Weibel's Conjecture for Twisted Algebraic K-theory

by

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To my brother, for showing me the axioms of a group.

To Rebecca, who taught me to choose completion over consistency.

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JDS

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SUMMARY

In this thesis, we prove Weibel's conjecture for some twistings of algebraic K-theory. It is based on the author's publication (1). The fundamental theorem for K_1 and K_0 states that for any ring R there is an exact sequence

$$0 \to \mathsf{K}_1(\mathrm{R}) \to \mathsf{K}_1(\mathrm{R}[t]) \oplus \mathsf{K}_1(\mathrm{R}[t^{-1}]) \to \mathsf{K}_1(\mathrm{R}[t^{\pm}]) \to \mathsf{K}_0(\mathrm{R}) \to 0.$$

We see K_0 can be defined using K_1. There is an analogous exact sequence, truncated on the right, for K_0. Bass defines $\mathsf{K}_{-1}(\mathrm{X})$ as the cokernel of the final morphism. He then iterates the construction to define a theory of negative K-groups (Sections XII.7 and XII.8 of Bass (2)).

Weibel's conjecture, originally posed in (3), asks if $\mathsf{K}_{-i}(\mathrm{R}) = 0$ for $i > \dim \mathrm{R}$ when R has finite Krull dimension. Kerz–Strunk–Tamme (4) prove Weibel's conjecture for any noetherian scheme of finite Krull dimension by establishing pro cdh-descent for algebraic K-theory, see the introduction for a historical summary of progress. Land–Tamme (5) show that a general class of localizing invariants satisfy pro cdh-descent. With this improvement, we extend Weibel's vanishing to some cases of twisted K-theory.

Theorem 6.1.2. *Let* X *be a quasi-compact quasi-separated scheme of Krull dimension* d *and* $\mathcal{A}$ *a sheaf of smooth proper connective* $\mathbb{E}_1$*-algebras on* X*, then* $\mathsf{K}_{-i}(\mathrm{Perf}_{\mathrm{X}})$ *vanishes for* $i > d$*.*

This extends Weibel's conjecture to a discrete Azumaya algebra over a scheme. Since the author's publication (1), Theorem D of Bachmann–Khan–Ravi–Sosnilo (6) recovers the

Azumaya algebra case by showing the vanishing holds across the $\mathbb{G}_m$-gerbe associated to the Azumaya algebra. Although it does not cover the smooth and proper setting, their result is quite general. It holds for ANS stacks of covering dimension d.

To an Azumaya algebra $\mathcal{A}$ of rank r^2 on X we can associate a Severi-Brauer variety P of relative dimension $r-1$ over X. Such a variety is étale-locally isomorphic over X to $\mathbb{P}_X^{r-1}$. In Quillen's work (7), he generalizes the projective bundle formula to Severi-Brauer varieties showing (for $i \geqslant 0$)

$$K_i(P) \cong \bigoplus_{n=0}^{r-1} K_i(\mathcal{A}^{\otimes n}).$$

At the root of this computation is a semi-orthogonal decomposition of Perf(P). Consequently, the computation lifts to the level of nonconnective K-theory spectra. Statements about the K-theory of Azumaya algebras can generally be extracted through this decomposition. In our case, the dimension of the Severi-Brauer variety jumps and so Weibel's conjecture, for our noncommutative algebra, does not follow from the commutative setting.

We could remedy this by characterizing a class of morphisms to X, which should include Severi-Brauer varieties, and then show the relative K-theory vanishes under $-d-1$. In Section 6.2, we show that smooth and proper morphisms (in fact, smooth and projective) are not sufficient. We warn the reader that we will use the overloaded words "smooth and proper" in both the scheme and algebra settings.

For $\mathbb{E}_1$-algebra objects, properness and smoothness are module and algebraic finiteness conditions, see Toën–Vaquié (8, Definition 2.4) for the dg-algebra definitions. Together, the two conditions characterize the dualizable objects in $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{L,c}_{st,cg})$, whose objects are ω-

compactly generated R-linear stable presentable ∞-categories. More surprisingly, the invertible objects of $\mathrm{Mod}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}})$ are exactly the module categories over derived Azumaya algebras, see Antieau–Gepner (9, Theorem 3.15). So Theorem 6.1.2 recovers the discrete Azumaya algebra case.

However, any connective derived Azumaya algebra is discrete. After base-changing to a field k, $\mathcal{A}_{\mathrm{k}} \cong \mathsf{H}_*\mathcal{A}_{\mathrm{k}}$ is a connective graded k-algebra and $\mathsf{H}_*\mathcal{A}_{\mathrm{k}} \otimes_{\mathrm{k}} (\mathsf{H}_*\mathcal{A}_{\mathrm{k}})^{\mathrm{op}}$ is Morita equivalent to k. So $\mathsf{H}_*\mathcal{A}_{\mathrm{k}}$ is discrete. The scope of Theorem 6.1.2 is not wasted as smooth proper connective $\mathbb{E}_1$-algebras can be nondiscrete, see Raedschelders–Stevenson for dg-algebra examples (10, Section 5).

In Chapter 1, we provide the necessary ∞-categorical background in great brevity. Chapter 2 brings monoidal structures into play. In Chapter 3, we give background on algebraic K-theory and provide some descent results for localizing invariants. Chapter 4 specializes to algebraic K-theory with coefficients. In particular, twisted algebraic K-theory is defined alongside Azumaya algebras. Chapter 5 builds control over negative twisted K-theory classes. Finally, Chapter 6 contains the main theorem of the thesis and provides a formal bound.

CHAPTER 1

∞-CATEGORICAL PRELIMINARIES

In this section, we provide some basic ∞-categorical background. We work with quasi-categories (see the well-known sources Joyal (11) and Lurie's two books (12) (13)). We will not build this background in detail. One should consult the sources for rigorous statements and *proofs.*

1.1 Quasi-category basics

The ∞-categories are the fibrant objects of the Joyal model structure on simplicial sets. They satisfy the following inner horn lifting property

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & C_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

with $0 < i < n$. The ith inner horn, Λ^n_i, is the full simplicial set on all vertices *except* i within Δ^n. Roughly speaking, this imposes a homotopy-coherent composition law on the morphism set C_1, as well as on the higher morphisms C_i. In general, we drop the $\bullet$-notation when discussing ∞-categories and reintroduce the subindex when necessary (*e.g. the* 1*-cells* L_1 *of our colimit diagram* $L \to C$...).

When we say "∞-category" in this document, we mean an $(\infty, 1)$ quasi-category. For $(\infty, 1)$-categories, when $n > 1$, this test diagram exists for inner *and outer* horns. This imposes homotopy invertibility on the n-cells when $n \geqslant 2$.

An ∞-category C which has horn lifts for all n and i is a *Kan complex* or ∞-groupoid. These are the quasi-categorical models for spaces. For two objects x, y of an ∞-category C, there is a mapping Kan complexs of morphisms, which we denote by $C(x, y)$. This is only defined up to equivalence of simplicial sets. For any ∞-category C, there is a maximally large sub ∞-groupoid, which we call ιC. This ∞-category is defined out of C by taking the full simplicial set on the homotopy equivalences within the mapping Kan complexes $C(X, Y)$. We also refer to this subgroupoid as the *moduli of objects ∞-groupoid.*

Every 1-category is an ∞-category via the *nerve functor.* This functor takes a 1-category C to the simplicial set $\mathcal{N}(C)_\bullet$ whose n-simplices are the set of 1-categorical functors $[n] \to C$. Every sSet-enriched category is also an ∞-category. These are also known as simplicial categories. The functor from sSet-enriched categories and sSet-enriched functors to ∞-categories is the *coherent nerve* or the *simplicial nerve.* It is corepresentable in the simplicial category of simplicial categories by an sSet-enriched Δ.

Set κ to be a regular uncountable cardinal. A simplicial set $X_\bullet$ is κ-small if the collection of nondegenerate simplices is a κ-small set. An ∞-category C is κ-small when it is categorically equivalent to a κ-small simplicial set (5.4.1.2 (12)).

The ∞-functors between the quasi-categories are *full*, in the sense that the entire mapping simplicial set is the functor simplicial set. With this mapping simplicial set, we can build an ∞-category of κ-small ∞-categories. To remain in the world of $(\infty, 1)$-categories, we take the largest Kan complex of this mapping simplicial set. This gives the κ-small fibrant objects a simplicial categorical structure. We can take the coherent nerve to build a quasi-category.

Let Cat_∞ be the resulting quasi-category when this construction is restricted to the κ-small quasi-categories.

A primary aim of the theory of ∞-categories is to homotopically enrich category theory over the theory of spaces while realizing the model theoretic concepts of homotopy limits and homotopy colimits more intrinsically. One consequence is that the ∞-category of spaces, Spc, plays the role that Set played in 1-category theory. An ∞-category of spaces can be constructed as the coherent nerve of the full simplicial subcategory on the κ-small Kan complexes. For C an ∞-category, the analogous ∞-category of presheaves is the functor ∞-category $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Spc})$. The Yoneda embedding is given, up to equivalence, by the mapping space functors $c \in C \mapsto C(-, c)$.

We are primarily interested in pointed ∞-categories. An ∞-category is *pointed* if it has an initial object, a final object, and a morphism from our choice of a final object to our choice of initial object. Universalities force this to be an equivalence.

1.2 Compact objects, Ind, and compact generation

A simplicial set Λ is κ-*filtered* if for any κ-small simplicial set D and any map $D \to \Lambda$ there is an extension $D \star * \to \Lambda$. Here $D \star *$ is the *cocone simplicial set* on D. In Lurie, it is also denoted by $D^{\triangleright}$. For two cardinals $\kappa < \daleth$, a $\daleth$-filtered simplicial set is also κ-filtered.

Definition 1.2.1. An object x of an ∞-category C is κ-*compact* if, for any κ-filtered colimit $L : \Lambda \to C$, the natural map of Kan complexes

$$\operatorname*{colim}_{\lambda \in \Lambda} C(x, L(\lambda)) \to C(x, \operatorname*{colim}_{\lambda \in \Lambda} L(\lambda))$$

is an equivalence. Let C^κ denote the full ∞-category on the collection of κ-compact objects and $\mathrm{Fun}^\kappa(\mathrm{C}, \mathrm{D})$ the full ∞-category on κ-compact object preserving functors. Note that when $\kappa < \daleth$, a κ-compact object is also $\daleth$-compact as there are fewer test diagrams for the larger cardinality. When we say compact, we mean ω-compact.

Example 1.2.2. Let X be a qcqs scheme. The category of chain complexes of quasi-coherent sheaves has a model structure where the weak equivalences are the quasi-isomorphisms and the degreewise surjections are fibrations, see Section 2.3 of Hovey (14). Localizing at the weak equivalences builds an ∞-category, $\mathrm{D}_{\mathrm{qcoh}}(\mathrm{X})$. Its homotopy category is the unbounded derived ∞-category. The compact objects of $\mathrm{D}_{\mathrm{qcoh}}(\mathrm{X})$ are the *perfect complexes.* A complex is perfect if it is locally quasi-isomorphic (or equivalent in the ∞-categorical setting) to a bounded complex of vector bundles.

The ∞-category $\mathrm{Ind}_\kappa(\mathrm{C})$ is defined in 5.3.5.1 (12) as the full ∞-category of presheaves $\mathrm{C}^{\mathrm{op}} \to \mathrm{Sp}$ which classify a right fibration $\tilde{\mathrm{C}} \to \mathrm{C}$ where $\tilde{\mathrm{C}}$ is κ-filtered. A more concrete description is the ∞-subcategory of $\mathrm{Pre}(\mathrm{C})$ which contains the compact objects and is closed under κ-filtered colimits. When we write Ind, absent of a cardinal, it is implied we mean ω.

Let D be an ∞-category with κ-filtered colimits. By left Kan extending, there is an equivalence of ∞-categories

$$\mathrm{Fun}(\mathrm{Ind}_\kappa(\mathrm{C})^\kappa, \mathrm{D}) \simeq \mathrm{Fun}^\kappa(\mathrm{Ind}_\kappa(\mathrm{C}), \mathrm{D}) \text{ (5.3.5.10 (12)).} \tag{1}$$

Later, we will see that $\mathrm{Ind}(\mathrm{C})^{\omega}$ is a model for the idempotent completion of C. In particular, when C is already idempotent complete there is an equivalence $\mathrm{C} \simeq \mathrm{Ind}(\mathrm{C})^{\omega}$.

Definition 1.2.3. An ∞-category C with all κ-filtered colimits is κ-*compactly generated* when the natural map $\mathrm{Ind}_{\kappa}(\mathrm{C}^{\kappa}) \to \mathrm{C}$ is an equivalence. Under the equivalence in Equation 1, this map corresponds to the inclusion $\mathrm{C}^{\kappa} \to \mathrm{C}$.

