcings, so the proof is similar to Lemma 1.43. □

To determine the chain condition of $\mathbb{M}(\tau, \mu, \kappa)$ we first recall the Δ-system lemma:

Lemma 1.56 (Δ-System Lemma)**.** *Let* $\theta < \kappa$ *where* θ *is regular and* κ *is inaccessible. Further, let* $\mathcal{F}$ *be a collection of sets, each set having size less than* θ*, where* $|\mathcal{F}| = \kappa$*. Then there is a subcollection* $\mathcal{F}' \subseteq \mathcal{F}$ *of size* κ *and* r *where* $x \cap y = r$ *for distinct* $x, y \in \mathcal{F}'$.

Lemma 1.57. $\mathbb{M}(\tau, \mu, \kappa)$ *is* κ*-Knaster.*

Proof. Let $((p_\alpha, r_\alpha) \colon \alpha < \kappa)$ be a collection of distinct conditions in $\mathbb{M}(\tau, \kappa)$. For notation let $\mathcal{F}_X = \{\mathrm{dom}(p_\alpha) \colon \alpha \in X\}$ and $\mathcal{G}_X = \{\mathrm{dom}(r_\alpha) \colon \alpha \in X\}$ where $X \subseteq \kappa$. The idea is to find an uncountable subset X of κ where $p_\alpha \cup p_\beta$ and $r_\alpha \cup r_\beta$ are well-defined functions for all $\alpha, \beta \in X$.

Each element of $\mathcal{F}_\kappa$ has size less than τ, and so the Δ-system lemma with $\theta = \tau$ implies that there is an unbounded $A \subseteq \kappa$ and set a such that $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = a$ for distinct $\alpha, \beta \in A$.

Further, each element of $\mathcal{G}_A$ has size less than μ, and so another application of the Δ-system lemma implies that there is an unbounded $B \subseteq A$ and set b such that $\mathrm{dom}(r_\alpha) \cap \mathrm{dom}(r_\beta) = b$ for distinct $\alpha, \beta \in B$.

Next, notice that there are $< \kappa$-many $p \in Add(\tau, \kappa)$ with domain a. Since B has size κ it follows that we may find $A' \subseteq B$ with size κ such that $p_\alpha \upharpoonright a = p_\beta \upharpoonright a$ for each $\alpha, \beta \in A'$. Finally, recall that if r is a right coordinate of a condition in $\mathbb{M}(\tau, \mu, \kappa)$, then $\Vdash_{Add(\tau,\alpha)} r(\alpha) \in \dot{Add}(\mu, 1)$ for each $\alpha \in \mathrm{dom}(r)$. Since $\Vdash_{Add(\tau,\alpha)} \dot{Add}(\mu, 1) \in V_\kappa$ for each $\alpha < \kappa$ and b has size less than μ, there are $< \kappa$-many such r with domain b. So we can thin $B' \subseteq A'$ with size κ such that $r_\alpha \upharpoonright b = r_\beta \upharpoonright b$ for $\alpha, \beta \in B'$. It follows that for $\alpha, \beta \in B'$, (p_α, r_α) and (p_β, r_β) are compatible with lower bound $(p_\alpha \cup p_\beta, r_\alpha \cup r_\beta)$. $\square$

These lemmas allow for the desired cardinal structure:

Lemma 1.58. *Let G be $\mathbb{M}(\tau, \mu, \kappa)$-generic. Then in $V[G]$ we have that*

1. *cardinals outside the interval (μ, κ) are preserved,*
2. *each $\alpha \in (\mu, \kappa)$ is collapsed to μ,*
3. *$2^\tau = \mu^+ = \kappa$.*

Proof of 1. Since $Add(\tau, \kappa) \times \mathbb{Q}$ projects onto $\mathbb{M}(\tau, \kappa)$ we can find a generic object $A \times Q$ for $Add(\tau, \kappa) \times \mathbb{Q}$ such that $V[G] \subseteq V[A \times Q]$. Using Easton's Lemma notice that a cardinal $\alpha \leq \mu$ is still a cardinal in $V[A \times Q]$ and therefore also in $V[G]$. Additionally, since $\mathbb{M}(\tau, \mu, \kappa)$ is κ-Knaster we preserve all cardinals $\mu \geq \kappa$. $\square$

Proof of 2. Recall that $Add(\tau, \alpha)$ forces $\alpha \leq 2^\tau$ and $Add(\mu, 1)$ adds a surjection from μ onto 2^τ. Since $\mathbb{M}(\tau, \mu, \kappa)$ projects onto $Add(\tau, \alpha) * \dot{Add}(\mu, 1)$ for $\alpha < \kappa$ it follows that in $V[G]$ there is a surjection from μ onto α. □

Proof of 3. Using Easton's Lemma, notice that $(2^\tau)^{V[A]} = (2^\tau)^{V[G]}$. Then Lemma 1.53 implies $V[G] \models 2^\tau = (2^\tau)^{V[A]} = \kappa$, and further that $V[G] \models \kappa = \mu^+$ because $\mathbb{M}(\tau, \mu, \kappa)$ collapses all ordinals in (μ, κ) to μ. □

Before we prove that ITP holds at τ^{++}, we need to show that generics for $\mathbb{M}(\tau, \kappa)$ can be prolonged to generics for $j(\mathbb{M}(\tau, \kappa))$ and further that the quotient forcing $j(\mathbb{M}(\tau, \kappa))/\mathbb{M}(\tau, \kappa)$ factors in a similar way to $\mathbb{M}(\tau, \kappa)$.

Lemma 1.59. *Let $j \colon V \to M$ be an elementary embedding with critical point κ and G be $\mathbb{M}(\tau, \kappa)$-generic over V. Then*

1. *In M there is a projection from $j(\mathbb{M}(\tau, \kappa))$ onto $\mathbb{M}(\tau, \kappa)$.*
2. *In $M[G]$ there is a τ^+-Knaster forcing $\mathbb{P}^*$ and a τ^+-closed forcing $\mathbb{Q}^*$ such that $\mathbb{P}^* \times \mathbb{Q}^*$ projects onto $j(\mathbb{M}(\tau, \kappa))/G$.*
3. *We can lift the embedding to $j \colon V[G] \to M[j(G)]$ where $j(G)$ is $j(\mathbb{M}(\tau, \kappa))$-generic over M.*

Proposition 1.60. *Assume that κ is supercompact and G is $\mathbb{M}(\tau, \kappa)$-generic over V. Then ITP holds at τ^{++} in $V[G]$.*

Proof. Let κ be supercompact in V and $\mu \geq \kappa$ be regular. Let $D = (d_x \colon x \in \mathcal{P}_\kappa(\mu))$ be a thin $\mathcal{P}_\kappa(\mu)$-list in $V[G]$. Let $\theta = |D|^{V[G]}$ and let $j \colon V \to M$ be a λ-supercompactness embedding with $\lambda = \max\{\mu, \theta\}$. By Lemma 1.59 we can lift this embedding to $j \colon V[G] \to M[j(G)]$.

Claim 1.61. *$D \in M[G]$.*

Proof of Claim. By definition of j we know that $V \models {}^\lambda M \subseteq M$ and since $\mathbb{M}(\tau, \kappa)$ is κ-cc we have further that $V[G] \models {}^\lambda M[G] \subseteq M[G]$. For each $x \in \mathcal{P}_\kappa(\mu)^{V[G]}$ observe that $d_x \in \mathcal{P}_\kappa(\mu)^{V[G]}$ and so we can think of d_x as a function from some $\alpha < \kappa$ into $\mu \subseteq M[G]$. This implies that $d_x \in M[G]$ for each $x \in \mathcal{P}_\kappa(\mu)^{V[G]}$. Therefore $D \in {}^\lambda M[G] \subseteq M[G]$ as desired. □

Claim 1.62. *There is a cofinal branch b for D in $M[j(G)]$.*

Proof of Claim. By elementarity $j(D) = (e_z \colon z \in j(\mathcal{P}_\kappa(\mu)))$ is a thin $j(\mathcal{P}_\kappa(\mu))$-list in $M[j(G)]$. Because j is a μ-supercompactness embedding we have that $j``\mu \in j(\mathcal{P}_\kappa(\mu))$. Further, observe that the function $j \restriction \mu \in M$ and so $b = \{\alpha < \mu \colon j(\alpha) \in e_{j``\mu}\} \in M[j(G)]$. For each $x \in \mathcal{P}_\kappa(\mu)$ we must show there is a $y \supseteq x$ such that $b \cap x = d_y \cap x$. For notation set $Lev(x) = \{d_y \cap x \colon y \supseteq x\}$. Since D is a thin list, we have that $|Lev(x)| < \kappa$ and so elementarity implies that $j(Lev(x)) = j``Lev(x) = \{e_{j(y)} \cap j(x) \colon y \supseteq x\}$. Now, observe that $e_{j``\mu} \cap j(x) \in j(Lev(x))$ by definition of $j(Lev(x))$ and so there is a $y \supseteq x$ such that $e_{j``\mu} \cap j(x) = e_{j(y)} \cap j(x)$. This implies that $b \cap x = d_y \cap x$ as desired. □

Our goal is to show that $b \in M[G] \subseteq V[G]$ and that b must have been ineffable in this model. By Lemma 1.59 let $H \times K$ be $\mathbb{P}^* \times \mathbb{Q}^*$-generic over $M[G]$ such that $M[j(G)] \subseteq M[G][K][H]$. We want to show that b remains sufficiently approximated in $M[G]$ and $M[G][K]$.

Claim 1.63. *b is thinly κ-approximated in $M[G]$.*

Proof of Claim. We need to show that $b \cap x \in M[G]$ for each $x \in \mathcal{P}_\kappa(\mu)^{M[G]}$. Since b is a cofinal branch through D we may find a $y \supseteq x$ such that $b \cap x = d_y \cap x \in M[G]$. To show that b is thinly κ-approximated, observe that since D is a thin list in $M[G]$ there $< \kappa$-many choices for $d_y \cap x$. So if $\dot{b}$ is a name for b then we can always force $\dot{b} \cap x = d_y \cap x$ for some y. In other words, there are $< \kappa$-many choices for $\dot{b} \cap x$. □

Claim 1.64. *b is τ^+-approximated in $M[G][K]$.*

Proof of Claim. Notice that $\mathcal{P}_{\tau^+}(\mu)^{M[G]} = \mathcal{P}_{\tau^+}(\mu)^{M[G][K]}$ because $\mathbb{Q}^*$ is τ^+-closed in $M[G]$. So, for $x \in \mathcal{P}_{\tau^+}(\mu)^{M[G][K]}$ the previous claim implies that $b \cap x \in M[G] \subseteq M[G][K]$ as desired. □

Using these two claims we are able to pull the b back to $M[G]$:

Claim 1.65. *$b \in M[G][K]$.*

Proof of Claim. By Easton's Lemma we know that $\mathbb{P}^* \times \mathbb{P}^*$ is τ^+-cc in $M[G][K]$. This implies that $\mathbb{P}^*$ has the τ^+-approximation property in $M[G][K]$. Since b is τ^+-approximated in $M[G][K]$ it follows that $b \in M[G][K]$. □

Claim 1.66. *$b \in M[G]$.*

Proof of Claim. In $M[G]$ notice that $2^\tau \geq \kappa$ and $\mathbb{Q}^*$ is τ^+-closed. It follows by the generalized Silver's Branch Lemma that $\mathbb{Q}^*$ has the thin κ-approximation property in $M[G]$. Since b is thinly κ-approximated in $M[G]$ it follows that $b \in M[G]$. □

We now just have to check that b is ineffable in $V[G]$. To see this, we need to show that $j``\mu \in j(S)$ where $S = \{x \in \mathcal{P}_\kappa(\mu) \colon b \cap x = d_x\}$. However, this follows because $j(b) \cap j``\mu = e_{j``\mu}$. (Note that $j(b)$ and $j(S)$ only make sense *after* we have shown $b \in V[G]$.) $\square$

1.9 Singular Cardinals and Prikry Forcing

First we recall the definition of the Singular Cardinal Hypothesis.

Definition 1.67. For a singular strong limit κ, we say that SCH holds at κ if $2^\kappa = \kappa^+$.

If we have any hope of getting the tree property everywhere we have to show that the tree property holds at the double successor of a singular strong limit cardinal. By a classical result of Specker, this presents some issues.

Proposition 1.68. *(Specker, 1949) If $\kappa^{<\kappa} = \kappa$ then the tree property fails at κ^+.*

Specker's result implies something stronger than the failure of the tree property at κ^+: the existence of a *special* κ^+-Aronszajn tree. By work of Jensen, this is known to be equivalent to $\square^*_\kappa$ holding. In any case, if we want to get the tree property to hold at the double successor of a singular strong limit κ, we must have that SCH fails at κ. This is traditionally done by blowing up $\mathcal{P}(\kappa)$ of a measurable κ and then singularizing κ with Prikry forcing. So, in order to get the tree property at κ^+ and κ^{++} simultaneously, we will have to modify the usual left coordinate of the Mitchell/CF forcing to include a Prikry part. We will be summarizing the information in Section 3 of (Unger, 2013).

Let $(\kappa_n \colon n < \omega)$ be an increasing sequence of regular cardinals, $\kappa = \kappa_0$, $\kappa_\omega = \sup_n \kappa_n$, and $\mu = \kappa_\omega^+$. Assume that κ is indestructibly supercompact. Further, let λ_0 be a measurable cardinal above μ and U^* a normal measure on λ_0. Let $\mathbb{A} = \mathrm{Add}(\kappa, \lambda_0)$. Since κ is indestructibly supercompact, in $V[\mathbb{A}]$ we may let U be the supercompactness measure on $\mathcal{P}_\kappa(\mu)$ and U_n be the projections of U to $\mathcal{P}_\kappa(\kappa_n)$. For notation, if x and y are sets of ordinals, let κ_y denote the set $\kappa \cap y$ and $x \prec y$ hold when $x \subseteq y$ and and $ot(x) < \kappa_y$. In $V[\mathbb{A}]$, define the diagonal Prikry forcing $\mathbb{I}$, originally developed in (Gitik and Sharon, 2008) and in (Neeman, 2009), as follows:

Definition 1.69. $\mathbb{I}$ has conditions of the form

$$p = (x_0, x_1, \ldots, x_{n-1}, A_n, A_{n+1}, \ldots)$$

where

1. $x_i \in \mathcal{P}_\kappa(\kappa_i)$ and $x_i \cap \kappa \in \kappa$ for $i < n$,
2. $x_i \prec x_{i+1}$ for $i < n-1$, and
3. $A_i \in U_i$ for $i \geq n$.

The string $(x_0, \ldots, x_{n-1})$ is the stem of p an denoted $\mathrm{stem}(p)$. Given another condition

$$q = (y_0, y_1, \ldots, y_{m-1}, B_m, B_{m+1} \ldots),$$

we say that $p \leq q$ if

1. $m \leq n$,
2. $\mathrm{stem}(p) \restriction m = \mathrm{stem}(q)$,

3. $A_i \subseteq B_i$ for $i \geq n$, and

4. $x_i \in B_i$ for $m \leq i < n$.

In other words, extensions of q lengthen the stem of q by choosing elements from the B_i's while also shrinking the B_i's. This forcing adds a generic sequence $(x_n : n < \omega) \in \prod_{n<\omega} \mathcal{P}_\kappa(\kappa_n)$ such that $\bigcup_{n<\omega} x_n = \kappa_\omega$. This generic sequence singularizes each κ_n to have cofinality ω and forces $\mu = \kappa^+$.

Lemma 1.70. *Importantly, $\mathbb{I}$ sastifies the Prikry property: for any statement φ and any $p \in \mathbb{I}$, there is a direct extension $q \leq^* p$ deciding φ.*

Definition 1.71. Given a formula φ and a stem h, write $\underline{h \Vdash^* \varphi}$ if there is a condition $p \in \mathbb{I}$ with stem h forcing φ.

Let $\dot{U}$ be an $\mathbb{A}$-name for U and for $\alpha < \lambda_0$, let $\mathbb{A}_\alpha = \mathrm{Add}(\kappa, \alpha)$. It is important that we are able to project our Prikry posets onto smaller Prikry posets, so we show that we can do this on a measure 1 set. The following is a generalization of work found in (Cummings and Foreman, 1998) and appears as Lemma 3.2 in (Unger, 2013).

Lemma 1.72. *There is a $B \subseteq \lambda_0$ of Mahlo cardinals with $B \in V$ such that*

1. *if g is $\mathbb{A}$-generic over V, then $\dot{U}[G] \cap V[g \restriction \alpha] \in V[g \restriction \alpha]$ and*

2. *$B \in U^*$.*

More specifically, B is the set of Mahlo cardinals of some club subset of λ_0. From this, for each $\alpha \in B$ and each $\mathbb{A}$-generic g over V, we can define supercompactness measures U^α on

$\mathcal{P}_\kappa(\lambda_0)$ in $V[g \restriction \alpha]$ and the diagonal Prikry forcing $\mathbb{I}_\alpha$ obtained from U^α as in Definition 1.69. The generic object for $\mathbb{A} * \dot{\mathbb{I}}$ induces generic objects for $\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha$ with $\alpha \in B$, so we have the following relationships between the Prikry posets and their regular open algebras.

Lemma 1.73. *The following holds:*

1. *For all $\alpha \in B$ there is a projection $\pi_\alpha \colon \mathbb{A} * \dot{\mathbb{I}} \to RO(\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha)$*
2. *For all $\alpha, \beta \in B$ with $\alpha < \beta$ there is a projection $\pi_{\alpha,\beta} \colon \mathbb{A}_\beta * \dot{\mathbb{I}}_\beta \to RO(\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha)$*
3. *$\mathbb{A} * \dot{\mathbb{I}}$ is μ-cc and $\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha$ is μ-cc for each $\alpha \in B$.*
4. *$\Vdash^V_{\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha} ((\mathbb{A} * \dot{\mathbb{I}})/(\mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha))^2$ is μ-cc for $\alpha \in B$.*

The final result from this lemma is Lemma 5.3 in (Unger, 2013).

Lemma 1.74. *Forcing with $\mathbb{A} * \dot{\mathbb{I}}$ yields the following cardinal structure:*

1. *κ is singular strong limit with $cf(\kappa) = \omega$,*
2. *$\kappa^+ = (\kappa^+_\omega)^V = \mu$, and*
3. *$2^\kappa = \lambda_0$.*

1.10 The Tree Property at Successive Cardinals

If we were interested in getting the tree property simultaneously at ω_2 and ω_3, then a natural idea would be to use the Mitchell forcings from Section 1.8 and iterate $\mathbb{M}(\omega, \omega_2)$ followed by $\mathbb{M}(\omega_1, \omega_3)$. In the generic extension, such an iteration would result in the tree property at ω_3, but it would unfortunately destroy the tree property at ω_2. The reason for this is that there is interference between the Cohen posets of $\mathbb{M}(\omega, \omega_2)$ and $\mathbb{M}(\omega_1, \omega_3)$. The definition below was first developed in (Abraham, 1983) as a way to deal with this interference. The advantage is that it allows consideration of Cohen posets from different models of set theory. It was later in (Cummings and Foreman, 1998) used as the fundamental building block to get the tree property simultaneously at an ω-sequence of cardinals. The following definition appears as Definition 2.1 in (Unger, 2014).

Definition 1.75. Let $V \subseteq W$ be models of set theory. Suppose that τ and κ are cardinals such that $W \models \tau$ is regular and κ is inaccessible. Let $\mathbb{P} = \mathrm{Add}(\tau, \kappa)_V$ and assume that $W \models \mathbb{P}$ is τ^+-cc and $< \tau$-distributive. Also, let $\mathbb{P} \upharpoonright \beta = \mathrm{Add}(\tau, \beta)_V$ for $\beta < \kappa$. Let $F \in W$ be a function from κ to $(V_\kappa)_W$. Define $\mathbb{R} = \mathbb{R}(\tau, \kappa, V, W, F)$ in W by recursion on $\beta \leq \kappa$ and set $\mathbb{R} = \mathbb{R} \upharpoonright \kappa$. Let $\mathbb{R} \upharpoonright 0$ be the trivial forcing. Otherwise, (p, q, f) is a condition in $\mathbb{R} \upharpoonright \beta$ when the following hold:

1. $p \in \mathbb{P} \upharpoonright \beta$,
2. q is a partial function on β and $|\mathrm{dom}(q)| \leq \tau$, and if $\alpha \in \mathrm{dom}(q)$, then
   (a) α is a successor ordinal,
   (b) $q(\alpha) \in W^{\mathbb{P} \upharpoonright \alpha}$, and
   (c) $\Vdash^W_{\mathbb{P} \upharpoonright \alpha} q(\alpha) \in \mathrm{Add}(\tau^+, 1)_{W^{\mathbb{P} \upharpoonright \alpha}}$,
3. f is a partial function on β and $|\mathrm{dom}(f)| \leq \tau$, and if $\alpha \in \mathrm{dom}(f)$, then
   (a) $\Vdash^W_{\mathbb{R} \upharpoonright \alpha} F(\alpha)$ is a canonically τ^+-directed closed forcing,
   (b) α is a limit ordinal,

(c) $f(\alpha) \in W^{\mathbb{R}\restriction\alpha}$, and

(d) $\Vdash^{W}_{\mathbb{R}\restriction\alpha} f(\alpha) \in F(\alpha)$.

We also define the ordering $(p_1, q_1, f_2) \leq (p_2, q_2, f_2)$ when the following hold:

1. $p_1 \leq_{\mathbb{P}\restriction\alpha} p_2$,

2. $\mathrm{dom}(q_2) \subseteq \mathrm{dom}(q_1)$ and if $\alpha \in \mathrm{dom}(q_2)$, then $p_1 \restriction \alpha \Vdash^{W}_{\mathbb{P}\restriction\alpha} q_1(\alpha) \leq q_2(\alpha)$,

3. $\mathrm{dom}(f_2) \subseteq \mathrm{dom}(f_1)$ and if $\alpha \in \mathrm{dom}(f_2)$, then $(p_1, q_1, f_1) \restriction \alpha \Vdash^{W}_{\mathbb{R}\restriction\alpha} f_1(\alpha) \leq f_2(\alpha)$.

Lemma 1.76. *This forcing satisfies the following properties. Each reference is from (Cummings and Foreman, 1998).*

1. *(Lemma 3.2)* $|\mathbb{R}| = \kappa$ *and* $\mathbb{R}$ *is* κ*-Knaster.*

2. *(Lemma 3.3) We have that* $\mathbb{R}$ *projects onto* $\mathbb{R} \restriction \alpha * F(\alpha)$, $\mathbb{P} \restriction \alpha * Add(\tau^+, 1)_{W^{\mathbb{P}\restriction\alpha}}$, *and* $\mathbb{P}$.

3. *(From Section 3.3) Let* $\mathbb{U}$ *be all conditions in* $\mathbb{R}$ *of the form* $(0, q, f)$ *with the ordering induced from* $\mathbb{R}$. *Then,* $\mathbb{U}$ *is canonically* τ^+*-directed closed,* κ*-cc, and the product forcing* $\mathbb{P} \times \mathbb{U}$ *projects onto* $\mathbb{R}$.

4. *(Variant of Lemma 3.6) If* $\theta \leq \tau$ *and* $\mathbb{P}$ *is canonically* θ*-directed closed in* W, *then* $\mathbb{R}$ *is canonically* θ*-directed closed in* W.

5. *(Lemma 3.11)* $\mathbb{U}$ *is* $\leq \tau$*-distributive in* $W^{\mathbb{P}}$.

6. *(Corollary 3.16)* $\mathbb{R}$ *is* $< \tau$*-distributive in* W.

7. *(Lemma 3.20) Let* G *be* $\mathbb{R}$*-generic over* W *and* $\mathbb{S}$ *be the quotient forcing of* $\mathbb{P} \times \mathbb{U}$ *defined in* $W[G]$. *It follows that* $W[G] \models$ *"*$\mathbb{S}$ *is* $< \tau^+$*-distributive,* τ*-closed, and* κ*-cc."*

8. *(From Section 3.3) Let G be $\mathbb{R}$-generic over W.*

   (a) *$W[G]$ and $W[g]$ have the same τ-sequences of ordinals, where g is the $\mathbb{P}$-generic induced by G.*

   (b) *$W[G] \models \tau^+$ is preserved and $2^\tau = \kappa = \tau^{++}$.*

   (c) *If X is a set of ordinals in $W[G]$ where $|X|^{W[G]} = \tau$ then there is a $Y \supseteq X$ in W where $|Y|^W = \tau$.*

9. *(Corollary 3.17) $W^{\mathbb{U}} \models \kappa = \tau^{++}$.*

10. *(From Section 3.5) $\mathbb{R}$ projects onto $\mathbb{R} \restriction \alpha$. In $W^{\mathbb{R} \restriction \alpha}$, there are forcings $\mathbb{P}^*, \mathbb{U}^*$ such that $\mathbb{P}^*$ is τ^+-cc, $\mathbb{U}^*$ is τ^+-closed, and $\mathbb{P}^* \times \mathbb{U}^*$ projects onto $\mathbb{R}/\mathbb{R} \restriction \alpha$.*

By iterating Definition 1.75, we get the main result of (Abraham, 1983). More precisely:

Lemma 1.77. *Let $\kappa_0 < \kappa_1$ where κ_0 is supercompact and κ_1 is weakly compact. Let F_0 be a Laver function for κ_0. Define $\mathbb{R}$ as $\mathbb{R}(\omega, \kappa_0, V, V, F_0)$. Working in $V[\mathbb{R}]$, define $\mathbb{R}'$ as $\mathbb{R}(\omega_1, \kappa_1, V, V[\mathbb{R}], id)$, where id is the identity map on κ_1. Then in $V[\mathbb{R} * \dot{\mathbb{R}}']$ we have the following:*

1. *ω_1 is preserved, $\kappa_0 = \omega_2$, and $\kappa_1 = \omega_3$,*
2. *$2^\omega = \omega_2$ and $2^{\omega_1} = \omega_3$, and*
3. *the tree property holds at ω_2 and ω_3.*

Finally, we collect some more facts from (Cummings and Foreman, 1998) that will be useful later. The reference is from the original paper.

Lemma 1.78 (Lemma 2.6)**.** *Let $\tau < \kappa$, and assume that $V \models$ "τ is regular and κ is inaccessible". Let $\mathbb{P} = Add(\tau, \eta)$. Let $W \supseteq V$ be a model of ZFC such that*

1. *κ and τ are cardinals in W,*
2. *if $X \in W$ is a set of ordinals such that $W \models |X| < \kappa$, then there is a $Y \supseteq X$ such that $Y \in V$ and $V \models |Y| < \kappa$.*

Then $\mathbb{P}$ is κ-Knaster in W.

