Large Cardinals and Anti-saturation Results on Ideals

by

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THESIS

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SUMMARY

This thesis consists primarily of anti-saturation results on ideals.

After a brief introduction covering historical context and necessary preliminaries, we extend an anti-saturation result for ideals on the successors of regular cardinals to inaccessibles:

Theorem. *Suppose* V *is a universe of* $\mathsf{ZFC}+\mathsf{GCH}$ *with an inaccessible cardinal* κ *admitting* κ*-complete, normal,* κ^+*-saturated ideals on* κ *concentrating on inaccessible cardinals below* κ*. Then there is a generic extension* $\mathsf{V}^{\mathbb{Q}}$ *in which there are no* κ*-complete,* κ^+*-saturated ideals on* κ *concentrating on inaccessible cardinals, but if* $\mathrm{I} \in \mathsf{V}$ *is a* κ*-complete, normal,* κ^+*-saturated ideal on* κ *concentrating on inaccessible cardinals, then* $\mathsf{V}^{\mathbb{Q}} \models$ "$\bar{\mathrm{I}}$ *is* κ^+*-presaturated*".

Theorem. *With the same assumptions and the same* $\mathbb{Q}$ *as above, if* $\delta \geq \kappa$ *is a inaccessible cardinal,* $\mathrm{I} \in \mathsf{V}$ *is a normal, fine,* δ*-presaturated ideal of uniform completeness* κ *on some algebra of sets* Z *such that:* $\mathcal{B}_\mathrm{I}$ *preserves the regularity of both* κ *and* δ*,* $\Vdash_{\mathcal{B}_\mathrm{I}} \delta^+ \leq |\dot{\mathsf{j}}_\mathrm{I}(\kappa)| < \dot{\mathsf{j}}_\mathrm{I}(\kappa)$*, and* $\mathcal{B}_\mathrm{I}$ *is proper on* $\mathrm{IA}_{<\delta^+}$*; then in* $\mathsf{V}^{\mathbb{Q}}$*,* $\bar{\mathrm{I}}$ *is not* δ^+*-saturated but* $\bar{\mathrm{I}}$ *is* δ^+*-presaturated.*

We then turn to mutually stationary sequences, and prove:

Theorem. *Let* $\langle \beta_\alpha \mid \alpha < \lambda \rangle$ *be a sequence generic for a Magidor forcing* $\mathbb{M}\left(\vec{\mathsf{U}}\right)$ *of length* λ*. Then there is a cofinite subset* K *of* $\langle \beta_\alpha \mid \alpha < \lambda, \alpha \text{ successor} \rangle$ *such that every family* $\mathsf{S}_\alpha \subseteq \beta_\alpha \in \mathsf{K}$ *of stationary sets is mutually stationary.*

Finally we analyze a Magidor-type forcing with interleaved collapses and guiding generics $\mathbb{MC}\left(\langle \mathsf{U}_\eta \mid \eta < \lambda \rangle\right)$, ultimately proving the following characterization of genericity theorem:

Theorem. *Let* $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle$ *and* $\vec{F} = \langle F_\eta \mid \eta < \lambda \rangle$ *be in* $V^{\mathbb{MC}(\langle U_\eta \mid \eta < \lambda \rangle)}$. *Then* $(\vec{\beta}, \vec{F})$ *is* $\mathbb{MC}\,(\langle U_\eta \mid \eta < \lambda \rangle)$*-generic over* V *precisely when at each limit ordinal* $\eta \leq \lambda$, $\vec{\beta} \upharpoonright \eta$ *meets* $\langle U_\zeta \mid \zeta < \eta \rangle$ *coboundedly often, and* $\langle F_\zeta \mid \zeta < \eta \rangle$ *meets the guiding generics coboundedly often.*

CHAPTER 1

INTRODUCTION

Ever since Cohen proved the independence of the Continuum Hypothesis from ZFC, set theorists have greatly expanded on his method of forcing to discover ever more complex phenomena in the multiverse of set theories.

A proof or disproof of the Continuum Hypothesis (CH), the statement that $2^{\aleph_0} = \aleph_1$, had been on mathematicians' radar since the nineteenth century, but Cohen finally proved its undecidability. In brief, Cohen's method of forcing starts with a universe V of ZFC and, using tools entirely within V, interprets a new universe with entirely new objects. Cohen's methods allow for the creation of a universe in which CH holds[1]. Furthermore, forcing with the combinatorial object $\mathsf{Add}(\omega, 1)$ can create a new universe $V^{\mathsf{Add}(\omega,1)}$ with a real number not present in V. And forcing with $\mathsf{Add}(\omega, \omega_2)$ can create real numbers on a scale hitherto undreamt of, so much that $V^{\mathsf{Add}(\omega,\omega_2)} \models |\mathcal{P}(\aleph_0)| \geq \aleph_2$.

The question then arose of whether similar independence results could hold at other cardinals, and for regular cardinals Easton quickly characterized the exact behavior of the powerset operator. A regular cardinal κ is one for which any $< \kappa$-sized union of $< \kappa$-sized sets is of size $< \kappa$. For instance (under ZFC) ω_1 is regular, as is $\omega_{\alpha+n}$ for any $n < \omega$, however $\aleph_\omega$ is

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not, as $\aleph_\omega = \bigcup_{n<\omega} \aleph_n$. So we say that $\aleph_\omega$ is singular, with cofinality $\mathrm{cf}(\aleph_\omega) = \omega$; likewise $\mathrm{cf}(\aleph_{\omega+\omega}) = \omega$ and $\mathrm{cf}(\aleph_{\omega_1}) = \omega_1$.

In brief, Easton showed, using Cohen's tools and a universe satisfying GCH (i.e. for every κ, $2^\kappa = \kappa^+$), that the following three properties of the powerset function are the *only* constraints to its behavior on regular cardinals:

1. (Cantor's Theorem) $2^\kappa > \kappa$
2. (König's Lemma) $\mathrm{cf}(2^\kappa) > \kappa$
3. (Weak Monotonicity) if $\kappa \leq \lambda$ then $2^\kappa \leq 2^\lambda$

For the powerset at singular cardinals, the situation is far more complicated and consistency results here require large cardinals. Here, we investigate SCH, the principle that if κ is a singular strong limit cardinal, then $2^\kappa = \kappa^+$. Unlike with GCH, Easton's method preserves SCH. As Silver showed, the powerset operator on large cardinals is much more restricted, in that if SCH fails, then the first cardinal at which SCH fails must have countable cofinality. Later results by Shelah's PCF theory gave even stronger restrictions. As for the necessity of large cardinals, Jensen's Covering Lemma proves that if SCH fails, then many large cardinal hypotheses must be consistent.

In short, a large cardinal κ is a cardinal so large that $V_\kappa \models$ ZFC, in particular one whose existence proves the consistency of ZFC. Since ω's existence proves the consistency of ZFC $-$ Infinity, ω may be considered *almost* a large cardinal. Several large cardinal axioms are generalizations of compactness-style phenomena holding at ω. For example, weak compactness, the principle that κ is inaccessible and $\kappa \to (\kappa)^2_2$, is a generalization of Ramsey's theorem that $\omega \to (\omega)^2_2$.

Combining Cohen's method of forcing with large cardinals will often partially preserve some of the large cardinal's properties in the extended universe. This following example is due to Mitchell: if κ is weakly compact, then every stationary subset of κ *reflects*, that is, if $S \subseteq \kappa$ is stationary, then there is some $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α. An appropriate forcing $\mathbb{P}$ in which $V^{\mathbb{P}} \models \kappa = \aleph_2$ will preserve this stationary reflection of κ; conversely, if $\aleph_2$ satisfies this stationary reflection, then there is a sub-universe in which $\aleph_2$ is weakly compact.

As for $\neg$SCH, Magidor argued that from a supercompact cardinal κ (a very large cardinal hypothesis), one can force κ to be singular and SCH to fail at κ; later works by Gitik, Mitchell, and Woodin optimized this result exactly to κ measurable with $o(\kappa) = \kappa^{++}$.

Sufficiently large cardinals, such as measurable, strong, strongly compact, and supercompact cardinals, come with a highly complete ultrafilter U that induces a nontrivial elementary self-embedding of the universe $j : V \to \mathrm{Ult}(V, U)$ into a well-founded ultrapower model of ZFC which is isomorphic to a transitive submodel M of V. For a measurable cardinal κ, M will itself be closed under κ-sequences from V, and $j(\kappa) > \kappa$. Various niceness properties of $j(\kappa)$ (e.g. cardinality lower bounds, being a V-cardinal) and how "close" M is to V, increase the consistency strength of the large cardinal hypothesis.

Combining the method of forcing and the ultrafilter-based embedding methods allow for the construction of generic ultrapowers, where the embeddings $j : V \to M$ lie in a forcing extension of V. Rather than filters, such constructions are typically phrased in terms of the dual notion of *ideal*: if I is an ideal on some set algebra $\mathcal{A}$ in some $\mathcal{P}(Z)$, then $\mathcal{A}/I$ is a valid notion of forcing which adds a filter U such that every $X \in \mathcal{A} \cap V$ is either in U or its complement is in U, and so we may construct $j : V \to \mathrm{Ult}(V, U)$. In the event that $\mathcal{A}/I$ only has antichains of size less than κ, then we say that I is κ-*saturated*. If $\kappa \leq |Z|^+$, then as a result $\mathcal{A}/I$ induces a well-founded generic ultrapower and preserves a large class of cardinals.

A major theme of this thesis is *anti-saturation* results, that say when an ideal is specifically *not* saturated.

Solovay's Splitting Theorem for stationary sets says that the nonstationary ideal on κ is not κ-saturated, and counts as the first major anti-saturation result for the nonstationary ideal on any regular cardinal κ. As for other results about the non-saturation of nonstationary ideals, Gitik and Shelah showed that when $\kappa > \omega_1$, NS_κ is not κ^+-saturated. However, it is consistent (from very large cardinals) that NS_{ω_1} is ω_2-saturated. Likewise, for $\lambda \geq \kappa \geq \omega_1$, the nonstationary ideal on $\mathcal{P}_\kappa(\lambda)$ (for $\lambda \geq \kappa$) is known not to be κ^+-saturated. Burke, Foreman, Gitik, Magidor, Matsubara, and Shelah proved this over much time and many papers.

Results on *any* ideal have an equally rich history, which we will cover in Section 3.1.

In proving that the nonstationary ideal on $\mathcal{P}_{\omega_1}(\lambda)$ is not $\lambda^{\mathrm{cf}(\lambda)}$-saturated for any singular λ, Foreman and Magidor introduced the notion of *mutually stationary* sequence, which acts as a generalization of stationarity, and of stationary reflection, for singular cardinals.

Paradoxically, although saturation results often have large cardinal consistency strength, mutual stationarity principles are a strong form of anti-saturation property and also have large cardinal consistency strength. For instance, a Prikry generic sequence added to a measurable cardinal exhibits a mutual stationarity property, and the measurability is known to be necessary. For mutual stationarity principles at accessible cardinals such as $\aleph_\omega$, even larger cardinals are necessary.

We now come to where this thesis's results fit into the above milieu.

Cox and Eskew have a recent anti-saturation result for successors of regular cardinals that still guarantees many of the corollaries of saturation, such as well-foundedness of the generic ultrapower and preservation of many cardinals. Chapter 3 generalizes their result to an inaccessible, in the form of Theorem 3.1.4 and Theorem 3.1.5.

Concerning mutual stationarity, we extend the mutual stationarity result for Prikry generic sequences to Magidor generic sequences, which act to singularize a cardinal to uncountable cofinality in a cardinal-preserving manner. This is the content of Chapter 4, in particular Section 4.4.

Finally, we develop a Magidor-like forcing $\mathbb{MC}\left(\vec{U}\right)$ that singularizes a regular cardinal κ without collapsing κ, and simultaneously collapses cardinals below κ using the method of *guiding generics* to turn κ into the extended universe's $\aleph_{\omega_1}$. This program takes up Chapter 5, and culminates in Lemma 5.1.21 and Theorem 5.1.24, which together form an exact characterization of genericity for $\mathbb{MC}\left(\vec{U}\right)$. Given the current approaches to achieving mutual stationarity results at accessible cardinals, Magidor-like forcings along the lines of $\mathbb{MC}\left(\vec{U}\right)$ seem necessary for achieving mutual stationarity results at $\aleph_\lambda$ for λ regular uncountable.

The paper is organized as follows. Chapter 2 describes our required preliminaries, including stationary sets, the basics of forcing, ideals & measures, Mitchell order, Prikry forcing and

Prikry forcing with collapses, and Magidor forcing. Chapter 3 first appeared on the arXiv (as (Schoem, 2019)) and covers our extension of Cox and Eskew's anti-saturation result to inaccessibles, in the form of Theorem Theorem 3.1.4 and Theorem 3.1.5. Chapter 4 outlines a brief history of mutual stationarity results, expounds on how mutual stationarity acts as an anti-saturation principle for nonstationary ideals, and expounds on indiscernibility-based methods pioneered by Koepke, and collated in 4.2; we then cover Lemma 4.3.4, a novel simultaneous indiscernibility result that shows in 4.4 that Magidor generic sequences exhibit mutual stationarity properties. Finally, Chapter 5 describes a Magidor-type forcing $\mathbb{MC}\left(\vec{U}\right)$ that turns a measurable κ into $\aleph_{\omega_1}$; we state and prove its relevant Prikry-type lemmas in Lemma 5.1.11 and Lemma 5.1.19 and a complete characterization of genericity in Lemma 5.1.21 and Theorem 5.1.24.

In the preliminaries up through The Prikry forcing section 2.5, we state well-known results, giving proofs or sketches as needed. In the preliminary Magidor forcing section 2.6, we flesh out most proofs, as the results are sometimes only implicit in the literature.

CHAPTER 2

PRELIMINARIES

We will cover the necessary background information for reading this thesis. Mostly we cite well-known results, giving proof sketches or proofs when doing so facilitates better understanding. In Section 2.6 on Magidor forcing, we give more extensive proofs, as results are not as well known.

Results from Sections 2.1, 2.2, 2.3, and 2.5 are largely found in (Jech, 2003). For forcing iterations in 2.2, we additionally draw from (Baumgartner, 1983); for Section 2.3, some material on ideals comes from (Foreman, 2010); for Section 2.4, we pull from (Mitchell, 2010) for Mitchell rank and order; for Prikry with collapses in Section 2.5, we also cite (Cummings, 2015); and for Section 2.6, we appeal to Magidor's original paper (Magidor, 1978) and Fuchs' recent work in (Fuchs, 2014).

2.1 Stationary Sets

Solovay's Splitting Theorem for stationary sets accounts for the first major anti-saturation result in set theory. We will elaborate on the anti-saturation in Section 2.3 and present the core concepts and behaviors surrounding closed unbounded sets and stationary sets here; further elaboration can be found in standard references such as (Jech, 2003), Chapter 8.

Definition 2.1.1. Let κ be a regular uncountable cardinal. Then a $C \subseteq \kappa$ is

- *closed* if whenever ρ is a limit ordinal below κ and $C \cap \rho$ is unbounded in ρ, then $\rho \in C$

- *unbounded* if whenever $\rho < \kappa$ there is some $\rho' \in C \setminus (\rho + 1)$ (i.e. some $\rho' > \rho$ such that $\rho' \in C$)
- *club* if both closed and unbounded

The intersection of any two clubs is club, and furthermore:

Proposition 2.1.2. *Let $\tau < \kappa$ and let $\langle C_\eta \mid \eta < \tau \rangle$ be a family of club subsets of κ. Then*

$$\bigcap_{\eta<\tau} C_\eta$$

is also club.

While the intersection of κ-many clubs is not necessarily club, there is an approximate version:

Definition 2.1.3 (Diagonal intersection)**.** Let $\langle X_\eta \mid \eta < \lambda \rangle$ be a family of sets of ordinals, with λ some ordinal. Then their *diagonal intersection* is

$$\mathop{\Large\triangle}_{\eta<\lambda} X_\eta := \{\beta \in \mathsf{Ord} | \forall \alpha < \beta \;\; \beta \in X_\alpha\}$$

Proposition 2.1.4. *Let $\langle C_\eta \mid \eta < \kappa \rangle$ be a family of club subsets of κ. Then*

$$\mathop{\Large\triangle}_{\eta<\kappa} C_\eta$$

is also club.

One may think of the club subsets of κ as the "large" subsets, in a way that will be made precise in Section 2.3. The "non-small" sets are then the stationary sets, and the "small" sets nonstationary:

Definition 2.1.5. Let $S \subseteq \kappa$. We say S is *stationary* if for every $C \subseteq \kappa$ club, $S \cap C \neq \emptyset$; and we say S is *nonstationary* if S is not stationary.

Since the intersection of two clubs is club, $S \cap C$ is also stationary for any stationary S and club C.

Stationary sets admit the following homogeneity result, which, as we will elaborate on in Section 2.3, is in a sense equivalent to Proposition 2.1.4:

Theorem 2.1.6 (Fodor's Theorem)**.** *Let S be stationary and let $f : S \to \kappa$ be regressive (i.e. for every $\alpha \in S$, $f(\alpha) < \alpha$). Then there is some $T \subseteq S$ stationary such that $f \upharpoonright T$ is constant.*

We think of stationary sets as "non-small" instead of "large" because any stationary set splits into two disjoint stationary sets. In fact, more is possible, as first shown in (Solovay, 1971):

Theorem 2.1.7 (Solovay's Splitting Theorem)**.** *For any S a stationary subset of κ, there exists a family $\langle T_\alpha \mid \alpha < \kappa \rangle$ of disjoint stationary subsets of S such that*

$$S = \bigsqcup_{\alpha<\kappa} T_\alpha$$

There are analogous notions of club and stationary on more general set algebras:

Definition 2.1.8. Let κ be a regular cardinal and let λ be any cardinal with $\lambda \geq \kappa$. We write $\mathcal{P}_\kappa(\lambda)$ to mean the collection of subsets of λ of cardinality less than κ. Then:

- A collection $C \subseteq \mathcal{P}_\kappa(\lambda)$ is *closed* if for any $\tau < \kappa$ and any $\subseteq$-increasing chain $\langle x_\alpha \mid \alpha < \tau \rangle$ of elements of C,
$$\bigcup_{\alpha<\tau} x_\alpha$$
is also in C
- A collection $C \subseteq \mathcal{P}_\kappa(\lambda)$ is *unbounded* if for any $x \in \mathcal{P}_\kappa(\lambda)$ there is some $y \in \mathcal{P}_\kappa(\lambda)$ with $y \supseteq x$ and $y \in C$
- If C is both closed and unbounded then C is *club*
- And $S \subseteq \mathcal{P}_\kappa(\lambda)$ is *stationary* if $S \cap C \neq \emptyset$ for every club C.

Similar intersection results hold:

Proposition 2.1.9. *If $\tau < \kappa$ and $\langle C_\alpha \mid \alpha < \tau \rangle$ is a family of clubs then*
$$\bigcap_{\alpha<\tau} C_\alpha$$
is also club.

Fodor's Theorem, with a modified notion of Δ, also holds:

Definition 2.1.10. Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ be a family of subsets of $\mathcal{P}_\kappa(\lambda)$. Then
$$\mathop{\Delta}_{\alpha<\kappa} X_\alpha := \left\{ x \in \mathcal{P}_\kappa(\lambda) \,\middle|\, x \in \bigcap_{a\in x} X_a \right\}$$

Proposition 2.1.11. *If $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a family of club sets on $\mathcal{P}_\kappa(\lambda)$ then*

$$\mathop{\Delta}_{\alpha<\kappa} C_\alpha$$

is also club.

Theorem 2.1.12 (Fodor's Theorem)**.** *If S is a stationary subset of $\mathcal{P}_\kappa(\lambda)$, $f : S \to \mathcal{P}_\kappa(\lambda)$, and f is regressive (i.e. $f(x) \in x$ for every nonempty $x \in S$) then there is a stationary $T \subseteq S$ such that $f \upharpoonright T$ is constant.*

2.2 Forcing

We will assume knowledge of forcing; for a detailed summary, see for instance (Jech, 2003), Chapter 13. Here we just recall the forcing and iterated forcing specifics necessary for our purposes.

Just to review, here is the core theorem of forcing:

Theorem 2.2.1 (Forcing and Generic Model Theorem)**.** *Let V be a universe of* ZFC *and let $(\mathbb{P}, <)$ be a notion of forcing. Then there is a universe $V^{\mathbb{P}} = V[G]$ of* ZFC *given by the addition of a new set G such that*

1. $\mathrm{Ord}^V = \mathrm{Ord}^{V[G]}$
2. *If $W \supseteq V$, $W \models \mathsf{ZFC}$ and $G \in W$, then $V[G] \subseteq W$*
3. *G is $\mathbb{P}$-generic over V, that is, for every dense subset (equivalently, every maximal antichain) D of $\mathbb{P}$ in V, $G \cap D \neq \emptyset$*

And the theory of $\mathsf{V}[\mathsf{G}]$ *may be understood by the forcing relation* $\Vdash_{\mathbb{P}}$*, entirely definable in* V*, because for each formula* φ*,*

$$\mathsf{V}[\mathsf{G}] \models \varphi \iff \exists \mathsf{p} \in \mathsf{G}\ \mathsf{p} \Vdash \varphi$$

Remark 2.2.2. Here are some notations and conventions:

- We adopt the convention that for p, $\mathsf{q} \in \mathbb{P}$, $\mathsf{q} \leq \mathsf{p}$ means that q is a *stronger* condition than p.
- If p, q are compatible, we write $\mathsf{p} \parallel \mathsf{q}$ to denote this, otherwise p, q are incompatible, denoted by $\mathsf{p} \perp \mathsf{q}$.
- If $\mathsf{p} \parallel \mathsf{q}$, we use $\mathsf{p} \wedge \mathsf{q}$ to denote the weakest r such that $\mathsf{r} \leq \mathsf{p}$ and $\mathsf{r} \leq \mathsf{q}$.
- For any p, q, we use $\mathsf{p} \vee \mathsf{q}$ to denote the strongest r such that $\mathsf{p} \leq \mathsf{r}$ and $\mathsf{q} \leq \mathsf{r}$.

If τ is an ordinal and $\tau \in \mathsf{Reg}^{\mathsf{V}}$ (i.e. τ is a regular cardinal in V), it is not necessarily the case that $\tau \in \mathsf{Reg}^{\mathsf{V}^{\mathbb{P}}}$ (or is even a cardinal). Certain nice combinatorial properties guarantee that τ remains a cardinal after forcing. We outline some key such properties:

Definition 2.2.3 (Chain condition, presaturation, and closure)**.** Let $(\mathbb{P}, \leq)$ be a poset and let τ be a cardinal. We say that:

(i) $\mathbb{P}$ has the τ-*chain condition* (is τ-cc) if every antichain of $\mathbb{P}$ is of size less than τ

(ii) $\mathbb{P}$ is τ-*directed closed* (i.e $<\tau$-directed closed, written τ-dc, $<\tau$-dc) if whenever $D \subseteq \mathbb{P}$ is a directed set[1] with $|D| < \tau$, there is a $q \in \mathbb{P}$ such that whenever $p \in D$, $q \leq p$

(iii) $\mathbb{P}$ is τ-*closed* (i.e. $<\tau$-closed) if whenever $\rho < \tau$ and $\langle p_\alpha \mid \alpha < \rho\rangle$ is a $\leq$-decreasing sequence in $\mathbb{P}$, there is a $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for all $\alpha < \rho$

(iv) $\mathbb{P}$ is τ-*distributive* if for every $\dot{f} : \check{\tau} \to \check{V}$ in $V^{\mathbb{P}}$, $\dot{f} \in \check{V}$; equivalently if the intersection of τ-many open dense subsets of $\mathbb{P}$ is dense[2]

(v) $\mathbb{P}$ is τ-*presaturated* (i.e. $<\tau$-presaturated) if for every $\lambda < \tau$ and every family $\langle A_\alpha \mid \alpha < \lambda\rangle$ of antichains, there are densely many $p \in \mathbb{P}$ such that for all α, $\{q \in A_\alpha \mid p \parallel q\}$ has cardinality $<\tau$. Note that τ-cc implies τ-presaturation.

(vi) $\mathbb{P}$ is τ-*preserving* if $V^{\mathbb{P}} \models$ "$\check{\tau}$ is a cardinal"

Proposition 2.2.4. *If* $\mathbb{P}$ *is* τ*-cc,* τ*-dc,* τ*-closed,* $<\tau$*-distributive,* τ*-distributive, or* τ*-presaturated, then* $\mathbb{P}$ *is* τ*-preserving.*

The proofs are standard; see (Jech, 2003), Chapters 14 and 15, for more details.

The usual posets for collapsing cardinals are the Lévy collapsing posets $\mathrm{Col}(\tau, \lambda)$ and $\mathrm{Col}(\tau, <\lambda)$. In brief, $\mathrm{Col}(\tau, \lambda)$ is a τ-directed closed, λ^+-cc forcing notion such that for every $\rho \in \mathrm{Card}^V \cap [\tau, \lambda]$, $|\rho|^{V^{\mathrm{Col}(\tau,\lambda)}} = \tau$ If λ is inaccessible, then $\mathrm{Col}(\tau, <\lambda)$ is a τ-directed closed, λ-cc forcing such that for every $\rho \in \mathrm{Card}^V \cap [\tau, \lambda)$, $|\rho|^{V^{\mathrm{Col}(\tau,\lambda)}} = \tau$.

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[1] that is, for all $p, q \in D$, there is an $r \in D$ such that $r \leq p, q$

[2] Note this is unlike the other properties here in that this is an $=\tau$ principle as opposed to a $<\tau$ principle. We write "$<\tau$-distributive" to mean "for all $\tau' < \tau$, $\mathbb{P}$ is τ'-distributive".

Often, we will wish to force with multiple forcings simultaneously, or in succession. If both $\mathbb{P}$ and $\mathbb{Q}$ are in V, then we will want to force with their *product*; if instead $\mathbb{Q} \in V^{\mathbb{P}}$, then we will use *iteration*.

(Jech, 2003), Chapter 15, is a standard reference for product forcing. For iteration forcing, this section draws from (Baumgartner, 1983).

Definition 2.2.5 (Product Forcing)**.** If $\mathbb{P}$ and $\mathbb{Q}$ are in V, then the poset $\mathbb{P} \times \mathbb{Q}$ consists of conditions of the form (p, q) where $p \in \mathbb{P}$ and $q \in \mathbb{Q}$, and $(p', q') \leq (p, q)$ if $p' \leq p$ and $q' \leq q$.

The following forcing theorem characterizes what a $\mathbb{P} \times \mathbb{Q}$-generic looks like:

Theorem 2.2.6 (Product Theorem)**.** *Let* $G \subseteq \mathbb{P} \times \mathbb{Q}$*. Then* G *is* $\mathbb{P} \times \mathbb{Q}$*-generic over* V *if and only if* $G = G_1 \times G_2$ *where* G_1 *is* $\mathbb{P}$*-generic over* V *and* G_2 *is* $\mathbb{Q}$*-generic over* $V^{\mathbb{P}}$*, and in such case* $V[G] = V[G_1][G_2]$.

Larger products are possible, but beyond our scope. As for properties preserved by product forcing:

Proposition 2.2.7. *Let* τ *be a regular cardinal, and suppose* $\mathbb{P}$ *and* $\mathbb{Q}$ *are both* τ*-closed. Then* $\mathbb{P} \times \mathbb{Q}$ *is also* τ*-closed.*

Proposition 2.2.8 (Easton's Lemma, c.f. (Jech, 2003), Lemma 15.19)**.** *Let* τ *be regular, and suppose* $\mathbb{P}$ *is* τ^+*-cc and* $\mathbb{Q}$ *is* τ^+*-closed. Then in* $V^{\mathbb{P}}$*,* $\mathbb{Q}$ *remains* τ*-distributive.*

In the event that $\mathbb{Q} \in V^{\mathbb{P}} \setminus V$, we may use the notion of two-step iteration:

Definition 2.2.9. The two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is given by conditions of the form $(p, \dot{q})$ where $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$, and $(p', \dot{q}') \leq (p, q)$ if $p' \leq p$ and $p' \Vdash_{\mathbb{P}} \dot{q}' \leq \dot{q}$

The following characterizes generic extensions of two-step iterations:

Proposition 2.2.10. *Let* $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}$*. Then* G *is* $\mathbb{P} * \dot{\mathbb{Q}}$*-generic over* V *if and only if* $G_1 := \{p \mid \exists \dot{q} \in \dot{\mathbb{Q}}\ (p, \dot{q}) \in G\}$ *is* $\mathbb{P}$*-generic over* V*; and* $G_2 := \{\dot{q} \mid \exists p \in G_1\ (p, \dot{q}) \in G\}$*, as interpreted in* $V[G_1]$*, is* $\mathbb{Q}$*-generic over* $V[G_1]$*. If so, we write* $G = G_1 * G_2$*.*

Iteration is better behaved than product:

Proposition 2.2.11. *If* $\mathbb{P}$ *is* τ*-cc and* $1_{\mathbb{P}} \Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ *is* τ*-cc then* $\mathbb{P} * \dot{\mathbb{Q}}$ *is* τ*-cc; the converse is also true.*

The same is also true for τ*-closed.*

Presaturation can be pushed downwards through a two-step iteration:

Lemma 2.2.12 (Lemma 2.12 of (Cox and Eskew, 2018))**.** *If* $\mathbb{P} * \dot{\mathbb{Q}}$ *is* κ*-presaturated then* $\mathbb{P}$ *is* κ*-presaturated and* $1_{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ *is* κ*-presaturated.*

Whether the converse holds is, to the author's knowledge, an open problem; this appears as Question 8.6 of (Cox and Eskew, 2018).

Longer iterations are possible:

Definition 2.2.13. Let τ be some ordinal. Then a τ-length iteration, written $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \tau \right\rangle$, is a notion of forcing for which:

1. $\mathbb{P}_0 \in V$

2. For all $\alpha > 0$, $\dot{\mathbb{Q}}_\alpha$ is a forcing in $V^{\mathbb{P}_\alpha}$ with $P_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$

3. If α is a limit ordinal, then $\mathbb{P}_\alpha$ is some α-sequence of $\mathbb{P}_\beta$-terms for $\beta < \alpha$

The behavior of a forcing iteration entirely depends on $\mathbb{P}_0$, the $\mathbb{Q}$'s, and what exactly happens at limit stages:

Definition 2.2.14.

1. A forcing iteration is said to be of finite (countable) support if for each α limit, only finitely (countably) many forcing terms drawn from the $\mathbb{P}_\beta$'s, $\beta < \alpha$, are allowed to be nontrivial.
2. A forcing iteration is of direct support at α if only boundedly in α-many forcing terms drawn from the $\mathbb{P}_\beta$'s, $\beta < \alpha$, are nontrivial.
3. A forcing iteration is of inverse support at α if (apart from any restrictions placed at limit stages below α) there are no restrictions placed on the forcing terms drawn from the $\mathbb{P}_\beta$'s, $\beta < \alpha$.
4. And a forcing iteration is of Easton support if the iteration is of direct support at regular cardinals, and of inverse support at singular limit ordinals.

We will care primarily about Easton support products, and have the following theorem about their preservation of closure and chain condition:

Proposition 2.2.15. *Any Easton support iteration of τ-closed forcings is τ-closed.*

Proposition 2.2.16. *If λ is a Mahlo cardinal (i.e. has stationarily many regular, equivalently inaccessible cardinals below), then any λ-length Easton support iteration of λ-cc forcings is λ-cc.*

2.3 Ideals, Measures, and Ultrapowers

We have already seen clubs and nonstationary sets in Section 2.1. Their properties generalize to the notions of filter and ideal. See (Jech, 2003), Chapter 7, for further elaboration on filters and ideals, and (Jech, 2003), Chapters 10, 12, and 17 for measurable cardinals and ultrapowers. We summarize the fundamentals here.

Definition 2.3.1. Let $\mathcal{F}$ be a collection of subsets of κ (or $\mathcal{P}_\kappa(\lambda)$). Then $\mathcal{F}$ is a *filter* if

- κ (or $\mathcal{P}_\kappa(\lambda)$) is in $\mathcal{F}$, and $\emptyset$ is not in $\mathcal{F}$
- $\mathcal{F}$ is upwards closed under $\subseteq$
- whenever A, B are in $\mathcal{F}$ then $A \cap B$ is also in $\mathcal{F}$

We say $\mathcal{F}$ is *ultra* (equivalently, maximal) if for every $A \subseteq \kappa$ (or $\mathcal{P}_\kappa(\lambda)$), either A is in $\mathcal{F}$, or A^c, its complement, is in $\mathcal{F}$.

And we say $\mathcal{F}$ is *nonprincipal* if there is no $X \subseteq \kappa$ (or $\mathcal{P}_\kappa(\lambda)$) such that for all $Y \in \mathcal{F}$, $X \subseteq Y$.

By the Axiom of Choice, every cardinal and every powerset algebra admits a nonprincipal ultrafilter.

The dual notion of a filter is an *ideal*:

Definition 2.3.2. Let $\mathcal{I}$ be a collection of subsets of κ (or $\mathcal{P}_\kappa(\lambda)$). Then $\mathcal{I}$ is an *ideal* if

- $\emptyset$ is in $\mathcal{I}$, and κ (or $\mathcal{P}_\kappa(\lambda)$) is not in $\mathcal{I}$
- $\mathcal{F}$ is downwards closed under $\subseteq$
- whenever A, B are in $\mathcal{I}$ then $A \cup B$ is also in $\mathcal{I}$

We say $\mathcal{I}$ is *prime* (equivalently, maximal) if for every $A \subseteq \kappa$ (or $\mathcal{P}_\kappa(\lambda)$), either A is in $\mathcal{I}$, or A^{C}, its complement, is in $\mathcal{I}$.

And we say $\mathcal{I}$ is *nonprincipal* if there is no $X \subseteq \kappa$ (or $\mathcal{P}_\kappa(\lambda)$) such that for all $Y \in \mathcal{I}$, $Y \subseteq X$.

Remark 2.3.3. Whenever $\mathcal{F}$ is a filter, its dual ideal is given by

$$\breve{\mathcal{F}} := \left\{ X^{\complement} \mid X \in \mathcal{F} \right\}$$

Likewise, if $\mathcal{I}$ is an ideal, its dual filter is given by

$$\breve{\mathcal{I}} := \left\{ X^{\complement} \mid X \in \mathcal{I} \right\}$$

For each κ (or $\mathcal{P}_\kappa(\lambda)$), the club filter $\mathcal{C}_\kappa$ ($\mathcal{C}_{\mathcal{P}_\kappa(\lambda)}$), consisting of all subsets on κ (or $\mathcal{P}_\kappa(\lambda)$) including a club, is a nonprincipal filter.

Likewise, the nonstationary ideal NS_κ ($\mathsf{NS}_{\mathcal{P}_\kappa(\lambda))}$) consisting of all nonstationary subsets of κ (or $\mathcal{P}_\kappa(\lambda)$) is a nonprincipal ideal, and furthermore, $\breve{\mathsf{NS}}_\kappa = \mathcal{C}_\kappa$ (and likewise for $\mathcal{P}_\kappa(\lambda)$)).

Remark 2.3.4. The stationary sets we may write as $(\mathsf{NS}_\kappa)^+$; in general, whenever $\mathcal{I}$ is an ideal, we write $\mathcal{I}^+$ to denote the collection of all S such that for every $C \in \breve{\mathcal{I}}$, $S \cap C \neq \emptyset$.

Both the nonstationary ideal and the club filter have much stronger closure properties:

Definition 2.3.5 (closure)**.** Let $\mathcal{F}$ be a filter and let τ be a cardinal. Then we say that $\mathcal{F}$ is τ-closed if for every $\rho < \tau$ and every family $\langle A_\alpha \mid \alpha < \rho \rangle$ of sets in $\mathcal{F}$,

$$\bigcap_{\alpha<\rho} A_\alpha$$

is also in $\mathcal{F}$.

Likewise, if $\mathcal{I}$ is an ideal, then $\mathcal{I}$ is τ-closed if for every $\rho < \tau$ and every family $\langle J_\alpha \mid \alpha < \rho \rangle$ of sets in $\mathcal{I}$,

$$\bigcap_{\alpha<\rho} J_\alpha$$

is also in $\mathcal{I}$.

Both $\mathcal{C}_\kappa$ ($\mathcal{C}_{\mathcal{P}_\kappa(\lambda)}$) and NS_κ ($\mathsf{NS}_{\mathcal{P}_\kappa(\lambda)}$)) are κ-closed by Proposition 2.1.2, and by Proposition 2.1.4, are additionally *normal*:

Definition 2.3.6. If $\mathcal{F}$ is a filter on κ (or $\mathcal{P}_\kappa(\lambda)$, then we say that $\mathcal{F}$ is *normal* if for any $\langle A_\alpha \mid \alpha < \kappa \rangle$ of sets in $\mathcal{F}$,

$$\Delta_{\alpha<\kappa} A_\alpha$$

is also in $\mathcal{F}$.