1.3 Accessible and presentable ∞-categories

Accessibility and presentability axiomatize a way in which an ∞-category can be generated by a small ∞-category. The notion of generation in this instance is freely adjoining κ-filtered colimits.

Definition 1.3.1 (5.4.2.1 (12))**.** An ∞-category C is κ-*accessible* if there is a κ-small ∞-category C_0 and an equivalence $\mathrm{Ind}_{\kappa}(\mathrm{C}_0) \simeq \mathrm{C}$.

Now we turn to presentability. An ∞-category C is κ-*presentable* if it is accessible and admits small colimits. Presentable ∞-categories can be equivalently defined as accessible localizations of presheaves on a small ∞-category (Theorem 5.5.1.1 (12)). They can also be viewed as Ind of an ∞-category which has small colimits. Define Pr^{L} to be the ∞-category of presentable ∞-categories and maps the full subcategory of colimit-preserving functors in $\iota\,\mathrm{Fun}(-,-)$. Denote this quasi-category by $\mathrm{Fun}^{\mathrm{L}}(\mathrm{C}, \mathrm{D})$.

There is a nice adjoint functor theorem for presentable ∞-categories.

Proposition 1.3.2 (5.5.2.9 (12))**.** *Let* $\mathsf{F} : \mathrm{C} \to \mathrm{D}$ *be any functor between presentable* ∞-*categories. Then*

1. *The functor* F *has a left adjoint when it preserves small colimits.*

2. *The functor* F *has a right adjoint when it preserves small limits and is accessible.*

The entire discussion could be formulated on the ∞-category of Pr^{R}, where objects are presentable ∞-categories and functors are the accessible right adjoints. The above proposition is a starting point for proving an antiequivalence $\mathrm{Pr}^{\mathrm{R}} \simeq \mathrm{Pr}^{\mathrm{L}}$.

1.4 Stability

Our ∞-categorical input into the K-theory machine will be stable ∞-categories.

Definition 1.4.1 (Proposition 1.1.3.4 of (13))**.** A pointed ∞-category C is *stable* when

1. C admits finite limits and colimits

2. A square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ W & \longrightarrow & Z \end{array}$$

is a pullback square if and only if it is a pushout square.

There is an earlier definition of (1.1.1.9 (13)) and the cited Proposition proves an equivalence of definitions (with the above being more general). For our cases, we will use Definition 1.4.1.

Definition 1.4.2. Let C be a pointed ∞-category with finite limits and colimits. The loop functor $\Omega : \mathrm{C} \to \mathrm{C}$ is given by sending an object x to the pullback

$$\begin{array}{ccc} \Omega x & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & x \end{array} .$$

The ∞-category of spectrum objects of C is given, up to equivalence, (see 1.4.2.24 (13)) by the limit

$$\mathrm{Sp}(\mathrm{C}) := \lim \cdots \xrightarrow{\Omega} \mathrm{C} \xrightarrow{\Omega} \mathrm{C} \xrightarrow{\Omega} \mathrm{C}$$

in Cat_∞. The objects of $\mathrm{Sp}(\mathrm{C})$ can be thought of as the infinite-loop objects of C. The functor Ω is now an equivalence on $\mathrm{Sp}(\mathrm{C})$. This is also a model in the equivalence class of *the stabilization of an ∞-category.* Any left exact functor $\mathsf{F} : \mathrm{C} \to \mathrm{D}$ from a stable ∞-category C to a presentable ∞-category factors through the ∞-category of spectrum objects $\mathrm{Sp}(\mathrm{D})$. The functor preserves Ω, and so it preserves the infinite loops spaces of C.

Definition 1.4.3. The *yoneda embedding* of an ∞-category is the limit-preserving map $\mathrm{C} \to \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathrm{Spc})$ into presheaves given on objects by $\mathsf{x} \mapsto \mathrm{C}(-, \mathsf{x})$. If C is stable, then this factors through the stabilization

$$\begin{tikzcd} \mathrm{C} \arrow[r, dashed] \arrow[dr] & \mathrm{Sp}(\mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathrm{Spc})) \simeq \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathrm{Sp}) \arrow[d, "- \circ \Omega^\infty"] \\ & \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathrm{Spc}). \end{tikzcd}$$

The dashed map is called the *spectral yoneda embedding.*

A functor between stable ∞-categories preserves finite limits iff it preserves finite colimits. We call these *exact functors.* For example, the spectral yoneda embedding is exact as C is stable. Let $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{C}, \mathrm{D})$ be the full quasi-category on the exact functors of $\mathrm{Fun}(\mathrm{C}, \mathrm{D})$. Consider the simplicial category of small stable ∞-categories with mapping simplicial set $\iota\,\mathrm{Fun}^{\mathrm{ex}}(-,-)$. Let $\mathrm{Cat}_\infty^{\mathrm{ex}}$ denote the simplicial nerve of this simplicial category.

An *idempotent morphism* of an ∞-category is a 1-morphism $\mathsf{f} : \Delta^1 \to \mathrm{C}$ with a 2-cell defining an equivalence $\mathsf{f} \circ \mathsf{f} \simeq \mathsf{f}$. An ∞-category C is *idempotent-complete* if its image under the yoneda embedding $\mathrm{C} \to \mathrm{Pre}(\mathrm{C}) := \mathrm{Fun}(\mathrm{C}^{\mathrm{op}}, \mathrm{Sp})$ is closed under retractions. Equivalently, the natural morphism $\mathrm{C} \to \mathrm{Ind}(\mathrm{C})^{\omega}$ is an equivalence. Let $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ be the full ∞-subcategory of $\mathrm{Cat}_{\infty}^{\mathrm{ex}}$ on the idempotent-complete stable ∞-categories. The fully faithful inclusion $\mathrm{Cat}_{\infty}^{\mathrm{perf}} \to \mathrm{Cat}_{\infty}^{\mathrm{ex}}$ admits a left adjoint called *idempotent completion*. We denote this functor by $\widehat{\mathrm{C}} \simeq \mathrm{Ind}(\mathrm{C})^{\omega}$. This makes $\mathrm{Cat}_{\infty}^{\mathrm{perf}}$ into a reflective localization of $\mathrm{Cat}_{\infty}^{\mathrm{ex}}$. Two objects C and D of $\mathrm{Cat}_{\infty}^{\mathrm{ex}}$ are *Morita equivalent* when there is an equivalence $\widehat{\mathsf{C}} \simeq \widehat{\mathsf{D}}$.

Let $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ be the full ∞-category of the stable ∞-categories of Pr^{L}. Let $\mathrm{Pr}_{\mathrm{st,cg}}^{\mathrm{L,c}}$ be the ∞-subcategory of $\mathrm{Pr}_{\mathrm{st}}^{\mathrm{L}}$ whose objects are the compactly generated stable presentable ∞-categories and functors which preserves colimits and compact objects. We can define this as a simplicial category as before and take the simplicial nerve to build this quasi-category.

Proposition 1.4.4 (Section 3.1 of Blumberg–Gepner–Tabuada (15) and 5.5.7.10 of (12))**.** *There is an adjunction*

$$\mathrm{Ind} : \mathrm{Cat}_{\infty}^{\mathrm{perf}} \rightleftarrows \mathrm{Pr}_{\mathrm{st,cg}}^{\mathrm{L,c}} : (-)^{\omega}$$

which is an equivalence, where Ind *is the left adjoint.*

CHAPTER 2

MONOIDAL STRUCTURES

2.1 Preliminaries

We record the definition of a symmetric monoidal ∞-category from Chapter 2 of (13). Let Set_*^ω be the 1-category of pointed finite sets with pointed morphisms. Set $\langle n \rangle$ to be the set $\{1, 2, \cdots n\}$ with a point $*$ attached. The inert morphisms of Set_*^ω are the cofibers of injections

$$\begin{tikzcd} \langle k \rangle \arrow[r, hook] \arrow[d] & \langle n \rangle \arrow[d, "\text{inert}"] \\ * \arrow[r] & \langle n-k \rangle. \end{tikzcd}$$

These are the stable squares of pointed sets. There is a special class of inert morphisms, the n dirichlet morphisms, $\rho_n^i : \langle n \rangle \to \langle 1 \rangle$. They are given by

$$\rho_n^i(j) = \begin{cases} * & \text{if } j \neq i \\ 1 & \text{if } j = i. \end{cases}$$

Definition 2.1.1 (Definition 2.0.0.7 (13))**.** A *symmetric monoidal ∞-category* is a coCartesian fibration of simplicial sets $C^\otimes \to \mathcal{N}(\mathrm{Fin}_*)$ with the Segal property that the n-product of the coCartesian lifts of ρ_n^i for $1 \leqslant i \leqslant n$ induces an equivalence on fibers $C^\otimes_{\langle n \rangle} \simeq (C^\otimes_{\langle 1 \rangle})^n$. The coCartesian structure defines associative multiplications on $C := C^\otimes_{\langle 1 \rangle}$ and the Segal condition codifies ∞-categorical symmetry for the multiplication. A *commutative algebra object* of C is a section of this coCartesian fibration which sends inert maps to coCartesian lifts of inert maps. We will denote the commutative algebra objects by CAlg(C).

There are natural ∞-categorical extensions of dualizability and invertibility.

Definition 2.1.2. An object X of a symmetric monoidal ∞-category C is *dualizable* if there is another object $\mathsf{X}^\vee$ and maps

$$\mathsf{X} \otimes \mathsf{X}^\vee \xrightarrow{\text{eval}} \mathbb{1}$$

$$\mathbb{1} \xrightarrow{\text{coev}} \mathsf{X}^\vee \otimes \mathsf{X}$$

such that the maps

$$\mathsf{X} \xrightarrow{\mathrm{id}_{\mathsf{X}} \otimes \text{coev}} \mathsf{X} \otimes \mathsf{X}^\vee \otimes \mathsf{X} \xrightarrow{\text{eval} \otimes \mathrm{id}_{\mathsf{X}}} \mathsf{X}$$

$$\mathsf{X}^\vee \xrightarrow{\text{coev} \otimes \mathrm{id}_{\mathsf{X}^\vee}} \mathsf{X}^\vee \otimes \mathsf{X} \otimes \mathsf{X}^\vee \xrightarrow{\mathrm{id}_{\mathsf{X}^\vee} \otimes \text{eval}} \mathsf{X}^\vee$$

are homotopic to the respective identity maps. An object is *invertible* if the above evaluation and coevaluation maps exist and the map

$$\mathbb{1} \xrightarrow{\text{coev}} \mathsf{X}^\vee \otimes \mathsf{X} \simeq \mathsf{X} \otimes \mathsf{X}^\vee \xrightarrow{\text{eval}} \mathbb{1}$$

is homotopic to the identity. Using tensor and symmetric monoidality, we see that an invertible object is dualizable.

2.2 Monoidal structures on ∞-categories of ∞-categories

We have described three different ∞-categories of stable ∞-categories: $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$, $\mathrm{Cat}^{\mathrm{ex}}_{\infty}$, and $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. These are related through the adjunctions

$$\mathrm{Cat}^{\mathrm{ex}}_{\infty} \underset{\widehat{(-)}}{\overset{\text{inclusion}}{\rightleftarrows}} \mathrm{Cat}^{\mathrm{perf}}_{\infty} \underset{\mathrm{Ind}(-)}{\overset{(-)^{\omega}}{\rightleftarrows}} \mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}.$$

We have written the left adjoints on the bottom. The last adjunction is an equivalence of ∞-categories. The rightmost ∞-category, $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$, is a (not full) ∞-subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. The latter lives fully inside of Pr^{L}, which has a closed symmetric monoidal structure.

Proposition 2.2.1. *There is a symmetric monoidal structure on* Pr^{L} *with the following properties (4.8.1.15 (13)).*

1. *Morphisms out of* $\mathrm{C} \otimes \mathrm{D}$ *are in equivalence with morphisms out of* $\mathrm{C} \times \mathrm{D}$ *which preserve colimits in each variable (4.8.1.14 (13)).*
2. *The unit is given by spaces.*
3. *It restricts to a symmetric monoidal structure on* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *with unit the* ∞*-category of spectra,* Sp *(Proposition 4.4 Ben-Zvi–Francis–Nadler(16)).*
4. *The internal mapping object, in both symmetric monoidal settings, is given by* $\mathrm{Fun}^{\mathrm{L}}(-,-)$ *giving a closed symmetric monoidal structure on* Pr^{L} *and* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ *(5.5.3.8 (12), 4.8.1.24 (13)).*

Using the criterion of 2.2.1.2 of (13), we can restrict this symmetric monoidal structure to one on the *full* ∞-subcategory of compactly-generated presentable stable ∞-categories. The following theorem of (15) restricts the structure to $\mathrm{Cat}_\infty^{\mathrm{perf}}$, equivalently $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$

Theorem 2.2.2 (3.1 of (15)). *The ∞-category $\mathrm{Cat}_\infty^{\mathrm{perf}}$ has a closed symmetric monoidal structure given by $\mathrm{C}\widehat{\otimes}\mathrm{D} = (\mathrm{Ind}(\mathrm{C}) \otimes \mathrm{Ind}(\mathrm{D}))^{\omega}$. The internal mapping object is $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{C}, \mathrm{D})$. The unit is the ∞-category Sp^{ω} of finite spectra. This further extends to a symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{ex}}$ given by idempotent-completing and then applying the monoidal product in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.*

We refer to the units of $\mathrm{Cat}_\infty^{\mathrm{perf}}$ and $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$ as Sp^{ω} and Sp respectively. The symmetric monoidal structures above induce $\mathbb{E}_\infty$-structures on the units. This is the smash product which is essentially unique.