Lemma 1.79 (Lemma 2.13)**.** *Let τ be regular and let $\mathbb{A} = Add(\tau, \eta)$ for some η. Let κ be inaccessible with $\tau < \kappa$. Then*

1. *If $\mathbb{Q}$ is κ-cc and $\mathbb{Q}$ is a projection of $\mathbb{P} \times \mathbb{U}$, where $\mathbb{P}$ is τ-cc and $\mathbb{U}$ is τ-closed, then $V[\mathbb{Q}]$ believes that $\mathbb{A}$ is κ-Knaster and $< \tau$-distributive.*
2. *Suppose that $V[\mathbb{Q}]$ believes that $\mathbb{Q}^*$ is a projection of $Add(\tau, \zeta)_V \times \mathbb{U}^*$ and that $\mathbb{U}^*$ is κ-closed. Then $V[\mathbb{Q} * \dot{\mathbb{Q}}^*] \models \mathbb{A}$ is κ-Knaster.*

CHAPTER 2

INDESTRUCTIBILITY OF ITP IN $\mathbb{M}(\tau, \kappa)$

Recall that in Section 1.8 we showed that ITP holds at τ^{++} after forcing with the Mitchell forcing $\mathbb{M}(\tau, \kappa)$. In this chapter we prove indestructibility results about which posets may be iterated with the Mitchell model and preserve ITP at τ^{++}. This is generalizing work of (Unger, 2012).

2.1 Generalized Branch Lemma

In this section we generalize Lemma 1.17. All tree property arguments involve appealing to various types of branch preservation lemmas, and so it is useful to also determine which could apply to ITP arguments as well. For example, Lemma 1.17 was used in (Neeman, 2014) to consistently get the tree property at $\aleph_{\omega+1}$ and $\aleph_n$ for each $n > 1$. Further, the proof of Lemma 1.17 was used in (Sinapova, 2012) in order to prove an important splitting lemma that allowed for the failure of the tree property at $\aleph_{\omega^2+1}$ with failure of SCH at $\aleph_{\omega^2}$. We start with a definition motivated from (Magidor and Shelah, 1996).

Definition 2.1. Let $\mathbb{Q}$ be a forcing and $\dot{b}$ be a $\mathbb{Q}$-name for a subset of an ordinal μ. Assume $\dot{b}$ is κ-approximated in V. Say that r_1, r_2 force contradictory information about $\dot{b}$ at $z \in \mathcal{P}_\kappa(\mu)$ if for each pair $(r_1', r_2') \leq_{\mathbb{Q}\times\mathbb{Q}} (r_1, r_2)$, if r_1' and r_2' each decide values for $\dot{b} \cap z$, they decide different values.

Lemma 2.2. *Let $\mathbb{Q}$ be a poset and $\dot{b}$ be a $\mathbb{Q}$-name for a κ-approximated subset of μ. Suppose r_1, r_2 force contradictory information about $\dot{b}$ at $x \in \mathcal{P}_\kappa(\mu)$. Then*

1. *For each $z \supseteq x$, r_1, r_2 force contradictory information about $\dot{b}$ at z,*
2. *For each pair $(r_1', r_2') \leq_{\mathbb{Q}\times\mathbb{Q}} (r_1, r_2)$, r_1', r_2' force contradictory information about $\dot{b}$ at x.*

Proof. We do each in turn:

1. Assume that $(r_1', r_2') \leq (r_1, r_2)$, $s_1, s_2 \in \mathcal{P}_\kappa(\mu)$, and $r_i' \Vdash s_i = \dot{b} \cap z$ for $i \in \{1,2\}$. This implies that $r_i' \Vdash s_i \cap x = \dot{b} \cap x$ for $i \in \{1,2\}$ and so $s_1 \cap x \neq s_2 \cap x$ by hypothesis. This implies $s_1 \neq s_2$ as desired.
2. This follows immediately from the fact that r_1, r_2 force contradictory information about $\dot{b}$ at x. □

Lemma 2.3. *Suppose that $\dot{b}$ is a $\mathbb{Q}$-name for a κ-approximated subset of μ and $\Vdash_{\mathbb{Q}} \dot{b} \notin V$. Then for every $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ there is $(r_1', r_2') \leq (r_1, r_2)$ and $z \in \mathcal{P}_\kappa(\mu)$ such that r_1', r_2' force contradictory information about $\dot{b}$ at z.*

Proof. Assume that the lemma is false. Fix $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ witnessing this. We claim that $(r_1, r_2) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{b}[\Gamma_L] = \dot{b}[\Gamma_R]$, where Γ_L is generic for the left coordinate and Γ_R is generic for the right coordinate. This is a contradiction because this implies that $\Vdash_{\mathbb{Q}} \dot{b} \in V$ by Lemma 1.6. Assume instead that $(r_1, r_2) \nVdash \dot{b}[\Gamma_L] = \dot{b}[\Gamma_R]$. This implies that we may find $(p, q) \leq (r_1, r_2)$ and $x \in \mathcal{P}_\kappa(\mu)$ such that $(p, q) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{b}[\Gamma_L] \cap x \neq \dot{b}[\Gamma_R] \cap x$. Because we assumed that the lemma is false, we may find $p' \leq p$ and $q' \leq q$ and $v \in \mathcal{P}_\kappa(\mu)$ such that $p' \Vdash_{\mathbb{Q}} \dot{b} \cap x = v$ and

$q' \Vdash_{\mathbb{Q}} \dot{b} \cap x = v$. However, if $p' \in G_L$ is $\mathbb{Q}$-generic over V and $q' \in G_R$ is $\mathbb{Q}$-generic over $V[G_L]$, this implies that $V[G_L][G_R] \models \dot{b}[G_L] \cap x = \dot{b}[G_R] \cap x$, a contradiction. $\square$

By combining Lemma 2.2 and Lemma 2.3 we have the following variant.

Lemma 2.4. *Suppose that $\dot{b}$ is a $\mathbb{Q}$-name for a κ-approximated subset of μ and $\Vdash_{\mathbb{Q}} \dot{b} \notin V$. Then for every $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ and $y \in \mathcal{P}_\kappa(\mu)$ there is $(r_1', r_2') \leq (r_1, r_2)$ and $z \supseteq y$ such that r_1', r_2' force contradictory information about $\dot{b}$ at z.*

We are now in a position to prove the generalized branch lemma.

Lemma 2.5 (Generalized Branch Lemma)**.** *Let κ be a regular cardinal. Assume $\mathbb{P}$ is τ^+-cc, $\mathbb{Q}$ is τ^+-closed for some $\tau < \kappa$ such that $2^\tau \geq \kappa$. Then in $V[\mathbb{P}]$ we have that $\mathbb{Q}$ has the thin κ-approximation property.*

Proof. In $V[\mathbb{P}]$ let $\dot{b}$ be a $\mathbb{Q}$-name for a subset of μ that is thinly κ-approximated in $V[\mathbb{P}]$. Assume instead that $\Vdash_{\mathbb{Q}} \dot{b} \notin V[\mathbb{P}]$. Work in V until further notice. Then we have the following claim:

Claim 2.6. *For $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$ define $D_{r_1, r_2} \subseteq \mathbb{P}$ to be the set of all conditions $p \in \mathbb{P}$ such that there is $(r_1', r_2') \leq_{\mathbb{Q} \times \mathbb{Q}} (r_1, r_2)$ and $z \in \mathcal{P}_\kappa(\mu)^V$ such that*

$$p \Vdash \text{“}r_1', r_2' \textit{ force contradictory information about } \dot{b} \textit{ at } z.\text{”}$$

Then D_{r_1, r_2} is dense.

Proof of Claim. Apply Lemma 2.3 in $V[\mathbb{P}]$ to show for each $q \in \mathbb{P}$ that

$$q \Vdash_{\mathbb{P}} \text{“}\exists (r_1', r_2') \leq (r_1, r_2)\ \exists z \in \mathcal{P}_\kappa(\mu)\ r_1', r_2' \text{ force contradictory information about } \dot{b} \text{ at } z.\text{”}$$

Then, choose $p' \leq q$, $(r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}$, and a $\mathbb{P}$-name σ for an element of $\mathcal{P}_\kappa(\mu)$ such that

$$p' \Vdash_{\mathbb{P}} \text{“}r_1', r_2' \text{ force contradictory information about } \dot{b} \text{ at } \sigma.\text{”}$$

Finally, fix a $\mathbb{P}$-generic G containing p' and notice that

$$V[G] \models \text{“}r_1', r_2' \text{ force contradictory information about } \dot{b} \text{ at } \sigma[G] \in \mathcal{P}_\kappa(\mu).\text{”}$$

Since $\mathbb{P}$ is κ-cc we may find a $z \supseteq \sigma[G]$ such that $z \in \mathcal{P}_\kappa(\mu)^V$. By Lemma 2.2, we have

$$V[G] \models \text{“}r_1', r_2' \text{ force contradictory information about } \dot{b} \text{ at } z.\text{”}$$

The claim is proven by choosing any $p \leq p'$ forcing this. □

We continue with another claim:

Claim 2.7. *For each $r \in \mathbb{Q}$ and $x \in \mathcal{P}_\kappa(\mu)$ there is a maximal antichain $A \subseteq \mathbb{P}$, conditions $r_0^*, r_1^* \in \mathbb{Q}$ extending r, and $x^* \in \mathcal{P}_\kappa(\mu)^V$ with $x^* \supseteq x$ such that for each $p \in A$*

$$p \Vdash \text{“}r_0^*, r_1^* \textit{ force contradictory information about } \dot{b} \textit{ at } x^*.\text{”}$$

Proof of Claim. Observe that this is not immediate from the first claim because (r_1', r_2') and x^* both depend on $p \in \mathbb{P}$. The goal is to remove this. We construct sequences $(A_\alpha, x_\alpha \colon \alpha < \tau^+)$ and $(r_i^\alpha \colon \alpha < \tau^+)$ for $i < 2$ by induction such that the following hold:

1. $(A_\alpha \colon \alpha < \tau^+)$ is a $\subseteq$-increasing sequence of antichains of $\mathbb{P}$,
2. $(x_\alpha \colon \alpha < \tau^+)$ is a $\subseteq$-increasing sequence of elements of $\mathcal{P}_\kappa(\mu)^V$, and
3. for each $i < 2$, $(r_i^\alpha \colon \alpha < \tau^+)$ is a decreasing sequence of elements of $\mathbb{Q}$.

Base Case: Fix $p_0 \in D_{r,r}$, letting $(r_1^0, r_2^0) \leq (r, r)$ and z witness this. Then set $A_0 = \{p_0\}$ and $x_0 = z \cup x$.

Successor Case: Assume that we have defined A_α, (r_0^α, r_1^α), and x_α. If A_α is a maximal antichain, then define $A_{\alpha+1} = A_\alpha$, $(r_0^{\alpha+1}, r_1^{\alpha+1}) = (r_0^\alpha, r_1^\alpha)$, and $x_{\alpha+1} = x_\alpha$. Otherwise, let $p \in \mathbb{P}$ be some element incompatible with each element of A_α. Next we fix $p' \leq p$ such that $p' \in D_{r_0^\alpha, r_1^\alpha}$. Let $(r_1^{\alpha+1}, r_2^{\alpha+1}) \leq (r_0^\alpha, r_1^\alpha)$ and z witness this. Then set $A_{\alpha+1} = A_\alpha \cup \{p'\}$ and $x_{\alpha+1} = z \cup x_\alpha$.

Limit Case: Assume that $\theta < \tau^+$ is a limit ordinal and we have defined $(A_\alpha, x_\alpha \colon \alpha < \theta)$ and $(r_i^\alpha \colon \alpha < \theta)$ for $i < 2$. Then set $A_\theta = \bigcup_{\alpha<\theta} A_\alpha$, (r_1^θ, r_2^θ) to be a lower bound for the sequence $((r_0^\alpha, r_1^\alpha) \colon \alpha < \theta)$, and $x_\theta = \bigcup_{\alpha<\theta} x_\alpha$. This completes the induction.

Finally, since antichains of $\mathbb{P}$ have size at most τ there must be an $\alpha < \tau^+$ such that A_α is a maximal antichain. Define $A = A_\alpha$, $(r_0^*, r_1^*) = (r_0^\alpha, r_1^\alpha)$, and $x^* = x_\alpha$. To see that this works, assume that $p \in A$. By the definition of A there must have been some $\beta < \alpha$ such that

$$p \Vdash \text{“}r_0^{\beta+1}, r_1^{\beta+1} \text{ force contradictory information about } \dot{b} \text{ at } x_{\beta+1}.\text{”}$$

Since $(r_0^*, r_1^*) \leq (r_0^{\beta+1}, r_1^{\beta+1})$ it follows by Lemma 2.2 that

$$p \Vdash \text{“}r_0^*, r_1^* \text{ force contradictory information about } \dot{b} \text{ at } x_{\beta+1}.\text{”}$$

Finally, since $x^* \supseteq x$ another application of Lemma 2.2 yields as desired that

$$p \Vdash \text{“}r_0^*, r_1^* \text{ force contradictory information about } \dot{b} \text{ at } x^*.\text{”} \qquad \square$$

To finish the proof we proceed in a manner similar to the proof of Lemma 1.16. Assume that θ is the least cardinal such that $2^\theta \geq \kappa$. Note that $\mathbb{Q}$ is θ^+-closed. We define sequences $(r_\sigma, i_\sigma, j_\sigma, A_\sigma \colon \sigma \in 2^{<\theta})$ and $(x_\alpha \colon \alpha < \theta)$ by induction with the following properties:

1. A_σ is a maximal antichain in $\mathbb{P}$,
2. $r_\sigma, i_\sigma, j_\sigma \in \mathbb{Q}$, where $i_\sigma = r_{\sigma\frown 0}$ and $j_\sigma = r_{\sigma\frown 1}$ for each $\sigma \in 2^{<\theta}$,
3. $p \Vdash$ “$r_{\sigma\frown 0}, r_{\sigma\frown 1}$ force contradictory information about $\dot{b}$ at $x_{|\sigma|}$” for each $p \in A_\sigma$,
4. if $\sigma_0 \subset \sigma_1$ then $r_{\sigma_1} \leq r_{\sigma_0}$, and $x_{|\sigma_0|} \subset x_{|\sigma_1|}$.

Base Case: Set $r_\varnothing = 1_{\mathbb{Q}}$ and apply the above claim to get a maximal antichain $A_\varnothing$, conditions $(i_\varnothing, j_\varnothing) \leq (r_\varnothing, r_\varnothing)$, and x_0 such that $p \Vdash$ “$i_\varnothing, j_\varnothing$ force contradictory information about $\dot{b}$ at x_0” for each $p \in A_\varnothing$.

Successor Case: Assume that $\alpha < \theta$ is an ordinal and that we have defined r_σ, i_σ, j_σ, A_σ, and x_α for each σ with length α. Set $r_{\sigma\frown 0} = i_\sigma$ and $r_{\sigma\frown 1} = j_\sigma$. Apply the above claim twice to get maximal antichains $A_{\sigma^\frown 0}$ and $A_{\sigma^\frown 1}$, pairs of conditions $(i_{\sigma^\frown 0}, j_{\sigma^\frown 0})$ and $(i_{\sigma^\frown 1}, j_{\sigma^\frown 1})$, and sets $y_{\sigma^\frown 0}, y_{\sigma^\frown 1} \supseteq x_\alpha$ such that

1. $i_{\sigma^\frown 0}, j_{\sigma^\frown 0} \leq r_{\sigma^\frown 0}$,

2. $i_{\sigma^\frown 1}, j_{\sigma^\frown 1} \leq r_{\sigma^\frown 1}$,

3. $p \Vdash$ "$i_{\sigma^\frown k}, j_{\sigma^\frown k}$ force contradictory information about $\dot{b}$ at $y_{\sigma^\frown k}$" for each $p \in A_{\sigma^\frown k}$ and each $k < 2$.

Since θ is least such that $2^\theta \geq \kappa$ it follows that $2^\alpha < \kappa$. We may therefore set $x_{\alpha+1} = \bigcup_{\sigma \in 2^\alpha, k<2} y_{\sigma^\frown k} \in \mathcal{P}_\kappa(\mu)$.

Limit Case: Assume $\lambda < \theta$ is a limit ordinal and we have defined r_σ, i_σ, j_σ, A_σ, and x_α for each σ with $|\sigma| = \alpha < \lambda$. For each $f \in 2^\lambda$, observe that $(r_{f \restriction i} \colon i < \lambda)$ is a decreasing sequence. So we may use the closure of $\mathbb{Q}$ to find a lower bound r_f. Next, for each $f \in 2^\lambda$ define $z_f = \bigcup_{\alpha<\lambda} x_{f\restriction\alpha}$. Apply the above claim to get a maximal antichain A_f, conditions i_f and j_f below r_f, and a set $y_f \supseteq z_f$ such that $p \Vdash$ "i_f, j_f force contradictory information about $\dot{b}$ at y_f" for each $p \in A_f$. Then set $x_\lambda = \bigcup_{f \in 2^\lambda} y_f$. This completes the construction.

With our sequences $(r_\sigma, A_\sigma \colon \sigma \in 2^{<\theta})$ and $(x_\alpha \colon \alpha < \theta)$, define $x = \bigcup_{\alpha<\theta} x_\alpha$ and let r_f be a lower bound for the sequence $(r_{f\restriction\alpha} \colon \alpha < \theta)$ for each $f \in 2^\theta$. Let G be $\mathbb{P}$-generic and work in $V[G]$ for the remainder of the proof. We show that for each distinct pair f and $g \in 2^\theta$, r_f and r_g force contradictory information about $\dot{b}$ at x. Indeed, if $i < \theta$ is least such that $f(i) \neq g(i)$, then $s = f \restriction i = g \restriction i$. Since A_s is a maximal antichain it follows that $G \cap A_s$ is nonempty. This implies that $r_{s^\frown 0}$ and $r_{s^\frown 1}$ force contradictory information about $\dot{b}$ at $x_{|s|}$. Lemma 2.2

implies that r_f and r_g force contradictory information about $\dot{b}$ at x. This implies that there are at least $2^\theta \geq \kappa$ distinct potential values of $\dot{b} \cap x$ contradicting that $\dot{b}$ is thinly κ-approximated in $V[G]$. $\square$

2.2 Indestructibility

With the work from the previous section we are now in the position to prove some indestructibility results about ITP at τ^{++}. In particular we show that forcing with Cohen forcing after Mitchell forcing over a sufficiently compact ground model does not destroy ITP at τ^{++}. Recall that $\mathbb{M}(\tau,\kappa)$ is the Mitchell poset defined in Section 1.8.

Theorem 2.8. *Let κ be supercompact and $\tau < \kappa$ be regular. If $\Vdash_{\mathbb{M}(\tau,\kappa)}$ "$\dot{\mathbb{Q}}$ is a τ^+-cc forcing of size τ^+" then $\Vdash_{\mathbb{M}(\tau,\kappa)*\dot{\mathbb{Q}}}$ "ITP holds at τ^{++}."*

Proof. Let G be $\mathbb{M}(\tau,\kappa)$-generic over V and let $\dot{D} \in V[G]$ be a $\mathbb{Q}$-name for a $\mathcal{P}_\kappa(\mu)$-list. Let $j\colon V \to M$ be a μ-supercompactness embedding with critical point κ. Since $j(\mathbb{M}(\tau,\kappa))$ projects onto $\mathbb{M}(\tau,\kappa)$ we are allowed to lift this embedding to $j\colon V[G] \to M[j(G)]$ where $j(G)$ is $j(\mathbb{M}(\tau,\kappa))$-generic over M. Note that this lifted embedding has critical point τ^{++}. Further, since $\mathbb{Q}$ has size τ^+ in $V[G]$ we may assume that $j(\mathbb{Q}) = \mathbb{Q}$. This implies that we may further lift the embedding to $j\colon V[G][x] \to M[j(G)][x]$ where x is $\mathbb{Q}$-generic over $V[j(G)]$.

Then arguments similar to the proof of Proposition 1.60 show that $D = \dot{D}[x] \in M[G][x]$ and further that D has an ineffable branch b in $M[j(G)][x]$. We have to pull b back to $M[G][x]$. To do this recall that by Lemma 1.59 the quotient $j(\mathbb{M}(\tau,\kappa))/G$ factors like the Mitchell poset and so $M[j(G)] \subseteq M[G][H][K]$, where H is a generic object for a τ^+-Knaster Cohen forcing in

$M[G]$ and K is a generic object for a τ^+-closed forcing in $M[G]$. Mutual genericity allows us to say $M[j(G)][x] \subseteq M[G][x][K][H]$. We may argue the following:

Claim 2.9. *b is thinly κ-approximated in $M[G]$.*

Proof. The same argument as in the proof of Proposition 1.60. We note that we are writing κ instead of τ^{++} because the generic object K collapses κ to τ^+. $\square$

Claim 2.10. *b is τ^+-approximated in $M[G][x]$ and $M[G][x][K]$.*

Proof. By the previous claim we have that b is τ^+-approximated in $M[G]$. Then, Lemma 1.18 implies that b is still τ^+-approximated in $M[G][x]$. Next, Easton's Lemma implies that K is generic for a $< \tau^+$-distributive forcing in $M[G][x]$, which gives us that $\mathcal{P}_{\tau^+}(\mu)^{M[G][x]} = \mathcal{P}_{\tau^+}(\mu)^{M[G][x][K]}$. So if $x \in \mathcal{P}_{\tau^+}(\mu)^{M[G][x][K]}$ we get that $b \cap x \in M[G][x] \subseteq M[G][x][K]$. $\square$

We finish the theorem by pulling back the branch:

Claim 2.11. $b \in M[G][x][K]$

Proof. This follows because H is generic for a Cohen forcing which is τ^+-Knaster and therefore has the τ^+-approximation property in $M[G][x][K]$. $\square$

Claim 2.12. $b \in M[G][x]$

Proof. Here we use Lemma 2.5 where the ground model is $M[G]$ and $\mathbb{Q}$ is the τ^+-cc forcing. $\square$

This final claim yields the result. $\square$

Theorem 2.13. *Let κ be supercompact and $\tau < \kappa$ be regular. If $\Vdash_{\mathbb{M}(\tau,\kappa)}$ "$\mathbb{Q} = Add(\tau, \theta)$ for some cardinal θ," then $\Vdash_{\mathbb{M}(\tau,\kappa)*\dot{\mathbb{Q}}}$ "ITP holds at τ^{++}."*

Proof. The proof is similar to the previous result so we summarize some of the key points. First, take an appropriate supercompactness embedding and lift it to $j\colon V[G][x] \to M[j(G)][x^*]$ where x^* is a generic object for $j(\mathbb{Q})$. We may factor $x^* = x \times y$, where x is generic for $\mathbb{Q}$ and y is generic for $j(\mathbb{Q})/x$. Further, observe that $j(\mathbb{Q})/x$ is just a Cohen forcing. So, like before we have that $M[j(G)][x^*] \subseteq M[G][x][K][H][y]$, where H, K are like above. To justify why we can move around the generic x, we observe that x is generic over $M[G][H]$ since $M[j(G)] \supseteq M[G][H]$. This implies that x and H are mutually generic. Further, x and K are mutually generic by Easton's Lemma. Then we argue that there is a branch $b \in M[G][x][K][H][y]$. Like above we argue that b is τ^+-approximated in $M[G][x][K]$ and use that $H \times y$ is generic for a τ^+-Knaster forcing to pull the branch back to $M[G][x][K]$. Finally, use Lemma 2.5 to pull the branch from $M[G][x][H_2]$ to $M[G][x]$. $\square$

CHAPTER 3

TREE PROPERTY AT κ^{+N} FOR $N \geq 1$

In this chapter we define a variant of the Cummings-Foreman iteration and show that, in the generic extension, the tree property holds at κ^{+n} for each $n \geq 1$ and κ is singular strong limit. One of the biggest differences between this forcing and the forcing in (Cummings and Foreman, 1998) is the first factor defined in the next section. More specifically, we need to interleave Prikry forcing with forcing defined in Section 1.10 in order to make κ singular.

After defining the first factor $\mathbb{Q}_0$, we prove some of its structural properties and define the Cummings-Foreman variant $\mathbb{R}_\omega$. The argument to get the tree property at κ^{+n} for $n \geq 2$ is similar in spirit to the original argument in (Cummings and Foreman, 1998), the biggest difference being the case when $n = 2$. To show the tree property at κ^+ we use techniques from (Sinapova, 2016) and the structural properties of the p-term forcings from Section 1.7.

3.1 The factor $\mathbb{Q}_0$

Let $(\kappa_n : n < \omega)$ be an increasing sequence of indestructibly supercompact cardinals, $\kappa = \kappa_0$, $\kappa_\omega = \sup_n \kappa_n$, and $\mu = \kappa_\omega^+$. Further, let $(\lambda_n : n < \omega)$ be another increasing sequence of supercompact cardinals with $\mu < \lambda_0$. Let $\lambda = \sup(\lambda_i)$. Fix Laver functions $(F_n : n < \omega)$ for the λ_n's. Using notation from Section 1.9, set $\mathbb{P}_0 = \mathbb{A} * \dot{\mathbb{I}}$ where $\mathbb{A} = Add(\kappa, \lambda_0)$ and $\mathbb{I}$ is the diagonal Prikry forcing. Further recall from this section that $\mathbb{P}_0$ projects onto forcings $\mathbb{A}_\beta * \dot{\mathbb{I}}_\beta$ where β comes from a measure one set B. Set $\mathbb{P}_{0,\beta} = \mathbb{A}_\beta * \dot{\mathbb{I}}_\beta$ for $\beta \in B$. If $\beta = \lambda_0$, set $\mathbb{P}_{0,\beta} := \mathbb{P}_0$.

Finally, say that $\beta \in B$ is a successor ordinal of B if $ot(B \cap \beta)$ is a successor ordinal. Otherwise, say that $\beta \in B$ is a limit ordinal of B.

We will begin by defining a modification of the forcing from Section 1.10 to be denoted $\mathbb{Q}_0$. This definition will be by induction, where we define forcings $\mathbb{Q}_0 \restriction \beta$ for $\beta \in B \cup \{\lambda_0\}$. We finally set $\mathbb{Q}_0 := \mathbb{Q}_0 \restriction \lambda_0$.

Definition 3.1. Let $\mathbb{Q}_0 \restriction 0$ be the trivial forcing. Otherwise, $((f, \dot{p}), r, g)$ is a condition in $\mathbb{Q}_0 \restriction \beta$ when the following hold:

1. $(f, \dot{p}) \in \mathbb{P}_{0,\beta}$
2. r is a partial function on β with $\mathrm{dom}(r)$ consisting of successor ordinals of B, and $|\mathrm{dom}(r)| < \mu$
3. if $\alpha \in \mathrm{dom}(r)$ then $\Vdash_{\mathbb{P}_{0,\alpha}} r(\alpha) \in \mathrm{Add}(\mu, 1)_{V[\mathbb{P}_{0,\alpha}]}$
4. g is a partial function on β with $\mathrm{dom}(g)$ consisting of limit ordinals of B, and $|g| < \mu$, and $\mathrm{dom}(g) \subseteq \{\alpha \colon \Vdash_{\mathbb{Q}_0 \restriction \alpha} F_0(\alpha)$ is a canonically μ-directed closed forcing$\}$
5. if $\alpha \in \mathrm{dom}(g)$, then $\Vdash_{\mathbb{Q}_0 \restriction \alpha} g(\alpha) \in F_0(\alpha)$

The ordering is defined by $(f_1, \dot{p}_1, r_1, g_1) \leq (f_2, \dot{p}_2, r_2, g_2)$ exactly when

1. $(f_1, \dot{p}_1) \leq_{\mathbb{P}_{0,\beta}} (f_2, \dot{p}_2)$
2. $\mathrm{dom}(r_1) \supseteq \mathrm{dom}(r_2)$ and for every $\alpha \in \mathrm{dom}(r_2)$, we have that $(f_1, \dot{p}_1) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} r_1(\alpha) \leq r_2(\alpha)$
3. $\mathrm{dom}(g_1) \supseteq \mathrm{dom}(g_2)$ and for every $\alpha \in \mathrm{dom}(g_2)$, we have that $(f_1, \dot{p}_1, r_1, q_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} g_1(\alpha) \leq g_2(\alpha)$.