We say that $\mathcal{I}$ is *normal* if its dual filter is normal.

And Fodor's Theorem generalizes to arbitrary filters and ideals:

Theorem 2.3.7 (Fodor's Theorem)**.** *Let $\mathcal{I}$ be a κ-complete ideal. Then $\mathcal{I}$ is normal if and only if for every $f : S \to \kappa$ ($\mathcal{P}_\kappa(\lambda)$) regressive with $S \in \mathcal{I}^+$, there is some $T \subseteq S$ with $T \in \mathcal{I}^+$ such that $f \restriction T$ is constant.*

As we have seen, the existence of nonprincipal ultrafilters (equivalently prime ideals) and κ-complete normal filters (ideals) are both theorems of ZFC. Combining the two, however, is a large cardinal hypothesis:

Definition 2.3.8. Let κ be a cardinal. Then κ is *measurable* if there is a normal measure, i.e. κ-complete normal nonprincipal ultrafilter, U on κ.

From a measurable cardinal, we may build an ultrapower of V as follows:

Definition 2.3.9. Let U be a normal measure on κ. Then $\mathrm{Ult}(V, U)$ is the proper class model of set theory for which:

- the elements are of the form $[f]_U$, the equivalence class of all $f : \kappa \to V$ where $[f]_U = [g]_U$ if for U-many α, $f(\alpha) = g(\alpha)$, i.e. $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$
- and $\in^*$, the binary membership relation, is given by $[f]_U \in^* [g]_U$ if for U-many α, $f(\alpha) \in g(\alpha)$

Theorem 2.3.10 (Łoś' Theorem)**.** *For any formula with parameters $\varphi(\overrightarrow{x})$, for any $\overrightarrow{[f]}$,* $\mathrm{Ult}(V, U) \models \varphi(\overrightarrow{[f]})$ *if and only if for U-many α, $V \models \varphi(\overrightarrow{f(\alpha)})$.*

Since U is countably complete, $\in^*$ in $\mathrm{Ult}(V, U)$ is well-founded, and therefore, by the Mostowski Collapsing Theorem, its transitive collapse M is a $\in$-model of set theory. By Łoś' Theorem, M is also a model of ZFC.

Further, M is a class model since the class map $\rho \mapsto [\alpha \mapsto \rho]_U$ maps to M-ordinals and is injective. By the κ-completeness, for all $\rho < \kappa$, $[\alpha \mapsto \rho]_U = \rho$; however $[\alpha \mapsto \kappa]_U > \kappa$. By normality, $\kappa \in M$ since $[\alpha \mapsto \alpha]_U = \kappa$.

There are some additional behavior properties of M that are standard and easy to prove, which we encapsulate in the following equivalent definition of measurable cardinal:

Proposition 2.3.11. *A cardinal κ is measurable if and only if there is a transitive class model $M \subseteq V$ and an elementary map $j : V \to M$ such that*

- *κ is the least ordinal for which $j(\kappa) > \kappa$*
- *M is closed under κ-sequences from V, i.e. $V \models M^\kappa \subseteq M$*

For the converse, if $j : V \to M$ is such an elementary embedding, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is a normal measure on κ.

Even if a nonprincipal filter (e.g. the club filter) is not maximal, we can use forcing to define a form of ultrapower. We summarize the core ideas and results we need of these generic ultrapowers; see (Foreman, 2010) for a more comprehensive overview.

Proposition 2.3.12. *Let $\mathcal{I}$ be a nonprincipal ideal on κ. Then the poset $\mathcal{B}_\mathcal{I}$, defined by*

$$\mathcal{B}_\mathcal{I} = \left\{ [A]_{\underset{\mathcal{I}}{\smile}} \mid A \in \mathcal{I}^+ \right\}$$

with partial order $[A]_{\underset{\mathcal{I}}{\smile}} \leq [B]_{\underset{\mathcal{I}}{\smile}}$ if $B \setminus A \in \mathcal{I}$, is a separative notion of forcing. Let G be $\mathcal{B}_\mathcal{I}$-generic over V. Then for every $X \in V$, $X \subseteq \kappa$, either $[X] \in G$ or $[X^\complement] \in G$, so G is a V-ultrafilter. Thus in $V[G]$ we may define $\mathrm{Ult}(V, G)$ and obtain an elementary embedding $j_\mathcal{I} : V \to \mathrm{Ult}(V, G)$.

Further, if $\mathcal{I}$ is κ-complete then G is V-κ-complete, and if $\mathcal{I}$ is normal then G is V-normal, and $G = \{X \subseteq \kappa \mid \kappa \in j(X)\}$.

We will variously refer to the generic ultrapower as $\mathrm{Ult}(V, G)$ or $\mathrm{Ult}(V, \mathcal{I})$, depending on context, and will also write j_I to denote $j_G : V \to \mathrm{Ult}(V, G)$ the generic ultrapower elementary embedding.

The generic ultrapower is not always well-founded; this is a special property that gets its own name:

Definition 2.3.13 (Definition 2.4 of (Foreman, 2010))**.** An ideal $\mathcal{I}$ is said to be *precipitous* if whenever G is a $\mathcal{B}_{\mathcal{I}}$-generic object over V, $\mathrm{Ult}(V, G)$ is well-founded.

A cardinal κ admits a precipitous ideal if and only if κ is measurable in some inner model; however, precipitous ideals need not be prime, and with large cardinals it is possible to force NS_κ to be precipitous.

We now turn to saturation, which in certain cases acts as a strong form of precipitousness:

Definition 2.3.14. Let μ be a cardinal and let $\mathcal{I}$ be an ideal. Then we say that $\mathcal{I}$ is μ-*saturated* if $\mathcal{B}_{\mathcal{I}}$ has the μ-chain condition; that is, if for any family $\langle J_\alpha \mid \alpha < \mu\rangle$ of $\mathcal{I}^+$-sets, there are $\alpha \neq \beta$ such that $J_\alpha \cap J_\beta \in \mathcal{I}^+$.

We say that the *saturation* of $\mathcal{I}$, written $\mathsf{sat}(\mathcal{I})$, is the least μ such that $\mathcal{I}$ is μ-saturated.

Clearly any ideal on κ is $(2^\kappa)^+$-saturated, and a prime ideal is 2-saturated.

As for why saturation acts as a strong form of precipitousness:

Theorem 2.3.15. *Let $\mathcal{I}$ be a κ-complete (nonprincipal) ideal on κ. Then if $\mathsf{sat}(\mathcal{I}) \leq \kappa^+$, then $\mathcal{I}$ is precipitous.*

Saturated ideals also lead to a measurable-like closure of the generic ultrapower:

Fact 2.3.16. If $\mathcal{I}$ is a κ-complete, κ^+-saturated ideal in V, and if G is a $\mathcal{B}_{\mathcal{I}}$-generic filter over V, then in $V[G]$, $\mathrm{Ult}(V, G)^\kappa \subseteq \mathrm{Ult}(V, G)$; that is, $\mathrm{Ult}(V, G)$ is closed under κ-sequences from $V[G]$.

This follows from Propositions 2.9 and 2.14 of (Foreman, 2010).

By the relevant chain condition, we see that if $\mathcal{I}$ is precipitous, then $\mathcal{I}$ also preserves $\mathtt{sat}(\mathcal{I})$; this motivates a principle intermediate between saturation and precipitousness:

Definition 2.3.17. Let $\mathcal{I}$ be a κ-complete (nonprincipal) ideal on κ, and let μ be a regular cardinal. Then $\mathcal{I}$ is μ-presaturated if $\mathcal{I}$ is precipitous and $\mathcal{B}_{\mathcal{I}}$ preserves μ.

We make the following caveat about presaturation for ideals versus presaturation for posets:

Caveat. In contrast to the definition for a general poset in Section 2.2, only for precipitous ideals I is it the case that I is μ-presaturated if and only if $\mathcal{B}_{\mathrm{I}}$ is μ-presaturated as a forcing poset. For precipitous ideals, this result appears as Theorem 4.2 of (Baumgartner and Taylor, 1982).

Solovay's Splitting theorem can now be rephrased as an anti-saturation result:

Fact 2.3.18. For every stationary S, $\mathcal{B}_{\mathsf{NS}_\kappa}$ is not κ-saturated below S.

This result can be argued purely using generic ultrapowers. The proof here is in large part courtesy of (Chen, 2014):

Proof sketch. Suppose for sake of contradiction that NS_κ is κ-saturated below S. Suppose that $S \subseteq \mathsf{Reg} \cap \kappa$; then

$$T := \{\alpha \in S | S \cap \alpha \in \mathsf{NS}_\alpha\}$$

is stationary in κ.

Then a V-generic filter G for $\mathcal{B}_{\mathsf{NS}_\kappa \upharpoonright T}$ is a V-κ-complete V-normal V-ultrafilter with well-founded ultrapower $j : V \to \mathrm{Ult}(V, G)$ that is closed under κ-sequences from $V[G]$. Since we forced with $\mathcal{B}_{\mathsf{NS}_\kappa \restriction \mathsf{T}}$, $\mathsf{T} \in \mathsf{G}$, hence $\kappa \in \mathsf{j}(\mathsf{T})$, and so $\mathsf{j}(\mathsf{S}) \cap \kappa = \mathsf{S}$ is no longer stationary in $\mathrm{Ult}(\mathsf{V}, \mathsf{G})$, hence is nonstationary in $\mathsf{V}[\mathsf{G}]$. But since NS_κ was assumed to be κ-saturated, our forcing has the κ-chain condition and hence S must be stationary in $\mathsf{V}[\mathsf{G}]$; this is a contradiction.

A similar argument holds if $\mathsf{S} \cap \mathrm{cof}(\lambda)$ is stationary for some $\lambda < \kappa$; for if we force in $\mathcal{B}_{\mathsf{NS}_\kappa}$ below $\mathsf{S} \cap \mathrm{cof}(\lambda)$, we get that $\kappa \in \mathsf{j}(\mathsf{S} \cap \mathrm{cof}(\lambda))$ and thus $\mathrm{cf}^{\mathrm{Ult}(\mathsf{V},\mathsf{G})}(\kappa) = \lambda$. But $\mathcal{B}_{\mathsf{NS}_\kappa}$ is κ-cc, so preserves the regularity of κ; this is a contradiction. $\square$

However, there are still useful arguments that can be written *just* from having that $\mathcal{B}_{\mathsf{NS}_\kappa}$ is precipitous. For example, this simplifies Silver's original argument in (Silver, 1977) that if SCH fails at a singular cardinal, then the first singular cardinal at which SCH fails must have countable cofinality. Precipitousness results concerning nonstationary ideals have a long history, and are well collated in (Cummings, 2010)) and (Foreman, 2010).

To compute saturation (and other) properties on ideals in generic extensions, we may use Foreman's Duality Theorems. Let $\mathbb{P}$ be a notion of forcing and let $\mathcal{I}$ be a precipitous ideal in V; we write $\overline{\mathcal{I}}$ to denote $\{\mathsf{A} \in \mathsf{V}^{\mathbb{P}} \mid \exists \mathsf{X} \in \mathcal{I} \ \mathsf{A} \subseteq \mathsf{X}\}$. Then Foreman's Duality Theorems (the many forms of which can be found in (Foreman, 2010), Chapter 7.4) allow for the computation of various (including saturation) properties of $\overline{\mathcal{I}}$. We will be using special cases of this fairly general formulation, which appears as Theorem 7.14 of (Foreman, 2010):

Theorem 2.3.19. *Let $\mathcal{I}$ be precipitous on a set algebra A, $\mathbb{P}$ is a poset, and there is a condition $\dot{\mathfrak{m}} \in \mathsf{j}_{\mathcal{I}}(\dot{\mathbb{P}})$ such that the embedding*

$$\mathrm{id} \times \dot{\mathsf{j}}_{\mathcal{I}} : (\mathsf{A}/\mathcal{I}) \times \mathbb{P} \to (\mathsf{A}/\mathcal{I}) * \mathsf{j}_{\mathcal{I}}(\dot{\mathbb{P}})/\dot{\mathfrak{m}}$$

is regular.

Then there exist conditions $\mathsf{R} \in \mathbb{P} * \dot{\mathsf{A}/\overline{\mathcal{I}}}$ *and* $\mathsf{S} \in (\mathsf{A}/\mathcal{I}) * \mathsf{j}_{\mathcal{I}}(\dot{\mathbb{P}})$ *such that*

$$\left(\mathbb{P} * \dot{\mathsf{A}/\overline{\mathcal{I}}}\right)/\mathsf{R} \cong \left(\mathsf{A}/\mathcal{I} * \mathsf{j}_{\mathcal{I}}(\dot{\mathbb{P}})\right)/\mathsf{S}$$

2.4 Mitchell order

By elementary arguments, for any normal measure U, $\mathsf{U} \notin \mathrm{Ult}(\mathsf{V}, \mathsf{U})$ (c.f. for instance (Jech, 2003), Chapter 17). Thus if U is the only normal measure on κ, then κ is nonmeasurable in $\mathrm{Ult}(\mathsf{V}, \mathsf{U})$. So if κ remains measurable in $\mathrm{Ult}(\mathsf{V}, \mathsf{U})$, then κ must admit two normal measures; further, by elementarity, U-many cardinals below κ are also measurable. So already the strength of "κ is measurable in $\mathrm{Ult}(\mathsf{V}, \mathsf{U})$" is much stronger than a single measurable.

Mitchell rank and Mitchell order give a precise notion of this additional large cardinal strength:

Definition 2.4.1 (Mitchell order)**.** Let κ be a cardinal, and let U_0, U_1 be normal measures on κ. Then we say that $\mathsf{U}_0 \prec \mathsf{U}_1$ if $\mathsf{U}_0 \in \mathrm{Ult}(\mathsf{V}, \mathsf{U}_1)$.

Mitchell order admits a notion of Mitchell rank:

Proposition 2.4.2. $\prec$ *is well-founded.*

Definition 2.4.3. If κ is a cardinal, then $\mathsf{o}(\kappa)$, the Mitchell order of κ, is the supremum of all valid order types of $\prec$-well-ordered sequences on κ.

If $\mathsf{o}(\kappa) = 2$ witnessed by $\mathsf{U}_0 \prec \mathsf{U}_1$, then by definition of $\mathrm{Ult}(\mathsf{V}, \mathsf{U}_1)$, there is some function r with domain κ such that $[\mathsf{r}]_{\mathsf{U}_1} = \mathsf{U}_0$.

In particular, by elementarity, U_1-many $\delta < \kappa$ are measurable, with measure $r(\delta) = \{A \cap \delta \mid A \in U_0\}$.

If furthermore $o(\kappa) = 3$ witnessed by $U_0 \prec U_1 \prec U_2$, then U_2-many $\delta < \kappa$ are measurable of Mitchell rank 2, and there is a function r on κ that can simultaneously represent both U_0 and U_1 for U_2, and U_0 for U_1.

For larger Mitchell ranks and orders, we have the following abstraction that generalizes the above r, which can be found in (Mitchell, 2010).

Definition 2.4.4 (Coherent system of measures)**.** A *coherent sequence of measures on* (a measurable) κ (with positive Mitchell order) is a function $\mathcal{U}$ for which:

1. $\mathcal{U} = \langle \mathcal{U}_{(\alpha,\eta)} \mid \alpha \leq \kappa$ is measurable, $\mathcal{U}_{(\alpha,\eta)}$ is a normal measure on α, and $\eta < o^{\mathcal{U}}(\alpha) \rangle$ where $o^{\mathcal{U}}(\alpha)$ is some ordinal
2. (Coherence) For all such α, η, if $U = \mathcal{U}_{(\alpha,\eta)}$ then $o^{j_U(\mathcal{U})}(\alpha) = \eta$ and $j_U(\mathcal{U})_{(\alpha,\zeta)} = \mathcal{U}_{(\alpha,\zeta)}$ for all $\zeta < \eta$.

Coherent systems will become important when we wish to introduce Magidor forcing.

2.5 Singularizing Cardinals via Prikry Forcing

While the Lévy collapses $\mathrm{Col}(\kappa, \lambda)$ and $\mathrm{Col}(\kappa, < \lambda)$ require no large cardinal strength (or only an inaccessible), being able to singularize a cardinal without collapsing any cardinals by means of forcing requires a ground model with remnants of measurability, as shown by Jensen's Covering Lemma (c.f. (Jech, 2003), Chapter 18). The first such forcing comes from work by

Prikry. A good summary can be found in (Jech, 2003), Chapter 21; as our results depend on generalizations of Prikry forcing, we summarize the salient points here.

Definition 2.5.1 (Prikry forcing)**.** Let U be a normal measure on κ. Then we define the forcing $\mathbb{P}(U)$ by:

1. Conditions are of the form (s, A) where $s \in [\kappa]^{<\omega}$ is strictly increasing, $A \in U$, and $\max(s) < \min(A)$
2. The condition $(t, B) \leq (s, A)$ if $t \supseteq s$, $B \subseteq A$, and for all $\rho \in t \setminus s$, $\rho \in A$

A generic filter G induces an ω-length sequence $\langle \beta_n \mid n < \omega \rangle$ cofinal below κ, given by

$$\langle \beta_n \mid n < \omega \rangle = \bigcup_{(s,A) \in G} s$$

Thus κ is no longer measurable, as $\mathrm{cf}^{V[G]}(\kappa) = \omega$. Remarkably, however, $\mathbb{P}(U)$ does not collapse cardinals, because the measure one components can decide the generic object's behavior. We make this precise:

Definition 2.5.2 (Direct and n-step extension)**.** Let $(t, B) \leq (s, A)$. We say that (t, B) is a *direct extension* of (s, A), written $(t, B) \leq^* (s, A)$, if $t = s$.

And we say that (t, B) is an n-*step* extension of (s, A) if $|t \setminus s| = n$.

As $(s, A) \parallel (s, B)$ for any A, B (as $(s, A \cap B) = (s, A) \wedge (s, B)$), incompatibility is entirely characterized by the stem and so $\mathbb{P}(U)$ is κ^+-cc.

As for preservation of cardinals below κ[1], we use the fact that $\leq^*$ is quite powerful.

Lemma 2.5.3 (Prikry property, open dense version)**.** *Let $p \in \mathbb{P}(U)$, and let D be an open dense subset of $\mathbb{P}(U)$. Then there is a $q \leq^* p$ and an $n < \omega$ such that every n-step extension of q is in D.*

Essentially, this follows from the normality of U. As a corollary, direct extensions are sufficient to decide sentences in the forcing language:

Corollary 2.5.4 (Prikry property, sentential version)**.** *Let $p \in \mathbb{P}(U)$ and let σ be a sentence of the forcing language. Then there is an $r \leq^* p$ such that r decides σ.*

Proof sketch. Since the collection of all conditions deciding σ is open dense, there is some $q \leq^* p$ and an n such that every n-step extension of q decides σ.

But exactly one of σ and $\neg\sigma$ is forced measure-one often by the n-step extensions of q, so there is an $r \leq^* q$ such that every n-step extension of r decides σ the same way (the fleshed-out argument proceeds by induction on n). But then by density, r must decide σ that same way. $\square$

Corollary 2.5.5. $\mathbb{P}(U)$ *adds no bounded subsets of κ, and thus preserves cardinals.*

Proof sketch. Let $p \Vdash \dot{X} \subseteq \tau$ for some $\tau < \kappa$. For each $\alpha < \tau$, there is some $p_\alpha \leq^* p$ deciding "$\alpha \in \dot{X}$"; then since U is κ-complete, the greatest lower bound q of $\langle p_\alpha \mid \alpha < \tau \rangle$ suffices to define $\dot{X}$ entirely within V. $\square$

---

[1] which is enough to show preservation of κ, as the limit of cardinals is also a cardinal

Prikry forcing is a key ingredient in proving the consistency of the failure of the Singular Cardinals Hypothesis, which follows from larger cardinals than a single measurable:

Fact 2.5.6. Let κ be supercompact. Then after forcing to make κ measurable such that $2^\kappa > \kappa^+$ and singularizing κ with Prikry forcing, SCH fails at κ.

The argument is originally due to Magidor; subsequent work by Gitik, Woodin, and Mitchell across several papers showed that the large cardinal hypothesis of κ measurable with $o(\kappa) = \kappa^{++}$ is necessary and sufficient.

As we will generalize in Section 2.6, Prikry forcing admits a neat characterization of genericity, courtesy of (Mathias, 1973):

Proposition 2.5.7. *If $\langle \beta_n \mid n < \omega \rangle$ is $\mathbb{P}(U)$-generic then for all $A \in U$, for cofinitely many n, $\beta_n \in A$.*

Conversely, suppose $\vec{\gamma} = \langle \gamma_n \mid n < \omega \rangle \in V^{\mathbb{P}(U)}$ is such that for every $A \in U$, for cofinitely many n, $\gamma_n \in A$. Let

$$H = \{(s, A) \mid s \textit{ a finite initial segment of } \vec{\gamma},\ A \in U,\ \textit{and } \forall n \geq \mathrm{lh}(s)\ \gamma_n \in A\}$$

Then H is $\mathbb{P}(U)$-generic over V.

If one wishes to simultaneously singularize κ while collapsing cardinals below to make κ into $\aleph_\omega$, then one may adjust the definition of Prikry forcing to interleave collapses between the Prikry points. To ensure a Prikry-type property as in Lemma 2.5.3, one may use the technique of *guiding generics*. We will generalize this to uncountable cofinality in Section 5.1.

This particular forcing is well documented in (Cummings, 2015).

Proposition 2.5.8. *Let κ be measurable with normal measure U and suppose that $2^\kappa = \kappa^+$. Let $j : V \to \mathrm{Ult}(V, U)$ be the measure embedding associated with U. Then there is a $K \in V$ such that K is $\mathtt{Col}(\kappa^+, < j(\kappa))^{\mathrm{Ult}(V,U)}$-generic over V.*

Proof. Observe that there are at most $(2^\kappa)^V$-many antichains of $\mathtt{Col}(\kappa^+, < j(\kappa))$ in the ultrapower; this is since each such antichain is of the form $[f]$, where f is a function with domain κ such that for all α, $f(\alpha)$ is an antichain of $\mathtt{Col}(\alpha^+, < \kappa)$.

Let $\langle A_\beta | \beta < 2^\kappa \rangle$ be a V-enumeration of all antichains of $\mathtt{Col}(\kappa^+, < j(\kappa))$ in the ultrapower. Since $(2^\kappa)^V = \kappa^+$, we have that $|j(\kappa)| = \kappa^+$, and thus there exactly κ^+-many antichains of $\mathtt{Col}(\kappa^+, < j(\kappa))$ in $\mathrm{Ult}(V, U)$. Since $\mathtt{Col}(\kappa^+, < j(\kappa))$ is $\kappa + 1$-closed and the ultrapower is closed under κ-sequences, we may iteratively build a descending chain $K' = \langle p_\beta | \beta < \kappa^+ \rangle$ such that for every maximal antichain A_β in the ultrapower, $p_\beta \in A_\beta$ or is stronger than some condition in A_β. Then K, the upwards closure of K', is as desired. $\square$

With such K, we may define a Prikry-type forcing with interleaved collapses chosen from K:

Definition 2.5.9 (Prikry with guided interleaved collapses)**.** Let U be a normal measure on κ, and let K be as in Proposition 2.5.8. Then we define $\mathbb{PC}(U)$ by:

1. Conditions are of the form (s, c, A, C) where
   - $s \in [\kappa]^{<\omega}$ is strictly increasing and $s(0) = \omega_1$
   - $\max(s) < \min(A)$

- $\mathrm{dom}(c) = \mathrm{dom}(s)$, and for each $n < \mathrm{lh}(s) - 1$, $c(n) \in \mathrm{Col}(s(n)^+, < s(n+1))$, and $c(\mathrm{lh}(s) - 1) \in \mathrm{Col}(s(\mathrm{lh}(s) - 1)^+, < \kappa)$
- C is a function with domain A, $C(\rho) \in \mathrm{Col}(\rho^+, < \kappa)$ for each $\rho \in A$, and $[C]_U \in K$

2. We say $(s', c', A', C') \leq (s, c, A, C)$ if
   - $s' \subseteq s$
   - for each $n < \mathrm{lh}(s)$, $c'(n) \leq c(n)$ in the relevant collapse forcing
   - for each $n \in [\mathrm{lh}(s), \mathrm{lh}(s'))$, $s'(n) \in A$ and $\sup(c(n)) < s'(n)$
   - $A' \subseteq A$
   - $[C'] \leq [C]$

   In the case that $s' = s$, we say $(s', c', A', C') \leq^* (s, c, A, C)$.

   In the case that $|s'| = |s| + n$, we say that (s', c', A', C') is an n-step extension of (s, c, A, C).

Conditions with the same s need not be compatible due to their collapse terms being incompatible. However, the use of the guiding generic still grants us a Prikry-type property:

Lemma 2.5.10. *For every $p \in \mathbb{PC}(U)$ and every dense set D there is a $q \leq^* p$ and an n such that every n-step extension of q is in D.*

Corollary 2.5.11. *Let G be $\mathbb{PC}(U)$-generic over V and let*

$$\langle \beta_n \mid n < \omega \rangle = \bigcup_{(s,c,A,C) \in G} s$$

Then each β_n and β_n^+ is preserved, as are all cardinals $\geq \kappa$.

Further, for every $\rho \in [\beta_n^+, \beta_{n+1}]$ *for some* n, $|\rho|^{V[G]} = \beta_n^+$*; hence* $\kappa = (\aleph_\omega)^{V[G]}$.

A proof is outlined in (Cummings, 2015).

Similar to Prikry forcing, we get the following generic object:

Proposition 2.5.12. *Let* G *be* $\mathbb{PC}(U)$*-generic over* V*; let*

$$\langle \beta_n \mid n < \omega \rangle = \bigcup_{(s,c,A,C) \in G} s$$

and for each n *let*

$$F_n = \{c(n) \mid \exists s, A, C \ (s, c, A, C) \in G \text{ and } n \in \mathrm{dom}(c)\}$$

Then

1. *for all* $A \in U$*, for cofinitely many* n*,* $\beta_n \in A$
2. F_n *is* $\mathrm{Col}(\beta_n^+, < \beta_{n+1})$*-generic over* V *and if* $C \in V$ *with* $[C] \in K$*, then for cofinitely many* n*,* $C(\beta_n) \in F_n$

Conversely:

Proposition 2.5.13. *Work in* $V^{\mathbb{PC}(U)}$*; let* $\vec{\gamma} = \langle \gamma_n \mid n < \omega \rangle$ *be a strictly increasing sequence with limit* κ *such that for each* $A \in U$*, for cofinitely many* n*,* $\gamma_n \in A$*. Let* G_n *be* $\mathrm{Col}(\gamma_n^+, < \gamma_{n+1})$*-generic over* V *such that for all* $C \in V$ *with* $[C] \in K$*, for cofinitely many* n*,* $C(\gamma_n) \in G_n$*.*

Let

$$H = \left\{ (s, c, A, C) \in \mathbb{PC}(U) \;\middle|\; \begin{array}{c} \forall n < lh(s)\ s(n) = \gamma_n \textit{ and } c(n) \in G_n, \\ \\ \textit{and } \forall n \geq lh(s)\ \gamma_n \in A \textit{ and } C(\gamma_n) \in G_n \end{array} \right\}$$

Then H *is* $\mathbb{PC}(U)$*-generic over* V.

The proof is essentially the argument found in (Sinapova and Unger, 2014).

2.6 Singularizing Cardinals via Magidor Forcing

Magidor forcing is a standard tool for singularizing cardinals to have uncountable cofinality without collapsing cardinals. The original approach for Magidor forcing found in (Magidor, 1978) uses a system of representatives r for normal measures below κ the measurable cardinal we wish to singularize to cofinality $\lambda < \kappa$. The relevant properties follow from a coherent system of measures:

Remark 2.6.1. Let $\mathcal{U}$ be a coherent system of normal measures at κ with $o^{\mathcal{U}}(\kappa) = \lambda$, and let $\alpha \leq \kappa$, $\eta < \tau < o^{\mathcal{U}}(\alpha)$.

Since $U_{\alpha,\eta} \in \mathrm{Ult}(V, U_{\alpha,\tau})$, we may fix some $r^{\tau}_{\alpha,\eta} : \kappa \to V$ such that $[r^{\tau}_{\alpha,\eta}]_{U_{\alpha,\tau}} = U_{\alpha,\eta}$.

By elementarity, $U_{\alpha,\tau}$-often below κ, $r^{\tau}_{\alpha,\eta}(\delta)$ will be a normal measure on δ, and will have Mitchell rank η among normal measures on δ. So we can further have that for $U_{\alpha,\tau}$-many $\eta < \kappa$, for all $\zeta < \eta$, $r^{\tau}_{\alpha,\zeta}(\delta) \prec r^{\tau}_{\alpha,\eta}(\delta)$.

And we can expect even stronger coherence as follows: For every $\alpha \leq \kappa$ and every $\tau < o^{\mathcal{U}}(\alpha)$, for $U_{\alpha,\tau}$-many δ, for every $\zeta < \eta < \tau$, we will have that

$$[r^{\eta}_{\alpha,\zeta} \restriction \delta]_{r^{\tau}_{\alpha,\eta}(\delta)} = r^{\tau}_{\alpha,\zeta}(\delta)$$

as a measure-one reflection of the statement that $[r^{\tau}_{\alpha,\eta}]_{U_{\alpha,\tau}} = U_{\alpha,\eta}$

Keeping this in mind, we will fix some measure-one sets witnessing useful coherence properties for r:

Remark 2.6.2.

- Fix $\vec{U} = \langle U_\xi \mid \xi < \lambda \rangle$ a Mitchell-order increasing sequence of normal measures on κ.
- Observe: for $\eta < \xi < \lambda$, we may let $r^{\xi}_{\eta} : \kappa \to V$ represent U_η in $\mathrm{Ult}(V, U_\xi)$. By elementarity, we have that for U_ξ-many δ, $r^{\xi}_{\eta}(\delta)$ is a normal measure on δ with Mitchell rank η, and furthermore this can be witnessed by r; i.e. $r^{\xi}_{\zeta}(\delta) \prec r^{\xi}_{\eta}(\delta)$ for all $\zeta < \eta$.
- Fix X_ξ to be the set of all such $\delta < \kappa$ as in the above bullet point.
- Fix $Y_0 = \kappa \cap \mathrm{Inacc}$.
- For U_ξ-many $\delta \in X_\xi$, we have a coherence property among the representatives, in the sense that whenever $\zeta < \eta < \xi$, $[r^{\eta}_{\zeta} \restriction \delta]_{r^{\xi}_{\eta}(\delta)} = r^{\xi}_{\zeta}(\delta)$.
- Fix Y_ξ to be the set of such $\delta < \kappa$ as in the above bullet point.

From now on, unless otherwise stated, we will assume our measure one sets are included in some Y_ξ, and any ordinal ρ strictly between λ and κ lives in some Y_ξ.

Definition 2.6.3 (Magidor forcing, $\mathbb{M}\left(\vec{U}\right)$: conditions)**.** Conditions are ordered pairs (f, A) where:

1. $\mathrm{dom}(f) \in [\lambda]^{<\omega}$ and $\mathrm{dom}(A) = \lambda \setminus \mathrm{dom}(f)$
2. For $\xi \in \mathrm{dom}(f)$, $f(\xi) \in Y_\xi$, $f(\xi) > \lambda$, and f is strictly increasing.
3. For $\xi \in \mathrm{dom}(A)$ with $\xi < \max \mathrm{dom}(f)$, let $\tau = \min\{\mathrm{dom}(f) \setminus (\xi + 1)\}$. Then $A(\xi) \in r_\xi^\tau(f(\tau))$. That is, if f already has already added an ordinal with index above ξ, then the measure that we'll be picking the value of $f(\xi)$ from in the future will be a normal measure on $f(\tau)$ (instead of κ) with Mitchell rank ξ.
4. For $\xi \in \mathrm{dom}(A)$ with $\xi > \max \mathrm{dom}(f)$, $A(\xi) \in U_\xi$. Letting $\tau = \max(\mathrm{dom}(f) \cap \xi)$, we further require that $A(\xi) \subseteq Y_\xi \setminus (f(\tau) + 1)$.

Definition 2.6.4 (Magidor forcing, $\mathbb{M}\left(\vec{U}\right)$: extension)**.** We say that $(g, B) \leq (f, A)$ if:

1. $g \supseteq f$
2. Whenever $\xi \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$, $g(\xi) \in A(\xi)$
3. Whenever $\xi \in \lambda \setminus \mathrm{dom}(g)$, $B(\xi) \subseteq A(\xi)$

In the case where $g = f$, we say that (f, B) is a *direct extension* of (f, A) and write $(f, B) \leq^* (f, A)$.

For notational convenience, if $p = (f, A)$ we will say that f is the *stem* of p, and write $\mathrm{stem}(p)$ to mean f. Additionally, we will say that A is the *measure component* of p and write $\mathrm{meas}(p)$ to mean A[1].

---

[1] We will use an alternative convention $p = (f^p, A^p)$ in line with (Gitik, 2010) in a future section, when stem and meas become too cumbersome.

Remark 2.6.5. There are several non-obvious reasons why we use $\langle Y_\xi \mid \xi < \lambda \rangle$. The clearest reason, however, is in ensuring that two-step extensions commute, in the following sense: suppose $(\emptyset, A)$ is a condition, $\zeta < \xi < \lambda$, $\rho \in Y_\zeta$, and $\sigma \in Y_\xi$. Then minimally extending to $((\xi, \sigma), A_\xi)$ ensures that $\rho \in A_\xi(\zeta) = A(\zeta) \cap \sigma \in r^\xi_\zeta(\sigma)$ so that for minimally extended choice of B, $((\zeta, \rho) \frown (\xi, \sigma), B)$ is below both $((\zeta, \rho), A_\zeta)$ and $((\xi, \sigma), A_\xi)$.

Here's some heuristic information on how $\mathbb{M}\left(\vec{U}\right)$ behaves:

- $\mathbb{M}\left(\vec{U}\right)$ adds a sequence $\langle \alpha_\eta \mid \eta < \lambda \rangle$ increasing, normal, and with supremum κ, changing the cofinality of κ into λ. We obtain $\langle \alpha_\eta \mid \eta < \lambda \rangle$ from a generic G by $\langle \alpha_\eta \mid \eta < \lambda \rangle = \bigcup_{(f,A) \in G} f$.
- A condition (f, A) in the generic filter has specified a finite subset of the above sequence where if $\eta \in \mathrm{dom}(f)$, then $f(\eta) = \alpha_\eta$. Further, if $\eta \in \mathrm{dom}(A)$, then (f, A) specifies that $\alpha_\eta \in A(\eta)$ (and in particular, α_η is a V-measurable cardinal with $\vec{U}$-rank η).

Lemma 2.6.6. $\mathbb{M}\left(\vec{U}\right)$ *has the* κ^+*-cc and thus preserves all cardinals larger than* κ*.*

Proof. As with Prikry forcing, incompatible conditions must have different stems so antichains are no larger than $\left|\kappa^{[\lambda]^{<\omega}}\right| = \kappa$. $\square$

Crucially, $\mathbb{M}\left(\vec{U}\right)$ itself factors into different Magidor forcings below each point, as follows:

Definition 2.6.7. Let $p = (f, A) \in \mathbb{M}\left(\vec{U}\right)$ and let ξ be a limit ordinal below λ. Then we may define the following:

- $(p)_\xi = (f, A)_\xi = (f \restriction (\xi + 1), A \restriction (\xi + 1))$

- $(p)^{\xi} = (f, A)^{\xi} = (f \restriction (\lambda \setminus (\xi + 1)), A \restriction (\lambda \setminus (\xi + 1)))$
- $\mathbb{M}\left(\vec{U}\right)_{(\xi,\beta)} = \{(f, A)_{\xi} \mid (f, A) \in \mathbb{M}\left(\vec{U}\right), \xi \in \mathrm{dom}(f), \text{ and } f(\xi) = \beta\}$
- $\mathbb{M}\left(\vec{U}\right)^{(\xi,\beta)} = \{(f, A)^{\xi} \mid (f, A) \in \mathbb{M}\left(\vec{U}\right), \xi \in \mathrm{dom}(f), \text{ and } f(\xi) = \beta\}$

Fact 2.6.8. Observe that, for ξ limit, by choice of Y_{ξ}'s, relative to the weakest A for which $p = ((\xi, \beta), A)$, we have that $\mathbb{M}\left(\vec{U}\right)/p \cong \mathbb{M}\left(\vec{U}\right)_{(\xi,\beta)} \times \mathbb{M}\left(\vec{U}\right)^{(\xi,\beta)}$, which are themselves Magidor forcings. In particular,

$$\mathbb{M}\left(\vec{U}\right)_{(\xi,\beta)} = \mathbb{M}\left(\left\langle r^{\xi}_{\zeta}(\beta) \mid \zeta < \xi \right\rangle\right)$$

and has the β^{+}-cc.