2.3 The R-linear setting

Recall the little k-cubes operad $\mathbb{E}_k$. An $\mathbb{E}_1$-algebra identifies with an $\mathbb{A}^\infty$-algebra while the colimit operad, $\mathbb{E}_\infty$, identifies the commutative algebra objects in ∞-categories. We let $\mathrm{Alg}(\mathrm{C})$ denote the $\mathbb{E}_1$-algebras of a symmetric monoidal ∞-category C. When C is equipped with a t-structure which is connectively symmetric monoidal, we let $\mathrm{Alg}^{\mathrm{cn}}(\mathrm{C}) := \mathrm{Alg}(\mathrm{C}^{\mathrm{cn}})$ denote the connective $\mathbb{E}_1$-algebras of C.

Let R be an $\mathbb{E}_k$-algebra of a symmetric monoidal ∞-category C, where C is Sp, $\mathrm{Cat}_\infty^{\mathrm{perf}}$, or $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$. Section 3.3 of (13) defines an ∞-category of R-bimodules. We denote this by $\mathrm{BMod}_{\mathrm{R}}(\mathrm{C})$. There are also ∞-categories of left R-modules and right R-modules, $\mathrm{LMod}_{\mathrm{R}}(\mathrm{C})$ and $\mathrm{RMod}_{\mathrm{R}}(\mathrm{C})$ respectively. For an $\mathbb{E}_\infty$-algebra, all three of these ∞-categories are equivalent

(Proposition 4.5.1.4, Corollary 4.5.1.5 (13)). We define the ∞-category of *perfect modules* over an $\mathbb{E}_1$-algebra R in C to be the ∞-category of ω-compact objects $\mathrm{Perf}(\mathrm{R}) := \mathrm{LMod}_{\mathrm{R}}(\mathrm{C})^{\omega}$. There is an improved identification of the perfect complexes within the derived ∞-category from Example 1.2.2.

Proposition 2.3.1 (7.1.1.16 (13)). *Let* R *be a discrete commutative ring. The Eilenberg-MacLane spectrum* HR *has an* $\mathbb{E}_\infty$*-structure and* $\mathrm{LMod}_{\mathrm{HR}}$ *is equivalent to the unbounded derived* ∞*-category of* R. *This identifies the full stable* ∞*-categories* $\mathrm{Perf}_{\mathrm{R}}$ *and* $\mathrm{Perf}_{\mathrm{HR}}$.

We have the following monoidal inheritance proposition.

Proposition 2.3.2 (7.1.2.6 (13)). *The functor* $\mathrm{R} \mapsto \mathrm{LMod}_{\mathrm{R}}$ *enhances to a fully faithful functor* $\mathrm{Alg}_{\mathbb{E}_k}(\mathrm{Sp}) \to \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$. *In particular, letting* k *go to* ∞ *gets us a fully faithful functor from the commutative ring spectra to the symmetric monoidal stable presentable* ∞*-categories. This functor factors through* $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$.

With Morita theory in mind, we provide linear settings in $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$, or $\mathrm{Cat}^{\mathrm{perf}}_{\infty}$. Let R be an $\mathbb{E}_\infty$-ring spectrum. By Proposition 2.3.2 and Theorem 2.2.2, the ∞-category $\mathrm{Mod}_{\mathrm{R}}$ is an $\mathbb{E}_\infty$-algebra in $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$. We can consider the ∞-category of modules $\mathrm{Mod}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}})$. Under the equivalence of Theorem 1.4.4, we have an equivalence $\mathrm{Mod}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}) \simeq \mathrm{Mod}_{\mathrm{Perf}_{\mathrm{R}}}(\mathrm{Cat}^{\mathrm{perf}}_{\infty})$. We refer to these ∞-categories of modules as $\mathrm{Cat}^{\omega}_{\mathrm{R}}$, using the notation of (9).

Let X be a quasi-compact quasi-separated scheme. By definition, X can be written as a cofiltered colimit of affine open subschemes, $\mathrm{X} \cong \operatorname*{colim}_{i \in I} \operatorname{Spec} S_i$. We can define the ∞-category of modules on X as the filtered limit along the pullbacks

$$\mathrm{Mod}_{\mathrm{X}} := \lim_{i \in I} \mathrm{Mod}_{S_i}.$$

The homotopy triangulated category of $\mathrm{Mod}_{\mathrm{X}}$ is equivalent to the unbounded derived category $D_{qc}(\mathrm{X})$. As $\mathrm{Pr}^{L,c}_{st,cg}$ is closed under small limits, $\mathrm{Mod}_{\mathrm{X}}$ is an object of $\mathrm{Pr}^{L,c}_{st,cg}$. It is a presentable stable ∞-category and has an $\mathbb{E}_\infty$-structure over the operation

$$\mathrm{Mod}_{\mathrm{X}} \otimes \mathrm{Mod}_{\mathrm{X}} \simeq \mathrm{Mod}_{\mathrm{X} \times \mathrm{X}} \xrightarrow{\Delta^*} \mathrm{Mod}_{\mathrm{X}}.$$

The first equivalence is a simple case of Theorem 1.2 of (16). The ω-compact objects are the perfect complexes on X. We define the X-*linear setting* to be $\mathrm{Mod}_{\mathrm{Mod}_{\mathrm{X}}}(\mathrm{Pr}^{L,c}_{st,cg})$.

CHAPTER 3

ALGEBRAIC K-THEORY

In this section, we build our perspective on K-theory.

3.1 K_0, higher K-theory, and universal additivity

Algebraic K-theory is designed to collapse all (appropriately defined) extensional information between any two objects of a category. Grothendieck built the group K_0 to encapsulate all additive functions from our input category taking values in a target abelian group. An *additive function* from an exact, abelian, Waldhausen, or stable category C is a function $\mathsf{f} : \mathrm{C} \to \mathrm{A}$ taking values in an abelian group which splits (or sums) across any "exact sequence" in each respective domain category. For an exact category, this property looks like

$$0 \to \mathsf{X} \to \mathsf{Z} \to \mathsf{Y} \to 0 \Rightarrow \mathsf{f}(\mathsf{Z}) = \mathsf{f}(\mathsf{X}) + \mathsf{f}(\mathsf{Y}) \text{ in A.}$$

To compute and contextualize K_0, there was a push for higher algebraic K-groups that resulted in a selection of constructions such as Waldhausen's $\mathsf{S}_\bullet$ and Quillen's $+$-construction (see Quillen (7) or Chapter IV of Weibel's K-Book (17) for an overview).

Definition 3.1.1. The connective algebraic K-theory of a stable ∞-category is given by the ∞-categorical $\mathsf{S}_\bullet$-construction. Let $\mathrm{Gap}(\Delta^n) := \mathrm{Fun}(\Delta^1, \Delta^n)$ be the arrow category of Δ^n. We can refer to its objects by (i,j) where $0 \leqslant i \leqslant j \leqslant n$. Let $\mathrm{Gap}(\Delta^n, \mathrm{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{Gap}(\Delta^n), \mathrm{C})$ which sends (k,k) to the point and squares

$$\begin{tikzcd} (j,k) \arrow[r] \arrow[d] & (j,l) \arrow[d] \\ (k,k) \arrow[r] & (k,l) \end{tikzcd}$$

to (co)fiber square of C. These functors can be equivalently defined as those which preserve the stable squares of $\mathrm{Gap}(\Delta^n)$. If a square of $\mathrm{Gap}(\Delta^n)$ is both a pushout and a pullback, it is sent to a pullback/pushout square of C. The ∞-categores $\mathrm{Gap}(\Delta^n, \mathrm{C})$ are equivalent to $\mathrm{Fun}(\Delta^{n-1}, \mathrm{C})$. The first equivalence evaluates every functor on the sequence

$$(0,1) \to (0,2) \to \cdots \to (0,n).$$

The other direction fleshes out an $n-1$ sequence by taking cofibers.

There is a simplicial structure present. We describe the structure on $\mathrm{C}^{\Delta^{n-1}}$. The 0th face map quotients the $n-1$ sequence by the constant sequence on the $(0,1)$ term, and then you forget the initial object (which is now the point) and reindex. For $0 < i$, the ∂_i map selects "the (i,i)-minor" of the functor. Under the equivalence with $\mathrm{C}^{\Delta^{n-1}}$, this deletes the $i-1$ object. The 0th degeneracy attaches a point to the start of the $n-1$ sequence. The higher degeneracy maps duplicate information in your $n-1$ sequence as usual, under an index shift by one. If we take the core of $\mathrm{Gap}(\Delta^n, \mathrm{C})$, we get a simplicial space $S_\bullet \mathrm{C} := \iota\mathrm{Gap}(\Delta^\bullet, \mathrm{C})$. The geometric realization is path-connected as $S_0\mathrm{C} \simeq *$. We define the i*th connective algebraic* K*-group* of C to be the group $\pi_i\Omega|S_\bullet C|$ and the *connective algebraic* K*-theory spectrum* of C to be $\mathsf{K}^{\mathrm{conn}}(C) := \Omega|S_\bullet C|$. We reserve the decorated "$\mathsf{K}^{\mathrm{conn}}$" for when we discuss the spectrum valued functor of connective K-theory (see Lemma 3.1.2 below). The K_i-groups will remain undecorated.

The following lemma can be found in Section 1.3 and 1.5 of Waldhausen (18). Here $S^n_\bullet C$ means we have applied the $S_\bullet$ construction n times. It can be applied to multisimplicial spaces. In the above definition, we implicitly used this fact to apply $S_\bullet$ to a quasi-category which is a simplicial set.

Lemma 3.1.2. *The assignment* $n \mapsto |S^n_\bullet(C)|$ *extends to a spectrum structure and is an* Ω*-spectrum. Its infinite-loop space is equivalent to* $\Omega|S_\bullet C|$*.*

As a corollary, all K-groups are abelian. We now want to briefly touch on connective K-theory's role as the *universal additive invariant.* We follow the treatment in Section 6 (15).

Definition 3.1.3. A cofiber sequence $C \xrightarrow{f} D \xrightarrow{g} E$ in $\mathrm{Cat}^{\mathrm{perf}}_\infty$ is split-exact when both functors have right adjoints, f is fully faithful, and the right adjoint h of g is fully faithful.

A functor $\mathcal{U} : \mathrm{Cat}^{\mathrm{perf}}_\infty \to \mathcal{A}$ taking values in a small stable ∞-category $\mathcal{A}$ is an *additive invariant* when it inverts Morita equivalences and for every split-exact sequence, the map

$$\mathcal{U}(C) \vee \mathcal{U}(E) \xrightarrow{f \vee h} \mathcal{U}(C)$$

is an equivalence. Connective K-theory is the universal example of an additive invariant, see Theorem 1.3 of (15). Another good universality statement is due to Barwick (19), which states that connective K-theory is homotopy initial among additive functors (defined in a suitable setting) which receive a natural transformation from ι, the moduli of objects functor.

3.2 Lower K-theory, nonconnective K-theory, and universal localization

One purpose of higher algebraic K-theory was to extend certain exact sequences of K_0-groups to the left, as we did not always have injectivity. In other situations, we did not have surjectivity *on the right.* So we would get a fiber sequence in the ∞-category of *connective* spectra for some Serre localizations $\mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}$ where the final map of the long exact sequences of stable homotopy groups $\mathsf{K}_0(\mathcal{B}) \to \mathsf{K}_0(\mathcal{B}/\mathcal{A})$ could have a nontrivial cokernel. The solution was to "right-derive" connective K-theory, extending the homology theory into non-connective spectra.