In the above, $(f_1, \dot{p}_1) \restriction \alpha$ denotes $\pi_\alpha(f_1, \dot{p}_1)$, where π_α is the projection from $\mathbb{P}_0$ to $RO(\mathbb{P}_{0,\alpha})$. Similarly, $(f_1, \dot{p}_1, r_1, q_1) \restriction \alpha$ is $(\pi_\alpha(f_1, \dot{p}_1), r_1 \restriction \alpha, q_1 \restriction \alpha)$.

3.2 Structural Properties of $\mathbb{Q}_0$

Proposition 3.2. *$|\mathbb{Q}_0| = \lambda_0$ and $\mathbb{Q}_0$ has the λ_0-Knaster property, and for all $\beta \in B$, $\mathbb{Q}_0 \restriction \beta$ is β-Knaster.*

Proof. $|\mathbb{Q}_0| = \lambda_0$ follows since $|\mathbb{P}_0| = \lambda_0$, $|\mathbb{P}_{0,\beta}| < \lambda_0$ for each $\beta \in B$ and for any α, there are less than λ_0 possibilities for $q(\alpha)$ or $f(\alpha)$. This follows since $\Vdash_{\mathbb{Q}_0 \restriction \alpha} g(\alpha), f(\alpha) \in V_{\lambda_0}$. The Knasterness part of the proposition follows from a Δ-system argument and since Prikry conditions with the same stem are compatible. $\square$

Proposition 3.3. *For $\alpha \in B$, $\mathbb{Q}_0$ can be projected to $\mathbb{P}_0$, $\mathbb{Q}_0 \restriction \alpha * F_0(\alpha)$, $\mathbb{P}_{0,\alpha} * Add(\mu, 1)_{V[\mathbb{P}_{0,\alpha}]}$ and $\mathbb{Q}_0 \restriction \alpha$.*

Proof. The projections are the following:

1. $\pi_1 \colon ((f, \dot{p}), r, g) \mapsto (f, \dot{p})$
2. $\pi_2 \colon ((f, \dot{p}), r, g) \mapsto ((f, \dot{p}, r, g) \restriction \alpha, g(\alpha))$
3. $\pi_3 \colon ((f, \dot{p}), r, g) \mapsto ((f, \dot{p}) \restriction \alpha, r(\alpha))$
4. $\pi_4 \colon ((f, \dot{p}), r, g) \mapsto (f, \dot{p}, r, g) \restriction \alpha$

We prove that π_2 and π_4 are projections and leave the rest to the reader. Recall that $(f, \dot{p}, r, q) \restriction \alpha$ is $(\pi_\alpha(f, \dot{p}), r \restriction \alpha, q \restriction \alpha)$. We start with π_4 and then deal with π_2. To check π_4 is order preserving, observe that if $(f_1, \dot{p}_1, r_1, g_1) \leq (f_2, \dot{p}_2, r_2, g_2)$, then $\pi_\alpha(f_1, \dot{p}_1) \leq \pi_\alpha(f_2, \dot{p}_2)$

because π_α is a projection. We still have that $\mathrm{dom}(r_1) \restriction \alpha \supseteq \mathrm{dom}(r_2) \restriction \alpha$ and for every $\beta \in \mathrm{dom}(r_2)$, $(f_1, \dot{p}_1) \restriction \beta \Vdash_{\mathbb{P}_{0,\beta}} r_1(\beta) \leq r_2(\beta)$. The similar condition holds for the g_i's and so it follows that $\pi_4(f_1, \dot{p}_1, r_1, g_1) \leq_{\mathbb{Q}_0 \restriction \alpha} \pi_4(f_2, \dot{p}_2, r_2, g_2)$.

Next we check the other condition for a projection. Assume that $(a, \dot{b}, c, d) \leq_{\mathbb{Q}_0 \restriction \alpha} \pi_4(f_2, \dot{p}_2, r_2, g_2)$. It follows that $(a, \dot{b}) \leq \pi_\alpha(f_2, \dot{p}_2)$, and since π_α is a projection we may find a $(f_1, \dot{p}_1) \leq (f_2, \dot{p}_2)$ such that $\pi_\alpha(f_1, \dot{p}_1) \leq (a, \dot{b})$. We define r_1 by first setting $\mathrm{dom}(r_1) = \mathrm{dom}(c) \cup \mathrm{dom}(r_2)$. Then, we let $r_1(\beta) = c(\beta)$ when $\beta \in \mathrm{dom}(c)$ and $r_1(\beta) = r_2(\beta)$ otherwise.

Finally, define g_1 by setting $\mathrm{dom}(g_1) = \mathrm{dom}(d) \cup \mathrm{dom}(g_2)$ and letting $g_1(\beta) = d(\beta)$ when $\beta \in \mathrm{dom}(d)$ and $g_1(\beta) = g_2(\beta)$ otherwise. It follows by construction that $(f_1, \dot{p}_1, r_1, g_1) \leq (f_2, \dot{p}_2, r_2, g_2)$ and that $\pi_4(f_1, \dot{p}_1, r_1, g_1) \leq (a, \dot{b}, c, d)$. So π_4 is a projection.

For π_2, assume that $(f_1, \dot{p}_1, r_1, g_1) \leq (f_2, \dot{p}_2, r_2, g_2)$. Since π_4 is a projection we know that $(f_1, \dot{p}_1, r_1, g_1) \restriction \alpha \leq (f_2, \dot{p}_2, r_2, g_2) \restriction \alpha$. Further, by definition of the ordering on $\mathbb{Q}_0$ we have that $(f_1, \dot{p}_1, r_1, g_1) \restriction \alpha \Vdash q_1(\alpha) \leq q_2(\alpha)$. So, $\pi_2(f_1, \dot{p}_1, r_1, g_1) \leq \pi_2(f_2, \dot{p}_2, r_2, g_2)$ by definition of a two-step iteration.

Next, assume that $((a, \dot{b}, c, d), e) \leq_{\mathbb{Q}_0 \restriction \alpha * F_0(\alpha)} ((f_2, \dot{p}_2, r_2, g_2) \restriction \alpha, g_2(\alpha))$. From the first coordinate, we know that $(a, \dot{b}, c, d) \leq (f_2, \dot{p}_2, r_2, g_2) \restriction \alpha$. Since π_4 is a projection, there is some $(f_1, \dot{p}_1, r_1, g_1) \leq_{\mathbb{Q}_0} (f_2, \dot{p}_2, r_2, g_2)$ such that $(f_1, \dot{p}_1, r_1, g_1) \restriction \alpha \leq_{\mathbb{Q}_0 \restriction \alpha} (a, \dot{b}, c, d)$. The idea is to modify g_1 to g_1^* by setting $g_1^* = g_1$ below α, by setting $g_1^*(\alpha) = e$, and by setting $g_1^* = g_2$ above α. Since $(a, \dot{b}, c, d) \Vdash_{\mathbb{Q}_0 \restriction \alpha} e \leq g_2(\alpha)$, it follows that $(f_1, \dot{p}_1, r_1, g_1^*) \leq_{\mathbb{Q}_0} (f_2, \dot{p}_2, r_2, g_2)$. But then we also have that $((f_1, \dot{p}_1, r_1, g_1^*) \restriction \alpha, g_1^*(\alpha)) \leq_{\mathbb{Q}_0 \restriction \alpha * F_0(\alpha)} ((a, \dot{b}, c, d), e)$ as desired. $\square$

Definition 3.4. Let $\mathbb{U}$ have conditions of the form $(0, 0, q, f) \in \mathbb{Q}_0$ with the ordering inherited from $\mathbb{Q}_0$.

Proposition 3.5. *$\mathbb{U}$ is μ-canonically directed closed and λ_0-cc (Knaster).*

Proof. The major difference between our new factor and the one from Definition 1.75 is the leftmost coordinate. Since our definition of $\mathbb{U}$ fixes the leftmost coordinate, we observe that this proposition should be true for the same reason that Lemma 1.76(3) holds. We present the directed closed argument from (Cummings and Foreman, 1998) for completeness.

Fix a directed set of conditions $\{(0, 0, q_\eta, f_\eta) \colon \eta < \theta\}$ for a cardinal $\theta < \mu$. We define a lower bound $(0, 0, r, g)$ as follows. Define $A_1 = \bigcup_{\eta<\theta} \operatorname{dom}(q_\eta)$. Notice that $|A_1| < \mu$. Define a function q with domain A_1. For $\alpha \in A_1$ consider $\{q_\eta(\alpha) \colon \eta < \theta\}$. If $\eta, \zeta < \theta$ then for some $\gamma < \theta$ we have that $(0, 0, q_\gamma, f_\gamma) \leq (0, 0, q_\eta, f_\eta), (0, 0, q_\zeta, f_\zeta)$, and so $\Vdash q_\gamma(\alpha) \leq q_\eta(\alpha), q_\zeta(\alpha)$. Then, there is a $\mathbb{P}_{0,\alpha}$-name for the directed set $\{q_\eta(\alpha) \colon \eta < \theta\} \subseteq \operatorname{Add}(\mu, 1)_{V[\mathbb{P}_{0,\alpha}]}$ and so we may let $r(\alpha)$ be a name forced to be the greatest lower bound to this directed set.

Next, define $A_2 = \bigcup_{\eta<\theta} \operatorname{dom}(f_\eta)$ and notice $|A_2| < \mu$. Observe that we may define a function g by induction on α with domain A_2 such that $(0, 0, r, g) \restriction \alpha \Vdash g(\alpha) \leq f_\eta(\alpha)$ for any α and η. The induction step is similar to the previous paragraph, where we say that $g(\alpha)$ is a name for the greatest lower bound of $\{f_\eta(\alpha) \colon \eta < \theta\}$ if that set is directed, and the trivial condition otherwise. It is not hard to see that $(0, 0, r, g)$ is in fact the greatest lower bound. $\square$

Lemma 3.6. *$\mathbb{Q}_0$ satisfies the following properties and cardinal structure:*

1. *$\mathbb{P}_0 \times \mathbb{U}$ projects onto $\mathbb{Q}_0$.*

2. *$\mathbb{P}_0 \times \mathbb{U}$ is λ_0-cc and all $< \mu$-sequences of ordinals in $V[\mathbb{P}_0 \times \mathbb{U}]$ are in $V[\mathbb{P}_0]$.*

3. *Assume G is $\mathbb{Q}_0$-generic over V and g is the $\mathbb{P}_0$-generic induced from G. If $X \in V[G]$ is a set of ordinals of size $< \mu$, then $X \in V[g]$.*

4. *$\mathbb{Q}_0$ collapses every cardinal between μ and λ_0 to μ.*

5. *$\mathbb{Q}_0$ preserves μ and forces $2^\kappa = \lambda_0 = \mu^+$.*

Proof. The proofs are similar to those found in Section 1.8 so we summarize the results.

1. The projection is the map $((f, \dot{p}), (0, 0, r, g)) \mapsto (f, \dot{p}, r, g)$.

2. The first part follows by Proposition 3.5 and since $\mathbb{P}_0$ is μ-cc. The second part follows from Easton's Lemma.

3. This follows immediately from the previous bullet point.

4. This follows because $\mathbb{Q}_0$ projects onto $\mathbb{P}_{0,\alpha} * \dot{Add}(\mu, 1)_{V[\mathbb{P}_{0,\alpha}]}$ for unboundedly many $\alpha < \lambda_0$.

5. $\mathbb{Q}_0$ preserves μ by the second bullet point and since $\mathbb{P}_0$ preserves μ. By the fourth bullet point we have that $\lambda_0 = \mu^+$. Finally, $(2^\kappa)^{V[\mathbb{Q}_0]} = (2^\kappa)^{V[\mathbb{P}_0]} = \lambda_0$ by the third bullet point and Lemma 1.74.

□

3.3 Quotients of $\mathbb{Q}_0$

We want to show that in $V[\mathbb{Q}_0 \restriction \alpha]$ the forcing $\mathbb{Q}_0/\mathbb{Q}_0 \restriction \alpha$ may be written as a projection of $\mathbb{P}_0/\mathbb{P}_{0,\beta} \times \mathbb{C}$ where $\mathbb{C}$ is sufficiently closed. This is motivated by work appearing in (Abraham,

1983) who showed in a simpler context that this may be done using certain types of projections. This definition is from (Cummings and Foreman, 1998).

Definition 3.7. A projection $\pi\colon \mathbb{P} \to \mathbb{Q}$ is good if for each $p \in \mathbb{P}$ and $q \leq \pi(p)$, there is a $p_1 \leq p$ such that $\pi(p_1) = q$ and, for all $r \leq p$, $\pi(r) \leq q \to r \leq p_1$. The element p_1 is denoted $Ext(p, q)$.

This notion of a good projection is to prove Lemma 3.13 below. Since $\mathbb{P}_0$-generic objects induce $\mathbb{P}_{0,\alpha}$-generic objects, we know that $RO(\mathbb{P}_{0,\alpha})$ is isomorphic to a complete subalgebra of $RO(\mathbb{P}_0)$. In other words, we have an injective complete homomorphism from $RO(\mathbb{P}_{0,\alpha})$ into $RO(\mathbb{P}_0)$. This maps induces a good projection from $RO(\mathbb{P}_0)$ to $RO(\mathbb{P}_{0,\alpha})$. This follows from general results about boolean algebras.

Lemma 3.8. *Let $\mathbb{B}$ and $\mathbb{C}$ be complete boolean algebras and assume that $i\colon \mathbb{B} \to \mathbb{C}$ is an injective complete homomorphism. Consider $\pi\colon \mathbb{C} \to \mathbb{B}$ defined by $c \mapsto \bigwedge\{b \in \mathbb{B}\colon c \leq i(b)\}$. Then π is a good projection.*

We need two very useful facts about the map π:

Claim 3.9. *For each $b \in \mathbb{B}$ and each $c \in \mathbb{C}$, we have that $\pi(i(b)) = b$ and $i(\pi(c)) \geq c$.*

Proof of Claim. For the first, notice that

$$\pi(i(b)) = \bigwedge\{b' \in \mathbb{B}\colon i(b) \leq i(b')\} = \bigwedge\{b' \in \mathbb{B}\colon b \leq b'\} = b.$$

For the second, notice that

$$i(\pi(c)) = i\left(\bigwedge\{b \in \mathbb{B} \colon c \leq i(b)\}\right) = \bigwedge\{i(b) \colon c \leq i(b)\} \geq c.$$

$\square$

Proof that π is a projection. First, note that $\pi(1_{\mathbb{C}}) = \pi(i(1_{\mathbb{B}})) = 1_{\mathbb{B}}$ by the previous claim. Next, if $c \leq c'$ then $\pi(c) \leq \pi(c')$ because $\{b \in \mathbb{B} \colon c' \leq i(b)\} \subseteq \{b \in \mathbb{B} \colon c \leq i(b)\}$. This shows π is order preserving. The final requirement for a projection follows when we prove that π is good. $\square$

Proof that π is good. Fix $c \in \mathbb{C}$ and $b \leq \pi(c)$. Let $Ext(c, b) = c \wedge i(b) \leq c$. To show that this witnesses that π is good, observe that $\pi(Ext(c, b)) = \pi(c \wedge i(b)) = \pi(c) \wedge b = b$. Further, if $r \leq c$ and $\pi(r) \leq b$, then $i(b) \geq i(\pi(r)) \geq r$ by the above claim. $\square$

Now let $\mathbb{Q}_0' \restriction \alpha$ be the forcing $\mathbb{Q}_0 \restriction \alpha$ defined in the previous section where we replace occurrences of $\mathbb{P}_{0,\alpha}$ with $RO(\mathbb{P}_{0,\alpha})$. These two forcings produce the same generic extension because $\mathbb{Q}_0 \restriction \alpha$ is a dense subset of $\mathbb{Q}_0' \restriction \alpha$. In what follows we think of conditions in $\mathbb{Q}_0$ or $\mathbb{Q}_0'$ as triples (p, q, f) where p is a condition of $\mathbb{P}_0$ or $RO(\mathbb{P}_0)$ depending on the context.

Lemma 3.10. *For $\alpha \in B$ the projections $\pi_2 \colon \mathbb{Q}_0' \to \mathbb{Q}_0' \restriction \alpha * F_0(\alpha)$ and $\pi_4 \colon \mathbb{Q}_0' \to \mathbb{Q}_0' \restriction \alpha$ are good. Here π_2 is the map $(p, q, f) \mapsto ((p, q, f) \restriction \alpha, f(\alpha))$ and π_4 is the map $(p, q, f) \mapsto (p, q, f) \restriction \alpha$.*

Proof. From the previous section we know that both of these are projections. For π_4, assume $(p', q', f') \leq \pi_4(p, q, f) = (\pi_\alpha(p), q \restriction \alpha, f \restriction \alpha)$. If $s = (p, q, f)$ and $t = (p', q', f')$, then $Ext(s, t) = (Ext(p, p'), q'^\frown(q \restriction \mathrm{dom}(q) \setminus \alpha), f'^\frown(f \restriction \mathrm{dom}(f) \setminus \alpha))$. Similarly, for π_2, assume $(p', q', f', g') \leq \pi_2(p, q, f) = (\pi_\alpha(p), q \restriction \alpha, f \restriction \alpha, f(\alpha))$. Note g' is a $\mathbb{Q}_0' \restriction \alpha$-name for an element of $F_0(\alpha)$. If $s = (p, q, f)$ and $t = (p', q', f', g')$, then $Ext(s, t) = (Ext(p, p'), q'^\frown(q \restriction \mathrm{dom}(q) \setminus \alpha), f'^\frown\{(\alpha, g')\}^\frown(f \restriction \mathrm{dom}(f) \setminus (\alpha + 1)))$. $\square$

One advantage of using good projections is that they allow us to modify the ordering on the forcing $\mathbb{P}/H$. The following is Lemma 2.10 in (Cummings and Foreman, 1998).

Lemma 3.11. *Assume $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a good projection and H is $\mathbb{Q}$-generic. Define the ordering $\leq^*$ on $\mathbb{P}/H$ by*

$$p \leq^* q \textit{ if and only if } \exists r \leq \pi(p)\, (r \in H \wedge Ext(p, r) \leq q)\,.$$

Then $(\mathbb{P}/H, \leq)$ and $(\mathbb{P}/H, \leq^*)$ are forcing equivalent over $V[H]$.

As an abbreviation, let $\mathbb{Q}_0 \restriction \alpha + 1 = \mathbb{Q}_0 \restriction \alpha * F_0(\alpha)$ and $\mathbb{Q}_0' \restriction \alpha + 1 = \mathbb{Q}_0' \restriction \alpha * F_0(\alpha)$. Like above we have that $\mathbb{Q}_0 \restriction \alpha + 1$ and $\mathbb{Q}_0' \restriction \alpha + 1$ produce the same generic extension.

Definition 3.12. Given $\alpha \in B$ and $i \in \{\alpha, \alpha + 1\}$ let G_i' be $\mathbb{Q}_0' \restriction i$-generic over V. Define the following:

1. $\mathbb{R}_i^* = \mathbb{Q}_0'/G_i'$ with the ordering $\leq^*$ defined above. Note that $\mathbb{R}_i^*$ and $\mathbb{Q}_0/G_i'$ are forcing equivalent.

2. $\mathbb{P}_i^* = \{p \in \mathbb{P}_0 \colon (p, 0, 0) \in \mathbb{R}_i^*\}$ ordered as a suborder of $\mathbb{P}_0$. Note that in either case $\mathbb{P}_i^*$ is isomorphic to $\mathbb{P}_0/\mathbb{P}_{0,\alpha}$.

3. $\mathbb{U}_i^* = \{(0, q, f) \in \mathbb{Q}_0' \colon (0, q, f) \in \mathbb{R}_i^*\}$ ordered as a suborder of $\mathbb{R}_i^*$.

It is not too hard to see that the identity map is a projection from $\mathbb{P}_i^* \times \mathbb{U}_i^*$ to $\mathbb{R}_i^*$. Further, like the poset $\mathbb{U}$ defined above, we have that $\mathbb{U}_i^*$ is sufficiently closed.

Lemma 3.13. *$\mathbb{U}_i^*$ is μ-closed in $V[\mathbb{Q}_0 \restriction i]$.*

Proof. The proof is similar to Lemma 2.18 in (Abraham, 1983). Working in $V[G_i']$, let $\theta < \mu$ and and $\tau \colon \theta \to \mathbb{U}_i^*$ be a decreasing sequence. If g_α is the generic for $\mathbb{P}_{0,\alpha}$ induced by G_i', we have that $\tau \in V[g_\alpha]$. Let $\dot{\tau}$ be a $\mathbb{P}_{0,\alpha}$-name for τ and let $t \in G_i'$ be such that $t \Vdash_{\mathbb{Q}_0' \restriction i} \dot{\tau}$ is a $\leq^*$-decreasing sequence in $\mathbb{U}_i^*$. It is enough to define a condition $(0, q^*, f^*)$ in V such that $t \Vdash_{\mathbb{Q}_0' \restriction i} (0, q^*, f^*)$ is a $\leq^*$-lower bound for $\dot{\tau}$. For notation, let $L(0, q, f) = q$ and $R(0, q, f) = f$. Now, define $\mathrm{dom}(q^*)$ and $\mathrm{dom}(f^*)$ in the following way:

$$\mathrm{dom}(q^*) = \{\beta > i \colon \exists\gamma < \theta, \exists p \in \mathbb{P}_{0,\alpha}\ p \Vdash \beta \in \mathrm{dom} L(\dot{\tau}(\gamma))\}$$

$$\mathrm{dom}(f^*) = \{\beta > i \colon \exists\gamma < \theta, \exists p \in \mathbb{P}_{0,\alpha}\ p \Vdash \beta \in \mathrm{dom} R(\dot{\tau}(\gamma))\}.$$

Since $\mathbb{P}_{0,\alpha}$ is μ-cc, it follows that $\mathrm{dom}(q^*)$ and $\mathrm{dom}(f^*)$ both have size less than μ. We define q^* and f^* by induction on $\beta \in \mathrm{dom}(q^*) \cup \mathrm{dom}(f^*)$ so that the statement $\dagger_\beta$ holds:

$$\dagger_\beta \text{ holds } \iff Ext((0, q^*, f^*) \restriction \beta, t) \Vdash_{\mathbb{Q}_0' \restriction \beta} \text{“for each } \gamma < \theta,\ \dot{\tau}(\gamma) \restriction \beta \in \Gamma_\beta.\text{”}$$

(Note that Γ_β is the canonical name for $\mathbb{Q}_0' \restriction \beta$-generic object.)

Towards that end, assume that we have defined $q^* \restriction \beta$ and $f^* \restriction \beta$ and that $\dagger_\beta$ holds. Let $\beta' > \beta$ be the next element of $\mathrm{dom}(q^*) \cup \mathrm{dom}(f^*)$. We need to define $q^*(\beta)$ and $f^*(\beta)$, as well as show that $\dagger_{\beta'}$ holds. Arguing as in (Abraham, 1983) we show the following:

Claim 3.14. *$Ext((0, q^*, f^*) \restriction \beta, t) \Vdash_{\mathbb{Q}'_0 \restriction \beta} (L(\dot\tau(\gamma))(\beta) \colon \gamma < \theta)$ is decreasing in $Add(\mu, 1)_{V[\mathbb{P}_{0,\beta}]}$.*

Claim 3.15. *$Ext((0, q^*, f^*) \restriction \beta, t) \Vdash_{\mathbb{Q}'_0 \restriction \beta} (R(\dot\tau(\gamma))(\beta) \colon \gamma < \theta)$ is decreasing in $F_0(\beta)$.*

Then $q^*(\beta)$ and $f^*(\beta)$ are chosen to be names that are forced by $Ext((0, q^*, f^*) \restriction \beta, t)$ to be lower bounds.

To show that $\dagger_{\beta'}$ holds, fix $\gamma < \theta$ and $r \in \mathbb{Q}'_0 \restriction \beta'$ where $r \leq Ext((0, q^*, f^*) \restriction \beta', t)$ and, for some $(0, q, f) \in \mathbb{Q}'_0$, $r \Vdash_{\mathbb{Q}' \restriction \beta'} \dot\tau(\gamma) = (0, q, f)$. Our goal is to show that $r \leq (0, q, f) \restriction \beta'$. Since $\dagger_\beta$ holds we know that $Ext((0, q^*, f^*) \restriction \beta, t) \Vdash \dot\tau(\gamma) \in \Gamma_\beta$, and since $Ext((0, q^*, f^*) \restriction \beta', t) \restriction \beta = Ext((0, q^*, f^*) \restriction \beta, t)$ it follows that $r \restriction \beta \leq (0, q, f) \restriction \beta$. If we let $r = (p', q', f')$ then we have that $p' \restriction \beta \Vdash q'(\beta) \leq q^*(\beta) \leq q(\beta)$ and further that $r \restriction \beta \Vdash f'(\beta) \leq f^*(\beta) \leq f(\beta)$. This implies $r \leq (0, q, f) \restriction \beta'$ as desired. The induction goes through and so this completes the definition of $(0, q^*, f^*)$.

Finally, since $\dagger_\beta$ holds for each $\beta \in \mathrm{dom}(q^*) \cup \mathrm{dom}(f^*)$ it follows that $Ext((0, q^*, f^*), t) \Vdash_{\mathbb{Q}'_0} \dot\tau(\gamma) \in \Gamma$ for each $\gamma < \theta$. This implies for each $\gamma < \theta$ that $t \Vdash (0, q^*, f^*) \leq^* \dot\tau(\gamma)$, as desired. $\square$

As an important consequence, if G is generic for $\mathbb{Q}_0$ and $G_{\alpha+1}$ is generic for $\mathbb{Q}_0 \restriction \alpha + 1$ induced from G, then we may write $V[G] \subseteq V[G_{\alpha+1}][h^* \times u^*]$ where h^* is generic for $\mathbb{P}_0/\mathbb{P}_{0,\alpha}$ and u^* is generic for a μ-closed forcing in $V[G_{\alpha+1}]$.

3.4 The forcing $\mathbb{R}_\omega$

Here we define an iteration $\mathbb{R}_\omega$ of the forcing $\mathbb{Q}_0$ defined earlier in this chapter with the forcings defined in Section 1.10.

Definition 3.16. Recalling the forcing $\mathbb{R}(\tau, \kappa, V, W, F)$ from Definition 1.75, we proceed in the same manner as (Cummings and Foreman, 1998):

1. Let $\mathbb{R}_1 = \mathbb{Q}_0$.

2. Let $\dot{F}_1$ be a $\mathbb{Q}_0$-name for a function on λ_1 such that $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = F_1(\alpha)$ when $F_1(\alpha)$ is a $\mathbb{Q}_0$-name and $\Vdash_{\mathbb{Q}_0} \dot{F}_1(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_1$ to be the canonical name for $\mathbb{R}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ where F_1^* is the interpretation of $\dot{F}_1$ in $V[\mathbb{Q}_0]$. Let $\mathbb{R}_2 = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.