Additionally,

$$\mathbb{M}\left(\vec{U}\right)^{(\xi,\beta)} = \mathbb{M}\left(\langle U'_{\chi} \mid \xi < \chi < \lambda\rangle\right)$$

where for $\xi < \chi < \lambda$, $U'_{\chi} = \{A \setminus (\beta + 1) \mid A \in U_{\chi}\}$. Furthermore, $\left(\mathbb{M}\left(\vec{U}\right)^{(\xi,\beta)}, \leq^{*}\right)$ is $< \beta^{+}$-directed closed by closure of the measures and size of the cardinals in each U'_{χ}.

Note that by Proposition 2.2.8, after forcing with $\mathbb{M}\left(\vec{U}\right)_{(\xi,\beta)}$, the partial order $\left(\mathbb{M}\left(\vec{U}\right)^{(\xi,\beta)}, \leq^{*}\right)$ remains $< \beta^{+}$-distributive.

A similar factoring applies to any finite stem, not just (ξ, β).

Magidor forcing admits an analogue of the Prikry lemma. Recall that in Prikry forcing, for every condition p and every dense subset D, there is an n and a $q \leq^{*} p$ such that every n-step extension of q is in D. For $\mathbb{M}\left(\vec{U}\right)$, this lemma takes on a different form; rather than n-step extensions, we care about a-step extensions. Whereas an n-step extension of Prikry tacked on n new ordinals at the end, an a-step extension will specify finitely many new elements of our generic sequence, with indices coming exactly from a:

Definition 2.6.9. Let $a \in \lambda^{<\omega}$. If $(f, A) \in \mathbb{M}\left(\vec{U}\right)$ and $a \subseteq \mathsf{dom}(A)$, we say that (g, B) is an a-step extension of (f, A) if $(g, B) \leq (f, A)$ and $\mathsf{dom}(g) = \mathsf{dom}(f) \sqcup a$.

Before we state and prove Prikry-type lemmas for Magidor forcing, it is worth formalizing how we may diagonalize over all possible extensions. We will want a notion of *minimal* extension, where we wish to extend e.g. (f, A) to $(f \frown \vec{\nu}, A')$ for some $\vec{\nu}$, and A' altered as little as possible from A:

Definition 2.6.10. If $p = (f, A)$, $\vec{\nu} : a \to \kappa$ for some a a finite subset of $\lambda \setminus \mathsf{dom}(f)$, and there is some B for which $(f \frown \vec{\nu}, B)$ is a condition below p, we write $p \frown \vec{\nu}$ to mean the weakest such $(f \frown \vec{\nu}, A')$ below p. Namely, for $\xi > \max \mathsf{dom}(f \frown \vec{\nu})$, $A'(\xi) = A(\xi) \setminus (f \frown \vec{\nu}(\xi) + 1)$, and for $\xi < \max \mathsf{dom}(f \frown \vec{\nu})$: for τ being the least domain element of $f \frown \vec{\nu}$ bigger than ξ:

- if $\tau \in \mathsf{dom}(\nu)$, we have that $A'(\xi) = A(\xi) \cap (\vec{\nu}(\tau))$; since $[\alpha \mapsto A(\xi) \cap \alpha]_{r^{\tau}_{\xi}(\nu(\tau))} = A(\xi)$ and $\nu(\tau) \in Y_{\tau}$, $A'(\xi) \in r^{\tau}_{\xi}(\nu(\tau))$ and is the $\subseteq$-largest such.
- if $\tau \in \mathsf{dom}(f)$, then $A'(\xi) = A(\xi)$.

Lemma 2.6.11 (Diagonalization Lemma). *Let* $p = (g, H) \in \mathbb{M}(\vec{U})$ *and let*

$$E = \left\{ \vec{\nu} : a \to \kappa \mid a \in [\mathsf{dom}(H)]^{<\omega}, \vec{\nu} \in \prod_{\alpha \in a} H(\alpha), \textit{ and } p \frown \vec{\nu} \textit{ is a condition below } p \right\}$$

Let $\langle p_{\vec{\nu}} \mid \vec{\nu} \in E \rangle$ *be some family of conditions such that* $p_{\vec{\nu}} \leq^* p \frown \vec{\nu}$. *Then there exists a* $q \leq^* p$ *such that for all* $\vec{\nu} \in E$ *for which* $q \frown \vec{\nu}$ *is a condition (i.e.* $\vec{\nu}$ *is increasing and lies in* $\prod_{\zeta \in \mathrm{dom}(\nu)} \mathrm{meas}(q)(\zeta)$*),* $q \frown \vec{\nu} \leq^* p_{\vec{\nu}}$.

Proof. For each $a \in [\mathrm{dom}(H)]^{<\omega}$, let $E_a = \{\vec{\nu} \in E \mid \mathrm{dom}(\vec{\nu}) = a\}$. We will show how to obtain, for each a, a $q_a \leq^* p$ such that for all $\vec{\nu} \in E_a$ for which $\vec{\nu}$ is a valid extension of q_a, $q_a \frown \vec{\nu} \leq^* p_{\vec{\nu}}$; we'll say that such q_a is good for p and E_a. Since $\leq^*$ is λ^+-directed closed, (as every involved measure is λ^+-complete), the greatest $\leq^*$-lower bound q of $\langle q_a \mid a \in [\mathrm{dom}(H)]^{<\omega} \rangle$ will be as desired.

To define q_a, we first define q_a for $|a| = 1$ and then will induct on $|a|$. Without loss of generality, we may assume that $g = \emptyset$; we will justify why we may assume this after the following claim.

Claim 2.6.12. *Suppose* $g = \emptyset$*; then for each* $\alpha < \lambda$ *there is some* $p_\alpha \leq^* p$ *(with* $p_\alpha = (\emptyset, H^\alpha)$*) that is good for* p *and* $E_{\{\alpha\}}$*, that is, for all* $\nu \in H^\alpha(\alpha)$*,* $p_\alpha \frown (\alpha, \nu) \leq^* p_{(\alpha,\nu)}$.

Proof of claim. Let $\alpha < \lambda$; it is given that for each $\nu \in H(\alpha)$, $p_{(\alpha,\nu)} := ((\alpha, \nu), H_\nu)$ is $\leq^*$-below $p \frown (\alpha, \nu)$.

We define $p_\alpha = (\emptyset, H^\alpha)$ over three separate cases, defining $H^\alpha(\beta)$ depending on where β falls relative to α.

Let $\beta < \lambda$. Then to define $H^\alpha(\beta)$:

Case 1: If $\beta > \alpha$, then let

$$H^\alpha(\beta) = \underset{\nu \in H(\alpha)}{\Delta} H_\nu(\beta)$$

Case 2: If $\beta < \alpha$, consider the map $\nu \mapsto H_\nu(\beta)$, mapping from $\nu \in H(\alpha)$ to measure one sets in $r^\alpha_\beta(\nu)$. By Łoś' Theorem and definition of r^α_β, $[\nu \mapsto H_\nu(\beta)]_{U_\alpha} \in U_\beta$; let $B_{\beta,\alpha} = [\nu \mapsto H_\nu(\beta)]_{U_\alpha}$. Also, since $[r^\beta_\alpha]_{U_\alpha} = U_\beta$, and since $B_{\beta,\alpha} = [\nu \mapsto B_{\beta,\alpha} \cap \nu]_{U_\alpha}$, we have that for some $C_{\beta,\alpha} \in U_\alpha$, for all $\nu \in C_{\beta,\alpha}$, $B_{\beta,\alpha} \cap \nu = H_\nu(\beta)$. Let

$$H^\alpha(\beta) = B_{\beta,\alpha} \cap H(\beta)$$

Our final case depends on the use of $C_{\beta,\alpha}$.

Case 3: Finally, for $\beta = \alpha$, let

$$H^\alpha(\alpha) = H(\alpha) \cap \bigcap_{\gamma<\alpha} C_{\gamma,\alpha}$$

which is measure one by the completeness of the relevant measure.

To finish the claim, let $\alpha \in \mathrm{dom}(H)$ and let $\nu \in H^\alpha(\alpha)$. Let $J = \mathrm{meas}(p_\alpha \frown (\alpha, \nu))$; our goal is to show that for each $\beta \neq \alpha$, $J(\beta) \subseteq H_\nu(\beta)$.

For $\beta > \alpha$, $J(\beta) = H^\alpha(\beta) \setminus (\nu + 1)$, which by the choice made in Case 1, is included in $H_\nu(\beta)$.

For $\beta < \alpha$, then by the choices made in Cases 2 and 3, $\nu \in C_{\beta,\alpha}$ and hence $J(\beta) = H^\alpha(\beta) \cap \nu \subseteq H_\nu(\beta)$.

Thus H^α is as desired. $\square$

Of course, we will want to argue for arbitrary g:

Claim 2.6.13. *For every* g *and every* $\alpha < \lambda$ *there is some* $p_\alpha \leq^* p$ *(with* $p_\alpha = (\emptyset, H^\alpha)$*) that is good for* p *and* $E_{\{\alpha\}}$*, that is, for all* $\nu \in H^\alpha(\alpha)$*,* $p_\alpha \frown (\alpha, \nu) \leq^* p_{(\alpha,\nu)}$*.*

Proof of claim. As in Claim 2.6.12, for each $\nu \in H(\alpha)$, let $p_{(\alpha,\nu)} \leq^* p \frown (\alpha, \nu)$, where $p_{(\alpha,\nu)} = (g, H_\nu)$.

In the event that $g = (\xi, \rho)$, we have two cases, depending on α:

Case 1: If $\alpha > \xi$, then we may argue as in Claim 2.6.12 for $p \restriction (\xi, \lambda) \in \mathbb{M}(\langle U_\gamma \mid \xi < \gamma < \lambda \rangle)$ to obtain some

$$(\emptyset, {}^\xi H^\alpha) \in \mathbb{M}(\langle U_\gamma \mid \xi < \gamma < \lambda \rangle)$$

such that whenever $\nu \in {}^\xi H^\alpha(\alpha)$, $(\emptyset, {}^\xi H^\alpha) \frown (\alpha, \nu) \leq^* p_{(\alpha,\nu)} \restriction (\xi, \lambda)$.

As for working below ξ, let $\zeta < \xi$. Then for each $\nu \in {}^\xi H^\alpha(\alpha)$, $H_\nu(\zeta) \in r^\xi_\zeta(\rho)$, and $|r^\xi_\zeta(\rho)| = 2^\rho < \kappa$. So by κ-completeness there is an $F_{\zeta,\alpha} \in U_\alpha$ and some $G_{\zeta,\alpha} \in r^\xi_\zeta(\rho)$ such that for all $\nu \in F_{\zeta,\alpha}$, $H_\nu(\zeta) = G_{\zeta,\alpha}$.

Then our desired $p_\alpha \leq^* p$ is given by

$$meas(p_\alpha)(\gamma) = \begin{cases} G_{\gamma,\alpha} & \gamma < \xi \\ {}^\xi H^\alpha(\gamma) \cap \bigcap_{\zeta<\xi} F_{\zeta,\alpha} & \gamma = \alpha \\ {}^\xi H^\alpha(\gamma) & \gamma > \xi,\ \gamma \neq \alpha \end{cases}$$

since whenever $\nu \in \mathrm{meas}(p_\alpha)(\nu)$, by construction we have that $(p_\alpha \frown (\alpha,\nu)) \restriction (\xi,\lambda) \leq^* p_{(\alpha,\nu)} \restriction (\xi,\lambda)$, and since $\nu \in F_{\zeta,\alpha}$ for all $\zeta < \xi$, $(p_\alpha \frown (\alpha,\nu)) \restriction \xi = \langle G_{\gamma,\alpha} \mid \gamma < \xi \rangle$ which is by construction a direct extension of $p_{(\alpha,\nu)} \restriction \xi$.

Case 2: If $\alpha < \xi$, then $H(\alpha) \in r^\xi_\alpha(\rho)$ and hence $H(\alpha) \subseteq \rho$. So working in $\mathbb{M}\left(\vec{U}\right)_{(\xi,\rho)}$, we may invoke Claim 2.6.12 for $(\emptyset, H) \restriction \xi$ to obtain $(\emptyset, G) \in \mathbb{M}\left(\vec{U}\right)_{(\xi,\rho)}$ such that for every $\nu \in G(\alpha)$, $(\emptyset, G) \frown (\alpha,\nu) \leq^* p_{(\alpha,\nu)} \restriction \xi$. But then since there are only ρ-many such ν, for each such ν and each $\beta > \xi$, $\bigcap_{\nu \in G(\alpha)} H_\nu(\beta) \in U_\beta$. So let

$$p_\alpha = \left((\xi,\rho), G \frown \left\langle \bigcap_{\nu \in G(\alpha)} H_\nu(\beta) \mid \xi < \beta < \lambda \right\rangle\right)$$

Then $p_\alpha \leq^* p$ and by our construction for $p_\alpha \restriction (\xi,\lambda)$, we have that $p_\alpha \frown (\alpha,\nu) \leq^* p_{(\alpha,\nu)}$ for every $\nu \in G(\alpha)$.

For larger g, note that $|g|$ is always finite. So, letting $\beta = \max \mathrm{dom}(g)$, we may recursively argue exactly as above for $p \restriction (\beta,\lambda)$ and $p \restriction \beta$. $\square$

Hence we now have, for each $|a| = 1$, some q_a good for p and E_a.

For the induction, suppose that we have that for each a of length n some $q_a \leq^* p$ that is good for p and E_a. Let b be a stem of length $n+1$, let $\alpha_0 = \min b$, and let $a = b \setminus \{\alpha_0\}$. Then by induction, for all $\nu \in H(\alpha_0)$, there is some $q'_\nu \leq^* p_{(\alpha_0,\nu)}$ such that q'_ν is good for $p_{(\alpha_0,\nu)}$ and E_a. By the above claim applied to the family $\langle q'_\nu \mid \nu \in H(\alpha_0) \rangle$, there is some q_b that is good for p and E_b. $\square$

Lemma 2.6.14 (Prikry-type lemma, open dense version)**.** *Let* D *be an open dense set and* $p = (g, H)$ *a condition. Then there is an* $a \in \lambda^{<\omega}$ *and an* $r \leq^* p$ *such that every* a*-step extension of* r *is in* D*.*

Proof. Let E be as in Lemma 2.6.11. For every $\vec{\nu} \in E$, let

$$p_{\vec{\nu}} = \begin{cases} \text{some chosen } q \leq^* p \frown \vec{\nu} \text{ with } q \in D & \text{if such } q \text{ exists} \\ p \frown \vec{\nu} & \text{otherwise} \end{cases}$$

Of course, by open density, there will *always* be some $q \leq p \frown \vec{\nu}$ with $q \in D$; the point is we want such q to be a direct extension only.

By Lemma 2.6.11, we obtain, for $\langle p_{\vec{\nu}} \mid \vec{\nu} \in E \rangle$, some $p' \leq^* p$ such that for all $\vec{\nu}$, $p' \frown \vec{\nu} \leq^* p_{\vec{\nu}}$. We now claim the following:

Claim 2.6.15. *For every* $a \in [\mathrm{dom}(H)]^{<\omega}$*, there is a* $q \leq^* p'$ *such that either every* a*-step extension of* q *is in* D*, or none are.*

Proof of claim. We proceed by induction on $|a|$.

If $a = \{i\}$, then let $B^+ = \{\nu \in \mathrm{meas}(p')(i) \mid p' \frown (i, \nu) \in D\}$, and $B^- = \{\nu \in \mathrm{meas}(p')(i) \mid p' \frown (i, \nu) \notin D\}$. Since $B^+ \sqcup B^- = \mathrm{meas}(p')(i)$, exactly one of these is measure one with respect to the relevant measure; let B' be whichever one it is. Let H^* be the function on λ where $H^*(i) = B'$ and when $j \neq i$, $H^*(j) = H(j)$. Let $q = (\mathrm{stem}(p'), H^*)$. To see that q is as desired, note that clearly $q \leq^* p'$, and if $B' = B^+$, then every $\{i\}$-step extension is in D. If instead $B' = B^-$, then let $\nu \in B^-$. Then $q \frown (i, \nu) \leq^* p' \frown (i, \nu) \leq^* p_{(i,\nu)}$ and since

$p' \frown (i, \nu) \notin D$, we must have had that $p_{(i,\nu)} = p \frown (i, \nu)$ which is also not in D. What's more, by definition of $p_{(i,\nu)}$, *no* direct extension of $p \frown (i, \nu)$ lands in D, either. In particular, no direct extension of $q \frown (i, \nu)$ is in D. Since ν ranged over all valid i-step extensions of q, we have shown that no $\{i\}$-step extension of q is in D.

Before we move to the general case, we note the following for the case $a = \{i, j\}$ with $i < j$. For each $\nu \in \mathit{meas}(p')(i)$, we may perform the same argument as above for $p' \frown (i, \nu)$ at j to obtain a $q_\nu \leq^* p' \frown (i, \nu)$ such that either every $\{j\}$-step extension of $p' \frown (i, \nu)$ is in D, or none are. We will additionally induct on this as well.

For $|a| > 1$, suppose the claim is true for every b-step extension with $|b| = n$. Let $|a| = n+1$, and let $i = \min(a)$. and suppose $\nu \in \mathit{meas}(p')(i)$. By our induction hypothesis, there is a $q_\nu \leq^* p' \frown (i, \nu)$ such that either every $a \setminus \{i\}$-step extension of q_ν is in D, or none are. As in the $|a| = 1$ case, let $B^+ = \{\nu \in \mathit{meas}(p')(i) \mid \forall\ a \setminus \{i\}\text{-step extensions } r \leq q_\nu,\ r \in D\}$ and let $B^- = \{\nu \in \mathit{meas}(p')(i) \mid \forall\ a \setminus \{i\}\text{-step extensions } r \leq q_\nu,\ r \notin D\}$. Since $B^+ \sqcup B^- = \mathit{meas}(p')(i)$, exactly one of B^+ and B^- is measure one; let B' be whichever one it is. Then the desired direct extension $q \leq^* p'$ will be given by $\mathit{meas}(q)(i) = B'$, $\mathit{meas}(q)(j) = \Delta_{\nu \in B'} \mathit{meas}(q_\nu)(j)$ whenever $j \in a \setminus \{i\}$, and $\mathit{meas}(q)(j) = \mathit{meas}(p')(j)$ otherwise.

To see that q is as desired, note that every a-step extension r of q can be viewed as a $\{i\}$-step extension followed by an $a \setminus \{i\}$-step extension; $\mathit{stem}(r)(i) \in B'$, and for every $j \in a \setminus \{i\}$, $\mathit{stem}(r)(j) \in \mathit{meas}(q_\nu)(j)$. Thus as in the $|a| = 1$ argument, and by choice of p', either every a-step extension of q is in D or none are.

And moreover, if $i' < i$, we can inductively argue for $p' \frown (i', \nu)$ for each $\nu \in \mathtt{meas}(p')(i')$ to obtain $q_\nu \leq^* p' \frown (i', \nu)$ such that either every a-step extension of q_ν is in D, or none are. □

By iterated use of the above claim, we fix a λ-length enumeration of $[\mathtt{dom}(H)]^{<\omega}$ and obtain a $\leq^*$-decreasing sequence $\langle p_a \mid a \subseteq [\mathtt{dom}(H)]^{<\omega} \rangle$ with $p_a \leq^* p'$ such that every a-step extension of p_a is in D or none are.

Let r be a $\leq^*$-lower bound of the p_a's, obtained by taking λ-sized intersections over the measure components of the p_a's. Then by construction of r, for any b, every b-step extension of r is below p_b so either every b-step extension of r is in D, or none are.

Since D is open dense, let $q \in D$ be below r, and let a be such that q is an a-step extension of r. Then since some a-step extension of q_a is in D, we have that all a-step extensions of q_a are in D and hence so are all a-step extensions of r. □

A careful reading of the above argument actually gives something stronger:

Lemma 2.6.16. *Let* D *be an open dense set,* $p = (g, H)$ *be a condition, and* $r \leq^* p$ *be as in the proof of Lemma* 2.6.14*. Then for every* $b \subseteq [\mathtt{dom}(H)]^{<\omega}$*, either every* b*-step extension of* r *is in* $\mathcal{D}$*, or none are.*

As with Prikry forcing, the open dense version translates into a sentential version for every sentence of the forcing language:

Lemma 2.6.17 (Prikry-type lemma, sentential version)**.** *Let* σ *be a sentence of the forcing language, and let* $(f, A) \in \mathbb{M}\left(\vec{U}\right)$*. Then there is an* $(f, B) \leq^* (f, A)$ *such that* (f, B) *decides* σ*.*

The proof may be done by induction: for such (f, A), let $(f, B) \leq^* (f, A)$ and $a \subseteq [\lambda]^{<\omega}$ be such that every a-step extension of (f, B) decides σ. The argument then proceeds by induction on $|a|$. But the crux of the argument is really a measure-one concentration:

Proof. Since the collection of conditions deciding σ is open dense, by Lemma 2.6.14, there is some $(f, A') \leq^* (f, A)$ and some a such that every a-step extension of (f, A') decides σ. But then, by a similar argument as in Claim 2.6.15, one of either σ or $\neg\sigma$ is forced measure one-often below (f, A'). So we may shrink the measures in A' to some measure-one family B so that every a-step extension of (f, B) decides σ the exact same way. But then (f, B) decides σ that way as well. $\square$

Further arguments along similar lines as in Lemma 2.6.14 yields an even stronger Prikry-type lemma in which we may change the initial part of a condition:

Lemma 2.6.18 (Prikry-type lemma, tail-change version). *Let* D *be an open dense set, let* $(f, A) \in \mathbb{M}(\vec{U})$*, and let* $\beta \in \mathrm{dom}(f)$*. Then there is an* $(f, B) \leq^* (f, A)$ *such that* $(f, B)_\beta = (f, A)_\beta$ *and for every* b*, if* (g, H) *is a* b*-step extension of* (f, B) *and* $(g, H) \in D$*, then every* $b \setminus (\mathrm{dom}(g) \cap \beta)$*-step extension of* $(g, H)_\beta \frown (f, B)^\beta$ *is also in* D*.*

This appears as Theorem 3.5 of (Fuchs, 2014), and we give a proof here:

Proof. For ease of notation, let $f(\beta) = \rho$. For each $r \leq (f, A)_\beta$ in $\mathbb{M}(\vec{U})_{(\beta,\rho)}$, by Lemma 2.6.16, let $p_r \leq^* r \frown (f, A)^\beta$ (in the full $\mathbb{M}(\vec{U})$) be such that for every b, either every b-step extension of p_r is in D, or none are. By construction, for each such r, $(p_r)_\beta \leq^* r$ and $(p_r)^\beta = (f, \mathrm{meas}(p_r))^\beta$.

Let $q_r = (f, A)_\beta \frown (p_r)^\beta$; note that $q_r \leq^* (f, A)$ by construction. Since there are at most 2^ρ-many such r and since $\leq^*$ on $\mathbb{M}(\vec{U})^{(\beta,\rho)}$ is ρ'-directed closed for ρ' the least inaccessible above ρ, let (f, B) be the greatest $\leq^*$-lower bound of the q_r's, where B is given by

$$B(\gamma) = \begin{cases} A(\gamma) & \gamma < \beta \\ \bigcap_r meas(q_r)(\gamma) & \gamma > \beta \end{cases}$$

To complete the argument, if $(g, H) \leq (f, B)$ and $(g, H) \in D$, let $r = (g, H)_\beta$ and let b be such that (g, H) is a b-step extension of (f, B). Then by construction, $(g, H) \leq r \frown (f, B)^\beta \leq p_r$ and by definition of p_r, every b-step extension of p_r (in particular, any $b \setminus (dom(g) \cap \beta)$-step extension of $r \frown (f, B)^\beta$) is also in D. $\square$

Lemma 2.6.19 (Prikry-type lemma, tail-change sentential version)**.** *Let* σ *be a sentence of the forcing language, let* $(f, A) \in \mathbb{M}(\vec{U})$*, and let* $\beta \in dom(f)$*. Then there is an* $(f, B) \leq^* (f, A)$ *such that* $(f, B)_\beta = (f, A)_\beta$ *and if* $(g, H) \leq (f, B)$ *and* (g, H) *decides* σ*, then* $(g, H)_\beta \frown (f, B)^\beta$ *also decides* σ *the same way.*

This appears as Lemma 4.5 of (Magidor, 1978), and has a short proof:

Proof. As the collection of conditions deciding σ is open dense, we may re-run the argument of Lemma 2.6.18, but while invoking Lemma 2.6.17 to ensure that each p_r as above also decides σ. Then define (f, B) as in the proof of Lemma 2.6.18. Then if $(g, H) \leq (f, B)$ and (g, H) decides σ, then let $r = (g, H)_\beta$. As before $(g, H) \leq r \frown (f, B)^\beta \leq p_r$ and p_r already decides σ. But then p_r must decide σ the same way as (g, H), and hence $r \frown (f, B)^\beta = (g, H)_\beta \frown (f, B)^\beta$ must also decide σ the same way as (g, H). $\square$

Observe that a $\mathbb{M}\left(\vec{U}\right)$-generic adds a sequence $\langle \alpha_\eta \mid \eta < \lambda \rangle = \bigcup_{(g,H) \in G} g$ strictly increasing, normal, and with supremum κ. As with Prikry forcing, Magidor forcing preserves cardinals:

Lemma 2.6.20. *Let $\delta \in V$ be a cardinal. Then δ is a cardinal in $V^{\mathbb{M}(\vec{U})}$; moreover if δ is V-regular, $\delta \neq \kappa$, and $\delta \neq \alpha_\eta$ for any limit η then δ remains regular.*

The proof is also in (Magidor, 1978). We prove it here, too:

Proof. Since $\mathbb{M}(\vec{U})$ is κ^+-cc and since κ is a limit of cardinals, we only need to check for $\delta < \kappa$.

In the event that $p \Vdash \delta \leq \alpha_0$, then already $\mathbb{M}(\vec{U})_{(0,\alpha_0)}$ is δ^+-directed closed and therefore cannot collapse δ.

Otherwise, without loss of generality (by expanding $\mathrm{dom}(\mathrm{stem}(p))$ if need be) let β be least such that $p \Vdash (\alpha_\beta)^+ \leq \delta$; by minimality, without loss of generality $p \Vdash \delta \leq \alpha_{\beta+1}$.

We will now show that any bounded subset of δ in $V^{\mathbb{M}(\vec{U})}$ is added by $\mathbb{M}(\vec{U})_{(\beta,\alpha_\beta)}$, which is δ-cc and therefore could not have collapsed (or singularized) δ. Let $p \Vdash \dot{X} \subseteq \delta$; for each $\gamma < \delta$, by Lemma 2.6.19, let $q_\gamma \leq^* p$ be such that $(q_\gamma)_\beta = p_\beta$ and if $r \leq q_\gamma$ and r decides the statement "$\gamma \in \dot{X}$", then $r_\beta \frown (q_\gamma)^\beta$ decides likewise. Let q be a $\leq^*$-lower bound of the q_γ's; by construction of q, any $\mathbb{M}(\vec{U})_{(\beta,\alpha_\beta)}$-generic including $(q)_\beta$ suffices to define $\dot{X}$, and thus X is added by $\mathbb{M}(\vec{U})_{(\beta,\alpha_\beta)}$. $\square$

As with Prikry forcing, $\mathbb{M}(\vec{U})$ admits a notion of geometricity that characterizes the genericity of the sequence $\langle \beta_\eta \mid \eta < \lambda \rangle$ added. Geometricity for a Magidor generic sequence differs from a Prikry generic sequence in that the geometricity reflects to smaller limit ordinals. We make this precise:

Definition 2.6.21. Suppose in some outer model W of V, some sequence $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle \in W$ is increasing, normal, and cofinal in κ.

Then we say that $\vec{\beta}$ is *geometric* if for every $\xi \leq \lambda$ limit, and every $\langle A_\eta \mid \eta < \xi \rangle$ in V with each $A_\eta \in r^\xi_\eta(\beta_\eta)$ (and if $\xi = \lambda$, then $A_\eta \in U_\eta$), for coboundedly many $\eta < \xi$, $\beta_\eta \in A_\eta$.

Lemma 2.6.22. *Let* G *be* $\mathbb{M}\left(\vec{U}\right)$*-generic over* V *and let*

$$\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle =: \bigcup_{(f,A) \in G} f$$

Then $\vec{\beta}$ *is geometric.*

Proof. When $\xi = \lambda$, for each A, the set

$$\mathcal{D} = \{(g, H) \mid \forall \eta > \max(\mathrm{dom}(g)),\ H(\eta) \subseteq A_\eta\}$$

is dense by measure-one set intersection; namely if $(g, H) \in \mathbb{M}\left(\vec{U}\right)$, then $(g, (H \restriction \max \mathrm{dom}(g)) \frown \langle H(\xi) \cap A(\xi) \mid \xi > \max \mathrm{dom}(g) \rangle) \in \mathcal{D}$. Thus if $(f, A) \in \mathcal{D} \cap G$, then by definition of $\leq$, for all $\xi > \max \mathrm{dom}(f)$, $\beta_\xi \in A(\xi)$.

For $\xi < \lambda$ limit, the same applies in $\mathbb{M}(\vec{U})_{(\xi,\beta_\xi)}$: within V, the set

$$\{(g, H) \mid g(\xi) = \beta_\xi \text{ and } \forall \zeta \in ((\max \mathrm{dom}(g)) \cap \xi, \xi)\ \ H(\eta) \subseteq A_\eta\}$$

is dense in $\mathbb{M}(\vec{U})_{(\xi,\beta_\xi)}$, hence is dense below the weakest condition p in the full $\mathbb{M}(\vec{U})$ with $\mathrm{stem}(p)(\xi) = \beta_\xi$. $\square$

The converse is also true in the following sense:

Theorem 2.6.23. *Suppose $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle$ is geometric. Let*

$$H = \left\{ (f, A) \in \mathbb{M}\left(\vec{U}\right) \;\middle|\; \begin{array}{l} \forall \eta \in \mathrm{dom}(f)\ f(\eta) = \beta_\eta \\ \textit{and } \forall \eta \in \mathrm{dom}(A)\ \beta_\eta \in A(\eta) \end{array} \right\}$$

Then H is $\mathbb{M}(\vec{U})$-generic over V.

A proof can be found in (Fuchs, 2014) that, at its core, makes use of Lemma 2.6.14 and Lemma 2.6.18. We present a more direct proof here, which better generalizes to more exotic Magidor-type forcings such as the one we describe in Chapter 5:

Proof. H is a filter, as by construction the empty condition $1_{\mathbb{M}(\vec{U})} \in H$ and H is clearly upwards closed. As for downwards directedness, if $p, q \in H$ then $\mathrm{stem}(p)$ and $\mathrm{stem}(q)$ are finite segments of the same sequence, hence $p \parallel q$ and the intersection of $\mathrm{meas}(p)$ and $\mathrm{meas}(q)$ contains every β_η for $\eta \notin \mathrm{dom}(p \cup q)$. Thus $p \wedge q \in H$.

Genericity is more complicated and proceeds by transfinite induction on the length of $\vec{U}$ and the factoring of Fact 2.6.8.

Our base case is when $\lambda = \omega$ and we are considering $\mathbb{M}(\langle U_n \mid n < \omega \rangle)$. Let $\mathcal{D} \in V$ be open dense in $\mathbb{M}(\langle U_n \mid n < \omega \rangle)$. For each $g \in V$ a valid $\mathbb{M}(\langle U_n \mid n < \omega \rangle)$-stem such that $\mathrm{dom}(g)$ is an initial subsequence of ω[1], by Lemma 2.6.14, let a_g, A_g be such that $(g, A_g) \leq^* 1_{\mathbb{M}(\vec{U})} \frown g$ and every a_g-step extension of (g, A_g) is in $\mathcal{D}$; without loss of generality, $a_g = [\max(\mathrm{dom}(g))+1, k_g]$ for some $k_g < \omega$. Let B be a measure one system given by

$$B(n) = \Delta_g A_g(n) := \left\{ \alpha < \kappa \mid \alpha \in \bigcap_{\max(\mathrm{ran}(g))<\alpha} A_g(n) \right\}$$

Since $\langle \beta_n \mid n < \omega \rangle$ is geometric, there is some j such that for all $n \geq j$, $\beta_n \in B(n)$; let $g = \langle \beta_i \mid i < j \rangle$. By definition of B, $(g, B \restriction [j, \omega)) \leq^* (g, A_g)$ and hence for all $n \geq j$, $\beta_n \in A_g(n)$. Let $p = (g, B \restriction [j, \omega)) \frown \langle \beta_l \mid j \leq l < k_g \rangle$. By definition, p is an a_g-step extension of (g, A_g) so lies in $\mathcal{D}$. Furthermore, by definition of H, $p \in H$ and thus $H \cap \mathcal{D} \neq \emptyset$.

For the successor step, suppose $\lambda = \bar{\lambda} + \omega$ for some limit $\bar{\lambda}$ such that the result is true for any Magidor forcing defined on a system of measures of length $\bar{\lambda}$. Let $\langle \beta_\eta \mid \eta < \lambda \rangle$ be geometric for $\mathbb{M}(\langle U_\eta \mid \eta < \lambda \rangle)$. Let $g = \left\langle (\bar{\lambda}, \beta_{\bar{\lambda}}) \right\rangle$ and in line with the factoring of Fact 2.6.8, consider $\mathbb{M}(\vec{U})_g \times \mathbb{M}(\vec{U})^g$. Then $\mathbb{M}\left(\vec{U}\right)_g = \mathbb{M}\left(\left\langle r^{\bar{\lambda}}_\eta(\beta_{\bar{\lambda}}) \mid \eta < \bar{\lambda} \right\rangle\right)$ is itself a Magidor forcing of length $\bar{\lambda}$, and by definition of geometricity, $\left\langle \beta_\eta \mid \eta < \bar{\lambda} \right\rangle$ is geometric over $\mathbb{M}\left(\vec{U}\right)_g$. Thus, by induction, the resulting H_g defined from $\left\langle \beta_\eta \mid \eta < \bar{\lambda} \right\rangle$ is $\mathbb{M}\left(\vec{U}\right)_g$-generic over V. Since $\left\langle \beta_{\bar{\lambda}+n} \mid n < \omega \right\rangle$ is geometric over $\mathbb{M}\left(\vec{U}\right)^g = \mathbb{M}\left(\left\langle U_{\bar{\lambda}+n} \mid n < \omega \right\rangle\right)$, by arguing exactly as in the base case, the filter H^g defined from $\left\langle \beta_{\bar{\lambda}+n} \mid n < \omega \right\rangle$ is $\mathbb{M}\left(\left\langle U_{\bar{\lambda}+n} \mid n < \omega \right\rangle\right)$-generic over V. However, since

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[1] Since $\lambda = \omega$, we may without loss of generality extend each g whose domain is a finite subsequence of ω to some g' whose domain is $[0, \max \mathrm{dom}(g)]$.

$\mathbb{M}\left(\vec{U}\right)_g$ is β^+_λ-cc, H^g is actually $\mathbb{M}\left(\vec{U}\right)^g$-generic over $V[H_g]$ and so by the Product Lemma, $H = H_g \times H^g$ and is $\mathbb{M}\left(\langle U_\eta \mid \eta < \lambda\rangle\right)$-generic over V.

For the limit case, suppose that $\lambda = \sup_{\rho<\tau} \lambda_\rho$ for some $\tau \leq \lambda$ regular, where each λ_ρ is a limit ordinal; our induction hypothesis is now that for all $\rho < \tau$, whenever $\vec{\beta}$ is of length λ_ρ and is geometric for some Magidor forcing $\mathbb{M}$ of length λ_ρ, the resulting filter H_ρ defined from $\vec{\beta}$ is $\mathbb{M}$-generic over V. Note furthermore that if $\rho < \rho'$ then by the Product Lemma and chain condition, $H_{\rho'} = H_\rho \times H'$ for some H_ρ $\mathbb{M}_{(\lambda_\rho, \beta_{\lambda_\rho})}$-generic over V and some H' $\mathbb{M}^{(\lambda_\rho, \beta_{\lambda_\rho})}$-generic over $V[H_\rho]$.