Theorem 3.2.1 (Fundamental theorem of K-theory)**.** *Let* R *be a discrete ring and* $\mathsf{i} > 0$*. There is an exact sequence of homotopy groups*

$$0 \to \mathsf{K}_\mathsf{i}(\mathsf{R}) \to \mathsf{K}_\mathsf{i}(\mathsf{R}[\mathsf{t}]) \oplus \mathsf{K}_\mathsf{i}(\mathsf{R}[\mathsf{t}^-]) \to \mathsf{K}_\mathsf{i}(\mathsf{R}[\mathsf{t}^\pm]) \to \mathsf{K}_{\mathsf{i}-1}(\mathsf{R}) \to 0.$$

The proof is below, see Equation † and the surrounding discussion. A truncated version of this exact sequence exists for $\mathsf{i} = 0$. Bass defined $\mathsf{K}_{-1}(\mathsf{R})$ to be the cokernel of the map $\mathsf{K}_0(\mathsf{R}[\mathsf{t}]) \oplus \mathsf{K}_0(\mathsf{R}[\mathsf{t}^-]) \to \mathsf{K}_0(\mathsf{R}[\mathsf{t}^\pm])$. Under this definition, the truncated exact sequence exists *again* for $\mathsf{i} = -1$, and so Bass defined a K_{-2} group in the same manner. The structure continues to repeat and so Bass inductively constructs a theory of negative K-groups.

Definition 3.2.2. For a category C of $\mathrm{Cat}_\infty^{\mathrm{perf}}$, we need a different definition. The main observation of Schlichting (20) and Blumberg–Gepner–Tabuada (15) is that we can build a K-theoretic suspension, internal to $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Let κ be an uncountable cardinal and define the flasque category $F_\kappa C$ to be $\mathrm{Ind}_\omega(C)^\kappa$ the κ-compact objects. This contains C fully faithfully as every object c is ω-compact. It is also closed under finite limits and colimits within $\mathrm{Ind}_\omega(C)$, and so is a small stable ∞-category. Form the cofiber

$$C \to F_\kappa C \to SC.$$

The ∞-category $F_\kappa C$ is closed under κ-small ω-filtered colimits, as κ-filtered colimits commute with κ-small limits. Most importantly, countable coproducts exist for $F_\kappa C$, the Eilenberg swindle applies, and $\mathsf{K}^{\mathrm{conn}}(F_\kappa C) \simeq *$. It follows that the induced map $\mathsf{K}^{\mathrm{conn}}(C) \to \Omega\mathsf{K}^{\mathrm{conn}}(SC)$ is an equivalence on connective covers. Define the nonconnective K-theory of C to be the stabilization

$$\mathsf{K}(C) := \operatorname*{colim}_i \Omega^i \mathsf{K}^{\mathrm{conn}}(S^i C).$$

We leave the nonconnective K-theory functor undecorated, as we no longer use $\mathsf{K}^{\mathrm{conn}}$.

We have the localization theorem from Thomason reworking Theorem 5 of (7) for compact objects.

Theorem 3.2.3 (Thm 7.4 of Thomason–Trobaugh(21)). *Let* X *be a quasi-compact quasi-separated scheme and* U *a quasi-compact open. Set* Y *to be the closed set* $X \setminus U$. *There is a fiber sequence of spectra*

$$\mathsf{K}(X \textit{ on } Y) \to \mathsf{K}(X) \to \mathsf{K}(U).$$

We have defined algebraic K-theory on $\mathrm{Cat}_\infty^{\mathrm{perf}}$. Under this definition, $\mathsf{K}(X \text{ on } Y)$ is the K-theory of the perfect complexes on X which are supported on Y. This can be defined as the fiber stable ∞-category of the pullback map $\mathrm{Perf}(X) \to \mathrm{Perf}(U)$. This property of localization, appropriately extended to all stable ∞-categories, is a defining factor of algebraic K-theory.

Definition 3.2.4. An *exact sequence* in $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$ is a square

$$\begin{tikzcd} C \arrow[r, "\mathsf{F}"] \arrow[d] & D \arrow[d, "\mathsf{G}"] \\ 0 \arrow[r] & E \end{tikzcd}$$

where F is fully faithful and the map $D/C \to E$ is an equivalence.

A sequence $C \to D \to E$ in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ is exact if $\mathrm{Ind}(C) \to \mathrm{Ind}(C) \to \mathrm{Ind}(D)$ is exact in $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$.

A *localizing invariant* $\mathcal{U}$ from $\mathrm{Cat}_\infty^{\mathrm{perf}}$ or $\mathrm{Pr}^{\mathrm{L,c}}_{\mathrm{st,cg}}$ taking values in a small stable ∞-category C is a functor $\mathcal{U}$ which preserves exact sequences. Nonconnective K-theory is the universal such example of a localizing invariant, see Theorem 1.3 of (15) again.

Remark 3.2.5. The definition of localizing invariants given in (15) includes commuting with filtered colimits. We use the term localizing invariant in the sense of (5), where commuting with filtered colimits is not required (those that do are sometimes called *finitary*). Our definition has also been referred to as weakly localizing.

3.3 R-linear localizing invariants and descent

The above discussions and constructions can be restricted to $\mathrm{Cat}_{\mathrm{R}}^{\omega}$ or $\mathrm{Cat}_{\mathrm{X}}^{\omega}$ for a commutative ring R or qcqs scheme X. This will be the context for our construction of twisted algebraic K-theory. Here we record some well-known sites and the corresponding descent theorems for localizing invariants. Every site we considered is cd-generated by a collection of squares. As a result, descent can be reduced to checking Mayer-Vietoris with respect to these squares. All of our examples of descent are essentially derived from localization.

Definition 3.3.1. A Nisnevich cover of a scheme X is a collection of étale maps $\{p_i : \tilde{U}_i \to X\}$ such that every closed point admits a lift. Diagramatically, for every closed point κ there must exist a j so that a dashed lift

$$\begin{tikzcd} & \tilde{U}_j \arrow[d, "p_j"] \\ \operatorname{Spec} k \arrow[r, "\kappa"'] \arrow[ur, dashed] & X \end{tikzcd}$$

exists. The big Nisnevich site on Sch is generated by the Nisnevich covers.

The following proposition holds for all (non-finitary) localizing invariants.

Proposition 3.3.2 (Proposition A.15 of (22)). *Any* X*-linear localizing invariant satisfies Nisnevich descent. This implies all localizing invariants satisfy Zariski descent.*

Remark 3.3.3. Algebraic K-theory famously fails to satisfy étale descent, where covers are simply jointly surjective étale maps. This can be seen on closed points with the étale cover $\mathbb{R} \to \mathbb{C}$. With étale descent, we would have a Čech spectral sequence

$$\mathsf{H}^{\mathsf{p}}(\mathbb{C}, \mathsf{K}_{\mathsf{q}}) \Rightarrow \mathsf{K}_{\mathsf{q}-\mathsf{p}}(\mathbb{R}).$$

Note that we do not have to sheafify for sheaf cohomology on the left-hand side, and that it vanishes when $\mathsf{p} > 0$. So étale descent would force an isomorphism $\mathsf{K}_{\mathsf{i}}(\mathbb{C}) \cong \mathsf{K}_{\mathsf{i}}(\mathbb{R})$. However, $\mathsf{K}_1(\mathbb{C}) \cong \mathbb{C}^{\times}$ and likewise $\mathsf{K}_1(\mathbb{R}) \cong \mathbb{R}^{\times}$. In fact, a more relevant statement can be made in the context of this thesis. All twisted K-theories (see the next chapter) would be trivial as all Azumaya algebras are étale locally Morita trivial.

Definition 3.3.4. The *cdh-topology* on the category of $\mathrm{Sch}_{/X}$ is cd-generated by the Nisnevich squares and the abstract blow-up squares, see Theorem 3.2.5 of Asok–Hoyois–Wendt (23). An abstract blow-up square is a pullback square of the form

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\rho"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

where i is a closed immersion and ρ is a proper morphism which is an isomorphism away from Y.

The following theorem is stated using the ∞-category of pro-spectra $\mathbf{Pro}(\mathrm{Sp})$, where an object is a small cofiltered diagram, $\mathsf{E} : \Lambda \to \mathrm{Sp}$, valued in spectra. We write $\{\mathsf{E}_{\mathsf{n}}\}$ for the corresponding pro-spectrum. If the brackets and index are omitted, then the pro-spectrum is considered constant. After adjusting equivalence class representatives, we may assume the cofiltered diagram is fixed when working with a finite diagram of pro-spectra. Any morphism can then be represented by a natural transformation of diagrams (also known as a level map). In this thesis, our pro-spectra will be indexed by $\mathbb{N}^{op}$. This originates from an $\mathbb{N}$-indexed diagram of n*th order nilpotent thickenings* of a closed immersion $Y \to X$. If $\mathcal{I}_Y$ is the corresponding ideal sheaf, this is the system of closed immersions given by the descending sequence of ideal sheaves

$$\subset \mathcal{I}_Y^n \subset \cdots \subset \mathcal{I}_Y^2 \subset \mathcal{I}_Y \subset \mathcal{O}_X.$$

We associate to a cdh square a pro-Mayer Vietoris property using these thickenings.

Definition 3.3.5. A square of pro-spectra

$$\begin{tikzcd} \{E_n\} \arrow[r] \arrow[d] & \{F_n\} \arrow[d] \\ \{X_n\} \arrow[r] & \{Y_n\} \end{tikzcd}$$

is *pro-cartesian* if and only if the induced map on the level-wise fiber pro-spectra is a weak equivalence (see Definition 2.27 of (5)).

A localizing invariant L is called k-*connective* if for any n-connective map of connective $\mathbb{E}_1$-ring spectra $A \to B$ the induced map of spectra $L(A) \to L(B)$ is $(\mathsf{n}+\mathsf{k})$-connective.

Theorem 3.3.6 (Land–Tamme (5))**.** *Given an abstract blow-up square (3.3.4) of schemes and a* k*-connective localizing invariant* L*, the square of pro-spectra*

$$\begin{tikzcd} \mathrm{L}(\mathsf{X}) \arrow[r] \arrow[d] & \mathrm{L}(\tilde{\mathsf{X}}) \arrow[d] \\ \{\mathrm{L}(\mathsf{Y}_{\mathsf{n}})\} \arrow[r] & \{\mathrm{L}(\mathsf{D}_{\mathsf{n}})\} \end{tikzcd}$$

is pro-cartesian. Here, Y_{n} *is the* n*th infinitesimal thickening of* Y *and the prosystem is indexed by* $\mathbb{N}^{\mathrm{op}}$*.*

CHAPTER 4

TWISTED K-THEORY AND K-THEORY WITH COEFFICIENTS

This chapter contains work which was published previously in Stapleton, J. Weibel's conjecture for twisted K-theory, Ann. of K-theory, 5(3): 621-637, 2020.

4.1 Azumaya algebras

Let R be a discrete commutative ring. A discrete R-algebra is *discretely proper* over R if it is projective as an R-module. An algebra is *faithful* over R if it is a compact generator of the ∞-category $\mathrm{Perf}_{\mathrm{R}}$.

Definition 4.1.1. An *Azumaya algebra* over R is a faithful proper R-algebra A such that the two-sided action map $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{A}^{\mathrm{op}} \to \mathrm{End}_{\mathrm{R}}(\mathrm{A})$ is an equivalence. We remark that tensor and endomorphisms are non-derived in this definition.

In Grothendieck's original papers (24) (25) (26), he globalizes this definition to schemes. There is a natural extension of the affine definition above. We prefer to use a different, but equivalent, definition.

Definition 4.1.2. A locally free sheaf of $\mathcal{O}_{\mathrm{X}}$-algebras $\mathcal{A}$ is a *sheaf of Azumaya algebras* if it is étale-locally isomorphic to $\mathrm{M}_{\mathfrak{n}}(\mathcal{O}_{\mathrm{X}})$ for some $\mathfrak{n}$.

An Azumaya algebra is a PGL_n-torsor over the étale topos of X and so, by Giraud (27), isomorphism classes are in bijection with $\mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)$. The central extension of sheaves of groups in the étale topology

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

leads to an exact sequence of nonabelian cohomology

$$\cdots \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m).$$

For $d \,|\, n$ we have a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}_n \arrow[r] & \mathrm{PGL}_n \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[u, equal] & \mathrm{GL}_d \arrow[r] \arrow[u] & \mathrm{PGL}_d \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

with the two right arrows given by block-summing the matrix along the diagonal n/d times. The Brauer group is the filtered colimit of cofibers

$$\mathsf{Br}(X) := \operatorname*{colim}_{d \xrightarrow{\text{div}} n} (\mathrm{cofib}(\mathrm{H}^1_{\text{ét}}(X, \mathrm{GL}_n) \to \mathrm{H}^1_{\text{ét}}(X, \mathrm{PGL}_n)))$$

along the partially-ordered set of the natural numbers under division. This is the group of Azumaya algebras modulo Morita equivalence with group operation given by tensor product (see (24)). We have an injection $\mathsf{Br}(X) \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$. When X is quasi-compact, this injection factors through the torsion subgroup. Let $\mathsf{Br}^{\mathrm{coh}}(X) := \mathsf{H}^2_{\text{ét}}(X, \mathbb{G}_m)_{\mathsf{tor}}$ denote the cohomological Brauer group. Grothendieck asked if the injection $\mathsf{Br}(X) \hookrightarrow \mathsf{Br}^{\mathrm{coh}}(X)$ is an isomorphism.