3. Similarly, for $n \geq 2$, let $\mathbb{R}_n = \mathbb{Q}_0 * \ldots * \dot{\mathbb{Q}}_{n-1}$ and let $\dot{F}_n$ be a $\mathbb{R}_n$-name for a function on λ_n such that $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = F_n(\alpha)$ when $F_n(\alpha)$ is a $\mathbb{R}_n$-name and $\Vdash_{\mathbb{R}_n} \dot{F}_n(\alpha) = 0$ otherwise. Then define $\dot{\mathbb{Q}}_n$ to be the canonical name for $\mathbb{R}(\lambda_{n-2}, \lambda_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$ where F_n^* is the interpretation of $\dot{F}_n$ in $V[\mathbb{R}_n]$.

4. Finally, let $\mathbb{R}_\omega$ be the inverse limit of $(\mathbb{R}_n : n < \omega)$.

For Definition 3.16 to actually make sense, we have to show that after forcing with $\mathbb{Q}_0$ we satisfy the hypotheses of Definition 1.75 to make $\mathbb{R}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ valid.

Lemma 3.17. *Let G be $\mathbb{Q}_0$-generic over V and g be the $\mathbb{P}_0$-generic induced by G. In $V[G]$ we have that*

1. *μ is regular and λ_1 is inaccessible,*

2. $Add(\mu, \lambda_1)_V$ *is* μ^+*-cc and* $< \mu$*-distributive.*

Proof. Since $|\mathbb{Q}_0| = \lambda_0 < \lambda_1$, it follows that λ_1 is still inaccessible in $V[G]$. To see that μ is regular in $V[G]$, assume otherwise and fix an unbounded sequence $f\colon \tau \to \mu$ with $\tau < \mu$ such that $f \in V[G]$. Since $\mathbb{U}$ is μ-closed, it follows that $f \in V[g]$. This contradicts Lemma 1.74 because μ is regular in $V[g]$.

The second part of this follows from Lemma 1.79(1), where $\tau = \mu$, $\kappa = \lambda_0$, $\mathbb{Q} = \mathbb{Q}_0$, and $\mathbb{U}$ as in the previous section. In particular we have that $\mathrm{Add}(\mu, \lambda_1)_V$ is λ_0-Knaster and $< \mu$-distributive in $V[G]$. Since $V[G] \models \lambda_0 = 2^\kappa = \mu^+$, the result follows. $\square$

It follows that the definition of $\mathbb{R}_2$ makes sense. Since the terms $\dot{\mathbb{Q}}_n$ for $n \geq 1$ are simply names for the factors in the original Cummings-Foreman paper, a proof by induction using the results from Section 1.10 and the results from earlier in Chapter 3 show the following:

Lemma 3.18 (Lemma 4.3 in (Cummings and Foreman, 1998))**.** *Let* $n \geq 1$*. Let* $\mathbb{R}_n = \mathbb{Q}_0 * \ldots \dot{\mathbb{Q}}_{n-1}$*, let* $\mathbb{P}_1 = Add(\mu, \lambda_1)_V$*, and let* $\mathbb{P}_n = Add(\lambda_{n-2}, \lambda_n)_{V[\mathbb{R}_{n-1}]}$ *for* $n \geq 2$*. Further, let* $\mathbb{U}_1 = \mathbb{U}(\mu, \lambda_1, V, V[\mathbb{Q}_0], F_1^*)$ *and* $\dot{\mathbb{U}}_n = \mathbb{U}_n(\lambda_{n-2}, \lambda_n, V[\mathbb{R}_{n-1}], V[\mathbb{R}_n], F_n^*)$ *for* $n \geq 2$*, where* $\mathbb{U}_n$ *is the poset* $\mathbb{U}$ *corresponding to* $\mathbb{Q}_n$ *defined in Section 1.10. We abuse notation and will occasionally denote* $\kappa = \lambda_{-2}$ *and* $\mu = \lambda_{-1}$*.*

1. *In* $V[\mathbb{R}_n]$*, we have* $2^{\lambda_{i-2}} = \lambda_i$ *for* $i < n$ *and the* λ_i*'s are still inaccessible for* $i \geq n$*.*

2. $V[\mathbb{R}_n] \models \mathbb{Q}_n$ *is* $< \lambda_{n-2}$*-distributive,* λ_n*-Knaster, and* $|\mathbb{Q}_n| = \lambda_n$*. If* $n \geq 2$*, then* $\mathbb{Q}_n$ *is also* λ_{n-3}*-closed in* $V[\mathbb{R}_n]$*.*

3. *All* $< \lambda_{n-2}$*-sequences of ordinals from* $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ *are in* $V[\mathbb{R}_{n-1} * \dot{\mathbb{P}}_{n-1}]$*.*

4. *In $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$, the cardinals κ, μ, and λ_i for $i \leq n-2$ are preserved.*

5. *$V[\mathbb{R}_n] \models$ "$\mathbb{Q}_n$ is a projection of $\mathbb{P}_n \times \mathbb{U}_n$," and we also have that $V[\mathbb{R}_n * \dot{\mathbb{P}}_n] \subseteq V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \subseteq V[\mathbb{R}_n * (\dot{\mathbb{P}}_n \times \dot{\mathbb{U}}_n)]$.*

6. *All λ_{n-2}-sequences of ordinals from $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ are in $V[\mathbb{R}_n * \dot{\mathbb{P}}_n]$.*

7. *λ_{n-1} is preserved in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$. In $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$, we have $2^{\lambda_{i-2}} = \lambda_i$ for $i \leq n$.*

8. *$Add(\lambda_{n-2}, \eta)_{V[\mathbb{R}_{n-1}]}$ is λ_{n-1}-Knaster in $V[\mathbb{R}_n * \dot{\mathbb{Q}}_n]$ for any ordinal η.*

9. *$V[\mathbb{R}_n * \dot{\mathbb{Q}}_n] \models$ "$Add(\lambda_{n-1}, \eta)_{V[\mathbb{R}_n]}$ is $< \lambda_{n-1}$-distributive and λ_n-Knaster" for any ordinal η.*

To show that the cardinal arithmetic works out after forcing with $\mathbb{R}_\omega$, we use the following lemma.

Lemma 3.19 (Lemma 4.4 in (Cummings and Foreman, 1998)). *Let G_ω be $\mathbb{R}_\omega$-generic and $X \in V[G_\omega]$ is a λ_n-sequence of ordinals. Then $X \in V[G_0][\cdots][G_n][G_{n+1}][g_{n+2}]$, where $G_0 * \cdots * G_n * G_{n+1} * g_{n+2}$ is the initial segment of G_ω which is V-generic for $\mathbb{Q}_0 * \cdots * \dot{\mathbb{Q}}_{n+1} * \dot{\mathbb{P}}_{n+2}$. In the case where $X \in V[G_\omega]$ is a μ-sequence of ordinals, it follows that $X \in V[G_0][g_1]$.*

Proof. For clarity, we show the case when $n = 0$. Since $\mathbb{R}_\omega/\mathbb{R}_4$ is λ_1-closed, it follows that $X \in V[G_0][G_1][G_2][G_3]$. Since $\mathbb{Q}_3$ is $< \lambda_1$-distributive, it follows $X \in V[G_0][G_1][G_2]$. We therefore have that $X \in V[G_0][G_1][g_2]$ since all λ_0-sequences of ordinals in $V[G_0][G_1][G_2]$ are in $V[G_0][G_1][g_2]$. $\square$

From the previous results we have the following:

Lemma 3.20. *After forcing with $\mathbb{R}_\omega$ we have the following cardinal structure:*

1. *$cf(\kappa) = \omega$,*
2. *$\mu = \kappa^+$,*
3. *$\lambda_n = \kappa^{+n+2}$ for all n,*
4. *$2^\kappa = \lambda_0$,*
5. *$2^\mu = \lambda_1$,*
6. *$2^{\lambda_n} = \lambda_{n+2}$ for all n.*

3.5 Tree property at κ^{++}

In this section we prove that the tree property holds in $V[\mathbb{R}_\omega]$ at $\lambda_0 = \kappa^{++}$. Importantly, we have to account for the presence of $\mathbb{Q}_0$. If T is a λ_0-tree, it follows by Lemma 3.19 that $T \in V[G_0][G_1][g_2]$. It is therefore enough to show that there is no λ_0-Aronszajn tree in $V[G_0][G_1][g_2]$.

3.5.1 Lifting the embedding

Recall that $\lambda = \sup \lambda_n$. Fix an elementary embedding $j \colon V \to M$ with $\mathrm{crit}(j) = \lambda_0$ such that $j(\lambda_0) > \lambda$, ${}^\lambda M \subseteq M$, and where $j(F_0)(\lambda_0)$ is the canonical $\mathbb{Q}_0$-name for $\mathbb{P}_2 \times \mathbb{U}_1$. Observe that $j(F_0)(\lambda_0)$ is a $\mathbb{Q}_0$-name for a λ_0-directed closed forcing in $V[\mathbb{Q}_0]$.

Our first task is to lift the embedding from $j \colon V \to M$ to $j \colon V[G_0][G_1][g_2] \to M[H_0][H_1][h_2]$. This argument is similar to the Six Stages listed in Section 4 of (Cummings and Foreman, 1998), but is fleshed out for completeness, for clarity, and to emphasize the parts where we are using the new factor $\mathbb{Q}_0$.

Claim 3.21. *There is a $V[G_0][g_2]$-generic filter $g_1 \times u_1$ for $\mathbb{P}_1 \times \mathbb{U}_1$ such that $g_1 \times u_1 \times g_2$ is generic for $\mathbb{P}_1 \times \mathbb{U}_1 \times \mathbb{P}_2$ over $V[G_0]$.*

Proof. Since $G_0 * G_1 * g_2$ is V-generic for $\mathbb{Q}_0 * \mathbb{Q}_1 * \mathbb{P}_2$, it follows that G_1 and g_2 are mutually generic over $V[G_0]$. So, in $V[G_0][g_2][G_1]$, we may consider the quotient forcing $\mathbb{S}$ of $\mathbb{P}_1 \times \mathbb{U}_1$ and $\mathbb{Q}_1$. If $g_1 \times u_1$ is generic for $\mathbb{S}$ over $V[G_0][g_2][G_1]$, then it's also generic for $\mathbb{P}_1 \times \mathbb{U}_1$ over $V[G_0][g_2]$. The product lemma implies that g_2 and $g_1 \times u_1$ are mutually generic over $V[G_0]$, and so $g_1 \times u_1 \times g_2$ is generic for $\mathbb{P}_1 \times \mathbb{U}_1 \times \mathbb{P}_2$ over $V[G_0]$. $\square$

Claim 3.22. *There is a H_0 such that H_0 is $V[g_1]$-generic for $j(\mathbb{Q}_0)$ with $H_0 \restriction \lambda_0 + 1 = G_0 * (g_2 \times u_1)$ and H_0 collapses λ_0 and λ_1 to cardinality μ. Further, we may lift to $j \colon V[G_0] \to M[H_0]$.*

Proof. In M, observe that $j(\mathbb{Q}_0) \restriction \lambda_0 = \mathbb{Q}_0$ and that elementarity implies $j(\mathbb{Q}_0)$ projects onto $\mathbb{Q}_0 * j(F_0)(\lambda_0) = \mathbb{Q}_0 * (\mathbb{U}_1 \times \mathbb{P}_2)$. Now, our choice of j implies $\mathbb{P}_2$ and $\mathbb{U}_1$ are really the forcings defined in $V[G_0]$ (also using V as a parameter for $\mathbb{U}_1$). However, the chain condition of $\mathbb{Q}_0$ and the closure of M imply $\mathbb{P}_2$ and $\mathbb{U}_1$ are the same in $V[G_0]$ as in $M[G_0]$ (using M instead of V in the definition for $\mathbb{U}_1$).

Next, the projection given by elementarity is in $V[G_0][g_1 \times u_1 \times g_2]$, so let H_0 be $V[G_0][g_1 \times u_1 \times g_2]$-generic for $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$. Facts about projections tell us that H_0 is $V[g_1]$-generic for $j(\mathbb{Q}_0)$ and is generated from $G_0 * (g_2 \times u_1)$. Also, we get that $H_0 \restriction \lambda_0 + 1 = G_0 * (g_2 \times u_1)$.

Further, we have that $H_0 \restriction \lambda_0 = G_0$. Since $\mathbb{Q}_0$ is λ_0-cc, it follows that we may lift to $j\colon V[G_0] \to M[H_0]$. $j(\mathbb{Q}_0)$ collapses all ordinals between $j(\mu) = \mu$ and $j(\lambda_0)$ to μ, and since $\lambda_0, \lambda_1 < j(\lambda_0)$, we have that H_0 collapses λ_0 and λ_1 to μ. $\square$

Claim 3.23. *There is a generic object h_1 for $j(\mathbb{P}_1)$ over $V[G_0][H_0]$ so that $j"g_1 \subseteq h_1$. This allows us to lift to $j\colon V[G_0][g_1] \to M[H_0][h_1]$.*

Proof. Observe that elementarity and sufficient closure of M implies that $j(\mathbb{P}_1) = \mathrm{Add}(\mu, j(\lambda_1))_V$. Observe also that there is a projection of $j(\mathbb{P}_1)$ onto $\mathbb{P}_1$. So, we may take an h_1 generic for $j(\mathbb{P}_1)/g_1$ over $V[G_0][H_0][g_1]$. This will satisfy $j"g_1 \subseteq h_1$ and let us lift the embedding j. $\square$

Claim 3.24. *There is a generic object x_1 for $j(\mathbb{U}_1)$ over $V[G_0][H_0][h_1]$ such that $j"u_1 \subseteq x_1$. Further, x_1 and h_1 are mutually generic over $M[H_0]$ by Easton's Lemma, so $h_1 \times x_1$ generates a filter H_1 generic for $j(\mathbb{Q}_1)$ over $M[H_0]$. We also have that $j"G_1 \subseteq H_1$, allowing us to lift to $j\colon V[G_0][G_1] \to M[H_0][H_1]$.*

Proof. Observe that in $M[G_0][g_2 \times u_1]$, $|u_1| \leq |\mathbb{U}_1| = \lambda_1 < \lambda$. Since M is closed under λ sequences and $\mathbb{Q}_0$, $\mathbb{P}_2$, and $\mathbb{U}_1$ have chain conditions smaller than λ (simply by cardinality considerations), it follows in $V[G_0][g_2 \times u_1]$ that ${}^{\lambda}M[G_0][g_2 \times u_1] \subseteq M[G_0][g_2 \times u_1]$. This implies that in $j"u_1 \in M[H_0]$ by choice of H_0.

Now, elementarity implies that, in $M[H_0]$, $j(\mathbb{U}_1)$ is $j(\mu^+)$-directed closed, where $j(\mu^+) = (\mu^+)^{M[H_0]}$ (note that μ is less than the critical point of j). We observed in Claim 3.22 that λ_1 is collapsed to μ in $M[H_0]$, and so it follows that there is a lower bound t in $j(\mathbb{U}_1)$ for $j"u_1$.

This allows us to fix a $V[G_0][H_0][h_0]$-generic filter x_1 for $j(\mathbb{U}_1)$ with $j"u_1 \subseteq x_1$. The choice of x_1 and the product lemma implies that $x_1 \times h_1$ is generic over $M[H_0]$. By elementarity, we have that $j(\mathbb{P}_1) \times j(\mathbb{U}_1)$ projects onto $j(\mathbb{Q}_1)$, and so it follows $x_1 \times h_1$ induces a filter H_1 on $j(\mathbb{Q}_1)$ over $M[H_0]$. Finally, to lift the embedding to $j\colon V[G_0][G_1] \to M[H_0][H_1]$ it is enough to show that $j``G_1 \subseteq H_1$.

To see this, observe that if we fix $(p, q, f) \in G_1$ then we may find $(p', q', f') \leq (p, q, f)$ such that $p' \in g_1$ and $(0, q', f') \in u_1$. Since $x_1 \times h_1$ induces the filter H_1 it follows that $(j(p'), 0, 0)$ and $(0, j(q'), j(f')$ are both in H_1. This implies that $(j(p'), j(q'), j(f')) = j((p', q', f')) \in H_1$ as well. Since $(p', q', f') \leq (p, q, f)$ it follows as desired that $j``G_1 \subseteq H_1$. □

Claim 3.25. *There is a generic object h_2 for $j(\mathbb{P}_2)$ over $V[G_0][H_0][h_1][x_1]$ such that $j"g_2 \subseteq h_2$. This lets us lift to $j\colon V[G_0][G_1][g_2] \to M[H_0][H_1][h_2]$.*

Proof. By elementarity notice that $j(\mathbb{P}_2) = \mathit{Add}(j(\lambda_0), j(\lambda_2))_{M[H_0]}$. It follows by choice of $j(F_0)(\lambda_0)$ that $j``g_2 \in M[H_0]$. This implies that there is a lower bound $p \in j(\mathbb{P}_2)$ for $j``g_2$. Then any generic filter h_2 containing the condition p allows us to lift the embedding j. □

3.5.2 Pulling back the branch

Let T be a λ_0-tree in $V[G_0][G_1][g_2]$. An argument similar to Lemma 1.21 shows that T has a branch b in $M[H_0][H_1][h_2]$. Since M is closed under λ sequences and the $\mathbb{Q}_0$, $\mathbb{Q}_1$, and $\mathbb{P}_2$ have chain conditions smaller than λ (by cardinality considerations), we have in $V[G_0][G_1][g_2]$ that ${}^{\lambda}M[G_0][G_1][g_2] \subseteq M[G_0][G_1][g_2]$. It follows then that $T \in M[G_0][G_1][g_2]$. We first start by making some structural observations about the codomain of our lifted embedding j.

Fact 3.26. *$M[H_0][h_1] = M[G_0][g_2 \times u_1][H_0^*][g_1 \times h_1^*]$, where H_0^* is a generic object for $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$ and h_1^* is a generic object for $j(\mathbb{P}_1)/g_1$.*

Proof. Follows since we may factor H_0 as $G_0 * (g_2 \times u_1) * H_0^*$ and h_1 as $g_1 \times h_1^*$. □

Fact 3.27. *$M[G_0][g_2 \times u_1][h_1] = M[G_0][g_1 \times u_1 \times g_2][h_1^*]$*

Proof. Follows since we may factor h_1 as $g_1 \times h_1^*$. □

Fact 3.28. *$M[H_0][h_1] \subseteq M[G_0][h_1 \times u_1 \times g_2][h_0^* \times u_0^*]$, where h_0^* is a generic object for $j(\mathbb{P}_0)/g_0$ and u_0^* is a generic object for $\mathbb{U}^*$ which is μ-closed in $M[G_0]$.*

Proof. Follows from Fact 3.26, Fact 3.27, and by Lemma 3.13 since there is a μ-closed forcing $\mathbb{U}^*$ such that $j(\mathbb{P}_0)/g_0 \times \mathbb{U}^*$ projects onto $j(\mathbb{Q}_0)/(G_0 * (g_2 \times u_1))$. □

Next we analyze the forcings required to get from $M[G_0][G_1][g_2]$ to $M[H_0][h_1]$.

Fact 3.29. *In $M[G_0][G_1][g_2]$, the quotient forcing $\mathbb{S} = (\mathbb{P}_1 \times \mathbb{U}_1)/\mathbb{Q}_1$ is μ-closed.*

Proof. By Lemma 1.76(7) we have that $\mathbb{S}$ is μ-closed in $M[G_0][G_1]$. Next, $\mathbb{P}_2$ is $< \lambda_0$-distributive in $M[G_0][G_1]$ by Lemma 3.18(9), and so by an Easton's Lemma variant we have that $\mathbb{S}$ is still μ-closed in $M[G_0][G_1][g_2]$. □

Fact 3.30. *All $< \mu$-sequences of ordinals from $M[G_0][h_1 \times u_1 \times g_2]$ are in $M[G_0]$.*

Proof. Assume that $f \in M[G_0][h_1 \times u_1 \times g_2]$ is a sequence of ordinals of length $\theta < \mu$. By Lemma 3.17 it follows that $j(\mathbb{P}_1)$ is λ_0-cc in $M[G_0]$. Since $\mathbb{P}_2 \times \mathbb{U}_1$ is λ_0-directed closed in $M[G_0]$ it follows by Easton's Lemma that $\mathbb{P}_2 \times \mathbb{U}_1$ is $< \lambda_0$-distributive in $M[G_0][h_1]$ and so f is in this model. By elementarity we have that $j(\mathbb{P}_1)$ is $< \mu$-distributive in $M[G_0]$ and so $f \in M[G_0]$ as desired. □

Fact 3.31. *$j(\mathbb{P}_1)/g_1$ is λ_0-Knaster and $< \mu$-distributive in $M[G_0][g_1 \times u_1 \times g_2]$.*

Proof. Fact 3.30 implies the distributivity part of the result, so it is enough to show Knasterness. Since $j(\mathbb{P}_1)/g_1$ is just adding Cohen subsets of μ we have by Lemma 1.79(1) that $j(\mathbb{P}_1)/g_1$ is λ_0-Knaster in $M[G_0]$. We use Lemma 1.78 to check that $j(\mathbb{P}_1)/g_1$ remains Knaster in $M[G_0][g_1 \times u_1 \times g_2]$. Towards that end assume that $X \in M[G_0][g_1 \times u_1 \times g_2]$ is a set of ordinals with $|X| < \lambda_0$. This implies that $|X| \leq \mu$ and so $X \in M[G_0][g_1]$. Since $\mathbb{P}_1$ is λ_0-cc in $M[G_0]$ we know there is a $Y \supseteq X$ in $M[G_0]$ such that $|Y| < \lambda_0$ in this model. Further since $\mathbb{Q}_0$ is λ_0-cc we have that there is a $Z \supseteq Y \supseteq X$ such that $|Z| < \lambda_0$ in M. The result follows. □

Claim 3.32. $b \in M[H_0][h_1]$

Proof. We know that $b \in M[H_0][H_1][h_2]$. Lemma 3.18(9) implies that $\mathbb{P}_2$ is $< \lambda_0$-distributive in $V[G_0][G_1]$ and so $j(\mathbb{P}_2)$ is $< j(\lambda_0)$-distributive in $M[H_0][H_1]$. Since b has length λ_0 it follows that $b \in M[H_0][H_1]$. By Claim 3.24 we have that $b \in M[H_0][h_1 \times x_1]$. By Lemma 1.76(5) we know $\mathbb{U}_1$ is $\leq \mu$-distributive in $V[G_0][g_1]$. Elementarity implies that $j(\mathbb{U}_1)$ is $\leq \mu$-distributive in $M[H_0][h_1]$. By elementarity we have that λ_0 is collapsed to μ in $M[H_0][h_1]$ and so it follows that $b \in M[H_0][h_1]$. □

Claim 3.33. $b \in M[G_0][h_1 \times u_1 \times g_2][u_0^*]$

Proof. Let $\mathbb{T} = j(\mathbb{P}_0)/g_0$. We have to argue that $\mathbb{T}$ does not add any branches. By Fact 3.28 and Claim 3.32, we know $b \in M[G_0][h_1 \times u_1 \times g_2][u_0^* \times h_0^*]$. Forcing with $\mathbb{U}^*$ in $M[G_0][h_1 \times u_1 \times g_2]$

collapses λ_0 to μ, and so by Lemma 1.14 it is enough to argue that $M[G_0][h_1 \times u_1 \times g_2][u_0^*] \models \mathbb{T}^2$ is μ-cc. We will work backwards and show that this follows from Lemma 1.73(4). First observe that $\mathbb{U}^*$ is μ-closed in $M[G_0][h_1 \times u_1 \times g_2]$ because all κ-sequences of elements of $M[G_0][h_1 \times u_1 \times g_2]$ are in $M[G_0]$. It follows by Easton's Lemma that it is enough to show $M[G_0][h_1 \times u_1 \times g_2] \models \mathbb{T}^2$ is μ-cc.

Next, because $\mathbb{U}_1 \times \mathbb{P}_2$ is λ_0-directed closed in $M[G_0]$ and $j(\mathbb{P}_1)$ is λ_0-cc in $M[G_0]$, it follows by Easton's Lemma that $u_1 \times g_2$ is generic for a μ-distributive forcing over $M[G_0][h_1]$. It is therefore enough to show that $M[G_0][h_1] \models \mathbb{T}^2$ is μ-cc because an antichain in $M[G_0][h_1 \times u_1 \times g_2]$ of $\mathbb{T}^2$ with size μ will be in $M[G_0][h_1]$ by distributivity.

Observe that h_1 and G_0 are mutually generic, and so we have that $M[G_0][h_1] = M[h_1][G_0]$. It is enough then to show that $M[h_1][g_0 \times u_0] \models \mathbb{T}^2$ is μ-cc because $\mathbb{P}_0 \times \mathbb{U}$ projects onto $\mathbb{Q}_0$.

Towards this end, observe that Lemma 1.73(4) implies that $\Vdash^M_{\mathbb{P}_0} \mathbb{T}^2$ is μ-cc. This implies that $M \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. $j(\mathbb{P}_1)$ is μ-closed in M, and so $M[h_1] \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. Finally, $\mathbb{U}$ is μ-closed in $M[h_1]$, so Easton's Lemma implies that $M[h_1][u_0] \models \mathbb{P}_0 * \mathbb{T}^2$ is μ-cc. So $M[h_1][u_0 \times g_0] \models \mathbb{T}^2$ is μ-cc, completing the claim. $\square$

Claim 3.34. $b \in M[G_0][h_1 \times u_1 \times g_2]$

Proof. Notice that forcing with $\mathbb{U}^*$ is μ-closed in $M[G_0][h_1 \times u_1 \times g_2]$ by Fact 3.30 and that $2^\kappa = \lambda_0$ in $M[G_0][h_1 \times u_1 \times g_2]$. It follows by Silver's branch lemma that $b \in M[G_0][h_1 \times u_1 \times g_2]$. $\square$

Claim 3.35. $b \in M[G_0][g_1 \times u_1 \times g_2]$

Proof. $j(\mathbb{P}_1)/g_1$ is λ_0-Knaster in $M[G_0][g_1 \times u_1 \times g_2]$ by Fact 3.31, and T is a tree of height λ_0 in this model. It follows that forcing with T will not add any new branches, and so $b \in M[G_0][g_1 \times u_1 \times g_2]$. □

Claim 3.36. $b \in M[G_0][G_1][g_2]$

Proof. By Fact 3.29 and since $2^\kappa = \lambda_0$ in $M[G_0][G_1][g_2]$, we have by Silver's branch lemma that forcing with $\mathbb{S}$ to get $M[G_0][g_1 \times u_1 \times g_2]$ does not add any branches. So $b \in M[G_0][G_1][g_2]$. □

3.6 Tree property at κ^{+n} for $n \geq 3$

The argument when $n \geq 3$ is almost exactly the same as the argument in (Cummings and Foreman, 1998). For brevity, let $V_n := V[G_0][G_1]\cdots[G_n]$ and $M_n := M[G_0][G_1]\cdots[G_n]$. The main difference between our iteration and the iteration in the Cummings-Foreman paper is the first factor $\mathbb{Q}_0$, and one of the main differences between our argument from the previous section and their argument was Claim 3.33. When $n \geq 3$, though, we may immediately lift our embedding to $j\colon V_{n-3} \to M_{n-3}$ because $\mathrm{crit}(j) = \lambda_n$ and $|\mathbb{R}_n| = \lambda_{n-1}$. This avoids the need for Claim 3.33 because we no longer have pull a branch from $M[\cdots H_0 \cdots]$ back to $M[\cdots G_0 \cdots]$, and so we no longer have to deal with the generic objects h_0^* and u_0^*.