Let $\mathcal{D}$ be open dense; for each $\mathbb{M}\left(\vec{U}\right)$-stem g, by Lemma 2.6.18, let A_g be such that:

- (g, A_g) is an $\mathbb{M}\left(\vec{U}\right)$-condition
- letting $\delta = \max \mathrm{dom}(g)$, $(g, A_g)_\delta = (1_{\mathbb{M}(\vec{U})} \frown g)_\delta$[1]
- (tail-change property) for every $b \subseteq [\mathrm{dom}(A_g)]^{<\omega}$, if (g', H) is a b-step extension of (g, A_g) and $(g', H) \in \mathcal{D}$ then every $b \setminus (\mathrm{dom}(g') \cap \delta)$-step extension of $(g', H)_\delta \frown (g, A_g)^\delta$ is also in $\mathcal{D}$

By Lemma 2.6.11, let $(\emptyset, B)$ diagonalize the family $\langle (g, A_g) \mid g \text{ a stem}\rangle$; that is, for every $g \in \prod_{\eta<\lambda} B(\eta)$, $(g, B \restriction (\lambda \setminus \mathrm{dom}(g))) \leq^* (g, A_g)$. But then by geometricity, let λ_ρ be the least limit ordinal such that for all $\xi \geq \lambda_\rho$, $\beta_\xi \in B(\xi)$.

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[1]Recall from Definition 2.6.7 that $(f, A)_\delta = (f \restriction \delta + 1, A \restriction \delta + 1)$.

But then

$$\mathcal{D}' := \left\{ (f, A)_{\lambda_\rho} \mid (f, A) \in \mathcal{D},\ (f, A)^{\lambda_\rho} \leq (\emptyset, B)^{\lambda_\rho},\ \text{and } f(\lambda_\rho) = \beta_{\lambda_\rho} \right\}$$

is open dense in $\mathbb{M}\left(\vec{U}\right)_{(\lambda_\rho, \beta_{\lambda_\rho})}$, and since $\vec{\beta} \restriction \lambda_\rho$ is geometric for $\mathbb{M}\left(\vec{U}\right)_{(\lambda_\rho, \beta_{\lambda_\rho})}$, by genericity of H_ρ let $(f, A) \in \mathbb{M}\left(\vec{U}\right)$ be such that $(f, A) \in \mathcal{D}$, $(f, A)^{\lambda_\rho} \leq (\emptyset, B)^{\lambda_\rho}$, and $(f, A)_{\lambda_\rho} \in H_\rho \cap \mathcal{D}'$. In particular, this means that $f \restriction \lambda_\rho \subseteq \vec{\beta} \restriction \lambda_\rho$; for all $\xi \in \mathrm{dom}(f)$ with $\xi \geq \lambda_\rho$, $f(\xi) \in B(\xi)$; and for all $\xi \in \mathrm{dom}(A)$, $\beta_\xi \in A(\xi)$.

Let $g = f \restriction (\lambda_\rho + 1)$; we claim that $(f, A) \leq (g, A_g)$. To see this, observe that by definition $g \subseteq f$. To see that $A(\xi) \subseteq A_g(\xi)$ for all relevant ξ, note that for $\xi < \lambda_\rho$, $A_g(\xi) = Y_\xi \cap g(\xi)$ which is maximal, hence $A(\xi) \subseteq A_g(\xi)$, and for $\xi > \lambda_\rho$, $A(\xi) \subseteq B(\xi)$ since $(f, A)^{\lambda_\rho} \leq (\emptyset, B)^{\lambda_\rho}$ and thus $A(\xi) \subseteq A_g(\xi)$. As for the stem extension, if $\xi \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$, then $\xi > \lambda_\rho$ and therefore, since $(f, A)^{\lambda_\rho} \leq (\emptyset, B)^{\lambda_\rho}$, $f(\xi) \in B(\xi)$ and hence $f(\xi) \in A_g(\xi)$.

But $(f, A) \in \mathcal{D}$! So let $b = \mathrm{dom}(f) \setminus \mathrm{dom}(g)$; then by the tail-change property of (g, A_g), every b-step extension of $(f, A)_{\lambda_\rho} \frown (g, A_g)^{\lambda_\rho} = (f, A)_{\lambda_\rho} \frown (\emptyset, A_g)^{\lambda_\rho}$ is also in $\mathcal{D}$. Let $(h, C) = ((f, A)_{\lambda_\rho} \frown (\emptyset, A_g)^{\lambda_\rho}) \frown (\vec{\beta} \restriction b)$. Then (h, C) is a b-step extension of $(f, A)_{\lambda_\rho} \frown (\emptyset, A_g)^{\lambda_\rho}$, so lies in $\mathcal{D}$; by construction of f, h is a finite subsegment of $\vec{\beta}$; and by construction of A, A_g, and B, for all relevant ξ, $\beta_\xi \in C(\xi)$. Therefore, $(h, C) \in H$, and also $(h, C) \in \mathcal{D}$ as desired. $\square$

We additionally have the following lemma that says geometric sequences actually meet V-measure one systems cofinitely instead of just coboundedly. This will help with arguing mutual stationarity results for Magidor generics, and may also help with restricted cofinality results below β_0, by a lemma from the original mutual stationarity paper of (Foreman and Magidor, 2001).

This also appears as Corollary 7.6 of (Fuchs, 2014); we give our own proof here.

Lemma 2.6.24. *Let* $\vec{\beta} = \langle \beta_\xi \mid \xi < \lambda \rangle$ *be geometric, and let* $Z \in \prod_{\xi<\lambda} U_\xi$ *with each* $Z(\xi) \subseteq Y_\xi$*. Then there is some* $a \in [\lambda]^{<\omega}$ *such that for all* $\xi \notin \mathrm{dom}(a)$*,* $\beta_\xi \in Z(\xi)$*; furthermore, there is some* Z' *such that* $(\vec{\beta} \restriction a, Z')$ *is a condition and for all* $\xi \in \mathrm{dom}(Z')$*,* $\beta_\xi \in Z'(\xi) \subseteq Z(\xi)$*.*

Proof. We build a by induction. By the definition of Definition 2.6.21 on $\vec{\beta}$ at λ, there is some $\zeta < \lambda$ such that for all $\xi \in [\zeta, \lambda)$, $\beta_\xi \in Z(\xi)$. If ζ may be chosen to be a limit ordinal (or 0), let ζ_0 be the least such limit ordinal (or 0) and let $a_0 = \emptyset$. Otherwise, let ζ' be the least such successor ordinal and let ζ_0 be the unique limit ordinal (or 0) and m_0 be the unique natural number such that $\zeta_0 + m_0 = \zeta'$. Let $a_0 = [\zeta_0, \zeta_0 + m_0)$.

We now proceed inductively; suppose we have defined ζ_n a limit ordinal and $a_n \in [\lambda]^{<\omega}$. By the definition of Definition 2.6.21 on $\vec{\beta}$ at ζ_n, there is some $\zeta < \zeta_n$ such that for all $\xi \in [\zeta, \zeta_n)$, $\beta_\xi \in Z(\xi) \cap \beta_{\zeta_n}$. If ζ may be made limit (or 0), let ζ_{n+1} be the least such, and let $a_{n+1} = \emptyset$. Otherwise, let ζ' be the least such successor ordinal, and let ζ_{n+1} be the unique limit ordinal (or 0) and let m_{n+1} be the unique natural number such that $\zeta_{n+1} + m_{n+1} = \zeta'$. Let $a_{n+1} = [\zeta_{n+1}, \zeta_{n+1} + m_{n+1})$.

Since Ord is well-ordered, the above must terminate at some n at which $\zeta_n = 0$. Let $a = \bigsqcup_{k\leq n} a_k$. Precisely by construction of a, for every $\xi \notin \mathrm{dom}(a)$, $\beta_\xi \in Z(\xi)$.

Even though $\vec{\beta} \restriction a$ is not a valid stem extension of $(\emptyset, Z)$, if we mimic the definition of $(\emptyset, Z) \frown (\vec{\beta} \restriction a)$ as in Definition 2.6.10, we obtain $(\vec{\beta} \restriction a, Z')$ with Z' as desired. $\square$

CHAPTER 3

DESTROYING SATURATION WHILE PRESERVING PRESATURATION

Our first anti-saturation result shows that it is consistent to, with forcing, destroy the saturation of a large class of ideals while preserving their presaturation. This answers an open question of (Cox and Eskew, 2018).

This chapter is a modified version of the preprint (Schoem, 2019) on the arXiv.

3.1 Background

Besides results about nonstationary ideals discussed in Section 1, one can also ask whether there is *any* ideal on κ that is κ-saturated, κ^+-saturated, any amount of presaturated, or even just precipitous. Results here are well-established and comprehensive.

The existence of exactly κ-saturated or κ^+-saturated ideals on inaccessible κ are equiconsistent with a measurable cardinal. This was first shown in (Kunen and Paris, 1971), with weakly compact being compatible with κ^+-saturation (and it was known since early work of Lévy and Silver that a κ-saturated ideal on κ prevents κ from being weakly compact). Subsequently, Boos showed that an exactly κ^+-saturated ideal on κ can exist at a non-weakly compact κ in (Boos, 1974).

As for successor cardinals, the consistency results are more striking. Certain arguments show that if κ carries a κ-saturated ideal, then κ must be weakly Mahlo, and hence not a successor. Proofs can be found in (Baumgartner et al., 1977) and (Ulam, 1997). However, κ^+-saturated ideals *can* occur at successor κ; the known ways to achieve this come from forcing over models with huge cardinals as done by Kunen in (Kunen, 1978) and Laver in (Laver, 1982).

Ideals on arbitrary sets Z project downwards to subsets Z' of Z, and it is natural to ask whether regularity of the inverse embedding implies nice saturation properties of the projected ideal:

Question 3.1.1 ((Foreman, 2010), Question 13 of Foreman). Let $n \in \omega$ and let $\mathcal{J}$ be an ideal on $Z \subseteq \mathcal{P}(\kappa^{+(n+1)})$. Let $\mathcal{I}$ be the projection of $\mathcal{J}$ from Z to some $Z' \subseteq \mathcal{P}(\kappa^{+n})$. Suppose that the canonical homomorphism from $\mathcal{P}(Z')/\mathcal{I}$ to $\mathcal{P}(Z)/\mathcal{J}$ is a regular embedding. Is $\mathcal{I}$ $\kappa^{+(n+1)}$-saturated?

The answer is no; prior work in (Cox and Zeman, 2014) established counterexamples. Later work by Cox and Eskew provided a template for finding counterexamples as follows. We observe that $\mathcal{I}$ a κ^{+n+1}-saturated ideal on κ^{+n} induces a wellfounded generic ultrapower and preserves κ^{+n+1}. So we will say that an ideal $\mathcal{I}$ on κ^{+n} is κ^{+n+1}-*presaturated* if $\mathcal{I}$ induces a wellfounded generic ultrapower and preserves κ^{+n+1}. Our template is then:

Fact 3.1.2 ((Cox and Eskew, 2018), corollary of Theorem 1.2). Any κ^{+n+1}-presaturated, non-κ^{+n+1} saturated ideal on κ^{+n} provides a counterexample to Question 3.1.1.

This makes finding presaturated, nonsaturated ideals an interesting project in its own right.

To construct such ideals for successor cardinals $\kappa = \mu^+$ (with μ regular and mild assumptions on cardinal arithmetic), (Cox and Eskew, 2018) generalized a forcing of (Baumgartner and Taylor, 1982) to add a club subset C of κ with $< \mu$-conditions. (The original version in

(Baumgartner and Taylor, 1982) was for $\mu = \omega$.) This C prevented κ^+-saturated ideals on κ from existing in the generic extension. At the same time, their forcing was *strongly proper*; with use of Foreman's Duality Theorem (Foreman, 2010), a powerful tool for computing properties of ideals in generic extensions, Cox and Eskew were then able to argue that their forcing preserved the κ^+-presaturation of a large class of ideals (including κ^+-saturated ideals) in the generic extension.

This results in the following:

Fact 3.1.3. Let V be a universe admitting κ-complete, κ^+-saturated ideals at κ the successor of a regular cardinal. Then there is a forcing $\mathbb{P}$ such that $V^{\mathbb{P}}$ admits no κ^+-saturated ideals, however if I is a κ^+-saturated ideal on κ in V, then in $V^{\mathbb{P}}$ there is an $S \in \bar{I}^+$[1] such that $\bar{I} \restriction S$ is κ^+-presaturated.

It remained open as to whether the above could be done for κ an inaccessible cardinal; this was the content of Question 8.5 of (Cox and Eskew, 2018) and further clarifications provided in (Cox and Schoem, 2018).

Theorem 3.1.4 and Theorem 3.1.5 establish that Question 3.1.1 is consistently false at κ inaccessible, by way of partially extending the arguments and results of Theorem 4.1 of (Cox and Eskew, 2018):

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[1] where $\bar{I}$ is the ideal induced in $V^{\mathbb{P}}$ by I and is defined by $\bar{I} = \{A \in \mathcal{P}^{V^{\mathbb{P}}}(\kappa) \mid \exists N \in I\ A \subseteq N\}$.

Theorem 3.1.4. *Suppose* V *is a universe of* ZFC *with an inaccessible cardinal* κ *admitting* κ*-complete, normal,* κ^+*-saturated ideals on* κ *concentrating on inaccessible cardinals below* κ *(i.e. such that* $\mathrm{Inacc}_\kappa \in \mathrm{I}^+$*). Then there is a poset* $\mathbb{Q}$ *such that:*

(i) $V^{\mathbb{Q}} \models$ "*there are no* κ*-complete,* κ^+*-saturated ideals on* κ *concentrating on inaccessible cardinals*"

(ii) If $\mathrm{I} \in V$ *is a* κ*-complete, normal,* κ^+*-saturated ideal on* κ *concentrating on inaccessible cardinals, then* $V^{\mathbb{Q}} \models$ "$\bar{\mathrm{I}}$ *is* κ^+*-presaturated*"

where $\bar{\mathrm{I}} = \{A \in \mathcal{P}^{V^{\mathbb{Q}}}(\kappa) \mid \exists N \in \mathrm{I}\ A \subseteq N\}$.

We can further generalize Theorem 3.1.4(ii) as follows:

Theorem 3.1.5. *With the same assumptions, there is a* $\mathbb{Q}$ *such that if* $\delta \geq \kappa$ *is an inaccessible cardinal,* $\mathrm{I} \in V$ *is normal, fine, precipitous,* δ^+*-presaturated ideal of uniform completeness* κ *on some algebra of sets* Z *such that:*

- $\mathcal{B}_\mathrm{I}$ *preserves the regularity of both* κ *and* δ*;*
- $\Vdash_{\mathcal{B}_\mathrm{I}} \delta^+ \leq |\dot{j}_\mathrm{I}(\kappa)| < \dot{j}_\mathrm{I}(\kappa)$ *where* $\dot{j}_\mathrm{I}$ *is a name for the generic elementary embedding* $j_\mathrm{I} : V \to M$ *added by* $\mathcal{B}_\mathrm{I} := \mathcal{P}(Z)/\mathrm{I}$*;*
- $\mathcal{B}_\mathrm{I}$ *is* δ^+*-proper on* $\mathrm{IA}_{<\delta^+}$*;*

then in $V^{\mathbb{Q}}$*,*

- $\bar{\mathrm{I}}$ *is not* δ^+*-saturated*
- *but* $\bar{\mathrm{I}}$ *is* δ^+*-presaturated*

where $\bar{I}$ *is as above.*

Here, $IA_{<\delta}$ is the collection of internally approachable structures of length $< \delta$; we will give a precise definition in Section 3.2.

Remark 3.1.6. It will turn out that the same $\mathbb{Q}$ will work for both Theorem 3.1.4 and Theorem 3.1.5.

Remark 3.1.7. In (Cox and Eskew, 2018), the analogous theorem (Theorem 4.1(2)) argued that there is an $S \in \bar{I}^+$ such that $\bar{I} \restriction S$ is not δ-saturated, but it is δ-presaturated.

The use of such an S was required there due to the forcing involved not being κ-cc.

This chapter is structured as follows. Section 3.2 presents the preliminary definitions and facts pertinent to this paper. Section 3.3 introduces the forcing iteration $\mathbb{Q}$ of Theorems 3.1.4(i), 3.1.4(ii), and 3.1.5. Section 3.4 shows that several saturated ideals are sundered from $V^{\mathbb{Q}}$. Section 3.5 proves that a portion of presaturated posets remain presaturated in $V^{\mathbb{Q}}$.

3.2 Chapter-specific preliminaries

Much of the necessary background can be found in Chapter 2; we cover more chapter-specific preliminaries here.

Forcing poset closure, and properness, relate to presaturation; we now summarize what properness is, and how both closure and properness relate to presaturation.

Let δ be regular uncountable, and let $H \supsetneq \delta$. Then we write $\mathcal{P}_\delta(H)$ for all subsets of H of size $< \delta$, and $\mathcal{P}^*_\delta(H)$ to denote the set of all $x \in \mathcal{P}_\delta(H)$ such that $x \cap \delta \in \delta$.

Definition 3.2.1. Let $\mathbb{P}$ be a notion of forcing, θ sufficiently large so that $\mathbb{P} \in H_\theta$, and $M \prec (H_\theta, \in, \mathbb{P})$.

We say that $p \in \mathbb{P}$ is an $(M, \mathbb{P})$-*master condition* if for every dense $D \in M$, $D \cap M$ is predense below p; equivalently, $p \Vdash_{\mathbb{P}} M[\dot{G}_{\mathbb{P}}] \cap V = M$.

Additionally, we say that p is an $(M, \mathbb{P})$-*strong master condition* if for every $p' \leq p$, there is some $p'_M \in M \cap \mathbb{P}$ such that every extension of p'_M in $M \cap \mathbb{P}$ is compatible with p'. [1]

Further, $\mathbb{P}$ is *(strongly) proper with respect to* M if every $p \in M \cap \mathbb{P}$ has a $q \leq p$ such that q is an $(M, \mathbb{P})$-(strong) master condition.

We say that $\mathbb{P}$ is *(strongly)* δ*-proper on a stationary set* if there is a stationary subset S of $\mathcal{P}^*_\delta(H_\theta)$ such that for every $M \in S$, $M \prec (H_\theta, \in, \mathbb{P})$ and $\mathbb{P}$ is (strongly) proper with respect to M.

Note that $\{M \in \mathcal{P}^*_\delta(H_\theta) \mid M \prec (H_\theta, \in, \mathbb{P})\}$ is a club subset of $\mathcal{P}^*_\delta(H_\theta)$; so a forcing being δ-proper on a stationary set really only depends on the properness condition.

The definitions of $\mathcal{P}^*_\delta(H)$ and 3.2.1 are as in (Cox and Eskew, 2018).

Fact 3.2.2. If $\mathbb{P}$ is δ-proper on a stationary set, then $\mathbb{P}$ is δ-presaturated.

This fact appears as Fact 2.8 of (Cox and Eskew, 2018), with proof; their proof, in turn, generalizes a result of Foreman and Magidor in the case of $\delta = \omega_1$ (namely, Proposition 3.2 of (Foreman and Magidor, 1995)).

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[1] It is straightforward to see that strong master conditions are also master conditions.

For the posets we will be working with, we will have a specific stationary subset witnessing δ-properness:

Definition 3.2.3. For δ regular and $\theta >> \delta$, we say that $\mathrm{IA}_{<\delta} \subseteq \mathcal{P}^*_\delta(\mathsf{H}_\theta)$, the "internally approachable sets of length $< \delta$", is the collection of all $\mathsf{M} \in \mathcal{P}^*_\delta(\mathsf{H}_\theta)$, with $|\mathsf{M}| = |\mathsf{M} \cap \delta|$, that are *internally approachable*, i.e. such that there is a $\zeta < \delta$ and a continuous $\subseteq$-increasing sequence $\langle \mathsf{N}_\alpha \mid \alpha < \zeta \rangle$ whose union is M, such that $\vec{\mathsf{N}} \restriction \alpha \in \mathsf{M}$ for all $\alpha < \zeta$.

In a sense, internal approachability is preserved by any generic extension, as noted in (Cox and Eskew, 2018):

Fact 3.2.4. Suppose $\mathbb{P}$ is a poset, $\mathsf{M} \prec (\mathsf{H}_\theta, \in, \mathbb{P})$, $\langle \mathsf{N}_\alpha \mid \alpha < \zeta \rangle$ witnesses that $\mathsf{M} \in \mathrm{IA}_{<\delta}$, and G is $(\mathsf{V}, \mathbb{P})$-generic. Then in $\mathsf{V}[\mathsf{G}]$, $\langle \mathsf{N}_\alpha[\mathsf{G}] \mid \alpha < \zeta \rangle$ witnesses that $\mathsf{M}[\mathsf{G}] \in \mathrm{IA}_{<\delta}$. (Without loss of generality, we may assume that $\mathbb{P} \in \mathsf{N}_0$.)

It is a standard fact that $\mathrm{IA}_{<\delta}$ is stationary. The following lemma makes clear its utility:

Lemma 3.2.5. *Let δ be regular and uncountable. Then:*

(i) If $\mathbb{P}$ is δ-cc and $\mathsf{M} \prec (\mathsf{H}_\theta, \in, \mathbb{P})$ is an element of $\mathcal{P}^*_\delta(\mathsf{H}_\theta)$ (i.e. if $\mathsf{M} \cap \delta \in \delta$), then $1_{\mathbb{P}}$ is an $(\mathsf{M}, \mathbb{P})$-master condition; in particular $\mathbb{P}$ is δ-proper on (a club subset of) $\mathcal{P}^*_\delta(\mathsf{H}_\theta)$.

(ii) If $\mathbb{Q}$ is $< \delta$-closed then $\mathbb{Q}$ is δ-proper on $\mathrm{IA}_{<\delta}$.

(iii) If $\mathbb{P}$ is δ-proper on $\mathrm{IA}_{<\delta}$ and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is δ-cc or $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is $< \delta$-closed then $\mathbb{P} * \dot{\mathbb{Q}}$ is δ-proper on $\mathrm{IA}_{<\delta}$.

This is roughly Fact 2.9 out of (Cox and Eskew, 2018). The following proof is largely reproduced from (Cox and Eskew, 2018) as well.

Proof. For part (i), let $A \in M$ be a maximal antichain in $\mathbb{P}$. Since $|A| < \delta$ and $M \cap \delta \in \delta$, we have that $A \subseteq M$. Thus $1_{\mathbb{P}} \Vdash M[\dot{G}] \cap \check{V} = M$, so $1_{\mathbb{P}}$ is a master condition for M.

Part (ii) is due to (Foreman and Magidor, 1995).

As for part (iii), let G be $\mathbb{P}$-generic over V. Suppose that $M \prec (H_\theta, \in, \mathbb{P} * \dot{\mathbb{Q}})$ and $M \in IA_{<\delta}$. By Fact 3.2.4, combined with (i) and (ii), $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is proper with respect to $M[\dot{G}]$. Hence $\mathbb{P} * \dot{\mathbb{Q}}$ is proper with respect to M. □

Presaturation comes with some partial cofinality preservation:

Fact 3.2.6. If $\mathbb{P}$ is λ-presaturated for λ regular then

$$\Vdash_{\mathbb{P}} \mathsf{cof}^{V}(\geq \lambda) = \mathsf{cof}^{V[\dot{G}]}(\geq \lambda)$$

The above fact has a partial converse. We will not make use of it, but it is another known way to argue that certain iterations of presaturated forcings are presaturated:

Fact 3.2.7. If $\mathbb{P}$ is $\lambda^{+\omega}$-cc for some regular $\lambda \geq \omega_1$ and

$$\forall n \in \omega \ \Vdash_{\mathbb{P}} \mathsf{cf}^{V[\dot{G}]}\left(\left(\lambda^{+n}\right)^{V}\right) \geq \lambda$$

then $\mathbb{P}$ is λ-presaturated.

This appears as Fact 2.11 in (Cox and Eskew, 2018), which in turn is a generalization of Theorem 4.3 of (Baumgartner and Taylor, 1982).

In line with Section 2.3, if $\mathrm{I} \in V$ is an ideal on κ and $\mathbb{P}$ is a notion of forcing understood from context, then we will write $\bar{\mathrm{I}} := \left\{ N \in \mathcal{P}^{V^{\mathbb{P}}}(\kappa) \mid \exists A \in \mathrm{I}\ N \subseteq A \right\}$.

To compute presaturation properties of ideals, these are the two forms of Foreman's Duality Theorem that we will use:

Lemma 3.2.8. *For a* κ*-complete,* κ^+ *saturated* $\mathrm{I} \in V$ *and a* κ*-cc poset* $\mathbb{Q} \in V$*,* $\bar{\mathrm{I}}$ *is* κ^+*-saturated in* $V^{\mathbb{Q}}$ *if and only if* $\Vdash_{\mathcal{B}_{\mathrm{I}}} \dot{j}_{\mathrm{I}}(\mathbb{Q})$ *is* κ^+*-cc.*

This appears as Corollary 7.21 in (Foreman, 2010).

Theorem 3.2.9. *Let* I *be a* κ*-complete normal precipitous ideal in* V *and* $\mathbb{Q}$ *be a* κ*-cc poset. Then* $\bar{\mathrm{I}}$ *is precipitous and there is a canonical isomorphism witnessing that*

$$\mathcal{B}\left(\mathbb{Q} * \mathcal{B}_{\bar{\mathrm{I}}}\right) \cong \mathcal{B}\left(\mathcal{B}_{\mathrm{I}} * \dot{j}_{\mathrm{I}}\left(\mathbb{Q}\right)\right)$$

where $\mathcal{B}(\mathbb{P})$ *refers to the Boolean completion of* $\mathbb{P}$*.*

This statement appears in (Cox and Eskew, 2018) as Fact 2.24, and is a corollary of Theorem 7.14 of (Foreman, 2010).

3.3 Iterating the Generalized Baumgartner-Taylor Poset

Through the rest of this paper, fix κ to be a Mahlo cardinal. We do this because by assuming that V admits a κ-complete, normal, κ^+-saturated ideal on κ concentrating on inaccessible (equivalently regular) cardinals below κ, we have that κ is Mahlo.

Over cardinals below κ, we will define a forcing iteration that will destroy κ^+-saturation but preserve κ^+-presaturation for ideals on κ, concentrating on inaccessibles by adding, for each

$\mu < \kappa$, μ inaccessible, a club subset C_μ of μ^+ using $< \mu$-conditions. This club C_μ will fail to contain certain ground model sets, in the sense that if $X \in V$ and $|X| \geq \mu$ then $X \not\subseteq C_\mu$.

Towards this end:

Definition 3.3.1. Let $\mu < \kappa$ be a regular cardinal such that $|[\mu^+]^{<\mu}| = \mu$. Let $\mathbb{P}(\mu)$ be the collection of all conditions (s, f) such that:

(1) $s \in [\mu^+ \setminus \mu]^{<\mu}$

(2) $f : s \to [\mu^+ \setminus \mu]^{<\mu}$ and if $\xi, \xi' \in s$ with $\xi < \xi'$ then $f(\xi) \subseteq \xi'$.

We say $(s, f) \leq (t, g)$ if $s \supseteq t$ and whenever $\xi \in t$, $f(\xi) \supseteq g(\xi)$.

For each $(s, f) \in \mathbb{P}(\mu)$, s can be thought of as approximating $\dot{C}_\mu$, in the sense that $(s, f) \Vdash s \subseteq \dot{C}_\mu$ (in fact, we will later define $C_\mu = \bigcup_{(s,f)\in G} s$, for G a $\mathbb{P}(\mu)$-generic filter over V).

Additionally, f can be thought of as "banning" certain ordinals from ever appearing in $\dot{C}_\mu$, in the sense that if $\alpha \in s$, $\beta > \alpha$, and $f(\alpha) \ni \beta$, then:

- it must be the case that $s \cap (\alpha, \beta] = \emptyset$. Otherwise, if $\gamma \in s \cap (\alpha, \beta]$, we would have that $\beta \in f(\alpha)$ and $\beta \notin \gamma$. Hence $f(\alpha) \not\subseteq \gamma$, contradicting conditionhood of (s, f).
- Additionally, $(s, f) \Vdash \dot{C}_\mu \cap (\alpha, \beta] = \emptyset$. This is since for every $(t, g) \leq (s, f)$, $\beta \in g(\alpha)$; hence $t \cap (\alpha, \beta] = \emptyset$.

Lemma 3.3.2. *If μ is a regular cardinal, then $\mathbb{P}(\mu)$ has the following properties:*

(1) $|\mathbb{P}(\mu)| = \mu^+$ hence $\mathbb{P}(\mu)$ has the μ^{++}-cc.

(2) $\mathbb{P}(\mu)$ is $< \mu$-directed closed.

(3) *If* $\theta \geq \mu^{++}$, $M \prec (H_\theta, \in, \mu^+)$, *and* $M \cap \mu^+ \in \mu^+ \cap \mathsf{cof}(\mu)$, *then* $\mathbb{P}(\mu)$ *is strongly proper for* M. *Hence* $\mathbb{P}(\mu)$ *preserves* μ^+.

(4) *If* G *is* $\mathbb{P}(\mu)$*-generic over* V, *then in* $V[G]$, *we have that*

$$C_\mu := \bigcup_{(s,f) \in G} s$$

is a club subset of μ^+ *such that if* $X \in V$ *and* $|X|^V \geq \mu$, *then* $X \not\subseteq C_\mu$.

(5) $\mathbb{P}(\mu)$ *is not* μ^+*-cc below any condition.*

Proof. The proofs are exactly as in Lemma 4.4 in (Cox and Eskew, 2018).

For the sake of clarity, we will prove (3) and (4).

To see that (3) holds, let $\theta \geq \mu^{++}$, $M \prec (H_\theta, \in, \mu^+)$, and $M \cap \mu^+ \in \mu^+ \cap \mathsf{cof}(\mu)$; suppose that $(s, f) \in \mathbb{P}(\mu) \cap M$. Observe that $\mu^{<\mu} = \mu$ and $M \prec (H_\theta, \in, \mu^+, \mu)$. Let $\delta = M \cap \mu^+$; since $(\mu^+)^{<\mu} = \mu^+$ as witnessed in H_θ, we have that there is a bijection $\phi : \mu^+ \to [\mu^+]^{<\mu}$ such that $\phi \in M$. Without loss of generality, we may assume that for each $\beta < \mu^+$ with $\mathsf{cf}(\beta) = \mu$, $\phi \restriction \beta$ surjects onto $[\beta]^{<\mu}$.

We wish to show that ${}^{<\mu}(M \cap \mu^+) \subseteq M$. Let $\delta = M \cap \mu^+$ and suppose that $b \in [\delta]^{<\mu}$. Since $\mathsf{cf}(\delta) = \mu$, we have that $\sup b < \delta$. But then by choice of ϕ, there is an $\alpha < \sup b$ such that $\phi(\alpha) = \beta$, and since $\sup b < \delta$, $\alpha \in M$. Thus $b \in M$, and so we have shown

$${}^{<\mu}(M \cap \mu^+) \subseteq M$$

Since $|s| < \mu \subseteq M \cap \mu^+$, we thus have that $s \subseteq M$ and hence $M \cap \mu^+ \notin s = \mathrm{dom}(f)$. Further, if $\xi \in s$ then $f(\xi) \in M \cap [\mu^+]^{<\mu}$; since $\mu \subseteq M$ and θ is sufficiently large, $f(\xi) \subseteq M \cap \mu^+$.

Thus the following condition (s', f') extends (s, f):

$$(s', f') := \left(s \frown \left(M \cap \mu^+\right), f \frown \left(M \cap \mu^+ \mapsto \{M \cap \mu^+\}\right)\right)$$

We now must argue that (s', f') is a strong master condition for $(M, \mathbb{P}(\mu))$. Let $(t, h) \leq (s', f')$. Then $t_M := t \cap M$ is a $< \mu$-sized subset of $M \cap \mu^+$, hence $t_M \in M$. Further, since $(t, h) \leq (s', f')$, we have that $M \cap \mu^+ \in t$. Hence, as (t, h) is a condition in $\mathbb{P}(\mu)$ (namely, by part (2) of Definition 3.3.1), $(h \restriction t_M) : t_M \to [M \cap \mu^+]^{<\mu}$. Thus $(t_M, h \restriction t_M) \in M \cap \mathbb{P}(\mu)$.

To complete the proof of strong properness, let $(u, g) \in M \cap \mathbb{P}(\mu)$, $(u, g) \leq (t_M, h \restriction t_M)$. Then let $F : u \cup t \to [\mu^+]^{<\mu}$, $F(\xi) = g(\xi)$ if $\xi \in u$, and $F(\xi) = h(\xi)$ otherwise. Then $(u \cup t, F) \in \mathbb{P}(\mu)$ and $(u \cup t, F) \leq (u, g), (t, h)$.

Since (u, g) was arbitrary, we have shown that every extension of $(t_M, h \restriction t_M)$ in $\mathbb{P}(\mu) \cap M$ is compatible with (t, h). Thus (s', f') is a strong master condition. This completes our proof of (3).

To see that (4) holds, we have three things to show:

(i) C_μ is unbounded in μ^+

(ii) C_μ is closed

(iii) If $X \in V$ and $|X|^V \geq \mu$ then $X \nsubseteq C_\mu$

To see (i), let $(s,f) \in \mathbb{P}(\mu)$ and let $\alpha < \mu^+$. By definition of $\mathbb{P}(\mu)$, $|s| < \mu$ and for each $\beta \in s$, $f(\beta)$ is a $< \mu$-sized subset of μ^+. Hence $\sup_{\beta \in s} \sup f(\beta) < \mu^+$, so let δ be such that $\sup_{\beta \in s} \sup f(\beta) < \delta < \mu^+$. Then

$$p := (s \frown \delta, f \frown (\delta \mapsto \emptyset))$$

is a condition below (s,f) such that $p \Vdash \delta \in \dot{C}_\mu$; thus C_μ is unbounded.

To see (ii), we argue contrapositively. Let $\beta \in \mu^+ \setminus (\mu+1)$ and suppose $(s,f) \in \mathbb{P}(\mu)$ is such that $(s,f) \Vdash \check{\beta} \notin \dot{C}_\mu$. We will argue that $(s,f) \Vdash \check{\beta} \notin \mathrm{Lim}(\dot{C}_\mu)$. Observe that there must be an $\alpha \in s \cap \beta$ such that $f(\alpha) \not\subseteq \beta$; for otherwise, we would have that for all $\alpha \in s \cap \beta$, $f(\alpha) \subseteq \beta$, hence $(s \frown \beta, f \frown (\beta \mapsto \emptyset))$ would be a condition below (s,f) forcing $\beta \in \dot{C}_\mu$. By conditionhood of (s,f), there is a unique such α and α is the largest element of $s \cap \beta$. Additionally, no extension (t,g) of (s,f) can have that $t \cap (\alpha, \beta) \neq \emptyset$, and hence $(s,f) \Vdash$ "$\check{\alpha}$ is the largest element of $\dot{C}_\mu \cap \check{\beta}$". Thus $(s,f) \Vdash \check{\beta} \notin \mathrm{Lim}(\dot{C}_\mu)$.

To see (iii), let $X \in V$ with $|X|^V \geq \mu$ and let $(s,f) \in \mathbb{P}(\mu)$. Observe that without loss of generality we may assume that $X \subseteq \mu^+ \setminus (\mu+1)$. Further, by taking an initial segment of X we may assume that $\mathrm{otp}(X) = \mu$ and hence that $\mathrm{cf}(\sup(X)) = \mu$. Since $|s| < \mu$ and $\sup(X)$ has cofinality μ, $s \cap \sup(X)$ is bounded below $\sup(X)$.

Now we have two cases. If there is a $\xi \in s \cap \sup(X)$ such that $f(\xi) \not\subseteq \sup(X)$, let $\rho \in f(\xi) \setminus \sup(X)$. Then $(s,f) \Vdash \dot{C}_\mu \cap (\xi, \rho] = \emptyset$ and hence $(s,f) \Vdash$ "$\dot{C}_\mu \cap \check{X}$ is bounded below $\sup(\check{X})$". Thus $X \not\subseteq C_\mu$.

Otherwise, let $\zeta = \sup\{\sup f(\xi) \mid \xi \in s \cap \sup(X)\}$. Since each $f(\xi) \subseteq \sup(X)$ and μ is regular, $\zeta < \sup(X)$. Let $p = (s \frown \zeta, f \frown (\zeta \mapsto \{\sup(X)\}))$. Then $p \leq (s, f)$ and $p \Vdash \max(\dot{C}_\mu \cap \sup(X)) = \zeta$. Hence $p \Vdash X \not\subseteq \dot{C}_\mu$. Thus $X \not\subseteq C_\mu$. This completes our proof of (4). $\square$

Definition 3.3.3. We define an Easton support iteration forcing $\mathbb{Q} = \left\langle \mathbb{Q}_\mu * \dot{\mathbb{C}}(\mu) \mid \mu < \kappa \right\rangle$ as follows:

For each $\mu < \kappa$, if μ is inaccessible in $V^{\mathbb{Q}_\mu}$, let $\mathbb{C}(\mu) = \mathbb{P}(\mu)$ as above, and otherwise let $\mathbb{C}(\mu)$ be the trivial forcing.