This map is not generally surjective. Edidin–Hassett–Kresch–Vistoli (28) give a non-separated counterexample by connecting the image of the Brauer group to quotient stacks. There are two ways to proceed in addressing the question. The first is to provide a class of schemes for when this holds. In (29), de Jong publishes a proof of O. Gabber that $\mathsf{Br}(X) \cong \mathsf{Br}^{\mathrm{coh}}(X)$ when X is equipped with an ample line bundle. Along with reproving Gabber's result for affines, Lieblich (30) shows that for a regular scheme with dimension less than or equal to 2 there are isomorphisms $\mathsf{Br}(X) \cong \mathsf{Br}^{\mathrm{coh}}(X) \cong \mathsf{H}^2_{\text{ét}}(X, \mathbb{G}_m)$.

4.2 Derived Azumaya algebras

A second way of addressing the question is to enlarge the class of objects considered. In Lieblich's thesis (30), he compactifies the moduli of Azumaya algebras by defining a derived Azumaya algebra.

Definition 4.2.1. A *derived Azumaya algebra* $\mathcal{A}$ over a connective $\mathbb{E}_\infty$-ring spectrum R is an $\mathbb{E}_1$-algebra $\mathcal{A}$ in the ∞-category $\mathrm{Mod}_{\mathrm{R}}$ such that the map

$$\mathcal{A} \otimes_{\mathrm{R}} \mathcal{A}^{\mathrm{op}} \to \mathrm{End}_{\mathrm{R}}(\mathcal{A})$$

is an equivalence of $\mathbb{E}_1$-algebras over R.

The map in this definition is the natural left module structure of the derived enveloping algebra on $\mathcal{A}$. We also have an étale triviality result.

Theorem 4.2.2 (Theorem 5.11 of (9))**.** *Let* R *be a connective* $\mathbb{E}_\infty$ *ring spectrum and* $\mathcal{A}$ *a derived Azumaya algebra over* R*. Then there is a faithfully flat étale* R*-algebra* S *such that* $\mathcal{A} \otimes_{\mathrm{R}} \mathrm{S}$ *is Morita equivalent to* S*.*

An étale R-algebra S is defined as an étale discrete $\pi_0\mathrm{R}$-algebra $\pi_0\mathrm{S}$ along with induced isomorphisms $\pi_i\mathrm{R}\otimes_{\pi_0\mathrm{R}}\pi_0\mathrm{S} \cong \pi_i\mathrm{S}$. There is a similar derived notion of properness and smoothness. In the discrete setting, the notion of separability played the role of smoothness.

Definition 4.2.3. An $\mathbb{E}_1$-algebra $\mathcal{A}$ over R is *proper* if the underlying R-module is compact. It is *smooth* over R if $\mathcal{A}$ is proper as a $\mathcal{A} \otimes_{\mathrm{R}} \mathcal{A}^{\mathrm{op}}$ left module.

After Lieblich, Toën (31) and (later) Antieau–Gepner (9) consider the analogous problem posed by Grothendieck in the dg-algebra and $\mathbb{E}_\infty$-algebra settings, respectively. Antieau–Gepner construct an étale sheaf **Br** in the ∞-topos $\mathrm{Shv}_{\mathrm{R}}^{\text{ét}}$. For any étale sheaf F, we can now associate a Brauer space $\mathbf{Br}(\mathrm{F})$. For X a quasi-compact quasi-separated scheme, they show $\pi_0(\mathbf{Br}(\mathrm{X})) \cong \mathsf{H}^1_{\text{ét}}(\mathrm{X}, \mathbb{Z}) \times \mathsf{H}^2_{\text{ét}}(\mathrm{X}, \mathbb{G}_m)$ and every such Brauer class is **algebraic**. Now for any (possibly nontorsion) $\alpha \in \mathsf{H}^2_{\text{ét}}(\mathrm{X}, \mathbb{G}_m)$ there is an associated derived Azumaya algebra $\mathcal{A}$. Derived Azumaya algebras also have a symmetric monoidal definition on $\mathrm{Cat}_{\mathrm{R}}^{\omega}$.

Theorem 4.2.4 (3.15 of (9), Proposition 1.5 of (31))**.** *An object* C *of* $\mathrm{Cat}_{\mathrm{R}}^{\omega}$ *is dualizable if and only if* $\mathrm{C} \simeq \mathrm{Mod}_{\mathcal{A}}$ *for a smooth and proper* R*-algebra* $\mathcal{A}$*. An object* C *of* $\mathrm{Cat}_{\mathrm{R}}^{\omega}$ *is invertible if and only if* $\mathrm{C} \simeq \mathrm{Mod}_{\mathcal{A}}$ *for a derived Azumaya* R*-algebra* $\mathcal{A}$*.*

We have analogous global results for a qcqs scheme and $\mathrm{Cat}_{\mathrm{X}}^{\omega}$. An important strength of smooth and proper R-algebras A is the ability to detect perfect complexes A after forgetting to R.

Proposition 4.2.5. *An algebra $\mathcal{A}$ is smooth and proper over a base commutative ring* R *if and only if there exists a square*

$$\begin{array}{ccc} \mathrm{Perf}_{\mathcal{A}} & \longrightarrow & \mathrm{Perf}_{\mathrm{R}} \\ \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathcal{A}} & \longrightarrow & \mathrm{Mod}_{\mathrm{R}} \end{array}$$

and, furthermore, this square is cartesian.

4.3 Twisted algebraic $\mathbb{K}$-theory

One of the primary constructions of algebraic topology is homology and cohomology. Let's describe the classical situation of twisted homology. We leave cohomology behind for clarity of exposition. In the framework of spectra, integral homology is, by design, corepresentable. We take the homotopy groups of the mapping spectrum $\mathrm{Map}_{\mathrm{Sp}}(\mathrm{H}\mathbb{Z}, -)$. We can vary the corepresenting discrete algebras HR to vary the coefficients of our homology. There is a more subtle way we can involve coefficients. For this, we need to fix a CW complex X for homological evaluation.

A *local system of* R*-modules* over a path-connected CW complex X is a group homomorphism $\mathcal{M} : \pi_1(\mathrm{X}) \to \mathrm{End}_{\mathrm{Mod}_{\mathrm{R}}}(\mathrm{M})$, see Steenrod (32). In loc. cited, Steenrod provides a direct construction of the CW homology of X taking values in the local system $\mathcal{M}$. By the definition, a simply connected space $\widetilde{\mathrm{X}}$ can have no nontrivial local systems. In fact, the universal cover $\widetilde{\mathrm{X}}$ of X can be used to produce an alternative but equivalent definition of local homology.

Consider the singular chain complex on $\widetilde{X}$, $C_n(\widetilde{X})$. The action of $\pi_1(X)$ on $\widetilde{X}$ by deck transformations translates, by precomposition, to a right action on $C_n(\widetilde{X})$. A morphism $\pi_1(X) \to \mathrm{End}_{\mathrm{Mod}_R}(M)$ determines a left $R[\pi_1(X)]$-module structure on M. We can then define the tensor $C_n(\widetilde{X}) \otimes_{R[\pi_1(X)]} M$. This has the impact of killing the deck transformation action (viewed as simaltaneously acting on both sides). It is equivalent to the definition of homology in a local system given by Steenrod.

What are the analogous twistings for algebraic K-theory of a qcqs scheme X? Based on our working definition, we should twist Perf_X by some X-linear object C of Cat_X^{ω}. Étale covers are the analogue of covering spaces in topology, so we need C to be étale Morita trivial. By Theorem 6.16 of (9), such a Perf_X-linear category would have a global compact generator. The endomorphism $\mathbb{E}_1$-algebra is then an R-algebra which is étale locally Morita equivalent to the base, equivalently, this is a derived Azumaya algebra over R.

Remark 4.3.1. Classical twisted algebraic K-theory began with a cocycle representative of a class $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. This can be used to define an unbounded derived ∞-category of α-twisted quasi-coherent sheaves on X, which is unique in α up to equivalence of ∞-categories. Algebraic K-theory of the compact objects is a definition for α-twisted K-theory. Again, by Corollary 6.20 of (9), all such Brauer classes α have a derived Azumaya algebra $\mathcal{A}$ associated to them. In particular, this association induces an equivalence $\mathrm{Perf}_{\mathcal{A}} \simeq \mathrm{Perf}_X^{\alpha}$. So all classical twisted K-theory is a twisting by some coefficient $\mathbb{E}_1$-algebra.

Our main theorem extends beyond the twisted setting to algebraic K-theory with coefficients in a dualizable $\mathbb{E}_1$-algebra (e.g. a smooth proper connective $\mathbb{E}_1$-algebra). For our purposes, *we will still refer* to this as twisted algebraic K-theory

Definition 4.3.2. Let $\mathcal{A}$ be an $\mathbb{E}_1$ ring spectrum over an $\mathbb{E}_\infty$ ring spectrum R. The $\mathcal{A}$-*twisted algebraic* K-*theory* functor is given on $\mathrm{Cat}_{\mathrm{R}}^{\omega}$ by

$$\mathsf{K}^{\mathcal{A}}(-) := \mathsf{K}(- \otimes_{\mathrm{Cat}_{\mathrm{R}}^{\omega}} \mathrm{Perf}_{\mathcal{A}}).$$

We want a definition of twisted K-theory for the global setting, $\mathrm{Cat}_{\mathrm{X}}^{\omega}$. Let $\mathrm{Alg}_{\mathrm{X}} := \mathrm{Alg}(\mathrm{Mod}_{\mathrm{X}})$ be the $\mathbb{E}_1$-algebras of the symmetric monoidal $\mathrm{Mod}_{\mathrm{X}}$ and $\mathrm{Alg}_{\mathrm{X}}^{\mathrm{cn}} := \mathrm{Alg}(\mathrm{Mod}_{\mathrm{X}}^{\mathrm{cn}})$. When X is a quasi-compact quasi-separated scheme of finite Krull dimension, an algebra of $\mathrm{Mod}_{\mathrm{X}}^{\mathrm{cn}}$ (equivalently, $\mathrm{D}_{\mathsf{qc}}^{\mathrm{cn}}(\mathsf{X})$) is equivalent to a sheaf of connective $\mathbb{E}_1$-algebras, over $\mathcal{O}_{\mathrm{X}}$, taking values in $\mathrm{Mod}_{\mathrm{X}}$. This follows by hypercompletion, which implies sheaves of connected chain complexes of quasi-coherent modules are equivalent to connected chain complexes of quasi-coherent sheaves.

We say $\mathcal{A}$ is a sheaf of connective $\mathbb{E}_1$-algebras over $\mathcal{O}_{\mathrm{X}}$ when we mean a sheaf taking values in $\mathrm{Alg}^{\mathrm{cn}}$ over an open U. Adding an adjective like "a smooth proper connective $\mathbb{E}_1$-algebra over X" enforces the condition locally.

Definition 4.3.3. For X a quasi-compact quasi-separated d-dimensional scheme, and $\mathcal{A}$ a sheaf of smooth proper $\mathbb{E}_1$-algebras over X, we define the $\mathcal{A}$-twisted algebraic K-theory functor to be

$$\mathsf{K}^{\mathcal{A}}(-) : \mathrm{Cat}_{\mathrm{X}}^{\omega} \to \mathrm{Sp} \text{ where } \mathsf{K}^{\mathcal{A}}(\mathrm{C}) := \mathsf{K}(\mathrm{C} \otimes \mathrm{Mod}_{\mathcal{A}}(\mathrm{Perf}_{\mathrm{X}})).$$

4.4 Lower nil-invariance of twisted K-theory

We prove some basic properties of $\mathcal{A}$-twisted K-theory, often assuming $\mathcal{A}$ is connective. We will not use smooth and properness until the later sections.

Lemma 4.4.1. *Let $\mathcal{A}$, S be connective $\mathbb{E}_1$ ring spectra over a discrete commutative ring* R. *Then the natural maps induce isomorphisms*

$$\begin{tikzcd}
 & \mathsf{K}_i^{\mathcal{A}}(\mathrm{S}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathsf{K}_i^{\mathcal{A}}(\pi_0(\mathrm{S})) \arrow[dr, "\cong"'] & & \mathsf{K}_i^{\pi_0(\mathcal{A})}(\mathrm{S}) \arrow[dl, "\cong"] \\
 & \mathsf{K}_i^{\pi_0(\mathcal{A})}(\pi_0(\mathrm{S})) &
\end{tikzcd}$$

for $i \leqslant 0$.