3.7 Generalization of Sinapova's Forcing

In this section and the following section we use the framework developed in (Sinapova, 2016) to argue that the tree property holds at κ^+. Much of the argument is the same, so we aim to describe the relevant forcing, prove some relevant structural properties about that forcing, and summarize how the argument in (Sinapova, 2016) is done in this particular circumstance.

Consider the μ-closed forcing $\mathbb{Q} = \mathbb{U} \times \mathbb{P}_1$, where $\mathbb{P}_1 = \mathrm{Add}(\mu, \lambda_1)_V$ and $\mathbb{U}$ are the conditions of the form $(0, 0, q, f) \in \mathbb{Q}_0$ with the induced suborder from $\mathbb{Q}_0$. One of the main differences between (Sinapova, 2016) and our situation is the presence of the poset $\mathbb{P}_1$.

Definition 3.37. Let $\dot{p}$ be a name for a condition in $\dot{\mathbb{I}}$. Define $\mathbb{R}_{\dot{p}}$ to have underlying set $\mathbb{Q}_0 \times \mathbb{P}_1$ with the following (modified) ordering:

Declare that $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2)$ exactly when

1. $a_1 \leq_{\mathbb{P}_1} a_2$
2. $(f_1, \dot{p}_1) \leq_{\mathbb{P}_0} (f_2, \dot{p}_2)$
3. $\mathrm{dom}(r_1) \supseteq \mathrm{dom}(r_2)$ and for every $\alpha \in \mathrm{dom}(r_2)$, we have that

   $(f_1, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} r_1(\alpha) \leq r_2(\alpha)$
4. $\mathrm{dom}(g_1) \supseteq \mathrm{dom}(g_2)$ and for every $\alpha \in \mathrm{dom}(g_2)$, we have that

   $(f_1, \dot{p}, r_1, g_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} g_1(\alpha) \leq g_2(\alpha)$.

Lemma 3.38. *$\mathbb{P}_0 \times \mathbb{Q}$ projects to $\mathbb{R}_{\dot{p}}$, witnessed by the identity.*

Proof. It is straightforward to see that the identity map is order preserving. To show the other requirement, suppose that $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2)$. We define $\overline{r}$ by setting $\mathrm{dom}(\overline{r}) = \mathrm{dom}(r_1)$ and define $\overline{r}(\alpha)$ by mixing names so that

$$\Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) \leq r_2(\alpha) \text{ and } (f_1, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) = r_1(\alpha).$$

More specifically, we may let

$$\overline{r}(\alpha) = \{\langle \tau, q\rangle \colon q \leq \langle f_1, \dot{p}\rangle \restriction \alpha \text{ and } q \Vdash_{\mathbb{P}_{0,\alpha}} \tau \in r_1(\alpha)\} \cup \{\langle \tau, q\rangle \colon q \perp \langle f_1, \dot{p}\rangle \restriction \alpha \text{ and } q \Vdash_{\mathbb{P}_{0,\alpha}} \tau \in r_2(\alpha)\}.$$

Next, by induction, we define $\overline{g}(\alpha)$ by induction so that

$$(0, 0, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) \leq g_2(\alpha) \text{ and } (f_1, \dot{p}, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) = g_1(\alpha).$$

The construction during the induction step is the similar to the construction of $\overline{r}(\alpha)$ above. But then, by definition, we have as desired that

1. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\mathbb{P}_0 \times \mathbb{Q}} (f_2, \dot{p}_2, r_2, g_2, a_2)$ and
2. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\dot{p}} (f_1, \dot{p}_1, r_1, g_1, a_1)$.

□

Lemma 3.39. *Let* $s^* = (0, \dot{p}, 0, 0, 0) \in \mathbb{Q}_0 \times \mathbb{P}_1$*. Then* $\mathbb{R}_{\dot{p}}/s^* = \{s \in \mathbb{R}_{\dot{p}} |\ s \leq_{\dot{p}} s^*\}$ *projects to* $(\mathbb{Q}_0 \times \mathbb{P}_1)/s^* = \{s \in \mathbb{Q}_0 \times \mathbb{P}_1 |\ s \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} s^*\}$ *witnessed by the identity.*

Proof. The proof is similar to the previous lemma. Since the last three coordinates of s^* are trivial, notice that $s \leq_{\dot{p}} s^*$ iff $s \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} s^*$. The identity is order preserving, so it is enough to check the nontrivial condition for projections:

Suppose $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} (f_2, \dot{p}_2, r_2, g_2, a_2)$. Define $\overline{r}(\alpha)$ and $\overline{g}(\alpha)$ similar to the previous lemma so that the following hold:

- $\Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) \leq r_2(\alpha)$,

- $(f_1, \dot{p}_1) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{r}(\alpha) = r_1(\alpha)$,

- $(0, 0, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) \leq g_2(\alpha)$, and

- $(f_1, \dot{p}_1, \overline{r}, \overline{g}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{g}(\alpha) = g_1(\alpha)$.

It follows as desired that

1. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2)$ and

2. $(f_1, \dot{p}_1, \overline{r}, \overline{g}, a_1) \leq_{\mathbb{Q}_0 \times \mathbb{P}_1} (f_1, \dot{p}_1, r_1, g_1, a_1)$. □

Definition 3.40. Let $\mathcal{A}$ be $\mathbb{A}$-generic over V. Let $p = \dot{p}_{\mathcal{A}}$. Define $\mathbb{Q}_p$ to have underlying set $\mathbb{Q}$ with the ordering $(0, 0, q_1, f_1, p_1) \leq_{\mathbb{Q}_p} (0, 0, q_2, f_2, p_2)$ exactly when

1. $p_1 \leq_{\mathbb{P}_1} p_2$

2. $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$ and $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$

3. there is an $a \in \mathcal{A}$ such that for every $\alpha \in \mathrm{dom}(q_2)$ and for every $\alpha \in \mathrm{dom}(f_2)$, we have that

   (a) $(a, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} q_1(\alpha) \leq q_2(\alpha)$

   (b) $(a, \dot{p}, q_1, f_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} f_1(\alpha) \leq f_2(\alpha)$

Lemma 3.41. *$\mathbb{Q}_p$ is κ-closed*

Proof. Assume that $\{(0, 0, q_i, f_i, p_i) \colon i < \theta\}$ is decreasing for $\theta < \kappa$. For each $i < \theta$ there is an $a_i \in \mathcal{A}$ such that (3) holds in the above definition. Since $\mathbb{A}$ is κ-directed closed, it follows that $a = \bigcup_{i<\theta} a_i \in \mathcal{A}$. Since $\mathbb{P}_1$ is μ-closed, we may let $\overline{p}$ be a lower bound of the p_i's. Next, observe that for each α and each $i < j < \theta$, we have $(a, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} q_j(\alpha) \leq q_i(\alpha)$. Since the $q_i(\alpha)$'s are forced to be in a μ-closed forcing, it follows that there is a name $\overline{q}(\alpha)$ such that $(a, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} \overline{q}(\alpha) \leq q_i(\alpha)$ for each $i < \theta$. Finally, define $\overline{f}(\alpha)$ by induction on α so that $(a, \dot{p}, \overline{q}, \overline{f}) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} \overline{f}(\alpha) \leq f_i(\alpha)$. This is possible since the $f_i(\alpha)$'s are forced to be in a μ-closed forcing as well. Then, $(0, 0, \overline{q}, \overline{f}, \overline{p})$ is a lower bound for the initial decreasing sequence, as desired. $\square$

Lemma 3.42. *$\mathbb{R}_p$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$*

Proof. In $V[\mathcal{A}]$, we argue that $\pi\colon \mathbb{R}_p/\mathcal{A} \to \mathbb{I} \times \mathbb{Q}_p$ defined by $(f_1, \dot{p}_1, r_1, g_1, a_1) \mapsto (\dot{p}_1^{\mathcal{A}}, r_1, g_1, a_1)$ is a dense embedding. Since π is onto, it is enough to show that π is order preserving and that $s \perp_{\mathbb{R}_p/\mathcal{A}} s'$ implies $\pi(s) \perp_{\mathbb{I} \times \mathbb{Q}_p} \pi(s')$.

Assume that $(f_1, \dot{p}_1, r_1, g_1, a_1) \leq_{\mathbb{R}_p/\mathcal{A}} (f_2, \dot{p}_2, r_2, g_2, a_2)$. Since $(f_1, \dot{p}_1) \leq_{\mathbb{P}_0} (f_2, \dot{p}_2)$ and $f_1 \in \mathcal{A}$, it follows that $\dot{p}_1^{\mathcal{A}} \leq_{\mathbb{I}} \dot{p}_2^{\mathcal{A}}$. It follows in turn that $(0, 0, r_1, g_1, a_1) \leq_{\mathbb{Q}_p} (0, 0, r_2, g_2, a_2)$.

Next, assume that $\pi(f_1, \dot{p}_1, r_1, g_1, a_1)$ and $\pi(f_2, \dot{p}_2, r_2, g_2, a_2)$ were compatible in $\mathbb{I} \times \mathbb{Q}_p$. Since π is onto, we may let $\pi(f, \dot{q}, r, g, a)$ witness this. Let $\overline{a}_1 \in \mathcal{A}$ witness that $(0, 0, r, g, a) \leq_{\mathbb{Q}_p} (0, 0, r_1, g_1, a_1)$ and $\overline{a}_2 \in \mathcal{A}$ witness that $(0, 0, r, g, a) \leq_{\mathbb{Q}_p} (0, 0, r_2, g_2, a_2)$. Let $\overline{a} \in \mathcal{A}$ be such that $\overline{a} \leq \overline{a}_1, \overline{a}_2$. By further extending $\overline{a}$, we may assume that $\overline{a} \Vdash_{\mathbb{A}} \dot{q} \leq \dot{p}_1, \dot{p}_2$. Finally, let $\overline{f} \in \mathcal{A}$ extend $\overline{a}$, f_1, and f_2. It follows that $(\overline{f}, \dot{q}, r, g, a)$ witnesses that $(f_1, \dot{p}_1, r_1, g_1, a_1)$ and $(f_2, \dot{p}_2, r_2, g_2, a_2)$ are compatible in $\mathbb{R}_p/\mathcal{A}$. Therefore, π is a dense embedding. $\square$

3.7.1 Splittings

One of the main technical tasks in (Sinapova, 2016) is proving Proposition 3.4, which is crucial in defining the branch in the forcing extension by the Mitchell poset. In this section we give the relevant definitions and summarize the results necessary to prove this result. First we define the following generic objects:

1. $G_0 \times g_1$ which is $\mathbb{Q}_0 \times \mathbb{P}_1$-generic,
2. $\mathcal{A}$ which is $\mathbb{A}$-generic induced from $G_0 \times g_1$,
3. $\mathcal{I}$ which is $\mathbb{I}$-generic over $V[\mathcal{A}]$ induced by $G_0 \times g_1$,
4. G^* which is $(\mathbb{P}_0 \times \mathbb{Q})/(G_0 \times g_1)$-generic,
5. G_q which is $\mathbb{R}_{\dot{q}}/(G_0 \times g_1)$-generic induced by G^*, where $q \in \mathcal{I}$,
6. $\mathcal{Q}$ which is $\mathbb{Q}$-generic induced by G^*, and
7. $\mathcal{Q}_p$ which is $\mathbb{Q}_p$-generic over $V[\mathcal{A}]$ induced by G^*, where $p \in \mathbb{I}$.

Given a μ-tree T in $V[G_0][g_1]$ we may consider a variety of names for this tree. We let $\tau \in V[\mathcal{A}]$ be an $(\mathbb{Q}_0 \times \mathbb{P}_1)/\mathcal{A}$-name for the tree. Further, this name induces an $\mathbb{I}$-name $\dot{T} \in V[\mathcal{A}][\mathcal{Q}]$ for the tree. By the coherence of these names we know that $q \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} u <_{\dot{T}} v$ if and only if there is an $a \in \mathcal{A}$ and $r \in \mathcal{Q}$ where $(a, \dot{q}, r) \Vdash^{V[\mathcal{A}]}_{(\mathbb{Q}_0 \times \mathbb{P}_1)/\mathcal{A}} u <_\tau v$.

Let $\dot{b} \in V[\mathcal{A}]$ be a $(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}$-name for the branch given by (Neeman, 2009) where we assume that

$$1 \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} \dot{b} \text{ is a branch through } \tau.$$

The following are Definitions 3.2 and 3.3 in (Sinapova, 2016), respectively.

Definition 3.43. Let h be a stem. Say there is an h-splitting at a node u if there is a $p \in \mathbb{I}$ with $\mathrm{stem}(p) = h$ and $r \in \mathbb{Q}$ such that $(p, r) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} u \in \dot{b}$ and nodes u_1, u_2 of higher levels and conditions r_1, r_2 such that for $k = 1$ or 2,

1. $r_k \leq_{\mathbb{Q}} r$, $r_k \in \mathbb{Q}_p$,
2. $(p, r_k) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} u_k \in \dot{b}$, and
3. $p \Vdash^{V[\mathcal{A}][\mathbb{Q}]}_{\mathbb{I}} u_1 \perp_{\dot{T}} u_2$.

Definition 3.44. Let h be a stem. Say that $\dagger_h$ holds if in $V[\mathcal{A}][\mathbb{Q}]$ there is an unbounded $J \subseteq \mu$, an ordinal $\xi < \kappa$, and $(p_\alpha | \alpha \in J)$, where each $p_\alpha \in \mathbb{I}$ is a condition with stem h. Further by setting $u_\alpha = (\alpha, \xi)$ we have:

1. for all $\alpha < \beta$ from J, $p_\alpha \wedge p_\beta \Vdash_{\mathbb{I}} u_\alpha <_{\dot{T}} u_\beta$;
2. for all $\alpha \in J$, $p_\alpha \Vdash_{\mathbb{I}} u_\alpha \in \dot{b}$.

Let $\alpha_h = \sup\{\alpha < \mu \colon$ there is an h-splitting at $Lev_\alpha(T)\}$. The key lemma involving $\dagger_h$ is the following:

Lemma 3.45. *If h is any stem where $\dagger_h$ holds then $\alpha_h < \mu$.*

The proof of this lemma amounts to inductively applying a splitting lemma in $V[\mathcal{A}][\mathbb{Q}]$ to construct conditions $(p_\sigma, r_\sigma, v_\sigma \colon \sigma \in \kappa_\omega^{<\omega})$ so that for every $\sigma \in \kappa_\omega^{<\omega}$,

1. $(p_\sigma, r_\sigma) \in \mathbb{I} \times \mathbb{Q}$,

2. $v_\sigma \in T$,

3. $\operatorname{stem}(p_\sigma) = h$ and $r_\sigma \in Q_{p_\sigma}$,

4. $(p_{f\restriction n}, r_{f\restriction n} \colon n < \omega)$ is a decreasing sequence for every $f \in \kappa_\omega^\omega$,

5. $(p_{\sigma^\frown i}, r_{\sigma^\frown i}) \Vdash_{\mathbb{I}\times\mathbb{Q}} v_{\sigma^\frown i} \in \dot{b}$ for every $i < \kappa_\omega$,

6. $p_{\sigma^\frown i} \wedge p_{\sigma^\frown j} \Vdash_{\mathbb{I}} v_{\sigma^\frown i} \perp_{\dot{T}} v_{\sigma^\frown j}$ for every $i < j$.

A proof of Lemma 3.45 and of this splitting lemma is done in detail in Section 4.3 in a more complicated setting.

3.8 Tree property at κ^+

In this section we finish the result by proving the following:

Theorem 3.46. *In $V[G_\omega]$, the tree property holds at κ^+.*

Proof. Assume that T is a μ-tree in $V[G_\omega]$. Working in $V[\mathbb{R}_3]$, the forcing $\mathbb{R}_\omega/\mathbb{R}_3$ is λ_0-closed and so $T \in V[G_0][G_1][G_2]$. $\mathbb{Q}_2$ is $< \lambda_0$-distributive in $V[G_0][G_1]$, and so $T \in V[G_0][G_1]$. It follows that $T \in V[G_0][g_1]$, where g_1 is $\mathbb{P}_1$-generic over $V[G_1]$. So, it is enough to show that there is no μ-Aronszajn tree in $V[G_0][g_1]$. Assume for the sake of contradiction that T is a μ-Aronszajn tree in $V[G_0][g_1]$.

By construction of $\mathbb{Q}_0$ we know that $\mathbb{Q}_0$ is the projection of $\mathbb{P}_0 \times \mathbb{U}$, where $\mathbb{U}$ is μ-directed closed in V. Since $\mathbb{P}_1$ is μ-directed closed in V, we also have that the product $\mathbb{Q} = \mathbb{U} \times \mathbb{P}_1$ is μ-closed. Then, if we let $(\mathcal{A} * \mathfrak{I}) \times \mathfrak{Q}$ be $\mathbb{P}_0 \times \mathbb{Q}$-generic, we know from the previous section that μ has the tree property in $V[(\mathcal{A} * \mathfrak{I}) \times \mathfrak{Q}]$.

In order to define a branch for T in $V[G_0][g_1]$, we use Lemma 3.45 to set $\alpha = \sup\{\alpha_h \mid \dagger_h$ holds$\}$. By counting the number of Prikry stems we have $\alpha < \mu$. Next, fix $u \in Lev_\alpha(T)$ and $s^* \in G^*$ with $s^* \Vdash_{(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}} u \in \dot{b}$. In $V[G_0][g_1]$, define

$$d = \{v \mid u <_T v \text{ and there is an } s \in G_0 \times g_1 \text{ such that } s \leq_{\mathbb{P}_0 \times \mathbb{Q}} s^* \text{ and } s \Vdash_{(\mathbb{P}_0 \times \mathbb{Q})/\mathcal{A}} v \in \dot{b}\}.$$

It is enough to show that the downwards closure of d is a branch through T. Since d is unbounded, this amounts to showing $d \cap Lev_\beta(T)$ is a singleton for each $\beta \geq \alpha$. We refer the reader to Lemma 4.21 for a proof given in a slightly different but more complicated setting. $\square$

CHAPTER 4

ITP AT κ^+ AND κ^{++}

In this chapter we show that after forcing with the poset $\mathbb{R}$ from (Sinapova, 2016), ITP holds at κ^+ and κ^{++} with κ is singular strong limit. Since the Prikry forcing is interleaved with the Mitchell forcing, the argument for ITP at κ^{++} is more straightforward and is similar the proof of Proposition 1.60 above. The argument for ITP at κ^+ is more involved and requires generalizing an argument from (Sinapova, 2016). In particular we give an analysis of a family of intermediate models that allow us to define a branch in the model $V[\mathbb{R}]$. This analysis is an application of the p-term forcing framework developed in Section 1.7.

Regarding large cardinal assumptions, in the original paper Sinapova assumed the existence of an ω-sequence of supercompact cardinals with a weakly compact above them. Since our goal is ITP at κ^{++} we instead will assume that the weakly compact cardinal is in fact supercompact. We note though that the same hypothesis of an ω-sequence of supercompacts is used to get ITP at κ^+.

4.1 Sinapova's Forcing

We give the definition of the relevant forcings found in (Sinapova, 2016) that will be used for the remainder of the chapter.

Definition 4.1. In V let $(\kappa_n : n < \omega)$ be an increasing sequence of indestructibly supercompact cardinals with $\kappa_\omega = \sup_n \kappa_n$, $\kappa = \kappa_0$, and $\mu = \kappa_\omega^+$. Let $\lambda > \mu$ be a supercompact cardinal. Let $\mathbb{P} = \mathbb{A} * \dot{\mathbb{I}}$ and $\mathbb{P}_\alpha = \mathbb{A}_\alpha * \dot{\mathbb{I}}_\alpha$ for $\alpha \in B$. Define $\mathbb{R}$ to be of the form $(f, \dot{p}, r)$ where:

1. $(f, \dot{p}) \in \mathbb{P}$,
2. r is a partial function with $\mathrm{dom}(r) \subseteq B$, $|\mathrm{dom}(r)| < \mu$, and
3. for each $\alpha \in \mathrm{dom}(r)$, $r(\alpha)$ is a $\mathbb{P}_\alpha$-name for a condition in $Add(\mu, 1)_{V[\mathbb{P}_\alpha]}$

The ordering on $\mathbb{R}$ is the natural one.

From (Sinapova, 2016), (Unger, 2013), and (Cummings and Foreman, 1998) we have the following:

Theorem 4.2. *In $V[\mathbb{R}]$ we have that*

1. *κ is singular strong limit with cf$(\kappa) = \omega$,*
2. *$\kappa^+ = \mu$,*
3. *$2^\kappa = \kappa^{++} = \lambda$, and*
4. *the tree property holds at κ^+ and κ^{++}.*

4.2 ITP at κ^{++}

In this section we prove the following:

Theorem 4.3. *After forcing with $\mathbb{R}$, ITP holds at κ^{++}.*

The argument is similar in spirit to the proof of Proposition 1.60 that ITP holds at $\aleph_2$ after forcing with the Mitchell Forcing assuming a supercompact.

Proof. Assume that $\theta \geq \kappa^{++}$. Our goal is to show that $ITP(\kappa^{++}, \theta)$ holds after forcing with $\mathbb{R}$. Let G be $\mathbb{R}$-generic over V and let D be a thin $\mathcal{P}_{\kappa^{++}}(\theta)$-list in $V[G]$. The plan is to argue that D has an ineffable branch in some outer model $V[G^*]$ and then pull this back to $V[G]$. Towards this end, let $j \colon V \to M$ be a θ-supercompactness embedding with $\mathrm{crit}(j) = \lambda$. Since $j(\mathbb{R})$ projects onto $\mathbb{R}$, we may lift this to $j \colon V[G] \to M[j(G)]$.

Lemma 4.4. *D has an ineffable branch b in $M[j(G)]$.*

Proof. As in the proof of Proposition 1.60 above, we technically can only show that D is a cofinal branch in $M[j(G)]$. However, this is enough in order to pull back the branch. As before, after we pull the branch back to $M[G]$ it follows by standard arguments that the branch is ineffable. $\square$

We have that $j(\mathbb{R})$ is a projection of $\mathbb{R} * (\mathbb{P}^* \times \mathbb{Q}^*)$ where $\mathbb{P}^* = j(\mathbb{P})/\mathbb{P}$ and $\mathbb{Q}^*$ is μ-closed. If b is the branch for D, then we have that $b \in M[G][P^*][Q^*]$ where P^* and Q^* are suitable generics for $\mathbb{P}^*$ and $\mathbb{Q}^*$, respectively. We pull b back to $V[G]$ by first arguing that it is sufficiently approximated:

Lemma 4.5. *b is thinly κ^{++}-approximated in $M[G]$.*

Proof. The proof is the same as the Claim 1.63 from Proposition 1.60. $\square$

Lemma 4.6. *b is μ-approximated in $M[G][Q^*]$.*

Proof. This follows because $\mathbb{Q}^*$ is μ-closed in $M[G]$ and so $\mathcal{P}_\mu(\theta)^{M[G][Q^*]} \subseteq \mathcal{P}_\mu(\theta)^{M[G]}$. $\square$

Now we are able to pull back b to $V[G]$:

Lemma 4.7. $b \in M[G][Q^*]$.

Proof. Follows since b is μ-approximated in $M[G][Q^*]$ and since $(\mathbb{P}^*)^2$ is μ-cc by Lemma 1.73. $\square$

Lemma 4.8. $b \in M[G]$.

Proof. Since $\mathbb{Q}^*$ is μ-closed and $2^\kappa = \lambda$ in $M[G]$, we have that $\mathbb{Q}^*$ has the thin κ^{++}-approximation property in $M[G]$. The result follows since b is thinly κ^{++}-approximated in $M[G]$. $\square$

Finally, observe that $S = \{x \in \mathcal{P}_{\kappa^{++}}(\theta) \colon b \cap x = d_x\}$ is defined using b and so $S \in M[G]$ as well. So S witnesses that b is an ineffable branch in $V[G]$, as desired. $\square$

4.3 ITP at κ^+

We need to show that $ITP(\kappa^+, \theta)$ holds for $\theta \geq \kappa^+$ after forcing with $\mathbb{R}$. Adopting notation similar in spirit to Chapter 3, we define $\mathbb{Q}$ as the μ-directed closed forcing of all conditions $(0, 0, r) \in \mathbb{R}$ with the induced ordering. From this we define $\mathbb{R}^* = \mathbb{P} \times \mathbb{Q}$ with the product ordering. Next, given a name $\dot{p}$ for a condition $\dot{\mathbb{I}}$, define $\mathbb{R}_{\dot{p}}$ to be the poset with same underlying set as $\mathbb{R}$ but with the following ordering: $(a, \dot{p}_1, b) \leq_{\dot{p}} (a', \dot{p}_2, b')$ if and only if

1. $(a, \dot{p}_1) \leq (a', \dot{p}_2)$, and
2. $\mathrm{dom}(b') \supseteq \mathrm{dom}(b)$ and $(a, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_\alpha} r_1(\alpha) \leq r_2(\alpha)$ for each $\alpha \in \mathrm{dom}(b)$.

Arguments from (Sinapova, 2016) (with appropriate lemma references) show the following:

1. (Lemma 2.3) The identity map is a projection from $\mathbb{R}^*$ to $\mathbb{R}_{\dot{p}}$.

2. (Lemma 2.7) For an $\mathbb{A}$-generic filter $\mathcal{A}$, we have that there is a forcing $\mathbb{Q}_{\dot{p}}$ such that $\mathbb{R}_{\dot{p}}$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$.

Similar to Subsection 3.7.1 we define the following generic objects:

1. G which is $\mathbb{R}$-generic,
2. $\mathcal{A}$ which is $\mathbb{A}$-generic induced from G,
3. $\mathcal{I}$ which is $\mathbb{I}$-generic over $V[\mathcal{A}]$ induced by G,
4. G^* which is $\mathbb{R}^*/G$-generic,
5. G_q which is $\mathbb{R}_{\dot{q}}/G$-generic induced by G^*, where $q \in \mathcal{I}$,
6. $\mathcal{Q}$ which is $\mathbb{Q}$-generic induced by G^*, and
7. $\mathcal{Q}_p$ which is $\mathbb{Q}_p$-generic over $V[\mathcal{A}]$ induced by G^*, where $p \in \mathbb{I}$.

Lemma 4.9. $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}]} = \mathcal{P}_\mu(\theta)^{V[\mathcal{A}][\mathcal{Q}]}$

Proof. Note that $\mathbb{Q}$ is μ-closed in V and $\mathbb{A}$ is κ^+-cc (and therefore μ-cc) in V. Easton's Lemma implies that $\mathbb{Q}$ is $< \mu$-distributive in $V[\mathcal{A}]$, giving the result. $\square$

Lemma 4.10. *Without loss of generality we may assume that our thin list is indexed by* $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}]}$.