Proposition 3.3.4. *If $\nu \leq \kappa$ is regular in V, then ν is still regular in $V^{\mathbb{Q}_\nu}$.*

Proof. This breaks into three cases:

1. $\nu = \tau^+$, for τ a regular cardinal
2. $\nu = \lambda^+$, for λ a singular cardinal
3. ν is inaccessible

If $\nu = \tau^+$ where τ is regular, we may decompose $\mathbb{Q}_\nu$ as

$$\mathbb{Q}_\tau * \dot{\mathbb{C}}(\tau)$$

Since τ is regular, $|\mathbb{Q}_\tau| \leq \tau$ hence is ν-cc. Thus $\mathbb{Q}_\tau$ preserves ν. Either $\dot{\mathbb{C}}(\tau)$ is trivial, or is $\dot{\mathbb{P}}(\tau)$, so by Lemma 3.3.2(3), $\dot{\mathbb{C}}(\tau)$ preserves ν. Thus $\dot{\mathbb{Q}}_\nu$ preserves ν.

If $\nu = \lambda^+$ where λ is singular, we have that $\mathbb{Q}_\nu = \mathbb{Q}_\lambda$ since none of the ordinals in $[\lambda, \nu)$ are inanccessible. Here, the situation is more complicated, since now $|\mathbb{Q}_\lambda|$ might be equal to $\lambda^{\mathrm{cf}(\lambda)} \geq \nu$. So we must verify more directly that ν is preserved.

So observe that if ν is collapsed, then $V^{\mathbb{Q}_\lambda} \models |\nu| \leq |\lambda|$ and since λ is singular, we would have a $\mathbb{Q}_\lambda$-name $\dot{f} : \check{\delta} \to \check{\nu}$ for a cofinal sequence in $\check{\nu}$ for some regular cardinal $\delta < \lambda$.

But we may decompose $\mathbb{Q}_\lambda$ into

$$\mathbb{Q}_\delta * \dot{\mathbb{C}}(\delta) * \dot{\mathbb{Q}}_{>\delta^+}$$

Now, $\dot{\mathbb{Q}}_{>\delta^+}$ is $< \delta^+$-directed closed, so $\dot{\mathbb{Q}}_{>\delta}$ could not have added such an f. Additionally, $\dot{\mathbb{C}}(\delta)$ satisfies the δ^{++}-cc, hence is ν-cc. Thus $\dot{\mathbb{C}}(\delta)$ also could not have added f. Finally, $|\mathbb{Q}_\delta| = \delta$ so $\mathbb{Q}_\delta$ satisfies the δ^+-cc, hence is also ν-cc. Thus $\mathbb{Q}_\delta$ could not have added such an f either.

As in the successor of a regular case, $\dot{\mathbb{C}}(\nu)$ and $\dot{\mathbb{Q}}_{\geq\nu}$ preserve ν as well.

And in the case where ν is inaccessible, suppose that in $V^{\mathbb{Q}_\nu}$ that $\mathrm{cf}(\check{\nu}) = \check{\delta} < \check{\nu}$. Then $\mathbb{Q}_\nu$ decomposes, as in the successor of a singular case, into

$$\mathbb{Q}_\delta * \dot{\mathbb{C}}(\delta) * \dot{\mathbb{Q}}_{>\delta^+}$$

The analysis is exactly as in the successor of a singular case. $\square$

This shows that whenever ν is regular in V, ν remains regular in $V^{\mathbb{Q}_\nu}$. When ν is inaccessible, we will now write $\mathbb{P}(\nu)$ rather than $\mathbb{C}(\nu)$.

Corollary 3.3.5. $\mathbb{Q}$ *preserves cardinals.*

Proof. Since κ is Mahlo, $\mathbb{Q} = \mathbb{Q}_\kappa$ is κ-cc hence preserves κ preserves cardinals $\geq \kappa$.

For $\nu < \kappa$ regular, we have that $\mathbb{Q} = \mathbb{Q}_\nu * \dot{\mathbb{C}}(\nu) * \dot{\mathbb{Q}}_{>\nu}$. By the preceding proposition, $\mathbb{Q}_\nu$ preserves ν. Either $\dot{\mathbb{C}}(\nu)$ is trivial or is $\dot{\mathbb{P}}(\nu)$, and so Lemma 3.3.2(3), $\dot{\mathbb{C}}(\nu)$ preserves ν. And by Lemma 3.3.2(2), $\dot{\mathbb{Q}}_{>\nu}$ is $< \nu^+$-directed closed hence preserves ν. □

3.4 Destroying Saturation

Since $\mathbb{Q}$ projects to each $\mathbb{Q}_\mu * \dot{\mathbb{P}}(\mu)$, $\mu < \kappa$ inaccessible, we may, for each such μ, let G_μ be the restriction of the $\mathbb{Q}$-generic G to $\mathbb{P}(\mu)$ and define $C_\mu = \{\xi \mid \exists (s, f) \in G_\mu \ \xi \in s\}$. By Lemma 3.3.2(4), C_μ is a club subset of μ^+ in $V^{\mathbb{Q}_\mu * \dot{\mathbb{P}}(\mu)}$ and for every $X \in V^{\mathbb{Q}_\mu}$ such that $X \subseteq [\mu, \mu^+)$ and X has $V^{\mathbb{Q}_\mu}$-cardinality $\geq \mu$, $X \not\subseteq C_\mu$.

As a warmup to proving Theorem 3.1.4(i), we argue the following:

Proposition 3.4.1. *Suppose that* $\mathrm{I} \in V$ *is* κ*-complete, normal,* κ^+*-saturated, and concentrates on* Inacc_κ*. Then in* $V^{\mathbb{Q}}$*,* $\bar{\mathrm{I}}$ *is not* κ^+*-saturated.*

Before we prove this, it will be helpful to isolate a lemma on what $j_{\mathrm{I}}(\mathbb{Q})$ looks like in $\mathrm{Ult}(V, \mathrm{I})$:

Lemma 3.4.2. *Let* I *be a* κ*-complete, normal, fine precipitous ideal concentrating on inaccessibles. Then in* $\mathrm{Ult}(V, \mathrm{I})$*,* $j_{\mathrm{I}}(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$*, where* $\dot{\mathbb{R}}$ *is a name for an Easton support iteration* $\left\langle \mathbb{R}_\alpha * \dot{\mathbb{C}}(\alpha) \mid \alpha \in [\kappa, j_{\mathrm{I}}(\kappa)) \right\rangle$*, such that if* α *is* $\mathrm{Ult}(V, \mathrm{I})$*-inaccessible,* $\mathbb{C}(\alpha) = \mathbb{P}(\alpha)$*, and* $\mathbb{C}(\alpha)$ *is the trivial forcing otherwise.*

Proof. This follows from the elementarity of j_{I}, and since I concentrates on inaccessibles, κ is inaccessible in $\mathrm{Ult}(V, \mathrm{I})$. □

And we remark on how $\mathrm{Ult}(V, I)$ computes κ^+:

Lemma 3.4.3. *Let* I *be as in Lemma 3.4.2, with* G *a* $\mathcal{B}_I$*-generic over* V*. Then if* I *is* κ^+*-saturated, then* $(\kappa^+)^{\mathrm{Ult}(V,I)} = (\kappa^+)^{V[G]}$.

Thus for κ^+-saturated ideals, we will just write κ^+ to mean $(\kappa^+)^V = (\kappa^+)^{V[G]} = (\kappa^+)^{\mathrm{Ult}(V,I)}$.

Proof. Since I is κ^+-presaturated, forcing with $\mathcal{B}_I$ preserves κ^+, hence $(\kappa^+)^V = (\kappa^+)^{V[G]} \geq (\kappa^+)^{\mathrm{Ult}(V,I)}$.

To see that $(\kappa^+)^{V[G]} = (\kappa^+)^{\mathrm{Ult}(V,I)}$, suppose that $\alpha \in [\kappa, (\kappa^+)^{V[G]})$. Let this be witnessed by some κ-sequence $f : \kappa \to \alpha$ a bijection. Since I is κ^+-saturated, we have from Fact 2.3.16 that $\mathrm{Ult}(V, I)$ is closed under κ-sequences from $V[G]$, and so $f \in \mathrm{Ult}(V, I)$. Thus α is not a cardinal in $\mathrm{Ult}(V, I)$. $\square$

Remark 3.4.4. As with ultrapowers from a measurable cardinal, we will have that if I is a κ-complete normal precipitous ideal in V, then in $V^{\mathcal{B}_I}$, $|j_I(\kappa)| = 2^\kappa$. However, by elementarity, in $\mathrm{Ult}(V, I)$, $j_I(\kappa)$ is inaccessible.

Remark 3.4.5. This is unlike a λ-complete, λ^+-saturated ideal J on λ a successor cardinal; for λ a successor cardinal, we would have that $j_J(\lambda) = \lambda^+$. The argument can be found in (Foreman, 2010).

Proof of Proposition 3.4.1. By Lemma 3.4.2, in $V^{\mathcal{B}_I}$, $\dot{j}_I(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is an Easton support iteration $\left\langle \mathbb{R}_\alpha * \dot{\mathbb{C}}(\alpha) \mid \alpha \in [\kappa, j_I(\kappa)) \right\rangle$ as in the lemma.

Since I concentrates on inaccessibles below κ, κ is still inaccessible in $\mathrm{Ult}(V, I)$. Thus $\mathbb{C}(\kappa) = \mathbb{P}(\kappa)$ which is not κ^+-cc. So $j_I(\mathbb{Q})$ is not κ^+-saturated.

So by Lemma 3.2.8, in $V^{\mathbb{Q}}$, $\bar{I}$ is not κ^+-saturated. $\square$

We now prove Theorem 3.1.4(i).

Proof of Theorem 3.1.4(i). Let G be $\mathbb{Q}$-generic, and suppose that in $V[G]$ there is a κ-complete, κ^+-saturated ideal $\mathcal{J}$ on κ concentrating on inaccessible cardinals below κ.

Let U be $P(\kappa)/\mathcal{J}$-generic over $V[G]$, and let $j : V[G] \to \mathrm{Ult}(V[G], U)$ be the generic ultrapower.

Let $N = \bigcup_{\alpha \in \mathrm{ORD}} j(V_\alpha)$. Then $j(\mathbb{Q}) \in N$ and hence $\mathrm{Ult}(V[G], U) = N[g']$ for some $g' \in V[G * U]$ which is $j(\mathbb{Q})$-generic over N.

Observe that κ is still inaccessible in $N[g']$ by inaccessibility in $V[G]$, by being the critical point of j, and since $\mathcal{J}$ concentrates on inaccessibles. Since $j(\kappa) > \kappa$ and $j(\kappa)$ is a cardinal in $N[g']$, $j(\kappa) > (\kappa^+)^{N[g']} \geq (\kappa^+)^{V[G]}$ (by κ-closure and κ^+-saturation of $\mathcal{J}$). Further, by the usual ultrapower argument, $|j(\kappa)| = 2^\kappa$.

So $j(\kappa)$ is not a cardinal in V, but by Fact 2.3.16, $N[g']$ is closed under κ-sequences from $V[G]$.

Work in $N[g']$. Let g' be the projection of $j(\mathbb{Q})$ to $\mathbb{P}(\kappa)$, and let

$$C_\kappa = \bigcup_{(s,f) \in g'} s$$

Then

$$N[g'] \models C_\kappa \text{ is club in } \kappa^+ \text{ and } \forall X \in N |X|^N \geq \kappa,\ X \not\subseteq C_\kappa \tag{3.1}$$

Since $V[G * U]$ is a κ^+-cc extension of V, we may let $D \in V$ be such that in $V[G * U]$, D is a club subset of C_κ. Let $E \subseteq D$ be in V, $(\mathrm{o.t.}(E))^V = \kappa$, $\alpha = \sup E$; since $\mathrm{cf}(\alpha) = \kappa$, let $\phi : \kappa \to \alpha$ be a normal increasing sequence.

Let $E' = \mathrm{lim}(E) \cap \mathrm{ran}(\phi)$.

Then $E' \subseteq D$ and $|E'|^V = \kappa$ since κ is inaccessible. Further, $j(\phi) \in N$ and $j(\phi) \restriction \kappa : \kappa \to j''\alpha$ is also in N.

Thus $\mathrm{ran}(j(\phi) \restriction \kappa) \in N$ and $j''E' \subseteq \mathrm{ran}(j(\phi) \restriction \kappa) \subseteq j''\alpha$.

But $j''E' = \mathrm{ran}(j(\phi) \restriction \kappa) \cap j(E') \in N$; and since $E' = \{\beta \in \mathrm{ran}(\phi) \mid j(\beta) \in j(E')\}$, we have that E' is a subset of C_κ with $|E'|^N = \kappa$ and $E' \subseteq [\kappa, \kappa^+)$.

This contradicts Equation 3.1, and hence $\mathcal{J}$ cannot be κ^+-saturated. □

3.5 Preserving Presaturation

We now prove Theorem 3.1.4(ii).

Proof of Theorem 3.1.4(ii). Let $I \in V$ be a κ-complete, normal, κ^+-saturated ideal in V concentrating on inaccessibles. Work in $V^{\mathcal{B}_I}$ and let U be the generic ultrafilter. Since I is κ-complete, $\mathrm{crit}(\dot{j}_I) = \kappa$ and $\dot{j}_I \restriction \kappa = \mathrm{id}$.

Thus, in $\mathrm{Ult}(V, U)$, by Lemma 3.4.2, $\dot{j}_I(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathbb{P}(\kappa)} * \dot{\mathbb{R}}$, where $\dot{\mathbb{R}}$ is an Easton support iteration $\langle \mathbb{R}_\alpha * \mathbb{C}(\alpha) \mid \alpha \in [\kappa^+, j_I(\kappa)) \rangle$, such that if α is inaccessible, $\mathbb{C}(\alpha) = \mathbb{P}(\alpha)$, and $\mathbb{C}(\alpha)$ is the trivial forcing otherwise.

We will argue that $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is κ^+-proper on a stationary set, and hence is κ^+-presaturated.

Observe that $\mathcal{B}_I$ is κ^+-cc. Hence, in $V[U]$ so also in $\mathrm{Ult}(V, U)$, $\mathbb{Q}$ is still κ^+-cc. Thus, in $\mathrm{Ult}(V, U)$, $\mathcal{B}_I * \mathbb{Q}$ is κ^+-cc and hence is κ^+-proper on $\mathcal{P}^*_{\kappa^+}(H_\theta)$ for all sufficiently large θ.

The difficulty comes in assuring $\mathbb{P}(\kappa) * \dot{\mathbb{R}}$ preserves the properness on a stationary set. We will do this by arguing that $\mathbb{P}(\kappa) * \dot{\mathbb{R}}$ is forced by $\mathcal{B}_I * \mathbb{Q}$ to be κ^+-proper on $\dot{IA}_{<\kappa^+}$. Once we have that, since $\mathcal{B}_I * \mathbb{Q}$ is κ^+-cc and forces $\mathbb{P}(\kappa) * \dot{\mathbb{R}}$ is κ^+-proper on a stationary set, the full forcing $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is then κ^+-proper on a stationary set.

Work in $\mathrm{Ult}(V, U)^{\mathbb{Q}}$. Here, $\mathbb{P}(\kappa)$ is proper on

$$\mathcal{S} := \{M \prec (H_\theta, \in, \kappa^+) \mid |M| = |M \cap \kappa^+| = \kappa \text{ and } M \cap \kappa^+ \in \mathsf{cof}(\kappa)\}$$

and by the $< \kappa^+$-directed closedness of $\dot{\mathbb{R}}$ and Fact 3.2.4, $\Vdash_{\mathbb{P}(\kappa)} \dot{\mathbb{R}}$ proper on $\check{IA}_{<\kappa^+}$. But not only is $\mathcal{S}$ stationary, $\mathcal{S}$ is a club subset of $\mathcal{P}^*_{\kappa^+}(H_\theta) \restriction \mathsf{cof}(\kappa)$, and hence $\mathcal{S} \cap IA_{<\kappa^+}$ is also stationary.

Thus, by Lemma 3.2.5, after forcing with $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$, we have that $\mathbb{P}(\kappa) * \dot{\mathbb{R}}$ is κ^+-proper on the stationary set $\mathcal{S} \cap IA_{<\delta^+}$.

Therefore, $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is κ^+-proper on a stationary subset of $\mathcal{P}^*_{\kappa^+}(H_\theta)^V$, hence is κ^+-presaturated. But by Theorem 3.2.9, $\mathcal{B}_I * \dot{j}_I(\mathbb{Q}) \cong \mathbb{Q} * \dot{\mathcal{B}}_{\bar{I}}$; then by Lemma 2.2.12, $\bar{I}$ is κ^+-presaturated. $\square$

A more general argument will prove Theorem 3.1.5:

Proof of Theorem 3.1.5. In V, let $\delta \geq \kappa$ be an inaccessible cardinal, and let I be a normal, precipitous, fine, δ^+-presaturated ideal of uniform completeness κ on some algebra of sets Z

such that $\mathcal{B}_I$ preserves the inaccessibility of κ and δ in $\mathrm{Ult}(V, I)$; $\Vdash_{\mathcal{B}_I} \delta^+ \leq |\dot{j}_I(\kappa)| < \dot{j}(\kappa)$; and $\mathcal{B}_I$ is δ^+-proper on $IA_{<\delta^+}$.

Let $\mathbb{Q}$ be the forcing from Definition 3.3.3. Recall that $\mathbb{Q}$ is κ-cc since $\mathbb{Q}$ is an Easton support iteration of κ-cc posets and κ is Mahlo.

We wish to show that $\mathcal{B}_{\bar{I}}$ is not δ^+-saturated, but is δ^+-presaturated.

Since $\mathcal{B}_I$ is precipitous, by Theorem 3.2.9,

$$\mathcal{B}\left(\mathcal{B}_I * \dot{j}_I(\mathbb{Q})\right) \cong \mathcal{B}\left(\mathbb{Q} * \mathcal{B}_{\bar{I}}\right) \tag{3.2}$$

Also, since I is κ-complete, $\mathtt{crit}(j_I) = \kappa$ and thus $j_I \restriction \kappa = \mathtt{id}$. Since $\mathcal{B}_I$ preserves the regularity of κ, we get that $\dot{j}_I(\mathbb{Q}) \restriction \kappa = \mathbb{Q}$.

Therefore, $j_I(\mathbb{Q}) = \mathbb{Q} * \langle \mathbb{R}_\alpha * \mathbb{C}(\alpha) \mid \alpha \in [\kappa, j_I(\kappa)) \rangle$, where each $\mathbb{R}_\alpha$ is an Easton support iteration, such that if α is inaccessible, $\mathbb{C}(\alpha) = \mathbb{P}(\alpha)$, and $\mathbb{C}(\alpha)$ is the trivial forcing otherwise.

Since $\mathrm{Ult}(V, I) \models$ "δ inaccessible", we get that $\mathbb{C}(\delta) = \mathbb{P}(\delta)$ which is not δ^+-cc. Thus $\left(\mathbb{Q} * \mathcal{B}_{\bar{I}}\right)$ is not δ^+-cc, and since $\mathbb{Q}$ is clearly δ^+-cc, $\mathcal{B}_{\bar{I}}$ cannot be δ^+-saturated.

As for the δ^+-presaturation of $\mathcal{B}_{\bar{I}}$, by Equation 3.2 it suffices to show that $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is δ^+-presaturated.

Work in $\mathrm{Ult}(V, I)$. Since $\mathcal{B}_I$ preserves the regularity of κ, we decompose $\dot{j}_I(\mathbb{Q})$ as

$$\dot{j}_I(\mathbb{Q}) = \mathbb{Q} * \left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right) * \left(\dot{j}_I(\mathbb{Q}) \restriction [\delta, j_I(\kappa))\right)$$

and further $\dot{j}_I(\mathbb{Q})(\delta) = \dot{\mathbb{P}}(\delta)$, so we decompose $\left(\dot{j}_I(\mathbb{Q}) \restriction [\delta, j_I(\kappa))\right)$ as $\left(\dot{\mathbb{P}}(\delta) * \dot{j}_I(\mathbb{Q}) \restriction [\delta^+, j_I(\kappa))\right)$.

So we may further decompose $\dot{j}_I(\mathbb{Q})$ as

$$\dot{j}_I(\mathbb{Q}) = \mathbb{Q} * \left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right) * \dot{\mathbb{P}}(\delta) * \left(\dot{j}_I(\mathbb{Q}) \restriction [\delta^+, j_I(\kappa))\right)$$

and we will argue the following items in $\mathrm{Ult}(V, I)$:

- $\mathbb{Q}$ is δ^+-cc
- $\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)$ is δ^+-cc
- $\dot{\mathbb{P}}(\delta)$ is δ^+-proper on a stationary set S such that $S \cap IA_{<\delta^+}$ is stationary
- $\left(\dot{j}_I(\mathbb{Q}) \restriction [\delta^+, j_I(\kappa))\right)$ is δ^+-directed closed and thus is δ-proper on $IA_{<\delta^+}$

in such a way that we may conclude that $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is δ^+-proper on a stationary set.

We have that $\mathbb{Q}$ is κ-cc, hence is δ^+-cc.

If $\delta = \kappa$, then $\left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right)$ is trivial so is δ^+-cc. Otherwise, $\delta > \kappa$, and since δ is inaccessible in $\mathrm{Ult}(V, I)$, $\left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right)$ is a δ-length direct limit iteration of posets of size $< \delta$. Thus $\left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right)$ has size δ, and so is δ^+-cc.

Therefore $\mathcal{B}_I * \mathbb{Q} * \left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa, \delta)\right)$ is δ^+-cc, so by Lemma , is δ^+-proper on $\mathcal{P}^*_{\delta^+}(H_\theta)$.

As in the proof of Theorem 3.1.4(ii), $\mathbb{P}(\delta)$ is proper on a club subset of $\mathcal{P}^*_{\delta^+}(H_\theta) \cap \mathsf{cof}(\delta)$ and so $\mathcal{B}_I * \mathbb{Q} * \left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa^+, \delta)\right) * \mathbb{P}(\delta)$ is δ^+-proper on the stationary set $IA_{<\delta^+} \cap \mathsf{cof}(\delta)$. Finally, $\left(\dot{j}_I(\mathbb{Q}) \restriction [\delta^+, j_I(\kappa))\right)$ is δ^+-directed closed and therefore is proper on $IA_{<\delta^+}$, which by Fact 3.2.4, is absolute between $V^{\mathcal{B}_I}$ and $V^{\mathcal{B}_I * \mathbb{Q} * \left(\dot{j}_I(\mathbb{Q}) \restriction [\kappa^+, \delta)\right) * \mathbb{P}(\delta)}$.

Thus $\mathcal{B}_I * \dot{j}_I(\mathbb{Q})$ is δ^+-proper on the stationary subset $IA_{<\delta^+} \cap \mathtt{cof}(\delta)$, and therefore is δ^+-presaturated. But then $\mathbb{Q} * \mathcal{B}_{\bar{I}}$ is δ^+-presaturated as well, and therefore by Lemma 2.2.12, $V^{\mathbb{Q}} \models$ "$\mathcal{B}_{\bar{I}}$ is δ^+-presaturated". $\square$

CHAPTER 4

MUTUAL STATIONARITY

Foreman and Magidor first introduced mutual stationarity in (Foreman and Magidor, 2001) to argue for the non-saturation of certain nonstationary ideals. Mutual stationarity generalizes stationarity to singular cardinals, and leads to interesting avenues of study in its own right. We will first define mutual stationarity and summarize prior results, with additional commentary and detail. This section culminates in the following mutual stationarity property for Magidor generic sequences: that if $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ is a Magidor generic sequence for some $\lambda < \kappa$ over some V, then there is a cofinite subset of the Magidor generic sequence K such that any family $S_\alpha \subseteq \kappa_\alpha \in K$ of statioary sets is mutually stationary. We spell this out in Section 4.4, with the core results being Theorem 4.4.5 and Theorem 4.4.6.

4.1 Background

To see that mutual stationarity generalizes stationarity, we recall that the following property precisely characterizes stationarity:

Proposition 4.1.1. *Let κ be a regular cardinal, and let $S \subseteq \kappa$. Then S is stationary if and only if whenever $\mathcal{U}$ is an algebra on κ (equivalently on some $\lambda \geq \kappa$), there is a $\mathcal{B} \prec \mathcal{U}$ such that* $\sup(\mathcal{B} \cap \kappa) \in S$.

With this in mind, we may generalize this to sets on larger sequences of cardinals:

Definition 4.1.2. Suppose $\mathsf{K} := \langle \kappa_\alpha \mid \alpha < \mu \rangle$ is an increasing sequence of regular cardinals with supremum λ of cofinality μ.

We say that a family $\langle S_\alpha \subseteq \kappa_\alpha \mid \alpha < \mu \rangle$ is *mutually stationary* if whenever $\mathcal{U}$ is an algebra on λ, there is a $\mathcal{B} \prec \mathcal{U}$ such that for all $\alpha \in \mathcal{B} \cap \mathsf{K}$, $\sup(\mathcal{B} \cap \kappa_\alpha) \in S_\alpha$. (Note that any mutually stationary sequence of sets is necessarily composed of stationary sets.)

We say $\mathsf{MS}(\kappa_\alpha \mid \alpha < \mu)$ holds if every family $\langle S_\alpha \subseteq \kappa_\alpha \mid \alpha < \mu \rangle$ of stationary sets is mutually stationary.

If we restrict our attention to stationary sets of cofinality δ above, we analogously write $\mathsf{MS}(\langle \kappa_\alpha \mid \alpha < \mu \rangle, \mathsf{cof}(\delta))$.

By Proposition 4.1.1, any mutually stationary family consists of stationary sets; and the questions of whether there *is* a mutually stationary family satisfying certain conditions, and whether any amount of $\mathsf{MS}(\circ)$, holds, are both independently interesting questions. We mostly focus on the consistency of $\mathsf{MS}(\circ)$-principles in this chapter.

Many mutual stationarity proofs are analogous to, or generalizations of, the following:

Theorem 4.1.3. *Let $\langle \kappa_\alpha \mid \alpha < \lambda, \alpha$ successor$\rangle$ be an increasing sequence of measurable cardinals with limit κ, and with $\lambda < \kappa$.*

Then $\mathsf{MS}(\langle \kappa_\alpha \mid \alpha < \lambda, \alpha$ successor$\rangle)$ holds.

This is stated without proof in (Foreman and Magidor, 2001). The proof is fairly straightforward, and proceeds via the use of *indiscernibles*:

Definition 4.1.4. Let $\mathcal{M}$ be a structure, I a linear order. Then a sequence $\langle a_i \mid i \in I\rangle \subseteq \mathcal{M}$ is *indiscernible*[1]if for every formula ϕ in the signature of $\mathcal{M}$ and every $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$,

$$\mathcal{M} \models \phi(a_{i_1}, \ldots, a_{i_n}) \iff \mathcal{M} \models \phi(a_{j_1}, \ldots, a_{j_n})$$

Proof of Theorem 4.1.3. Let $\mathcal{A}$ be an algebra on κ; augment $\mathcal{A}$ to $\mathcal{A}'$ with terms for each S_α. For each α, let H_α be a measure one (with respect to a normal measure on κ_α) family of $\mathcal{A}'$-order-indiscernibles for $\mathcal{A}' \cap \kappa_\alpha$. Since $\mathrm{Lim}(H_\alpha)$, the collection of limit points of H_α, is then a club, let $\gamma_\alpha \in S_\alpha \cap \mathrm{Lim}(H_\alpha)$, let $H'_\alpha = H_\alpha \cap \gamma_\alpha$, and let $\mathcal{M} = \mathrm{Hull}_{\mathcal{A}'}\left(\bigcup_{\alpha<\lambda} H'_\alpha \mid \alpha < \lambda\right)$.

Clearly for each α, $\mathcal{M} \cap \kappa_\alpha \geq \gamma_\alpha$. To see that this is an equality, suppose that for some Skolem term F and some $x \in \left[\bigcup_{\alpha<\lambda} H'_\alpha \mid \alpha < \lambda\right]^{<\omega}$, $F(x) = \beta$; let α be a successor ordinal such that $\beta \in [\kappa_{\alpha-1}, \kappa_\alpha)$.

Since H_α is unbounded in κ_α, let $\delta \in H_\alpha \cap (\max\{\gamma_\alpha, \beta\}, \kappa_\alpha)$. Since x is finite and H_α is unbounded in γ_α, let $\zeta \in H_\alpha \cap (\max(x \cap \gamma_\alpha), \gamma_\alpha)$.

By choice of δ, we have that $\mathcal{A}' \models F(x) < \delta$; but by how we chose δ and ζ, and since $x \cap [\zeta, \kappa_\alpha) = \emptyset$, the elements of x bear the exact same order relation to ζ as they do to δ. [2] Therefore, since $\mathcal{M} \models F(x) < \delta$ and x, δ, ζ are order indiscernibles for $\mathcal{M}$, we have that $\mathcal{M} \models F(x) < \zeta$ and $\zeta < \gamma_\alpha$. $\square$

A largely similar argument works for the points of a Prikry generic sequence:

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[1]Some authors use the term *order-indiscernible* instead for this concept.

[2]In time, we will give this a precise name; see Definition 4.2.1 in 4.2.

Theorem 4.1.5 (Theorem 5.4 of (Cummings et al., 2006))**.** *Let κ be measurable with normal measure U, let $\mathbb{P}(U)$ be Prikry forcing at κ, and let $\langle \kappa_n \mid n < \omega \rangle$ be Prikry generic. Then in the generic extension, there is some $m < \omega$ such that*

$$MS(\langle \kappa_n \mid m \leq n < \omega \rangle)$$

holds.

This theorem originally appeared in (Cummings et al., 2006). But, in ways we will see in 4.2, Theorem 4.1.5 was anticipated much earlier by results in (Koepke, 1984).

Proof sketch. With the use of the Prikry property, the argument of Theorem 4.1.3 readily adapts to a U-measure one set B of order indiscernibles below κ; then $(\emptyset, B \cap \mathrm{Lim}(B))$ forces the desired mutual stationarity property. We will further refine and elaborate on this argument in 4.2. □

The following lemmas make working with mutually stationary sequences and mutual stationarity principles easier.

Lemma 4.1.6 (Folklore)**.**

1. *Any subsequence of a mutually stationary family of sets is also mutually stationary.*
2. *If $\langle S_\alpha \mid \alpha < \mu \rangle$ is mutually stationary with each $S_\alpha \subseteq \kappa_\alpha \cap \mathrm{cof}(\leq \nu)$ and $\nu < \kappa_0$ then the mutual stationarity can be witnessed by an elementary substructure of size ν.*

(Foreman and Magidor, 2001) cites this as implicit in (Baumgartner, 1991).

As a corollary, mutual stationarity principles of restricted cofinality (under mild assumptions) are preserved under finite changes:

Lemma 4.1.7 (Lemma 23 of (Foreman and Magidor, 2001)). *Let* $\langle S_\alpha \mid \alpha < \mu \rangle$ *be mutually stationary with each* $S_\alpha \subseteq \kappa_\alpha \cap \mathsf{cof}(\leq \nu)$*, with* $\nu < \kappa_0$*. Suppose* $\lambda_0, \ldots, \lambda_n$ *are all greater than* ν*, not equal to any* κ_α*, and* $S_{\lambda_i} \subseteq \lambda_i \cap \mathsf{cof}(\leq \nu)$*.*

Then

$$\langle S_\alpha \mid \alpha < \mu \rangle \frown \langle S_{\lambda_i} \mid i < n \rangle$$

is also mutually stationary.

As for consistency results:

Theorem 4.1.8 (Theorem 7 of (Foreman and Magidor, 2001)). *Let* $\langle \kappa_\alpha \mid \alpha < \mu \rangle$ *be any family of regular cardinals. Then*

$$\mathsf{MS}(\langle \kappa_\alpha \mid \alpha < \mu \rangle, \mathsf{cof}(\omega))$$

Theorem 4.1.9 (Theorem 24 of (Foreman and Magidor, 2001)). *In* L*, for all* $k \in \omega$*,*

$$\neg\mathsf{MS}(\langle \aleph_n \mid n > k \rangle, \mathsf{cof}(\omega_k))$$

Further inner model theoretic results showed that mutual stationarity principles for fixed uncountable cofinality requires large cardinals incompatible with L:

Theorem 4.1.10 (Theorem 1.4 and Corollary 1.5 of (Koepke and Welch, 2007))**.** *If* $k < \omega$ *and* $\mathsf{MS}(\langle \aleph_n \mid n > k \rangle, \mathsf{cof}(\omega_k))$*, then there is an inner model in which each* $\aleph_n^V$ *has stationarily many measurable cardinals of Mitchell order* ω_{n-2}*.*

As of writing, the best known argument in the other direction comes courtesy of (Ben-Neria, 2019):

Theorem 4.1.11 (Theorem 1.3 of (Ben-Neria, 2019))**.** *Assume* GCH *and let* $\langle \kappa_n \mid n < \omega \rangle$ *be a sequence of cardinals with* $\kappa_0 = \omega$*, and with limit* κ_ω *such that each* κ_n *is* κ_ω^+*-supercompact. Then after forcing with a full-support iteration of* $\mathsf{Col}(\kappa_n, < \kappa_{n+1})$*, for each* $k < \omega$*,*

$$\mathsf{MS}(\langle \aleph_n \mid n > k \rangle, \mathsf{cof}(\omega_k))$$

holds.

Mutual stationarity results at every other $\aleph_n$ require much weaker large cardinal assumptions. For instance:

Theorem 4.1.12 (Theorem 1.6 of (Koepke, 2007); Theorem 1 of (Koepke and Welch, 2006))**.** *The principle*

$$\mathsf{MS}(\langle \aleph_{2n+3} \mid n < \omega \rangle, \mathsf{cof}(\omega_1))$$

and the existence of a measurable cardinal are equiconsistent.

and for larger cofinality:

Theorem 4.1.13 (Theorem 1.4 of (Ben-Neria, 2019)). *It is consistent, relative to a sequence $\langle \kappa_n \mid n < \omega \rangle$ with each κ_n being κ_n^+-supercompact with a κ_{n-1}^+-Mitchell sequence of such measures, that every sequence $S_n \subseteq \aleph_{2n+3} \cap \mathrm{cof}(< \aleph_{2n+2})$ of stationary subsets is mutually stationary.*

In particular, for all $k < \omega$,

$$\mathsf{MS}(\langle \aleph_{2n+3} \mid k < 2n+3 < \omega \rangle, \mathrm{cof}(\omega_k))$$

This family of consistency results demonstrate that MS principles has large cardinal strength, often follows directly from large cardinals and, failing that, in a forcing extension, and fails in L. However, relationships between square principles and mutual stationarity are not straightforward. The arguments of (Koepke and Welch, 2007) use global $\square$ in the Dodd-Jensen core model K to construct inner models for large cardinals from MS principles, but relationships between MS principles and failures of $\square$ remain uninvestigated.

We conclude our background section by elaborating on how we may view mutual stationarity principles as a strong form of anti-saturation.

Its introduction implicitly takes this view in the form of the following (a weaker version of which appears as Theorem 7 and Corollary 8 in (Foreman and Magidor, 2001)):

Theorem 4.1.14. *Assume δ is the successor of a regular cardinal and λ is a singular cardinal with $\mathrm{cf}(\lambda) = \mu > \delta$. Let $\langle \kappa_\alpha \mid \alpha < \mu \rangle$ be a cofinal sequence of cardinals in λ and suppose $\mathsf{MS}(\langle \kappa_\alpha \mid \alpha < \mu \rangle, \mathrm{cof}(< \delta))$. Then the nonstationary ideal on $\mathcal{P}_\delta(\lambda)$ is not λ^μ-saturated.*

Under $\neg\mathsf{SCH}_\lambda$ with λ of uncountable cofinality, this would give a more impressive anti-saturation result for the nonstationary ideal on $(\mathcal{P}_\delta(\lambda))$ than previously known. Thus compatibility of mutual stationarity principles with $\neg\mathsf{SCH}$ are of independent interest.

Proof of Theorem 4.1.14. Let $\langle S_\alpha \subseteq \kappa_\alpha \cap \mathsf{cof}(<\delta) \mid \alpha < \mu\rangle$ be a sequence of stationary sets. By Solovay's Splitting theorem, for each α let $\left\langle T^\alpha_\beta \mid \beta < \kappa_\alpha \right\rangle$ be a pairwise disjoint sequence of stationary subsets of S_α.

For each $f \in \prod_{\alpha<\mu} \kappa_\alpha$, let $S_f = \{M \in \mathcal{P}_\delta(\lambda) \mid \alpha \in M \implies \sup(M \cap \kappa_\alpha) \in T^\alpha_{f(\alpha)}\}$.