Proof. We have the following isomorphisms of discrete rings

$$\pi_0(\mathcal{A} \otimes_{\mathsf{R}} \mathrm{S}) \cong \pi_0(\mathcal{A} \otimes_{\mathsf{R}} \pi_0(\mathrm{S})) \cong \pi_0(\pi_0(\mathcal{A}) \otimes_{\mathsf{R}} \mathrm{S}) \cong \pi_0(\pi_0(\mathcal{A}) \otimes_{\mathsf{R}} \pi_0(\mathrm{S})).$$

The lemma follows since $\mathsf{K}_i(\mathrm{R}) \cong \mathsf{K}_i(\pi_0(\mathrm{R}))$ for $i \leqslant 0$ (see Theorem 9.53 of (15)) and by symmetric monoidality of the functor $\mathrm{Alg}_{/\mathrm{R}} \xrightarrow{\mathrm{Perf}(-)} \mathrm{Cat}^{\omega}_{\mathrm{R}}$. □

The previous proposition suggests we can work discretely and then transfer the results to the derived setting. This is true to some extent. However, taking π_0 of a connective $\mathbb{E}_1$ ring spectrum does not preserve smoothness or properness, which are necessary for our proof of Proposition 5.3.1. We will also need reduction invariance for low dimensional K-groups.

Proposition 4.4.2. *Let* R *be a discrete commutative ring and* $\mathcal{A}$ *a connective* $\mathbb{E}_1$ *ring spectrum over* R*. Let* S *be a commutative* R*-algebra and* I *be a nilpotent ideal of* S*. Then the induced morphism* $\mathsf{K}^{\mathcal{A}}_i(\mathrm{S}) \xrightarrow{\cong} \mathsf{K}^{\mathcal{A}}_i(\mathrm{S}/\mathrm{I})$ *is an isomorphism for* $i \leqslant 0$*.*

Proof. By naturality of the fundamental exact sequence of twisted K-theory (see (Equation †) and the surrounding discussion at the beginning of Section 3), we can restrict the proof to $\mathsf{K}^{\mathcal{A}}_0$. By Lemma 4.4.1, we can assume $\mathcal{A}$ is a discrete R-algebra. Let $\varphi : \mathrm{S} \twoheadrightarrow \mathrm{S}/\mathrm{I}$ be the surjection. After $-\otimes_{\mathrm{R}}\mathcal{A}$ we have a surjection $(\ker\varphi)\otimes_{\mathrm{R}}\mathcal{A} \twoheadrightarrow \ker(\varphi\otimes_{\mathrm{R}}\mathcal{A})$. The nonunital ring $(\ker\varphi)\otimes_{\mathrm{R}}\mathcal{A}$ is nilpotent. So $\ker(\varphi\otimes_{\mathrm{R}}\mathcal{A})$ is nilpotent as well. The proposition follows from nil-invariance of K_0. □

A Zariski descent spectral sequence argument gives us a global result.

Corollary 4.4.3. *Let* X *be a quasi-compact quasi-separated scheme of finite Krull dimension* d *and* $\mathcal{A}$ *be an* $\mathbb{E}_1$*-algebra in* $\mathrm{Alg}^{\mathrm{cn}}_{\mathrm{X}}$*. The natural morphism* $f : \mathrm{X}_{\mathrm{red}} \to \mathrm{X}$ *induces isomorphisms*

$$\mathsf{K}^{f^*\mathcal{A}}_{-i}(\mathrm{X}_{\mathrm{red}}) \cong \mathsf{K}^{\mathcal{A}}_{-i}(\mathrm{X})$$

for $i \geqslant d$.

Proof. We have descent spectral sequences

$$E_2^{p,q} = H^p_{Zar}(X, \underline{K_q^{\mathcal{A}}}) \Rightarrow K^{\mathcal{A}}_{q-p}(X) \text{ and}$$

$$E_2^{p,q} = H^p_{Zar}(X, \underline{f_* K_q^{f^*(\mathcal{A})}}) \Rightarrow K^{f^*\mathcal{A}}_{q-p}(X_{red})$$

both with differential $d_2 = (2,1)$. We let $\underline{F}$ denote the Zariski sheafification of the presheaf F. The spectral sequences agree for $q \leqslant 0$ and vanish for $p > d$. □

In Theorem 6.1.4, we extend our main theorem across smooth affine morphisms. We will need reduction invariance in this setting.

Definition 4.4.4. For $f : S \to X$ a morphism of quasi-compact quasi-separated schemes and $\mathcal{A}$ an $\mathbb{E}_1$-algebra of Alg_X^{cn}, the *relative* $\mathcal{A}$*-twisted* K*-theory of* f is

$$K^{\mathcal{A}}(f) := \mathrm{fib}(K^{\mathcal{A}}(X) \xrightarrow{f^*} K^{\mathcal{A}}(S)).$$

As defined, $K^{\mathcal{A}}(f)$ is a spectrum. There is an associated presheaf of spectra on the base scheme X given by $U \mapsto K^{\mathcal{A}}(f_{|U})$. This presheaf sits in a fiber sequence

$$K^{\mathcal{A}}(f) \to K^{\mathcal{A}} \to K_S^{\mathcal{A}}$$

where the presheaf $K_S^{\mathcal{A}}$ is also defined by pullback along f. Both presheaves $K^{\mathcal{A}}$ and $K_S^{\mathcal{A}}$ satisfy Nisnevich descent and so $K^{\mathcal{A}}(f)$ does as well.

Corollary 4.4.5. *Let* $f : S \to X$ *be a smooth affine morphism of quasi-compact quasi-separated schemes with finite Krull dimension. Suppose* X *has Krull dimension* d*. Let* $\mathcal{A}$ *be an* $\mathbb{E}_1$*-algebra of* $\mathrm{Alg}_X^{\mathrm{cn}}$*. Then the commutative diagram*

$$\begin{tikzcd} S_{red} \arrow[r, "f_{red}"] \arrow[d] & X_{red} \arrow[d, "g"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

induces an isomorphism of relative twisted K*-theory groups*

$$K_{-i}^{g^*\mathcal{A}}(f_{red}) \cong K_{-i}^{\mathcal{A}}(f)$$

for $i \geqslant d + 1$.

Proof. We have two descent spectral sequences

$$E_2^{p,q} = H^p_{Zar}(X, \underline{K_q^{\mathcal{A}}(f)}) \Rightarrow K_{q-p}^{\mathcal{A}}(f)(X) \text{ and}$$

$$E_2^{p,q} = H^p_{Zar}(X, \underline{g_* K_q^{g^*\mathcal{A}}(f_{red})}) \Rightarrow K_{q-p}^{g^*\mathcal{A}}(f_{red})(X_{red})$$

with differential of degree $d = (2, 1)$ and $\underline{F}$ the Zariski sheafification of a presheaf F. For an open affine $\operatorname{Spec} U \to X$ with pullback $\operatorname{Spec} T \to S$ we examine the morphism of long exact sequences when $q \leqslant 0$

$$\begin{tikzcd}
\cdots \arrow[r] & K_q^{\mathcal{A}}(U) \arrow[r] \arrow[d, "\cong"] & K_q^{\mathcal{A}}(T) \arrow[r] \arrow[d, "\cong"] & K_{q-1}^{\mathcal{A}}(f) \arrow[r] \arrow[d] & K_{q-1}^{\mathcal{A}}(U) \arrow[r] \arrow[d, "\cong"] & K_{q-1}^{\mathcal{A}}(T) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & K_q^{\mathcal{A}}(U_{red}) \arrow[r] & K_q^{\mathcal{A}}(T_{red}) \arrow[r] & K_{q-1}^{\mathcal{A}}(f_{red}) \arrow[r] & K_{q-1}^{\mathcal{A}}(U_{red}) \arrow[r] & K_{q-1}^{\mathcal{A}}(T_{red}) \arrow[r] & \cdots
\end{tikzcd}$$

By the 5-lemma, this induces sheaf isomorphisms $\underline{g_* K_q^{g^*\mathcal{A}}(f_{red})} \cong \underline{K_q^{\mathcal{A}}(f)}$ for $q < 0$ and cohomology vanishes for $p > d$. $\square$

CHAPTER 5

BLOWING UP NEGATIVE TWISTED K-THEORY CLASSES

This chapter contains work which was published previously in Stapleton, J. Weibel's conjecture for twisted K-theory, Ann. of K-theory, 5(3): 621-637, 2020.

The term "blowing up" in the title is a misnomer for any proper birational morphism. Throughout this section, X is a quasi-separated quasi-compact scheme of finite Krull dimension d, and $\mathcal{A}$ is a sheaf of smooth proper connective $\mathbb{E}_1$-algebras on X. We drop a notational remark here. When we are truncating $\mathbb{E}_1$-algebras and working with their modules, we use π_* and drop the $\bullet$-notation of complexes. We reserve H_* and $\bullet$ for when we are specifically working with the underlying complex on the scheme.

5.1 Fundamental theorem for twisted K-theory

We first construct geometric cycles for negative twisted K-theory classes on X using a classical argument of Bass (see XII.7 of (2)) which works for a general additive invariant. We have an open cover

$$\begin{array}{ccc} X[t^{\pm}] & \xrightarrow{f} & X[t^{-}] \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle j} \\ X[t] & \xrightarrow{k} & \mathbb{P}^1_X. \end{array}$$

Since twisted K-theory satisfies Zariski descent, there is an associated Mayer-Vietoris sequence of homotopy groups

$$\cdots \longrightarrow \mathsf{K}^{\mathcal{A}}_{-n}(\mathbb{P}^1_X) \xrightarrow{(j^*k^*)} \mathsf{K}^{\mathcal{A}}_{-n}(X[t]) \oplus \mathsf{K}^{\mathcal{A}}_{-n}(X[t^-]) \xrightarrow{f^*-g^*} \mathsf{K}^{\mathcal{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} \mathsf{K}^{\mathcal{A}}_{-n-1}(\mathbb{P}^1_X) \longrightarrow \cdots .$$

As an additive invariant, $\mathsf{K}^{\mathcal{A}}(\mathbb{P}^1_X) \simeq \mathsf{K}^{\mathcal{A}}(X) \oplus \mathsf{K}^{\mathcal{A}}(X)$ splits as a $\mathsf{K}^{\mathcal{A}}(X)$-module with generators

$$[\mathcal{O} \otimes_{\mathcal{O}_X} \mathcal{A}] = [\mathcal{A}] \text{ and } [\mathcal{O}(1) \otimes_{\mathcal{O}_X} \mathcal{A}] = [\mathcal{A}(1)]$$

corresponding to the Beilinson semiorthogonal decomposition. Adjusting the generators to $[\mathcal{A}]$ and $[\mathcal{A}] - [\mathcal{A}(1)]$, we can identify the map (j^*, k^*) as it is a map of $\mathsf{K}^{\mathcal{A}}(X)$-modules. The second generator vanishes under each restriction. This identifies the map as

$$\mathsf{K}^{\mathcal{A}}(\mathbb{P}^1_X) \simeq \mathsf{K}^{\mathcal{A}}(X)[\mathcal{A}] \oplus \mathsf{K}^{\mathcal{A}}(X)([\mathcal{A}] - [\mathcal{A}(1)]) \xrightarrow{\Delta \oplus 0} \mathsf{K}^{\mathcal{A}}_{-n}(X[t]) \oplus \mathsf{K}^{\mathcal{A}}_{-n}(X[t^-])$$

with Δ the diagonal map corresponding to pulling back along the projections $X[t] \to X$ and $X[t^-] \to X$. As Δ is an embedding the long exact sequence splits as

$$0 \longrightarrow \mathsf{K}^{\mathcal{A}}_{-n}(X) \xrightarrow{\Delta} \mathsf{K}^{\mathcal{A}}_{-n}(X[t]) \oplus \mathsf{K}^{\mathcal{A}}_{-n}(X[t^-]) \xrightarrow{\pm} \mathsf{K}^{\mathcal{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} \mathsf{K}^{\mathcal{A}}_{-n-1}(X) \longrightarrow 0 . \tag{$\dagger$}$$

After iterating the complex

$$\mathsf{K}^{\mathcal{A}}_{-n}(X[t]) \to \mathsf{K}^{\mathcal{A}}_{-n}(X[t^{\pm}]) \twoheadrightarrow \mathsf{K}^{\mathcal{A}}_{-n-1}(X),$$

we can piece together a complex

$$K_0^{\mathcal{A}}(\mathbb{A}_X^{n+1}) \to K_0^{\mathcal{A}}(\mathbb{G}_{m,X}^{n+1}) \twoheadrightarrow K_{-n-1}^{\mathcal{A}}(X).$$

Negative twisted K-theory classes have geometric representations as twisted perfect modules on $\mathbb{G}_{m,X}^i$. There is even a sufficient geometric criterion implying a given representative is 0; it is the restriction of a twisted perfect module on $\mathbb{A}_X^i$.

5.2 Extensions across open immersions

We now provide basic results on extending algebras and modules across open immersions. We will want to retain quasi-coherence. For our purposes, the main example is $\mathbb{G}_{m,X}^n \to \mathbb{A}^n$.