Proof. It suffices to show that $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}]}$ is stationary in $V[G]$. To see this, observe first that $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}][\mathcal{Q}]}$ is stationary in $V[G^*]$ because $\mathbb{I}$ is μ-cc in $V[\mathcal{A}][\mathcal{Q}]$. By the previous lemma and since stationarity is downwards absolute, we have that $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}]}$ is stationary in $V[G]$. $\square$

So going forward we will denote $\mathcal{P}_\mu(\theta)^{V[\mathcal{A}]}$ as $\mathcal{P}_\mu(\theta)$. Let $D = (d_x : x \in \mathcal{P}_\mu(\theta))$ be a thin list in $V[G]$ and $(\dot{d}_x : x \in \mathcal{P}_\mu(\theta))$ be an $\mathbb{I}$-name for the thin list in $V[\mathcal{A}][\mathcal{Q}]$.

Definition 4.11. For $z \in \mathcal{P}_\mu(\theta)$, define $C_z = \{x \mid z \subseteq x\}$.

Now, in (Cummings et al., 2020a) they show that ITP holds at κ^+ in $V[\mathbb{A} * \dot{\mathbb{I}}]$. Since $\mathbb{Q}$ is μ-directed closed, we know that each κ_n is still supercompact in $V[\mathbb{Q}]$ and so the same argument gives that ITP holds at κ^+ in $V[\mathbb{Q}][\mathbb{A} * \dot{\mathbb{I}}]$. This allows us to show something stronger:

Lemma 4.12. *In $V[\mathcal{A}][\mathcal{Q}]$, there is a stationary set $T \subseteq \mathcal{P}_\mu(\theta)$, there are $z^* \in \mathcal{P}_\mu(\theta)$ and conditions $(p_z : z \in T \cap C_{z^*})$ with stem $\overline{h}$, such that*

1. *if $z^* \subseteq y \subseteq z$ with $y, z \in T$, then $p_y \wedge p_z \Vdash_{\mathbb{I}} \dot{d}_y = \dot{d}_z \restriction y$, and*
2. *there is some $y \in T$ such that p_y forces $\{z : p_z \in \dot{\mathcal{J}}\}$ is stationary.*

Proof. Working in $V[\mathcal{A}][\mathcal{Q}]$ let $\dot{b}$ be an $\mathbb{I}$-name for the ineffable branch. Since $\dot{b}$ is forced to be ineffable it follows that $S = \{x : \exists p \in \mathbb{I}, p \Vdash \dot{b} \restriction x = \dot{d}_x\}$ is stationary. For each $x \in S$ fix some $p_x \in \mathbb{I}$ witnessing this. Since there are κ_ω-many stems there is a stationary $T \subseteq S$ and a stem $\overline{h}$ such that p_x has stem $\overline{h}$ for each $x \in T$. Since conditions in $\mathbb{I}$ with the same stem are compatible we have that (1) holds. Further (2) holds because $\mathbb{I}$ is μ-cc in $V[\mathcal{A}][\mathcal{Q}]$. $\square$

In the original paper they actually proved Lemma 4.12 directly in $V[\mathbb{A}]$ and used it to define a branch in $V[\mathbb{A} * \dot{\mathbb{I}}]$. We therefore note that the existence of an ineffable branch in $V[\mathbb{Q}][\mathbb{A} * \dot{\mathbb{I}}]$ and Lemma 4.12 are equivalent.

In what follows, we use the following conventions. Throughout the remainder of the section, $\Vdash$ may refer to either $\Vdash^{\mathcal{A}}_{\mathbb{I} \times \mathbb{Q}}$ or $\Vdash^{\mathcal{A} \times \mathcal{Q}}_{\mathbb{I}}$ or $\Vdash^{\mathcal{A} \times \mathcal{Q}_p}_{\mathbb{I}}$.

- For $x \in \mathcal{P}_\mu(\theta)$, say that $p \Vdash x \in \dot{b}$ if and only if $p \Vdash \dot{b} \restriction x = \dot{d}_x$.
- For $x, y \in \mathcal{P}_\mu(\theta)$, say that $p \Vdash x \perp y$ if and only if $p \Vdash$ "there is no $z \supseteq x, y$ such that $\dot{d}_z \restriction x = \dot{d}_x$ and $\dot{d}_z \restriction y = \dot{d}_y$."

Let b be the ineffable branch in $V[G^*]$ given by Lemma 4.12. Let $\dot{b} \in V[\mathcal{A}]$ be an $\mathbb{R}^*/\mathcal{A}$-name for the ineffable branch. In particular, assume that

$$1 \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} \text{"}\dot{b} \text{ is an ineffable branch witnessed by } \dot{T}.\text{"}$$

Definition 4.13. Given a stem h, $\underline{\dagger_h \text{ holds}}$ if in $V[\mathcal{A}][\mathcal{Q}]$, there is a stationary set T, there are $z^* \in \mathcal{P}_\mu(\theta)$ and conditions $(p_z \colon z \in T \cap C_{z^*})$ with stem h, such that

1. if $z^* \subseteq y \subseteq z$ with $y, z \in T$, then $p_y \wedge p_z \Vdash_{\mathbb{I}} \dot{d}_y = \dot{d}_z \restriction y$, and
2. there is some $y \in T$ such that p_y forces $\{z \colon p_z \in \dot{\mathcal{J}}\}$ is stationary.

More precisely, it is shown in (Cummings et al., 2020a) that the following holds:

Lemma 4.14. *By density, any stem can be extended to a stem h for which $\dagger_h$ holds.*

Definition 4.15. Given a stem h, $\underline{\text{there is an } h\text{-splitting at } x \in \mathcal{P}_\mu(\theta)}$ if there are $(p, r) \in \mathbb{I} \times \mathcal{Q}$, $x_0, x_1 \supset x$, and r_0, r_1 extending r so that for $i < 2$,

1. $\text{stem}(p) = h$,
2. $(p, r) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x \in \dot{b}$,
3. $r_i \in \mathcal{Q}_p$,

4. $(p, r_i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x_i \in \dot{b}$,

5. $p \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} x_0 \perp x_1$.

Definition 4.16. In $V[\mathcal{A}][\mathcal{Q}]$, given a stem h such that $\dagger_h$ holds, define E_h to be all $x \in \mathcal{P}_\mu(\theta)$ such that $(p, r) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x \in \dot{b}$ for some $(p, r) \in \mathbb{I} \times \mathcal{Q}$ with stem$(p) = h$.

The following lemma is a generalization of Proposition 3.4 in (Sinapova, 2016). The proof is similar in spirit, although the conventions $x \in \dot{b}$ and $x \perp y$ have different meanings here.

Lemma 4.17. *For a stem h, define $B_h = \{x \mid$ there is an h-splitting at $x\}$. Then, in $V[\mathcal{A}][\mathcal{Q}]$, $\dagger_h$ implies there is some z such that $B_h \cap C_z$ is empty.*

Proof. Fix your favorite stem h and assume the lemma is false. Then we may find $\overline{r} \in \mathcal{Q}$ forcing $B_h \cap C_z \neq \varnothing$ for each z, and that $\dot{T}$ and $(\dot{p}_z : z \in \dot{T})$ are $\mathbb{Q}$-names in $V[\mathcal{A}]$ witnessing $\dagger_h$. (Note: technically the definition of $\dagger_h$ requires an additional z^*, but we omit this for simplicity.) Our first goal is to prove the following Splitting Lemma:

Lemma 4.18 (Splitting)**.** *Assume stem$(q) = h$ and $r \leq_{\mathbb{Q}} \overline{r}$ with $r \in \mathcal{Q}_q$. Then there is a sequence $(p_i, r_i, x_i : i < \kappa_\omega)$ where $(p_i : i < \kappa_\omega)$ are conditions with stem h, and for $i < \kappa_\omega$,*

1. *$x_i \in \mathcal{P}_\mu(\theta)$,*

2. *$(p_i, r_i) \leq_{\mathbb{I}\times\mathbb{Q}} (q, r)$,*

3. *$r_i \in \mathcal{Q}_{p_i}$,*

4. *$(p_i, r_i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x_i \in \dot{b}$, and*

5. *for $j \neq i$, $p_i \wedge p_j \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} x_i \perp x_j$.*

Proof of Lemma 4.18. Start by letting $\mathcal{Q}'$ be $\mathbb{Q}/\mathcal{Q}_q$-generic over $V[\mathcal{A}][\mathcal{Q}_q]$ where $r \in \mathcal{Q}'$. We continue in $V[\mathcal{A}][\mathcal{Q}']$ where E_h, T, and $(p_z \colon z \in T)$ denote the interpretations of these names in $V[\mathcal{A}][\mathcal{Q}']$. We need to prove the following:

Claim 4.19. *For $x \in E_h$ there are $p \leq q$ with stem h, $r_0, r_1 \in \mathcal{Q}_p$, and $x_0, x_1 \supset x$ such that for $i < 2$,*

1. $(p, r_i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x_i \in \dot{b}$, *and*
2. $p \Vdash^{V[\mathcal{A}][\mathcal{Q}']}_{\mathbb{I}}$ "$x_0 \perp x_1$ *and* $\dot{d}_{x_0} \restriction x = \dot{d}_x = \dot{d}_{x_1} \restriction x$."

Proof of claim. Let $(p, t) \in \mathbb{I} \times \mathcal{Q}'$ witnessing that $x \in E_h$. In other words, $\mathrm{stem}(p) = h$ and $(p, t) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x \in \dot{b}$. By our hypothesis, we know that there is some $y \in B_h \cap C_x$. By definition, we may find some $(p', t') \in \mathbb{I} \times \mathcal{Q}'$ where $\mathrm{stem}(p') = h$ witnessing this, along with conditions r_0, r_1 extending t', and $x_0, x_1 \supset y$ such that for $i < 2$,

- $r_i \in \mathcal{Q}_{p'}$,
- $(p', r_i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x_i \in \dot{b}$,
- $p' \Vdash^{V[\mathcal{A}][\mathcal{Q}']}_{\mathbb{I}}$ "$x_0 \perp x_1$ and $\dot{d}_{x_0} \restriction y = \dot{d}_y = \dot{d}_{x_1} \restriction y$."

We may assume that p' extends p and q because they have the same stem. Observe that $p \Vdash^{V[\mathcal{A}][\mathcal{Q}']}_{\mathbb{I}} \dot{d}_y \restriction x = \dot{d}_x$, and so it follows readily that p', x_0, and x_1 are as required. $\square$

Continuing with the proof of the lemma, first assume without loss of generality that r satisfies the conclusion of the claim. It follows in $V[\mathcal{A}][\mathcal{Q}_q]$ that there is some club $C \subseteq \mathcal{P}_\mu(\theta)$

where, for $x \in C$ and $u \subset x$, if u is forced into $\dot{E}_h$ by a condition extending r, then there are $u \subset v_0, v_1 \subset x$ witnessing the conclusion of the claim for u.

Working in $V[\mathcal{A}][\mathcal{Q}']$ construct terms $(p^i, z_i, x_i, t_i \colon i < \kappa_\omega)$ such that for each $i < \kappa_\omega$:

1. $z_i \in T$,
2. $x_i \in C$ and $z_i \subset x_i \subseteq z_{i+1}$.
3. $t_i \in \mathcal{Q}'$, $t_i \leq_{\mathbb{Q}} r$, and $t_i \Vdash^{V[\mathcal{A}]}_{\mathbb{Q}}$ "$z_i \in \dot{T}$ and $p^i = \dot{p}_{z_i}$."

This sequence is in $V[\mathcal{A}]$ by Easton's Lemma. Further we have that

1. $p^i \wedge p^j \Vdash_{\mathbb{I}} \dot{d}_{z_i} = \dot{d}_{z_j} \restriction z_i$ for $i < j$, and
2. $(p^i, t_i) \Vdash_{\mathbb{I} \times \mathbb{Q}} z_i \in \dot{b}$ and so $t_i \Vdash_{\mathbb{Q}} z_i \in \dot{E}_h$ for $i < \kappa_\omega$.

Working in $V[\mathcal{A}][\mathcal{Q}_q]$, for each $i < \kappa_\omega$, find q_i extending q and p^i where $\mathrm{stem}(q_i) = h$, and sets $v_0^i, v_1^i \subset x_i \subseteq z_{i+1}$ such that for some $r_0^i, r_1^i \in \mathcal{Q}_{q_i}$:

1. $(q_i, r_0^i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} v_0^i \in \dot{b}$,
2. $(q_i, r_1^i) \Vdash^{V[\mathcal{A}]}_{\mathbb{I} \times \mathbb{Q}} v_1^i \in \dot{b}$, and
3. $q_i \Vdash_{\mathbb{I}}$ "$v_0^i \perp v_1^i$ and $\dot{d}_{v_1^i} \restriction z_i = \dot{d}_{z_i} = \dot{d}_{v_2^i} \restriction z_i$".

To do this we have to have that $z_i \subset x_i \subseteq z_{i+1}$ and $x_i \in C$. Using the Prikry property, fix $p_i \leq^* q_i \wedge p^{i+1}$ deciding both "$\dot{d}_{z_{i+1}} \restriction v_0^i = \dot{d}_{v_0^i}$" and "$\dot{d}_{z_{i+1}} \restriction v_1^i = \dot{d}_{v_1^i}$". Since p_i forces $v_0^i \perp v_1^i$, without loss of generality assume that "$\dot{d}_{z_{i+1}} \restriction v_0^i = \dot{d}_{v_0^i}$" is decided to be false. Let $r_i = r_0^i$ and $v_i = v_0^i$. Finally, for $i < j < \kappa_\omega$, it follows that

$$p_i \wedge p_j \Vdash_{\mathbb{I}} \dot{d}_{v_j} \restriction v_i \neq \dot{d}_{v_i}.$$

By construction we know that $v_i \subseteq v_j$, and so this implies that $p_i \wedge p_j \Vdash_{\mathbb{I}} v_i \perp v_j$. The sequence $(p_i, r_i, v_i \colon i < \kappa_\omega)$ yields the splitting lemma. $\square$

Going forward, we work in $V[\mathcal{A}][\mathcal{Q}]$ and repeatedly apply the splitting lemma to construct a sequence $(p_\sigma, r_\sigma, x_\sigma \colon \sigma \in \kappa_\omega^{<\omega})$ so that for every $\sigma \in \kappa_\omega^{<\omega}$,

1. $(p_\sigma, r_\sigma) \in \mathbb{I} \times \mathbb{Q}$ and $x_\sigma \in \mathcal{P}_\mu(\theta)$,
2. $\mathrm{stem}(p_\sigma) = h$, $r_\sigma \in \mathcal{Q}_{p_\sigma}$, and $r_\sigma \leq \overline{r}$,
3. $((p_{f\restriction n}, r_{f\restriction n}) \colon n < \omega)$ is a $\leq_{\mathbb{I}\times\mathbb{Q}}$-decreasing sequence for $f \in \kappa_\omega^\omega$,
4. $(p_\sigma, r_\sigma) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} x_\sigma \in \dot{b}$,
5. $p_{\sigma^\frown i} \wedge p_{\sigma^\frown j} \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} x_{\sigma^\frown i} \perp x_{\sigma^\frown j}$ for distinct $i, j < \kappa_\omega$.

Now, define $x = \displaystyle\bigcup_{\sigma \in \kappa_\omega^{<\omega}} x_\sigma \in \mathcal{P}_\mu(\theta)$ and fix some $x^* \supseteq x$ with $1 \Vdash^{V[\mathcal{A}][\mathcal{Q}]}_{\mathbb{I}} x^* \in \dot{b}$. For each $f \in \kappa_\omega^\omega$ define $p_f = \displaystyle\bigwedge_{n<\omega} p_{f\restriction n} \in \mathbb{I}$. (Note that p_f has stem h.) Further, find $r_f \in \mathcal{Q}_{p_f}$ with $(p_f, r_f) \leq_{\mathbb{I}\times\mathbb{Q}} (p_{f\restriction n}, r_{f\restriction n})$ for $n < \omega$. Using this, find $(p_f^*, r_f^*) \leq_{\mathbb{I}\times\mathbb{Q}} (p_f, r_f)$ and $s_f \in \mathcal{P}_\mu(\theta)$ such that $r_f^* \in \mathbb{Q}_{p_f}$ and $(p_f^*, r_f^*) \Vdash^{V[\mathcal{A}]}_{\mathbb{I}\times\mathbb{Q}} \dot{d}_{x^*} = s_f = \dot{b} \restriction x^*$. Since our list is thin, it follows that there are distinct $f, g \in \kappa_\omega^\omega$ such that $s = s_f = s_g$ and $\mathrm{stem}(p_f^*) = \mathrm{stem}(p_g^*)$. Let n be the least value such that $i = f(n) \neq g(n) = j$.

Finally, set $p = p_f^* \wedge p_g^*$ and observe that $r_f^*, r_g^* \in \mathcal{Q}_p$. Then, if $\mathcal{I}^*$ is $\mathbb{I}$-generic over $V[\mathcal{A}][\mathcal{Q}_p]$ with $p \in \mathcal{I}^*$, we have in $V[\mathcal{A}][\mathcal{Q}_p][\mathcal{I}^*]$ that

- $r_f^* \Vdash_{\mathbb{Q}/\mathcal{Q}_p} \dot{d}_{x^*} = s$ and $x_{\sigma^\frown i} \in \dot{b}$,
- $r_g^* \Vdash_{\mathbb{Q}/\mathcal{Q}_p} \dot{d}_{x^*} = s$ and $x_{\sigma^\frown j} \in \dot{b}$, and

- $x_{\sigma^\frown i} \perp x_{\sigma^\frown j}$.

However, this is a contradiction; indeed, the first bullet point implies that $d_{x_{\sigma^\frown i}} = d_{x^*} \cap x_{\sigma^\frown i}$, the second bullet point implies that $d_{x_{\sigma^\frown j}} = d_{x^*} \cap x_{\sigma^\frown j}$, and both of these combined together imply that the third bullet point is false. $\square$

For each h such that $\dagger_h$ holds, let z_h be such that $B_h \cap C_{z_h}$ is empty. Define $a = \bigcup\{z_h \colon \dagger_h$ holds$\} \in \mathcal{P}_\mu(\theta)$. Find $s^* \in G^*$ and $z \supset a$ such that $z \in \mathcal{P}_\mu(\theta)$ and $s^* \Vdash_{\mathbb{R}^*/\mathcal{A}} \dot{b} \restriction z = \dot{d}_z$. In $V[G]$, define

$$d = \bigcup\{d_v \in \mathcal{P}_\mu(\theta) \colon v \supset z \text{ and } (\exists t \in G)\ t \leq_{\mathbb{R}^*} s^*, t \Vdash_{\mathbb{R}^*/\mathcal{A}} v \in \dot{b}\}.$$

Lemma 4.20. *If $s \in \mathbb{R}^*/G$, then there is $p \in \mathcal{I}$ such that $s \in G_p$.*

Proof. This is Lemma 3.8 in (Sinapova, 2016) and is similar to the proof of Lemma 1.50. $\square$

Lemma 4.21. *d is an ineffable branch in $V[G]$.*

Proof. In $V[G^*]$, we have that $A = \{x \in T \colon p_x \in \mathcal{I}\}$ stationary and that $A \subseteq \{x \colon b \restriction x = d_x\}$. Observe that it is enough to show that $A \cap \{x \colon z \subseteq x\} \subseteq \{x \colon d \restriction x = d_x\}$. Indeed, this would imply that d is ineffable in $V[G^*]$. But then $\{x \colon d \restriction x = d_x\} \in V[G]$ because it is definable from d and further is stationary in $V[G]$ by downwards absoluteness. Towards this end, let $x \in A$ with $z \subseteq x$. We must show that $d \restriction x = d_x$.

If $\alpha \in d_x$, then since $x \in A$ we know that $d_x = b \restriction x$. So we may find some $t \in G^* \subseteq G$ with $t \leq_{\mathbb{R}^*} s^*$ forcing this. But then by definition of d it follows that $d_x \subseteq d$, implying that $\alpha \in d \restriction x$.

Going the other direction, assume towards a contradiction that $\alpha \in d \restriction x$ but $\alpha \notin d_x$. Working in $V[G]$, fix some $v \supseteq z$ and $s_1 \leq_{\mathbb{R}^*} s^*$, $s_1 \in G$, such that $\alpha \in d_v$ and that $s_1 \Vdash_{\mathbb{R}^*/\mathcal{A}} v \in \dot{b}$. We can also find $s_2 \leq s^*$ in $G^* \subseteq G$ such that $s_2 \Vdash_{\mathbb{R}^*/\mathcal{A}} x \in \dot{b}$. Denote $s_1 = (a_1, p_1, r_1)$ and $s_2 = (a_2, p_2, r_2)$. By the above lemma, we may find $q \in \mathcal{I}$ such that $s_1, s_2 \in G_q$. Since $\alpha \notin d_x$ and $\alpha \in d_v$, it follows that $p_1 \wedge p_2 \Vdash_{\mathbb{I}}^{V[\mathcal{A}][\mathcal{Q}_q]} x \perp v$.

We use this to show that there must have been an h-splitting that occurred at z. This contradiction yields the result. Without loss of generality we may assume that $q \leq p_1$, $q \leq p_2$, and $\dagger_h$ holds where $h = \text{stem}(q)$. Since $q \in \mathcal{I}$, we may also assume s^* forces $z \in E_h$ by sufficiently extending the Prikry part of s^*. Then, if $s_1' = (a_1, q, r_1)$ and $s_2' = (a_2, q, r_2)$, it follows that q, s_1', s_2', x, v witness an h-splitting at z. However, this is a contradiction because $z \supset a$. $\square$

CHAPTER 5

ITP AT κ^{+N} FOR $N \geq 1$

In this chapter we show that ITP holds at κ^{+n} for each $n \geq 1$ after forcing with $\mathbb{R}_\omega$ defined in Chapter 3. This extends the work of (Unger, 2014) and (Fontanella, 2013) where it was shown that ITP holds at $\aleph_n$ for each $n \geq 2$ after forcing with the Cummings-Foreman iteration. As before, two of the biggest differences in the argument occur when showing that ITP holds at κ^+ and κ^{++} because we use a Prikry poset to make κ singular strong limit.

Although ITP at κ^{++} will be similar to the argument from Subsection 3.5.1, we will now be required to lift our embedding to $j\colon V[\mathbb{R}_\omega] \to M[j(\mathbb{R}_\omega)]$. The reason for this is that ITP involves consideration of $\mathcal{P}_\lambda(\theta)$-lists with arbitrarily large θ, whereas arguing that the tree property holds involves trees with a fixed height. The argument for ITP at κ^+ will be similar to the arguments from Section 3.8 and Chapter 4 where we use the p-term forcing framework from Section 1.7 to define an ineffable branch in $V[\mathbb{R}_\omega]$ for a given thin $\mathcal{P}_{\kappa^+}(\theta)$-list.

5.1 ITP at κ^{++}

Assume that $(\kappa_n\colon n < \omega)$ is an increasing sequence of indestructibly supercompact cardinals, with $\kappa = \kappa_0$, $\kappa_\omega = \sup_n \kappa_n$, and $\mu = \kappa_\omega^+$. Further assume that $(\lambda_n\colon n < \omega)$ is increasing sequence of supercompact cardinals with $\lambda_0 > \mu$, and set $\lambda = \sup_n \lambda_n$. In this section we prove the ITP holds in $V[\mathbb{R}_\omega]$ at $\lambda_0 = \kappa^{++}$.

5.1.1 Lifting the embedding

Let G_ω be $\mathbb{R}_\omega$-generic. Fix an elementary embedding $j\colon V \to M$ with $\text{crit}(j) = \lambda_0$ such that $j(\lambda_0) > \lambda^+$, ${}^{\lambda^+}M \subseteq M$, and where $j(F_0)(\lambda_0)$ is a $\mathbb{Q}_0$-name for $\mathbb{U}_1 \times \mathbb{P}_2 \times \mathbb{U}_2^T \times \mathbb{A}$, where

1. $\mathbb{P}_2$ is the forcing $Add(\lambda_0, \lambda_2)_{V[G_0]}$,
2. $\mathbb{U}_1$ is the forcing $\mathbb{U}$ corresponding to $\mathbb{Q}_1$ defined in Section 1.10,
3. $\mathbb{U}_2^T$ is the forcing $\mathcal{A}(\mathbb{Q}_1, \dot{\mathbb{U}}_2)$,
4. $\mathbb{A}$ is the forcing $\mathcal{A}(\mathbb{Q}_1 * \dot{\mathbb{Q}}_2, \mathbb{R}_\omega/\mathbb{R}_3)$.

Here $\mathcal{A}(\cdot)$ denotes the 1-term forcing given by Definition 1.40. Notice that the $j(F_0)(\lambda_0)$ chosen here includes the factor $\mathbb{U}_2^T \times \mathbb{A}$ which does not appear in Chapter 3. As mentioned at the beginning of the chapter, our goal is to lift the embedding to $j\colon V[G_\omega] \to M[j(G_\omega)]$. The lifting is a combination of work from Chapter 3 and an additional argument appearing in the work of Unger and Fontanella. More specifically, we perform the argument from Subsection 3.5.1 to lift our embedding to $j\colon V[G_0][G_1] \to M[H_0][H_1]$. Further, let h_2 be the generic object for $j(\mathbb{P}_2)$ given by Claim 3.25. Before we continue we include a reference to the following fact about $j(\mathbb{Q}_0)$.

Fact 5.1. *$j(\mathbb{Q}_0)$ collapses all ordinals between μ and $j(\lambda_0)$ to μ.*

Claim 5.2. *There is a generic object x_2 for $j(\mathbb{U}_2)$ over $M[H_0][H_1]$ such that $h_2 \times x_2$ generates a generic filter H_2 for $j(\mathbb{Q}_2)$ over $M[H_0][H_1]$. Further, this filter allows us to lift to $j\colon V[G_0][G_1][G_2] \to M[H_0][H_1][H_2]$.*

Proof. If u_2^T is the generic for $\mathbb{U}_2^T$ induced by G_ω, then u_2^T and $j``u_2^T$ have size $\mu$ in $M[H_0]$. This follows by Fact 5.1 and by the size of $\mathbb{U}_2^T$. Elementarity and Lemma 1.41 imply that $j(\mathbb{U}_2^T)$ is $j(\lambda_1)$-directed closed in $M[H_0]$ and so we may find a lower bound $p \in j(\mathbb{U}_2^T)$ for $j``u_2^T$. Then let x_2^T be generic for $j(\mathbb{U}_2^T)$ containing p. By construction we have that $j``u_2^T \subseteq x_2^T$. Recall that each condition of $\mathbb{U}_2^T$ is a $\mathbb{Q}_1$-name in $V[G_0]$ for an element of $\dot{\mathbb{U}}_2$, and so it follows that that $u_2 = \{\tau[G_1] \colon \tau \in u_2^T\}$ and $x_2 = \{\tau[H_1] \colon \tau \in x_2^T\}$ are generic for $\mathbb{U}_2$ and $j(\mathbb{U}_2)$ over $V[G_0][G_1]$ and $M[H_0][H_1]$, respectively, and further that $j``u_2 \subseteq x_2$.