We claim that the family $\left\langle S_f \mid f \in \prod_{\alpha<\mu} \kappa_\alpha \right\rangle$ is an antichain for the nonstationary ideal on $\mathcal{P}_\delta(\lambda)$. To see this, since $\left\langle T^\alpha_{f(\alpha)} \mid \alpha < \mu \right\rangle$ is mutually stationary, each S_f is stationary in $\mathcal{P}_\delta(\lambda)$. To argue that $S_f \cap S_g$ is nonstationary for distinct f, g, fix $f \neq g$. Then there is some α for which $f(\alpha) \neq g(\alpha)$; hence for such α, $S^\alpha_{f(\alpha)} \cap S^\alpha_{g(\alpha)} = \emptyset$. Thus $S_f \cap S_g$ misses the club $\{M \in \mathcal{P}_\delta(\lambda) \mid \alpha \in M\}$.

$\left\langle S_f \mid f \in \prod_{\alpha<\mu} \kappa_\alpha \right\rangle$ is an antichain for the nonstationary ideal on $\mathcal{P}_\delta(\lambda)$, of cardinality $\prod_{\alpha<\mu} \kappa_\alpha = \lambda^\mu$. $\square$

However, very little is known about mutual stationarity principles with $\neg\mathsf{SCH}$.

The argument of (Koepke, 2007) does not depend on cardinal arithmetic, so should readily achieve $\mathsf{MS}(\langle \aleph_{2n+3} \mid n < \omega; \mathsf{cof}(\leq \omega_1)\rangle) + \neg\mathsf{SCH}_{\aleph_\omega})$.

The arguments of (Ben-Neria, 2019) assume GCH, but suggest a framework for achieving mutual stationarity principles with the failure of GCH.

4.2 The Koepke Approach

In this section, we flesh out Koepke's argument for Theorem 4.1.5, that Prikry generics achieve MS on a tail. Koepke's argument uses *mutual Ramseyness*, an alternative and more general way to think about the proof of Theorem 4.1.5. As our arguments on Magidor generics also utilize mutual Ramseyness, we will explain here what mutual Ramseyness is, why mutual Ramseyness holds for Prikry generics on a tail, and why that implies mutual stationarity principles.

Definition 4.2.1 (Definition 2.2 of (Koepke, 2007))**.** Suppose that κ is a limit cardinal with $\mathsf{cf}(\kappa) = \lambda$, and let $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be a normal cofinal sequence of cardinals below κ.

1. Let $x \in [\kappa]^{<\omega}$. Then $\mathsf{type}(x) = \langle |x \cap \kappa_\alpha| \mid \alpha \in \lambda \rangle$.

   We'll say that $t \in [\omega]^\lambda$ is a type if for some x, $t = \mathsf{type}(x)$.

   We'll additionally say that the *instances* of t are $\mathsf{inst}(t) := \{x \in [\kappa]^{<\omega} \mid \mathsf{type}(x) = t\}$

2. We say that a sequence of sets $\langle I_\alpha \mid \alpha \in \lambda \cap \mathsf{Succ} \rangle$ with each $I_\alpha \subseteq \kappa_\alpha$ is *mutually homogeneous for a function* $F : [\kappa]^{<\omega} \to \kappa$ if for every $x, y \in \left[\bigcup_{\alpha \in \lambda \cap \mathsf{Succ}} I_\alpha\right]^{<\omega}$, we have that $F(x) = F(y)$ whenever $\mathsf{type}(x) = \mathsf{type}(y)$ and $x \cap (F(x)+1) = y \cap (F(x)+1)$.

3. We say that $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ is *mutually Ramsey* if for all $F : [\kappa]^{<\omega} \to \kappa$, there is a mutually homogeneous for F sequence $\langle I_\alpha \mid \alpha \in \lambda \cap \mathsf{Succ} \rangle$ such that $|I_\alpha| = \kappa_\alpha$.

Remark 4.2.2. 1. For $x \in [\kappa]^{<\omega}$, $\mathsf{type}(x)$ is the (non-decreasing, eventually stabilizing) sequence that tells you, at point α, how many elements of x there are below κ_α.

2. In the definition of mutually homogeneous, the restriction $x \cap (F(x)+1) = y \cap (F(x)+1)$ is equivalent to $F(x) < \min(x \Delta y)$ (the symmetric difference).

   The idea here is that a sequence is mutually homogeneous for F if F's values on the (finite subsets picked from the) sequence depend only on type and a generalization of regressiveness. In a sense, this is exactly the behavior one would want out of a Skolem function on a structure with many large order-indiscernibles, which is exactly how Theorem 4.1.3 and Theorem 4.1.5 proceed.

Our use case for mutually homogeneous sequences is to build order indiscernibles for a given model $\mathcal{M}$. This is implicit in (Koepke, 2007):

Definition 4.2.3. Let $\theta \geq \kappa$, and let $\mathcal{M}$ be a model of domain V_θ (or H_θ or θ) with signature of size less than κ_0. Suppose that $\mathcal{M}$ has a term F which is a Skolem function for $\mathcal{M}$ (i.e. $F : [\mathrm{dom}(\mathcal{M})]^{<\omega} \to \theta$ and for every $X \subseteq \kappa$, $X \subseteq F"[X]^{<\omega} \prec \mathcal{M}$). Then we say that $\mathcal{I} := \langle I_\alpha \mid \alpha \in \lambda \cap \mathrm{Succ}\rangle$ is *mutually homogeneous for* $\mathcal{M}$ *and* F if

- $\mathcal{I}$ is mutually homogeneous for F
- for each formula ϕ in the signature of $\mathcal{M}$, if $x, y \in \left[\bigcup_{\alpha \in \lambda \cap \mathrm{Succ}} I_\alpha\right]^{<\omega}$ are of the appropriate arity for ϕ and $\mathrm{type}(x) = \mathrm{type}(y)$, then $\mathcal{M} \models \phi(x) \iff \mathcal{M} \models \phi(y)$
- for each term definable in $\mathcal{M}$ viewed as a map $g : [\kappa]^{<\omega} \to \kappa$, $\mathcal{I}$ is mutually homogeneous for g; that is, if $x, y \in \left[\bigcup_{\alpha \in \lambda \cap \mathrm{Succ}} I_\alpha\right]^{<\omega}$ with $\mathrm{type}(x) = \mathrm{type}(y)$ and $x \cap (g(x)+1) = y \cap (g(x)+1)$ then $g(x) = g(y)$

Remark 4.2.4. If $\langle \kappa_\alpha \mid \alpha \in \lambda \cap \mathsf{Succ}\rangle$ is mutually Ramsey such that each $I_\alpha \subseteq \kappa_\alpha$ may be chosen to be club (or measure one for some appropriate notion of "measure one") in κ_α, then constructing a mutually homogeneous system for $\mathcal{M}$ is straightforward, albeit with the following adjustment needed for the formulas ϕ:

We may view each formula ϕ restricted to parameters chosen from $[\kappa]^{<\omega}$ as a function $\phi : [\kappa]^{<\omega} \to \{0, 1, 2\}$ defined by:

$$\phi(x) = \begin{cases} 0 & |x| \text{ is inappropriate for use in } \phi \\ 1 & \mathcal{M} \models \neg\phi(x) \\ 2 & \mathcal{M} \models \phi(x) \end{cases}$$

Then as long as we have that $\langle I_\alpha \mid \alpha \in \lambda \cap \mathsf{Succ}\rangle$ is mutually homogeneous for ϕ and that $0, 1, 2 \notin I_0$, we have that $y \cap (\phi(x)+1) = \emptyset$ for every $x, y \in [\kappa]^{<\omega}$; therefore, if $\mathsf{type}(x) = \mathsf{type}(y)$ then $\phi(x) = \phi(y)$.

So to obtain a mutually homogeneous system for $\mathcal{M}$, we construct mutually homogeneous systems of clubs (or measure one sets) for F and for each ϕ and each g as in Definition 4.2.3, of which there will be less than κ_0-many, and take their intersection.

Theorem 4.1.5, the result on Prikry generics in (Cummings et al., 2006), fundamentally relies only on arguments involving such $F : [\kappa]^{<\omega} \to \kappa$. Koepke has arguments pertaining to such F and in a much earlier paper (Koepke, 1984) proved the following:

Theorem 4.2.5 ((Koepke, 1984); also Proposition 2.3 of (Koepke, 2007))**.** *Let* $V[G]$ *be a Prikry generic model for* κ *over* V*, and let* $\langle \kappa_n \mid n < \omega \rangle$ *be a Prikry generic sequence for* κ*. Then in* $V[G]$*, for some* m*,* $\langle \kappa_n \mid m \leq n < \omega \rangle$ *is mutually Ramsey. Furthermore, the sequence of* I_n*'s witnessing this in* $V[G]$ *may be chosen to be stationary subsets of each* κ_n *consisting only of inaccessibles chosen from some* σ*-closed filter.*

The proof is fundamentally exactly as in (Cummings et al., 2006). The fact that each I_n may consist only of inaccessibles comes from observing that measure-one (hence stationarily) many cardinals below a measurable are Mahlo.

Corollary 4.2.6. *If* $\langle \kappa_n \mid n < \omega \rangle$ *is a Prikry generic sequence over* V *in* $V[G]$*, then in* $V[G]$*,* $MS(\langle \kappa_n \mid n < \omega \rangle)$ *holds on a tail.*

Proof. Let $\vec{S} = \langle S_n \mid n < \omega \rangle$ be a family of stationary sets on $\langle \kappa_n \mid n < \omega \rangle$; without loss of generality, all of $\langle \kappa_n \mid n < \omega \rangle$ is mutually Ramsey.

Let $\mathcal{M}$ be a structure on κ with Skolem function F and, without loss of generality, terms for each S_n; by mutual Ramseyness of κ, let $\langle I_n \mid n < \omega \rangle$ be a mutually homogeneous for $\mathcal{M}$ and F family of stationary sets on each κ_n.

Since each I_n is unbounded in κ_n, let $\gamma_n \in S_n \cap \mathrm{Lim}(I_n)$, and let

$$\mathcal{U} = F'' \left[\bigcup_{n<\omega} I_n \cap \gamma_n \right]^{<\omega}$$

Since F is a Skolem function and since $\gamma_n \in \mathrm{Lim}(I_n)$, $\mathcal{U} \prec \mathcal{M}$ and $\sup(\mathcal{U} \cap \kappa_n) \geq \gamma_n$.

To see that $\sup(\mathcal{U} \cap \kappa_n) \leq \gamma_n$, let $x \in [\bigcup_{n<\omega} I_n \cap \gamma_n]^{<\omega}$ be such that $F(x) = \alpha \in [\kappa_{n-1}, \kappa_n)$ for some n.

Since I_n is unbounded in κ_n, let $\delta \in I_n \cap (\max\{\gamma_n, \alpha\}, \kappa_n)$; since x is finite and I_n is unbounded in γ_n, let $\zeta \in I_n \cap (\max(x \cap \gamma_n), \gamma_n)$.

We now have that $\mathcal{M} \models F(x) < \delta$ by choice of δ. Additionally, by choice of δ and ζ, and since $x \cap [\zeta, \kappa_n) = \emptyset$, the elements of x bear the same order relation to ζ as they do to δ, and in particular $\mathtt{type}(x \cup \{\delta\}) = \mathtt{type}(x \cup \{\zeta\})$.

Thus by mutual homogeneity applied to the formula "$F(\vec{a}) < b$", $\mathcal{M} \models F(x) < \zeta$ and observe that this implies $F(x) < \gamma_n$.

Therefore $\sup(\mathcal{U} \cap \kappa_n) = \gamma_n$ as desired. $\square$

4.3 An Indiscernibility Result Applicable to Magidor Forcing

To argue for the mutual stationarity property for Magidor generic sequences, we need an as of yet unpublished simultaneous homogeneity result for multiple normal measures on κ. We first argue an "easy" version for $[\kappa]^{<\omega}$ that pulls at most one ordinal from each measure on κ, and then generalize to larger sequences.

Definition 4.3.1. Let $\langle X_i \mid 0 \leq i < n \rangle$ be sets of ordinals. Then we write

$$\overset{\uparrow}{\prod_{0 \leq i < n}} X_i$$

to denote all n-length increasing sequences $\beta_0 < \cdots < \beta_{n-1}$ where $\beta_i \in X_i$.

Lemma 4.3.2. *Suppose λ is a regular cardinal below κ and $\overrightarrow{U} = \langle U_\alpha \mid \alpha < \lambda \rangle$ is a sequence of normal measures on κ and suppose that $f : [\kappa]^{<\omega} \to \kappa$ is regressive, i.e. for all x, $f(x) < \min(x)$.*

Then there is an $\overrightarrow{H} = \langle H_\alpha \mid \alpha < \lambda \rangle$ where each $H_\alpha \in U_\alpha$, such that for every n and every $\alpha_0 < \cdots < \alpha_{n-1}$,

$$f \restriction \prod_{0 \le i < n}^{\uparrow} H_{\alpha_i}$$

is constant.

Proof. We proceed by induction on n, for all such f simultaneously. The need for all functions simultaneously will be apparent during the inductive step.

If $n = 1$, then we need ${}^1\overrightarrow{H} = \left\langle {}^1H_\alpha \mid \alpha < \lambda \right\rangle$ such that for each α, $f \restriction {}^1H_\alpha$ is constant. This follows readily by Fodor's theorem, since all the U_α's are κ-complete and normal.

For the induction hypothesis, suppose that for every $g : [\kappa]^k \to \kappa$ regressive, there is an ${}^k\overrightarrow{H}$ such that for each $\alpha_0 < \cdots < \alpha_{k-1}$, $g \restriction \prod^{\uparrow}_{0 \le i < k} {}^kH_{\alpha_i}$ is constant.

For each $\beta_0 < \kappa$, let ${}_{\beta_0}f : [\kappa \setminus \{\beta_0\}]^k \to \kappa$, ${}_{\beta_0}f(X) = f(\{\beta_0\} \sqcup X)$.

Since f is regressive, so is ${}_{\beta_0}f$; so by the induction hypothesis, for each β_0, there is a sequence ${}^k_{\beta_0}\overrightarrow{H}$ such that for each $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, ${}_{\beta_0}f \restriction \prod^{\uparrow}_{1 \le i < k+1} {}^k_{\beta_0}H_{\alpha_i}$ is constant, say with value $\gamma_{\alpha_1,\ldots,\alpha_k}(\beta_0)$.

Let $g_{\alpha_1,\ldots,\alpha_k} : \kappa \to \kappa$, $g_{\alpha_1,\ldots,\alpha_k}(\beta) = \gamma_{\alpha_1,\ldots,\alpha_k}(\beta)$; observe that $g_{\alpha_1,\ldots,\alpha_k}$ is regressive.

Fix an $\alpha < \lambda$; we now wish to define ${}^{k+1}H_\alpha$.

By regressiveness of $g_{\alpha_1,\ldots,\alpha_k}$, for each $\alpha_1 < \cdots < \alpha_k$ in $[\lambda \setminus (\alpha+1)]^{<\omega}$, there is an $A^\alpha_{\alpha_1,\ldots,\alpha_k} \in U_\alpha$ such that $g_{\alpha_1,\ldots,\alpha_k}$ is constant on $A^\alpha_{\alpha_1,\ldots,\alpha_k}$, say with value $\delta^\alpha_{\alpha_1,\ldots,\alpha_k}$.

Let

$$B_\alpha = \left(\bigcap_{\alpha<\alpha_1<\cdots<\alpha_k<\lambda} A^{\alpha}_{\alpha_1,\ldots,\alpha_k} \right)$$

For each $\beta < \kappa$, let ${}^k_\beta H'_\alpha = {}^k_\beta H_\alpha \cap B_\alpha$ and define

$${}^{k+1}H_\alpha = \Delta_{\beta<\kappa} {}^k_\beta H'_\alpha$$

Then ${}^{k+1}H_\alpha$ is in U_α, and we now verify that ${}^{k+1}\overrightarrow{H}$ has the appropriate homogeneity. Suppose that $\alpha_0 < \cdots < \alpha_k$. We wish to verify that for every $\beta_i \in {}^{k+1}H_{\alpha_i}$, $\beta_i < \beta_{i+1}$, we have $f(\{\beta_0, \ldots, \beta_k\}) = \delta^{\alpha_0}_{\alpha_1,\ldots,\alpha_k}$. To see this, observe that for each possible choice of β_0, $f(\{\beta_0, \ldots, \beta_k\}) = {}_{\beta_0}f(\{\beta_1, \ldots, \beta_k\})$; since $\beta_i < \beta_{i+1}$, for $i \geq 1$ we have that $\beta_i \in {}_{\beta_0}^{k}H_{\alpha_i}$. Hence $f(\{\beta_0, \ldots, \beta_k\}) = \gamma_{\alpha_1,\ldots,\alpha_k}(\beta_0)$. But since $\beta_0 \in B_{\alpha_0}$, we have that in fact $\gamma_{\alpha_1,\ldots,\alpha_k}(\beta_0) = \delta^{\alpha_0}_{\alpha_1,\ldots,\alpha_k}$ as desired.

In the end, our desired H_α is $\bigcap_{n<\omega} {}^n H_\alpha$. □

While Lemma 4.3.2 has some utility, we will need the following more general multi-arity version:

Definition 4.3.3. Let $\langle X_i \mid 0 \leq i < n \rangle$ be sets of ordinals and let $k_0, \ldots, k_{n-1}$ be natural numbers. Then we write

$$\prod^{\uparrow}_{0\leq i<n} X_i^{k_i}$$

to denote the collection of all $\overrightarrow{\beta_0}, \ldots, \overrightarrow{\beta_{n-1}}$ which are $k_0, \ldots, k_{n-1}$-ary increasing sequences such that $\overrightarrow{\beta_i} \subseteq X_i$ and $\max \overrightarrow{\beta_i} < \min \overrightarrow{\beta_{i+1}}$.

Lemma 4.3.4. *Suppose* λ *is a regular cardinal below* κ *and* $\overrightarrow{U} = \langle U_\alpha \mid \alpha < \lambda \rangle$ *is a system of normal measures on* κ *and suppose that* $f : [\kappa]^{<\omega} \to \kappa$ *is regressive.*

Then there is an $\overrightarrow{H} = \langle H_\alpha \mid \alpha < \lambda \rangle$ *where each* $H_\alpha \in U_\alpha$*, such that for every* n *and for every* $k_0, \ldots, k_{n-1}$*, for every* $\alpha_0 < \cdots < \alpha_{n-1}$*,*

$$f \restriction \prod_{0 \le i < n}^{\uparrow} H_{\alpha_i}^{k_i}$$

is constant.

There was nothing saying that a normal measure *can't* repeat itself in the statement of Lemma 4.3.2. That leads to the following short proof, where we repeat the same normal measure ω-many times consecutively:

Proof. Apply Lemma 4.3.2 to the sequence $\vec{U}' = \left\langle U'_{\omega\cdot\alpha+k} \mid k < \omega, \alpha < \lambda \right\rangle$ where $U'_{\omega\cdot\alpha+k} = U_\alpha$ for all k.

This gives a sequence $\left\langle H'_{\omega\cdot\alpha+k} \mid k < \omega, \alpha < \lambda \right\rangle$ with the homogeneity result for $\vec{U}'$ as in Lemma 4.3.2. To obtain homogeneity as desired for $\vec{U}$, we will have that $H_\alpha = \bigcap_{k<\omega} H'_{\omega\cdot\alpha+k}$. $\square$

4.4 Mutual Stationarity for Magidor Generics

We now generalize Koepke's approach to Magidor forcing. Before we do so, we have an ancillary lemma that will come into play. Note that Prikry forcing preserves the stationarity of sets of size less than κ, because Prikry forcing adds no new bounded subsets of κ. We need a different argument for Magidor forcing.

For this section, let $\vec{U} = \langle U_\alpha \mid \alpha < \lambda \rangle$ be a Mitchell order increasing sequence of normal measures on κ of length λ, for some $\lambda < \kappa$. Recall that $V^{\mathbb{M}(\vec{U})}$ denotes a generic extension after forcing with $\mathbb{M}\left(\vec{U}\right)$, as defined in Section 2.2.

Lemma 4.4.1. *Let δ be a cardinal in V, let and suppose that in $V^{\mathbb{M}(\vec{U})}$, $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ denotes the Magidor generic sequence added by $\mathbb{M}\left(\vec{U}\right)$. Suppose that $\delta = \kappa_{\alpha+1}$ for some α. Let $S \subseteq \delta$ be stationary in V. Then S is stationary in $V^{\mathbb{M}(\vec{U})}$.*

Proof. It is enough to show that S is stationary after forcing with $\mathbb{M}\left(\vec{U}\right)_{(\alpha+1,\delta)} \times \mathbb{M}\left(\vec{U}\right)^{(\alpha+1,\delta)}$.

To see that S is stationary in $V^{\mathbb{M}(\vec{U})_{(\alpha+1,\delta)}}$, let $1 \leq n < \omega$ be such that $\alpha + 1 = \beta + n$ for some β limit (or 0). Then densely often (by extending the stem such that $\beta + k$ is in the domain of the stem, for all $k \leq n$), we may assume we're forcing with $\mathbb{M}\left(\vec{U}\right)_{(\beta,\xi)}$ for some ξ, which is ξ^+-cc, and hence is δ-cc as well. Thus S is stationary in $V^{\mathbb{M}(\vec{U})_{(\alpha+1,\delta)}}$.

Since the direct extension order on $\mathbb{M}\left(\vec{U}\right)^{(\alpha+1,\delta)}$ remains δ-closed after forcing with $\mathbb{M}\left(\vec{U}\right)_{(\alpha+1,\delta)}$, S remains stationary in the full generic extension. $\square$

Note that since mutually stationary sequences of sets are only defined for families of regular cardinals, we don't have to concern ourselves with sets cofinal in the cardinals singularized by the Magidor forcing.

Theorem 4.4.2. *Let $\lambda < \kappa$ and let G be Magidor generic over V with $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ the induced Magidor generic sequence. In $V[G]$, let $f : [\kappa]^{<\omega} \to \kappa$.*

Then there are $\delta < \lambda$ *and* $\langle A_\alpha \mid \delta < \alpha < \lambda, \alpha$ *successor*$\rangle$ *such that each* A_α *is stationary in* κ_α *and on which* f *is mutually homogeneous, that is, whenever* $x, y \subseteq [\bigcup_{\delta<\alpha<\lambda} A_\alpha]^{<\omega}$ *with* $\mathtt{type}(x) = \mathtt{type}(y)$ *and* $x \cap (f(x)+1) = y \cap (f(x)+1)$, *we have that* $f(x) = f(y)$.

Moreover, each $A_\alpha \in r^\lambda_\alpha(\kappa_\alpha)$, *where* $r^\lambda_\alpha(\kappa_\alpha)$ *is a Mitchell-rank* α *measure on* κ_α *and* r *is the coherent system of representatives for* $\langle U_\alpha \mid \alpha < \lambda\rangle$.

Proof. Work in V; let $(g, H) \Vdash \dot{f} : [\kappa]^{<\omega} \to \kappa$. For each stem g' extending g, and each $x \in [\kappa]^{<\omega}$ strictly increasing with $\min(x) > \max \mathtt{ran}(g)$, let $F(g', x)$ be defined by

$$F(g', x) = \begin{cases} \beta & \exists E\ (g', E) \Vdash \dot{f}(x) = \beta \\ \emptyset & \text{otherwise} \end{cases}$$

Since $\vec{U}$ is a family of normal measures on κ, applying Lemma 4.3.4 to F yields a family $X \in \prod_{\max \mathtt{dom}(g)<\alpha<\lambda} U_\alpha$ of order indiscernibles for F. What that precisely means is that whenever g_1, g_2 are finite extensions of g with the same domain such that, above $\max \mathtt{dom}(g)$, $g_i \restriction (\max \mathtt{dom}(g), \lambda) = \left\langle \kappa^i_{\alpha_0}, \ldots, \kappa^i_{\alpha_n} \right\rangle$ such that each $\kappa^i_{\alpha_j} \in X(\alpha_j)$, and x_1, x_2 are in $[\kappa]^{<\omega}$ with $\min(x_i) > \max \mathtt{ran}(g)$ such that $x_i = \left\langle x^i_{\beta_0}, \ldots, x^i_{\beta_m} \right\rangle$ where $x^i_{\beta_j} \in X(\beta_j)$, and x_1 bears the same order relations to $\mathtt{ran}(g_1)$ as x_2 bears to $\mathtt{ran}(g_2)$, and $(\mathtt{ran}(g_1) \cup x_1) \cap (F(g_1, x_1)+1) = (\mathtt{ran}(g_2) \cup x_2) \cap (F(g_1, x_1)+1)$[1], then $F(g_1, x_1) = F(g_2, x_2)$. We may further constrain X so that for $\beta > \max \mathtt{dom}(g)$, $X(\beta) \subseteq H(\beta)$.

---

[1] Recall that $x \cap (h(x)+1) = y \cap (h(x)+1)$ amounts to saying that $h(x) < \min(x \Delta y)$, so amounts to asserting that h is regressive on an appropriate subcollection of κ.

Let Y with $\mathrm{dom}(Y) = \mathrm{dom}(X)$ be such that for $\beta < \max \mathrm{dom}(g)$, $Y(\beta) = H(\beta)$; and for $\beta > \max \mathrm{dom}(g)$, $Y(\beta)$ is the collection of stationary reflection points in each $X(\beta)$, i.e.

$$Y(\beta) = X(\beta) \cap \{\gamma < \kappa \mid \gamma \cap X(\beta) \text{ stationary in } \gamma\}$$

For each $\beta > \max \mathrm{dom}(g)$, $Y(\beta) \in U_\beta$; this is a consequence of the fact that the collection of stationary reflection points below a measurable cardinal is measure one. Thus (g, Y) is in fact a condition.

We wish to verify that (g, Y) forces the desired property for $\dot{f}$; by the Generic Model Theorem, it suffices to show the desired property is true in $V[G]$ whenever $(g, Y) \in G$. Let G be $\mathbb{M}\left(\vec{U}\right)$-generic over V with $(g, Y) \in G$; let $\vec{\kappa} = \langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be the G-generic Magidor sequence. This buys for us that for all $\alpha < \lambda$ with $\alpha > \max \mathrm{dom}(g)$, $\kappa_\alpha \in Y(\alpha)$. Let $\delta = \max \mathrm{dom}(g)$ and let $\langle A_\alpha \mid \delta < \alpha < \lambda, \alpha \text{ successor} \rangle$ be given by $A_\alpha = X(\alpha) \cap \kappa_\alpha$. Each A_α is V-stationary in κ_α since $\kappa_\alpha \in Y(\alpha)$, and by Lemma 4.4.1, A_α remains stationary in κ_α.

From here on out, unless otherwise noted, we work in $V[G]$. We will verify that $\langle A_\alpha \mid \delta < \alpha < \lambda, \alpha \text{ successor} \rangle$ witnesses the desired conclusion.

Suppose $x, y \subseteq \left[\bigcup_{\delta<\alpha<\lambda,\ \alpha \text{ successor}} A_\alpha\right]^{<\omega}$ are such that $\mathrm{type}(x) = \mathrm{type}(y)$ and $x \cap (\dot{f}[G](x) + 1) = y \cap (\dot{f}[G](x) + 1)$. Let β be least such that $x, y \subseteq \kappa_\beta$; since x, y finite, we have that $\beta = \alpha + 1$ for some α. Furthermore, since x, y are finite, $\mathrm{type}(x) = \mathrm{type}(y)$, and $x \cap (\dot{f}[G](x) + 1) = y \cap (\dot{f}[G](x) + 1)$, we may extend (g, Y) to some $(g', Y') \in G$ such that:

- $\beta \in \mathrm{dom}(g')$ and $g'(\beta) = \kappa_\beta$

- since $\mathrm{lh}(x) = \mathrm{lh}(y) = n$, if we write $x = \langle x_i \mid i < n \rangle$ and $y = \langle y_i \mid i < n \rangle$ in increasing order, then for each i such that $x_i, y_i < \kappa_\alpha$, there exists some $\gamma \in \mathrm{dom}(g')$ such that $x_i, y_i \in (\kappa_\gamma, \kappa_{\gamma+1})$
- (g', Y') decides the values of both $\dot{f}(x)$ and $\dot{f}(y)$

But $\mathrm{type}(x) = \mathrm{type}(y)$, and $\min(x)$ and $\min(y)$ are both above κ_δ. Thus x bears the exact same order relations to the elements of $\mathrm{ran}(g')$ as does y. Furthermore, since $x \cap (\dot{f}[G](x)+1) = y \cap (\dot{f}[G](x)+1)$ is true in $V[G]$, we have that $(\mathrm{ran}(g') \cup x) \cap (F(g', x) + 1) = (\mathrm{ran}(g') \cup y) \cap (F(g', x) + 1)$ in V. Since x, y, and $g' \restriction (\delta, \lambda)$ were all picked from X, by choice of X we have that $F(g', x) = F(g', y)$.

But then by definition of $F(g', x)$ and $F(g', y)$, without loss of generality there is some E such that $(g', E) \Vdash \dot{f}(x) = F(g', x) = F(g', y) = \dot{f}(y)$. While (g', E) is not necessarily in G, we have that $(g', E) \parallel (g', Y')$ and both (g', E) and (g', Y') decide the values of $\dot{f}(x)$ and $\dot{f}(y)$. But then by compatibility, (g', E) and (g', Y') must decide the values of $\dot{f}(x)$ and $\dot{f}(y)$ the same way, and so $(g', Y') \Vdash \dot{f}(x) = \dot{f}(y)$. But then $(g', Y') \in G$ so $V[G] \models f(x) = f(y)$. $\square$

Theorem 4.4.2 gives a per-function mutual homogeneity on a tail of the Magidor generic sequence, but to achieve mutual stationarity, we need a uniform version. Fortunately, Theorem 4.4.2 can be uniformized below a single condition, which then forces that the entire Magidor sequence is mutually Ramsey. This isn't particular to the complexities of mutual homogeneity or Ramseyness, but rather arises from the forcing apparatus itself:

Lemma 4.4.3. *There is a single* $p \in \mathbb{MC}\left(\vec{u}\right)$ *such that*

$$p \Vdash \text{“}\langle \kappa_\alpha \mid \alpha < \lambda, \ \dot{\alpha} \textit{ successor}\rangle \textit{ is mutually Ramsey as witnessed by } \check{A}_\alpha \in r^\lambda_\alpha\left(\dot{\kappa_\alpha}\right)\text{"}$$

Proof. For the sake of contradiction, suppose otherwise. By unpacking the definitions of $\nVdash$ and of mutually Ramsey, we have that below every p there is a q such that

$$\begin{aligned} q \Vdash \text{“}\exists \dot{f} : [\kappa]^{<\omega} \to \kappa \text{ for which there is no family of } \check{A}_\alpha \in r^\lambda_\alpha\left(\dot{\kappa_\alpha}\right) \text{ on successors} \\ \text{witnessing mutual homogeneity for } f\text{"} \end{aligned} \tag{4.1}$$

Without loss of generality, each such q has a name $\dot{f}_q$ such that q forces $\dot{f}_q$ to witness the criterion in Equation 4.1.

By choosing a maximal antichain A of such q and then by applying the Mixing Lemma to $\langle f_q \mid q \in A\rangle$, there is a name $\dot{g}$ such that

$$\begin{aligned} 1_{\mathbb{M}(\vec{u})} \Vdash \text{“}\dot{g} : [\kappa]^{<\omega} \to \kappa \text{ admits no family of } \check{A}_\alpha \in r^\lambda_\alpha\left(\dot{\kappa_\alpha}\right) \text{ on successors} \\ \text{witnessing mutual homogeneity for } \dot{g}\text{"} \end{aligned} \tag{4.2}$$

But then by the argument of Theorem 4.4.2, since $\mathsf{stem}\left(1_{\mathbb{M}(\vec{u})}\right) = \emptyset$, there is a condition $(\emptyset, Y)$ such that

$$(\emptyset, Y) \Vdash \text{“}\exists A_\alpha \mid \alpha < \dot{\lambda} \text{ successor}, \ A_\alpha \in r^\lambda_\alpha\left(\dot{\kappa_\alpha}\right), \ \text{ that is mutually homogeneous for } g\text{"}$$

This contradicts Equation 4.2. $\square$

Due to Lemma 2.6.24, we can adjust our Magidor generic sequence on a finite set to meet whatever condition we want, including the p from Lemma 4.4.3:

Lemma 4.4.4. *Let* $q \in \mathbb{M}\left(\vec{U}\right)$*, and let* G *be* $\mathbb{M}\left(\vec{U}\right)$*-generic over* V*. Then there is a* G' *also* $\mathbb{M}\left(\vec{U}\right)$*-generic over* V *such that* $q \in G'$ *and* $V[G] = V[G']$*.*

Proof. Let $q = (g, H)$, and let $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be the Magidor generic sequence induced by G. By Lemma 2.6.24, there exists a finite $b \subseteq \mathrm{dom}(H)$ such that for all $\alpha \in \mathrm{dom}(H) \setminus b$, $\kappa_\alpha \in H(\alpha)$. Thus, let $q' = (g', H')$ be a b-step extension of q such that for all $\alpha \in \mathrm{dom}(H')$, $\kappa_\alpha \in H'(\alpha)$.

Then let $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle$ be given by $\kappa'_\alpha = g'(\alpha)$ if $\alpha \in \mathrm{dom}(g)$, and $\kappa'_\alpha = \kappa_\alpha$ otherwise. Since $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ is geometric, so is $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle$, and so by Theorem 2.6.23, let G' be the generic filter induced by $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle$.

By construction of G', $q' \in G'$ since $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle \upharpoonright \mathrm{dom}(g') = g'$ and for all $\alpha \in \mathrm{dom}(H')$, $\kappa'_\alpha \in H'(\alpha)$. Thus $q \in G'$ as $q' \leq q$. But since $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ and $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle$ differ only on a finite subsequence, both are equidefinable from each other and therefore $V[G'] = V[G]$. $\square$

As a consequence, we have that Magidor generic sequences are mutually Ramsey on a *cofinite* set:

Theorem 4.4.5. *Let* G *be* $\mathbb{M}\left(\vec{U}\right)$*-generic over* V*, and let* $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ *be its induced generic sequence. Then there is a cofinite subset* K *of* $\{\alpha < \lambda \mid \alpha \text{ successor}\}$ *such that* $\langle \kappa_\alpha \mid \alpha \in K \rangle$ *is mutually Ramsey as witnessed by a family of* $A_\alpha \in r^\lambda_\alpha(\kappa_\alpha)$*.*

Proof. Let $p \in \mathbb{M}\left(\vec{U}\right)$ be as in Lemma 4.4.3, and by Lemma 4.4.4, let G' be generic such that $p \in G'$ and $V[G] = V[G']$.

Then by Lemma 4.4.3, the $V[G']$-generic sequence $\langle \kappa'_\alpha \mid \alpha < \lambda \rangle$ is mutually Ramsey on the κ'_α for which α is a successor, as witnessed by elements of $r^\lambda_\alpha(\kappa'_\alpha)$.

But by the proof of Lemma 4.4.4, $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ and $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ agree on a cofinite subsequence K'. Then $K = K' \cap \{\alpha \in \mathtt{Ord} \mid \alpha \text{ successor}\}$ is as desired. □

As for Prikry generics, (c.f. Corollary 4.2.6), from the mutual Ramseyness we obtain mutual stationarity:

Theorem 4.4.6. *Let G be $\mathbb{M}\left(\vec{U}\right)$-generic over V, let $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be its induced generic sequence, and let K be as in Theorem 4.4.5. Then*

$$\mathsf{MS}(\langle \kappa_\alpha \mid \kappa_\alpha \in K \rangle)$$

holds.

Proof. Let $\langle S_\alpha \mid \alpha \in K \rangle$ be a system of stationary sets with each $S_\alpha \subseteq \kappa_\alpha$, let $\mathcal{M}$ be a structure κ with countable signature and with Skolem function F. Since $\mathcal{M}$ has only countably many functions in its signature, let $\langle A_\alpha \mid \alpha \in K \rangle$ be mutually homogeneous for $\mathcal{M}$ and F with each A_α stationary in κ_α.