Lemma 5.2.1. *Let* $j : U \to X$ *be an open immersion of quasi-compact quasi-separated schemes with finite Krull dimension. Let* $\mathcal{A}$ *be a sheaf of proper connective* $\mathbb{E}_1$*-algebras on* X *and* $j^*\mathcal{A}$ *its restriction. Let* $\mathcal{N}$ *be a* $j^*\mathcal{A}$*-module, discrete over* $\mathcal{O}_U$*, which is finitely generated as an* $\mathcal{O}_U$*-module. Then there exists a discrete* $\mathcal{A}$*-module* $\mathcal{M}$*, finitely generated over* $\mathcal{O}_X$*, such that* $j^*\mathcal{M} \cong \mathcal{N}$.

Proof. We induct on affine opens, so we assume $X = U \cup \operatorname{Spec} R$. By sheafification, we can reduce to an open $V := \operatorname{Spec} R \cap U$ of an affine $\operatorname{Spec} R$.

The $j^*\mathcal{A}$-module structure on $\mathcal{N}$ comes from forgetting along the truncation $j^*\mathcal{A} \to \pi_0(j^*\mathcal{A})$ and the natural $\pi_0(j^*\mathcal{A})$-module structure. Under restriction,

$$j^*\pi_0(\mathcal{A}) \cong \pi_0(j^*\mathcal{A}).$$

Using these reductions, let A be a finite R-algebra with $\mathcal{A} = \tilde{A}$. Let V be a quasi-compact open of Spec R, and N be a finitely generated left A_V-module. We have an isomorphism $N \cong j^* j_* N$ and $j_* N$ is has a left $\mathcal{A}$-module structure.

Cover our open V with finitely many open affines of the form $V = \cup_{i=1}^m \operatorname{Spec} R[1/f_i]$. We have a surjection for each i

$$A[1/f_i]^{n_i} \xrightarrow{\begin{pmatrix} x_{i1}/f_i^{\alpha_1} & \cdots & x_{in_i}/f_i^{\alpha_n} \end{pmatrix}} N[1/f_i]$$

with all x_{ij} elements of Spec R. Consider the finitely generated A left submodule generated by the image of the left A-module map

$$A^{\Sigma n_i} \xrightarrow{\begin{pmatrix} x_{11} & x_{21} & \cdots & x_{mn_m} \end{pmatrix}} N.$$

This restricts to N_V. $\square$

5.3 Blowing-up negative twisted K-classes

Proposition 5.3.1. *Let* $X = \operatorname{Spec} R$ *be a reduced affine scheme of finite Krull dimension and* $\mathcal{A}$ *a smooth proper connective* $\mathbb{E}_1$*-algebra on* R*. Let* $\gamma \in K_{-i}^{\mathcal{A}}(X)$ *for* $i > 0$*. Then there is a projective birational morphism* $\rho : \tilde{X} \to X$ *so that* $\rho^*\gamma = 0 \in K_{-i}^{\mathcal{A}}(\tilde{X})$*.*

Proof. We fix a diagram of schemes over X

$$\begin{tikzcd} \mathbb{G}^i_{m,X} \arrow[rr, "j"] \arrow[dr, "\pi_1"'] & & \mathbb{A}^i_X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}.$$

For any morphism $f : Y_1 \to Y_2$, we let $\tilde{f} : \mathbb{G}^i_{m,Y_1} \to \mathbb{G}^i_{m,Y_2}$ denote the pullback. Lift γ to a $K_0^{\mathcal{A}}(\mathbb{G}^i_{m,X})$-class $[P] - [Q]$, with P, Q some choice of $\pi_1^*\mathcal{A}$-twisted perfect modules on $\mathbb{G}^i_{m,X}$. Without loss of generality, we work with P.

The Induction Step:

We induct on the range of homotopy groups of P. As $\pi_1^*\mathcal{A}$ is a sheaf of proper $\mathbb{E}_1$-algebras, P forgets to a perfect complex $P_\bullet$ on $\mathbb{G}^i_{m,X}$ by Lemma 4.2.5. We may choose $P_\bullet$ to be strict perfect without changing the quasi-isomorphism class. After some (de)suspension, we may assume $P_\bullet$ is connective as this only alters the K_0-class by ± 1. For the lowest nontrivial differential of $P_\bullet$, d_1, we utilize part (iv) of Lemma 6.5 of (4) (with the morphism $\mathbb{G}^i_{m,X} \to X$) to construct a projective birational morphism $\rho : X_1 \to X$ so that $\operatorname{coker}(\tilde{\rho}^* d_1)$ $(= H_0(\tilde{\rho}^* P_\bullet))$ has tor-dimension $\leqslant 1$ over X_1. Consider the following distinguished triangle of $\tilde{\rho}^*\pi_1^*\mathcal{A}$-modules on $\mathbb{G}^i_{m,X_1}$, we drop the bullet notation as we are in the twisted setting now

$$F \to \tilde{\rho}^* P \to \pi_0(\tilde{\rho}^* P) \cong \operatorname{coker} \tilde{\rho}^* d_1.$$

In Lemma 5.3.2 below, we cover the base induction step, when the homotopy is concentrated in a single degree. Using this, construct a projective birational morphism $\phi : X_2 \to X_1$ such that $L\tilde{\phi}^*\pi_0(\tilde{\rho}^*P)$ is a twisted perfect module and is the restriction of a twisted perfect module from $\mathbb{A}^i_{X_2}$. By two out of three, $L\tilde{\phi}^*F$ is twisted perfect and $[\tilde{\phi}^*\tilde{\rho}^*P] = [L\tilde{\phi}^*F] + [L\tilde{\phi}^*\pi_0(\tilde{\rho}^*P)]$ in $K_0^{\mathcal{A}}(\mathbb{G}^i_{m,X_2})$. We then repeat the entire induction step with $L\tilde{\phi}^*F$.

We need to guarantee the induction will terminate, which is the purpose of the first projective birational morphism of each step. Since $\mathrm{coker}(\tilde{\rho}^*d_1)$ has tor-dimension $\leqslant 1$ over X_1,

$$L\tilde{\phi}^*\mathrm{coker}(\tilde{\rho}^*d_1) = \tilde{\phi}^*\mathrm{coker}(\tilde{\rho}^*d_1) \cong \mathrm{coker}(\tilde{\phi}^*\tilde{\rho}^*d_1).$$

The first equality guarantees $L\tilde{\phi}^*F$ will have no homotopy groups outside the original range of P. Both equivalences guarantee $\pi_0(L\tilde{\phi}^*F) = 0$, so the homotopy range of $L\tilde{\phi}^*F$ is strictly smaller than $\tilde{\phi}^*\tilde{\rho}^*P$.

At the end of induction, we will have decomposed the pullback of the twisted K_0-class $[\tilde{\rho}^*P]$ into a summand of n twisted K_0-classes, each of which vanishes when we map into negative K-theory. Proposition 5.3.1 follows from the next lemma. □

Lemma 5.3.2. *Let* X *be a quasi-compact quasi-separated scheme of finite Krull dimension which is quasi-projective over an affine scheme. Let* $\mathcal{A}$ *be a sheaf of smooth proper connective* $\mathbb{E}_1$*-algebras on* X*. Let* $\mathcal{N}$ *be a* $\pi_1^*\mathcal{A}$*-module, which is a coherent discrete sheaf on* $\mathbb{G}^i_{m,X}$*. Then there exists a blow-up* $\phi : \tilde{X} \to X$ *so that* $\tilde{\phi}^*\mathcal{N}$ *is perfect over* $\tilde{\phi}^*\pi_1^*\mathcal{A}$ *on* $\mathbb{G}_{m,\tilde{X}}$ *and is the restriction of a perfect module, twisted over the pullback of* $\mathcal{A}$*, to* $\mathbb{A}^i_{\tilde{X}}$*.*

Proof. Using Lemma 5.2.1, extend $\mathcal{N}$ from $\mathbb{G}^i_{m,X}$ to a $\pi_2^*\mathcal{A}$-module $\mathcal{M}$, coherent over $\mathbb{A}^i_X$. Using the ample family, choose a resolution in $\mathcal{O}_{\mathbb{A}^i_X}$-modules of the form

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{M} \to 0$$

where $\mathcal{F}$ is a vector bundle and $\mathcal{K}$ is the coherent kernel. As X is reduced, $\mathcal{K}$ is flat over some nonempty open set U of X. By platification par éclatement (see Theorem 5.2.2 of Raynaud–Gruson (33)), there is a U-admissable blow-up $\phi : \tilde{X} \to X$ so that the strict transform of $\mathcal{K}$ along the pullback morphism $p : \mathbb{A}^i_{\tilde{X}} \to \mathbb{A}^i_X$ is flat over $\tilde{X}$.

We now show the pullback $p^*\mathcal{M}$ is a perfect $p^*\pi_2^*\mathcal{A}$-module. Let $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\tilde{X}}$ be the inclusion of the open set and Z the closed complement. For any sheaf of modules $\mathcal{G}$ on $\mathbb{A}^i_{\tilde{X}}$, we let $\mathcal{G}_Z$ denote the subsheaf of sections supported on Z. We have a short exact sequence natural in $\mathcal{G}$

$$0 \to \mathcal{G}_Z \to \mathcal{G} \to j^{st}\mathcal{G} \to 0.$$

We also obtain the following exact sequence of sheaves of abelian groups via pullback

$$0 \to \mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}) \to p^*\mathcal{K} \to p^*\mathcal{F} \to p^*\mathcal{M} \to 0.$$

To make our notation clearer, we set $\mathcal{T} = \mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_X})$. We flesh both these exact sequences out into a (nonexact) commutative diagram of $p^{-1}\mathcal{O}_{\mathbb{A}^i_X}$-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{T}_Z \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & j^{st}\mathcal{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{K})_Z \arrow[r] \arrow[d] & p^*\mathcal{K} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{F})_Z \arrow[r] \arrow[d] & p^*\mathcal{F} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{M})_Z \arrow[r] \arrow[d] & p^*\mathcal{M} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}.$$

We observe that every row and the middle column is exact. The first map in the left column is an injection and the last map in the right column is a surjection. Since $p^*\mathcal{F}$ is flat, we have $(p^*\mathcal{F})_Z = 0$. This induces a lifting of the injection

$$\begin{tikzcd}
\mathcal{T}_Z \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\
(p^*\mathcal{K})_Z \arrow[r] \arrow[ur, dashed] & p^*\mathcal{K}
\end{tikzcd}.$$

We finish the proof by showing $j^*\mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}) = 0$. Since $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\tilde{X}}$ is flat, the sheaf is isomorphic to $\mathcal{T}or_1^{\mathbb{A}^i_U}(j^*p^{-1}\mathcal{M}, j^*\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}})$ and $j^*\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}} \cong \mathcal{O}_{\mathbb{A}^i_U}$. Our big diagram can be rewritten as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{T}_Z \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathcal{T} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{K})_Z \arrow[r] \arrow[d] & p^*\mathcal{K} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & p^*\mathcal{F} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{M})_Z \arrow[r] \arrow[d] & p^*\mathcal{M} \arrow[r] \arrow[d] & j^{st}p^*\mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and we can glue together to get a flat resolution of $p^*\mathcal{M}$ as an $\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}$-module

$$0 \to j^{st}p^*\mathcal{K} \to p^*\mathcal{F} \to p^*\mathcal{M} \to 0$$

implying globally finite Tor-amplitude. It remains to show the complex is pseudo-coherent. This follows since $\mathbb{A}^i_{\tilde{X}}$ is Noetherian and $p^*\mathcal{M}$ is coherent. Since $p^*\pi_2^*\mathcal{A}$ is a sheaf of smooth $\mathbb{E}_1$-algebras over $\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}$, the module $p^*\mathcal{M}$ is perfect over $p^*\pi_2^*\mathcal{A}$ by Lemma 4.2.5. By commutativity, $p^*\mathcal{M}$ restricts to $\tilde{\phi}^*\mathcal{N}$ on $\mathbb{G}^i_{m,\tilde{X}}$. This completes the proof of Proposition 5.3.1. $\square$

To extend our result across a smooth affine morphism, we will need a relative version of Proposition 5.3.1.

Corollary 5.3.3. *Let* $f : S \to X$ *be a smooth affine morphism of quasi-compact quasi-separated schemes of finite Krull dimension with* X *reduced and quasi-projective over a noetherian base ring. Let* $\mathcal{A}$ *be a sheaf of smooth proper connective* $\mathbb{E}_1$*-algebras over* X *and consider a negative twisted* K*-theory class* $\gamma \in K_i^{\mathcal{A}}(S)$ *for* $i < 0$*. Then there exists a projective birational morphism* $\rho : \tilde{X} \to X$ *such that, under the pullback of the pullback morphism,* $\rho_S^* \gamma = 0$*.*

Proof. We will briefly check that we can run the induction argument in the proof of Proposition 5.3.1. The assumptions of this corollary are invariant under pullback along projective birational morphisms $\tilde{X} \to X$. We need to ensure we can select projective birational morphisms to our base X. Lemma 6.5 of Kerz–Strunk–Tamme (4) is stated in a relative setting. The proof also relies on platification par éclatement. This can still be applied in our relative setting as X is reduced (see Proposition 5 of Kerz–Strunk (34)). □

CHAPTER 6

THE MAIN THEOREM

This chapter contains work which was published previously in Stapleton, J. Weibel's conjecture for twisted K-theory, Ann. of K-theory, 5(3): 621-637, 2020.