Let H_2 be the generic filter for $j(\mathbb{Q}_2)$ generated by $h_2 \times x_2$. To see that we may lift to $j \colon V[G_0][G_1][G_2] \to M[H_0][H_1][H_2]$ it is enough to show that $j``G_2 \subseteq H_2$. Towards that end, if $(p, q, f) \in G_2$ then we may find $(p', q', f') \leq (p, q, f)$ such that $p' \in g_2$ and $(0, q', f') \in u_2$. Since $x_2 \times h_2$ induces the filter H_2 it follows that $(j(p'), 0, 0)$ and $(0, j(q'), j(f')$ are both in H_2. This implies that $(j(p'), j(q'), j(f')) = j((p', q', f')) \in H_2$ as well. Since $(p', q', f') \leq (p, q, f)$ it follows as desired that $j``G_2 \subseteq H_2$. $\square$

Claim 5.3. *There is a generic object H_∞ for $j(\mathbb{R}_\omega)/(H_0 * H_1 * H_2)$ over $M[H_0][H_1][H_2]$ that allows us to lift the embedding to $j \colon V[G_\omega] \to M[j(G_\omega)]$.*

Proof. The proof is similar to Claim 5.2. If G_∞^T is the generic for $\mathbb{A}$ induced by G_ω, then Fact 5.1 implies that $j``G_\infty^T$ has size $\mu$ in $M[H_0]$. Elementarity implies that $j(\mathbb{A})$ is $j(\lambda_0)$-directed closed in $M[H_0]$ and so we may find a lower bound $p \in j(\mathbb{A})$ for $j``G_\infty^T$. If H_∞^T is generic for $j(\mathbb{A})$ containing p, then we have that $j``G_\infty^T \subseteq H_\infty^T$. As before this induces a generic object H_∞ for $j(\mathbb{R}_\omega)/(H_0 * H_1 * H_2)$ allowing us to lift to $j \colon V[G_\omega] \to M[j(G_\omega)]$. $\square$

5.1.2 Pulling back the branch

Let $D = (d_x \colon x \in \mathcal{P}_{\lambda_0}(\theta)) \in V[G_\omega]$ be a thin $\mathcal{P}_{\lambda_0}(\theta)$-list and assume that $\theta \geq \lambda_0$. Without loss of generality we may assume that lifted elementary embedding j from the previous section has the further property that $j(\lambda_0) > \theta$. Our goal is to show that D has an ineffable branch in $V[G_\omega]$. As usual, the plan is to argue that D has an ineffable branch in some outer model and then pull this back to $V[G_\omega]$. Towards that end we have the following:

Lemma 5.4. *D has an ineffable branch b in $M[j(G_\omega)]$.*

Proof. As in Proposition 1.60 or Lemma 4.4, we define $b = \{\alpha < \mu \colon j(\alpha) \in j(d)_{j``\mu}\}$. Then, we argue that b is a cofinal branch for D. This will be enough to pull the branch b back to $M[G_\omega]$. Afterwards we check that b is ineffable by showing that the set $S = \{x \in \mathcal{P}_{\lambda_0}(\theta) \colon b \cap x = d_x\}$ is measure one (and therefore stationary). □

Next, we use the closure of j to show that b is actually in an initial segment of $M[j(G_\omega)]$.

Lemma 5.5. *All $< j(\lambda_0)$-sequences in $M[j(G_\omega)]$ are in $M[H_0][h_1]$. In particular, $b \in M[H_0][h_1]$.*

Proof. The proof is similar to Claim 3.32 where instead we use that $j(\mathbb{R}_\omega)/(H_0 * H_1)$ is $< j(\lambda_0)$-distributive in $M[H_0][H_1]$. Then $b \in M[H_0][h_1]$ because b has size $\theta < j(\lambda_0)$ and M is closed under θ-sequences. □

With this in mind, we now want to analyze the forcings to get from $M[G_\omega]$ to $M[H_0][h_1]$. We start with some remarks about the codomain of our lifted embedding. They are analogous to the results in Subsection 3.5.2 so we sketch the proofs.

Fact 5.6. *$M[H_0][h_1] = M[G_0][g_2 \times u_1 \times u_2^T \times G_\infty^T][H_0^*][g_1 \times h_1^*]$, where H_0^* is a generic object for $j(\mathbb{Q}_0)/(\mathbb{Q}_0 * j(F_0)(\lambda_0))$ and h_1^* is a generic object for $j(\mathbb{P}_1)/g_1$.*

Proof. Follows from the definitions of H_0 and h_1. □

Fact 5.7. *$M[G_0][g_2 \times u_1 \times u_2^T \times G_\infty^T][h_1] = M[G_0][g_1 \times u_1 \times g_2 \times u_2^T \times G_\infty^T][h_1^*]$*

Proof. Follows from the definition of h_1. □

Fact 5.8. *$M[H_0][h_1] \subseteq M[G_0][h_1 \times u_1 \times g_2 \times u_2^T \times G_\infty^T][h_0^* \times u_0^*]$, where h_0^* is a generic object for $j(\mathbb{P}_0)/g_0$ and u_0^* is a generic object for $\mathbb{U}^*$ which is μ-closed in $M[G_0]$.*

Proof. Follows from Fact 5.6, Fact 5.7, and Lemma 3.13. □

Fact 5.9. *The forcing $\mathbb{S}$ to go from $M[G_\omega]$ to $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$ is μ-closed and $\leq \mu$-distributive in $M[G_\omega]$.*

Proof. Similar to Fact 3.29, the forcing $\mathbb{S}$ comprises of a product of the quotient forcings defined in Lemma 1.76(7). This forcing remains closed in $M[G_\omega]$ because the forcing $\mathbb{R}_\omega/\mathbb{R}_2$ does not add any $< \mu$-sequences. The distributivity follows by an argument similar to Fact 3.30 that all μ-sequences from $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$ are actually in $M[G_0][g_1]$. □

Fact 5.10. *$j(\mathbb{P}_1)/g_1$ is λ_0-Knaster and $< \mu$-distributive in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$.*

Proof. Similar to the proof of Fact 3.31. □

As in the proof of Proposition 1.60, we show that b is sufficiently approximated in the models to get from $M[G_\omega]$ to $M[H_0][h_1]$ and use this to pull back the branch.

Lemma 5.11. *b is thinly λ_0-approximated in $M[G_\omega]$.*

Proof. Follows from the same argument as Claim 1.63. □

Lemma 5.12. *b is thinly λ_0-approximated in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$,*

Proof. Follows from Fact 5.9 and Lemma 5.11. □

Lemma 5.13. *b is thinly λ_0-approximated in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1]$.*

Proof. Fact 5.10 implies $j(\mathbb{P}_1)/g_1$ is λ_0-cc in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$. So by Lemma 1.18 we have that b is still thinly λ_0-approximated after forcing with $j(\mathbb{P}_1)/g_1$. □

Lemma 5.14. *b is μ-approximated in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1][u_0^*]$.*

Proof. Follows from the same argument as Claim 1.64. □

We are now in the position to pull the branch back to $M[G_\omega]$.

Lemma 5.15. $b \in M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1][u_0^*]$

Proof. We know that $b \in M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1][u_0^*][g_0^*]$ where g_0^* is $j(\mathbb{P}_0)/g_0$-generic. By an argument similar to Claim 3.33 we have that $(j(\mathbb{P}_0)/g_0)^2$ is μ-cc in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1][u_0^*]$. The result follows by Lemma 5.14 since b is μ-approximated in this model. □

Lemma 5.16. $b \in M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1]$

Proof. $\mathbb{U}^*$ has the thin λ_0-approximation property in this model since $2^\kappa = \lambda_0$ and $\mathbb{U}^*$ is μ-closed in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times h_1]$. The result follows by Lemma 5.13 since b is thinly λ_0-approximated in this model. □

Lemma 5.17. $b \in M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$

Proof. $j(\mathbb{P}_1)/g_1$ has the λ_0-approximation in $M[G_0][u_1 \times g_2 \times u_2^T \times G_\infty^T \times g_1]$ by Fact 5.10. The result follows by Lemma 5.12 since b is λ_0-approximated in this model. □

Lemma 5.18. $b \in M[G_\omega]$

Proof. $\mathbb{S}$ has the thin λ_0-approximation property in $M[G_\omega]$ by Fact 5.9 and since $2^\kappa = \lambda_0$. The result follows by Lemma 5.11 since b is thinly λ_0-approximated in this model. □

5.2 ITP at κ^{+n} for $n \geq 3$

The argument when $n \geq 3$ is almost exactly the same as the argument in (Unger, 2014). Similar to Chapter 3, the main difference between our iteration and the iteration in the paper by Unger is the first factor $\mathbb{Q}_0$. Further, one of the main differences in the argument here is dealing with the generic objects h_0^* and u_0^*. When showing that ITP holds at λ_i for $i \geq 1$, in the process of lifting our embedding to $j\colon V[\mathbb{R}_\omega] \to M[j(\mathbb{R}_\omega)]$ we may start by immediately lifting our embedding to $j\colon V[\mathbb{R}_i] \to M[\mathbb{R}_i]$ because $\mathrm{crit}(j) = \lambda_i$ and $|\mathbb{R}_i| = \lambda_{i-1}$. So, since we no longer have to pull a branch from $M[\cdots H_0 \cdots]$ back to $M[\cdots G_0 \cdots]$ we avoid interaction with the generic objects h_0^* and u_0^*.

5.3 ITP at κ^+

When we argued that the tree property holds at κ^+ after forcing with $\mathbb{R}_\omega$, recall that we reduced this to showing that the there are no κ^+-Aronszajn trees in $V[G_0][g_1]$. In this section we need to argue slightly differently because we are working with $\mathcal{P}_\mu(\theta)$-lists where θ can be arbitrarily large. We first argue that $\mathbb{R}_\omega$ is the projection of a Prikry poset and a closed forcing. This allows us to mimic the arguments found in Section 3.8 and Chapter 4.

Lemma 5.19. *There is a forcing $\mathbb{T}$ which is a μ-closed forcing in V such that $\mathbb{R}_\omega$ is the projection of $\mathbb{P}_0 \times \mathbb{U} \times \mathbb{P}_1 \times \mathbb{T}$. Further, we may assume without loss of generality that the projection is the identity map.*

Proof. We may write $\mathbb{R}_\omega = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1 * (\mathbb{R}_\omega \dot{/} \mathbb{R}_2)$. Since $V[\mathbb{Q}_0 * \dot{\mathbb{Q}}_1] \models (\mathbb{R}_\omega/\mathbb{R}_2)$ is μ-closed, Lemma 1.41 implies that $\mathbb{T}_1 = \mathcal{A}(\mathbb{Q}_0 * \dot{\mathbb{Q}}_1, (\mathbb{R}_\omega \dot{/} \mathbb{R}_2))$ is μ-closed in V. Further, facts about term forcing imply that the identity map is a projection from $(\mathbb{Q}_0 * \dot{\mathbb{Q}}_1) \times \mathbb{T}_1$ to $\mathbb{R}_\omega$.

Next, recall that, in $V[\mathbb{Q}_0]$, the identity map is a projection from $\mathbb{P}_1 \times \mathbb{U}_1$ to $\mathbb{Q}_1$, where $\mathbb{P}_1 = Add(\mu, \lambda_1)_V$. Identifying $\mathbb{Q}_0 * (\check{\mathbb{P}}_1 \times \dot{\mathbb{U}}_1)$ with $(\mathbb{Q}_0 \times \mathbb{P}_1) * \dot{\mathbb{U}}_1$, this implies that the identity map is a projection from $(\mathbb{Q}_0 \times \mathbb{P}_1) * \dot{\mathbb{U}}_1$ to $\mathbb{Q}_0 * \dot{\mathbb{Q}}_1$.

Claim 5.20. $V[\mathbb{Q}_0 \times \mathbb{P}_1] \models \mathbb{U}_1$ *is μ-closed.*

Proof. We have by definition that $V[\mathbb{Q}_0] \models \mathbb{U}_1$ is μ^+-closed. Further, we have in $V[\mathbb{Q}_0]$ that $\mathbb{P}_1$ is $< \mu$-distributive. By the Easton's Lemma Variant (Lemma 1.3) using $V[\mathbb{Q}_0]$ as the ground model we have that $V[\mathbb{Q}_0 \times \mathbb{P}_1] \models \mathbb{U}_1$ is μ-closed. $\square$

Again, Lemma 1.41 implies that $\mathbb{T}_0 = \mathcal{A}(\mathbb{Q}_0 \times \mathbb{P}_1, \dot{\mathbb{U}}_1)$ is μ-closed in V and we have that the identity map is a projection from $\mathbb{Q}_0 \times \mathbb{P}_1 \times \mathbb{T}_0$ to $(\mathbb{Q}_0 \times \mathbb{P}_1) * \dot{\mathbb{U}}_1$.

Finally, since there is a projection from $\mathbb{P}_0 \times \mathbb{U}$ to $\mathbb{Q}_0$, by setting $\mathbb{T} = \mathbb{T}_0 \times \mathbb{T}_1$ we have that $\mathbb{R}_\omega$ is the projection of $\mathbb{P}_0 \times \mathbb{U} \times \mathbb{P}_1 \times \mathbb{T}$. Note that $\mathbb{T}$ is μ-closed in V since each term is μ-closed in V. $\square$

This factoring allows us to define variants of the forcings $\mathbb{R}^*$, $\mathbb{Q}$, $\mathbb{R}_{\dot{p}}$, and $\mathbb{Q}_p$ from Chapter 4.

Definition 5.21. Define $\mathbb{Q} = \mathbb{U} \times \mathbb{P}_1 \times \mathbb{T}$ from the previous lemma and set $\mathbb{R}^* = (\mathbb{A} * \dot{\mathbb{I}}) \times \mathbb{Q}$ with the product ordering. Note that $\mathbb{R}^*$ has the same underlying set as $\mathbb{R}_\omega$ but has a different ordering.

Definition 5.22. Let $\dot{p}$ be a name for a condition in $\dot{\mathbb{I}}$. Define $\mathbb{R}_{\dot{p}}$ to have the same underlying set as $\mathbb{R}^*$ with the following ordering: Declare that $(f_1, \dot{p}_1, r_1, g_1, a_1, b_1) \leq_{\dot{p}} (f_2, \dot{p}_2, r_2, g_2, a_2, b_2)$ exactly when

1. $a_1 \leq_{\mathbb{P}_1} a_2$
2. $b_1 \leq_{\mathbb{T}} b_2$
3. $(f_1, \dot{p}_1) \leq_{\mathbb{P}_0} (f_2, \dot{p}_2)$
4. $\mathrm{dom}(r_1) \supseteq \mathrm{dom}(r_2)$ and for every $\alpha \in \mathrm{dom}(r_2)$, we have that

   $(f_1, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} r_1(\alpha) \leq r_2(\alpha)$
5. $\mathrm{dom}(g_1) \supseteq \mathrm{dom}(g_2)$ and for every $\alpha \in \mathrm{dom}(g_2)$, we have that

   $(f_1, \dot{p}, r_1, g_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} g_1(\alpha) \leq g_2(\alpha)$.

Definition 5.23. Let $\mathcal{A}$ be $\mathbb{A}$-generic over V. Let $p = \dot{p}_{\mathcal{A}}$. Define $\mathbb{Q}_p$ to have the same underlying set as $\mathbb{Q}$ with the ordering $(0, 0, q_1, f_1, a_1, b_1) \leq_{\mathbb{Q}_p} (0, 0, q_2, f_2, a_2, b_2)$ exactly when

1. $a_1 \leq_{\mathbb{P}_1} a_2$
2. $b_1 \leq_{\mathbb{T}} b_2$
3. $\mathrm{dom}(q_1) \supseteq \mathrm{dom}(q_2)$ and $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_2)$

4. there is an $a \in \mathcal{A}$ such that for every $\alpha \in \mathrm{dom}(q_2)$ and for every $\alpha \in \mathrm{dom}(f_2)$, we have that

   (a) $(a, \dot{p}) \restriction \alpha \Vdash_{\mathbb{P}_{0,\alpha}} q_1(\alpha) \leq q_2(\alpha)$

   (b) $(a, \dot{p}, q_1, f_1) \restriction \alpha \Vdash_{\mathbb{Q}_0 \restriction \alpha} f_1(\alpha) \leq f_2(\alpha)$

Lemma 5.24 (Structural Lemma)**.** *The following relationships hold between $\mathbb{R}_\omega$, $\mathbb{R}^*$, $\mathbb{R}_{\dot{p}}$, $\mathbb{Q}$, and $\mathbb{Q}_p$:*

1. *The identity map is a projection from $\mathbb{R}^*$ to $\mathbb{R}_{\dot{p}}$.*
2. *If $s^* = (0, \dot{p}, 0, 0, 0, 0) \in \mathbb{R}_\omega$, then the identity map is a projection from $\mathbb{R}_{\dot{p}}/s^*$ to $\mathbb{R}_\omega/s^*$, where $\mathbb{R}_{\dot{p}}/s^* = \{s \in \mathbb{R}_{\dot{p}} \mid s \leq_{\dot{p}} s^*\}$ and $\mathbb{R}_\omega/s^* = \{s \in \mathbb{R}_\omega \mid s \leq_{\mathbb{R}_\omega} s^*\}$*
3. *$\mathbb{Q}_p$ is κ-closed.*
4. *$\mathbb{R}_{\dot{p}}$ is isomorphic to $\mathbb{A} * (\dot{\mathbb{I}} \times \dot{\mathbb{Q}}_p)$*
5. *$\mathbb{Q}$ projects onto $\mathbb{Q}_p$ and if $q \leq p$ then $\mathbb{Q}_p$ projects to $\mathbb{Q}_q$.*

Theorem 5.25. *After forcing with $\mathbb{R}_\omega$, ITP holds at κ^+.*

Proof. The goal is to perform the argument from Chapter 4 showing that ITP holds at κ^+. In other words, take an $\mathbb{I}$-name $\dot{D} \in V[G_\omega] \subseteq V[\mathcal{A} \times \mathcal{Q}]$ for a $\mathcal{P}_\mu(\theta)$-list where $\mathcal{A}$ is generic for $\mathbb{A}$ and $\mathcal{Q}$ is generic for $\mathbb{Q}$. We have that there is an ineffable branch in $V[\mathcal{A} \times \mathcal{Q}][\mathcal{I}]$ where $\mathcal{I}$ is generic for the Diagonal Prikry forcing in $V[\mathcal{A} \times \mathcal{Q}]$. It follows by Lemma 5.24 that we are in exactly the same situation as in Chapter 4. So, the same arguments allow us to pull the branch back to $V[G_\omega]$. Importantly, in the proof of Lemma 4.17 it is necessary that $\mathbb{Q}$ is μ-closed in V and that $\mathbb{Q}$ is $< \mu$-distributive and countably closed in $V[\mathcal{A}]$. $\square$

CHAPTER 6

ITP AT κ^+ AND κ^{++} WITH A FORCING SUITABLE FOR COLLAPSES

In this chapter we show - with a forcing different from Chapter 4 - that we can force to get ITP at κ^+ and κ^{++}. In particular this forcing no longer interleaves the Prikry forcing with the Mitchell forcing. One of the advantages of this approach is that it can be more suitable to making κ small. The reason is that it is easier to add collapses between the Prikry points after the gap between κ and 2^κ becomes accessible.

Additionally, we give an alternative argument to get the strong tree property at κ^{++}. Although this is weaker than ITP at κ^{++}, the author thinks that this approach will be more suitable to add collapses to get the ITP simultaneously at an ω-sequence of cardinals. This argument is similar in spirit to (Cummings et al., 2020b), avoiding the use of a technical branch preservation lemma presented in the next section.

As before assume in V that $(\kappa_n : n < \omega)$ is an increasing sequence of (indestructibly) supercompact cardinals with $\kappa_\omega = \sup_n \kappa_n$, $\kappa = \kappa_0$, and $\mu = \kappa_\omega^+$. Assume that $\lambda > \mu$ is supercompact. Let $\mathbb{M}(\kappa, \mu, \lambda)$ be the Mitchell forcing from Section 1.8 and G be a generic object for $\mathbb{M}(\kappa, \mu, \lambda)$. Further, in $V[G]$ assume $\mathbb{P}$ is the diagonal Prikry forcing from Section 1.9 and H is $\mathbb{P}$-generic over $V[G]$.

6.1 Strong Tree Property at κ^{++}

In this section we give an argument to show that the strong tree property holds at κ^{++}. Towards that end, note that since $\mathbb{P}$ is μ-cc it is enough to assume that $\mathcal{P}_\lambda(\theta) = \mathcal{P}_\lambda(\theta)^{V[G]}$. So, working in $V[G]$ let $(\dot{d}_x \colon x \in \mathcal{P}_\lambda(\theta))$ be a $\mathbb{P}$-name for a thin $\mathcal{P}_\lambda(\theta)$-list and for each $x \in \mathcal{P}_\lambda(\theta)$ let $(\dot{\sigma}^x_\alpha \colon \alpha < \mu)$ be a $\mathbb{P}$-name for the set $\{d_y \cap x \colon y \supseteq x\}$.

Let $j \colon V \to M$ be a θ-supercompactness embedding and lift this to $j \colon V[G] \to M[G^*]$. We assume that j is defined in $V[G][K \times A]$ for a $\mathbb{Q} \times \mathbb{A}$-generic filter $K \times A$. Here the poset $\mathbb{Q}$ is the usual μ-closed forcing and $\mathbb{A}$ is the κ^+-Knaster Cohen forcing.

Lemma 6.1. *In $V[G]$ there is a cofinal $J \subseteq \mathcal{P}_\lambda(\theta)$, a function $f \colon J \to \mu$, and a stem h such that for $x \subseteq y$ in J,*

$$h \Vdash^* \dot{\sigma}^x_{f(x)} = \dot{\sigma}^y_{f(y)} \restriction x.$$

Proof. Note that $j``\theta \in j(\mathcal{P}_\lambda(\theta))$ and so elementarity implies that for each $x \in \mathcal{P}_\lambda(\theta)$ there is a $p_x \in j(\mathbb{P})$ and $\xi_x < \mu$ such that $p_x \Vdash j(\dot{d})_{j``\theta} \restriction j(x) = j(\dot{\sigma})^{j(x)}_{\xi_x}$. Notice that in $M[G^*]$ we have that μ is preserved and both λ and θ are collapsed to have cofinality μ. Since there are κ_ω-many stems it follows there is a cofinal $J \subseteq \mathcal{P}_\lambda(\theta)$ and stem $\overline{h}$ such that $x \in J$ if and only if $\overline{h} \Vdash^* j(\dot{d})_{j``\theta} \restriction j(x) = j(\dot{\sigma})^{j(x)}_{\xi_x}$. Further, define $f \colon J \to \mu$ by $f(x) = \xi_x$.

Claim 6.2. $J, f \in V[G][K]$

Proof of Claim. Assume that $d \subseteq \mathcal{P}_\lambda(\theta)$ with $|d| < \mu$ in $V[G]$. Set $d' = J \cap d$. We need to show that $f \restriction d' \in V[G]$. To see this is enough, notice that $\mathbb{A}$ has the μ-approximation property in $V[G][K]$ and so we would have $f \in V[G][K]$ and further that $\mathrm{dom}(f) = J \in V[G][K]$. Now,

μ is regular in $V[G]$ and so $\bigcup d \in \mathcal{P}_\lambda(\theta)^{V[G]}$. Since J is a cofinal subset of $\mathcal{P}_\lambda(\theta)$, there is a $z \supseteq \bigcup d$ with $z \in J$. Then elementarity implies that $(f \restriction d')(x) = \xi$ if and only if $x \in d$ and $\overline{h} \Vdash^* \dot{\sigma}^x_\xi = \dot{\sigma}^z_{f(z)} \restriction x$. So $f \restriction d'$ is definable from $d, f(z), \overline{h} \in V[G]$ and the relation $\Vdash^*$ and so $f \restriction d' \in V[G]$. $\square$

Claim 6.3. *For each $x \in \mathcal{P}_\lambda(\theta)$, if $D_x = \{y \colon y \subseteq x\}$, then $J \restriction D_x, f \restriction D_x \in V[G]$.*

Proof of Claim. Since $J \subseteq \mathcal{P}_\lambda(\theta)$ is cofinal, fix $z \supseteq x$ with $z \in J$. For $y \subseteq x$, we have that $y \in J$ if and only if there is (a unique) $\xi < \mu$ such that $\overline{h} \Vdash^* \dot{\sigma}^y_\xi = \dot{\sigma}^z_{f(z)} \restriction y$. This implies that $J \restriction D_x, f \restriction D_x \in V[G]$ as desired. $\square$

Working in $V[G][K][A]$, define $J_h = \{x \in J \colon \exists \xi < \mu, h \Vdash^* j(\dot{d})_{j``\theta} \restriction j(x) = j(\dot{\sigma})^{j(x)}_\xi\}$ for each stem $h \sqsupseteq \overline{h}$. Similarly, for $x \in J_h$, define $f_h(x)$ to be the unique ξ witnessing this. Like above, if J_h is a cofinal subset of $\mathcal{P}_\lambda(\theta)$ then both $J_h, f_h \in V[G][K]$. Since there are κ_ω-many stems, we may find a $\overline{z} \in \mathcal{P}_\lambda(\theta)$ such that if J_h is not cofinal then there is no $z \in J_h$ such that $\overline{z} \subseteq z$. In $V[G][K]$ let $\dot{J}_h$ and $\dot{f}_h$ be $\mathbb{A}$-names for J_h and f_h. If $\mathcal{C}_h$ is all the possible values for $\dot{J}_h$, then it follows that $|\mathcal{C}_h| \leq \kappa$. For each cofinal J_h, let $(C_{h,i} \colon i < \kappa)$ be an enumeration of $\mathcal{C}_h$. Further, for each h and $i < \kappa$ let $a_{h,i} \in \mathbb{A}$ force that $C_{h,i} = \dot{J}_h$ and define $f_{h,i} \colon C_{h,i} \to \mu$ by letting $f_{h,i}(x)$ be the unique ξ such that $a_{h,i} \Vdash^{V[G][K]}_{\mathbb{A}} \left(h \Vdash^* j(\dot{d})_{j``\theta} \restriction j(x) = j(\dot{\sigma})^{j(x)}_\xi\right)$.

In $V[G]$, for each stem h let $\dot{\mathcal{C}}_h$ be $\mathbb{Q}$-names for the $\mathcal{C}_h$. For each h such that $\dot{C}_h \neq \varnothing$, fix $\mathbb{Q}$-names $\dot{C}_{h,i}$ and $\dot{f}_{h,i}$ for each $i < \kappa$. Assume towards a contradiction that $\Vdash_{\mathbb{Q}} (\dot{C}_{h,i}, \dot{f}_{h,i}) \notin V[G]$ for each h and $i < \kappa$.

Write q_1, q_2 <u>force contradictory information about $\dot{f}_{h,i}(x)$</u> if they decide $x \in \dot{C}_{h,i}$, one of the conditions forces that the statement is true, and $(q_1, q_2) \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{f}_{h,i}[\Gamma_L](x) \neq \dot{f}_{h,i}[\Gamma_R](x)$.