For each relevant α, since $\mathtt{Lim}(A_\alpha)$ is club in κ_α let $\gamma_\alpha \in S_\alpha \cap \mathtt{Lim}(A_\alpha)$, and let

$$\mathcal{U} = F''\left[\bigcup_{\delta<\alpha<\kappa,\ \alpha \text{ successor}} A_\alpha \cap \gamma_\alpha\right]^{<\omega}$$

Then since F is a Skolem function, $\mathcal{U} \prec \mathcal{M}$, and by the exact same logic as in Corollary 4.2.6, for each relevant α, $\sup(\mathcal{U} \cap \kappa_\alpha) = \gamma_\alpha$ for every $\alpha \in \mathsf{K}$. $\square$

CHAPTER 5

A MAGIDOR FORCING WITH COLLAPSES

Section 4.1 covered prior results on mutual stationarity principles at $\aleph_\omega$. To push mutual stationarity principles down to larger accessible cardinals, especially of uncountable cofinality, we want tools to singularize a large cardinal κ while also collapsing below to make κ into $\aleph_\lambda$ for some λ.

To that end, we now describe a Magidor-type forcing with interleaved guided collapses $\mathbb{MC}\left(\vec{U}\right)$ that singularizes a measurable cardinal κ to cofinality λ and collapses cardinals below. We state and prove Prikry-type lemmas (Lemma 5.1.11 and Lemma 5.1.19).

The main result of ths chapter is a characterization of genericity for $\mathbb{MC}\left(\vec{U}\right)$, spanning Lemma 5.1.21 and Theorem 5.1.24.

In the event that $\lambda = \omega_1$, we will further get that κ becomes the new $\aleph_{\omega_1}$.

5.1 Defining the Forcing

Throughout this section, suppose GCH. We fix $\vec{U} = \langle U_\xi | \xi \leq \lambda \rangle$ a Mitchell order-increasing system of measure, $\lambda < \kappa$, $\langle Y_\xi \mid \xi < \lambda \rangle$, and $\left\langle r^\alpha_\beta \middle| \beta < \alpha < \lambda \right\rangle$ as in Section 2.6[1].

Work in $\mathrm{Ult}(V, U_\lambda)$ and consider $\mathtt{Col}(\kappa^+, < j_{U_\lambda}(\kappa)) = [\mathtt{Col}(\alpha^+, < \kappa)]_{U_\lambda}$.

Lemma 5.1.1. *There is a* $K \in V$ *which is* $\mathtt{Col}(\kappa^+, < j_{U_\lambda}(\kappa))$*-generic over* $\mathrm{Ult}(V, U_\lambda)$.

---

[1] In brief, this assures that the ordinals in each Y_ξ is measurable of Mitchell order ξ, as witnessed by the r-sequence of measure representatives, and with reflection in the r-measures.

Proof. This is Proposition 2.5.8. $\square$

Lemma 5.1.2. *Furthermore, for every $\alpha \leq \lambda$ there is a $K_\alpha \in V$ which is $\mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))$-generic over* $\mathrm{Ult}(V, U_\alpha)$*, such that:*

1. *whenever $\alpha < \beta \leq \lambda$, $K_\alpha \in \mathrm{Ult}(V, U_\beta)$*
2. *there is a family of functions $\left\langle k_\alpha^\beta \middle| \alpha < \beta < \lambda \right\rangle$ such that $K_\alpha = [\eta \mapsto k_\alpha^\beta(\eta)]_{U_\beta}$*
3. *in particular, for some $Z_\beta \in U_\beta$, for every $\eta \in Z_\beta$, $k_\alpha^\beta(\eta)$ is a $\mathrm{Col}(\eta^+, < j_{r_\alpha^\beta(\eta)}(\eta))$-generic over* $\mathrm{Ult}(V, r_\alpha^\beta(\eta))$ *and $[K_\alpha \restriction \eta]_{r_\alpha^\beta(\eta)} = k_\alpha^\beta(\eta)$; furthermore, this reflects downwards through r.*

Proof. The existence of K_α can be argued exactly as in Lemma 5.1.1.

As for obtaining that $K_\alpha \in \mathrm{Ult}(V, U_\beta)$, observe that since $U_\alpha \in \mathrm{Ult}(V, U_\beta)$ (and by GCH), we simply have that $|j_{U_\alpha}(\kappa)|^{\mathrm{Ult}(V,U_\beta)} = 2^\kappa = \kappa^+$ (by mapping κ^κ onto $j_{U_\alpha}(\kappa)$ by $f \mapsto [f]_{U_\alpha}$, see for instance (Jech, 2003), Chapter 17, for details). Furthermore, since $U_\alpha \in \mathrm{Ult}(V, U_\beta)$, not only is $\mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V,U_\alpha)}$ in $\mathrm{Ult}(V, U_\beta)$, but so is the entire enumerated family $\left\langle A_\rho \middle| \rho < (2^\kappa)^{\mathrm{Ult}(V,U_\beta)} \right\rangle$ of $\mathrm{Ult}(V, U_\alpha)$-antichains of $\mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V,U_\alpha)}$.

Recall that since $\mathrm{Ult}(V, U_\alpha)$ is closed under κ-sequences, in V we enumerate $K'_\alpha = \langle p_\alpha \mid \alpha < \kappa^+ \rangle$ a descending chain of $\mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V,U_\alpha)}$ such that for every A_ρ, either $p_\rho \in A_\rho$ or p_ρ is below some element of A_ρ. Since K'_α is definable entirely from the enumerated antichains and U_α, $K'_\alpha \in \mathrm{Ult}(V, U_\beta)$, and hence so is K_α the upwards closure of K'_α.

The second and third items are measure-one reflections of the first. $\square$

Remark 5.1.3. Our only use of GCH is to construct the sequence $\langle K_\alpha \mid \alpha \leq \lambda\rangle$ via the appropriate closure of measure ultrapowers. In the absence of GCH, stronger large cardinal hypotheses suffice to have ultrapowers with the appropriate closure.

The above coherent family of ultrapower collapse-generics allows us to define a Magidor-style forcing with interleaved collapses, where the collapses are "guided" by the ultrapower generics. In summary, conditions will similar to Magidor forcing to add a sequence $\langle \kappa_\alpha | \alpha < \lambda\rangle$ with supremum κ, but:

- κ_0 will be the new λ^{++},
- between any two adjacent Magidor points κ_α and $\kappa_{\alpha+1}$ we will also force with $\mathsf{Col}(\kappa_\alpha^+, < \kappa_{\alpha+1})$,
- extension will be as in Magidor forcing, but additionally we may strengthen collapses, and we don't allow already-collapsed ordinals to become future Magidor points,
- when we add new points to the growing finite subfamily of the Magidor sequence, we must also choose new collapse terms to be below some condition below the relevant guiding generic in a sense;
- and the conditions are restricted in such a way as to ensure that no Magidor sequence point gets collapsed, and that the poset still has a Prikry-type property.

Definition 5.1.4. $\mathbb{MC}\left(\vec{U}\right)$: conditions are of the form (f, c, A, C) where:

1. $\mathsf{dom}(f), \mathsf{dom}(c) \in [\lambda \sqcup \{-1\}]^{<\omega}$, $\mathsf{dom}(f) = \mathsf{dom}(c)$, and $\mathsf{dom}(A) = \mathsf{dom}(C) = \lambda \setminus \mathsf{dom}(f)$, and $f(-1) = \lambda^+$. We will write $\mathsf{dom}^+(f)$ to mean $\mathsf{dom}(f) \cap \mathsf{Ord}$, and similarly for $\mathsf{dom}(c)$.
2. For $\xi \in \mathsf{dom}^+(f)$, $f(\xi) \in Z_\xi$, $f(\xi) > \lambda^+$, and f is strictly increasing

3. for $\xi \in \mathtt{dom}(A)$ with $\xi < \max \mathtt{dom}^+(f)$, let $\tau = \min\{\mathtt{dom}^+(f) \setminus (\xi+1)\}$. Then $A(\xi) \in r^\tau_\xi(f(\tau))$.

4. For $\xi \in \mathtt{dom}(A)$ with $\xi > \max \mathtt{dom}^+(f)$, $A(\xi) \in U_\xi$; with $\tau = \max(\mathtt{dom}^+(f) \cap \xi)$, we further require that $A(\xi) \subseteq Z_\xi \setminus (f(\tau)+1)$.

5. For each $\alpha \in \mathtt{dom}(c)$, $c(\alpha)$, if $(\mathtt{dom}(f) \setminus (\alpha+1)) \neq \emptyset$, is a $\mathtt{Col}(f(\alpha)^+, < f(\min(\mathtt{dom}(f) \setminus (\alpha+1))))$-condition; otherwise is a $\mathtt{Col}(f(\alpha)^+, < \kappa)$-condition

6. for each $\alpha \in \mathtt{dom}(A)$, for each $\beta \in \mathtt{dom}(A) \cap \alpha$, $A(\alpha) \cap \sup c(\beta) = \emptyset$, that is, no collapsed ordinals will ever become Magidor points;

7. C acts as a measure-one system of future choices for c; that is, for each relevant α, $C(\alpha)$ is a function with domain $A(\alpha)$ such that

   - if $\delta \in A(\alpha)$, then $C(\alpha)(\delta) \in \mathtt{Col}(\delta^+, < \kappa)$ if $\delta > \max \mathtt{dom}(f)$; otherwise, letting $\xi = \min(\mathtt{dom}(f) \setminus (\alpha+1))$, is in $\mathtt{Col}(\delta^+, < f(\xi))$
   - if $\alpha > \max \mathtt{dom}(f)$, then $[C(\alpha)]_{U_\alpha} \in K_\alpha$; otherwise, letting $\xi = \min(\mathtt{dom}(f) \setminus (\alpha+1))$, we have that $[C(\alpha)]_{r^\xi_\alpha(f(\xi))} \in k^\xi_\alpha(f(\xi))$.

We say that the condition $(f', c', A', C') \leq (f, c, A, C)$ if

1. $f' \supseteq f$ and if $\alpha \in \mathtt{dom}^+(f') \setminus \mathtt{dom}^+(f)$, then $f'(\alpha) \in A(\alpha)$
2. for each $\alpha \in \mathtt{dom}(c)$, $c'(\alpha) \leq c(\alpha)$ in the relevant collapse forcing
3. for each $\alpha \in \mathtt{dom}^+(c') \setminus \mathtt{dom}^+(c)$, $c'(\alpha) \leq C(\alpha)(f'(\alpha))$
4. for all $\alpha \in \mathtt{dom}(A')$, $A'(\alpha) \subseteq A(\alpha)$

5. for all $\alpha \in \mathrm{dom}(C')$, $C'(\alpha)(\delta) \leq C(\alpha)(\delta)$ for all $\delta \in A'(\alpha)$

As for our definition of direct extension, we will say that $(f', c', A', C') \leq^* (f, c, A, C)$ if $f' = f$; note that we may still strengthen collapse terms in a direct extension.

Remark 5.1.5. For ease of notation, and in accordance with conventions found in (Gitik, 2010), we may write a condition $p \in \mathbb{MC}\left(\vec{U}\right)$ as

$$p = (f^p, c^p, A^p, C^p)$$

We'll say that the *stem* of p is f^p.

Observe that $\mathbb{MC}\left(\vec{U}\right)$ has the κ^+-cc essentially for the same reasons as classical Magidor forcing. Note that two conditions with the same Magidor stem need not be compatible due to their collapse terms being incompatible:

Proposition 5.1.6. $\mathbb{MC}\left(\vec{U}\right)$ *has the* κ^+*-cc.*

Proof. Observe that if p, q are conditions such that both $f^p = f^q$ and $c^p = c^q$, then $p \parallel q$; simply strengthen A^p, A^q, C^p, C^q accordingly. Thus there are only κ-many choices of incompatible stems and collapses. $\square$

As with Magidor forcing, we have notions of how to project conditions, and a factoring property:

Definition 5.1.7. Let $p = (f, c, A, C) \in \mathbb{MC}\left(\vec{U}\right)$ and let $\xi < \lambda$. Then we may define the following:

- $(p)_\xi = (f, c, A, C)_\xi = (f \restriction (\xi+1), c \restriction (\xi+1), A \restriction (\xi+1), C \restriction (\xi+1))$
- $(p)^\xi = (f, c, A, C)^\xi = (f \restriction (\lambda \setminus (\xi+1)), c \restriction (\lambda \setminus (\xi+1)), A \restriction (\lambda \setminus (\xi+1)), C \restriction (\lambda \setminus (\xi+1)))$
- $\mathbb{MC}\left(\vec{U}\right)_{(\xi,\beta)} = \left\{(f, c, A, C)_\xi \mid (f, A) \in \mathbb{MC}\left(\vec{U}\right), \xi \in \mathrm{dom}(f), \text{ and } f(\xi) = \beta\right\}$
- $\mathbb{MC}\left(\vec{U}\right)^{(\xi,\beta)} = \left\{(f, c, A, C)^\xi \mid (f, A) \in \mathbb{MC}\left(\vec{U}\right), \xi \in \mathrm{dom}(f), \text{ and } f(\xi) = \beta\right\}$

Fact 5.1.8. As with vanilla Magidor forcing (c.f. Fact 2.6.8), due to our use of the reflected guiding generics $\langle k^\xi_\alpha | \alpha < \xi, \xi < \lambda\rangle$, for each ξ a limit point below λ and each $\beta \in Y_\xi$, relative to the weakest p for which $f^p(\xi) = \beta$,

$$\mathbb{MC}\left(\vec{U}\right)/p = \mathbb{MC}\left(\vec{U}\right)_{(\xi,\beta)} \times \mathbb{MC}\left(\vec{U}\right)^{(\xi,\beta)}$$

so $\mathbb{MC}\left(\vec{U}\right)$ factors.

In particular,

$$\mathbb{MC}\left(\vec{U}\right)_{(\xi,\beta)} = \mathbb{MC}\left(\left\langle r^\xi_\alpha(\beta) \mid \alpha < \xi\right\rangle\right)$$

with guiding generics $\left\langle k^\xi_\alpha(\beta) \mid \alpha < \xi\right\rangle$, and

$$\mathbb{MC}\left(\vec{U}\right)^{(\xi,\beta)} = \mathbb{MC}\left(\langle U_\gamma \mid \xi < \gamma < \lambda\rangle\right)$$

(restricted to stems above ρ^+) with guiding generics $\langle K_\gamma \mid \gamma < \lambda\rangle$.

As with $\mathbb{M}\left(\vec{U}\right)$, due to the factoring of Fact 5.1.8, many preservation arguments that work for $\mathbb{MC}\left(\vec{U}\right)$ still reflect downwards. The factoring will also allow us to inductively prove a characterization of genericity for $\mathbb{MC}\left(\vec{U}\right)$ in Theorem 5.1.24.

The Diagonalization Lemma for $\mathbb{MC}\left(\vec{U}\right)$ requires more assumptions than with $\mathbb{M}(\vec{U})$; due to the introduction of collapse terms, not every indexed family of step extensions below p can be diagonalized.

However, a Prikry-type property still holds, and through its proof we can extract a partial Diagonalization Lemma.

As with Magidor forcing, the step extensions in question for the Prikry lemma are a-step extensions:

Definition 5.1.9. Let $a \in [\lambda \cup \{-1\}]^{<\omega}$, let $b \subseteq a \cup \{-1\}$, and let p, q be conditions with $q \leq p$. We'll say that q is an a-step extension of p if $\mathrm{dom}(f^q) = \mathrm{dom}(f^p) \sqcup a$.

Note that a-step extensions, like direct extensions, may additionally extend the c-terms.

Also as with Magidor forcing, we want a notion of minimal extension $p \frown \vec{\nu}$ for which $f^{p \frown \vec{\nu}} = f^p \frown \vec{\nu}$:

Definition 5.1.10. Let p be a condition, let $\alpha \in \mathrm{dom}(A^q)$, and let $\nu \in A^q(\alpha)$. Then we may define the condition $p \frown (\alpha, \nu)$ as the weakest extension q of p such $f^q(\alpha) = \nu$, namely:

- $f^{p \frown (\alpha,\nu)} = f^p \frown (\alpha, \nu)$
- $c^{p \frown (\alpha,\nu)} = c^p \frown (\alpha, C^p(\alpha)(\nu))$
- For $\beta > \alpha$, $A^{p \frown (\alpha,\nu)}(\beta) = A^p(\beta) \setminus \sup(C^p(\alpha)(\nu))$ and $C^{p \frown (\alpha,\nu)}(\beta) = C^p(\beta) \restriction A^{p \frown (\alpha,\nu)}(\beta)$
- For $\beta < \alpha$, $A^{p \frown (\alpha,\nu)}(\beta) = A^p(\beta) \cap \nu \cap \{\mu \in Z_\beta \cap \nu \mid \sup C^p(\beta)(\mu) < \nu\}$; by Lemma 5.1.2(3), this is in $r^\alpha_\beta(\nu)$

- and for $\beta < \alpha$, $C^{p\frown(\alpha,\nu)}(\beta) = C^p(\beta) \restriction A^{p\frown(\alpha,\nu)}(\beta)$; by Lemma 5.1.2(3), $[C^{p\frown(\alpha,\nu)}(\beta)]_{r^\alpha_\beta(\nu)} \in k^\alpha_\beta(\nu)$ as needed

For longer ν, we define, for each $a \in [\mathrm{dom}(A^p)]^{<\omega}$ and each strictly increasing $\vec{\nu} \in \prod_{\alpha\in a} A^p(\alpha)$, $p \frown \vec{\nu}$ similarly.

Lemma 5.1.11 (Prikry-type lemma, open dense version)**.** *If* p *is a condition and* $\mathcal{D}$ *is an open dense set, then there is a direct extension* $r \leq^* p$ *and an* $a \in [\lambda]^{<\omega}$ *such that every* a*-step extension of* r *is in* $\mathcal{D}$*.*

Before we prove this, we will introduce some terminology for families of direct extensions of a fixed condition:

Definition 5.1.12. Let $p \in \mathbb{MC}\left(\vec{U}\right)$, let $\mathcal{D}$ be an open dense set, and let

$$E = \left\{ \vec{\nu} : a \to \kappa \,\middle|\, a \in [\mathrm{dom}(A^p)]^{<\omega}, \vec{\nu} \in \prod_{\alpha\in a} A^p(\alpha), \text{ and } p \frown \vec{\nu} \text{ is a condition below } p \right\}$$

and for each $a \in [\lambda]^{<\omega}$ such that there are valid a-step extensions of p, let

$$E_a = \{\vec{\nu} \in E \mid \mathrm{dom}(\vec{\nu}) = a\}$$

Let $\langle p_{\vec{\nu}} | \vec{\nu} \in E_a(\text{ or } E)\rangle$ be an E_a (or E)-indexed family of conditions. We say that such a family *meets* $\mathcal{D}$ *below* p *if possible* if for each $\vec{\nu} \in E_a$ (or E), $p_{\vec{\nu}} \leq^* p \frown \vec{\nu}$ and if there is some $r \leq^* p_{\vec{\nu}}$ such that $r \in \mathcal{D}$, then $p_{\vec{\nu}} \in \mathcal{D}$ as well.

Proof of Lemma 5.1.11. We proceed in a series of claims. Let E, and E_a for each $a \in [\mathrm{dom}(A^p)]^{<\omega}$, be as in Definition 5.1.12; note that for each $\alpha \in \mathrm{dom}(A^p)$, $E_{\{\alpha\}} = A^p(\alpha)$.

Claim 5.1.13. *For each* $\alpha \in \mathrm{dom}(A^p)$, *there is a family* $\langle p_{(\alpha,\nu)} | \nu \in A^p(\alpha) \rangle$ *that meets* $\mathcal{D}$ *below* p *if possible such that*

$$[\nu \mapsto c^{p_{(\alpha,\nu)}}(\alpha)] \in K_\alpha$$

(or $k^\beta_\alpha(\rho)$ *if applicable).*

Proof of Claim 5.1.13. Without loss of generality, we argue for $\alpha > \max \mathrm{dom}(f^p)$; for smaller α, due to Fact 5.1.8, we may factor and work over $\mathbb{MC}\left(\vec{U}\right)_{(\alpha,f^p(\alpha))}$ and $\mathbb{MC}\left(\vec{U}\right)^{(\alpha,f^p(\alpha))}$.

Let $\mathcal{D}^* \subseteq \mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V,U_\alpha)}$ be given by

$$\mathcal{D}^* = \left\{ [\nu \mapsto c^{p_{(\alpha,\nu)}}(\alpha)]_{U_\alpha} \middle| \langle p_{(\alpha,\nu)} | \nu \in A^p(\alpha) \rangle \text{ meets } \mathcal{D} \text{ below } p \text{ if possible} \right\}$$

Then $\mathcal{D}^*$ is dense below $[C^p(\alpha)]$. To see this, let $[b] \in \mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V,U_\alpha)}$ be below $[C^p(\alpha)]$; simply because $[b] \leq [C^p(\alpha)]$, (on a measure one collection of $\nu \in A^p(\alpha)$) we may straightaway let $p_{(\alpha,\nu)}$ be such that $p_{(\alpha,\nu)} \leq^* p \frown (\alpha, \nu)$, $c^{p_{(\alpha,\nu)}}(\alpha) \leq b(\nu)$, and if some $r \leq^* p_{(\alpha,\nu)}$ is in $\mathcal{D}$, then $p_{(\alpha,\nu)}$ is already chosen to be in $\mathcal{D}$. Then $\langle p_{(\alpha,\nu)} | \nu \in A^p(\alpha) \rangle$ meets $\mathcal{D}$ below p if possible and by construction, $[\nu \mapsto c^{p_{(\alpha,\nu)}}(\alpha)] \leq [b]$.

Thus by the genericity of K_α, $\mathcal{D}^* \cap K_\alpha \neq \emptyset$ and therefore there is some $\langle p_{(\alpha,\nu)} | \nu \in A^p(\alpha) \rangle$ that meets $\mathcal{D}$ below p if possible such that $[\nu \mapsto c^{p_{(\alpha,\nu)}}(\alpha)]_{U_\alpha} \in K_\alpha$. □

Having the family's collapse terms at α in K_α in the ultrapower allows us to diagonalize:

Claim 5.1.14. *Let* $\langle p_{(\alpha,\nu)} | \nu \in A^p(\alpha) \rangle$ *be as in Claim* 5.1.13*. Then there is a* $p_\alpha \leq^* p$ *such that* $\langle p_\alpha \frown (\alpha,\nu) | \nu \in A^{p_\alpha}(\alpha)\rangle$ *meets* $\mathcal{D}$ *below* p *if possible.*

Proof of Claim 5.1.14. Without loss of generality, we argue for p such that $f^p = \langle(-1,\lambda)\rangle$; we can handle larger stems by factoring and measure one concentrations as per Fact 5.1.8. In this reduced case, it suffices to define A^{p_α}, C^{p_α}, and $c^{p_\alpha}(-1)$.

Observe that for each $\nu \in A^p(\alpha)$, $c^{p_{(\alpha,\nu)}}(-1) \in \mathrm{Col}(\lambda^+, <\nu)$ which is of cardinality ν. Thus we may view the map $\nu \mapsto c^{p_{(\alpha,\nu)}}(-1)$ as a regressive map on $A^p(\alpha)$, and so by the normality of U_α and Fodor's Theorem, there is some $A' \in U_\alpha$, $A' \subseteq A^p(\alpha)$, and some $c' \in \mathrm{Col}(\lambda^+, <\kappa)$ such that for all $\nu \in A'$, $c^{p_{(\alpha,\nu)}}(-1) = c'$.

Let

$$c^{p_\alpha}(-1) = c'$$

For $\beta > \alpha$, observe that for each $\nu \in A'$, $[C^{p_{(\alpha,\nu)}}(\beta)]_{U_\beta} \in K_\beta$, and thus $\{[C^{p_{(\alpha,\nu)}}(\beta)]_{U_\beta} \mid \nu \in A'\}$ is a κ-sized downwards directed subset of K_β. Thus, by the κ^+-distributvity $\mathrm{Col}(\kappa^+, < j_{U_\beta}(\kappa))^{\mathrm{Ult}(V,U_\beta)}$, we may find some $[C']$ such that $[C']_{U_\beta} \leq [C^{p_{(\alpha,\nu)}}(\beta)]_{U_\beta}$ as witnessed by $W_\nu \in U_\beta$, for all $\nu \in A'$ [1]. We let

$$A^{p_\alpha}(\beta) = A' \cap \Delta_{\nu\in A'} W_\nu$$

---

[1] This is a widely applicable forcing result: let $\mathbb{P}$ be κ^+-distributive, let G be $\mathbb{P}$-generic, and let $\langle p_\alpha \mid \alpha < \kappa\rangle$ be a downwards directed subset of G. Then there is a $p \in G$ that is a lower bound of $\langle p_\alpha \mid \alpha < \kappa\rangle$. To see this, note that by distributivity, $\langle p_\alpha \mid \alpha < \kappa\rangle$ is in the ground model and $D := \{q \mid \forall\alpha\ q \leq p_\alpha \text{ or } \exists\alpha\ p \perp p_\alpha\}$ is dense. Let $p \in D \cap G$. Since each $p_\alpha \in G$, $p \parallel p_\alpha$ for all α and thus $p \leq p_\alpha$ for all α.

and let

$$C^{p_\alpha}(\beta) = C' \restriction A^{p_\alpha}(\beta)$$

For $\beta < \alpha$, recall that each $A^{p(\alpha,\nu)}(\beta) \in r^\alpha_\beta(\nu)$, and each $C^{p(\alpha,\nu)}(\beta)$ is a map with domain $A^{p(\alpha,\nu)}(\beta)$ such that

- for each $\mu \in A^{p(\alpha,\nu)}(\beta)$, $C^{p(\alpha,\nu)}(\beta)(\mu) \in \mathrm{Col}(\mu^+, < \nu)$
- $[C^{p(\alpha,\nu)}(\beta)]_{r^\alpha_\beta(\nu)} \in k^\alpha_\beta(\nu)$

Let $B_{\beta,\alpha} = [\nu \mapsto A^{p(\alpha,\nu)}(\beta)]_{U_\alpha} \in U_\beta$ and let $X_{\beta,\alpha} = [\nu \mapsto C^{p(\alpha,\nu)}(\beta)]_{U_\alpha}$. By elementarity and the measure coherence, $B_{\beta,\alpha} \in U_\beta$; by the guiding generic coherence of Lemma 5.1.2, $[X_{\beta,\alpha}]_{U_\beta} \in K_\beta$.

So let $F_{\beta,\alpha}$ be a U_α-measure-one collection of $\nu \in A'$ such that

- $B_{\beta,\alpha} \cap \nu = A^{p(\alpha,\nu)}(\beta)$
- $X_{\beta,\alpha} \restriction A^{p(\alpha,\nu)}(\beta) = C^{p(\alpha,\nu)}(\beta)$

Let

$$A^{p_\alpha}(\beta) = B_{\beta,\alpha} \cap A^p(\alpha)$$

and let

$$C^{p_\alpha}(\beta) = X_{\beta,\alpha} \restriction A^{p_\alpha}(\beta)$$

We will need $F_{\beta,\alpha}$ for the next case.

For $\beta = \alpha$, by the result of Claim 5.1.13, $[\nu \mapsto c^{p(\alpha,\nu)}(\alpha)]_{U_\alpha} \in K_\alpha$ so let C'' be such that $[C''] = [\nu \mapsto c^{p(\alpha,\nu)}(\alpha)]$ and let A'' be a measure one set witnessing this equality.

Let

$$A^{p_\alpha}(\alpha) = A'' \cap A' \cap \bigcap_{\beta<\alpha} F_{\beta,\alpha}$$

This ensures that for every $\nu \in A^{p_\alpha}(\alpha)$, $(p_\alpha \frown (\alpha, \nu)) \restriction \alpha$ behaves as described above.

Let

$$C^{p_\alpha}(\alpha) = C''$$

Then p_α is as desired. $\square$

Claim 5.1.15. *For all* $a \in [\mathrm{dom}(A^p)]^{<\omega}$*, there is a* $p_a \leq^* p$ *such that*

$$\langle p_a \frown \vec{\nu} | p_a \frown \vec{\nu} \textit{ is a valid condition}\rangle$$

meets $\mathcal{D}$ *below* p *if possible.*

Proof of Claim 5.1.15. We induct on $|a|$; Claim 5.1.13 and Claim 5.1.14 show how to find such p_a for $|a| = 1$.

As for the induction, suppose without loss of generality that $f^p = \langle(-1, \lambda)\rangle$ as before; suppose the above statement is true for every a with $|a| = n$. Let $\alpha = \min(a)$ and let $b = a \setminus \{\alpha\}$. Then by the induction hypothesis, for (measure one many) $\mu \in A^p(\alpha)$, there is some $q_\mu \leq^* p \frown (\alpha, \mu)$ such that

$$\langle q_\mu \frown \vec{\nu} | q_\mu \frown \vec{\nu} \text{ is a valid condition}\rangle$$

meets $\mathcal{D}$ below p if possible. By the same argument as in Claim 5.1.13, we may assume that $[\mu \mapsto c^{q_\mu}(\alpha) \mid \alpha \in A^p(\alpha)]_{U_\alpha} \in K_\alpha$, and thus by the same argument as in Claim 5.1.14, we have some $q \leq^* p$ such that $q \frown (\alpha, \mu) \leq^* q_\mu$ whenever $\mu \in A^q(\alpha)$ and $\langle q \frown (\alpha, \mu) | \mu \in A^q(\alpha)\rangle$ meets $\mathcal{D}$ below p if possible[1].

But then by construction, $\langle q \frown \vec{\nu} | \vec{\nu} \in E_a\rangle$ meets $\mathcal{D}$ below p if possible, and so q is our desired p_a. $\square$

By enumerating over all a for which p has valid a-step extensions (that is, $a \in [\mathrm{dom}(A^p)]^{<\omega}$), we may additionally construct the $\langle q_a \mid a \in [\mathrm{dom}(A^p)]^{<\omega}\rangle$ such that for each $\alpha \in \mathrm{dom}(f^p)$, $\{c^{q_a}(\alpha) \mid a \in [\mathrm{dom}(A^p)]^{<\omega}\}$ forms a directed system in the relevant collapse forcing of size less than λ^+.

Therefore, by invoking the λ^+-closure of each relevant measure to shrink the measure terms A^{q_a}, the λ^+-directed closure of each relevant collapse forcing to shrink the c^{q_a}'s, and the κ^+-directed closure of $\mathrm{Col}(\kappa^+, < j_{U_\alpha}(\kappa))^{\mathrm{Ult}(V, U_\alpha)}$ on the C^{q_a}'s, we may let q be a lower bound of $\langle q_a | a \in [\mathrm{dom}(A^p)]^{<\omega}\rangle$.

By Claim 5.1.15, we have that

$$\left\langle q \frown \vec{\nu} \middle| \nu \in \left[\prod A^q(\alpha)\right]^{<\omega} \text{ increasing}\right\rangle$$

---

[1]Note that the same Fodor's Theorem argument works to concentrate $A^q(\alpha)$ so that for every $\mu \in A^q(\alpha)$, the map $\mu \mapsto c^{q_\mu}(-1) \in \mathrm{Col}(\lambda^+, < \mu)$ is constant.

meets $\mathcal{D}$ below q if possible. That means that for each applicable $\vec{\nu}$, if some direct extension of $q \frown \vec{\nu}$ lies in $\mathcal{D}$ then $q \frown \vec{\nu} \in \mathcal{D}$ already.

We will now find homogeneous families of such ν, in the following sense. Let $h : [\prod A^q(\alpha)]^{<\omega} \to \{0,1\}$[1],

$$h(\vec{\nu}) = \begin{cases} 1 & q \frown \vec{\nu} \in \mathcal{D} \\ 0 & \text{otherwise} \end{cases}$$

By Lemma 4.3.2[2], there is some $r \leq^* q$ with $c^r = c^q$ such that for every $a \in [\mathrm{dom}(A^r)]^{<\omega}$,

$$h \upharpoonright \prod_{\alpha \in a}^{\uparrow} A^r(\alpha)$$

is constant.

But then r is finally as desired, as there is some a so that every a-step extension of r is in $\mathcal{D}$. To see this, observe that by density, there must be some $\vec{\nu}$ such that some direct extension of $r \frown \vec{\nu} \in \mathcal{D}$; but then $r \frown \vec{\nu} \in \mathcal{D}$, and since $r \frown \vec{\nu} \leq^* q \frown \vec{\nu}$, by construction of q, $q \frown \vec{\nu} \in \mathcal{D}$, so $h(\vec{\nu}) = 1$. Let $a = \mathrm{dom}(\vec{\nu})$; by the homogeneity of h, every a-step extension of r is in $\mathcal{D}$. $\square$

Combining elements of the aforementioned proof yields the following restricted Diagonalization Lemma:

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[1] h as defined is not exactly a map on this domain, but rather is a map on increasing finite sequences drawn from the measure one family of $A^q(\alpha)$'s.

[2] Technically, Lemma 4.3.2 applies independently to h on each sub-interval between points in $\mathrm{dom}(f^q) \cup \{\kappa\}$, taking reflected measures as needed.

Lemma 5.1.16 (Diagonalization Lemma). *Let* $p \in \mathbb{MC}\left(\vec{U}\right)$*; let*

$$E = \left\{ \vec{\nu} : a \to \kappa \,\middle|\, a \in [\mathrm{dom}(A^p)]^{<\omega}, \vec{\nu} \in \prod_{\alpha \in a} A^p(\alpha), \text{ and } p \frown \vec{\nu} \text{ is a condition below } p \right\}$$

Let $\langle p_{\vec{\nu}} | \vec{\nu} \in E \rangle$ *be a family such that:*

1. $p_{\vec{\nu}} \leq^* p \frown \vec{\nu}$
2. *for all* a*, for all* $\alpha \in \mathrm{dom}(a)$*, there is a* $t : A^p(\alpha) \to \{p_{\vec{\nu}} \mid \vec{\nu} \in E\}$ *such that* $t(\mu)$ *is some* $\vec{\nu}$ *such that* $\mathrm{dom}(\vec{\nu}) = a$*,* $\vec{\nu}(\alpha) = \mu$*, and*

$$[\mu \mapsto c^{p_{t(\mu)}}(\alpha)] \in K_\alpha (\text{ or } k^\beta_\alpha(\rho))$$

Then there is some $q \leq^* p$ *such that whenever* $q \frown \vec{\nu}$ *is a valid condition,* $q \frown \vec{\nu} \leq^* p_{\vec{\nu}}$

And in the course of proving Lemma 5.1.11, we obtain simultaneous homogeneity for all step extensions:

Lemma 5.1.17. *Let* $\mathcal{D}$ *be open dense, let* p *be a condition, and let* $r \leq^* p$ *be as in the proof of Lemma* 5.1.11*. Then for every* $b \in [\lambda]^{<\omega}$ *for which* p *admits* b*-step extensions, either every* b*-step extension of* r *is in* $\mathcal{D}$*, or none are.*

And as with vanilla Magidor forcing, we obtain the sentential version of the Prikry lemma. The proof is functionally the same as in Lemma 2.6.17:

Lemma 5.1.18 (Prikry-type lemma, sentential version). *Let* $p \in \mathbb{MC}\left(\vec{U}\right)$*, and let* σ *be a sentence of the forcing language. Then there is some* $r \leq^* p$ *such that* r *decides* σ*.*

Proof. Since the collection of conditions deciding σ is open dense, by Lemma 5.1.11, there is some $q \leq^* p$ and some a such that every a-step extension of q decides σ.

We argue for the case of $a = \{\alpha\}$; the rest are by induction and proceed much like in vanilla Magidor forcing (c.f. Lemma 2.6.17).

Let

$$A_\sigma = \{\nu \in A^q(\alpha) \mid q \frown (\alpha, \nu) \Vdash \sigma\}$$

and let

$$A_{\neg\sigma} = \{\nu \in A^q(\alpha) \mid q \frown (\alpha, \nu) \Vdash \sigma\}$$

Exactly one of A_σ and $A_{\neg\sigma}$ is measure one; if without loss of generality A_σ is measure one, then let

$$A'(\beta) = \begin{cases} A(\beta) & \beta \neq \alpha \\ \\ A_\sigma & \beta = \alpha \end{cases}$$

and for each $\beta \in \mathrm{dom}(A')$, let $C'(\beta) = C(\beta) \restriction A'(\beta)$. Then every a-step extension of (f^r, c^r, A', C') forces σ, and thus $(f^r, c^r, A', C') \Vdash \sigma$. Note that no extension is needed to c^r; thus the induction steps may proceed only on the A°-component with appropriate domain restriction on the C°-component. $\square$

Further arguments along the lines of Lemma 2.6.18 yield a tail-change version:

Lemma 5.1.19 (Prikry-type lemma, tail-change version)**.** *If* p *is a condition,* $\mathcal{D}$ *is an open dense set, and* $\beta, \beta + 1 \in \mathrm{dom}(f^p)$*, then there is a direct extension* $q \leq^* p$ *such that* $(q)_{\beta+1} =$

$(p)_{\beta+1}$ *and whenever* $a \in [\mathrm{dom}(A^p)]^{<\omega}$, *if* q' *is an* a*-step extension of* q *and* $q' \in \mathcal{D}$, *then every* $a \setminus (\mathrm{dom}(f^{q'}) \cap (\beta+1))$*-step extension of* $(q')_{\beta+1} \frown (q)^{\beta+1}$ *is also in* $\mathcal{D}$.