6.1 Twisted Weibel's conjecture

We now prove Theorem 6.1.2 and an extension across a smooth affine morphism. We begin with the base induction step for both theorems. Kerz–Strunk (34), using a result of Grothendieck and a spectral sequence argument, reduce $-\mathsf{d}$-vanishing of a Zariski sheaf of spectra on a d-dimensional quasi-compact quasi-separated scheme to the equivalent vanishing statement on each of the local rings.

Proposition 6.1.1. *Let* R *be a regular noetherian ring of Krull dimension* d *over a local Artinian ring* k*. Let* $\mathcal{A}$ *be a smooth proper connective* $\mathbb{E}_1$*-algebra over* R*, then* $\mathsf{K}_i^{\mathcal{A}}(\mathrm{R}) = 0$ *for* $i < 0$*.*

Proof. By Proposition 4.4.5, we may assume k is a field. Proposition A.4 of (10) shows that the t-structure on $\mathsf{D}(\mathcal{A})$ restricts to a t-structure on $\mathrm{Perf}(\mathcal{A})$, which is observably bounded. The heart is the category of modules over $\pi_0(\mathcal{A})$. As $\pi_0(\mathcal{A})$ is finite-dimensional over k, this is a noetherian abelian category. By Theorem 1.2 of Antieau–Gepner–Heller (35)), the negative K-theory vanishes. □

Theorem 6.1.2. *Let* X *be a quasi-compact quasi-separated scheme of Krull dimension* d *and* $\mathcal{A}$ *a sheaf of smooth proper connective* $\mathbb{E}_1$*-algebras on* X*, then* $K^{\mathcal{A}}_{-i}(X)$ *vanishes for* $i > d$.

Proof. Proposition 6.1.1 covers the base case so assume $d > 0$. By the Kerz–Strunk spectral sequence argument and Proposition 4.4.3, we may assume X is a noetherian reduced affine scheme.

Choose a negative $K^{\mathcal{A}}$-theory class $\gamma \in K^{\mathcal{A}}_{-i}(X)$ for $i \geqslant \dim X + 1$. Using Proposition 5.3.1, construct a projective birational morphism that kills γ and extend it to an abstract blow-up square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} .$$

By (5, Theorem A.8), there is a Mayer-Vietoris exact sequence of pro-groups

$$\cdots \longrightarrow \{K^{\mathcal{A}}_{-i+1}(E_n)\} \longrightarrow K^{\mathcal{A}}_{-i}(X) \longrightarrow K^{\mathcal{A}}_{-i}(\tilde{X}) \oplus \{K^{\mathcal{A}}_{-i}(Y_n)\} \longrightarrow \{K^{\mathcal{A}}_{-i}(E_n)\} \longrightarrow \cdots .$$

When $i \geqslant \dim X + 1$, by induction every nonconstant pro-group vanishes and $K^{\mathcal{A}}_{-i}(X) \cong K^{\mathcal{A}}_{-i}(\tilde{X})$ showing $\gamma = 0$. ☐

By (9, Theorem 3.15), we recover Weibel's vanishing for discrete Azumaya algebras.

Corollary 6.1.3. *For* X *a quasi-compact quasi-separated scheme of Krull dimension* d *and* $\mathcal{A}$ *a sheaf of discrete Azumaya algebras,* $K^{\mathcal{A}}_{-i}(X) = 0$ *for* $i > d$.

The next result nearly covers the K-regularity portion of Weibel's conjecture, but we are missing the boundary case $K^{\mathcal{A}}_{-d}(X) \cong K^{\mathcal{A}}_{-d}(\mathbb{A}^n_X)$.

Theorem 6.1.4. *Let* $f : S \to X$ *be a smooth affine morphism of quasi-compact quasi-separated schemes of finite Krull dimension and* $\mathcal{A}$ *a sheaf of smooth proper connective* $\mathbb{E}_1$*-algebras on* X*. Then* $K^{\mathcal{A}}_{-i}(f) = 0$ *for* $i > \dim X + 1$.

Proof. The base case is covered by Proposition 6.1.1 and our reductions are analagous to those in the proof of Theorem 6.1.2. So assume X is a reduced affine scheme of dimension d. Choose $\gamma \in K^{\mathcal{A}}_{-i}(S)$ with $i \geqslant d$. Using Corollary 5.3.3, construct a projective birational morphism $\rho : \tilde{X} \to X$ that kills γ. We then build a morphism of abstract blow-up squares

$$\begin{tikzcd}
D \arrow[rr] \arrow[dr] \arrow[dd] & & \tilde{S} \arrow[dr] \arrow[dd] & \\
& E \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
V \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\
& Y \arrow[rr] & & X
\end{tikzcd}$$

By Theorem 3.3.6, we again get a long exact sequence of pro-groups corresponding to the back square

$$\cdots \longrightarrow \{K^{\mathcal{A}}_{-i+1}(D_n)\} \longrightarrow K^{\mathcal{A}}_{-i}(S) \longrightarrow K^{\mathcal{A}}_{-i}(\tilde{S}) \oplus \{K^{\mathcal{A}}_{-i}(V_n)\} \longrightarrow \{K^{\mathcal{A}}_{-i}(D_n)\} \longrightarrow \cdots .$$

When $i \geqslant \dim X + 1$, every nonconstant pro-group vanishes by induction and we have an isomorphism $K^{\mathcal{A}}_{-i}(S) \cong K^{\mathcal{A}}_{-i}(\tilde{S})$ implying $\gamma = 0$. □

6.2 Formal bounds on the main theorem

The conditions on the morphism in Corollary 5.3.3 are more general than those of Theorem 6.1.4. We might hope to generalize Theorem 6.1.4 to a smooth quasi-projective or smooth projective map of noetherian schemes. Although the induction step is present, both base cases fail. Consider the descent spectral sequence

$$E_2^{p,q} := H^p(X, \underline{K_q}) \Rightarrow K_{q-p}(X) \text{ with } d_2 = (2,1)$$

If $\dim X \leqslant 3$, then

$$E_3^{2,1} = E_\infty^{2,1} = \mathrm{coker}(H^0(X,\mathbb{Z}) \xrightarrow{d_2} H^2(X, \mathcal{O}_X^*))$$

contributes to $K_{-1}(X)$. The differential is zero as the edge morphism

$$K_0(X) \xrightarrow{\text{rank}} E_\infty^{0,0}$$

identifies $E_\infty^{0,0}$ with the rank component of K_0, implying $E_2^{0,0} = E_\infty^{0,0}$. We now construct a family of examples for schemes X with nontrivial $H^2(X, \mathcal{O}_X^*)$. Let X_{red} be quasi-projective smooth over a field k and form the cartesian diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X_{red} \arrow[d] \\ \mathrm{Spec}(k[t]/(t^2)) \arrow[r] & \mathrm{Spec}\, k \end{tikzcd}.$$

The pullback X will be our counterexample. We have an isomorphism

$$\mathcal{O}_X^* \cong g_*(\mathcal{O}_{X_{red}}^*) \oplus g_*(\mathcal{O}_{X_{red}})$$

of sheaves of abelian groups on X with $g : X_{red} \to X$ the pullback of the reduction morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2)$. Locally, $(R[t]/(t^2))^\times$ consists of all elements of the form $u + v \cdot t$ where $u \in R^\times$ and $v \in R$. Sheaf cohomology commutes with coproducts so this turns into an isomorphism

$$H^2(X, \mathcal{O}_X^*) \cong H^2(X, g_*(\mathcal{O}_{X_{red}}^*)) \oplus H^2(X, g_*(\mathcal{O}_{X_{red}})) \cong H^2(X_{red}, \mathcal{O}_{X_{red}}^*) \oplus H^2(X_{red}, \mathcal{O}_{X_{red}}).$$

Now the problem reduces to finding a surface or 3-fold X_{red} with nontrivial degree 2 sheaf cohomology. Take a smooth quartic in $\mathbb{P}_k^3$ for a counterexample which is smooth and proper. Here is a counterexample which is smooth and quasi-affine. Let $(A, \mathfrak{m})$ be a 3-dimensional local ring which is smooth over a field k. Take $X = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ to be the punctured spectrum. Then $H^2(X, \mathcal{O}_X) \cong H_{\mathfrak{m}}^3(A)$, which is the injective hull of the residue field $A/\mathfrak{m}$.

Appendix

On the following page, Figure 1 is a screenshot of the "Policies for Authors" page, with url `https://msp.org/publications/policies/`. As of August 22, 2021, the about page of the Annals of K-theory journal, `https://msp.org/akt/about/journal/about.html`, points to the previous url under a button labelled "Policies for Authors".

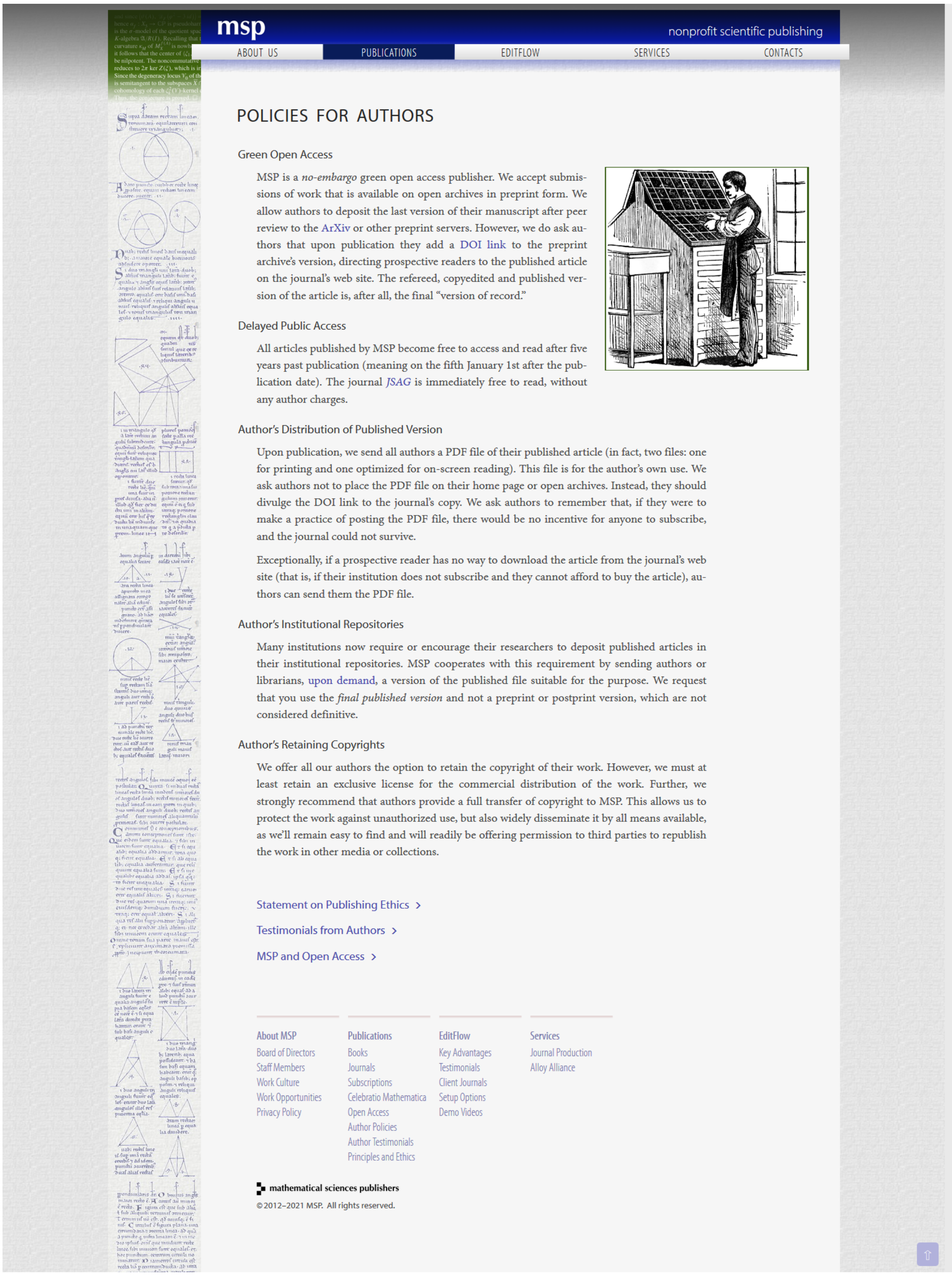
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Figure 1: A screenshot of the webpage `https://msp.org/publications/policies/`.

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