Claim 6.4 (Splitting)**.** *In $V[G]$, if $q_1, q_2 \in \mathbb{Q}$, $h \sqsupseteq \overline{h}$, $i < \kappa$, and $x \in \mathcal{P}_\lambda(\theta)$, then there are $y \supseteq x$ and $(q_1', q_2') \leq (q_1, q_2)$ such that q_1', q_2' force contradictory information about $\dot{f}_{h,i}(y)$.*

Proof of Claim. Assume that (q_1, q_2), $\dot{f}_{h,i}$, and x witness that the claim is false. We claim that $(q_1, q_2) \Vdash \dot{f}_{h,i}[\Gamma_L] = \dot{f}_{h,i}[\Gamma_R]$. If not, find a y and $(q_1', q_2') \leq (q_1, q_2)$ such that $(q_1', q_2') \Vdash \dot{f}_{h,i}[\Gamma_L](y) \neq \dot{f}_{h,i}[\Gamma_R](y)$. By hypothesis, we may find $(q_1'', q_2'') \leq (q_1', q_2')$, $z \supseteq x \cup y$, and $\xi_y^1, \xi_y^2, \xi_z < \mu$ such that both q_1'', q_2'' decide $z \in \dot{C}_{h,i}$, $(q_1'', q_2'') \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{f}_{h,i}[\Gamma_L](z) = \xi_z = \dot{f}_{h,i}[\Gamma_R](z)$, and $(q_1'', q_2'') \Vdash_{\mathbb{Q}\times\mathbb{Q}} \dot{f}_{h,i}[\Gamma_L](y) = \xi_y^1 \neq \xi_y^2 = \dot{f}_{h,i}[\Gamma_R](y)$. For $j \in \{1, 2\}$, since $y \subseteq z$ we have that $q_j'' \Vdash \dot{f}_{h,i}(y) = \delta$ if and only if $h \Vdash^* \sigma_\delta^y = \sigma_{\xi_z}^z \restriction y$. However, this implies that $\xi_y^1 = \xi_y^2$, a contradiction. $\square$

In $V[G]$, for each $h \sqsupseteq \overline{h}$ and $i < \kappa$, use the splitting to construct $((q_\sigma, x_{\sigma,h,i}) \colon \sigma \in 2^{<\kappa}, h \sqsupseteq \overline{h}, i < \kappa)$ where $q_{\sigma\frown 0}$ and $q_{\sigma\frown 1}$ force contradictory information about $\dot{f}_{h,i}(x_{\sigma,h,i})$. Set $x^* = \bigcup_{\sigma,h,i} x_{\sigma,h,i} \in \mathcal{P}_\lambda(\theta)$. For $g \in 2^\kappa$, use the closure of $\mathbb{Q}$ to find a lower bound $p_g \leq q_{g\restriction i}$ for each $i < \kappa$. Then choose $q_g' \leq p_g$, $h_g \sqsupseteq \overline{h}$, $i_g < \kappa$, and $\xi_g < \mu$ such that

1. $q_g' \Vdash x^* \in \dot{C}_{h_g,i_g}$,
2. for $x \subseteq x^*$, $q_g' \Vdash \dot{f}_{h_g,i_g}(x) = \delta$ if and only if $h_g \Vdash^* \sigma_\delta^x = \sigma_{\xi_g}^{x^*} \restriction x$.

Since $2^\kappa > \mu$, the contradiction comes after observing that there are distinct $f, g \in 2^\kappa$ such that $h_f = h_g$, $i_f = i_g$, and $\xi_f = \xi_g$. $\square$

Fix J, f, and the stem h from the previous lemma. Assume that $|h| = n$. Arguments found in (Neeman, 2009) or (Cummings et al., 2020a) give us the following two results:

Lemma 6.5. *In $V[G]$ there are $z \in \mathcal{P}_\lambda(\theta)$ and U_n-measure one sets A_x for $x \in J \cap \{y \colon y \supseteq z\}$ such that for $z \subseteq x \subseteq y$, both $x, y \in J$, and for all $s \in A_x \cap A_y$,*

$$h^\frown s \Vdash^* \dot{\sigma}^x_{f(x)} = \dot{\sigma}^y_{f(y)} \restriction x.$$

Lemma 6.6. *In $V[G]$ there are $z^* \in \mathcal{P}_\lambda(\theta)$ and conditions p_x for $x \in J \cap \{y \colon y \supseteq z^*\}$ with stem h such that for $z^* \subseteq x \subseteq y$, both $x, y \in J$,*

$$p_x \wedge p_y \Vdash \dot{\sigma}^x_{f(x)} = \dot{\sigma}^y_{f(y)} \restriction x.$$

Lemma 6.7. *In $V[G][H]$ if $I = \{x \colon p_x \in H\}$ is cofinal subset of $\mathcal{P}_\lambda(\theta)$, then there is a cofinal branch through our list.*

Proof. Working in $V[G][H]$, for each $x \in I$ let $g(x) \supseteq x$ be such that $\dot{\sigma}^x_{f(x)}[H] = d_{g(x)} \cap x$ and define $b = \bigcup_{x \in I}(d_{g(x)} \cap x)$. To see that this is a cofinal branch, fix $y \in \mathcal{P}_\lambda(\theta)$ and use the hypothesis to find $z \supseteq y$ such that $z \in I$. It is enough to show that $b \cap z = d_{g(z)} \cap z$ as this implies that $b \cap y = (b \cap z) \cap y = d_{g(z)} \cap y$. By definition $d_{g(z)} \cap z \subseteq b \cap z$. Conversely, if $\alpha \in b \cap z$, fix some $x \in I$ such that $\alpha \in d_{g(x)} \cap x$. Since I is cofinal, fix $w \in I$ such that $w \supseteq g(x) \cup g(z) \supseteq x \cup z$. By the previous lemma, we have that $(d_{g(w)} \cap w) \cap x = d_{g(x)} \cap x$ and

$(d_{g(w)} \cap w) \cap z = d_{g(z)} \cap z$. The first equality implies that $\alpha \in d_{g(w)}$. Since $\alpha \in z$, then second equality implies that $\alpha \in d_{g(z)} \cap z$ as desired. $\square$

Lemma 6.8. *In $V[G][H]$ the list $(d_x : x \in \mathcal{P}_\lambda(\theta))$ has a cofinal branch.*

Proof. Assume otherwise and fix $q \in H$ forcing this. Without loss of generality we may assume that the stem of q is h. By the previous lemma we have that $q \Vdash$ "$\dot{I} = \{x : p_x \in \Gamma\}$ is not cofinal." Since $\mathbb{P}$ is μ-cc, we may find a $z \in \mathcal{P}_\lambda(\theta)$ such that $q \Vdash$ "$z \not\subseteq x$ for each $x \in \dot{I}$." Since $J \cap \{y : y \supseteq z^*\}$ is cofinal, we may find an $x \in J \cap \{y : y \supseteq z^*\}$ such that $x \supseteq z$. Since q and p_x both have stem h, we know that $q \wedge p_x$ exists. However, $q \wedge p_x \Vdash p_x \notin \Gamma$ because $q \Vdash x \notin \dot{I}$, whereas $q \wedge p_x \Vdash p_x \in \Gamma$ because $p_x \Vdash p_x \in \Gamma$. This contradiction yields the result. $\square$

6.2 ITP at κ^{++}

In this section we prove that ITP holds at λ in $V[G][H]$. Similar to Proposition 1.60 we will take an elementary embedding, lift it to $j : V[\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}}] \to M[j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})]$ and then pull the branch back to $V[G][H]$.

Towards that end, working in $V[G][H]$ let $\theta \geq \lambda$ and let D be a thin $\mathcal{P}_\lambda(\theta)$-list. Let $j : V \to M$ be a θ-supercompactness embedding with critical point λ. Since the forcing $\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}}$ is λ-cc we may lift the embedding to $j : V[G][H] \to M[G^*][H^*]$ for some $j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})$-generic object $G^* * H^*$.

Lemma 6.9. *D has an ineffable branch $b \in M[G^*][H^*]$.*

Proof. The same argument as Claim 1.62. $\square$

Lemma 6.10. *b is λ-approximated in $M[G]$.*

Proof. The same argument as Claim 1.63. □

Lemma 6.11. *b is λ-approximated in $M[G][H]$.*

Proof. Since $\mathbb{P}$ is μ-cc in $M[G]$ the result follows from Lemma 1.18 and the previous result. □

The difficulty here is that the quotient forcing $j(\mathbb{M}(\kappa,\mu,\lambda) * \dot{\mathbb{P}})/G * H$ does not factor as nicely as the quotient of the Mitchell forcing. However, we can show that the forcing $j(\mathbb{M}(\kappa,\mu,\lambda) * \dot{\mathbb{P}})/G * H$ is λ-approximated. By Lemma 6.11 this shows that $b \in M[G][H]$ as desired. The following argument is originally from (Sinapova and Unger, 2018) and is presented for completeness. We start two definitions this paper that will help decide whether a condition may be forced into the quotient.

Definition 6.12. Assume that s' is a stem. We say <u>s' is compatible with $r \in \mathbb{P}$</u> if there is a $r' \leq r$ with $\text{stem}(r') = s'$.

Definition 6.13. Assume that $r \in \mathbb{P}$, $\text{stem}(r) = s$, and s' is a stem with $s' \sqsupseteq s$. Then we say <u>points in s' above s are constrained by r</u> if $r' \leq r$ for some $r' \in \mathbb{P}$ with $\text{stem}(r') = s'$.

The following originally appeared in (Sinapova and Unger, 2018).

Lemma 6.14. *In $M[G]$ assume that $\overline{r} \in \mathbb{P}$, $m \in j(\mathbb{M}(\kappa,\mu,\lambda))/G$, and $\dot{r}$ is forced into $j(\mathbb{P})$. Further, assume*

1. *m decides stem($\dot{r}$),*

2. $stem(\overline{r}) \sqsupseteq stem(\dot{r})$, and

3. $m \Vdash$ "points in $stem(\overline{r})$ above $stem(\dot{r})$ are constrained by $\dot{r}$.

Then there is a $\overline{r}' \leq^* \overline{r}$ such that $\overline{r}'$ forces $(m, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * H$.

We now are in a position to prove the following:

Lemma 6.15. *$j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * H$ is λ-approximated in $M[G][H]$.*

Proof. Assume that $d \in \mathcal{P}_\lambda(\theta)^{M[G^*][H^*]}$ and that $d \cap x \in M[G][H]$ for each $x \in \mathcal{P}_\lambda(\theta)^{M[G][H]}$. Assume instead that $\Vdash \dot{d} \notin M[G][H]$, where $\dot{d}$ is a name for d. Recall that $\mathbb{M}(\kappa, \mu, \lambda)$ has the underlying set $\mathbb{A} \times \mathbb{Q}$ where $\mathbb{A} = Add(\kappa, \lambda)$ and $\mathbb{Q}$ is μ-closed in V. In $M[G]$ let $\mathbb{Q}^*$ be μ-closed forcing given by Lemma 1.59.

We start with a modification of Lemma 6.14.

Claim 6.16. *In $M[G]$ assume that $\overline{r} \in \mathbb{P}$ and $(p, f, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda))/G$. Further, assume that*

1. (p, f) decides $stem(\dot{r})$,

2. $stem(\overline{r}) \sqsupseteq stem(\dot{r})$,

3. $\overline{r} \Vdash (p, f, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$, and

4. $(p, f) \Vdash$ "points in $stem(\overline{r})$ above $stem(\dot{r})$ are constrained by $\dot{r}$."

Then there is a $\overline{r}' \leq^* \overline{r}$ and $f' \leq_{j(\mathbb{Q})} f$ such that

1. $\overline{r}' \Vdash (p, f', \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$,

2. for each h compatible with $\overline{r}'$, we may find a $p' \in \mathbb{A}$ with $p' \leq p$ and $(p', f') \Vdash$ "points in h above $stem(\dot{r})$ are constrained by $\dot{r}$."

Proof of Claim. Let $(h_i \colon i < \kappa_\omega)$ be an enumeration of the stems extending $\mathrm{stem}(\overline{r})$. We construct a decreasing sequence $(f_i \colon i < \kappa_\omega)$ below f of elements of $\mathbb{Q}^*$ in the following way: given $(f_i \colon i < \alpha)$ for $\alpha < \kappa_\omega$, use that $\mathbb{Q}^*$ is μ-closed to find a lower bound f' for this sequence. If $(p, f') \Vdash h_\alpha$ is incompatible with $\dot{r}$, then let $f_\alpha = f'$. Otherwise, we may find a $f_\alpha \in \mathbb{Q}^*$ and $p' \leq p$ such that $(p', f_\alpha) \Vdash h_\alpha$ is compatible with $\dot{r}$. Note that this implies that $(p', f_\alpha) \Vdash$ "points in h_α above $\mathrm{stem}(\dot{r})$ are constrained by $\dot{r}$." This completes the construction.

Now let $f' \in \mathbb{Q}^*$ be a lower bound for the sequence $(f_i \colon i < \kappa_\omega)$. By hypothesis and by Lemma 6.14 we may find a $\overline{r}' \leq^* \overline{r}$ forcing $(p, f', \dot{r})$ into the quotient. The second part of the conclusion of the claim follows by the construction of the sequence $(f_i \colon i < \kappa_\omega)$. $\square$

Using this we prove the following key claim:

Claim 6.17. *In $M[G][H]$ there is $(p, f, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * H$ such that, for each*

1. $x \in \mathcal{P}_\lambda(\theta)^{M[G][H]}$,
2. $y \subseteq x$ *also in $M[G][H]$, and*
3. $(p', f', \dot{r}')$ *extending $(p, f, \dot{r})$ in $j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * H$,*

if

1. $f' \leq f$ *and*
2. $(p', f', \dot{r}') \Vdash \dot{d} \cap x = y$,

then we can find $f'' \leq f'$ where $(p, f'', \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * H$ and $(p, f'', \dot{r}) \Vdash \dot{d} \cap x = y$ as well.

Proof of Claim. Assume the claim is false. The idea is to recursively define a sequence of terms. This will be done by repeatedly applying the following subclaim:

Subclaim 6.18. *In $M[G]$ suppose that $\overline{r}$ forces that Claim 6.17 is false. Further assume $\overline{r}$ forces $(p, f, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$. Then for $i < 2$ there are terms $\overline{r}^* \leq \overline{r}$, $p_i, f^*, \dot{r}_i, \dot{x}, \dot{y}_i$ such that*

1. *$f^* \leq_{j(\mathbb{Q})} f'$,*
2. *$stem(\overline{r}^*) = stem(\dot{r}_1) \sqsupseteq stem(\dot{r}_0)$,*
3. *$(p_0, f^*) \Vdash$ "points in $stem(\overline{r}^*)$ above $stem(\dot{r}_0)$ are constrained by $\dot{r}_0$,"*
4. *$(p_i, f^*, \dot{r}_i)$ extends $(p, f^*, \dot{r})$ for each i,*
5. *$\overline{r}^*$ forces*
   (a) *$(p_i, f^*, \dot{r}_i) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$ for each i,*
   (b) *$(p_i, f^*, \dot{r}_i) \Vdash \dot{d} \cap \dot{x} = \dot{y}_i$ for each i,*
   (c) *$\dot{y}_0 \neq \dot{y}_1$.*

Proof of Subclaim. Since we assume that Claim 6.17 is false, there is a condition $(p', f', \dot{r}_0)$, $f' \leq f$, names $\dot{x}, \dot{y}_0$, and $\overline{r}'$ extending $\overline{r}$ such that $\overline{r}'$ forces

1. $(p', f', \dot{r}_0) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$ and extends $(p, f, \dot{r})$,
2. $(p', f', \dot{r}_0) \Vdash \dot{d} \cap \dot{x} = \dot{y}_0$, and
3. for any $f^* \leq f'$, if $(p, f^*, \dot{r}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$ then $(p, f^*, \dot{r})$ does not force $\dot{d} \cap \dot{x} = \dot{y}_0$.

Without loss of generality we can assume $\mathrm{stem}(\overline{r}') \sqsupseteq \mathrm{stem}(\dot{r}_0)$. Then, let $(p_0, f_0) \leq (p', f')$ where $(p_0, f_0) \Vdash$ "points in $\mathrm{stem}(\overline{r})$ above $\mathrm{stem}(\dot{r}_0)$ are constrained by $\dot{r}_0$." Lemma 6.14 implies that we can find $\overline{r}'' \leq^* \overline{r}'$ where $\overline{r}'' \Vdash (p_0, f_0, \dot{r}_0) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$. Applying Claim 6.16 to $\overline{r}''$ and $(p, f_0, \dot{r})$, we can find $\overline{r}''' \leq^* \overline{r}''$ and $f'' \leq f_0$ such that

1. $\overline{r}''' \Vdash (p, f'', \dot{r}_0) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$, and
2. for each h compatible with $\overline{r}'''$, we may find a $p' \in \mathbb{A}$ with $p' \leq p$ and $(p', f'') \Vdash$ "points in h above $\mathrm{stem}(\dot{r}_0)$ are constrained by $\dot{r}_0$."

Then, let $\overline{r}^* \leq \overline{r}'''$, p_1, $f^* \leq f''$, and $\dot{y}_1$ be such that $\overline{r}^*$ forces

1. $(p_1, f^*, \dot{r}_1) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$ and extends $(p, f'', \dot{r})$,
2. $(p_1, f^*, \dot{r}_1) \Vdash \dot{d} \cap \dot{x} = \dot{y}_1$,
3. $\dot{y}_0 \neq \dot{y}_1$.

It is not difficult to check that $\overline{r}^*$, p_i, f^*, $\dot{r}_i$, $\dot{x}$, and $\dot{y}_i$ are as desired. $\square$

Now, working in $M[G]$ we may fix $\overline{r} \in \mathbb{P}$ forcing that Claim 6.17 is false. Using our subclaim, we define by recursion a sequence of terms $(\overline{r}_\alpha, p_{\alpha,i}, f_\alpha, \dot{r}_{\alpha,i}, \dot{x}_\alpha, \dot{y}_{\alpha,i}, \dot{y}_\alpha, : \alpha < \mu, i < 2)$ so that $(f_\alpha \colon \alpha < \mu)$ is a decreasing sequence of conditions in $j(\mathbb{Q})$, and $\overline{r}_\alpha \in \mathbb{P}$ forces the following:

1. if $\beta < \gamma < \alpha$, then $\dot{x}_\beta \in \mathcal{P}_\lambda(\theta)$ and $\dot{x}_\beta \subseteq \dot{x}_\gamma$,
2. $(p_{\alpha,i}, f_\alpha, \dot{r}_{\alpha,i}) \in j(\mathbb{M}(\kappa, \mu, \lambda) * \dot{\mathbb{P}})/G * \dot{H}$,
3. $(p_{\alpha,i}, f_\alpha, \dot{r}_{\alpha,i}) \Vdash \dot{d} \cap \bigcup_{\beta<\alpha} \dot{x}_\beta = \dot{y}_\alpha$,
4. $(p_{\alpha,i}, f_\alpha, \dot{r}_{\alpha,i}) \Vdash \dot{d} \cap \dot{x}_\alpha = \dot{y}_{\alpha,i}$,

5. $\dot{y}_{\alpha,0} \neq \dot{y}_{\alpha,1}$,

6. $(p_{\alpha,i}, f_\alpha)$ decides $\text{stem}(\dot{r}_{\alpha,i})$ and $\text{stem}(\overline{r}_\alpha)$ extends $\text{stem}(\dot{r}_{\alpha,i})$.

Since $\mathbb{A}^2$ is μ-cc, it follows that we can find $\alpha < \alpha' < \mu$ such that

1. $\text{stem}(\overline{r}_\alpha) = \text{stem}(\overline{r}_{\alpha'})$

2. $\text{stem}(\dot{r}_{\alpha,0}) = \text{stem}(\dot{r}_{\alpha',0})$ and $\text{stem}(\dot{r}_{\alpha,1}) = \text{stem}(\dot{r}_{\alpha',1})$

3. $p_{\alpha,i}$ is compatible with $p_{\alpha',i}$ for $i < 2$.

Next, for $i < 2$ let $q_i \leq p_{\alpha,i}, p_{\alpha',i}$ witness this and let $\dot{r}_i$ name the weakest extension of $\dot{r}_{\alpha,i}$ and $\dot{r}_{\alpha',i}$. Using this we may directly extend $\overline{r}_\alpha$ and $\overline{r}_{\alpha'}$ to a condition r forcing that $(q_i, f_{\alpha'}, \dot{r}_i) \in j(\mathbb{M}(\kappa,\mu,\lambda) * \dot{\mathbb{P}})/G * \dot{H}$ for $i < 2$. Finally, fix a generic H' containing r. In $M[G][H']$, for each $i < 2$ we have that $y_{\alpha',i} \cap x_\alpha = y_{\alpha,i}$. However, we also have for $i < 2$ that $y_{\alpha',i} \cap \bigcup_{\beta<\alpha'} x_\beta = y_{\alpha'}$. This implies that $y_{\alpha,0} = y_{\alpha,1}$, a contradiction. $\square$

Now back to the proof of Lemma 6.15. Given our assumptions about d, we know the set $\mathbb{D}$ of pairs in $\left[j(\mathbb{M}(\kappa,\mu,\lambda) * \dot{\mathbb{P}})/G * H\right]^2$ deciding different values of $\dot{d} \cap x$ for some $x \in \mathcal{P}_\lambda(\theta)^{M[G][H]}$ is dense. Let $(p, f, \dot{r}) \in M$ be given by Claim 6.17. It follows that, not only is the set $\mathbb{D}$ below $(p, f, \dot{r})$ dense, but we may find a condition in $\mathbb{D}$ by only extending the second coordinate. We further note that any such extension also satisfies Claim 6.17.

Working in $M[G]$ let $\overline{r} \in \mathbb{P}$ such that $\overline{r} \Vdash^{M[G]}_{\mathbb{P}}$ "$(p, f, \dot{r})$ satisfies Claim 6.17." Construct a sequence $(f_s, A_s \colon s \in 2^{<\kappa})$ such that the following holds:

1. $f_s \in j(\mathbb{Q})$ for each s,

2. A_s is a maximal antichain below $\overline{r}$,

3. if $\sigma_0 \subset \sigma_1$ then $f_{\sigma_1} \leq f_{\sigma_0}$, and

4. for each $r' \in A_s$, there is a $x_{r',s} \in \mathcal{P}_\lambda(\theta)$ such that $r' \Vdash$ "$(p, f_{s^\frown 0}, \dot{r})$ and $(p, f_{s^\frown 1}, \dot{r})$ decide different values of $\dot{d} \cap x_{r',s}$."

Then, let $x = \bigcup\{x_{r',s} \colon s \in 2^{<\kappa}, r' \in A_s\} \in \mathcal{P}_\lambda(\theta)$. Recall that the quotient $j(\mathbb{M}(\kappa,\mu,\lambda))/G$ projects onto $Add(\kappa,1)$ and so we may take a name $\dot{c}$ for this Cohen subset of κ added by $j(\mathbb{M}(\kappa,\mu,\lambda))/G$. (We view $\dot{c}$ as a function from κ to 2.) By definition of the f_s's we know that $(f_{\dot{c}\restriction i} \colon i < \kappa)$ is a decreasing sequence in $M[G][Add(\kappa,1)]$. Using the chain condition of $Add(\kappa,1)$ we may define $f^* \in M[G]$ so that $(p, f^*, \dot{r})$ is forced into $j(\mathbb{M}(\kappa,\mu,\lambda) * \dot{\mathbb{P}})/G * H$, and f^* is forced to be a lower bound of $(f_{\dot{c}\restriction i} \colon i < \kappa)$. However, this leads to a contradiction. In particular, working in $M[G][H]$, if we extend $(p, f^*, \dot{r})$ to a condition deciding the value of $\dot{d} \cap x$, then since $f^* \leq f_{c\restriction i}$ for each $i < \kappa$ we may define c in $M[G][H]$. $\square$

6.3 ITP at κ^+

The argument is similar to Chapter 4 that ITP at κ^+ holds after forcing with the Sinapova forcing. More specifically, if $\dot{D} \in V[G]$ is a $\mathbb{P}$-name for a thin $\mathcal{P}_\mu(\theta)$-list, then we can write $V[G] \subseteq V[\mathcal{A} \times \mathcal{Q}]$ where $\mathcal{A}$ is a generic object for $Add(\kappa,\lambda)$ and $\mathcal{Q}$ is a generic object for term forcing that is μ-closed in V. Just as in Chapter 4 we may define the posets $\mathbb{Q}_p$ and get the same structural relationships between models $V[G][\mathcal{I}]$ and $V[\mathcal{A} \times \mathcal{Q}][\mathcal{I}]$ with the intermediate models $V[\mathcal{A} \times \mathcal{Q}_p][\mathcal{I}]$. Similar arguments give Lemma 4.18 and this allows us to define an ineffable branch in $V[G][\mathcal{I}]$ like Lemma 4.21.

CHAPTER 7

CONCLUSION

This thesis makes progress towards getting the tree property and the super tree property to hold everywhere simultaneously. There are a few natural directions one can go for further research. First would be the following:

Question 7.1. *Is it possible to add collapses to get the tree property to hold simultaneously at* $\aleph_{\omega^2+n}$ *for* $n \geq 1$ *while* $\aleph_{\omega^2}$ *is singular strong limit?*

Part of the difficulty is that when adding collapses to get the tree property at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$, one crucially uses that the guiding generic is a generic object for a forcing with closure greater than the size of the tree. So if we plan to add collapses to a Cummings-Foreman variant we need to give a different argument (or use a different method) to get around this. More ambitiously, we can ask the following:

Question 7.2. *Is it possible for any of the results in this thesis to hold at* $\aleph_\omega$ *when* $\aleph_\omega$ *is singular strong limit?*

At the moment it is not known if it is consistent that $\aleph_{\omega+1}$ has the tree property but SCH fails at $\aleph_\omega$.

Another line of research one could take is an analysis of the arguments to get the (super) tree property at κ^+ when κ is singular strong limit found in Chapters 3 or 4. Recall that the flavor of these arguments are different than the case with the double successor because κ^+ was not formerly a large cardinal - so we have no access to an embedding j to lift. Instead we rely on an analysis of the intermediate models between $V[\mathbb{M}]$ and $V[\mathbb{P} \times \mathbb{Q}]$, done in a more abstract setting with p-term forcings in Section 1.7. As a question, one may ask the following:

Question 7.3. *Which properties of $\mathbb{P}$ are required in order to define a branch in $V[\mathbb{M}]$?*

Regarding the ITP we may ask if would be to extend the results of Chapter 4 or Chapter 6 in the following way:

Question 7.4. *Is it possible to add collapses to have $\aleph_{\omega^2}$ singular strong limit with ITP holding simultaneously at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$?*

More ambitiously, we may ask the following:

Question 7.5. *Is it possible to add collapses to have ITP to hold simultaneously at $\aleph_{\omega^2+n}$ for $n \geq 1$?*

Approaching Question 7.4 would most likely involve a variant of the forcing from Chapter 6 where the Prikry forcing is outside the Mitchell forcing. We also note that answering Question 7.5 would have to solve the difficulty with the guiding generic mentioned after Question 7.1.

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VITA

Education

- 2016 – Present — Ph.D., Mathematics — University of Illinois at Chicago — Chicago, IL
- 2016 – 2019 — M.S., Mathematics — University of Illinois at Chicago — Chicago, IL
- 2012 – 2016 — B.S., Mathematics — Duquesne University — Pittsburgh, PA

Research

- 2020 — Simultaneous Instances of the Tree Property and ITP — Submitted
- 2021 — Simultaneous Instances of the Super Tree Property — In Preparation

Employment

- Fall 2021 — Assistant Professor of Mathematics — Culver-Stockton College
- 2019 - 2020 — Sullivan Visiting Professor of Mathematics — St. Ignatius College Prep
- 2016 - Present — Teaching Assistant — University of Illinois at Chicago