The proof is almost identical to the analogue for $\mathbb{M}\left(\vec{U}\right)$ (c.f. Lemma 2.6.18). The core difference here is in assuring that the collapse terms are compatible.

Proof. For ease of notation, let $f^p(\beta) = \zeta$ and let $f^p(\beta+1) = \rho$. Recall that both ζ and ρ are inaccessible, and so $\zeta < 2^\zeta < \rho$.

For each $r \leq (p)_{(\beta+1,\rho)}$ in $\mathbb{MC}\left(\vec{U}\right)_{(\beta+1,\rho)}$, by Lemma 5.1.17, let $p_r \leq^* r \frown (p)^{\beta+1}$ (in the full $\mathbb{MC}\left(\vec{U}\right)$) be such that for every b, either every b-step extension of p_r is in $\mathcal{D}$, or none are. By construction, for each r, $(p_r)_{\beta+1} \leq^* r$ and $(p_r)^{\beta+1} = (f^p, c^{p_r}, A^{p_r}, C^{p_r})^{\beta+1}$.

We will additionally need that the c^{p_r}'s are all compatible on $[\beta+1, \lambda)$. If there are ρ-many such r, then since $\leq^*$ on $\mathbb{MC}\left(\vec{U}\right)^{(\beta+1,\rho)}$ is ρ^+-directed closed, we may recursively construct the p_r's along a well-ordering of all such r of order type ρ, so that the sequence of $(p_r)^{(\beta+1,\rho)}$'s forms a ρ-length descending chain.

We claim that there are ρ-many such r. To see this, note that our choices for f^r consist of elements of $[\zeta]^{<\omega}$, of which there are $\zeta < \rho$-many, and choices for C^r and A^r range over measures and families of collapses up to and including ζ, of which there are $2^\zeta < \rho$-many. Finally, for c^r: below β, collapse terms are bounded in ζ, so there are at most ζ-many, and at β, $c^r(\beta) \in \mathrm{Col}(\zeta^+, <\rho)$ and so there are ρ-many. Thus, there are ρ-many such r.

Thus the family of $(p_r)^{\beta+1}$'s forms a ρ-length descending chain in $\leq^*$, the ρ^+-closed direct extension order on $\mathbb{MC}\left(\vec{U}\right)^{(\beta+1,\rho)}$. For each r, let $q_r = (p)_{\beta+1} \frown (p_r)^{\beta+1}$; then by construction there is a $\leq^*$-lower bound q of the q_r's.

But then q is as desired. To see this, suppose $\mathsf{q}' \leq \mathsf{q}$ and $\mathsf{q}' \in \mathcal{D}$. Let $\mathsf{r} = (\mathsf{q}')_{\beta+1}$, and let b be such that q' is a b-step extension of q. Then by construction, $\mathsf{q}' \leq \mathsf{r} \frown (\mathsf{q})^{\beta+1} \leq \mathsf{p}_\mathsf{r}$, and by definition of p_r, every b-step extension of p_r, and hence any $\mathsf{b} \setminus \mathsf{dom}(\mathsf{f}^\mathsf{r})$-step extension of $\mathsf{r} \frown (\mathsf{q})^{\beta+1}$, is also in $\mathcal{D}$. $\square$

Likewise, we have a similar sentential version with tail changes:

Lemma 5.1.20 (Prikry-type lemma, tail-change sentential version)**.** *Let σ be a sentence of the forcing language, let $\mathsf{p} \in \mathbb{MC}\left(\vec{\mathsf{U}}\right)$, and let $\beta, \beta+1 \in \mathsf{dom}(\mathsf{f}^\mathsf{p})$. Then there is some $\mathsf{q} \leq^* \mathsf{p}$ such that $(\mathsf{q})_{\beta+1} = (\mathsf{p})_{\beta+1}$ and if $\mathsf{q}' \leq \mathsf{q}$ is such that q' decides σ, then $(\mathsf{q}')_{\beta+1} \frown (\mathsf{q})^{\beta+1}$ decides σ the same way.*

This proof is identical to the analogous argument for $\mathbb{M}\left(\vec{\mathsf{U}}\right)$ (c.f. Lemma 2.6.19):

Proof. For ease of notation, let $\mathsf{f}^\mathsf{p}(\beta + 1) = \rho$. As the collection of conditions deciding σ is open dense, we may re-run the argument of Lemma 5.1.19, but while invoking Lemma 5.1.18 to additionally ensure that each p_r as above also decides σ.

Since ensuring that each p_r decides σ only involves shrinking the A° and C° components, the family of $(\mathsf{p}_\mathsf{r})^{\beta+1}$'s still forms a ρ-length descending chain, and we may define q a $\leq^*$-lower bound of the q_r's as in the proof of Lemma 5.1.19.

But then if $\mathsf{q}' \leq \mathsf{q}$ and q decides σ, then if we let $\mathsf{r} = (\mathsf{q}')_{\beta+1}$, we have that $\mathsf{q}' \leq \mathsf{r} \frown (\mathsf{q})^{\beta+1} \leq \mathsf{p}_\mathsf{r}$; since p_r already decides σ, q' and q must decide σ the same way, and hence $\mathsf{r} \frown (\mathsf{q})^{\beta+1}$ must also decide σ the same way as q'. $\square$

As for cardinal arithmetic in $V^{\mathbb{MC}(\vec{U})}$, an $\mathbb{MC}\left(\vec{U}\right)$-generic object adds a Magidor sequence $\langle \beta_\eta | \eta < \lambda \rangle$, and collapses cardinals in $(\beta_\eta^+, \beta_{\eta+1})$ for every $\eta \in \lambda \cup \{-1\}$:

Lemma 5.1.21. *Let* G *be* $\mathbb{MC}\left(\vec{U}\right)$*-generic. Let*

$$\vec{\beta} = \langle \beta_\eta | \eta < \lambda \rangle = \bigcup_{p \in G} f^p$$

and for each $\eta < \lambda$*, let*

$$F_\eta = \bigcup_{p \in G, \eta \in \mathrm{dom}(f^p)} c^p(\eta)$$

Then

1. $\vec{\beta}$ *is increasing, normal, and has supremum* κ
2. *for all* $\langle A_\eta | \eta < \lambda \rangle \in V$ *with each* $A_\eta \in U_\eta$*, for coboundedly many* $\zeta < \lambda$*,* $\beta_\zeta \in A_\zeta$
3. *for all* η*,* F_η *is* $\mathrm{Col}(\beta_\eta^+, < \beta_{\eta+1})$*-generic over* V
4. *for all* $\langle C_\eta | \eta < \lambda \rangle$ *in* V *with each* C_η *a map from a measure one set in* U_η *to collapses such that each* $[C_\eta] \in K_\eta$*: for coboundedly many* $\zeta < \lambda$*,* $C_\zeta(\beta_\zeta) \in F_\zeta$
5. *and the above items reflect down to each limit ordinal* $\eta < \lambda$ *with respect to the reflected measures* r *and reflected guiding generics* k*. More precisely, if* η *is a limit ordinal below* λ*, then:*
   (a) $\langle \beta_\zeta \mid \zeta < \eta \rangle$ *is increasing, normal, and has supremum* β_η*;*
   (b) *for each* $\langle A_\zeta \mid \zeta < \eta \rangle$ *in* V *with each* A_η *in* $r^\eta_\zeta(\beta_\eta)$*, for coboundedly many* ζ*,* $\beta_\zeta \in A_\zeta$*;*

(c) for all $\langle C_\zeta \mid \zeta < \eta \rangle$ *in* V *with each* C_ζ *a map from a measure one set in* $r^\eta_\zeta(\beta_\eta)$ *to collapses such that each* $[C_\eta] \in k^\eta_\zeta(\beta_\eta)$*: for coboundedly many* $\zeta < \eta$*,* $C_\zeta(\beta_\zeta) \in F_\zeta$

Proof. Items 1, 2, 5a, and 5b are exactly as in the standard density and/or factoring arguments for Magidor forcings (c.f. Lemma 2.6.22). For completeness, we argue Item 2. For each such $\langle A_\eta \mid \eta < \lambda \rangle$ with each $A_\eta \in U_\eta$, the set

$$\mathcal{D}_2 = \{p \mid \forall \eta > \max \mathrm{dom}(f^p)\ A^p(\eta) \subseteq A_\eta\}$$

is open dense; simply take, for each such p, the weakest direct extension $q \leq^* p$ where for each such η, $A^q(\eta) = A^p(\eta) \cap A_\eta$. Then if $r \in \mathcal{D}_2 \cap G$, by definition of β, $\beta_\eta \in A_\eta$ for all $\eta > \max \mathrm{dom}(f^r)$.

In the case of 5b, we factor to $\mathbb{MC}\left(\vec{U}\right)_{(\eta,\beta_\eta)} = \mathbb{MC}\left(\langle r^\eta_\zeta(\beta_\eta) \mid \zeta < \eta \rangle\right)$, which is its own Magidor forcing with guided interleaved collapses, and argue similarly as for Item 2.

Item 3 follows since below the weakest p for which $f^p(\eta) = \beta_\eta$ and $f^p(\eta+1) = \beta_{\eta+1}$, the map

$$q \mapsto c^q(\eta) \text{ with domain } q \leq p$$

is a projection map from $\mathbb{MC}\left(\vec{U}\right)/p$ to $\mathrm{Col}(\beta_\eta^+, < \beta_{\eta+1})$.

Item 4 follows from the fact that given such $\langle C_\eta | \eta < \lambda \rangle$, since the guiding generics are filters,

$$\mathcal{D}_4 = \left\{ p \in \mathbb{MC}\left(\vec{U}\right) \;\middle|\; \begin{array}{l} \forall \eta \in \mathrm{dom}(A^p) \setminus (\max \mathrm{dom}(f^p) + 1)\ [C^p(\eta)]_{U_\eta} \leq [C_\eta]_{U_\eta} \\ \text{as witnessed by } A^p(\eta) \end{array} \right\}$$

is dense, as whenever p is a condition and $\eta > \max \mathrm{dom}(f^p)$, $[C^p(\eta)] \in K_\eta$ hence $[C^p(\eta)] \parallel [C_\eta]$. So let $r \in \mathcal{D}_4 \cap G$. For each $\eta > \max \mathrm{dom}(f^r)$ such that $\beta_\eta \in A^r(\eta)$, by genericity, there must be some $r' \leq r$ such that $r' \in G$ and both η and $\eta+1$ are in $\mathrm{dom}(f^{r'})$. But then by genericity, $c^{r'}(\eta) \in F_\eta$, and since $r' \leq r$, we get that $F_\eta \ni c^{r'}(\eta) \leq C^r(\beta_\eta) \leq C_\eta(\beta_\eta) \in F_\eta$.

Finally, Item 5c follows from reflection and factoring: let η be a limit point and work in $\mathbb{MC}\left(\vec{U}\right)_{(\eta,\beta_\eta)}$. Then

$$\mathcal{D}_{5c} = \left\{ p \in \mathbb{MC}\left(\vec{U}\right)_{(\eta,\beta_\eta)} \;\middle|\; \begin{array}{l} \forall \zeta \in \mathrm{dom}(A^p) \setminus (\max \mathrm{dom}(f^p)+1)\ [C^p(\zeta)]_{r^\eta_\zeta(\beta_\eta)} \leq [C_\eta]_{r^\eta_\zeta(\beta_\eta)} \\ \text{as witnessed by } A^p(\eta) \end{array} \right\}$$

is dense in $\mathbb{MC}\left(\vec{U}\right)_{(\eta,\beta_\eta)}$, and we may argue as in Item 4. □

As a result, we have the following cardinal arithmetic in $V^{\mathbb{MC}(\vec{U})}$:

Lemma 5.1.22. *Let δ be a V-cardinal and let $\langle \beta_\eta | \eta < \lambda \rangle$ be the Magidor sequence added by some $\mathbb{MC}\left(\vec{U}\right)$-generic G. Then:*

1. *If $\delta \in (\beta_\eta^+, \beta_{\eta+1})$ for some η, then δ is collapsed to size β_η^+*
2. *If $\delta = \beta_\eta$ for some η or $\delta = \kappa$, then δ is preserved*
3. *If $\delta = \beta_{\eta+1}$ for some η or $\delta = \beta_0$ or $\delta \leq \lambda^+$ or $\delta > \kappa$, then δ is $V[G]$-regular.*

Thus if λ is below the first fixed point of the map $\theta \mapsto \aleph_\theta$, then in $V[G]$, $\kappa = (\aleph_\lambda)^{V[G]}$. In particular, if $\lambda = \omega_1$ then $\kappa = (\aleph_{\omega_1})^{V[G]}$.

The collapsing part is easy to see from Lemma 5.1.21, as G adds $\mathsf{Col}(\beta_\eta^+, <\beta_{\eta+1})$-generics for every η. Much as in vanilla Magidor forcing, the preservation argument emulates Lemma 2.6.20 with the help of Lemma 5.1.20.

As for a characterization of genericity along the lines of Theorem 2.6.23, we have that the Magidor generic sequence is geometric, along with a geometricity notion for the collapses, as follows:

Definition 5.1.23 (Geometricity)**.** A sequence $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle$ of ordinals cofinal in κ and a family of $\mathsf{Col}(\beta_\eta^+, <\beta_{\eta+1})$-filters $\langle F_\eta \mid \eta < \lambda \rangle$ living in some outer model of V is *geometric* if the conclusions of Lemma 5.1.21 hold for $\vec{\beta}$ and $\langle F_\eta \mid \eta < \lambda \rangle$.

Theorem 5.1.24. *Suppose in some outer model of* V *that* $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda \rangle$ *is increasing, normal, and has limit* κ*, and for each* η*, suppose* F_η *be* $\mathsf{Col}(\beta_\eta^+, <\beta_{\eta+1})$*-filters. Suppose furthermore that* $\vec{\beta}$*,* $\langle F_\eta \mid \eta < \lambda \rangle$ *is geometric (note that this means each* F_η *is* $\mathsf{Col}(\beta_\eta^+, <\beta_{\eta+1})$*-generic over* V[1]*).*

Define

$$H = \left\{ (f, c, A, C) \in \mathbb{MC}\left(\vec{U}\right) \;\middle|\; \begin{array}{l} \forall \eta \in \mathrm{dom}(f)\ f(\eta) = \beta_\eta, \\ \forall \eta \in \mathrm{dom}(f)\ c(\eta) \in F_\eta, \\ \textit{and } \forall \eta \in \mathrm{dom}(A),\ \beta_\eta \in A(\eta) \textit{ and } C(\eta)(\beta_\eta) \in F_\eta \end{array} \right\}$$

---

[1] *Note that since* $\mathsf{Col}(\beta_\eta^+, <\beta_{\eta+1})$ *is* $\beta_{\eta+1}$*-cc, we have that whenever* $\eta < \eta'$*,* $F_{\eta'}$ *is* $\mathsf{Col}(\beta_{\eta'}^+, <\beta_{\eta'+1})$*-generic over* $V[F_\eta]$*. So no additional work is needed to attain mutual genericity for the* F_η*'s.*

Then H *is* $\mathbb{MC}\left(\vec{\mathsf{U}}\right)$*-generic over* V*.*

To emphasize the complexity of this proof here, note that (Fuchs, 2014) proves a similar characterization of genericity for vanilla Magidor forcing $\mathbb{M}\left(\vec{\mathsf{U}}\right)$. The crux of the proof below is in having a good Diagonalization Lemma, such as Lemma 5.1.16, and a Prikry-type tail-change lemma as in Lemma 5.1.19.

Proof. As with the characterization of genericity for Magidor forcing (c.f. Theorem 2.6.23), the proof is by induction on the length of $\vec{\mathsf{U}}$.

Our base case is when $\lambda = \omega$, and our forcing is $\mathbb{MC}(\langle \mathsf{U}_n \mid n < \omega\rangle)$; let $\langle \beta_n \mid n < \omega\rangle$, and F_n for each $n < \omega$, be such that the conclusions of Lemma 5.1.21 apply. Let $\mathcal{D} \subseteq \mathbb{MC}(\langle \mathsf{U}_n \mid n < \omega\rangle)$ be open dense. Without loss of generality, we work over g's that are valid stems with domain $[-1, m]$ for some $m < \omega$. By Lemma 5.1.19, fix some condition $p_g \leq 1_{\mathbb{MC}(\vec{\mathsf{U}})} \frown g$ such that $(p_g)_{\max \mathrm{dom}(g)} = (1_{\mathbb{MC}(\vec{\mathsf{U}})} \frown g)_{\max \mathrm{dom}(g)}$ and if $r \leq p_g$ is a b-step extension and $r \in \mathcal{D}$, then every b-step extension of $(r)_m \frown (p_g)^m$ is also in $\mathcal{D}$.

For each g, c^{p_g} is trivial by construction, hence satisfies the hypothesis of our Diagonalization Lemma (Lemma 5.1.16). Thus by Lemma 5.1.16, there is a condition p with $f^p = ((-1, \lambda))$ such that for every g for which $p \frown g \leq p$, $p \frown g \leq^* p_g$.

What we actually need is a $q \leq^* p$ with the following property: for any g, $(1_{\mathbb{MC}(\vec{\mathsf{U}})} \frown g) \frown (q)^{\max \mathrm{dom}(g)} \leq^* p_g$. To attain this, we define $f^q = f^p$ and $c^q = c^p$. As for A^q and C^q, let

$$B(n) = A^p(n) \cap \bigtriangleup_g A^{p_g}(n) = A^p(n) \cap \{\alpha < \kappa \mid \forall g\ |g| < n, \max \mathrm{ran}(g) < \alpha \implies \alpha \in A^{p_g}(n)\}$$

For all n observe that $\{[C^{p_g}(n)] \mid |g| < n\}$ forms a κ-sized directed set in a κ^+-distributive collapse forcing. So let $C'(n)$ be such that $[C'(n)]$ is a lower bound of the $[C^{p_g}(n)]$'s as witnessed by W_n. Then let $A^q(n) = B(n) \cap W_n$, and let $C^q(n) = C' \upharpoonright A^q(n)$. Then by construction, for any g, $(1_{\mathbb{MC}(\vec{u})} \frown g) \frown (q)^{\max \mathrm{dom}(g)} \leq^* p_g$.

By geometricity for $\vec{\beta}$ (namely Item 2) and for the collapses (Item 4, there is an $m \in \omega$ such that for all $n \geq m$, $\beta_n \in A^q(n)$ and $C^q(n)(\beta_n) \in F_n$.

Let $g = \vec{\beta} \upharpoonright [-1, m-1]$. By construction, $(1_{\mathbb{MC}(\vec{u})} \frown g) \frown (q)^{m-1} \leq p_g$. Since $\mathcal{D}$ is dense, let $r \leq (1_{\mathbb{MC}(\vec{u})} \frown g) \frown (q)^{m-1}$ be such that $r \in \mathcal{D}$. By geometricitity, item 3, the projection of $\mathcal{D}$ to $\mathrm{Col}(\beta_n^+, < \beta_{n+1})$, for each $n < m$, is open dense in $\mathrm{Col}(\beta_n^+, < \beta_{n+1})$, so we may without loss of generality have that for each $n < m$, $c^r(n) \in F_n$. Also without loss of generality, let $b = \mathrm{dom}(f^r) \setminus [-1, m-1] = [m, m+k]$ for some $k < \omega$. Then by definition of p_g, *every* b-step extension of $(r)_{m-1} \frown (q)^{m-1}$ is in $\mathcal{D}$, so let $r' = \left((r)_{m-1} \frown (q)^{m-1}\right) \frown \vec{\beta} \upharpoonright [m, m+k]$. Then $r' \in \mathcal{D}$.

But then $r' \in H$. To see this, note that by construction, for all $n > m+k$, $\beta_n \in A^q(n) = A^{r'}(n)$ and $C^q(n)(\beta_n) = C^{r'}(n)(\beta_n) =\in F_n$. Also by construction, $f^{r'} \upharpoonright m = g = \vec{\beta} \upharpoonright m$ and $c^{r'} \upharpoonright m = c^r \upharpoonright m \in \prod_{n<m} F_n$. And for $n \in [m, m+k]$, we explicitly defned $f^{r'}(n) = \beta_n$, and $c^{r'}(n) = C^q(n)(\beta_n) \in F_n$.

Therefore $r' \in H \cap \mathcal{D}$, and so this concludes the base case.

For the successor step, suppose that $\lambda = \bar{\lambda} + \omega$ and suppose the result is true for any forcing of the form $\mathbb{MC}\left(\langle U'_\eta \mid \eta < \bar{\lambda}\rangle\right)$. Let $\vec{U} = \langle U_\eta \mid \eta < \lambda\rangle$, and let $\vec{\beta} = \langle \beta_\eta \mid \eta < \lambda\rangle$ and $\langle F_\eta \mid \eta < \lambda\rangle$ be geometric for $\mathbb{MC}\left(\vec{U}\right)$, with induced filter H. Let $g = \left\langle(\bar{\lambda}, \beta_{\bar{\lambda}})\right\rangle$, and in line with the factoring of Fact 5.1.8, consider $\mathbb{MC}\left(\vec{U}\right)_g \times \mathbb{MC}\left(\vec{U}\right)^g$. Then the induction hypothesis applies to $\mathbb{MC}\left(\vec{U}\right)_g = \mathbb{MC}\left(\left\langle r^{\bar\lambda}_\eta(\beta_{\bar\lambda}) \mid \eta < \lambda\right\rangle\right)$, and since $\left\langle \beta_\eta \mid \eta < \bar\lambda\right\rangle$ and $\left\langle F_\eta \mid \eta < \bar\lambda\right\rangle$ are geometric for $\mathbb{MC}\left(\vec{U}\right)_g$, the resulting forcing H_g defined from $\left\langle \beta_\eta \mid \eta < \bar\lambda\right\rangle$ and $\left\langle F_\eta \mid \eta < \bar\lambda\right\rangle$ is generic over V. Since $\mathbb{MC}\left(\vec{U}\right)^g = \mathbb{MC}\left(\langle U_{\bar\lambda+n} \mid n < \omega\rangle\right)$, by arguing exactly as in the base case, the filter H^g defined from $\left\langle \beta_{\bar\lambda+n} \mid n < \omega\right\rangle$ and $\left\langle F_{\bar\lambda+n} \mid n < \omega\right\rangle$ is $\mathbb{MC}\left(\vec{U}\right)^g$-generic over V. But since $\mathbb{MC}\left(\vec{U}\right)_g$ is $\beta^+_{\bar\lambda}$-cc, and $\mathbb{MC}\left(\vec{U}\right)^g$ is $\beta^+_{\bar\lambda}$-distributive, H^g is actually $\mathbb{MC}\left(\vec{U}\right)^g$-generic over $V[H_g]$ and so $H = H_g \times Hg$ is $\mathbb{MC}\left(\vec{U}\right)$-generic over V. This concludes the successor step.

For the limit step, let $\lambda = \sup_{\rho<\tau} \lambda_\rho$ with each λ_ρ a limit ordinal. Our induction hypothesis is now that for each $\rho < \tau$ and for each $\langle \beta_\eta \mid \eta < \lambda_\rho\rangle$ and $\langle F_\eta \mid \eta < \lambda_\rho\rangle$ geometric for some Magidor collapse forcing $\mathbb{MC}\left(\langle U_\eta \mid \eta < \lambda_\rho\rangle\right)$, the resulting filter H_ρ is $\mathbb{MC}\left(\langle U_\eta \mid \eta < \lambda_\rho\rangle\right)$-generic over V; note that if $\rho < \rho'$ then by the Product Lemma and appropriate chain condition, $H_\rho = H_{\rho'} \times H'$ for some $H_{\rho'}$ $\mathbb{MC}\left(\langle U_\eta \mid \eta \le \lambda_{\rho'}\rangle\right)_{(\lambda_\rho,\beta_{\lambda_\rho})}$-generic over V and some H' $\mathbb{MC}\left(\langle U_\eta \mid \eta \le \lambda_{\rho'}\rangle\right)^{(\lambda_\rho,\beta_{\lambda_\rho})}$-generic over $V[H_\rho]$.

Let $\langle \beta_\eta \mid \eta < \lambda\rangle$ and $\langle F_\eta \mid \eta < \lambda\rangle$ be geometric for some Magidor collapse forcing $\mathbb{MC}\left(\langle U_\eta \mid \eta \le \lambda\rangle\right)$. Let $\mathcal{D}$ be dense in $\mathbb{MC}\left(\langle U_\eta \mid \eta \le \lambda\rangle\right)$. By applying Lemma 5.1.19 to the trivial condition, for each g a stem such that $\max \mathrm{dom}(g) = \delta + 1$ for some δ such that $\delta \in \mathrm{dom}(g)$, take $p_g \le^* 1_{\mathbb{MC}(\langle U_\eta|\eta<\lambda_\rho\rangle)} \frown g$ such that

- $(p_g)_{\delta+1} = \left(1_{\mathbb{MC}(\langle U_\eta|\eta<\lambda\rangle)}\right)_{\delta+1} \frown g$; in particular c^{p_g} consists only of trivial collapse terms
- (tail-change property) for every $b \subseteq [\mathrm{dom}(A^{p_g})]^{<\omega}$, if q is a b-step extension of p_g and $q \in \mathcal{D}$ then every $b \setminus (\mathrm{dom}(f^q) \cap (\delta+1)$-step extension of $(q)_{\delta+1} \frown (p_g)^{\delta+1}$ is also in $\mathcal{D}$

(Note that before we used the notation $\vec{v}$ for g.)

Since each c^{p_g} consists only of trivial collapse terms, the family of p_g's satisfies the hypotheses of Lemma 5.1.16. So by Lemma 5.1.16, let p be such that $f^p = \langle(-1, \lambda)\rangle$ and whenever g is a valid stem by which to extend p, $p \frown g \leq^* p_g$.

By the exact same logic as in the base case, we may further refine A^p and C^p to a $q \leq^* p$ such that c^q consists of trivial collapse terms and whenever g is a stem with $\delta+1 = \max \mathrm{dom}(g)$ and $\delta \in \mathrm{dom}(g)$, then $\left(1_{\mathbb{MC}(\langle U_\eta | \eta<\lambda\rangle)} \frown g\right)_{\delta+1} \frown (q)^{\delta+1} \leq^* p_g$.

By geometricity, let ρ be an ordinal such that for all $\eta \geq \lambda_\rho$, $\beta_\eta \in A^q(\eta)$ and $C^q(\eta)(\beta_\eta) \in F_\eta$.

By the induction hypothesis applied to $\langle \beta_\eta \mid \eta < \lambda_\rho\rangle$ and $\langle F_\eta \mid \eta < \lambda_\rho\rangle$, we induce H_ρ a $\mathbb{MC}\left(\left\langle r_\eta^{\lambda_\rho}(\beta_{\lambda_\rho}) \mid \eta < \lambda_\rho\right\rangle\right)$-generic filter over V.

Let $\mathcal{D}'$ be a subset of $\mathbb{MC}\left(\left\langle r_\eta^{\lambda_\rho}(\beta_{\lambda_\rho}) \mid \eta < \lambda_\rho\right\rangle\right) \times \mathrm{Col}(\beta^+_{\lambda_\rho}, <\beta_{\lambda_\rho+1})$ defined by

$$\mathcal{D}' = \left\{(r)_{\lambda_\rho} \times c^r(\lambda_\rho) \mid r \in \mathcal{D}, (r)^{\lambda_\rho} \leq (q)^{\lambda_\rho}, \text{ and } f^r(\lambda_\rho) = \beta_{\lambda_\rho} \text{ and } f^r(\lambda_\rho + 1) = \beta_{\lambda_\rho+1}\right\}$$

(Recall that $(r)_{\lambda_\rho}$ is essentially $r \restriction (\lambda_\rho + 1)$; see Definition 5.1.7 for a complete description.)

Since $c^q \restriction (\lambda_\rho + 1)$ consists only of trivial collapse terms, $\mathcal{D}'$ is open dense in $\mathbb{MC}\left(\left\langle r_\eta^{\lambda_\rho}(\beta_{\lambda_\rho}) \mid \eta < \lambda_\rho\right\rangle\right) \times \mathrm{Col}(\beta^+_{\lambda_\rho}, <\beta_{\lambda_\rho+1})$. So let $r \in \mathcal{D}$ be such that $r \leq q$, $c^r(\lambda_\rho) \in F_{\lambda_\rho}$, and $(r)_{\lambda_\rho} \in H_\rho \cap \mathcal{D}'$. Note that by definition, $f^r(\lambda_\rho) = \beta_{\lambda_\rho}$ and $f^r(\lambda_\rho + 1) = \beta_{\lambda_\rho+1}$. Since $(r)_{\lambda_\rho} \in H_\rho$, by the induction hypothesis we have that whenever $\xi < \lambda_\rho$, either $f^r(\xi) = \beta_\xi$ and $c^r(\xi) \in F_\xi$, or $\beta_\xi \in A^r(\xi)$ and $C^r(\xi)(\beta_\xi) \in F_\xi$. Let $g = f^r \restriction (\lambda_\rho + 2)$.

Claim 5.1.25. $r \leq p_g$.

Proof of claim. By definition of g, $f^r \supseteq g = f^{p_g}$; since c^{p_g} consists only of trivial collapse terms, $c^r \restriction \mathrm{dom}(c^{p_g}) \leq c^{p_g}$.

Taken together with the facts that $(r)^{\lambda_\rho} \leq (q)^{\lambda_\rho}$ and $\lambda_\rho+1 = \max \mathrm{dom}(g)$ and $\lambda_\rho \in \mathrm{dom}(g)$, we further have that

$$r \leq \left(1_{\mathbb{MC}(\langle u_\eta | \eta<\lambda\rangle)} \frown g\right)_{\lambda_\rho+1} \frown (q)^{\lambda_\rho+1} \leq^* p_g$$

□

Furthermore, $r \in \mathcal{D}$, so letting $b = \mathrm{dom}(r) \setminus \mathrm{dom}(g)$, by the tail-change property of p_g, every b-step extension of $(r)_{\lambda_\rho+1} \frown (p_g)^{\lambda_\rho+1}$ is also in $\mathcal{D}$.

Let $r' = ((r)_{\lambda_\rho+1} \frown (q)^{\lambda_\rho+1}) \frown (\vec{\beta} \restriction b)$. By definition of q, $(r)_{\lambda_\rho+1} \frown (q)^{\lambda_\rho+1} \leq p_g$, so r' is also a b-step extension of $(r)_{\lambda_\rho+1} \frown (p_g)^{\lambda_\rho+1}$. Therefore $r' \in \mathcal{D}$. As soon as we show that $r' \in H$, we have completed the limit step.

Claim 5.1.26. $r' \in H$.

Proof of claim. By construction, $f^{r'}$ is a finite subsegment of $\vec{\beta}$, as $f^{r'} \restriction (\lambda_\rho+2) = f^r \restriction (\lambda_\rho+2) = g$ which is a finite subsegment of $\vec{\beta}$, and $f^{r'} \restriction [\lambda_\rho + 2, \lambda) = \vec{\beta} \restriction b$.

For $c^{r'}$: let $\xi \in \mathrm{dom}(c^{r'})$. If $\xi \leq \lambda_\rho + 1$, then $c^{r'}(\xi) = c^r(\xi) \in F_\xi$ by definition of r. And if $\xi > \lambda_\rho + 1$, then $c^{r'}(\xi) = C^q(\xi)(\beta_\xi) \in F_\xi$ by construction of q.

As for $A^{r'}$, for $\xi < \lambda_\rho$, we have that $A^{r'} \restriction \lambda_\rho = A^r$ so $\beta_\xi \in A^{r'}(\xi)$. And for $\xi > \lambda_\rho$, $A^{r'}(\xi) = A^q(\xi)$ so by definition of q and λ_ρ, $\beta_\xi \in A^{r'}(\xi)$.

Finally, for $C^{r'}$, if $\xi < \lambda_\rho$ then $C^{r'}(\xi)(\beta_\xi) = C^r(\xi)(\beta_\xi) \in F_\xi$ by construction of r, and if $\xi > \lambda_\rho$ then $C^{r'}(\xi)(\beta_\xi) = C^q(\xi)(\beta_\xi) \in F_\xi$.

Therefore by definition of H, $r' \in H$. $\square$

This completes the limit step, as we have now found an $r' \in H \cap \mathcal{D}$.

Therefore, by induction, for any limit λ, H is $\mathbb{MC}\left(\langle U_\eta \mid \eta < \lambda\rangle\right)$-generic over V. $\square$

5.2 Future Directions

Investigating Magidor-like forcings with collapses, and their characterizations of genericity, was motivated by the question of mutual stationarity properties at $\aleph_{\omega_1}$.

In particular, one might hazard the following mutual stationarity property at every other $\aleph$ below $\aleph_{\omega_1}$:

Conjecture 5.2.1. Suppose $V \models$ ZFC + GCH, and let $\vec{U}$ be an (ω_1)-length Mitchell order increasing system of measures on κ, with U_{ω_1} Mitchell order above $\vec{U}$ giving rise to guiding generics for $\vec{U}$.

Then in $V^{\mathbb{MC}(\vec{U})}$,

$$MS(\langle \aleph_{2n+3} \mid n < \omega\rangle \frown \langle \aleph_{\alpha+2n} \mid \alpha < \omega_1 \text{ limit}, 0 < n < \omega\rangle \,; \omega_1)$$

holds.

One possible idea would be to emulate the proof of $MS(\langle \aleph_{2n+3} \mid n < \omega\rangle \,; \omega_1)$ in (Koepke, 2007).

Future work could include mutual stationarity principles of larger fixed cofinality or at every $\aleph_\alpha$ below $\aleph_{\omega_1}$, perhaps using supercompact versions of $\mathbb{MC}\left(\vec{\mathrm{U}}\right)$.

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APPENDIX

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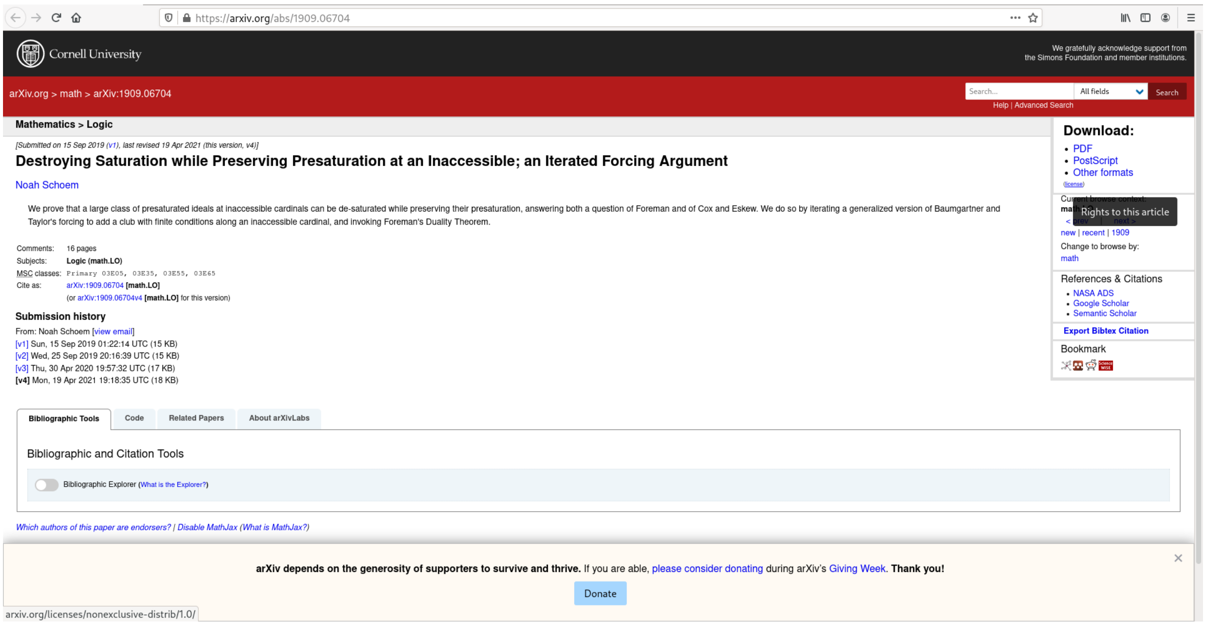
APPENDIX (Continued)



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Destroying Saturation while Preserving Presaturation at an Inaccessible; an Iterated Forcing Argument

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We prove that a large class of presaturated ideals at inaccessible cardinals can be de-saturated while preserving their presaturation, answering both a question of Foreman and of Cox and Eskew. We do so by iterating a generalized version of Baumgartner and Taylor's forcing to add a club with finite conditions along an inaccessible cardinal, and invoking Foreman's Duality Theorem.

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